Set Theory and Relations Functions

UNIT-1

* **Set :-** A set is any well defined collection of distinct or distinguisable elements by a well defined collection means that there exists a rule with the help of which we should be able to say whether any given object belongs to or not to the collection under some specific rule.

The set of students in a class.
The set of months in a year.
$A = \{1, 2, 3\}$.

Generally capital letters A, B, C, X, Y, Z etc are used to denote sets and its elements by lowercase letters a, b, c, x, y, z - - etc.

* **Elements of a set :-**

If $A = \{1, 2, 3\}$ the objects in a set are called elements or members.
The distinct elements means no element is repeated.
The Distinguishable means that given any object or element is either in the set or not in the set. so the elements of a set must be distinct and distinguishable.

The symbol $\in$ (Epsilon) is used to indicate "belongs to" if 1 is the element of set A then symbolically we write

$$1 \in A$$

The symbol $\notin$ is used to indicate "does not belongs to" if 4 is not an element of set A then symbolically we write -

$$4 \notin A$$

* **Symbols :-**

N = Natural Numbers $= \{1, 2, 3, \ldots\}$
I or Z = set of Integers $= \{\ldots -2, -1, 0, 1, 2, \ldots\}$
Z^+ = set of Positive Integers $= \{0, 1, 2, \ldots\}$
Z^- = set of Negative Integers $= \{\ldots, -2, -1\}$.
R = set of Real Numbers.
Q = Set of Rational Numbers (where $\frac{p}{q}$, $q \neq 0$)
C = set of Complex Numbers.

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* Set Representation :-

Set are represented by two forms:-

- Roster or Enumeration or Tabular form.
- Set builder form.

- **Roster form** :- In this method the elements of a set are described by listing or written them within braces seperated by comma (,).

E.g. The set of vowel in English alphabet can be written as

$V = \{a, e, i, o, u\}$

The set of Natural Number.

$N = \{1, 2, 3, ----\}$.

- **Set builder form** :- In this method the elements of a set are represented by stating a property that uniquely characterised them.

E.g. Let consider the sets

$A = \{1, 2, 3\}$ $B = \{2, 4, 6, 8 ---\}$ $C = \{a, e, i, o, u\}$

$A = \{x : x \in N \text{ \& } x < 4\}$.

$B = \{x : x = 2n \text{ \& } n \in N\}$

OR

$= \{x : x$ is an even natural number$\}$.

OR

$= \{x : x$ is an even positive integer$\}$.

$C = \{x : x$ is a vowel in English alphabet$\}$.

* Types of Set :-

- **Cardinality of set (OR) Cardinal No. of set** :-

The number of distinct elements in a set is called cardinal no. of set.

The number of distinct elements in set A is called cardinal no. of set it is represented as -

$O(A)$ (OR) $n(A)$ (OR) $|A|$.

$A = \{1, 2, 3\}$

$\boxed{n(A) = 3.}$

- **Finite Set** :- A set is finite if it contains finite no. of

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The set of rivers in a India.

- Infinite set :- A set which is not finite is called Infinite set.

  e.g. set of Natural Numbers.

- Singelton set :- A set which contains only 1 element is called singelton set.

  e.g. $A = \{ x : 4 < x < 6 \ \& \ x \in N \}$

  $A = \{ \phi \}$.

- Null set (OR) Empty (OR) Void set :- A set which contain no or zero elements is called Null set.

  This set is denoted by ϕ (OR) { }.

  Cardinality of null set is always zero.

- Equality of sets (OR) Equal sets :- Two sets A and B are said to be equal if every element of set A is an element of B and also every element of set B is an element of A. The equality of two sets A and B are denoted by -

  $\boxed{A = B}$

  symbolically, $A = B$ if and only if $\boxed{x \in A \iff x \in B}$

- Equivalence set :- If the elements of 1 set can be put for one to one maping with the elements of another set then two sets are called equivalence set.

  It is denoted by $\sim$ (OR) $\equiv$ (OR) $\cong$.

  (OR)

  If the elements of sets are same then it is known as equivalent set

* Subset :- Let A and B be <u>two non-empty sets</u>. The set A is subset of B if and only if every element of set A is an element of set B.

Subset is denoted by " $\subseteq$ ".

Symbolically, $A \subseteq B$ iff $x \in A \implies x \in B$.

e.g. $A = \{1, 2\}$

$B = \{1, 2, 3\}$.

Here, $A \subseteq B$

A is contained in B.

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* **SuperSet :-** It is denoted by "$\supseteq$"
  If A is subset of B, then, B is superset of A.
  $B \supseteq A$.

* **Properties of Subset :-**
  - ϕ is the subset of every set.
  - If $A \subseteq B$ & $B \subseteq A$, then $A = B$.
  - If $A \subseteq B$ & $B \subseteq C$ then $A \subseteq C$. (A is also subset of C).

* **Power set :-** The set of all possible subsets of set A is called Power set of set A. Denoted by "P(A)".

e.g. $A = \{1, 2\}$

$P(A) = \{\{\phi, \{1\}, \{2\}, \{1,2\}\}$.

$A = \{1, 2, 3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}; \{1,2\}, \{1,3\}; \{2,3\}, \{1,2,3\}\}$.

<u>NOTE</u> :- If a set has 'n' elements then '2^n' subsets of that set are possible

* <u>Proof</u> :- Let Set A has n elements
  So, no. of subsets taking one element = $^nC_1$
  No. of subsets taking two element = $^nC_2$
  No. of subsets taking three element = $^nC_3$.
  " " " "
  " " " "
  No. of subsets taking n element = ^nC_n.

  We know that ϕ is subset of every set.
  So. Total no. of subsets

$$= 1 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n.$$
$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n.$$
$$= (1+1)^n$$
$$= (2)^n.$$

<u>Prob</u>. Find P(A) for the following sets.

1) $A = \{a\}$. $P(A) = \{\phi, \{a\}\}$
2) $A = \{\phi, \{\phi\}\}$. $P(A) = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}, \{\{\phi\}\}\}$
3) $A = \{\phi\}$. $P(A) = \{\phi, \{\phi\}\}$
4) $A = \{\{a\}\}$ $P(A) = \{\phi, \{\{a\}\}\}$
5) $A = \phi$ $P(A) = \{\phi\}$

- **Union :-** Let A and B be two ~~sets~~ non-empty sets the union of A and B is the set of all elements which are either in A or in B or in both A and B. and the union of A and B is denoted by '$A \cup B$'. It is also known as Joint or Logical sum of A and B. Symbolically,

$$A \cup B = \{ x : x \in A \text{ or } x \in B \}.$$

e.g. $A = \{1, 2, 3\}$ $\quad B = \{1, 2, 3, 4, 5\}$.

$A \cup B = \{1, 2, 3, 4, 5\}$.

Properties of Union of Set :-

★ **Commutative Property :-** Union of sets is commutative i.e. $A \cup B = B \cup A$.

Proof : Let $x \in (A \cup B)$

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in (B \cup A)$

So. $(A \cup B) \subseteq (B \cup A)$ —— (1).

Let $x \in (B \cup A)$

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in (A \cup B)$

So, $(B \cup A) \subseteq (A \cup B)$ —— (2)

from eq^n (1) & (2)

$\boxed{A \cup B = B \cup A}$

★ **Associative Property :-** Union of sets is Associative. i.e $A \cup (B \cup C) = (A \cup B) \cup C$.

Proof :- Let $x \in (A \cup (B \cup C))$

$\Rightarrow x \in A$ or $x \in (B \cup C)$

$\Rightarrow x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow x \in (A \cup B)$ or $x \in C$

$\Rightarrow x \in ((A \cup B) \cup C)$

So, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ —— (1)

Let $x \in ((A \cup B) \cup C)$

$\Rightarrow x \in (A \cup B)$ or $x \in C$.

$\Rightarrow (x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow x \in A$ or $x \in B \cup C$

$\Rightarrow x \in A \cup (B \cup C)$

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So, $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ —— (2)

From eqn (1) & (2),

$$\boxed{A \cup (B \cup C) = (A \cup B) \cup C}$$

★ Idempotent Property :- Union of sets is idempotent i.e. $A \cup A = A$.

Proof :- Let, $x \in A \cup A$

$\Rightarrow x \in A$ or $x \in A$

$\Rightarrow x \in A$.

So, $(A \cup A) \subseteq A$ —— (1)

Let $x \in A$.

$\Rightarrow x \in A$ or $x \in A$.

$\Rightarrow x \in (A \cup A)$

So, $A \subseteq (A \cup A)$ —— (2)

From eqn (1) & (2) $\boxed{A \cup A = A}$

★ Let A and B be two non-empty sets.

$A = \{1, 2, 3\}$

$B = \{1, 2, 4, 5\}$

$A \cup B = \{1, 2, 3, 4, 5\}$.

$\Rightarrow A \subseteq (A \cup B)$

$\Rightarrow B \subseteq (A \cup B)$

★ $A \cup \phi = A$

$A \cup U = U$

★ If $A \subseteq B$ then $A \cup B = B$.

Let $x \in A \cup B$

$\Rightarrow x \in A$ or $x \in B$.

$\Rightarrow x \in B$ or $x \in B$ ($\because A \subseteq B$ then $x \in A \Rightarrow x \in B$)

$\Rightarrow x \in B$.

So, $A \cup B \subseteq B$ —— (1)

We know that, $B \subseteq A \cup B$. —— (2)

From (1) & (2)

$$\boxed{A \cup B = B}$$

e.g. $A = \{1, 2\}$

$B = \{1, 2, 3\}$

is the set of elements which belongs to both A and B.
(common to both A and B).
The intersection of A and B is denoted by '$A \cap B$.'
Symbolically,
$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

e.g. $A = \{1, 2, 3\}$
$B = \{1, 3, 5\}$.
$A \cap B = \{1, 3\}$.

Properties of Intersection of Sets :-

Commutative Property :- Intersection of sets is Commutative.
i.e. $A \cap B = B \cap A$.

Proof :- Let $x \in A \cap B$
$\Rightarrow x \in A$ and $x \in B$.
$\Rightarrow x \in B$ and $x \in A$.
$\Rightarrow x \in (B \cap A)$
So, $A \cap B \subseteq B \cap A$. —— (1)

Let $x \in B \cap A$
$\Rightarrow x \in B$ and $x \in A$
$\Rightarrow x \in A$ and $x \in B$
$\Rightarrow x \in (A \cap B)$
So, $B \cap A \subseteq A \cap B$ —— (2)

From eqn (1) & (2) -
$$\boxed{A \cap B = B \cap A}$$

Associative Property :- Intersection of sets is Associative.
i.e. $A \cap (B \cap C) = (A \cap B) \cap C$.

Proof :- Let $x \in A \cap (B \cap C)$
$\Rightarrow x \in A$ and $x \in B \cap C$
$\Rightarrow x \in A$ and ($x \in B$ and $x \in C$)
$\Rightarrow x \in A \cap B$ and $x \in C$.
$\Rightarrow x \in (A \cap B) \cap C$.
So, $A \cap (B \cap C) \subseteq (A \cap B) \cap C$. —— (1)

Let $x \in (A \cap B) \cap C$.
$\Rightarrow x \in (A \cap B)$ and $x \in C$.
$\Rightarrow$ ($x \in A$ and $x \in B$) and $x \in C$.
$\Rightarrow x \in A$ and $x \in (B \cap C)$
$\Rightarrow x \in A \cap (B \cap C)$
So $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ —— (2).

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From eq^n 1 & 2 –

$$\boxed{A \cap (B \cap C) = (A \cap B) \cap C}$$

* Idempotent Property :- Intersection of sets is Idempotent
  i.e. $A \cap A = A$.

Proof :- Let $x \in A \cap A$

$\Rightarrow x \in A$ and $x \in A$.

$\Rightarrow x \in A$.

So, $A \cap A \subseteq A$ —— (1)

Let $x \in A$.

$\Rightarrow x \in A$ and $x \in A$.

$\Rightarrow x \in (A \cap A)$

So, $A \subseteq A \cap A$ —— (2)

From eq^n (1) & (2)

$$\boxed{A \cap A = A}$$

* $A = \{1, 2, 3\}$
  $B = \{1, 3, 5\}$.

$A \cap B = \{1, 3\}$.

$\Rightarrow A \cap B \subseteq A$

$\Rightarrow A \cap B \subseteq B$.

* $A \cap \phi = \phi$

  $A \cap U = A$.

NOTE: $A \cap B \subseteq A \cup B$.

- Compliment of Sets :- Let U be the universal set and A be any subset of universal set the compliment of A is a set containing elements of universal set which do not belong to A.

The compliment of A is denoted by A' (OR) A^c (OR) $\bar{A}$.

e.g. $U = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 4, 6\}$.

$A' = \{2, 3, 5\}$.

Symbolically, $A' = \{x : x \in U$ and $x \notin A\}$.

NOTE: $x \in A^c$ (OR) $x \notin A$ are same.

those elements which belongs to A but does not belongs to B.
It is denoted by '$A-B$' (or) 'A/B'.
Symbolically,

$$A-B=\{x : x\in A \text{ and } x\notin B\}$$

e.g. $A=\{1,2,3\}$
$B=\{2,3,4,5\}$.
$A-B=\{1\}$
$B-A=\{4,5\}$.

NOTE:- $A-B = A-(A\cap B)$
$B-A = B-(A\cap B)$
$\boxed{A-B \neq B-A}$

* Symmetric Difference of sets :- Let A and B be two sets then symmetric difference of A and B is a set containing all the elements that belong to A or B but not both.
It is denoted by '$A\oplus B$'.
e.g. $A=\{1,2,3,4,5\}$
$B=\{4,5,6\}$
$A\oplus B=\{1,2,3,6\}$.

$A\oplus B=(A\cup B)-(A\cap B)$
$A\cup B=\{1,2,3,4,5,6\}$
$A\cap B=\{4,5\}$
$A\oplus B=(A\cup B)-(A\cap B)=\{1,2,3,6\}$.

* Disjoint sets :- Let A and B be two sets then, if there is no common element b/w A and B then, they are said to be disjoint sets.
e.g. $A=\{1,2,3\}$
$B=\{4,5,6\}$.

If $A\cap B=\Phi$ Then A and B are disjoint sets.

* Proper Subset :- Let A and B be two non-empty sets and $A\subseteq B$ if there is atleast one element in B which does not belong to A then, A is called proper subset of B and it is denoted by '$\subset$' ($A\subset B$)

* Venn Diagram :- Venn Diagram is a pictorial representation of sets which are used to show relationships ~~b/w two~~ among sets.
The universal set is represented by interior of a rectangle and its subsets are represented by circular areas within the rectangle.

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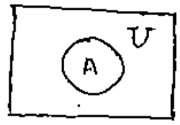
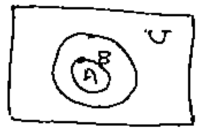
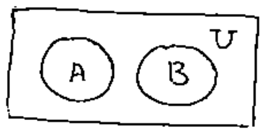
$A \subseteq U$



$A \subseteq B$

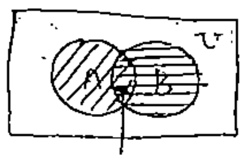


Disjoint sets

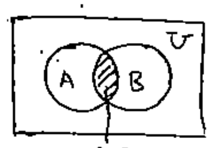


$A \cap B \neq \phi$

$A \cup B$
$A \cap B$.

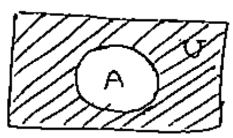


$A \cap B$



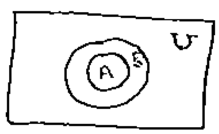
$A \cap B$

A^c



$A - B$

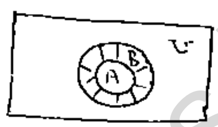
$A \subseteq B$



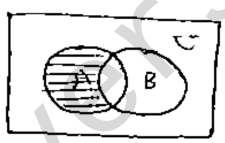
No shaded area means
So, $A - B = \phi$.

$B - A$.

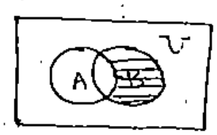
$A \subseteq B$



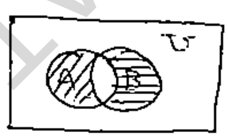
$A - B$



$B - A$



$A \oplus B$

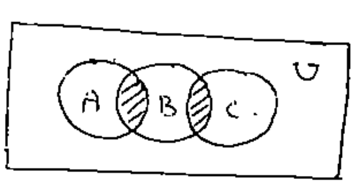


NOTE:

$\boxed{A \oplus B = (A - B) \cup (B - A)}$

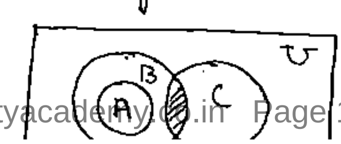
Q. Draw a venn Diagram of sets A, B & C where - A and B have common elements, B and C have common elements but A and C are disjoint.

Soln



Q A ⊆ B, set A and C are disjoint but B and C have elements in common.

Soln.



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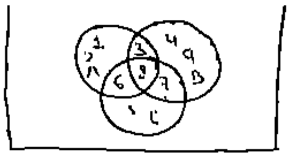

1) $A - (B \cap C)$
2) $(A \cup B) - C$
3) $B \oplus C$
4) $A - (B - C)$
5) $A - (B \oplus C)$
6) $A \cap (B \oplus C)$

$A = \{1, 2, 6, 3, 8$

$B = \{3, 4, 7, 8, 9$

$C = \{5, 6, 7, 8\}$.

$B \cap C = \{7, 8\}$

$A \cup B = \{1, 2, 3, 4, 6$

$B \oplus C = (B \cup C) - ($
or $(B - C) \cup ($

$B - C = \{3, 4, 9\}$

1) $A - (B \cap C) = \{1, 2, 3, 8, 6\}$
2) $(A \cup B) - C = \{1, 2, 3, 4, 9\}$
3) $(B \oplus C) = \{6, 3, 4, 5, 9\}$
4) $A - (B - C) = \{1, 2, 6, 8\}$.
5) $A - (B \oplus C) = \{1, 2, 8\}$.
6) $A \cap (B \oplus C) = \{6, 3\}$.

* Algebra of Sets :-

- Idempotent Law :- 1) $A \cup A = A$
  2) $A \cap A = A$.
- Commutative Law :- 1) $A \cup B = B \cup A$
  2) $A \cap B = B \cap A$.
- Associative Law :- 1) $A \cup (B \cup C) = (A \cup B) \cup C$.
  2) $A \cap (B \cap C) = (A \cap B) \cap C$.
- Identity Law :- 1) $A \cup \phi = A$
  2) $A \cap U = A$.
- Bound Law :- 1) $A \cup U = U$
  2) $A \cap \phi = \phi$.
- Compliment Law :- 1) $U^c = \phi$
  2) $\phi^c = U$
  3) $A \cup A^c = U$
  4) $A \cap A^c = \phi$.
- Involution Law :- $(A^c)^c = A$.
- Distributive Law :-

Q Prove that union of sets is distributive over intersect of sets and vice-versa.

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Prove :- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof :- Let $x \in (A \cup (B \cap C))$

$\Rightarrow x \in A$ or $x \in (B \cap C)$

$\Rightarrow x \in A$ or $x \in B$ and $x \in C$.

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$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$

$\Rightarrow (x \in A \cup B) \text{ and } (x \in A \cup C)$.

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$ —— ①

So, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Let, $x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C)$

$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$

$\Rightarrow x \in A \text{ or } x \in B \text{ and } x \in C$.

$\Rightarrow x \in A \text{ or } x \in B \cap C$.

$\Rightarrow x \in A \cup (B \cap C)$

So, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ —— ②

from eqⁿ ① & ②

$\Rightarrow \boxed{A \cup (B \cap C) = (A \cup B) \cap (A \cup C).}$

Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: Let $x \in A \cap (B \cup C)$.

$\Rightarrow x \in A \text{ and } x \in (B \cup C)$

$\Rightarrow x \in A \text{ and } x \in B \text{ or } x \in C$.

$\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$

$\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C)$.

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

So, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ —— ①

Let, $(A \cap B) \cup (A \cap C) \ni x$.

$\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C)$

$\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$

$\Rightarrow x \in A \text{ and } (x \in B \text{ or } x \in C)$

$\Rightarrow x \in A \text{ and } x \in (B \cup C)$

$\Rightarrow x \in A \cap (B \cup C)$

So, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ —— ②

from eqⁿ ① & ②

$\Rightarrow \boxed{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

DeMorgan's Law :- 1) $(A \cup B)^c = A^c \cap B^c$

2) $(A \cap B)^c = A^c \cup B^c$

<u>Prove:-</u> $(A \cup B)^c = A^c \cap B^c$

<u>Proof:-</u> Let $x \in (A \cup B)^c$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ **and** $x \notin B$

$\Rightarrow x \in A^c$ and $x \in B^c$

$\Rightarrow x \in (A^c \cap B^c)$

So, $(A \cup B)^c \subseteq (A^c \cap B^c)$ — 1

$\{ x \in A^c$, $x \notin A \}$.

{ In negativity for union we use [illegible] & for intersection we [illegible] }

Let, $x \in A^c \cap B^c$

$\Rightarrow x \in A^c$ and $x \in B^c$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \in (A \cup B)^c$

So, $(A^c \cap B^c) \subseteq (A \cup B)^c$ — 2

From eqn 1 & 2 –

$$\Rightarrow \boxed{(A \cup B)^c = (A^c \cap B^c)}$$

<u>Prove:</u> $(A \cap B)^c = A^c \cup B^c$

<u>Proof:</u> Let $x \in (A \cap B)^c$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A^c$ or $x \in B^c$

$\Rightarrow x \in A^c \cup B^c$

So, $(A \cap B)^c \subseteq (A^c \cup B^c)$ — 1

Let, $x \in A^c \cup B^c$

$\Rightarrow x \in A^c$ or $x \in B^c$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \in (A \cap B)^c$

So, $(A^c \cup B^c) \subseteq (A \cap B)^c$ — 2

From eqn 1 & 2

$$\Rightarrow \boxed{(A \cap B)^c = A^c \cup B^c}$$

Q Prove that $(A-B)-C = A-(B\cup C)$.

Let $x \in (A-B)-C$

$\Rightarrow x \in (A-B)$ and $x \notin C$.

$\Rightarrow (x \in A$ and $x \notin B)$ and $x \notin C$.

$\Rightarrow x \in A$ and $(x \notin B$ and $x \notin C)$

$\Rightarrow x \in A$ and $(x \notin (B\cup C))$

$\Rightarrow x \in A-(B\cup C)$

So, $(A-B)-C \subseteq A-(B\cup C)$ —— (1)

Let, $x \in A-(B\cup C)$

$\Rightarrow x \in A$ and $x \notin (B\cup C)$

$\Rightarrow x \in A$ and $(x \notin B$ and $x \notin C)$

$\Rightarrow (x \in A$ and $x \notin B)$ and $x \notin C$

$\Rightarrow x \in A-B$ and $x \notin C$

$\Rightarrow x \in (A-B)-C$.

So, $A-(B\cup C) \subseteq (A-B)-C$. —— (2)

From eqn (1) & (2)

$\Rightarrow \boxed{(A-B)-C = A-(B\cup C)}$

Q Prove that $(A-B)\cap(B-A) = \phi$.

Let, $x \in (A-B)\cap(B-A)$

$\Rightarrow x \in (A-B)$ and $x \in (B-A)$

$\Rightarrow (x \in A$ and $x \notin B)$ and $(x \in B$ and $x \notin A)$

$\Rightarrow (x \in A$ and $x \notin A)$ and $(x \in B$ and $x \notin B)$.

$\Rightarrow x \in \phi$ and $x \in \phi$.

$\Rightarrow x \in \phi$.

So, $(A-B)\cap(B-A) \subseteq \phi$. —— (1)

~~Let, x~~ We know that ϕ is subset of every set

$\phi \subseteq (A-B)\cap(B-A)$ —— (2)

From (1) & (2)

$\Rightarrow \boxed{(A-B)\cap(B-A) = \phi.}$

Q Prove that $A-(A\cap B) = (A-B)$

Let, $x \in A-(A\cap B)$

$\Rightarrow x \in A$ and $x \notin (A\cap B)$

$\Rightarrow x \in A$ and $(x \notin A$ or $x \notin B)$

$\Rightarrow (x \in A$ and $x \notin A)$ or $(x \in A$ and $x \notin B)$

$\Rightarrow x \in \phi$ or $x \in (A-B)$

$\Rightarrow x \in \phi \cup (A-B)$

$\Rightarrow x \in (A-B)$

So, $A-(A\cap B) \subseteq (A-B)$ —— (1)

$\Rightarrow \cancel{x \in A \text{ and } x \notin B}$

$\Rightarrow x \in \phi \cup (A - B)$

$\Rightarrow x \in \phi$ or $x \in (A - B)$.

$\Rightarrow x \in \phi$ or $x \in A$ and $x \notin B$.

$\Rightarrow (x \in A$ and $x \notin A)$ or $(x \in A$ and $x \notin B)$.

$\Rightarrow x \in A$ and $(x \notin A$ or $x \notin B)$

$\Rightarrow x \in A$ and $x \notin (A \cap B)$

$\Rightarrow x \in A - (A \cap B)$.

So, $(A - B) \subseteq A - (A \cap B)$ —— (2)

$\Rightarrow \boxed{A - (A \cap B) = A - B}$ From (1) & (2).

Q If A and B are two sets then $(A \cap B) \cup (A \cap \sim B)$ and $A \cap (\sim A \cup B)$ are equal to ?

$\sim B$ = compliment
$\sim (A \cup B) = (A \cup B)'$

$(A \cap B) \cup (A \cap \sim B)$.

$\Rightarrow A \cap (B \cup \sim B)$ [By Distributive Law].

$\Rightarrow A \cap U$ [By compliment Law].

$\Rightarrow A$ [By Identity Law].

$A \cap (\sim A \cup B)$

$\Rightarrow (A \cap \sim A) \cup (A \cap B)$ [Distributive Law]

$\Rightarrow \phi \cup (A \cap B)$ [By compliment Law]

$\Rightarrow A \cap B$ [By Identity Law].

Q If A and B are two subsets of universal set then prove the following -

1) $A - B = B - A$ if & only if $A = B$.

2) $A - B = A$ iff $A \cap B = \phi$.

1) (i) If $A = B$ then $A - B = B - A$.

Let $x \in A - B$

$\Rightarrow x \in A$ and $x \notin B$.

$\Rightarrow x \in B$ and $x \notin A$.

$\Rightarrow x \in (B - A)$

So, $(A - B) \subseteq B - A$ —— (1).

Let $x \in (B - A)$

$\Rightarrow x \in B$ and $x \notin A$.

$\Rightarrow x \in A$ and $x \notin B$.

$\Rightarrow x \in (A - B)$

So, $(B - A) \subseteq (A - B)$ —— (2)

From (1) & (2).

