SICP 1.13 Solution

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https://github.com/jixianyan/SICP-Solutions

Problem:

$$Fib(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ Fib(n-1) + Fib(n-2) & \text{otherwise} \end{cases}$$

Prove that $Fib(n)$ is the closest integer to $\varphi^n/\sqrt{5}$, where $\varphi = (1+\sqrt{5})/2$. Hint: Let $\psi = (1-\sqrt{5})/2$. Use induction and the definition of the Fibonacci numbers(see Section 1.2.2) to prove that $Fib(n) = (\varphi^n - \psi^n)/\sqrt{5}$.

Proof:

First, prove that

$$Fib(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}} \tag{1}$$

Assume that

$$Fib(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}} \tag{2}$$

Then

$$\begin{aligned} Fib(n+1) &= \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}} \\ &= \frac{\varphi^n \cdot \varphi - \psi^n \cdot \psi}{\sqrt{5}} \\ &= \frac{\varphi^n \cdot \frac{1+\sqrt{5}}{2} - \psi^n \cdot \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \\ &= \frac{1}{2} \cdot \left(\frac{\varphi^n - \psi^n + \sqrt{5} \cdot (\varphi^n + \psi^n)}{\sqrt{5}} \right) \\ &= \frac{1}{2} \cdot Fib(n) + \frac{\varphi^n + \psi^n}{2} \end{aligned} \tag{3}$$

$$\begin{aligned} Fib(n+2) &= \frac{\varphi^{n+2} - \psi^{n+2}}{\sqrt{5}} \\ &= \frac{\varphi^n \cdot \varphi^2 - \psi^n \cdot \psi^2}{\sqrt{5}} \\ &= \frac{\varphi^n \cdot (\frac{1+\sqrt{5}}{2})^2 - \psi^n \cdot (\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}} \\ &= \frac{\varphi^n \cdot \frac{1+5+2\sqrt{5}}{4} - \psi^n \cdot \frac{1+5-2\sqrt{5}}{4}}{\sqrt{5}} \\ &= \frac{1}{2} \left(\frac{3\,(\varphi^n - \psi^n) + \sqrt{5}\,(\varphi^n + \psi^n)}{\sqrt{5}} \right) \\ &= \frac{3}{2} \cdot Fib(n) + \frac{\varphi^n + \psi^n}{2} \end{aligned} \tag{4}$$

It can be proved that

$$Fib(n+2) = Fib(n+1) + Fib(n) \tag{5}$$

And because

$$\frac{\varphi^0 - \psi^0}{\sqrt{5}} = 0 = Fib\,(0) \tag{6}$$

$$\frac{\varphi^1 - \psi^1}{\sqrt{5}} = 1 = Fib\,(1) \tag{7}$$

From this it is clear that (1) is true.
Thus $Fib(n)$ can spilt into the form of the difference of two numbers:

$$Fib\,(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}} = \frac{\varphi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}} \tag{8}$$

And

$$\frac{1}{\sqrt{5}} < \frac{1}{2}\,|\psi| = \left|\frac{1-\sqrt{5}}{2}\right| < 1 \tag{9}$$

Therefore,

$$\left|\frac{\psi^n}{\sqrt{5}}\right| < \frac{1}{2} \tag{10}$$

i.e.

$$\left|Fib(n) - \frac{\varphi^n}{\sqrt{5}}\right| = \frac{\psi^n}{\sqrt{5}} < \frac{1}{2} \tag{11}$$

Therefore, $Fib(n)$ is the closest integer to $\varphi^n/\sqrt{5}$.

Q.E.D