$\Rightarrow \boxed{(A - B) = (B - A)}$

(11) If $(A-B) = (B-A)$ then $A = B$.

$x \in (B-A)$

$\Rightarrow x \in B$ and $x \notin A$. —①

$x \in (A-B)$

$\Rightarrow x \in A$ and $x \notin B$. —②

Eqⁿ ① & ② can be equal only when $A = B$.

(2) (i) ~~If A∩B=φ then~~ If $A - B = A$ then $A \cap B = \phi$

~~$x \in A \cap B$.~~

~~$\Rightarrow x \in A$ and $x \notin B$.~~

Let, $A \cap B \neq \phi$.

$\Rightarrow x \in (A \cap B)$

$\Rightarrow x \in (A-B) \cap B$. $\because [A - B = A]$

$\Rightarrow x \in A-B$ and $x \in B$.

$\Rightarrow (x \in A$ and $x \notin B)$, and $x \in B$.

$\Rightarrow x \in A$ and $(x \notin B$ and $x \in B)$

$\Rightarrow x \in A$ and $x \notin \phi$.

$\Rightarrow x \in A \cap \phi$.

$\Rightarrow x \in \phi$.

So, $A \cap B \subseteq \phi$. —① But this contradicts our assumption $A \cap B = \phi$.

But, we know that ϕ is subset of every set ~~②~~

$\phi \subseteq A \cap B$. —②

from eqⁿ ① & ②

$\boxed{A \cap B = \phi}$

(ii) If $A \cap B = \phi$ then $A - B = A$.

Let, $x \in A - B$.

$\Rightarrow x \in A$ and $x \notin B$.

$\Rightarrow x \in A$ $(\because A \cap B = \phi)$.

So, $A - B \subseteq A$ — ①

Let, $x \in A$

$\Rightarrow x \in A$ and $x \notin B$. $[\because A \cap B = \phi]$

$\Rightarrow x \in (A-B)$

So, $A \subseteq A - B$ — ②

From ① & ②

$\boxed{A - B = A}$

(OR)

If $(A \cap B) = \phi$ then $A - B = A$.

Let, $x \in (A-B)$
$\Rightarrow x \in (A-B) \cup \phi$
$\Rightarrow x \in (A-B)$ or $x \in (A \cap B)$
$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \in B)$
$\Rightarrow (x \in A$ and $x \in B')$ or $(x \in A$ and $x \in B)$.
$\Rightarrow x \in A$ and $(x \in B'$ ~~and~~ or $x \in B)$
$\Rightarrow x \in A$ and $x \in (B \cup B')$
$\Rightarrow x \in A$ and $x \in U$
$\Rightarrow x \in (A \cap U)$
$\Rightarrow x \in A$
So, $A - B \subseteq A$ —— ①.

* Cartesian Product :-

$A \times B = \{(x, y) : x \in A$ and $y \in B\}$.

e.g. ordered pair . $A = \{1, 2, 3\}$
$B = \{a, b\}$.

$A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$.
$B \times A = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

Ordered Pair :- It is a pair of objects formed by using the two components in specified order

In the ordered pair (x, y), x is the first component and y is the second component.

Consider two sets A and B, then cartesian product of A and B is denoted by '$A \times B$' and it is the set of all possible ordered pairs (x, y) with $x \in A$ and $y \in B$.

Symbolically,
$A \times B = \{(x,y) : x \in A$ and $y \in B\}$.

NOTE: $A \times B \neq B \times A$.

* Multiset :- Multisets are sets where an element appea more than ones.

e.g. $A = \{1, 1, 1, 2, 2, 3\}$.
multiset A can also be written as-
$A = \{3 \cdot 1, 2 \cdot 2, 1 \cdot 3\}$.

Multiplicity :- The multiplicity of an element in a multiset is defined to be the no. of times an element appears in the multiset.

3 is the multiplicity of 1.
2 " " " " 2.
1 " " " " 3.

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NOTE: [illegible] is a special type of multiset in which the multiplicity of every element is one or zero

- Cardinality of Multiset :- cardinality of Multiset is equal to the cardinality of corresponding set

(OR)

cardinality of Multiset is defined as the cardinality of its corresponding set. Assuming that all the elements are distinct in the ~~set~~ multiset.

So cardinality of multiset A = Cardinality of corresponding set A.

e.g. $A = \{1, 1, 1, 2, 2, 3\}$

$A = \{1, 2, 3\}$.

cardinality of Multiset $A = 3$.

- Operation in Multisets :-

let, $A = \{3 \cdot a,\ 2 \cdot b,\ 1 \cdot c\}$.

$B = \{2 \cdot a,\ 1 \cdot b,\ 1 \cdot d\}$.

union of multisets ($\cup$) :- $(A \cup B) = \{3 \cdot a,\ 2 \cdot b,\ 1 \cdot c,\ 1 \cdot d\}$

Intersection of Multisets ($\cap$) :- $A \cap B = \{2 \cdot a,\ 1 \cdot b\}$

Difference ($-$) :- $A - B = \{1 \cdot a,\ 1 \cdot b,\ 1 \cdot c\}$

Sum ($+$) :- $B - A = \{1 \cdot d\}$.

$A + B = \{5 \cdot a,\ 3 \cdot b,\ 1 \cdot c,\ 1 \cdot d\}$.

Union :- $A \cup B$ is the multiset where the multiplicity of an element is maximum of its multiplicities in A and B.

eg $A \cup B = \{3 \cdot a,\ 2 \cdot b,\ 1 \cdot c,\ 1 \cdot d\}$.

Intersection :- $A \cap B$ is the multiset where the multiplicity of an common element is minimum of its multiplicities in A and B.

eg $A \cap B = \{2 \cdot a,\ 1 \cdot b\}$.

Difference :- $A - B$ is the multiset where the multiplicity of an element is equal to multiplicity of element in A minus multiplicity of element in B, if the difference is positive but it is equal to zero if the difference is zero and negative.

e.g $A - B = \{1 \cdot a,\ 1 \cdot b,\ 1 \cdot c\}$

$B - A = \{1 \cdot d\}$.

Sum :- $A + B$ is the multiset where multiplicity of an element is equal to the sum of multiplicities of an element in both multisets A and B.

Eg $A + B = \{5 \cdot a,\ 3 \cdot b,\ 1 \cdot c,\ 1 \cdot d\}$

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$Q = \{3\cdot a,\ 3\cdot b,\ 2\cdot d\}$

find 1) $P \cup Q$, 2) $P \cap Q$, 3) $P - Q$, 4) $Q - P$, 5) $P+$

1) $P \cup Q = \{4\cdot a,\ 3\cdot b,\ 1\cdot c,\ 2\cdot d\}$.

2) $P \cap Q = \{3\cdot a,\ 3\cdot b\}$

3) $P - Q = \{1\cdot a,\ 1\cdot c\}$.

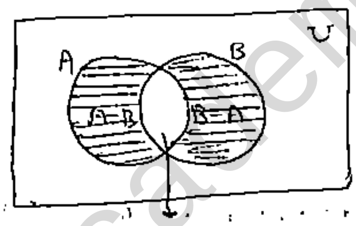
4) $Q - P = \{2\cdot d\}$.

5) $P + Q = \{7\cdot a,\ 6\cdot b,\ 1\cdot c,\ 2\cdot d\}$.

* Set Inclusion Exclusion Principle :-

(OR)

Counting Principle :-



$A \cap B$.

$n(A) = n(A-B) + n(A \cap B)$ — (1)

$n(B) = n(B-A) + n(A \cap B)$ — (2)

$n(A \cup B) = n(A-B) + n(B-A) + n(A \cap B)$ — (3)

Put the values of $n(A-B)$ & $n(B-A)$ from eq^n (1) & (2) in eq^n (3) -

$n(A \cup B) = n(A) - n(A \cap B) + n(B) - \cancel{n(A \cap B)} + \cancel{n(A \cap B)}$

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$$

Corollary :-

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

Proof: Let $B \cup C = D$.

$n(A \cup D) = n(A) + n(D) - n(A \cap D)$.

$= n(A) + n(B \cup C) - n[A \cap (B \cup C)]$

$= n(A) + n(B) + n(C) - n(B \cap C) - n[(A \cap B) \cup (A \cap C)]$

$= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) + n[(A \cap B) \cap (A \cap C)]$

$= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$.

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Q 40 lecturers were interviewed for a job, 25 were mathematicians, 28 were physicist and 7 were niether. How many lecturers were mathematicians and physicists.

→ Let, M be the set of mathematicians and:
P be the set of Physicists.

Given :- $n(M) = 25$
$n(P) = 28$.

Find :- $n(M \cap P) = ?$

Given :- $n(M \cup P) = 40 - 7 = 33$

$n(M \cup P) = n(M) + n(P) - n(M \cap P)$ {By set incl. ex. principle}

$n(M \cap P) = n(M) + n(P) - n(M \cup P)$
$= 25 + 28 - 33$
$= 53 - 33$

$\boxed{n(M \cap P) = 20}$

So, 20 lectureres were both mathematicians and physicists.

Q In a survey of 600 TV viewers given the following information.
1) 385 watch cricket matches.
2) 295 watch hockey matches.
3) 215 watch football matches.
4) 145 watch cricket & football matches both.
5) 170 watch cricket & hockey matches both.
6) 150 watch hockey & football matches both.
7) 150 does not watch any of the three games.

Find 1) How many people watch all three kind of matches.
2) How many people watch exactly one sport.

→ Let, C be the set of cricket viewers.
H be the set of Hockey viewers.
F be the set of football viewers.

Given :- $n(C) = 385$
$n(H) = 295$
$n(F) = 215$
$n(C \cap F) = 145$
$n(C \cap H) = 170$
$n(H \cap F) = 150$.
$n(C \cup H \cup F) = 600 - 150 = 450$

Now, $n(C \cup H \cup F) = n(C) + n(H) + n(F) - n(C \cap H) - n(C \cap F) - n(H \cap F) + n(C \cap H \cap F)$.

$450 = 385 + 295 + 215 - 170 - 145 - 150 + n(C \cap H \cap F)$
$450 = 895 - 465 + n(C \cap H \cap F)$
$450 = 430 + n(C \cap H \cap F)$

(By set Inc. Exc. Principle).

$\boxed{n(C \cap H \cap F) = 20}$

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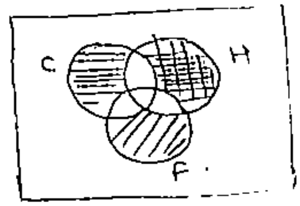
2) only cricket viewers.



$n(C) - n(C \cap H) - n(C \cap H \cap F) - n(C \cap F)$

$= 385 - 170 + 20 - 145$

$= 90.$

Only Hockey viewers.

$n(H) - n(C \cap H) + n(C \cap H \cap F) - n(H \cap F)$

$= 295 - 170 + 20 - 150.$

$= -5$

only Football viewers.

$n(F) - n(C \cap F) + n(C \cap H \cap F) - n(H \cap F)$

$= 215 - 145 + 20 - 150.$

$= -60.$

No. of viewers exactly watch = n(exactly cricket) + n(exactly Hockey) + n(exactly Football).

$= 90 - 5 - 60.$

$= 25$

Q Among first 500 +ve integers. Find.

1) The no. of Integers which are not divisible by 2 nor by 3 nor by 5.

2) The no. of integers which are exactly divisible by one of them.

→ Let ; A be the set of integers divisible by 2

B be the set of integers divisible by 3.

C be the set of integers divisible by 5.

$n(A) = \left\lfloor \frac{500}{2} \right\rfloor = 250.$

$n(B) = \left\lfloor \frac{500}{3} \right\rfloor = 166.$

$n(C) = \left\lfloor \frac{500}{5} \right\rfloor = 100.$

$n(A \cap B) = \left\lfloor \frac{500}{2 \times 3} \right\rfloor = 83$

$n(B \cap C) = \left\lfloor \frac{500}{3 \times 5} \right\rfloor = 33.$

$n(A \cap C) = \left\lfloor \frac{500}{2 \times 5} \right\rfloor = 50.$

$n(A \cap B \cap C) = \left\lfloor \frac{500}{2 \times 3 \times 5} \right\rfloor = 16.$

1) $n(A' \cap B' \cap C') = ?$

$n(A' \cap B' \cap C') = n(A \cup B \cup C)'$

$= 500 - n(A \cup B \cup C)$

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$n(A\cup B\cup C) = n(A)+n(B)+n(C) - n(A\cap B) - n(A\cap C) - n(B\cap C) + n(A\cap B\cap C)$

$= 250+166+100-83-33-50+16$

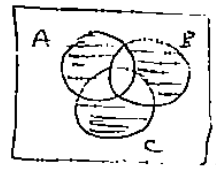
$= 366$.

$n(A'\cap B'\cap C') = n(U) - n(A\cup B\cup C)$

$= 500-366$

$= 134$

(2) Only divisible by 2.



$n(A) - n(A\cap B) - n(A\cap C) + n(A\cap B\cap C)$

$= 250 - 83 - 33 + 16$

$= 150$

Only divisible by 3.

$n(B) - n(A\cap B) - n(B\cap C) + n(A\cap B\cap C)$

$= 166 - 83 - 50 + 16$

$= 49$.

Only divisible by 5

$n(C) - n(A\cap C) - n(B\cap C) + n(A\cap B\cap C)$

$= 100 - 33 - 50 + 16$.

$= 33$

No. of Integers exactly divisible by one of them

$= n(\text{exactly divisible by } 2) + n(3) + n(5)$

$= 150 + 49 + 33$

$= 232$

Q 75 children went to a circus where they attend a magic show, comedy show and animal show. 20 of them attended all three shows and 55 attended atleast two shows. Each show cost 5 Rs. The total money collected 700 ₹. Find the no. of children who didnot attend any of three shows.

→ Let, M be the set of Magic show.
C be the set of comedy show.
A be the set of Animal show.

Total money collected = ₹ 700.

Cost per show = ₹ 5

Total no. of shows attended $= \frac{700}{5} = 140$

$n(M\cap C\cap A) = 20$.

No. of children attended atleast two shows $= 55$.

⇒ No. of children attended exactly two shows + No. of childrens attended all shows $= 55$.

⇒ No. of children attended Exactly 2 shows $+ 20 = 55$.

⇒ No. of children attended exactly 2 shows $= 35$

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No. of children attended exactly 1 show = 140 − (35×2 + 20×3)

= 140 − (70 + 60)

= 140 − 130

= 10

No. of children did not attend any of 3 shows

= 75 − (35 + 20 + 10).

= 75 − (65)

= 10.

* Countable and Uncountable Sets :-

Countable Infinite Set :- An infinite set A is said to be countably infinite or denumerable if it is equivalent to set N (set of Natural No.'s) i.e. if there exist a one to one maping with natural ...

$f : N \to A$. eg set of ...

Countable set :- A set A is called countable when it is either

- Empty or Null set i.e. $A = \phi$. or
- Finite set or
- Countable Infinite set.

Natural	1	2	3	4	5	6	7	8	...
Even	2	4	6	8	10	12	14	16	...

Q Show that the set of +ve rational no.'s is countable

$p+q=2$, $p+q=3$, $p+q=4$.

$$\begin{array}{cccccc}
1/1 & 2/1 & 3/1 & 4/1 & 5/1 & \cdots \\
1/2 & 2/2 & 3/2 & 4/2 & 5/2 & \cdots \\
1/3 & 2/3 & 3/3 & 4/3 & 5/3 & \cdots \\
1/4 & 2/4 & 3/4 & & & \cdots \\
1/5 & & & & & \\
\vdots & & & & & \\
1/i & 2/i & 3/i & 4/i & 5/i & \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
1 & 1/2 & 2 & 3 & 1/3 & 1/4 & 2/3 & 3/2 & \cdots
\end{array}$$

Q $A = \{ n^7 : n \in N \}$. Cardinality.

$A = \{ 1^7, 2^7, 3^7, 4^7 \ldots \}$

$$\begin{array}{ccccc}
1 & 2 & 3 & 4 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
1^7 & 2^7 & 3^7 & 4^7 & \cdots
\end{array}$$

n(A) = This set is countable infinite.

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* Relation :- Let A and B be two non-empty sets then R is a relation from A to B if $R \subseteq A \times B$ and it is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$. It is denoted by $a R b$ or $(a,b) \in R$. and it is read as "a is related to b".

If there is no relation b/w a and b then $a \not R b$ or $(a, b) \notin R$. and it is read is "a is not related to b".

Symbolically,

$$R = \{(a,b) : a \in A \ \& \ b \in B \text{ and } a R b \text{ or } (a,b) \in R\}.$$

Q e.g. $A = \{1, 2, 3, 4\}$, $B = \{1, 2\}$. A relation R is defined from A to B ie $a R b$ iff $a \times b$ = even no. Find relation R.

→ $R \subseteq A \times B$.

$A = \{1, 2, 3, 4\}$, $B = \{1, 2\}$.

$A \times B = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2), (4,1), (4,2)\}$.

$R = \{(1,2), (2,1), (2,2), (3,2), (4,1), (4,2)\}$.

Q $A = \{2, 4, 6, 8\}$, $B = \{1, 2, 3\}$. Find the relation R such that $a R b$ iff $a \geq b$.

→ $A = \{2, 4, 6, 8\}$, $B = \{1, 2, 3\}$.

$A \times B = \{(2,1), (2,2), (2,3), (4,1), (4,2), (4,3), (6,1), (6,2), (6,3), (8,1), (8,2), (8,3)\}$

$R = \{(2,1), (2,2), (4,1), (4,2), (4,3), (6,1), (6,2), (8,1), (6,3), (8,2), (8,3)\}$.

Q $A = \{1, 2, 3, 4\}$ A relation R is defined on set A such that $a R b$ iff a is divisible by b.

→ $A = \{1, 2, 3, 4\}$

$A = \{1, 2, 3, 4\}$.

$A \times A = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$.

$R = \{(1,1), (2,1), (2,2), (3,1), (3,3), (4,1), (4,2), (4,4)\}$.

* Domain and Range of a Relation :-

Domain of a Relation :- It is the set consisting of all first elements of ordered pairs of relation R. It is denoted by $D(R)$ or $Dom(R)$.

Symbolically,

$$D(R) = \{a : a \in A \text{ and } a R b\}.$$

NOTE: $D(R) \subseteq A$

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second elements of ordered pairs of relation R. It is denoted by R(R) or Ran(R).
Symbolically,

$R(R) = \{ b : b \in B \text{ and } aRb \}$.

NOTE: $R(R) \subseteq B$.

e.g. $A = \{1, 2, 3, 4\}$, $B = \{1, 2\}$.

Domain → $D(R) = \{1, 2, 3, 4\}$

Range → $R(R) = \{1, 2\}$.

$A = \{2, 4, 6, 8\}$, $B = \{1, 2, 3\}$.

Domain → $D(R) = \{2, 4, 6, 8\}$.

Range → $R(R) = \{1, 2, 3\}$

$A = \{1, 2, 3, 4\}$. {(s

Domain → $D(R) = \{1, 2, 3, 4\}$.

Range → $R(R) = \{1, 2, 3, 4\}$.

Q let $A = \{a, b, c\}$, $B = \{a, b, c, d\}$. R and S are two relati ~~such that~~ defined from A to B such that $R = \{(a,b), (b,c), (c,d), (a,a)\}$ and $S = \{(a,a), (a,d), (b,d)\}$. find domain and range $(R \cup S)$, $(R \cap S)$ and $(R - S)$.

→ $R \cup S = \{(a,a), (a,b), (b,c), (c,d), (a,d), (b,d)\}$

$R \cap S = \{(a,a)\}$

$R - S = \{(a,b), (b,c), (c,d)\}$.

$D(R \cup S) = \{\cancel{(a,a)}, \cancel{(a,b)}, \cancel{(b,c)}, \cancel{(c,d)}, \cancel{(a,d)}, \cancel{(b,d)}\}$.
$= \{a, b, c\}$

$R(R \cup S) = \{a, b, c, d\}$.

$D(R \cap S) = \{a\}$.

$R(R \cap S) = \{a\}$

$D(R - S) = \{a, b, c\}$

$R(R - S) = \{b, c, d\}$.

* Types of Relation :-

• Universal relation :- A relation R is called universal relation from A to B if $R = A \times B$.

For one set:

A relation R is called universal relation defined on set A if $R = A \times A$.

e.g. $A = \{1, 2\}$

$R = \{(1,1), (1,2), (2,1), (2,2)\}$

- **Identity Relation (OR) Diagonal Relation :-** A relation R is called identity relation on set A if every element of set is related to itself only.

It is denoted by 'I_A' or 'Δ_A' or 'Δ'.

$$I_A = \{(a, a) : \forall\, a \in A \text{ and } aRa\}.$$

e.g. $A = \{1, 2, 3\}$

$I_A = \{(1,1), (2,2), (3,3)\}$.

- **Inverse Relation :-** A relation defined from B to A is called Inverse relation if a relation R is defined from A to B. It is denoted by 'R^{-1}'.

Symbolically,

$$R^{-1} = \{(b,a) : b \in B,\ a \in A \text{ and } aRb\}.$$

e.g. $R = \{(1,1), (1,2), (1,3), (3,2)\}$

$R^{-1} = \{(1,1), (2,1), (3,1), (2,3)\}$.

- **Empty Relation :- (OR) NULL (OR) VOID Relation :-** A relation R is called empty relation if $R = \phi$.

e.g. $A = \{1, 2, 3\}$.

and $a R b$ iff $a+b > 6$.

Here, $R = \phi$.

- **Compliment of Relation :-** If a relation is defined from A to B then compliment of R is set of ordered pairs such that

$$R^c = \{(a,b) : (a,b) \notin R\}.$$

$R^c = (A \times A) - R$. [Defined on one set]

$R^c = (A \times B) - R$. [Defined on two set].

It is denoted by R^c or R'.

e.g. $A = \{1, 2, 3\}$

$B = \{4, 5\}$

$R = \{(1,4), (\ ,4), (3,5)\}$.

$A \times B = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$.

$R^c = (A \times B) - R$.

$R^c = \{(1,5), (2,4), (2,5)\}$

$R: A \to B \qquad R = \{(a,b) : aRb\}$

$S: B \to C \qquad S = \{(b,c) : bSc\}$

Let, R be the relation from A to B and S be the relation from B to C then composition of relation R and S is a relation consisting of odered pairs (a,c) where $a \in A$ and $c \in C$ provided that there exists some $b \in B$ such that aRb and bRc. It is denoted by '$R \circ S$'

Symbolically,

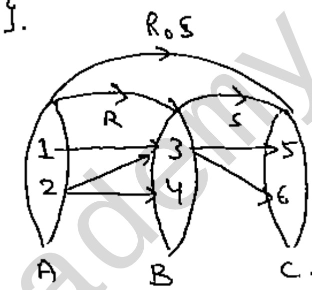
$$R \circ S = \{(a,c) : \exists b \in B, aRb \text{ and } bSc\}.$$

<u>e.g.</u> $A = \{1,2\}, \quad B = \{3,4\}, \quad C = \{5,6\}.$

$R = \{(1,3), (2,3), (2,4)\}$

$S = \{(3,5), (3,6)\}.$

$R \circ S = \{(1,5), (1,6), (2,5), (2,6)\}.$



<u>Q</u>. R and S relations are defined on set $A = \{1,2,3,4\}$ and

$R = \{(1,2), (2,2), (2,3), (2,4), (3,2), (4,3)\}.$

$S = \{(2,2), (2,3), (3,2), (3,3), (3,4), (4,2)\}$. find $R \circ S$ & $S \circ R$

→ $R \circ S = \{(1,2), (1,3), (2,2), (2,3), (2,4), (3,3), (3,2), (4$
$(4,3), (4,4)\}.$

$S \circ R = \{(2,2), (2,3), (2,4), (3,3), (3,4), (3,2), (4,2), (4,3), (4,4$

<u>Q</u> Prove that inverse of $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

→ Let, $(a,b) \in (R \cup S)^{-1}$

$\Rightarrow (b,a) \in (R \cup S).$

$\Rightarrow (b,a) \in R$ or $(b,a) \in S.$

$\Rightarrow (a,b) \in R^{-1}$ or $(a,b) \in S^{-1}.$

$\Rightarrow (a,b) \in (R^{-1} \cup S^{-1}).$ ——(1)

So, $(R \cup S)^{-1} \subseteq (R^{-1} \cup S^{-1})$

Let, $(a,b) \in (R^{-1} \cup S^{-1})$

$\Rightarrow (a,b) \in R^{-1}$ or $(a,b) \in S^{-1}$

$\Rightarrow (b,a) \in R$ or $(b,a) \in S.$

$\Rightarrow (b,a) \in (R \cup S)$

$\Rightarrow (a,b) \in (R \cup S)^{-1}$

So, $(R^{-1} \cup S^{-1}) \subseteq (R \cup S)^{-1}$ ——(2)

From eqn (1) & (2).

$$\Rightarrow \boxed{(R \cup S)^{-1} = R^{-1} \cup S^{-1}}$$

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Q. Prove that $(R \cap S)^{-1} = (R^{-1} \cap S^{-1})$.

$\Rightarrow$ Let $(a, b) \in (R \cap S)^{-1}$

$\Rightarrow (b, a) \in (R \cap S)$

$\Rightarrow (b, a) \in R$ and $(b, a) \in S$.

$\Rightarrow (a, b) \in R^{-1}$ and $(a, b) \in S^{-1}$

$\Rightarrow (a, b) \in R^{-1} \cap S^{-1}$.

So, $(R \cap S)^{-1} \subseteq R^{-1} \cap S^{-1}$. — (1)

Let, $(a, b) \in (R^{-1} \cap S^{-1})$

$\Rightarrow (a, b) \in R^{-1}$ and $(a, b) \in S^{-1}$

$\Rightarrow (b, a) \in R$ and $(b, a) \in S$.

$\Rightarrow (b, a) \in (R \cap S)$

$\Rightarrow (a, b) \in (R \cap S)^{-1}$

So, $(R^{-1} \cap S^{-1}) \subseteq (R \cap S)^{-1}$ — (2)

From eq^n (1) & (2)

$$\boxed{(R \cap S)^{-1} = (R^{-1} \cap S^{-1})}$$

Q. If $R \subseteq S$ then, $R^{-1} \subseteq S^{-1}$.

Given, $R \subseteq S$.

Let, $(a, b) \in R^{-1}$

$\Rightarrow (b, a) \in R$.

$\Rightarrow (b, a) \in S$ $\quad (\because R \subseteq S)$

$\Rightarrow (a, b) \in S^{-1}$

So, $R^{-1} \subseteq S^{-1}$.

Q. Prove that composition of relation follows the associative law.

Proof :- $\boxed{(R_o S)_o T = R_o (S_o T)}$

Let, R be a relation from A to B. S be a relation from B to C. and T be a relation from C to D.

Let $(a, d) \in (R_o S)_o T$.

$\Rightarrow$ (such that) $\exists c \in C$, $(a, c) \in (R_o S)$ and $(c, d) \in T$ — (1)

Now, $(a, c) \in R_o S$.

$\Rightarrow \exists b \in B$, $(a, b) \in R$ and $(b, c) \in S$ — (2)

From eq^n (1) & (2),

$(b, c) \in S$ and $(c, d) \in T$.

$\Rightarrow (b, d) \in (S_o T)$ — (3).

From eq^n (2) & (3).

$(a, b) \in R$ and $(b, d) \in (S_o T)$

$\Rightarrow (a, d) \in R_o (S_o T)$ — (4)

So, $(R_o S)_o T \subseteq R_o (S_o T)$ — (5)

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Conversely, Let, $(a,d) \in R_o(S_oT)$

$\Rightarrow \exists b \in B$ such that $(a,b) \in R$ and $(b,d) \in (S_oT)$ — ⑥

Now, $(b,d) \in S_oT$.

$\Rightarrow \exists c \in C$, $(b,c) \in S$ and $(c,d) \in T$ — ⑦

from eqn ⑥ & ⑦

$(a,b) \in R$ and $(b,c) \in S$

$\Rightarrow (a,c) \in R_oS$ — ⑧

from eqn ⑦ & ⑧

$(a,c) \in R_oS$ and $(c,d) \in T$

$\Rightarrow (a,d) \in (R_oS)_oT$ — ⑨

so, $R_o(S_oT) \subseteq (R_oS)_oT$ — ⑩

from eqn ⑤ and ⑩

$$\boxed{[(R_oS)_oT] = R_o(S_oT)}$$

Q Prove that $(R_oS)^{-1} = (S^{-1}{}_o R^{-1})$

→ Let, R be a relation from A to B.
and S be a relation from B to C.

Let, $(c,a) \in (R_oS)^{-1}$

$\Rightarrow (a,c) \in (R_oS)$

$\Rightarrow \exists b \in B$, $(a,b) \in R$ and $(b,c) \in S$ — ①

from eqn ①.

$(a,b) \in R$.

$\Rightarrow (b,a) \in R^{-1}$ — ②

from eqn ①

$(b,c) \in S$.

$\Rightarrow (c,b) \in S^{-1}$ — ③

from eqn ③ and ②

$(c,b) \in S^{-1}$ and $(b,a) \in R^{-1}$

$\Rightarrow (c,a) \in S^{-1}{}_o R^{-1}$ — ④

so, $(R_oS)^{-1} \subseteq (S^{-1}{}_o R^{-1})$ — ⑤

Conversely, Let, $(c,a) \in (S^{-1}{}_o R^{-1})$

$\Rightarrow \exists b \in B$, $(c,b) \in S^{-1}$ and $(b,a) \in R^{-1}$ — ⑥

from eqn ⑥

$(c,b) \in S^{-1}$.

$\Rightarrow (b,c) \in S$ — ⑦.

from eqn ⑥.

$(b,a) \in R^{-1}$

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$\Rightarrow \quad (a,b) \in R$ — (8)

from eqⁿ (7) & (8)

$(a,b) \in R$ and $(b,c) \in S$

$\Rightarrow \quad (a,c) \in R \circ S.$

$\Rightarrow \quad (c,a) \in (R\circ S)^{-1}$ — (9)

So, $S^{-1} \circ R^{-1} \subseteq (R\circ S)^{-1}$ — (10)

from eqⁿ (5) and (10).

$$\boxed{(R\circ S)^{-1} = S^{-1} \circ R^{-1}}$$

Q/ Let S, R and T are relations on set $A = \{0,1,2,3\}$ defined by $R = \{(a,b) : a+b = 3\}$, $S = \{(a,b) : 3$ is divisor of $a+b\}$ and $T = \{(a,b) : \max(a,b) = 3\}$. Find.

i) R^2 $(R\circ R)$. ii) $R\circ S$ iii) $R\circ T$ iv) $S\circ T$, v) $(R\circ S)\circ T$

$A = \{0,1,2,3\}$. $\qquad \left[R^n = R^{n-1} \circ R\right]$

$R = \{(0,3), (3,0), (1,2), (2,1)\}$.

$S = \{(0,3), (3,0), (1,2), (2,1), (3,3)\}$.

$T = \{(0,3), (3,0), (1,3), (3,1), (2,3), (3,2), (3,3)\}$.

i) R^2 $(R\circ R) = \{(0,0), (3,3), (1,1), (2,2)\}$

ii) $R\circ S = \{(0,0), (0,3), (3,3), (1,1), (2,2)\}$

iii) $R\circ T = \{(0,0), (0,1), (0,2), (0,3), (3,3), (1,3), (2,3)\}$

iv) $S\circ T = \{(0,0), (0,1), (0,2), (0,3), (3,3), (1,3), (2,3), (3,0), (3,1), (3,2)\}$.

v) $(R\circ S)\circ T = \{(0,3), (0,0), (3,0), (3,3), (1,3), (2,3), (3,2), (3,1), (0,1), (0,2)\}$.

* Properties of Relation :-

• *Reflexive Relation* :- A binary relation ~~on~~ R on set A is said to be reflexive relation if every element of set A is related to itself. i.e. $\boxed{\forall\, a \in A,\ aRa \text{ or } (a,a) \in R}$

e.g. $A = \{1,2,3\}$.

$R = \{(1,1), (1,2), (2,2), (2,3), (3,3)\}$.

R is reflexive relation since, $(1,1) \in R$. $(2,2) \in R$ $(3,3) \in R$.

$|A| = n$. (cardinality of set A)

Total binary relations $= 2^{n^2}$

Total reflexive relations $= 2^{n(n-1)}$

Total symmetric relations $= 2^{n(n+1)/2}$

NOTE: Every identity relation is a reflexive relation but every reflexive relation is not identity relation.

• Irreflexive Relation :- A relation R is defined on set A is said to be Irreflexive if there is no element in A which is related to itself.

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e.g. $A = \{1,2,3\}$.

$R = \{(1,2), ~~(\ldots)~~, (2,3)\}$. , R is irreflexive relation

since, $(1,1) \notin R$.

$(2,2) \notin R$.

$(3,3) \notin R$.

- Non Reflexive Relation :- A relation R is defined on set A is said to be non-reflexive relation if it is neither reflexive nor in-reflexive i.e. some elements are related to itself but there exist atleast one element which is not related to itself.

e.g $A = \{1,2,3\}$.

$R = \{(1,1), (2,2), (1,3), (2,3), (3,1)\}$.

R is Non-Reflexive

since. $(3,3) \notin R$.

- Symmetric Relation :- A relation R is defined on set A is said to be symmetric relation if whenever aRb then bRa.

i.e. $\boxed{(a,b) \in R \implies (b,a) \in R.}$

e.g $A = \{1,2,3\}$.

$R = \{(1,1), (1,2), (2,1), (2,3), (3,2)\}$.

R is symmetric relation.

since, $(1,1) \in R$.

$(1,2) \in R \implies (2,1) \in R$.

$(2,3) \in R \implies (3,2) \in R$.

Therefore, it is symmetric relation.

NOTE: A relation R is symmetric if $R = R'$.

- Asymmetric Relation :- A relation R on set A is said to be asymmetric relation if $(a,b) \in R \implies (b,a) \notin R$.

i.e $\boxed{(a,b) \in R \implies (b,a) \notin R.}$

{ $p \to q$; if then sign.

e.g. $A = \{1,2,3\}$

$R = \{(1,2), (3,2)\}$.

R is assymmetric relation.

since, $(1,2) \in R \implies (2,1) \notin R$.

$(3,2) \in R \implies (2,3) \notin R$.

- Anti Symmetric Relation :- A relation R is defined on set A is said to be anti-symmetric relation if $\boxed{(a,b) \in R \text{ and } (b,a) \in R \implies a = b.}$

~~eg~~. contrapositive definition of Anti-symmetric Relation :-

$\boxed{\text{If } \forall (a,b) \in R \text{ and } (b,a) \notin R. \implies a \neq b.}$

E.g. $A = \{1, 2, 3\}$.
$R = \{(1,1), (1,2)\}$.

{ Divide is always a antisymmetric relation }

R is antisymmetric relation.
since, $(1,1) \in R$.

R is contrapositive anti-symmetric relation.
Since, $(1,2) \in R$
$(2,1) \notin R$.
and also, $1 \neq 2$.

- **Transitive Relation** :- A relation R is defined on set A is said to be transitive relation if

$$\boxed{(a,b) \in R \text{ and } (b,c) \in R \Rightarrow (a,c) \in R.}$$

E.g. $A = \{1, 2, 3\}$.
$R = \{(1,2), (2,3), (1,3)\}$.
R is transitive relation
since, $(1,2) \in R$. and $(2,3) \in R$.
$\Rightarrow (1,3) \in R$.

- **Equivalence Relation** :- A relation R is defined on set A is said to be equivalence relation if it is reflexive, symmetric and transitive. i.e. equivalence relation satisfies following properties.
1) $(a,a) \in R$, $\forall a \in A$. (Reflexive)
2) $(a,b) \in R \Rightarrow (b,a) \in R$. (Symmetric)
3) $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$. (Transitive).

Q Determine whether the relation R on set $A = \{-3, -2, -1, 0, 1, 2, 3\}$ is Reflexive, symmetric, irreflexive, antisymmetric, asymmetric, and transitive. where $(a,b) \in R$ iff i) $a = 1$, ii) $a = 2b$, iii) $a + b$ = even.

→ $A = \{-3, -2, -1, 0, 1, 2, 3\}$.

$R_1 = \{(1,-3), (1,-2), (1,-1), (1,0), (1,1), (1,2), (1,3)\}$.

1) Every element of A is ~~not~~ related to itself therefore, it is not reflexive relation.
∵ $(2,2) \notin R$, $(3,3) \notin R$. etc.

2) $(1,-3) \in R \Rightarrow (-3,1) \notin R$. Therefore, it is not ~~sy~~ symmetric relation.

3) since $(1,1) \in R$. Therefore it is not irreflexive relation.

4) $(1,1) \in R$ and $1 = 1$. ; $(1,-3) \in R \Rightarrow (-3,1) \notin R$.
$1 \neq -3$.

Therefore, it is antisymmetric relation.

5) $(1,1) \in R$. Therefore it is not asymmetric relation.

6) $(1,1) \in R$, $(1,-1) \in R$ ... Therefore, it is transitive relation.

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$R_2 = \{(-2,-1), (2,0), (0,0)\}$. $[a-2b]$

1) Every element is not related to itself. Therefore, it is not a reflexive relation.
   $(-2,-2)$ and $(2,2) \notin R$ so, not a reflexive relation.
2) $(-2,-1) \in R \Rightarrow (-1,-2) \notin R$. Therefore, it is not symmetric relation.
3) $(0,0) \in R$. Therefore, it is not irreflexive relation.
4) $(0,0) \in R$, $(2,1) \in R$, $(1,2) \notin R \Rightarrow 1 \neq 2$. so, it is antisymmetric relation.
5) Since, $(0,0) \in R$. so, it is not asymmetric relation.
6) Not transitive relation.

$[a+b = \text{even}]$

$R_3 = \{(1,1), (2,2), (3,3), (-1,-1), (-2,-2), (-3,-3), (-1,-3), (1,3), (0,2), (0,-2), (1,-1), (2,-2), (3,-3), (-3,-1), (-2,2), (2,0), (-2,0), (-1,1), (-3,3), (3,1), (1,-3), (-3,1), (-1,3), (3,-1)\}$.

1) It is not reflexive.
2) symmetric relation.
3) Not reflexive relation.
4) Not antisymmetric.
5) Not Asymmetric
6) Transitive relation.

Q Prove that the relation R "(a-b) is divisible by 6" $\forall a, b \in Z^+$. is an equivalence relation.

→ Imp { congruent Relation:- $\boxed{a \equiv b \bmod m.}$ $\Rightarrow$ a-b divisible by m. }

Equivalence relation satisfies when reflexive symmetric and transitive relation satisfies.

1) 0 is divisible by 6.
   $\Rightarrow$ $(a-a)$ is divisible by 6, $\forall a \in Z^+$. $\Rightarrow (a,a) \in R$.
   so, it ~~follow~~ reflexive relation.
2) let $(a, b) \in R$.
   $\Rightarrow (a-b)$ is divisible by 6.
   $\Rightarrow -(a-b)$ is divisible by 6.
   $\Rightarrow (b-a)$ is divisible by 6.
   $\Rightarrow (b,a) \in R$.
   It follows symmetric property.
3) Let. $(a,b) \in R$ and $(b,c) \in R$.
   $\Rightarrow (a-b)$ is divisible by 6 and $(b-c)$ is divisible by 6.
   $\Rightarrow (a-b)+(b-c)$ is divisible by 6.
   $\Rightarrow (a-c)$ is divisible by 6.
   $\Rightarrow (a,c) \in R$.
   So. It follows transitive relation. So, R is an equivalence relation

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Q State whether following relations are equivalent or not on the set of lines in a planes.

1) $x \parallel y$
2) $x \perp y$.

For Equivalence relation reflexive, symmetric and transitive relation must satisfy.

1) $x \parallel y$.

Reflexive :- Every line is parallel to itself. ie. $x \parallel x$. so, it is reflexive.

symmetric : $x \parallel y \Rightarrow y \parallel x$. It follows symmetric relation.

Transitive: $x \parallel y$ and $y \parallel z \Rightarrow x \parallel z$. It follows transitive relation. Hence, it is an equivalence relation.

2) $x \perp y$.

Reflexive :- No line is $\perp$ to itself. so, it does not follow reflexive property.

Hence, it is not an equivalence relation.

Q i) Show that $R = \{(a,b) : a \equiv b \bmod m\}$ is an equivalence relation on z^+.

ii) Show also if $x_1 \equiv y_1$ and $x_2 \equiv y_2$ then $(x_1 + x_2) \equiv (y_1 + y_2)$.

(ii) $x_1 \equiv y_1 \Rightarrow (x_1 - y_1)$ is divisible by m —(1)

and $x_2 \equiv y_2 \Rightarrow (x_2 - y_2)$ is divisible by m —(2)

Adding eqn (1) & (2) -

$\Rightarrow ((x_1 - y_1) + (x_2 - y_2))$ is divisible by m.

$\Rightarrow (x_1 + x_2) - (y_1 + y_2)$ is divisible by m.

$\Rightarrow (x_1 + x_2) \equiv (y_1 + y_2) \bmod m$.

$\Rightarrow (x_1 + x_2) \equiv (y_1 + y_2)$.

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Q Show that relation ... relation on collection of sets.

→ For equivalence relation reflexive, symmetric and transitive relation must satisfy.

For Reflexive Property – If $(a,a) \in R$.
then, $a \subseteq a \Rightarrow a \subseteq a$.
So, it is reflexive.

For symmetric property – If $(a,b) \in R$, then $a \subseteq b$.
then. $b \subseteq a$ i.e $(b,a) \notin R$.

It does not follow symmetric property.

So, it is not an equivalence relation.

Also, it is not an equivalence relation on collection of sets.

Q Check whether $R = \{(a,b) : a \geq b\}$ on set of real numbers is an equivalence relation or not.

→ $R = \{(a,b) : a \geq b\}$.

For Reflexive Property :-
If $(a,a) \in R$, i.e. $a \geq a$.
$\Rightarrow a \geq a$.
so, it is reflexive.

For symmetric Property :-
If $(a,b) \in R \Rightarrow$ i.e. $a \geq b$.
but, $b \ngeq a \Rightarrow (b,a) \notin R$.

So, it is not an symmetric property.

Therefore, it is not an equivalence relation.

Q Prove that if R is an equivalence relation then R^{-1} is also an equivalence on set A.

→ Given R is an equivalence relation.

1) For Reflexive Property :- If R is reflexive relation.
then if $(a,a) \in R$.
$\Rightarrow (a,a) \in R^{-1}$
R^{-1} is an eq. reflexive relation.

2) For symmetric Property :- If R is symmetric relation.
then if $(a,b) \in R$ and $(b,a) \in R$.
then $(b,a) \in R^{-1}$ and $(a,b) \in R^{-1}$.

So, R^{-1} is an symmetric relation.

3) for transitive property :- since R is transitive relation.
then $(a,b) \in R$, $(b,c) \in R \Rightarrow (a,c) \in R$.

so, then $(b,a) \in R^{-1}$ and $(c,b) \in R^{-1} \Rightarrow (c,a) \in R^{-1}$

so, R^{-1} is a transitive relation.

Therefore, R^{-1} is an equivalence relation.

Q If R and S are two equivalence relations then $R \cap S$ is also an equivalence relation.

→ Given R and S is an equivalence relation.

1) for Reflexive relation :- If $(a,a) \in R$.
and $(a,a) \in S$.
then, $(a,a) \in R \cap S$.

So, it is reflexive.

2) for symmetric property :- $(a,b) \in R$ & $(b,a) \in R$.
$(a,b) \in S$ & $(b,a) \in S$

For $(R \cap S)$: $(a,b) \in R \cap S$ & $(b,a) \in R \cap S$.

So, it is symmetric relation.

3) for transitive property :- $(a,b) \in R$ & $(b,c) \in R$ & $(a,c) \in R$.
$\Rightarrow (a,b) \in S$ & $(b,c) \in S$ & $(a,c) \in S$.

then, $(a,b) \in R \cap S$ & $(b,c) \in R \cap S \Rightarrow (a,c) \in R \cap S$.

So, it is transitive relation.

Therefore, it is an equivalence relation.

* **Equivalence Class :-** Let R be an equivalence relation defined on set A the set of all elements that are related to an element $a \in A$ is called equivalence class of that element a.

If $a \in A$

then, $\boxed{[a] = \{b : aRb\}}$

It is represented by $[a]_R$ or $[a]$.

e.g. $A = \{1,2,3\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$

$[1] = \{1,2\}$

$[2] = \{1,2\}$

$[3] = \{3\}$

Properties :-

- $a \in [a]_R$, $\forall a \in A$.
- $[a] = [b]$ iff $(a,b) \in R$.
- $[a] \neq [b]$ or $[a] \cap [b] = \phi$.

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elements of set A under equivalence relation
R are is called Quotient set of A by R.
It is denoted by $\boxed{A/R}$ (A is induced by R).

$A = \{1, 2, 3\}$.

$R = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$.

$[1] = \{1,2\}$.

$[2] = \{1,2\}$

$[3] = \{3\}$

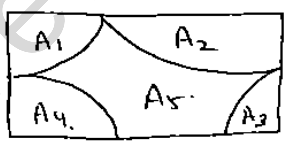
$A/R = \{[2]_R, [3]_R\}$.

* Partition :- The partition of A set is a collection of mutually disjoint, non-empty subsets of A whose union is A.

~~Partition~~ so, P is called partition of A if-

- P contains non-empty subset of A. i.e. $A_1, A_2, A_3, \ldots A_n$ where, $\boxed{A_i \subseteq A}$ for $i = \{1, 2, 3, \ldots n\}$.
- Union of all subsets under partition P gives set A. i.e. $A_1 \cup A_2 \cup A_3 \cup A_4 \cup \ldots \cup A_n = A$.
  or $\boxed{\bigcup_{i=1}^{n} A_i = A}$
- The elements of partition P are pair wise disjoint i.e. $|A_i \cap A_j \neq \Phi|$ or $[A_i = A_j]$

Theorem:- Every equivalence relation on a set generates a unique partition of set. The blocks of this partition form equivalence classes.



Q Let R be defined on set $A = \{1,2,3,4\}$ as $R = \{(a,b) : a-b = e\ldots$
find all the equivalence classes.

→ $A = \{1, 2, 3, 4\}$.

$R = \{(1,1), (1,3), (2,2), (2,4), (3,1), (3,3), (4,2), (4,4)\}$.

$[1] = \{1, 3\}$.

$[2] = \{2, 4\}$.

$[3] = \{1, 3\}$.

$[4] = \{2, 4\}$.

v.imp Q. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and Let R be the relation of $A \times A$ defined by $(a,b) \, R \, (c,d)$ iff $a+d = b+c$.

1) Prove that R is an equivalence relation.
2) Find $[2, 5]_R$

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(1) ① Reflexive :- $(a+b) = (a+b)$

$\Rightarrow (a,b)\ R\ (a,b)$.

So, it is reflexive.

② Symmetric :- Let $(a,b)\ R\ (c,d)$.

$\Rightarrow a+d = b+c$.

$\Rightarrow b+c = a+d$.

$\Rightarrow c+b = d+a$.

$\Rightarrow (c,d)\ R\ (a,b)$.

So, it is symmetric.

③ Transitive :- Let, $(a,b)\ R\ (c,d)$.

$\Rightarrow a+d = b+c$. —①.

and, $(c,d)\ R\ (e,f)$.

$\Rightarrow (c+f) = (d+e)$ —②.

Adding eqn ① & ②.

$a+\cancel{d}+\cancel{c}+f = e+\cancel{d}+b+\cancel{c}$.

$a+f = b+e$.

$\Rightarrow (a,b)\ R\ (e,f)$

So, it is transitive.

$\Rightarrow$ It is an equivalence relation.

(2) $[2,5]_R = ?$

$(2,5)\ R\ (x,y)$

$\Rightarrow 2+y = x+5$

$\Rightarrow y = x+3$.

$x=1$, $y=4$

$x=2$, $y=5$

$x=3$, $y=6$.

$x=4$, $y=7$.

$x=5$, $y=8$.

$x=6$, $y=9$

$[2,5]_R = \{(1,4), (2,5), (3,6), (4,7)\ (5,8), (6,9)\}$.

* Closure of Relation :-

Reflexive Closure :- Let R be a relation on set A then $R \cup I_A$ is called reflexive closure of R.

where, $I_A = \{(a,a) ; \forall\ a \in A\}$

$\boxed{R_F = R \cup I_A}$

eg. Let, $A = \{1,2,3\}$

$R = \{(1,1), (1,2), (2,3)\}$.

$R_F = \{(1,1), (1,2), (2,3), (2,2), (3,3)\}$

$R \cup R^{-1}$ is called symmetric closure of R where, R^{-1} is the Inverse of relation R.

$$\boxed{R_S = R \cup R^{-1}}$$

e.g. $A = \{1, 2, 3\}$.

$R = \{(1,1), (1,2), (2,3)\}$.

$R^{-1} = \{(1,1), (2,1), (3,2)\}$.

$R_S = \{(1,1), (1,2), (2,3), (2,1), (3,2)\}$.

Transitive closure :- Let R be a relation on set A then transitive closure R^* or R^+ is equal to

$$R^* = R \cup R^2 \cup R^3 \cup R^4 \cdots \cup R^n.$$

or $$\boxed{R^* = \bigcup_{i=1}^{n} R^i}$$

where, 'n' is the number of elements in the set.

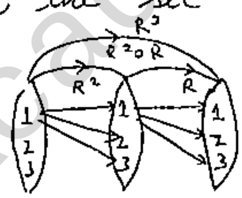
e.g. $A = \{1, 2, 3\}$.

$R = \{(1,1), (1,2), (2,3)\}$.

$R_0 R = \{(1,2), (1,3), (1,1)\}$.

$R^2{}_0 R = \{(1,1), (1,2), (1,3)\}$.

$R^3 = \{(1,1), (1,2), (1,3)\}$.



* Pictorial Representation of Relation :-

A relation can be expressed in following ways-

- Relation as a Matrix.
- Relation as a Directed Graph (Di-graph).
- Relation as an Arrow Diagram.
- Relation as a Table.

• Relation as a Matrix :-

Let, $A = \{a_1, a_2, \cdots a_i, \cdots a_m\}$.

$B = \{b_1, b_2, \cdots b_j, \cdots b_n\}$.

are finite sets & contain m and n elements respectively.

Let, R be a relation from set A to set B then the relation matrix of R is a $m \times n$ matrix defined by -

$$M_R = [m_{ij}] = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$$

e.g. $A = \{1, 2, 3\}$.

$R = \{(1,1), (1,2), (2,3), (3,3), (2,1)\}$.

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$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

This matrix is also called <u>zero one</u> matrix or <u>Boolean</u> matrix or <u>adjacency</u> matrix of relation of R.

*(1) If $\boxed{m_{ii} = 1}$ then relation is Reflexive i.e. main diagonal of matrix contain 1.

*(2) If $\boxed{m_{ij} = m_{ji}}$ then relation is symmetric.

or $\boxed{M_R = M_R^T}$ (T = Transpose of Matrix).

Q Let $N = \{1, 2, 3, ---\}$ and the relation R is defined in $N \times N$ as follows. $(a,b) R (c,d)$ iff $ad = bc$. Then show that R is an equivalence relation or not.

→ 1) <u>Reflexive</u> :-
$ab = ba$.
$\Rightarrow (a,b) R (a,b)$
So, it is reflexive.

2) <u>Symmetric</u> :- Let, $(a,b) R (c,d)$
$ad = bc$.
$bc = ad$.
$cb = \cancel{da}\ da$
$\Rightarrow (c,d) R (a,b)$
So, it is symmetric.

3) <u>Transitive</u> : Let $(a,b) R (c,d)$
$ad = bc$. — ①
& $(c,d) R (e,f)$
$cf = de$. — ②
multiply eqn ① & ②
$a\cancel{d} \cdot \cancel{c}f = b\cancel{c} \cdot \cancel{d}e$
$a \cdot f = b \cdot e$.
$a \cdot f = e \cdot b$.
$(a,b) R (e \cdot f)$
So, it is transitive.

Therefore, it is an equivalence relation.

*(3) A relation R is transitive if
$\boxed{M_R^2 + M_R = M_R}$

Join and Meet of Relation Matrix :-

Join $(\vee) \Rightarrow$ OR

Meet $(\wedge) \Rightarrow$ AND.

Join :- It is denoted by '$\vee$'.

$1 \vee 1 = 1$
$1 \vee 0 = 1$
$0 \vee 1 = 1$
$0 \vee 0 = 0$.

Join of two matrixes can be obtained by performing join of corresponding elements of matrices.

Union of two Matrices :- $\boxed{M_{R \cup S} = M_R \vee R_S.}$

Meet :- It is denoted by '$\wedge$'.

$1 \wedge 1 = 1$
$1 \wedge 0 = 0$
$0 \wedge 1 = 0$
$0 \wedge 0 = 0$.

Meet ~~Join~~ of two matrices can be obtained by performing ~~g~~ meet of corresponding elements of matrices.

Intersection of Relation :- $\boxed{M_{R \cap S} = M_R \wedge M_S}$

eg
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

e.g
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Transpose of a Relation :- $\boxed{M_{R^{-1}} = M_R^T.}$

Q. Set $A = \{a, b, c\}$ has two relation R & S represented by following matrices.

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \& \quad M_S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ find $M_{R \cup S}$ or, $M_{R \cap S}$, $M_{R \circ S}$, $M_{R \oplus S}$.

→ $M_{R \cup S} = M_R \vee M_S.$
$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$M_{R \cap S} = M_R \wedge M_S$
$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$M_{R\circ S} = M_R \cdot M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

{ Non-zero element should be 1. }

$$= \begin{bmatrix} 1+0+0 & 1+0+0 & 1+0+1 \\ 0+0+0 & 0+0+0 & 0+0+1 \\ 1+1+0 & 1+1+0 & 1+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_{R\oplus S} = M_{R\cup S} - M_{R\cap S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

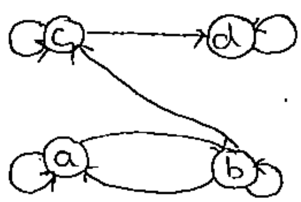
{ XOR: Same input - 0. Diff input - 1. }

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- **Relation as a Directed Graph (Di-Graph) :-**

e.g. $A = \{a, b, c, d\}$

$R = \{(a,a), (a,b), (b,a), (b,b), (b,c), (c,c), (c,d), (d,d)\}$



* **Partial order relation (POR)** :- A relation R defined on set A is said to be POR if it is -

- Reflexive ⇒ [$(a,a) \in R$, $\forall a \in A$]
- Antisymmetric ⇒ [$(a,b) \in R$ and $(b,a) \in R \Rightarrow a = b$]
- Transitive ⇒ [$(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$]

Any POR relation can be represented as - '$\preceq$'

* **Poset** :- The set A along with POR is called Poset or Partial ordered set.

It is represented by $(A, \preceq)$

e.g. The relation "$\leq$" is a POR on the set of integers 'Z'.

$(Z, \leq)$.

1) Reflexive - $a \leq a$, $\forall a \in Z$.
2) Antisymmetric - $a \leq b$ and $b \leq a \Rightarrow a = b$.
3) Transitive - $a \leq b$ and $b \leq c \Rightarrow a \leq c$.

$\leq$ is a POR

$\geq$ is a POR

$\subset$ is a POR

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defined on set A is said to be Total order relation if and only if it is.

- Partial order Relation.
- Either $(a,b) \in R$ or $(b,a) \in R$, $\forall\ a, b \in A$ (It is calle comparability).

e.g. $R = \{(a,b) : a \le b\}$ over the set of integers is a TOR

Reflexive → : $a \le a \in R$.
Antisymmetric → $a \le b$ and $b \le a \Rightarrow a = b$.
Transitive → $a \le b$ and $b \le c \Rightarrow a \le c$.
It is POR.

* Function :-

Let X and Y be two non-empty sets. A function f from X to Y is a rule that assigns to each element $x \in X$, a unique element $y \in Y$.
It is denoted by $\boxed{f : X \to Y.}$
function is also called Mapping, transformation or correspondence.
If $x \in X$, $y \in Y$ and $f : X \to Y$ then we write $f(x) = Y$.
If $\boxed{X}$ has $\boxed{m}$ elements and $\boxed{Y}$ has $\boxed{n}$ elements.

$f : X \to Y$ n^m (No. of function)
$f : Y \to X$ m^n
$f : X \to X$ m^m

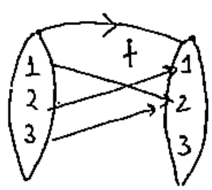
Note: Total no. of relation from one set to another set $(X \to Y)$
Total no. of relations from one set to one set (same) $(X \to X)$

No. of relations > No. of functions. (Always).
$\boxed{2^{m \times n} > n^m}$

* Domain and Codomain of a function :-

If $f : X \to Y$
X should be domain & Y should be codomain

$A = \{1, 2, 3\}$.



A A

$f : A \to A$.
Domain $= A = \{1, 2, 3\}$
Co-domain $= A = \{1, 2, 3\}$.
y is image of x.
x is preimage of y.

$f(1) = 2$.
$f(2) = 1$
$f(3) = 2$.

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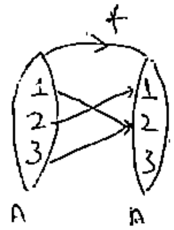
* Range of a function :- Range is subset of Codomain.

Range $\subseteq$ Codomain.

e.g



Codomain = {1, 2, 3} = A.

Range = {1, 2}.

y is image of x.

x is preimage of y.

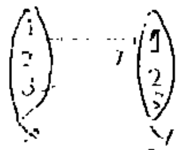
Q. Let A = {1, 2, 3, 4}, B = {a, b, c, d}, & f : A → B such that f = {(1,a), (2,c), (3,d), ~~4,d~~ (4,d)}. Find domain, codomain and range.

Domain = A = {1, 2, 3, 4}.

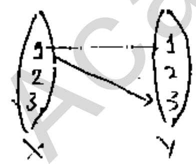
Codomain = B = {a, b, c, d}.

Range = {a, c, d}.

NOTE :



Not a function.
(No element should be left in X)

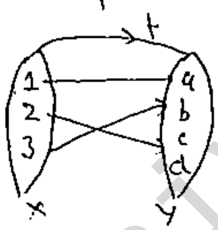


Not a function
(No two images of single element is not possible)

* Types of function :-

• One to one (Injective) function :- Let $f : X \to Y$ then f is called one to one function if for distinct elements of X there are distinct images in Y. i.e. f is one to one iff $\boxed{f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$ $\forall\ x_1, x_2 \in X$.

e.g



{(1,a), (2,c), (3,b)}

e.g Let $f(x) = 5x+1$ on set of integer.

Let $f(x_1) = f(x_2)$.

$\Rightarrow 5x_1 + \cancel{1} = 5x_2 + \cancel{1}$

$\Rightarrow \cancel{5}x_1 = \cancel{5}x_2$

$\Rightarrow \boxed{x_1 = x_2}$.

• Onto (surjective) function :- Let $f : X \to Y$ then f is called onto function if for every element $y \in Y$ there is an element $x \in X$ such that $\boxed{f(x) = y}$ (or) function f is onto if $\boxed{\text{codomain} = \text{Range}}$.

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eg. $f(x) = 5x+1$ over the set of real numbers. $f: R \to R$.

$\Rightarrow f(x) = 5x+1$

$\Rightarrow y = 5x+1$

$\Rightarrow \boxed{x = \frac{y-1}{5}}$ $\forall y \in R \Rightarrow x \in R$. (there exist $\exists x \in R$

$f(x) = 5x+1$ over the set of integer no's $f: Z \to Z$.

$\Rightarrow f(x) = 5x+1$

$\Rightarrow y = 5x+1$

$\Rightarrow \boxed{x = \frac{y-1}{5}}$ $\forall y \in Z \Rightarrow x \notin Z$. (there does not exist $\exists x \in Z$).

- One to one onto (Bijective) function :- A function reaches one to one and onto both is called bijective function.

Let $f(x) = 5x+1$ on set of real no's.

$\Rightarrow$ Let $f(x_1) = f(x_2)$

$\Rightarrow 5x_1 + \cancel{1} = 5x_2 + \cancel{1}$

$\Rightarrow \cancel{5}x_1 = \cancel{5}x_2$

$\Rightarrow \boxed{x_1 = x_2}$

Let, $f(x) = 5x+1$.

$\Rightarrow y = 5x+1$

$\boxed{x = \frac{y-1}{5}}$ $\forall y \in R \Rightarrow$ there exist $\exists x \in R$

- Into function :- A function which is not onto is called into function.

- Identity function :- Let $f: X \to X$ function f is called identity function if $\boxed{f(x) = x}$ $\forall x \in X$.

If image of every element is element itself then it is called Identity function.

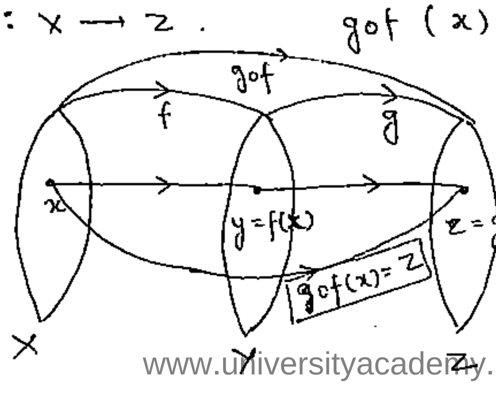
It is denoted by I_X.

* Composition of Function :-

$f: X \to Y$. $f(x) = y$

$g: Y \to Z$ $g(y) = z$.

$g \circ f: X \to Z$. $g \circ f(x) = z$.



$z = g(y) = g(f(x)) = g \circ f(x)$.

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- composition of function is not commutative.
  i.e. $\boxed{fog \neq gof}$
- composition of function is Associative.
  i.e. $\boxed{(hog)of = ho(gof)}$

Proof Let, $f(x) = y$, $g(y) = z$, $h(z) = w$

Let, $f: x \to y$, $g: y \to z$, $h: z \to W$.

LHS $(hog)of$

$\Rightarrow (hog)of(x)$

$\Rightarrow (hog)f(x)$

$\Rightarrow (hog)y$

$\Rightarrow h(g(y))$

$\Rightarrow h(z)$

$\Rightarrow w$

RHS $ho(gof)$

$\Rightarrow ho(gof)(x)$

$\Rightarrow ho(g(f(x)))$

$\Rightarrow ho(g(y))$

$\Rightarrow h(z)$

$\Rightarrow w$

$\boxed{LHS = RHS}$ (Hence Proved). composition of function is Associative.

Q. $f(x) = 3x+2$, $g(x) = 4x+8$ on the set of real no's. Find gof & fog.

$fog(x) = f(g(x))$
$= 3(g(x))+2$
$= 3(4x+8)+2$
$= 12x+24+2$

$\boxed{fog(x) = 12x+26}$

$gof(x) = g(f(x))$
$= 4(f(x))+8$
$= 4(3x+2)+8$
$= 12x+8+8$

$\boxed{gof(x) = 12x+16}$

Q Let h, g & f be functions from N to N. such that $f(x) = x+1$, $g(x) = 2x$ & $h(x) = \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd} \end{cases}$.

Find fof, gof, fog, goh, hog, (fog)oh, go(fog)oh.

$\rightarrow$ $fof(x) = f(f(x))$
$= \cancel{f(f(x)+1)}\ f(x)+1$
$= x+1+1$

$fof(x) = x+2$

$gof(x) = g(f(x))$
$= 2f(x)$
$= 2\cdot(x+1) = 2x+2$

$fog(x) = f(g(x))$
$= g(x)+1 = 2x+1$

$$g \circ h = g(h(x))$$
$$= 2(h(x)).$$

$\Rightarrow$ if x is even.

$g \circ h = 0$.

$\Rightarrow$ if x is odd.

$g \circ h = 2 \times 1 = 2$.

$$h \circ g(x) = h(g(x))$$
$$= h(2x)$$
$$h \circ g(x) = 0. \quad \{ \forall x \text{ is even} \}.$$

$$(f \circ g) \circ h(x) = \begin{cases} (f \circ g)(h(x)) & \text{if } x \text{ is even} \\ (f \circ g)(h(x)) & \text{if } x \text{ is odd} \end{cases}$$

$$= \begin{cases} (f \circ g)\, 0 & \text{if } x \text{ is even} \\ (f \circ g)\, 1 & \text{if } x \text{ is odd.} \end{cases}$$

$$= \begin{cases} f \circ (g(0)) & \text{if } x \text{ is even} \\ f \circ (g(1)) & \text{if } x \text{ is odd.} \end{cases}$$

$$= \begin{cases} f(0) & \text{if } x \text{ is zero.} \\ f(2) & \text{if } x \text{ is } 2. \end{cases}$$

$$= \begin{cases} 1 \\ 3 \end{cases}$$

$$g \circ (f \circ g) \circ h = \begin{cases} g \circ (f \circ g)(h(x)) & \text{if } x \text{ is even} \\ g \circ (f \circ g)(h(x)) & \text{if } x \text{ is odd.} \end{cases}$$

$$= \begin{cases} g \circ f \circ g(0) & \text{if } x \text{ is even} \\ g \circ f \circ (g(1)) & \text{if } x \text{ is odd.} \end{cases}$$

$$= \begin{cases} g \circ f(0) & \text{if } x \text{ is even} \\ g \circ f(2). & \text{if } x \text{ is odd.} \end{cases}$$

$$= \begin{cases} g(1) & \text{if } x \text{ is even} \\ g(3) & \text{if } x \text{ is odd.} \end{cases}$$

$$= \begin{cases} 2 \\ 6. \end{cases}$$

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* **Inverse of a Function** :- Let f be a <u>bijective function</u> from X to Y the inverse function of f is the function that assigns an element $y \in Y$ a unique element $x \in X$ such that $f(x) = y$.
Inverse of f denoted by f^{-1}.
i.e. [Iff $f(x) = y$ then $f^{-1}(y) = x$.]

e.g $f(x) = \frac{7-2x}{3}$.

$\Rightarrow y = \frac{7-2x}{3} \Rightarrow 3y = 7-2x$.

$\Rightarrow 2x = 7-3y$

$\Rightarrow x = \frac{7-3y}{2} \Rightarrow \boxed{f^{-1}(y) = \frac{7-3y}{2}}$

* **Theorem** :- If $f: X \to Y$ & $g: Y \to X$ then g is called inverse of f i.e. $\boxed{g = f^{-1}}$
iff [$fog = I_Y$ & $gof = I_X$.]

Proof :- Functions f & g should be bijective functions.
$f: X \to Y$ $\forall y \in Y$ there exist $\exists x \in X$ such that $f(x) = y$
$g: Y \to X$ $\forall x \in X$ there exist $\exists y \in Y$ such that $g(y) = x$

$gof : X \to X$ & $fog : Y \to Y$

Now, $gof(x) = g(f(x))$ & $fog(y) = f(g(y))$
$= g(y)$ $= f(x)$.
$gof(x) = x$. $fog(y) = y$.
$\Rightarrow \boxed{gof(x) = I_X}$ $\Rightarrow \boxed{fog(y) = I_Y.}$

Q Let $f: R \to R$ & $g: R \to R$ where R is the set of real no.s. Find fog & gof where $f(x) = x^2 - 2$ & $g(x) = x + 4$. State whether these functions are injective, surjective & bijective.

→ $fog(x) = f(g(x))$
$= (x+4)^2 - 2$.
$= x^2 + 16 + 8x - 2$.
$\boxed{fog(x) = x^2 + 8x + 14.}$

$gof(x) = g(f(x))$
$= (x^2 - 2) + 4$
$\boxed{gof(x) = x^2 + 2}$.

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let, $fog(x_1) = fog(x_2)$

$\Rightarrow x_1^2 + 8x_1 + \cancel{14} = x_2^2 + 8x_2 + \cancel{14}$

$\Rightarrow x_1(x_1+8) = x_2(x_2+8)$

Let, $x_1 = 2$, $x_2 = -10$

$\Rightarrow 2(2+8) = -10(-10+8)$

$\Rightarrow 20 = +20$.

$\Rightarrow \boxed{x_1 \neq x_2}$

It is not one to one function.

<u>for surjective</u>

$y = x^2 + 8x + 14$.

$y = x^2 + 8x + 14 + 2 - 2$

$\Rightarrow y = (x+4)^2 - 2$.

$\Rightarrow (x+4)^2 = y+2$. <u>or</u> $x+4 = \sqrt{y+2}$

$x = \sqrt{y+2} - 4$

$\Rightarrow (x+4)^2 > 0$.

for $y < -2$

$\Rightarrow y+2 < 0$

So, it is not onto function

It is also not bijective function.

<u>for injective</u>

Let, $gof(x_1) = gof(x_2)$

$\Rightarrow x_1^2 + \cancel{2} = x_2^2 + \cancel{2}$

$\Rightarrow x_1^2 = x_2^2$.

$\Rightarrow x_1 = \pm x_2$. $\Rightarrow \boxed{x_1 \neq x_2}$

It is not one to one function

<u>for surjective</u> :-

Let, $gof = x^2 + 2$

$\Rightarrow y = x^2 + 2$

$\Rightarrow x^2 = y - 2$.

If $y < 2$ therefore there is no such x.

So, it is not onto.

So, It is also not bijective.

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Q $f: Z^+ \to Z^+$, $f(x) = x^4$.
$f: Z \to Z$, $f(x) = x + x^2$.
$f: R \to R$, $f(x) = \sin x$.
$f: R \to R$, $f(x) = |x|$

Whether these functions are injective or not.

1) Let, $f(x_1) = f(x_2)$

$x_1^4 = x_2^4$

$x_1^4 = \pm x_2$

but, $f: Z^+ \to Z^+$

$\Rightarrow \boxed{x_1 = x_2}$ for Z^+ set it is one to one.

2) Let, $f(x_1) = f(x_2)$

$\Rightarrow x_1 + x_1^2 = x_2 + x_2^2$.

$\Rightarrow x_1(\cancel{1} + x_1) = x_2(1 + x_2)$

Let, $x_1 = 1$ & $x_2 = -2$.

$\Rightarrow 1(1+1) = -2(1-2)$

$\Rightarrow 2 = 2$

$\Rightarrow \boxed{x_1 \neq x_2}$. So, it is not one to one.

3) Let, $f(x_1) = f(x_2)$

$\Rightarrow \sin x_1 = \sin x_2$.

Let, $x_1 = 0$, $x_2 = \pi$.

$\Rightarrow \sin 0 = \sin \pi$

$\Rightarrow 0 = 0$.

$\Rightarrow \boxed{x_1 \neq x_2}$ So, it is not one to one.

4) Let, $f(x_1) = f(x_2)$

$\Rightarrow |x_1| = |x_2|$.

$\Rightarrow \boxed{x_1 = \pm x_2}$ So, it is not one to one.

Q State whether following functions are onto or not.

1) $f: R \to R$. $f(x) = e^x$.

2) $f: R \to R$ $f(x) = \sin x$.

3) $f: Z \to Z$ $f(x) = x^2 + 14x - 51$.

1) $f(x) = e^x$.

$\Rightarrow y = e^x$.

$\Rightarrow \boxed{x = \log y}$

$\therefore$ log is not defined for $-$ve real no.'s. So, for $-$ve value of y x does not exist. So, it is not onto.

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2) $y = \sin x$.

$\boxed{x = \sin^{-1} y}$

For $-1 \le y \le 1$ sin does not have any value.

so, it is not onto.

3) $y = x^2 + 14x - 51$.

$y = x^2 + 14x - 51 + 49 - 49$.

$y = (x+7)^2 - 100$.

$(x+7)^2 = y + 100$.

$\Rightarrow (x+7)^2 > 0$.

for $y < -100$.

$\Rightarrow y + 100 < 0$.

so, it is not onto.

* Theorem :- If $f : X \to Y$ & $g : Y \to Z$ are invertible function then $gof : X \to Z$ is also invertible function and

$\boxed{(gof)^{-1} = f^{-1} \circ g^{-1}}$.

Proof since f and g are invertible therefore they are bijective functions. so, $gof : X \to Z$ is bijective. Hence, gof is also invertible.

$f : X \to Y$ $\forall y \in Y$ there exist $\exists x \in X$ such that $f(x) = y \Rightarrow f^{-1}(y) =$

$g : Y \to Z$ $\forall z \in Z$ there exist $\exists y \in Y$ such that $g(y) = z \Rightarrow g^{-1}(z) =$

since, $gof : X \to Z$ is bijective and invertible.

$\Rightarrow gof(x) = z$.

$\Rightarrow \boxed{(gof)^{-1}(z) = x}$

(OR)

$gof(x) = g(f(x))$

$= g(y)$

$gof(x) = z$

$\Rightarrow \boxed{(gof)^{-1}(z) = x.}$ — ①

$f^{-1} \circ g^{-1}(z) = f^{-1}(g^{-1}(z))$

$= f^{-1}(y)$

$\Rightarrow \boxed{f^{-1} \circ g^{-1}(z) = x.}$ — ② From eq^n ① & ②.

$\therefore \boxed{(gof)^{-1}(z) = (f^{-1} \circ g^{-1})(z)}$

Q show that the function $f(x) = x^5$ and $g(x) = x^{1/5}$ are inverse of each other over the set of real numbers.

$\rightarrow$ $fog(x) = f(g(x))$

$$fog(x) = (x^{\frac{1}{5}})^5$$

$$\Rightarrow \boxed{fog(x) = x.} \Rightarrow \boxed{fog = I_x}$$

$$gof(x) = g(f(x))$$
$$= (x^5)^{1/5}$$

$$\Rightarrow \boxed{gof(x) = x.} \Rightarrow \boxed{gof = I_x}$$

Since, $fog = gof = I_x$

Therefore, they are inverse of each other.

Q Let, $X = \{1, 2, 3\}$ a function $f: X \to X$ is defined as $f = \{(1,2), (2,1), (3,3)\}$. Find f^2, f^3, f^{-1}.

→ $f^{-1} = \{(2,1), (1,2), (3,3)\}$.

$f(1) = 2$

$f(2) = 1$

$f(3) = 3$

$$f^2 = fof \Rightarrow fof(1) = f(f(1)) = f(2) = 1.$$
$$fof(2) = f(f(2)) = f(1) = 2.$$
$$fof(3) = f(f(3)) = f(3) = 3.$$

$$f^2 = fof = \{(1,1), (2,2), (3,3)\}.$$

$$f^3 = f^2of \Rightarrow f^2of(1) = f^2(f(1)) = f^2(2) = 2.$$
$$\Rightarrow f^2of(2) = f^2(f(2)) = f^2(1) = 1.$$
$$\Rightarrow f^2of(3) = f^2(f(3)) = f^2(3) = 3.$$

$$f^3 = f^2of = \{(1,2), (2,1), (3,3)\}.$$

* Permutation Function :-

Let, A be a finite set the set of all bijective functions from A to A is called the set of permutation functions from A to A. If set A contains n elements then there are $n!$ different permutations on A.

The matrix form to represent a permutation function on a finite set is to list all the elements of the set across the top row and the images of each element exactly below it.

Consider $A = \{x_1, x_2, x_3, ---- x_n\}$.

Let, 'P' be a permutation on set A. then

$$P = \begin{pmatrix} x_1 & x_2 & x_3 & ----- & x_n \\ P(x_1) & P(x_2) & P(x_3) & ----- & P(x_n) \end{pmatrix}$$

Where, $P(x_1), P(x_2), P(x_3) --- P(x_n) \in A$. ($P(x_1), P(x_2) \ldots P(x_n)$ are Images.)

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$P = 2! = 2.$

$P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$,

$P = \{P_1, P_2\}.$

- Inverse of Permutation :-

$A = \{1, 2, 3, 4\}.$ (OR) $P(1) = 2,\ P(2) = 3,\ P(3) = 1,\ P(4) = 4$

$P^{-1}(2) = 1,\ P^{-1}(3) = 2,\ P^{-1}(1) = 3,\ P^{-1}(4) = 4.$

$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} 2 & 3 & 1 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

$P^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}.$

- Composition or Product of Permutation :-

Let P_1 & P_2 be two permutations then composition of P_1 & P_2 is denoted by $P_1 \circ P_2$ is another permutation defined by

$\boxed{P_1 \circ P_2(x) = P_1(P_2(x))}$ where, $x \in X$.

e.g. $A = \{a, b, c\}.$

$P_1 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$, $P_2 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}.$

$P_1 \circ P_2 = \begin{pmatrix} a & b & c \\ b & c & a \\ b & a & c \end{pmatrix}$

$P_1 \circ P_2 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$

$P_1 \circ P_2(a) = P_1(P_2(a)) = P_1(b) = b.$

$P_1 \circ P_2(b) = P_1(P_2(b)) = P_1(c) = a.$

$P_1 \circ P_2(c) = P_1(P_2(c)) = P_1(a) = c.$

$P_2 \circ P_1 = \begin{pmatrix} a & b & c \\ c & b & a \\ a & c & b \end{pmatrix}$

$P_2 \circ P_1 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}.$

Q $A = \{5, 6, 7\}.$ $P_1 = \begin{pmatrix} 5 & 6 & 7 \\ 6 & 5 & 7 \end{pmatrix}$, $P_2 = \begin{pmatrix} 5 & 6 & 7 \\ 7 & 5 & 6 \end{pmatrix}$

find P^{-1}, P_2^{-1}, $P_1 \circ P_2$, $P_1^{-1} \circ P_2$, $P_2^{-1} \circ P_1^{-1}$.

$P_1 \circ P_2 = \begin{pmatrix} 5 & 6 & 7 \\ 7 & 5 & 6 \\ 7 & 6 & 5 \end{pmatrix}$

$$P_1 \circ P_2 = \begin{pmatrix} 5 & 6 & 7 \\ 7 & 6 & 5 \end{pmatrix}$$

$P_1(5) = 6$, $P_1(6) = 5$, $P_1(7) = 7$

$P_1^{-1}(6) = 5$, $P_1^{-1}(5) = 6$, $P_1^{-1}(7) = 7$.

$$P_1^{-1} = \begin{pmatrix} 5 & 6 & 7 \\ 6 & 5 & 7 \end{pmatrix}$$

$$P_2^{-1} = \begin{pmatrix} 5 & 6 & 7 \\ 6 & 7 & 5 \end{pmatrix}$$

$$P_1^{-1} \circ P_2 = \begin{pmatrix} 5 & 6 & 7 \\ 7 & 6 & 5 \end{pmatrix}$$

$$P_2^{-1} \circ P_1^{-1} = \begin{pmatrix} 5 & 6 & 7 \\ 6 & 5 & 7 \\ 7 & 6 & 5 \end{pmatrix}$$

$$P_2^{-1} \circ P_1^{-1} = \begin{pmatrix} 5 & 6 & 7 \\ 7 & 6 & 5 \end{pmatrix}$$

* Mathematical Induction

:- Mathematical Induction is a technique of proving a preposition over the positive Integers.

A Formal statement of principle of mathematical Induction can be stated as follows.

Let, $P(n)$ be a statement over the +ve Integers then

- Inductive Base : show that $P(1)$ is true for $n \geq 1$.
  $P(n_0)$ is true for $n \geq n_0$.
- Inductive Hypothesis : Let $P(k)$ is true for $n = k$. (arbitrary k)
- Inductive Step : verify $P(k+1)$ is true for $n = k+1$ using Inductive Hypothesis.

Q Show that sum of square of first n +ve Integers is $\frac{n(n+1)(2n+1)}{6}$.

→ Let, $P(n) : 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

1) P(n) for $n = 1$. $P(1)$. LHS $= 1^2 = 1$

Inductive Base :-

RHS $= \frac{n(n+1)(2n+1)}{6}$

$= \frac{1(2)(3)}{6} = \frac{\cancel{6}}{\cancel{6}} 1$

RHS $= 1$.

LHS = RHS.

P(1) is verified.

Let $P(k) = 1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$ is true.

3) Inductive step :-

Now, P(k+1) For $n = k+1$

$$LHS = 1^2 + 2^2 + 3^2 + \cdots + (k+1)^2$$

$$= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{[By using step 2]}$$

$$= (k+1)\left[\frac{k(2k+1) + 6(k+1)}{6}\right]$$

$$= \frac{(k+1)}{6}\left[2k^2 + k + 6k + 6\right]$$

$$= \frac{(k+1)}{6}\left[2k^2 + 7k + 6\right]$$

$$= \frac{(k+1)}{6}\left[2k^2 + 4k + 3k + 6\right]$$

$$= \frac{(k+1)}{6}\left[2k(k+2) + 3(k+2)\right]$$

$$= \frac{(k+1)(2k+3)(k+2)}{6}$$

$\therefore$ P(k+1) is true . Proved.

Q Prove by Mathematical Induction $1 \cdot 1! + 2 \cdot 2! + \cdots n \cdot (n!) = (n+1)! - 1$

→ Let, $P(n) : 1 \cdot (1!) + 2 \cdot (2!) + \cdots + n(n!) = (n+1)! - 1$.

1.) Inductive Base:-

For, $n = 1$, $LHS = 1 \cdot (1!) = 1 \cdot 1 = 1$

$RHS = (n+1)! - 1 = 2! - 1 = 2 - 1 = 1$.

LHS = RHS.

Hence, P(1) is true.

2) Inductive Hypothesis!-

Let, $P(k) = 1(1!) + 2(2!) + \cdots + k(k!) = (k+1)! - 1$ is true.

3) Inductive step!-

$P(k+1) : 1(1!) + 2(2!) + \cdots + (k+1) \cdot (k+1)! = (k+2)! - 1$.

Now, For $n = k+1$.

$$LHS = 1(1!) + 2(2!) + \cdots + (k+1)(k+1)!$$

$$= 1 \cdot (1!) + 2(2!) + \cdots + k(k!) + (k+1)(k+1)!$$

$$= (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)!\left[1 + k + 1\right] - 1$$

$$= (k+1)!\,(k+2) - 1$$

$$= (k+2)! - 1$$

$= RHS$. $\therefore$ P(k+1) is true. Hence (P(n)) is true for natural no.'s.

Q $3 + 33 + 333 + \cdots + \underbrace{333\ldots3}_{n \text{ times}} = \dfrac{10^{n+1} - 9n - 10}{27}$

→ Let, $P(n) : 3 + 33 + 333 + \cdots + 333\ldots3 = \dfrac{10^{n+1} - 9n - 10}{27}$

1) Inductive Base :-

for, $n = 1$. $LHS = 3$.

$RHS = \dfrac{10^2 - 9 - 10}{27} = \dfrac{100 - 19}{27} = \dfrac{\cancel{81}^{3}}{\cancel{27}} = 3$.

$LHS = RHS$

∴ $P(1)$ is true.

2) Inductive Hypothesis :-

Let, $P(k) : 3 + 33 + 333 + \cdots + \underbrace{333\ldots3}_{k \text{ times}} = \dfrac{10^{k+1} - 9k - 10}{27}$

3) Inductive step :- for $n = k+1$.

$P(k+1) : 3 + 33 + \cdots + \underbrace{333\ldots3}_{(k+1) \text{ times}} = \dfrac{10^{k+2} - 9(k+1) - 10}{27}$.

$LHS = 3 + 33 + 333 + \cdots + \underbrace{333\ldots3}_{k \text{ times}} + \underbrace{333\ldots3}_{(k+1) \text{ times}}$.

$= \dfrac{10^{k+1} - 9k - 10}{27} + \dfrac{(333\ldots3)3}{3}$

$= \dfrac{10^{k+1} - 9k - 10}{27} + \dfrac{999\ldots9}{3}$

$= \dfrac{10^{k+1} - 9k - 10}{27} + \dfrac{10^{k+1} - 1}{3}$

$\left\{ \begin{array}{l} 9 = 10^1 - 1 \\ 99 = 10^2 - 1 \\ 999 = 10^3 - 1 \end{array} \right\}$

$= \dfrac{10^{k+1} - 9k - 10 + 9 \cdot 10^{k+1} - 9}{27}$

$= \dfrac{10^{k+1}(1 + 9) - 9(k+1) - 10}{27}$

$= \dfrac{10^{k+2} - 9(k+1) - 10}{27} = RHS$

∴ $P(k+1)$ is true. (Proved)

Q Prove that $11^{n+2} + 12^{2n+1}$ is divisible by 133.

→ 1) Inductive Base :-

for $n = 1$, $P(1) = 11^3 + 12^3 = 3059 = 133 \times 23$.

3059 is divisible by 133.

∴ $P(1)$ is divisible by 133.

2) Inductive Hypothesis :-

Let $P(k) : 11^{k+2} + 12^{2k+1}$ is divisible by 133.

3) Inductive step :-

$P(k+1) : 11^{k+3} + 12^{2k+3}$ is divisible by 133.

$= 11 \cdot 11^{k+2} + 12^2 \cdot 12^{k+2}$

$= 11 \cdot 11^{k+2} + 144 \cdot 12^{k+2}$

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$= 11 \cdot 11^{k+2} + 133 \cdot 12^{2k+1} + 11 \cdot 12^{2k+1}.$

$= 11 (11^{k+2} + 12^{2k+1}) + 133 \cdot 12^{2k+1}.$

$= 11 \cdot 133 m + 133 \cdot 12^{2k+1}.$ $\{ \therefore m \in Z^+ \}$.

$= (11 m + 12^{2k+1}) 133.$

$= 133 \cdot s$ $\{ \therefore s \in Z^+ \}$.

so, $P(k+1)$ is divisible by 133.

Q1 Prove that $8^m - 3^m$ is divisible by 5.
Q-2 Prove that $n^5 - n$ is divisible by 5.
Q-3 Prove that $n^3 + 2n$ is divisible by 3.

1) $P(m)$: $8^m - 3^m$, is divisible by 5.

1) Inductive Base:

for $m = 1$. $P(1) = 8^1 - 3^1 = 5$

$\therefore$ $P(1)$ is divisible by 5.

2) Inductive Hypothesis:-

let, $P(k)$: $8^k - 3^k$ is divisible by 5.

3) Inductive step:-

$P(k+1)$: $8^{k+1} - 3^{k+1}$ is divisible by 5.

$= 8 \cdot 8^k - 3 \cdot 3^k.$

$= (5+3) 8^k - 3 \cdot 3^k.$

$= 5 \cdot 8^k + 3 \cdot 8^k - 3 \cdot 3^k.$

$= 5 \cdot 8^k + 3 (8^k - 3^k).$

$= 5 \cdot 8^k + 3 \cdot 5 \cdot s$ $\{ \therefore s \in Z^+ . \}$

$= 5 (8^k + 3 s).$

$= 5 \cdot m.$ $\{ \therefore m \in Z^+ \}$.

$\therefore$ $P(k+1)$ is true ie. it is divisible by 5.

(2). $P(n)$: $n^5 - n$ is divisible by 5.

1) Inductive Base:-

for $n = 1$. , $P(1) = (1)^5 - 1 = 0$.

$\therefore$ $P(1)$ is divisible by 5.

2) Inductive Hypothesis:-

$P(k)$: $k^5 - k$ is divisible by 5.

3) Inductive Step:-

$P(k+1)$: $(k+1)^5 - (k+1)$ is divisible by 5.

$= ({}^5C_0 + {}^5C_1 \cdot k + {}^5C_2 \cdot k^2 + {}^5C_3 k^3 + {}^5C_4 k^4 + {}^5C_5 k^5) - (k +$

$= (\cancel{1} + 5k + 10k^2 + 10k^3 + 5k^4 + k^5) - k - \cancel{1}.$

$= (k^5 - k) + (5k^4 + 10k^3 + 10k^2 + 5k).$

$= 5m + 5(k^4 + 2k^3 + 2k^2 + k)$.

$= 5m + 5p$.

$= 5(m+p)$

$= 5s$ $\quad \{\because m \,\&\, p \in Z^+\}$.

$\therefore$ $P(k+1)$ is divisible by 5. $\{\because s \in Z^+\}$

3) $P(n)$: $n^3 + 2n$ is divisible by 3.

1) Inductive Base:-

For. $n=1$, $P(1)$: $1^3 + 2(1) = 3$

$\therefore$ $P(1)$ is divisible by 3.

2) Inductive Hypothesis:-

Let, $P(k)$: $k^3 + 2k$ is divisible by 3.

3) Inductive Step:-

$P(k+1)$: $(k+1)^3 + 2(k+1)$ is divisible by 3.

$= (k+1)(k+1)^2 + 2(k+1)$

$= k^3 + 1 + 3k^2 + 3 + 2k + 2$

$= (k^3 + 2k) + 3k^2 + 6$.

$= 3 \cdot m + 3(k^2 + 2)$

$= 3 \cdot m + 3 \cdot s$.

$= 3(m+s)$

$= 3p$.

$\therefore$ $P(k+1)$ is divisible by 3.

Q Using MI show that $2^n < n!$ for $n \geq 4$.

→ $P(n)$: $2^n < n!$

1) Inductive Base:-

For, $n=4$, $P(4)$: $2^4 < 4!$

$16 < 24$.

$\therefore$ $P(4)$ is true.

2). Inductive Hypothesis:-

Let, $P(k)$: $2^k < k!$ is true for $k \geq 4$.

3) Inductive Step:-

$P(k+1)$: $2^{k+1} < (k+1)!$ is true for $(k+1) \geq 4$.

Using Inductive Hypothesis.

$2^k < k!$

$\Rightarrow 2^k \cdot 2 < 2(k!)$

$\Rightarrow 2^{k+1} < 2(k!)$ — ①

Again, $k \geq 4$

$\Rightarrow (k+1) \geq 5$

$\Rightarrow (k+1) > 2$.

$\Rightarrow (k+1)(k!) > 2(k!)$.

$(k+1)! < 2(k!)$ ——— (2)

from eqn (1) & (2).

$\Rightarrow 2^{k+1} < (k+1)!$

$\therefore P(k+1)$ is true.

Q Prove that. $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$ for $n \geq 2$.

→ $P(n): \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$.

1) Inductive Base:-

For. $n=2$, $P(2) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1.70178$.

$\sqrt{2} = 1.414$.

$\Rightarrow 1.7017 > 1.414$.

$\therefore P(2)$ is true.

2) Inductive Hypothesis:-

$P(k): \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k}$. is true for $k \geq 2$

3) Inductive step:-

$P(k+1): \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ is true for $k+1 \geq$

Using Inductive Hypothesis-

$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k}$

$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$

$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \frac{\sqrt{k}\cdot\sqrt{k+1} + 1}{\sqrt{k+1}}$

$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \frac{\sqrt{k}\cdot\sqrt{k} + 1}{\sqrt{k+1}}$ $\{\because \sqrt{k+1} >$

$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \frac{k+1}{\sqrt{k+1}}$

$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$

$\therefore P(k+1)$ is true.

Q If a set has n elements then prove that no. of possible relations on set A is greater than no. of possible functions on set A. $[2^{n^2} > n^n]$.

→ $P(n): 2^{n^2} > n^n$.

1) Inductive Base:-

for, $n=1$, $2^{1^2} > 1^1 \Rightarrow 2 > 1$.

$\therefore P(1)$ is true.

2) <u>Inductive Hypothesis:-</u>

$P(k)$: $2k^2 > k^k$.

3) <u>Inductive step:-</u>

$P(k+1)$: $2^{(k+1)^2} > (k+1)^{k+1}$

* Methods of Proofs :-

$p \to q$ implication

• **Direct Proof:** In this method we will assume that hypothesis p is true in implication ($p \to q$). We can prove that q is true by using some rules and theorems. This will show $p \to q$ is true.

Q Show by using method of direct proof if n is even then n^2 is even.

→ Let p: n is even.

q: n^2 is even

$p \to q$: If n is even then n^2 is even.

Now, Let p is true ie. n is even.

$\Rightarrow n = 2k$, where $k \in Z$.

$\Rightarrow n^2 = (2k)^2$

$\Rightarrow n^2 = 4k^2$

$\Rightarrow n^2 = 2 \cdot 2k^2$

$\Rightarrow n^2 = 2 \cdot m$ (where, $m = 2k^2 \in Z$)

So, n^2 is even, ie. q is true.

Therefore, $p \to q$ is true.

Q Using method of direct proof show that sum of 2 odd integers is even.

→ Let, p: x and y are odd integers.

q: $(x+y)$ is even.

$p \to q$: If x & y are two odd integers than $(x+y)$ is even.

Now, Let, p is true. ie. x & y are odd integers.

$x = 2n_1 + 1$

$y = 2n_2 + 1$, $n_1, n_2 \in Z$.

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$= 2(m_1+n_2+1)$

$= 2(K+1)$. , $K \in Z$, where, $K=(n_1+n_2) \in Z$.

$\Rightarrow$ So, $x+y$ is even. i.e. q is true.

Therefore, $p \to q$ is true.

• Indirect Proof :-

Proof By contrapositive :- In this method we can prove $p \to q$ is true by showing $\sim q \to \sim p$ is true.

Q Prove that if n is integer and $3n+2$ is even then n is even.

$\to$ Let, p: $3n+2$ is even , $n \in Z$.

q: n is even

$p \to q$: If $3n+2$ is even where $n \in Z$ then n is even.

$\sim p$: $3n+2$ is odd, $n \in Z$.

$\sim q$: n is odd.

$\sim q \to \sim p$: If n is odd then $(3n+2)$ is odd where $n \in Z$.

Now, Let, $\sim q$ is true i.e. n is odd.

$n = 2K+1$, $K \in Z$

So, $3n+2 = 3(2K+1)+2$

$= 6K+5$

$= 6K+4+1$.

$= 2(3K+2)+1$

$= 2m+1$, where $m = 3K+2 \in Z$.

$\Rightarrow$ $3n+2$ is odd i.e. $\sim p$ is true.

Hence, $\sim q \to \sim p$ is true.

Therefore, $p \to q$ is true.

Proof By contradiction :- In this method we assume that statement is false i.e. if p is statement then assume that $\sim p$ is true. then by using some rules & theorems we will show that given statement p is true as well as false i.e. we will reach at a contradiction. Therefore, p must be true.

ImpQ Proof that $\sqrt{3}$ is irrational.

Let, $\sqrt{3}$ is rational.

$\Rightarrow$ $\sqrt{3} = \frac{p}{q}$, where, p & q are prime no's and $q \neq 0$. —— (1)

By squaring both sides-

$\Rightarrow$ $3 = \frac{p^2}{q^2}$

$\Rightarrow$ $p^2 = 3q^2$ —— (2)

$\Rightarrow$ 3 is a factor of 'p^2'.

$\Rightarrow$ 3 is a factor of 'p'.

Let, $p = 3n$, $n \in Z$.

Put, $p = 3n$ in eq^n (2).

$\Rightarrow (3n)^2 = 3q^2$

$\Rightarrow 9n^2 = 3q^2.$

$\Rightarrow q^2 = 3n^2.$

$\Rightarrow$ 3 is a factor of q^2

$\Rightarrow$ 3 is a factor of q.

$\therefore$ 3 is a common factor of p & q both which means p and q are not prime which leads to a contradiction. i.e. p and q are not prime no. so, $\sqrt{3}$ is not a rational number.
i.e. $\sqrt{3}$ is Irrational.

Q Prove that $5-3\sqrt{2}$ is irrational.

→ Let, $5-3\sqrt{2}$ is rational. — ①

$\Rightarrow 5-3\sqrt{2} = \frac{p}{q}$, where, p & q are prime & $q \neq 0$.

$\Rightarrow 3\sqrt{2} = 5 - \frac{p}{q}$

$\Rightarrow \sqrt{2} = \frac{5q-p}{3q}$

$\Rightarrow \sqrt{2} = \frac{r}{s}$, where, $r = 5q-p$ & $3q \neq 0$.

on squaring both sides-

$\Rightarrow 2 = \frac{r^2}{s^2}$

$\Rightarrow 2s^2 = r^2.$ ②

$\Rightarrow r^2 = 2s^2$

$\Rightarrow$ 2 is factor of r^2.

$\Rightarrow$ 2 is factor of r.

Let, $r = 2n$, $n \in Z$.

Put in eqn ②-

$\Rightarrow (2n)^2 = 2s^2.$

$\Rightarrow 4n^2 = 2s^2.$

$\Rightarrow 2n^2 = s^2.$

$\Rightarrow s^2 = 2n^2.$

$\Rightarrow$ 2 is factor of s^2

$\Rightarrow$ 2 is factor of s.

$\therefore$ 2 is a common factor of r & s both which means r & s are not prime which leads to a contradiction. i.e. r & s are not prime no. So, $\sqrt{2}$ is not a rational no.
i.e. $5-3\sqrt{2}$ is irrational.

By counter Example :- In this method we will give an example for which given statement is false. This ~~sto~~ example is called counter example and negation of that statement is true.

an counter example.

→ Let f & g be two functions such that $f(x) = x^2$

$g(x) = 2x+1$.

$f \circ g = f(g(x)) = f(2x+1)$

$= (2x+1)^2 = 4x^2 + 4x + 1$.

$g \circ f = g(f(x)) = g(x^2)$

$= 2x^2 + 1$.

So, composition of function is not commutative.

Q Find a counter example to a statement that every +ve integ can be written as sum of square of 3 integers.

→ $1 = 1^2 + 0^2 + 0^2$

$2 = 1^2 + 1^2 + 0^2$

$3 = 1^2 + 1^2 + 1^2$

$4 = 2^2 + 0^2 + 0^2$

$5 = 2^2 + 0^2 + 1^2$.

$6 = 2^2 + 1^2 + 1^2$

$7 = \times$.

⇒ 7 can not be written as sum of square of 3 integers so, 7 is the counter example to that statement and statement is fa

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UNIT-3

Partial order set, Lattice and Boolean Algebra.

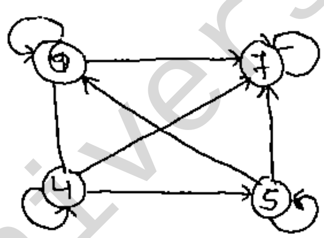
$\left\{ \begin{array}{l} a \preceq b \\ \text{or} \\ a \prec b \end{array} \right.$ [a is partially order related to b] }

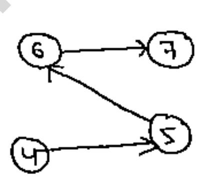
***** **Hasse Diagram of Poset :-** Let $(A, \preceq)$ is a poset and $(a, b) \in R$. i.e. $a \preceq b$.

It means 'a' is immediate predecessor of 'b' or 'b' is immediate successor of 'a'. 'B' is also called cover of 'a'.
It also means 'b' succeeds 'a' or 'a' preceeds 'b'.
If $a \preceq b$ but no element of set A lies b/w a & b then it is denoted by $\boxed{a << b}$.
a strictly succeeds 'b' that means there is no element in [illegible] or immediate.

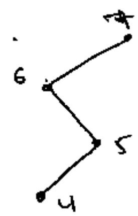
Hasse Diagram of Poset $(A, \preceq)$ is a directed graph whose vertices are elements of 'A' and there is directed path from a to b whenever $a << b$ [a strictly preceeds b]. In Hasse Diagram we will place 'b' higher to 'a' and draw a line b/w 'a' and 'b' to indicate succession.
As 'POR' is reflexive as well as transitive so, edges showing these properties will be removed.

e.g. Consider $A = \{4, 5, 6, 7\}$.
Let R be a relation '$\leq$' on A the directed graph of R is shown below. (Less than equal to)





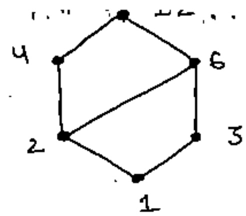
$\Rightarrow$



Or

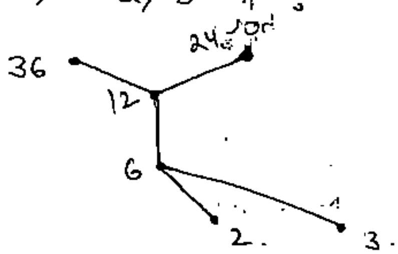
Q Let $A = \{1, 2, 3, 4, 6, 12\}$. Draw the Hasse Diagram of $(A, /)$.

$\Rightarrow$ $(A, /) \Rightarrow a / b$ [a divides b]
'a is divisible b', 'b' is not divisible by 'a'.

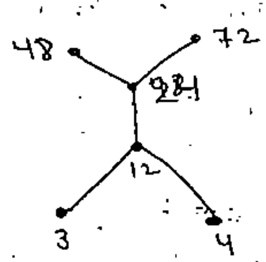


Q $A = \{2, 3, 6, 12, 24, 36\}$: $(A, /)$.

⇒ $(A, /)$ ⇒ a/b

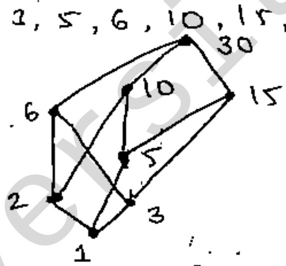


Q $A = \{3, 4, 12, 24, 48, 72\}$, $(A, /)$.



Q D_m is a set of factors of m. Draw the Hasse Di
of Poset $(D_m, /)$. where $m = 30$.

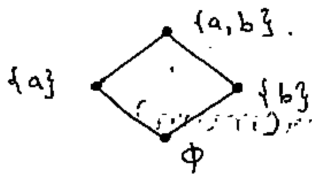
⇒ $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.



Q Draw Hasse Dia. of $(P(A), \subseteq)$. $A = \{a, b\}$.

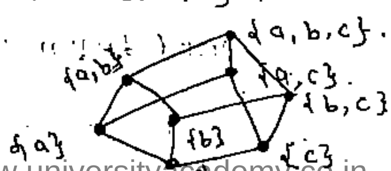
$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$.

⇒ $(\{\phi, \{a\}, \{b\}, \{a, b\}\}, \subseteq)$.



Q $A = \{a, b, c\}$, $(P(A), \subseteq)$.

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.



* Maximal and Minimal Elements :-

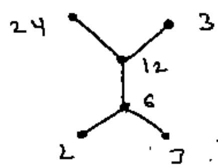
Minimal Element :- An element is called minimal element if no other element strictly preceeds it.

In Hasse Diagram an element is minimal element if no edge enters to this element from below.

Maximal Element :- An element is called maximal element if no other element strictly succeeds it.

In Hasse Diagram an element is maximal element if no edge leads to this element from above.

There is more than one maximal and minimal elements.

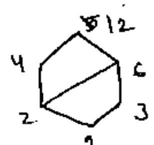


Minimal = 2, 3.
Maximal = 24, 36.

* First element (or) Least Element and Last Element (or) Greatest Element :-

- **First element (OR) Least Element :-** 'm' is first element if 'm' preceeds all the elements ie: $\boxed{m \preceq b, \forall b \in A}$
- **Last element (OR) Greatest element :-** 'n' is last element if 'n' succeeds all the elements ie. $\boxed{a \preceq n, \forall a \in A}$

eg.



First element = 1
Last element = 12

There is always only one least and one last element.

* Upper Bound and Lower Bound :-

- **Upper Bound :-** If x be a subset of poset $(A, \preceq)$, then M element $(M \in A)$ is called upper bound of subset X if M succeeds every element of A:
  ie $\boxed{M : \forall x \in X, x \preceq M, M \in A}$
- **Lower Bound :-** If X be a subset of $(A, \preceq)$, then $N \in A$, is called Lower bound of subset X. If N preeceds every element of A.
  ie. $\boxed{N : \forall x \in X, N \preceq x, N \in A}$

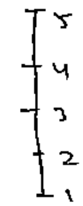
* Least Upper Bound (OR) (Superemum) [lub]

If an upper bound of x preeceds any onther upper bound of x then it is called lub or superemum.
It is denoted as 'lub(X) or sup(X)'.

* Greatest Lower Bound (glb) (OR) (Infemum)

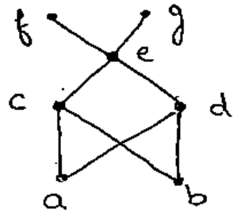
If an lower bound of x succeeds any other lower bound of x then it is called glb and infemum.

e.g. UB of $\{3,4\} = \{4,5\}$. $A = \{1,2,3,4,5\}$.



lub $= 4$.
LB of $\{3,4\} = \{1,2,3\}$.
glb $= 3$.

Q.



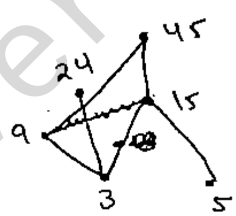
$A = \{c, d, e\}$.
Find upper bound and lower bound of A.

UB of c = c, e, f, g.
UB of d = d, e, f, g.
UB of e = e, f, g.
$\Rightarrow$ UB of A = $\{e, f, g\}$.
$\Rightarrow$ lub of A = e.

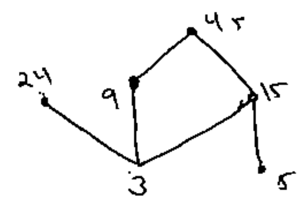
Minimal element = a, b.
Maximal element = f, g.
Least and Last element does not exist.

LB of c = c, a, b.
LB of d = d, a, b.
LB of e = c, d, a, b, e.
$\Rightarrow$ LB of A = $\{a, b\}$.
$\Rightarrow$ glb of A does not exist.

Q Draw the Hasse Diagram of Poset $(\{3, 5, 9, 15, 24, 45\}, |)$, find LB, UB, glb, lub of $\{3, 5\}$.



$\Rightarrow$



Minimal element = 3, 5
Maximal element = 24, 45.
Least and Last element does not exist.

UB of 3 = 9, 24, 15, 45
UB of 5 = 15, 45.
UB of $\{3,5\} = \{15, 45\}$
lub of $\{3,5\}$ = 15

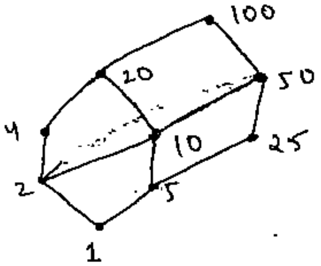
LB of 15 = 15, 5, 3.
LB of 45 = 45, 15, 5, 3
LB of $\{15, 45\}$ = 3, 5, 15
glb of $\{15, 45\}$ = 15.

LB of 3 = 3.
LB of 5 = 5
LB of $\{3,5\} = \phi$ $\Rightarrow$ glb of $\{3,5\} = \phi$.

Q. Draw a hasse Dia. of $(D_{100}, /)$. Find glb of $\{10,20\}$, lub of $\{10,20\}$, glb of $\{5,10,20,25\}$ & lub of $\{5,10,20,25\}$.

D_{100} = Set of factors of 100.

A = $(\{1, 2, 4, 5, 10, 20, 25, 50, 100\}, /)$.



UB of $\{10\}$ = 10, 20, 50, 100,

UB of $\{20\}$ = 100, 20,

LB of $\{10\}$ = 5, 2, 1, 10.

LB of $\{20\}$ = 4, 5, 2, 1, 10, 20.

LB of $\{5\}$ = 1, 5.

LB of $\{25\}$ = 5, 1, 25.

UB of $\{5\}$ = 5, 10, 20, 50, 100

UB of $\{25\}$ = 50, 100

UB of $\{10,20\}$ = $\{20, 100\}$.

lub of $\{10,20\}$ = 20.

LB of $\{10,20\}$ = $\{1, 2, 5, 10\}$.

glb of $\{10,20\}$ = 10.

glb of $\{5,10,20,25\}$ = $\{1, 5\}$

glb of $\{5,10,20,25\}$ = 5.

UB of $\{5,10,20,25\}$ = 100,

lub of $\{5,10,20,25\}$ = 100.

<u>NOTE</u>:- for $(D_m, /)$ where, m is factors of m.

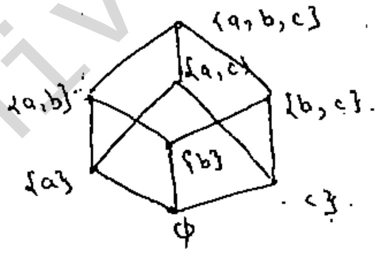
glb $\{a,b\}$: HCF $\{a,b\}$.

lub $\{a,b\}$ - LCM $\{a,b\}$.

Q. A = $\{a,b,c\}$, $(P(A), \subseteq)$, find Max., Min., greatest, least.

→ P(A) = $\{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{a,c\}\}$.

⇒ $(\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}, \subseteq)$.



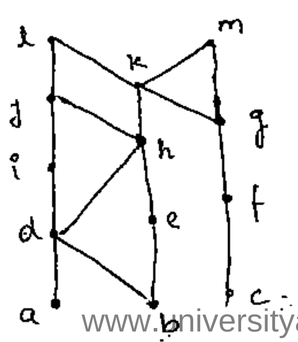
Maximal = $\{a,b,c\}$

Minimal = $\{\phi\}$.

Least = $\{\phi\}$

Greatest = $\{a,b,c\}$.

Q



Find min, max, greatest & least element.

lub of $\{a,b,c\}$

glb of $\{f,g,h\}$.

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Minimal Element = a, b, c.
Maximal Element = l, m.
Least and Greatest element does not exist.
UB of {a} = a, d, i, j, h, k, l, m.
UB of {b} = b, d, e, i, h, j, k, l, m.
UB of {c} = c, f, g, m, k, l.

UB of {a, b, c} = l, k, m.
lub ~~glb~~ of {a, b, c} = k.

LB of {f} = c, f.
LB of {g} = g, f, c.
LB of {h} = h, e, b, d, a.
LB of {f, g, h} does not exist.
and Hence, glb of {f, g, h} also does not exist.

* **Lattice** :- A ~~poset~~ lattice is a poset $(A, \leq)$ in which every subset (a, b) consisting of two elements has least upper bound (lub) and greatest lower ~~upper~~ bound (gl

$lub\{a, b\} = a \vee b \rightarrow$ Join of a & b. [$\vee$ - Join]
$glb\{a, b\} = a \wedge b \rightarrow$ Meet of a & b. [$\wedge$ - Meet]

Thus, lattice is a poset with two binary operations join & meet satisfying following properties.

Commutative Property :- → • $a \vee b = b \vee a$.
→ • $a \wedge b = b \wedge a$.

Associative Property :- → • $a \vee (b \vee c) = (a \vee b) \vee c$.
→ • $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

Absorption Law :- → • $a \vee (a \wedge b) = a$
→ • $a \wedge (a \vee b) = a$.

Lattice is denoted by $(L, \vee, \wedge)$.

$X \vee Y = X \cup Y$
$X \wedge Y = X \cap Y$
$\forall X, Y \in P(A)$

eg. 1) $(P(A), \cup, \cap)$ is a lattice under $\subseteq$, where $A = \{a, b\}$.
$\{a\} \vee \{b\} = \{a\} \cup \{b\} = \{a, b\}$.
$\{a\} \wedge \{b\} = \{a\} \cap \{b\} = \phi$.

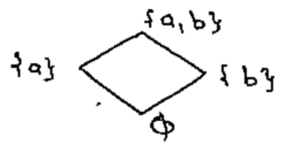
lub $\{a, b\} = \{a, b\}$.
glb $\{a, b\} = \phi$.

2) (Dm, lcm, gcd) is a lattice under '/' where,
lub(a, b) = lcm (a, b)
glb(a, b) = gcd (a, b)
where, a, b ∈ Dm.

e.g. $A = \{a, b\}$, $P(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$.

$\vee$	ϕ	$\{a\}$	$\{b\}$	$\{a,b\}$
ϕ	ϕ	$\{a\}$	$\{b\}$	$\{a,b\}$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$
$\{b\}$	$\{b\}$	$\{a,b\}$	$\{b\}$	$\{a,b\}$
$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$

[Join]



lub $= \{a,b\}$

$\wedge$	ϕ	$\{a\}$	$\{b\}$	$\{a,b\}$
ϕ	ϕ	ϕ	ϕ	ϕ
$\{a\}$	ϕ	$\{a\}$	ϕ	$\{a\}$
$\{b\}$	ϕ	ϕ	$\{b\}$	$\{b\}$
$\{a,b\}$	ϕ	$\{a\}$	$\{b\}$	$\{a,b\}$

[Meet]

glb $= \phi$.

Join and meet exist for every pair. so it is a lattice.

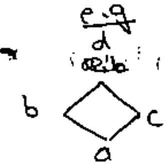
... lub any one does not exist. so, lattice ...

Here, 2 and 3 are not related to each other
$\Rightarrow$ glb & lb does not exist.
So, it is not a lattice.

It is a lattice. bcoz each element are related to each other. glb & lub exist.

- Properties :-
- Commutative :-
- Associative.
- Absorption.
- Idempotent law :- $\rightarrow a \vee a = a$
  $\rightarrow a \wedge a = a$.
- ... lattice :- $\rightarrow a \vee b = b$. if $a \le b$
  $\rightarrow a \wedge b = a$. if $a \le b$.

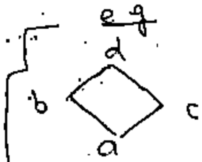
e.g



$a \vee d = d$
$a \wedge d = a$

- Isotonicity law or Property :-
  $a, b, c \in L$ (lattice). if $a \le b$.
  $\rightarrow a \vee c \le b \vee c$.
  $\rightarrow a \wedge c \le b \wedge c$.

e.g



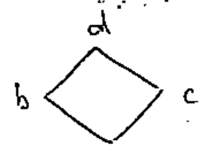
$a \wedge c = a$
$b \wedge c = a$
$\boxed{a \le a}$

$\boxed{a \le b}$
$a \vee c = c$
$b \vee c = d$
$\boxed{c \le d}$

- Distributive Inequality :-

$\rightarrow a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$
$\rightarrow a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$

e.g $a \vee (b \wedge c) = a \vee a = a$.
$(a \vee b) \wedge (a \vee c) = b \wedge c = a$ $\}$ $\boxed{a \le a}$



$a \wedge (b \vee c) = a \wedge d = a$. $\}$ $\boxed{a \ge a}$
$(a \wedge b) \vee (a \wedge c) = a \vee a = a$

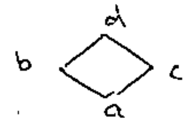
⇒ If $a \leq b$ and $a \leq c$

then, $a \leq b \vee c$, $a \leq b \wedge c$.

Eg $a \leq b$, $a \leq c$.

$b \vee c = d$.

$a \leq d$. , $a \leq a$.



⇒ If $b \leq a$ and $c \leq a$.

then $b \vee c \leq a$, $b \wedge c \leq a$.

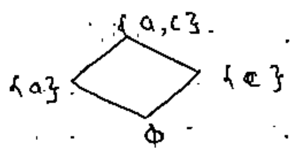
* **Sub-Lattice** :- Let, L be a lattice $(L, \leq)$ a non-empty subset 'S' of 'L' is called sublattice of L

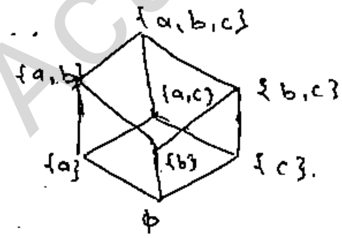
$\boxed{\text{If } a \vee b \in S \ \& \ a \wedge b \in S \text{ whenever } a, b \in S.}$

- Every sublattice is a lattice.
- Every singleton set is sublattice of 'L'
- If n divides m, then D_n is sub-lattice of D_m.

  e.g. If 25 divides 100, then D_{25} is sub lattice of D_{100}.

eg $A = \{a, b, c\}$. , $(P(A), \subseteq)$



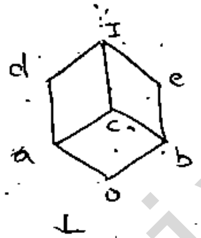


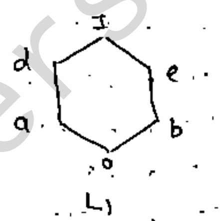
$\{a\} \vee \{c\} = \{a, c\}$

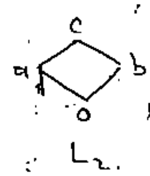
$\{a\} \wedge \{c\} = \phi$.

so, it is a sublattice.

eg







Find L_1 and L_2 are sublattice of L.

V	0	a	b	c
0	0	a	b	c
a	a	a	c	c
b	b	c	b	c
c	c	c	c	c

L_2 is a sublattice.

$a \vee b = c \in L_2$.

V	0	a	b	d	e	I
0	0	a	b	d	e	I
a	a	a	c	I	I	I
b	b	c	b	I	I	I
d	d	I	I	d	I	I
e	e	I	I	I	e	I
I	I	I	I	I	I	I

L_1 is not a sub lattice

$a \vee b = c \notin L_1$

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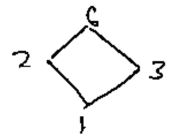
* Isomorphic Lattice :-

Mainly for Isomorphic lattice we check the no. of elements in two lattice are equal or not.
If not equal so, it is not a Isomorphic lattice.

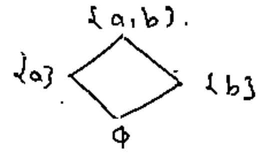
$f : L_1 \to L_2$

$f(a \vee b) = f(a) \vee f(b)$, $a, b \in L_1$.
$f(a \wedge b) = f(a) \wedge f(b)$

e.g.



$(D_6, /)$.
L_1



$(P(A), \subseteq)$.
L_2

Let, $f : L_1 \to L_2$ be a bijective function.
such that, $f(1) = \phi$, $f(2) = \{a\}$, $f(3) = \{b\}$, $f(6) = \{a,b\}$

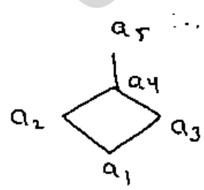
$f(1 \vee 2) = f(2) = \{a\}$. $\Rightarrow$ $f(1) \vee f(2) = \phi \vee \{a\} = \{a\}$
$f(1 \vee 3) = f(3) = \{b\}$ $\Rightarrow$ $f(1) \vee f(3) = \phi \vee \{b\} = \{b\}$.
$f(1 \vee 6) = f(6) = \{a,b\}$ $\Rightarrow$ $f(1) \vee f(6) = \phi \vee \{a,b\} = \{a,b\}$
$f(2 \vee 3) = f(6) = \{a,b\}$ $\Rightarrow$ $f(2) \vee f(3) = \{a\} \vee \{b\} = \{a,b\}$
$f(2 \vee 6) = f(6) = \{a,b\}$ $\Rightarrow$ $f(2) \vee f(6) = \{a\} \vee \{a,b\} = \{a,b\}$
$f(3 \vee 6) = f(6) = \{a,b\}$. $\Rightarrow$ $f(3) \vee f(6) = \{b\} \vee \{a,b\} = \{a,b\}$

LHS = RHS

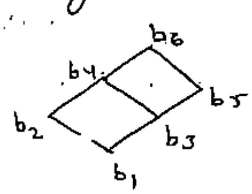
$f(1 \wedge 2) = f(1) = \phi$ $\Rightarrow$ $f(1) \wedge f(2) = \phi \wedge \{a\} = \phi$
$f(1 \wedge 3) = f(1) = \phi$ $\Rightarrow$ $f(1) \wedge f(3) = \phi \wedge \{b\} = \phi$
$f(1 \wedge 6) = f(1) = \phi$ $\Rightarrow$ $f(1) \wedge f(6) = \phi \wedge \{a,b\} = \phi$
$f(2 \wedge 3) = f(1) = \phi$ $\Rightarrow$ $f(2) \wedge f(3) = \{a\} \wedge \{b\} = \phi$
$f(2 \wedge 6) = f(1) = \phi$ $\Rightarrow$ $f(2) \wedge f(6) = \{a\} \wedge \{a,b\} = \phi$
$f(3 \wedge 6) = f(1) = \phi$ $\Rightarrow$ $f(3) \wedge f(6) = \{b\} \wedge \{a,b\} = \phi$

LHS = RHS

Q Consider the Lattice given below are they Isomorphic.



L_1



L_2.

Let, $f : L_1 \to L_2$ be a bijective function.
such that , $f(a_1) = b_1$, $f(a_2) = b_2$, $f(a_3) = b_3$, $f(a_4) = b_4$.
$f(a_5) = b_6$.

Here, no. of ~~two~~ elements of two lattice i.e. L_1 and L_2 are not equal so, they are not isomorphic

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* **Bounded Lattice :-** A lattice L is said to be bounded if it has a greatest or last (I or ~~Least~~ 1) element and least or first (0 or 0) element.

In such lattice we have:-

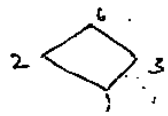
$$\begin{cases} a \vee I = I \\ a \wedge I = a. \end{cases}$$

$$\begin{cases} a \vee 0 = a. \\ a \wedge 0 = 0 \end{cases} \quad \forall \; a \in L, \quad 0 \leq a \leq I$$

E.g. $(D_m, /)$ is always bounded.

$0 = 1.$
$I = m.$

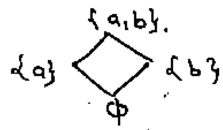
$(D_6, /)$



$0 = 1$
$I = 6.$

$(P(A), \subseteq)$ is always bounded.

$0 = \phi$
$I = A$



NOTE: $(Z, \leq)$ and $(N, \leq)$ is not a bounded lattice
or $(Z, \geq)$ and $(N, \geq)$ is not a bounded lattice
bcoz no. of Integers and Natural No.'s are not defined.

* **Complemented Lattice :-** (Firstly lattice must be bounded for complemented lattice).

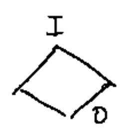
Let, L be a <u>bounded lattice</u> with first element 0 and last element I.

Let, $a \in L$ an element $a' \in L$ is called complement of 'a' if

$$\boxed{\begin{array}{l} a \vee a' = I \\ a \wedge a' = 0 \end{array}}$$

A lattice L is said to be complemented lattice if there exi complement of each element.

NOTE (1) $\boxed{\begin{array}{l} I \vee 0 = I \\ I \wedge 0 = 0 \end{array}}$



I and 0 are complements of each other.

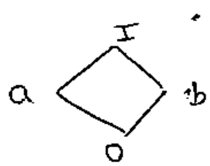
i.e. $\boxed{\begin{array}{l} I' = 0 \\ 0' = I \end{array}}$

(2) Complement of an element in a Lattice is not unique.

(3) Complement is symmetric.

$\boxed{a' = b \quad \text{then} \quad b' = a}$.

①



$a \vee b = I$, $I' = 0$
$a \wedge b = 0.$ $\quad 0' = I.$

So, $\boxed{a' = b.}$ $\Rightarrow$
and also $b' = a.$ bcoz it is symmetric

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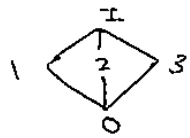
Q  find complement.

$I' = 0$

$0' = I$.

$a \vee a' = I$

$a \wedge a' = 0$

$1 \vee 2 = I$ $\quad$ $1 \vee 3 = I$

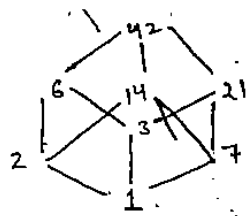
$1 \wedge 2 = 0$ $\quad$ $1 \wedge 3 = 0$

$\therefore$ $1' = 2, 3$. $\Rightarrow$ $2 \vee 3 = I$, $2 \wedge 3 = 0$ $\Rightarrow$ $2' = 1, 3$

$3' = 1, 2$.

Q $(D_{42}, /)$. Find complement of each element.

Factors of 42 = {1, 2, 3, 6, 7, 14, 21, 42}.



$I = 42$

$0 = 1$.

So, $1' = 42$.

$42' = 1$.

$2 \vee 21 = 42$
$2 \wedge 21 = 1$.

$2' = 21$
$21' = 2$.

$3 \vee 14 = 42$
$3 \wedge 14 = 1$

$3' = 14$
$14' = 3$.

In this only these element would be the complement if there is no relation b/w those 2 elements.

(OR)

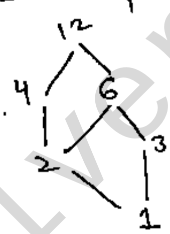
Make Join & Meet table and find I or 1 at place where Join and Meet are same

$6 \vee 7 = 42$
$6 \wedge 7 = 1$

$6' = 7$
$7' = 6$.

Q Is D_{12} a complemented lattice.

Factors of 12 = {1, 2, 3, 4, 6, 12}.



$I = 12$

$0 = 1$.

So, $12' = 1$

$1' = 12$

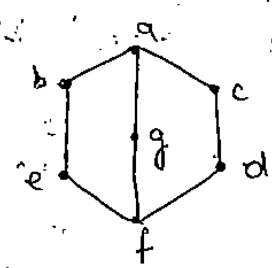
$2 \vee 3 = 6$.

$2 \wedge 3 = 1$.

complement of 2 does not exist. So, D_{12} is not a complemented lattice.

Q In the following lattice how many complement does the element 'e' have. Give all.

→



$I = a$

$0 = f$

$e \vee c = a$
$e \wedge c = f$

$e \vee d = a$
$e \wedge d = f$

e

$e \vee g = a$
$e \wedge g = f$

$e' = c, d, g$

Distributive Lattice :- A lattice L is said to be distributive lattice if for $a, b, c \in L$ following properties are satisfied.

- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

{These are distributive properties}

otherwise lattice L is non-distributive.

e.g. $(P(A), \subseteq)$ is a poset $\Rightarrow (P(A), \cup, \cap)$.

The power set P(A) of set A under two operations union and intersection is a distributive lattice. Since

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\quad \vee \Rightarrow \cup$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $\quad \wedge \Rightarrow \cap$

for, $A, B, C \in P(A)$.

Q If L be a bounded distributive lattice then prove that if a complement exists it is unique.

→ If let, a_1 & $a_2 \in L$ are complements of L.

Let, $a \in L$.

$$\left.\begin{aligned} a \vee a_1 &= I \\ a \wedge a_1 &= 0 \end{aligned}\right\} \text{— (1)}$$

$$\left.\begin{aligned} a \vee a_2 &= I \\ a \wedge a_2 &= 0 \end{aligned}\right\} \text{— (2)}$$

$a_1 = a_1 \vee 0$

$a_1 = a_1 \vee (a \wedge a_2)$ From eq^n (2)

$a_1 = (a_1 \vee a) \wedge (a_1 \vee a_2)$ [By distributive Lattice]

$a_1 = I \wedge (a_1 \vee a_2)$ [From eq^n - (1)]

$\boxed{a_1 = (a_1 \vee a_2)}$ — (3)

Again, $a_2 = a_2 \vee 0$.

$a_2 = a_2 \vee (a \wedge a_1)$ From eq^n (1)

$a_2 = (a_2 \vee a) \wedge (a_2 \vee a_1)$ [By distributive Lattice

$a_2 = I \wedge (a_2 \vee a_1)$ From eq^n (2)

$\boxed{a_2 = (a_2 \vee a_1)}$ — (4)

From eq^n (3) & (4)

$\Rightarrow \boxed{a_1 = a_2.}$

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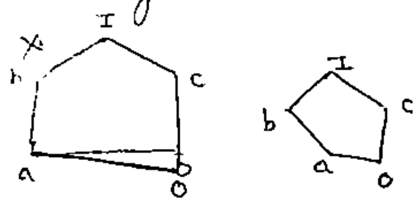
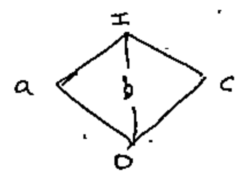
NOTE :- A lattice L is Non-Distributive if and only if it contains a sub lattice i.e. isomorphic to one of the two lattices given below-

Pentagon.



Diamond.



$a \vee (b \wedge c) = a \vee 0 = a$

$(a \vee b) \wedge (a \vee c) = b \wedge I = b$.

$\therefore$ $\boxed{a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)}$

$a \vee (b \wedge c) = a \vee 0 = a$

$(a \vee b) \wedge (a \vee c) = I \wedge I = I$.

$\boxed{a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)}$

Therefore, lattices isomorphic to these two lattices are also non-distributive.

* Complete lattice :-

A lattice L is called complete lattice if each of its non empty subset has a glb and lub.

Let A be the class of all subsets of some universal set U and a relation '$\leq$' is defined as

$$\boxed{x \leq y \Rightarrow x \subseteq y}$$

Such that $x \vee y = x \cup y$ &

$x \wedge y = x \cap y$

Every subset of A has a glb and a lub. So, A is complete lattice.

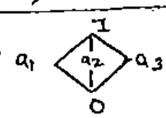
* Modular Lattice :-

A lattice L is called Modular if

$$\boxed{a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c}$$

$\boxed{\forall \ a, b, c \in L.}$

eg.



$a_1 \leq I$.

$a_1 \vee (a_2 \wedge I)$
$= a_1 \vee a_2$
$= I$.

$(a_1 \vee a_2) \wedge I$
$= I \wedge I$
$= I$.

NOTE :- $\vee \rightarrow +$, $\wedge \rightarrow \cdot$

* Boolean Algebra :-

(Demorgan's Law)

$(a \cdot b) + (a' + b') = 1$

$(a \cdot b) \cdot (a' + b') = 0$

$(a \cdot b)' = a' + b'$

$(a + b)' = a' \cdot b'$.

\# Boolean Algebra Properties.
\# SOP & POS form.
\# Max and Min term.
\# Logic Gates and its Implementations.
\# K-Map. 3 & 4 variables. with Don't care condition.
\# Simplification of Boolean Expressions.
\# Demorgan's Law.

From DLD

Prepositional Logic & Predicate Calcul

* **Preposition :-** Preposition is a declarative statement which is either true or false but not both.

It is usually denoted by lower case letters (p, q, r, - - - -
These are also called boolean variables, logic variables or prepositional variables.

e.g 1. Narendra Modi is the Prime Minister of India.
2. New Delhi is the capital of India.
3. Paris is in Australia.
4. 2+2 = 5
5. $x = 2$ is a solution of $x^2 = 4$.
6. Come Here.
7. Shut the door.
8. Where are you going?

First five are prepositions in which 1, 2 and 5 are true and 3 and 4 are false.
6, 7, 8 are not prepositions.

* **Compound Preposition :-** A compound preposition is formed by composition of two or more prepositions called sub-prepositions or components

NOTE: The preposition which can not be broken into sub-prepositions is called primitive preposition.

e.g. Sky is blue and clouds are white.
This prepositions contains two sub prepositions -
→ Sky is blue.
→ Clouds are white.

* **Fundamental Connectives :-**

connective word	Name of connective	Symbol.
NOT	Negation	~ or ' or ¬ or –
OR	Disjunction	$\vee$
AND	Conjunction	$\wedge$.
If - then	Implication (conditional)	$\longrightarrow$
If and only if	Biconditional	$\leftrightarrow$, $\rightleftarrows$

* Truth Table :-

- NOT (Negation).

p	~p
T	F
F	T

- Disjunction. (OR)

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

- Conjunction. (AND)

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

- Implication (If - Then).

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

In this it is only false when if condition is true & then condition is false.

$\underset{\text{True}}{p} \longrightarrow \underset{\text{False}}{q} \implies$ Ans is false

- Biconditional (If and only if).

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

It is only true when both p and q is true then it is true & both p and q are false then it is true.

Q Consider the following statements.

p : It is raining today.
q : It is cloudy today.
r : It is windy today.

Write following statements in symbolic form.

1) It is either raining or cloudy today.
   $p \vee q$.
2) Neither it is raining nor windy today.
   $\sim p \wedge \sim r$.

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$\sim(p \wedge \sim r)$

4) It is cloudy or windy today but not raining.

$(q \vee r) \wedge \sim p$.

5) It is false that, it is raining or cloudy but not windy.

$\sim((p \vee q) \wedge \sim r)$

6) It is windy else it is both raining and cloudy.

$r \vee (p \wedge q)$.

Q Let, p : It will rain tommorow. q : I shall go to market. Write meaning of following propositions.

1) $\sim p$ ~~and~~ $\wedge \sim q$.
Neither it will not rain tommorow ~~and~~ nor I shall go to market

2) $\sim p \vee \sim q$.
It will not rain tommorow then or I shall go to market.

3) $\sim q \wedge p$.
I shall not go to market and it will rain tommorow.

Q Construct the truth Table for $Y = \sim(p \vee q) \vee (\sim p \wedge \sim q)$.

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim p \wedge \sim q$	$\sim(p \vee q)$	Y
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	T	F	F	F
F	F	T	T	F	T	T.	T

Q Construct the truth table $((P \underset{A}{\rightarrow} Q) \vee R) \vee (P \rightarrow \underset{B}{Q} \rightarrow R)$.

P	Q	R	$P \rightarrow Q$	$(P \rightarrow Q) \vee R$	$P \rightarrow Q \rightarrow R$.	$A \vee B$
T	T	T	T	T	T	T
T	T	F	T	T	F	T
T	F	T	F	T	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	F	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

*** Logical Equivalent :-** If two prepositions $P(p, q, r, s, t \ldots)$ and $Q(p, q, r, s, t, \ldots)$ where $p, q, r \ldots$ are prepositional variables have always identical truth values. Then, P and Q are logical equivalence.

It is denoted as $P \equiv Q$.

e.g.

- $p \to q \equiv (\sim p \vee q)$
- $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
- $p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$.

p	q	$\sim p$	$p \to q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

p	q	$\sim q$	$\sim p$	A: $p \to q$	B: $q \to p$	C: $\sim p \vee q$	D: $\sim q \vee p$	$p \leftrightarrow q$	$A \cap B$	$C \cap D$
T	T	F	F	T	T	T	T	T	T	T
T	F	T	F	F	T	F	T	F	F	F
F	T	F	T	T	F	T	F	F	F	F
F	F	T	T	T	T	T	T	T	T	T

Q Establish following equivalence using Truth Table.

$(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$.

p	q	r	$p \vee q$	$(p \vee q) \to r$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Q $p \leftrightarrow q \equiv (p \vee q) \to (p \wedge q)$.

p	q	$p \vee q$	$p \wedge q$	$p \leftrightarrow q$	$(p \vee q) \to (p \wedge q)$
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	F	F
F	F	F	F	T	T

Q Let p, q & r are prepositions

p : Reema is not well.

q : Reema misses the major examination.

r : Reema passes the course.

Express each of the following prepositions as English sentences.

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1) $(p \to \sim r) \vee (q \to \sim r)$.

If Reema is not well then she does not passes the course or if she misses the major examination then she does not pass the course.

2) $(q \to (\sim r))$

If Reema misses the major examination then she does not pass the course.

3) $\sim q \leftrightarrow r$.

Reema does not misses the major examination if and only if she passes the course.

Q Write the following in symbolic form.

If either Ram takes Maths or Shyam takes science then hari will take Biology.

$(p \vee q) \to r$

Let, p : Ram takes Maths
q : Shyam takes science.
r : Hari will take Biology.

* Derived Connectives :- (NAND, NOR, X-OR).

NAND $p \uparrow q \equiv \sim(p \wedge q)$

NOR $p \downarrow q \equiv \sim(p \vee q)$

X-OR. $p \oplus q \equiv \sim(p \leftrightarrow q)$

p	q	$p \uparrow q$	$p \wedge q$	$\sim(p \wedge q)$	$p \downarrow q$	$p \vee q$	$\sim(p \vee q)$
T	T	F	T	F			
T	F	T	F	T			
F	T	T	F	T			
F	F	T	F	T			

p	q	$p \oplus q$	$p \leftrightarrow q$	$\sim(p \leftrightarrow q)$
T	T	F	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F

* Well-formed Formula :-

Imp

In prepositional logic a statement which is collection of variables or statements represented in well-defined form using parenthesis according to priority of operations is known as well-formed formula.

To generate well-formed formula recursively following rules are used -

⇒ Negation has priority over $\wedge$ & $\vee$. $(((\sim p) \wedge q) \vee r)$

- An atomic statement p is well-formed formula.
- If p is well formed formula then $\sim p$ is also well formed formula.
- If $(p \wedge q)$ are well formed formula then $(p \wedge q)$, $(p \vee q)$, $(p \to q)$ & $(p \leftrightarrow q)$ are also well formed formula.
- A statement consist of variables, parenthesis & connectives is recursively a well formed formula if & only if it can be obtained by applying above 3 rules.

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Q Which of the following are well-formed formula. If no, explain why?

1) $(p \to (p \vee q))$ Well-formed.

2) $(p \wedge q) \to (\sim q)$ Not well formed because conjunction is used with two variables.

3) $((p \leftrightarrow q) \to r)$ well formed.

4) $((p \wedge q) \leftrightarrow (q \to r) \to s)$ Not well formed. because in bi-conditional statement is not in parenthesis. so, not well formed bi-conditional statement.

* Converse, Inverse and contrapositive of conditional Statement :-

Let, $p \to q$ is a statement then-

Converse :- $q \to p$

Inverse :- $\sim p \to \sim q$..

Contrapositive :- $\sim q \to \sim p$.

p	q	$\sim p$	$\sim q$	$\sim p \to \sim q$	$q \to p$	$\sim q \to \sim p$	$p \to q$
T	T	F	F	T	T	T	T
T	F	F	T	T	T	F	F
F	T	T	F	F	F	T	T
F	F	T	T	T	T	T	T
				Inverse	converse	contrapositive	

Q Let $p \to q$: If ABCD is a square then it is a rectangle. write down converse, Inverse & contrapositive of this statement.

p : ABCD is a square. $\sim p$: ABCD is not a square.

q : ABCD is a rectangle, $\sim q$: ABCD is not a rectangle.

Converse :- $q \to p$ ⇒ If ABCD is a rectangle then it is a square.

Inverse :- $\sim p \to \sim q$ ⇒ If ABCD is not a square then it is not a rectangle.

Contrapositive :- $\sim q \to \sim p$ ⇒ If ABCD is not a rectangle then it is not a square.

Q If $x=2$ or $x=-2$ then $x^2=4$.

p : $x=2$

q : $x=-2$

r : $x^2=4$.

$(p \vee q) \to r$.

$\sim p$: $x \neq 2$..

$\sim q$: $x \neq -2$

$\sim r$: $x^2 \neq 4$.

converse :- $r \to (p \vee q)$ ⇒ If $x^2=4$ then $x=2$ or $x=-2$.

Inverse :- $\sim(p \vee q) \to \sim r$ ⇒ $(\sim p \wedge \sim q) \to \sim r$

If Neither $x=2$ nor $x=-2$ then $x^2 \neq 4$.

Contrapositive :- $\sim r \to \sim(p \vee q)$

If $x^2 \neq 4$ neither $x=2$ nor $x=-2$.

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then $x+4 < 6$. Write the statement s & its converse.

$\sim q : x < 2$ $\quad \sim q \Rightarrow \sim p$: If $x < 2$ then $x+4 < 6$.

$\sim p : x+4 < 6$. $\quad p : x+4 \not< 6$, $q : x \not< 2$.

s : If $x+4 \not< 6$ then $x \not< 2$. $\quad (\sim < \; = \; \geq)$ $\quad (\sim > \; = \; \leq)$.

Converse :- If $x \not< 2$ then $x+4 \not< 6$.

* Algebra of Prepositions :-

- Idempotant Law :- $\begin{cases} p \vee p \equiv p \\ p \wedge p \equiv p. \end{cases}$
- Commutative Law :- $\begin{cases} p \vee q \equiv q \vee p \\ p \wedge q \equiv q \wedge p. \end{cases}$
- Associative Law :- $\begin{cases} (p \vee q) \vee r \equiv p \vee (q \vee r) \\ (p \wedge q) \wedge r \equiv p \wedge (q \wedge r). \end{cases}$
- Absorption Law :- $\begin{cases} p \vee (p \wedge q) \equiv p \\ p \wedge (p \vee q) \equiv p. \end{cases}$
- Distributive Law :- $\begin{cases} p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \\ p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \end{cases}$
- Involution Law :- $\sim(\sim p) \equiv p$.
- Identity Law :-
  $p \vee T \equiv T$
  $\Rightarrow p \wedge T \equiv p$.
  $\Rightarrow p \vee F \equiv p$
  $p \wedge F \equiv F$
- Compliment Law :- $\sim T \equiv F$, $p \vee \sim p \equiv T$
  $\sim F \equiv T$, $p \wedge \sim p \equiv F$.
- Demorgan's Law :- $\sim(p \vee q) \equiv \sim p \wedge \sim q$
  $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

Q Using Algebra of Prepositions show that,

$$(p \leftrightarrow q) = (p \vee q) \rightarrow (p \wedge q).$$

$\rightarrow$ L.H.S $\quad p \leftrightarrow q$

$\Rightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

$\Rightarrow (\sim p \vee q) \wedge (\sim q \vee p)$

$\Rightarrow ((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p)$ [Distributive Law

$\Rightarrow ((\sim q \wedge \sim p) \vee (\sim q \wedge q)) \vee ((p \wedge \sim p) \vee (p \wedge q))$ [Distributive]

$\Rightarrow (((\sim p \wedge \sim q) \vee f) \vee (f \vee (q \wedge p)))$ [Using Compliment Law]

$\Rightarrow ((\sim p \vee \sim q) \vee (q \wedge p))$ [Identity Law]

$\Rightarrow (\sim(p \vee q)) \vee (q \wedge p))$ [Demorgan's Law]

$\Rightarrow \sim(p \vee q) \vee (p \wedge q)$

$\Rightarrow (p \vee q) \rightarrow (p \wedge q)$

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④ $(p \wedge q) \vee (p \wedge \sim q) \equiv p$

LHS $\equiv (p \wedge q) \vee (p \wedge \sim q)$

$\equiv p \wedge (q \vee \sim q)$ [Distributive Law]

$\equiv p \wedge T$ [Compliment Law]

$\equiv p$ [Identity Law]

⑤ $(\sim p \rightarrow (\sim p \rightarrow (\sim p \wedge q))) \equiv p \vee q$

LHS $\equiv (\sim p \rightarrow (\sim(\sim p) \vee (\sim p \wedge q)))$ $[p \rightarrow q \equiv \sim p \vee q]$

$\equiv (\sim p \rightarrow (p \vee (\sim p \wedge q)))$ [Invdution Law]

$\equiv (\sim p \rightarrow ((p \vee \sim p) \wedge (p \vee q)))$ [Distributive Law]

$\equiv (\sim p \rightarrow (T \wedge (p \vee q)))$ [complement Law]

$\equiv (\sim p \rightarrow (p \vee q))$ [Identity Law]

$\equiv \sim(\sim p) \vee (p \vee q)$ $[p \rightarrow q \equiv \sim p \vee q]$

$\equiv p \vee (p \vee q)$ [Involution Law]

$\equiv (p \vee p) \vee q$ [Associative Law]

$\equiv p \vee q$ [Idempotent Law]

⑥ $\sim(p \vee q) \vee (\sim p \wedge q) \equiv \sim p$

LHS $\equiv \sim(p \vee q) \vee (\sim p \wedge q)$

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$ [Demorgan's Law]

$\equiv \sim p \wedge [\sim q \vee q]$ [Distributive Law]

$\equiv \sim p \wedge T$ [Compliment Law]

$\equiv p$ [Identity Law]

* Negation of Compound Statement :-

① If $|A| = 0$ then A is a singular Matrix.

$\sim(p) \equiv \sim p$

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

$\sim(p \wedge q) \equiv \sim p \vee \sim q$.

$\sim(p \rightarrow q) \equiv \sim(\sim p \vee q) \equiv p \wedge \sim q$

$\sim(p \leftrightarrow q) \equiv p \leftrightarrow \sim q$ (or) $\sim p \leftrightarrow q$

Q- If $|A| = 0$ then A is a singular matrix.

p : $|A| = 0$.

q : A is a singular matrix.

$\sim(p \rightarrow q) \equiv p \wedge \sim q$

$|A| = 0$ and A is not a singular Matrix.

i) ΔABC is a Isosceles Δ, if & only if 2 sides are equal
ΔABC is not isosceles if & only if two sides of Δ are equal.

ii) No. is divisible by 3 if & only if sum of digit of no. is divisible by 3.
$\sim(p \leftrightarrow q) \equiv \sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$.
No. is divisible by 3 if and only if sum of digit of no. is not divisible by 3.

iii) If tomorrow is holiday then we will go either to market or cinema.

p : Tommorrow is Holiday.
q : we will go to market
r : we will go to cinema.

$\sim(p \rightarrow (q \vee r))$
$\equiv p \wedge \sim(q \vee r)$
$\equiv p \wedge \sim q \wedge \sim r$.

Tommorrow is Holiday and we will neither go to market and cinema..

* Tautology, contradiction (falsi) and Contingency (satisfiable) :-

- Tautology :- A compound preposition that is always true for all possible truth values of its prepositional variables is called tautology. i.e. it contains only 'T' in last column of its truth table.
e.g. The doctor is either male or female.

- Contradiction :- A compound preposition that is always false for all possible truth values of its prepositional variables is called contradiction. ie. it contains only 'F' in last column of its truth table.
e.g. x is even and x is odd.
It is also called 'Falsify'.

- Contingency :- A preposition that is neither tautology nor contradiction is called contingency. i.e. the last column contains both T and F in its truth Table. It is also called "satisfiable".
Satisfiable :- It contains atleast one 'T'.

Q. Determine which of the following is tautology, contradiction and contingency.

1) $((\sim p \to q) \to (q \to p))$. contingency.

2) $\sim(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \vee r))$ contradiction

3) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ Tautology

4) $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ contingency

5) $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ Tautology

1)

p	q	~p	~p→q	q→p	((~p→q)→(q→p))
T	T	F	T	T	T
T	F	F	T	T	T
F	T	T	T	F	F
F	F	T	F	T	T

2)

p	q	r	q∧r	B p∨q	A p∨(q∧r)	~A	C p∨r	B∧C	~A↔(B∧C)
T	T	T	T	T	T	F	T	T	F
T	T	F	F	T	T	F	T	T	F
T	F	T	F	T	T	F	T	T	F
T	F	F	F	T	T	F	T	T	F
F	T	T	T	T	T	F	T	T	F
F	T	F	F	T	F	T	F	F	F
F	F	T	F	F	F	T	T	F	F
F	F	F	F	F	F	T	F	F	F

3)

p	q	r	q→r	p→q	p→r	A p→(q→r)	B (p→q)→(p→r)	A→B
T	T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	F	T
T	F	T	T	F	T	T	T	T
T	F	F	T	F	F	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

4) 5)

p	q	r	~p	B p∨q	~q	~r	A ~q∨~r	(~p∧A)	C ~(~p∧A)	B∧C	D ~p∧q	E ~p∧~r
T	T	T	F	T	F	F	F	F	T	T	F	F
T	T	F	F	T	F	T	T	F	T	T	F	F
T	F	T	F	T	T	F	T	F	T	T	F	F
T	F	F	F	T	T	T	T	F	T	T	F	F
F	T	T	T	T	F	F	F	F	T	T	T	F
F	T	F	T	T	F	T	T	T	F	F	T	F
F	F	T	T	F	T	F	T	T	F	F	F	F
F	F	F	T	F	T	T	T	T	F	F	F	T

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F	T	F	F	T
F	T	F	F	T
F	T	F	F	T
F	T	F	F	T
T	T	F	F	T
T	T	T	T	T
F	F	T	T	T
T	T	T	T	T

Q find a formula A that uses the variables p, q & r such that A is a contradiction.

$\sim(p \vee (q \wedge r)) \longleftrightarrow ((p \vee q) \wedge (p \vee r))$

* **Logical Implication ($\Rightarrow$)** :- A preposition P is said to logically imply a preposition Q if implication $P \rightarrow Q$ is a tautology and it is denoted by $P \Rightarrow Q$.

Q Show that $(p \vee q) \wedge \sim q \rightarrow p$ is a logical implication

p	q	$\sim q$	$p \vee q$	$(p \vee q) \wedge \sim q$	$(p \vee q) \wedge \sim p \rightarrow p$.
T	T	F	T	F	T
T	F	T	T	T	T
F	T	F	T	F	T
F	F	T.	F	F	T

So, this implication is a logical implication and we can write $\boxed{(p \vee q) \wedge \sim q \Rightarrow p.}$

* **Dual** :- The dual of any compound preposition can be obtained by replacing $\vee$ and $\wedge$ by $\wedge$ and $\vee$ resp and T and F by F and T respectively.

eg
- $p \vee q$

  Dual $\equiv p \wedge q$
- $(p \vee q) \wedge \sim q$

  Dual $\equiv (p \wedge q) \vee \sim q$.
- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

  $\equiv (\sim p \vee q) \wedge (\sim q \vee p)$

  Dual $\equiv (\sim p \wedge q) \vee (\sim q \wedge p)$.

*. **Theorem of Duality** :- If $P(p, q, r ---) \equiv Q(p, q, r$

then, $P^*(p, q, r ---) \equiv Q^*(p, q, r, ---)$

where, P^* and Q^* are dual of P & Q respectively.

Q Show that following logical expressions are equiva by using duality theorem without using truth tabl

$$(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$$

$$\sim (p \vee q) \vee r \equiv (\sim p \vee r) \wedge (\sim q \vee r).$$

Dual of above statement -

$$\text{Dual} \equiv \sim (p \wedge q) \wedge r \equiv (\sim p \wedge r) \vee (\sim q \wedge r).$$

Taking LHS $\equiv \sim (p \wedge q) \wedge r$

$\equiv (\sim p \vee \sim q) \wedge r$. (DeMorgan's Law)

$\equiv (\sim p \wedge r) \vee (\sim q \wedge r)$ (Distributive Law)

$\equiv$ RHS.

Since, Dual are equal.

So, $\boxed{(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)}$

Q Given that the value of $p \to q$ is false. Determine the value of $(\bar{p} \vee \bar{q}) \to q$.

→ $p \to q$
$\;\;\;$ T $\;\;$ F.

$\equiv (\bar{p} \vee \bar{q}) \to q$

$\equiv (\bar{T} \vee \bar{F}) \to F$.

$\equiv (F \vee T) \to F$.

$\equiv T \to F$.

$\equiv F$

Q Show that the truth values of the following formula are independent of there components.

1) $(p \to q) \rightleftarrows (\sim p \vee q)$

$\equiv ((p \to q) \to (\sim p \vee q)) \wedge ((\sim p \vee q) \to (p \to q))$

$\equiv (\sim (p \to q) \vee (\sim p \vee q)) \wedge (\sim (\sim p \vee q) \vee (p \to q))$

$\equiv (\sim (\sim p \vee q) \vee (\sim p \vee q)) \wedge (\sim (\sim p \vee q) \vee (\sim p \vee q))$

$\equiv T \wedge T$

$\equiv T$ (Tautology).

2) $(p \wedge (p \to q)) \to q$

$\equiv (\sim (p \wedge (p \to q))) \vee q$

$\equiv (\sim (p \wedge (\sim p \vee q))) \vee q$

$\equiv (\sim ((p \wedge \sim p) \vee (p \wedge q))) \vee q$.

$\equiv (\sim (F \vee (p \wedge q))) \vee q$

$\equiv (\sim (F)) \vee q$

$\equiv T \vee q$

$\equiv T$.

3) $(P \to Q) \wedge (Q \to R) \to (P \to R)$

$\equiv \sim ((P \to Q) \wedge (Q \to R)) \vee (P \to R)$

$\equiv \sim ((\sim P \vee Q) \wedge (\sim Q \vee R)) \vee (\sim P \vee R)$

$\equiv ((P \wedge \sim Q) \vee (Q \wedge \sim R)) \vee (\sim P \vee R)$

$\equiv$ [illegible]

$\Rightarrow ((P \vee Q) \wedge (T) \wedge (P \vee \sim R) \wedge (\sim Q \vee \sim R)) \vee (\sim P \vee R)$

$\Rightarrow ((P \vee Q) \wedge (P \vee \sim R) \wedge (\sim Q \vee \sim R)) \vee (\sim P \vee R)$

$\Rightarrow ((P \vee Q \vee \sim P) \vee (P \vee Q \vee R)) \wedge ((P \vee \sim R \vee \sim P) \vee (P \vee \sim R \vee R)) \wedge ((\sim Q \vee \sim R \vee \sim P) \vee (\sim Q \vee \sim R \vee R$

$\Rightarrow ((T \vee Q) \vee (P \vee Q \vee R)) \wedge ((T \vee \sim R) \vee (P \vee T)) \wedge ((\sim Q \vee \sim R \vee \sim P) \vee (Q \vee T$

$\Rightarrow (T \vee (P \vee Q \vee R)) \wedge (T \vee T) \wedge ((\sim Q \vee \sim R \vee \sim P) \vee T)$

$\Rightarrow T \wedge T \wedge T \Rightarrow T$

* **Argument :-** An argument is an assertion that a given set of preposition $P_1, P_2, \ldots P_n$ gives another preposition Q. Here, $P_1, P_2, \ldots, P_n$ are called Premises and Q is called Conclusion or consequent such argument is denoted by.

$$\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ \vdots \\ \underline{P_n} \\ \therefore Q \end{array} \quad \text{or} \quad P_1, P_2, P_3, \ldots, P_n \vdash Q.$$

* **Valid Argument :-** An argument $P_1, P_2, P_3 \ldots P_n \vdash Q$ is said to be valid if Q is true whenever premises $P_1, P_2, \ldots P_n$ are true.

eg ① $p \to q,\ p \vdash q$

P p	C q	P $p \to q$
T	T	T
T	F	T
F	F	T
F	F	T

→ critical row (first row)

A valid argument since conclusion is true in critical row.

② $p \to q,\ q \vdash p$

C p	P q	P $p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

critical rows

It is not a valid argument since, 2nd critical row premises are true but conclusion is false.

A critical row: a row in which premises are true.

If in all critical rows premises are true & conclusion is true then it is valid argument.

NOTE :- We can say that $P_1, P_2, \ldots P_n \vdash Q$ is valid argument if and only if the premises $P_1, P_2, \ldots P_n$ are true as well as conclusion Q is true i.e.

$P_1 \wedge P_2 \wedge \ldots \wedge P_n \longrightarrow Q$ is a tautology.

For any preposition $P(p, q, r \ldots)$ and $Q(p, q, r, \ldots)$ The following three statements are equivalent -

- The argument $P(p, q, r \ldots) \vdash Q(p, q, r, \ldots)$ is valid.
- $P \to Q$ is a tautology.
- $P \Rightarrow Q$

Q Test the validity of following arguments.

1) $p \vee q,\ q \vee r,\ \sim r \vdash p$.
2) $p \leftrightarrow q,\ \sim p \vee q,\ \sim r \vdash \sim q$.
3) $p \to \sim q,\ r \to p,\ q \vdash \sim r$.

1) $p \vee q,\ q \vee r,\ \sim r \vdash p$.

C.

p	q	r	$\boxed{P}$ $\sim r$	$\boxed{P}$ $p \vee q$	$\boxed{P}$ $q \vee r$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	T	T
T	F	F	T	T	F
F	T	T	F	T	T
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	F

critical rows

Since, conclusion c is true in 2nd critical row & False in 6th critical row. So, this is not a valid argument.

2) $p \leftrightarrow q,\ \sim p \vee q,\ \sim r \vdash \sim q$.

p	q	r	$\sim p$	$\boxed{C}$ $\sim q$	$\boxed{P}$ $\sim r$	$\boxed{P}$ $p \leftrightarrow q$	$\boxed{P}$ $\sim p \vee q$
T	T	T	F	F	F	T	T
T	T	F	F	F	T	T	T
T	F	T	F	T	F	F	F
T	F	F	F	T	T	F	F
F	T	T	T	F	F	F	T
F	T	F	T	F	T	F	T
F	F	T	T	T	F	T	T
F	F	F	T	T	T	T	T

Critical rows.

Since, conclusion c is true in 8th row & False in 2nd row. So, this is not a valid argument.

3) $p \rightarrow \sim q,\ r \rightarrow p,\ q \vdash \sim r$.

p	$\boxed{P}$ q	r	$\sim q$	$\boxed{C}$ $\sim r$	$\boxed{P}$ $p \rightarrow \sim q$	$\boxed{P}$ $r \rightarrow p$
T	T	T	F	F	F	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	F	T	T	T	T
F	T	T	F	F	T	F
F	T	F	F	T	T	T
F	F	T	T	F	T	F
F	F	F	T	T	T	T

critical row

Since conclusion c is true in 6th row. So, it is a valid argument.

* Rules of Inference :-

- **Modus Ponens (Law of Detachment):**

$$\begin{array}{ll} p \rightarrow q & \text{is true} \\ \underline{p} & \text{is true} \\ \therefore q & \text{is true.} \end{array}$$

- **Modus Tollens :-**

$$\begin{array}{ll} p \rightarrow q & \text{is true} \\ \underline{\sim q} & \text{is true} \\ \therefore \sim p & \text{is true} \end{array}$$

- **Hypothetical syllogism :-**

$$\begin{array}{ll} p \rightarrow q & \text{is true} \\ \underline{q \rightarrow r} & \text{is true} \\ \therefore p \rightarrow r & \text{is true.} \end{array}$$

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- Disjunctive Syllogism :-

$p \vee q$	is true		$p \vee q$	is true
$\sim q$	is true		$\sim p$	is true
$\therefore p$	is true.		$\therefore q$	is true.

- Addition :-

  p is true

  $\therefore p \vee q$ is true.

- Simplification :-

  $p \wedge q$ is true.

  $\therefore p$ is true

- Conjunction :-

  p is true

  q is true

  $\therefore p \wedge q$ is true.

- Absorption Law :-

  $p \rightarrow q$ is true

  $\therefore p \rightarrow (p \wedge q)$ is true.

- Constructive Dilemma :-

  $(p \rightarrow q) \wedge (r \rightarrow s)$. is true

  $p \vee r$. is true.

  $\therefore q \vee s$ is true.

- Destructive Dilemma :-

  $(p \rightarrow q) \wedge (r \rightarrow s)$ is true

  $\sim q \vee \sim s$ is true

  $\therefore \sim p \vee \sim r$. is true.

Q Prove the argument $p, q \vdash (p \vee r) \wedge q$ is a valid argument without using Truth table.

→ 1) p (Given Premise)

2) q (Given Premise)

3) $p \vee r$ (Addition of (1))

4) $(p \vee r) \wedge q$ (conjunction using 2 & 3)

This is valid argument.

Q $p \vee (q \rightarrow p)$, $\sim p \wedge r \vdash \sim q$.

→ 1) $p \vee (q \rightarrow p)$ (Given Premise).

2) $\sim p \wedge r$ (Given Premise)

3) $\sim p$ (simplification of 2).

4) $q \rightarrow p$ (Disjunctive syllogism of 1 2 3.)

5) $\sim q$ (Modus Tollen of 3 & 4).

This is valid argument.

Q $p \to \sim q,\ r \to p,\ q \vdash \sim r$ ...

→ 1) $p \to \sim q$ (Given Premise)
2) $r \to p$ (Given Premise)
3) q (Given Premise).
4) $\sim p$ (Modus Tollen 1 & 3).
5) $\sim r$ (Modus Tollen using 2 & 4)

This is valid argument.

Q Check the validity of the arguments.

1) If I tried hard ~~&~~ ~~If I tried hard~~ & I have talent then I will become an engineer. If I become an engineer

Let, p: I tried hard.
q: I have talent.
r: I will become an engineer.

then I will be happy. Therefore, If will not be happy then I didn't try hard or I don't have talent.

s: I will be happy.

1) $(p \wedge q) \to r$ (Given Premise)
2) $r \to s$ (Given Premise).

$\therefore \sim s \to (\sim p \vee \sim q)$

1) $(p \wedge q) \to r$ (Given Premise)
2) $r \to s$ (Given Premise).
3) $(p \wedge q) \to s$ (Hypothetical syllogism using 1 & 2)
4) $\sim s \to (\sim(p \wedge q))$ (contrapositive) of 3)
5) $\sim s \to (\sim p \vee \sim q)$ (Demorgan's Law of 4).

2) If Rohit is married, he is sad. If he is sad then he does not watch TV. If he watches TV therefore Rohit is unmarried.

→ p: Rohit is married.
q: Rohit is sad.
r: Rohit watches TV.

$p \to q$
$q \to \sim r$
r

$\therefore \sim p$.

1) $p \to q$ (Given Premise)
2) $q \to \sim r$ (Given Premise)
3) r (Given Premise)
4) $p \to \sim r$. (Hypothetical syllogism of 1 & 2)
5) $\sim p$ (Modus Tollen using 3 & 4).

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Q Show that t is a valid conclusion from the given premises.

$\sim p \wedge q$, $r \rightarrow p$, $\sim r \rightarrow s$ and $s \rightarrow t$

1). $\sim p \wedge q$ (Given Premise)
2) $r \rightarrow p$ (Given Premise)
3) $\sim r \rightarrow s$ (Given Premise)
4) $s \rightarrow t$ (Given Premise).
5) $\sim p$ (simplification using 1)
6) $\sim r$ (Modus Tollen using 2 & 5).
7) $\sim r \rightarrow t$ (Hypothetical syllogism using 3 & 4).
8) t (Modus Ponen's using 6 & 7)

Q Show that s is a valid conclusion from given premise
$p \rightarrow q$, $q \rightarrow r$, $r \rightarrow (s$ ~~and~~ $\wedge t)$, p.

1) $p \rightarrow q$ (Given Premise)
2) $q \rightarrow r$ (Given Premise)
3) $r \rightarrow (s \wedge t)$ (Given Premise)
4) p (Given Premise)
5) $p \rightarrow r$ (Hypothetical syllogism using 1 & 2)
6) r (Modus Ponen using 4 & 5).
7) $p \rightarrow (s \wedge t)$. (Hypothetical syllogism using 3 & 5)
8) $s \wedge t$ (Modus Ponen using 3 & 6).
9) s. (simplification of 8)

Q. Translate the following in symbolic form and test the validity of the argument.
If 6 is even then 2 does not divides seven either 5 is not prime or 2 divides 7 but 5 is prime. Therefore, 6 is

→ p: 6 is even.
q: 2 does ~~not~~ divides seven.
r: 5 is prime.

$p \rightarrow \sim q$
$\sim r \vee q$
r
$\therefore \sim p$

$\sim r \vee q$
or
$[(p \rightarrow q) \equiv \sim p \vee q]$

1) $p \rightarrow \sim q$ (Given Premise)
2) $\sim r \vee q$ (Given Premise)
3) r. (Given Premise)
4) q (Disjunctive syllogism using 2 & 3)
5) $\sim p$. (Modus Tollens using 1 & 4).

Tom bought a washing machine or a TV. If he bought a TV then he likes to watch movie. He does not like movies therefore Tom bought a washing machine.

p : Tom bought washing machine
q : Tom bought a TV
r : Tom likes to watch movie.

$$\begin{array}{l} p \vee q \\ q \rightarrow r \\ \sim r \\ \hline \therefore p \end{array}$$

1)	$p \vee q$	(Given Premise)
2)	$q \rightarrow r$	(Given Premise)
3)	$\sim r$	(Given Premise)
4)	$\sim q$	(Modus Tollens 2 & 3)
5)	p	(Disjunctive syllogism of 1 & 4)

* Predicate Calculus : (First order Logic).

[illegible] logic [illegible] calculus is specialisation and Predicate [illegible] is Generalisation
[illegible] Logic is Generalisation of Propositional logic.
$\forall$ [illegible] Quantifier , $\exists$ Existential Quantifier.

First order Logic is the extension of prepositional logic by generalising and quantifying the prepositions over given universe of discourse.
In General in first order logic every Individual has a property 'P'. It is also called first order predicate calculus.

Predicate Calculus :-
Consider the following two statements
p : sparrow can fly.
q : Parrot can fly.
These two prepositions can be replaced by a single preposition i.e. " x can fly ".
In this sentence 'x' is subject and 'can fly' is predicate.
This single sentence generalise two prepositions or more. So this type of generalisation is known as predicate Calculus.
'x' is called predicate variable & sentence "x' can fly' is represented by p(x) i.e.

$P(x)$: x can fly.

Universe of Discourse (UD) :- It is the set of all possible values that can be substituted in place of predicate variable.

eg Let, $P(x)$: x can fly.

UD = set of all birds which can fly.

* Quantifiers :- There are two type of quantifiers.

- Universal Quantifier.
- Existential Quantifier.

Universal Quantifier :- Let $P(x)$ be a prepositional function defined on set A.

consider the expression $(\forall x \in A)\ P(x)$ or $\forall x\ P(x)$.

Here, the symbol "$\forall$" is read as "for all" or "for every" and it is called Universal Quantifier then, the above statement is read as

for every x belongs to A, $P(x)$ is true.

Existential Quantifier :- Let $P(x)$ be a prepositional function defined on set A

consider the expression $(\exists x \in A)\ P(x)$ or $\exists x\ P(x)$.

Here, the symbol "$\exists$" is read as "for some" or "for atleast one" or "There exists" and it is called Existential Quantifier then the above statement is read as

for some x belongs to A, $P(x)$ is true.

NOTE: The value of x for which $P(x)$ is false ... example.

Q. Express following in symbolic form using quantifiers.

1) Some chalk boards are green.
2) Every student has a laptop.
3) All chairs has four legs.
4) Faridabad is an industrial city.
5) Some integers are prime and some are composite.

1) $P(x)$: x is a chalk board.
$Q(x)$: x is green.

$\exists x\ (P(x) \wedge Q(x))$.

$P(x)$ ... use $\wedge$ ... for every ... $\rightarrow$ ...

2) $P(x)$: x is a student
$Q(x)$: x has a laptop.

$\forall x\ (P(x) \rightarrow Q(x))$.

3) $P(x)$: x is a chair
$Q(x)$: x has four legs.

$\forall x\ (P(x) \rightarrow Q(x))$.

4) $P(x)$: x is Faridabad city.
$Q(x)$: x is an Industrial city.
$\exists x \ (P(x) \wedge Q(x))$.

5) $P(x)$: x is an integer.
$Q(x)$: x is prime.
$R(x)$: x is composite.
$\exists x \ \{(P(x) \wedge Q(x)) \wedge \exists x \ (P(x) \wedge R(x))\}$

Q. Consider the function $|x| = \begin{cases} x & ; \text{ if } x \geq 0 \\ -x & ; \text{ if } x < 0 \end{cases}$

Determine the truth value of following preposition where the UD is set of real numbers.

1) $\forall x \ (x^2 \geq 0)$
2) $\forall x \ (|x| > 0)$.

→ UD = R.

1) $\forall x \ (x^2 \geq 0)$ This is true because the square of every real number is non-negative or positive.
2) $\forall x \ (|x| > 0)$ This is false because $x = 0$, $|0| = 0$.
$0 > 0$. which is not greater.

* Variant of Quantified Conditional statement :-

$\forall x \ (P(x) \longrightarrow Q(x))$.

1) Converse : $\forall x \ (Q(x) \longrightarrow P(x))$
2) Inverse : $\forall x \ (\sim P(x) \rightarrow \sim Q(x))$.
3) Contrapositive : $\forall x \ (\sim Q(x) \longrightarrow \sim P(x))$.

* Multiple Quantifiers :- A preposition having more than one variable can be quantified using multiple quantifiers.

Prepositions containing both Universal and Existential quantifiers can not be exchanged without altering the meaning of the preposition.

Let. $P(x,y)$: x loves y.

$\exists x \forall y \ P(x,y)$ → There is a person who loves everyone in this world.

$\forall y \exists x \ P(x,y)$ → Everyone in the world is loved by atleast one person.

NOTE: $\forall x \forall y$ is same as $\forall y \forall x$.
$\exists x \exists y$ is same as $\exists y \exists x$.

Q. x is father of mother of y.
UD = All the persons.

P(x) : x is a person.
F(x, y) : x is father of y.
M(x, y) : x is mother of y.

x is father of z
z is mother of y.

$\exists z \, (P(z) \wedge F(x, z) \wedge M(z, y))$.

* Negation of Quantified Preposition :-

$\sim(\forall x \, P(x)) \equiv \exists x \sim P(x)$
$\sim(\exists x \, P(x)) \equiv \forall x \sim P(x)$

Q Write the following preposition using quantifier then negate each quantified preposition.

1) Some students don't ~~leave~~ live in hostel.
P(x) : x is a student.
Q(x) : x don't live in hostel.

$\exists x \, (P(x) \wedge Q(x))$.

$\Rightarrow \sim(\exists x \, (P(x) \wedge Q(x)))$
$\forall x \sim(P(x) \wedge Q(x))$

$\equiv \forall x \, (\sim P(x) \vee \sim Q(x))$
$\equiv \forall x \, (P(x) \rightarrow \sim Q(x))$. All students live in hostle

2) Every bird can fly.
Q(x) : x can fly.
P(x) : x is a bird.

$\forall x \, (P(x) \longrightarrow Q(x))$
$\Rightarrow \sim(\forall x \, (P(x) \rightarrow Q(x)))$
$\equiv \exists x \sim(\sim P(x) \vee Q(x))$
$\equiv \exists x \, (P(x) \wedge \sim Q(x))$ Some birds can not fly.

3) All boys can run faster than all girls.
P(x) : x is a boy.
R(x, y) : x can run faster than y.
Q(y) : y is a girl.

$\forall x \forall y \, (P(x) \rightarrow R(x, y))$.
$\Rightarrow \sim(\forall x \forall y \, (P(x) \rightarrow R(x, y)))$
$\equiv \exists x \exists y \, (P(x) \wedge \sim R(x, y))$.
Some boys can not run faster than some girls.

Q Use quantifier to say that $\sqrt{3}$ is irrational.

$\sim(\exists x \, (x^2 = 3))$.
UD = set of rational no.'s.

* Rules of Inference :-

- Universal Specification :-

$$\frac{\forall x \quad P(x) \text{ is true}}{\therefore P(a)} \quad \text{where } a \text{ is particular member of UD.}$$

- Universal Generalisation :- If $P(a)$ is true for all 'a' in UD

$$\frac{P(a) \text{ is true}}{\therefore \forall x \; P(x) \text{ is true.}}$$

- Existential specification :-

$$\frac{\exists x \quad P(x) \text{ is true}}{\therefore P(a) \text{ is true}}$$

- Existential Generalisation :-

$$\frac{P(a) \text{ is true}}{\therefore \exists x \; P(x) \text{ is true.}}$$

- Universal Modus Ponen's :-

$$\forall x \; (P(x) \rightarrow Q(x)). \text{ is true}$$
$$\frac{P(a) \qquad \text{is true}}{\therefore Q(a) \qquad \text{is true.}}$$

- Universal Modus Tollen's :-

$$\forall x \; (P(x) \rightarrow Q(x)) \text{ is true}$$
$$\frac{\sim Q(a) \qquad \text{is true}}{\therefore \sim P(a) \qquad \text{is true.}}$$

Q If an number is odd then its square is odd. k is a particular no. that is odd. Therefore, k^2 is odd.

→ $P(x)$: x is odd.
$Q(x)$: x^2 is odd.

$$\forall x \; (P(x) \rightarrow Q(x)).$$
$$\frac{P(k)}{\therefore Q(k)} \qquad \text{where } k \in UD.$$

Above argument is valid by Universal Modus Ponen's Rule.

Q Leopards are dangerous animals there are leopards. Therefore, there are dangerous animals.

→ $P(x)$: x is a leopards.
$Q(x)$: x is dangerous animals.

$$\forall x \; (P(x) \rightarrow Q(x)).$$
$$\frac{\exists x \quad P(x)}{\therefore \exists x \; Q(x)}$$

1) $\forall x \; (P(x) \rightarrow Q(x))$ (Given Premise)
2) $\exists x \quad P(x)$ (Given Premise)
3) $P(a)$ (~~Universal~~ Existential specification using 2).
4) $Q(a)$ (Universal Modus Ponen's of 1 & 3)
5) $\exists x \; Q(x)$ (Existential Generalisation of 4).

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Therefore, all human being are mortal.

→ $P(x)$: x is an animal.
$Q(x)$: x is mortal.
$R(x)$: x is human being.

$\forall x \ (P(x) \rightarrow Q(x))$.
$\forall x \ (R(x) \rightarrow P(x))$.
$\therefore \forall x \ (R(x) \rightarrow Q(x))$.

1) $\forall x \ (P(x) \rightarrow Q(x))$ (Given Premise)
2) $\forall x \ (R(x) \rightarrow P(x))$ (Given Premise)
3) $P(a) \rightarrow Q(a)$ (Universal specification of 1)
4) $R(a) \rightarrow P(a)$ (Universal specification of 2)
5) $R(a) \rightarrow Q(a)$ (Hypothetical syllogism of 3 & 4)
6) $\forall x \ (R(x) \rightarrow Q(x))$ (Universal Generalisation of 5)

Q Let, $M(x)$: x is mammal.
$A(x)$: x is an animal.
$W(x)$: x is warm blooded.

Translate following into a formula.
1) Every mammal is warm blooded.

Translate into english sentence.
2) $\exists x \ (A(x) \wedge \sim W(x))$

1) $\forall x \ (M(x) \rightarrow W(x))$
2) Some animals are not warm blooded.

Q $q(x,y,z)$: $x+y=z$
Determine the truth value of following prepositions or statements.
1) $\forall x \forall y \exists z \ q(x,y,z)$ True (for all x & for all y there will some z)
2) $\exists z \forall x \forall y \ q(x,y,z)$. False. (for some z there is all x & y).

Q Show that $\sim P(a,b)$ follows logically form.
$\forall x \forall y \ (P(x,y) \rightarrow W(x,y))$ and $\sim W(a,b)$.

1) $\forall x \forall y \ (P(x,y) \rightarrow W(x,y))$ (Given Premise)
2) $\sim W(a,b)$ (Give Premise).
3) $P(a,b) \rightarrow W(a,b)$ (Universal specification of 1)
4) $\sim P(a,b)$. (Modus Tollen's of 2 & 3)

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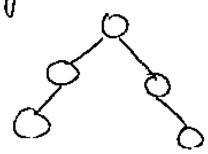
UNIT-5

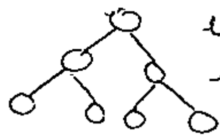
Graph Theory and Recurrence Relation

* Properties of Tree :-

- A tree having 'n' nodes have $\boxed{n-1}$ branches.



- In a tree at 'i' level there will be 2^i nodes



level 0
level 1
level 2.

$$n = 2^0 + 2^1 + 2^2 + 2^3 + --- + 2^h.$$

$$\boxed{n = 2^{h+1} - 1}$$

$$\Rightarrow n+1 = 2^{h+1}$$

Taking log on both sides

$$\Rightarrow \log_2 (n+1) = h+1$$

$$\Rightarrow \boxed{h = \log_2 (n+1) - 1}$$

* **Recurrence Relations** :- For a sequence of numbers or numeric function ($a_0, a_1, a_2, a_3, ----a_r----$) an equation relating a_r, for any r, to one or more of the a_i's, $i<r$ is called recurrence relations.

Recurrence Relation are also called difference equation because they can be written in terms of the difference b/w the consecutive terms of a sequence.

eg $a_r = 4 \cdot a_{r-1}$, $r \geq 1$.
$a_r = a_{r-1} - 2a_{r-2}$, $r \geq 2$.

* **Order of Recurrence Relation** :- The difference b/w the highest and lowest subscripts of a_r or $f(x)$ or y_n or a_n is called the order of recurrence relation.

eg

1) $a_r = 4 \cdot a_{r-1}$ $\qquad r-(r-1) = 1.$
2) $a_r = a_{r-1} - 2a_{r-2}$, $r \geq 2$, $\qquad r-(r-2) = 2$
3) $a_n = 2a_{n-1} - a_{n-2}$, $n \geq 2$, $\qquad n-(n-2) = 2$
4) $f(x+2) + 8f(x+1) + 2f(x) = 5.$ $\qquad x+2-x = 2.$

* **Degree of Recurrence Relation** :- The highest power of a_r or $f(x)$ or y_n is called degree of recurrence relation

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e.g. $a^2_{r+2} + a_{r+1} - a_r = 0$.

Degree = 2.

Q find the first four terms of $a_k = k(a_{k-1})^2, \forall k \geq 1$, $a_0 = 1$.

→ $a_k = k(a_{k-1})^2$.

Put, $k = 1$.

$a_1 = 1(a_0)^2$.

$a_1 = 1.(1)^2$

$\boxed{a_1 = 1.}$

Put, $k = 2$.

$a_2 = 2(a_1)^2$

$= 2.(1)^2$.

$\boxed{a_2 = 2}$

Put, $k = 3$.

$a_3 = 3(a_2)^2$

$= 3.(2)^2$

$\boxed{a_3 = 12}$

* Linear Recurrence Relation with constant Coeffici

If the degree of the recurrence relation is 1 then the recurrence relation is called linear recurrence relation.

Generally,

$$\boxed{c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)}$$

Where, c_i ~~$(0 \leq i \leq n)$~~ is constant & $(0 \leq i < k)$ and $f(n)$ is a function which satisfies the equation is called linear recurrence relation.

If $f(n) = 0$ then recurrence relation is linear Homoge recurrence relation.

& if $f(n) \neq 0$ then recurrence relation is called non homogen recurrence relation.

e.g. $a_n = 5 \cdot a_{n-1}$. Linear Homogenous.

$a_n = a_{n-1} \cdot a_{n-2}$ Non Linear

$a_n - a_{n-1} = 5$. Linear Non-Homogeneous.

* Solution of Recurrence Relation :-

An explicit formula which satisfy the recurrence relation with initial condition is called solution to the recurrence relation.

There are three methods -

- Iteration Method.
- Characteristic Roots Method.
- Using Generating Function.

- Iteration Method :-

Q $a_k = 2 \cdot a_{k-1}$, $a_0 = 1$.

$a_k = 2 \cdot 2 \cdot a_{k-2}$

$a_k = 2^2 a_{k-2}$

$a_k = 2^2 \cdot 2 \cdot a_{k-3}$

$a_k = 2^3 \cdot a_{k-3}$

$\vdots$

$a^k = 2^k \cdot a_{k-k}$.

$a^k = 2^k \cdot a_0$.

$a^k = 2^k \cdot (1)$.

$\boxed{a^k = 2^k}$

G.P

$\frac{a \cdot (r^n - 1)}{r - 1}$

Q $a_n = a_{n-1} + n$, $a_1 = 4$.

$a_n = a_{n-2} + n + (n-1)$

$a_n = a_{n-2} + (n-1) + n$.

$a_n = a_{n-3} + (n-2) + (n-1) + n$.

$\vdots$

$a_n = a_{n-k} + \cancel{(n-1) + (n-2) + \dots + (n-(k-1))}$

$a_n = a_{n-k} + (n-(k-1)) + (n-(k-2)) + \dots + (n-2) + (n-1) + n$.

Replace $k = n-1$

$a_n = a_{n-(n-1)} + (n-(n-1-1)) + (n-(n-1-2)) + \dots + (n-2) + (n-1) + n$

$= a_1 + 2 + 3 + \dots + (n-2) + (n-1) + n$.

$= 4 + 2 + 3 + \dots + (n-2) + (n-1) + n$.

$= 3 + 1 + 2 + 3 + \dots + (n-2) + (n-1) + n$

$$\boxed{a_n = 3 + \frac{n \cdot (n-1)}{2}}$$

Q $a_n = n \cdot a_{n-1}$. , $a_0 = 5$.

$a_n = n(n-1)\, a_{n-2}$.

$a_n = n(n-1)(n-2)\, a_{n-3}$

$a_n = n(n-1)(n-2) \dots (n-(n-1))$ next to a_{n-n}.

$a_n = n(n-1)(n-2) \dots \cancel{a_1}\ 1 \cdot a_0$.

$a_n = n!\ 5$

$\boxed{a_n = 5n!}$

Q $a_n = 2a_{n-1} + 1$, $a_1 = 7$.

$a_n = 2 \cdot (2a_{n-2} + 1) + 1$

$a_n = 2(2(2a_{n-3} + 1) + 1) + 1$

$a_n = 2(2(2a_{n-3} + 1)) + 1) + 1$

$a_n = 2^3 a_{n-3} + 2^2 + 2 + 1$.

$a_n = 2^k a_{n-k} + 2^{k-1} + 2^{k-2} + 2^{k-3} + \dots + 2^2 + 2 + 1$.

$(n-k = 1)$

$(k = n-1)$

Replace, $k = n-1$

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$a_n = 2^{n-1} a_{(n-n+1)} + 2^{n-2} + 2^{n-4} + \cdots + 2^2 + 2 + 1$

$= 2^{n-1} a_1 + 2^{n-2} + 2^{n-4} + \cdots + 2^2 + 2 + 1.$

$= 7 \cdot 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1.$

$= 7 \cdot 2^{n-1} + 1 \frac{(2^{n-1} - 1)}{2-1}$

$= 7 \cdot 2^{n-1} + 2^{n-1} - 1.$

$= 2^{n-1}(7+1) - 1$

$= 8 \cdot 2^{n-1} - 1.$

$= 2^3 \cdot 2^{n-1} - 1. \quad = 2^{3+n-1} - 1.$

$$\boxed{a_n = 2^{n+2} - 1}$$

* Methods of characteristics of Roots :-

Solution of Linear Homogenous Recurrence Relation when $f(n) = 0$.

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0 \quad \text{—— (1)}$$

1) Put, $a_n = \alpha^n$ in eq^n (1)

$$\alpha^n + c_1 \alpha^{n-1} + c_2 \alpha^{n-2} + \cdots + c_k \alpha^{n-k} = 0 \quad \text{—— (2)}$$

2) Divide eq^n (2) by α^{n-k}.

$$\frac{\alpha^n}{\alpha^{n-k}} + c_1 \frac{\alpha^{n-1}}{\alpha^{n-k}} + c_2 \frac{\alpha^{n-2}}{\alpha^{n-k}} + \cdots + c_k \frac{\alpha^{n-k}}{\alpha^{n-k}} = 0$$

$$\Rightarrow \boxed{\alpha^k + c_1 \alpha^{k-1} + c_2 \alpha^{k-2} + \cdots + c_k = 0} \quad \text{—— (3)}$$

Equation (3) is called characteristic eq^n and this eq^n has k no. of roots and these roots are characteristic roots.

Let, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ be the characteristic roots.

1) <u>If roots are real</u>.

$\Rightarrow$ If <u>roots</u> are <u>distinct</u>.

$$\boxed{a_n = b_1 \alpha_1^n + b_2 \alpha_2^n + b_3 \alpha_3^n + \cdots + b_k \alpha_k^n}$$

$\Rightarrow$ If <u>some roots</u> are <u>equal</u>.

Let, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha, \ \alpha_4, \alpha_5, \ldots, \alpha_k$

$$\boxed{a_n = (b_1 + b_2 n + b_3 n^2)\alpha^n + b_4 \alpha_4^n + \cdots + b_k \alpha_k^n}$$

2) <u>If roots are Imaginary</u>.

$\Rightarrow$ If <u>roots</u> are <u>distinct</u>.

$$\boxed{a_n = b_1(\alpha_1 + i\alpha_2)^n + b_2(\alpha_1 - i\alpha_2)^n}$$

$\Rightarrow$ If <u>roots</u> are <u>equal</u>. (If roots are repeated twice the complete pair repeated twice)

$$\boxed{a_n = (b_1 + b_2 n)(\alpha_1 + i\alpha_2)^n + (b_3 + b_4 n)(\alpha_1 - i\alpha_2)^n}$$

where, b_1, b_2, b_3, b_4 are constant

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Q $a_r - 3a_{r-1} + 2a_{r-2} = 0$ — ① $a_0 = 1$, $a_1 = 4$.

Put, $a_r = \alpha^r$ in eqⁿ ①

$$\alpha^r - 3\alpha^{r-1} + 2\alpha^{r-2} = 0 \quad \text{— ②}$$

Divide eqⁿ ② by α^{r-2}.

$$\frac{\alpha^r}{\alpha^{r-2}} - 3\frac{\alpha^{r-1}}{\alpha^{r-2}} + 2\frac{\alpha^{r-2}}{\alpha^{r-2}} = 0$$

$$\alpha^2 - 3\alpha + 2 = 0. \quad \text{— ③}$$

$$\alpha^2 - 2\alpha - \alpha + 2 = 0$$

$$\alpha(\alpha-2) - 1(\alpha-2) = 0$$

$$(\alpha-2)(\alpha-1) = 0$$

$\alpha = 2, 1$ (Roots are real & distinct).

Then solⁿ

$$a_r = b_1 \cdot 1^r + b_2 \cdot 2^r.$$

$$\boxed{a_r = b_1 + b_2 \cdot 2^r} \quad \text{— ④}$$

Now, Put, $r = 0$ in eqⁿ ④

$$a_0 = b_1 + b_2 2^0$$

$$b_1 + b_2 = 1. \quad \text{— ⑤}$$

Now, Put, $r = 1$ in eqⁿ ④

$$a_1 = b_1 + 2b_2.$$

$$b_1 + 2b_2 = 4 \quad \text{— ⑥}$$

$$\cancel{b_1} + b_2 = 1$$
$$-\cancel{b_1} - 2b_2 = -4$$
$$-b_2 = -3$$

$$\boxed{b_2 = 3}$$

$$\boxed{b_1 = -2}$$

Now Put b_1 & b_2 in eqⁿ ④

$$a_r = -2 + 3 \cdot 2^r.$$

$$\Rightarrow \boxed{a_r = 3 \cdot 2^r - 2}$$

Q $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$ — ① $a_0 = 1$, $a_1 = 4$, $a_2 = 8$.

Put $a_n = \alpha^n$ in eqⁿ ①

$$\alpha^n - 7\alpha^{n-1} + 16\alpha^{n-2} - 12\alpha^{n-3} = 0 \quad \text{— ②}$$

Divide eqⁿ ② α^{n-3} ②.

$$\alpha^3 - 7\alpha^2 + 16\alpha - 12 = 0 \quad \text{— ③}$$

$$\alpha^2(\alpha-2) - 5\alpha(\alpha-2) + 6(\alpha-2) = 0$$

$$\Rightarrow (\alpha-2)(\alpha^2 - 5\alpha + 6) = 0$$

$$\Rightarrow (\alpha-2)(\alpha-2)(\alpha-3) = 0$$

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$\alpha = 2, 2, 3$. $\therefore$ 2 Roots are equal & 1 distinct.

Then soln

$$\boxed{a_n = b_1 3^n + (b_2 + b_3 n) \cdot 2^n} \quad \text{—④}$$

Now Put, $n = 0$ in eqn ④

$a_0 = b_1 + b_2$.

$b_1 + b_2 = 1$. —— ⑤

Now, Put, $n = 1$ in eqn ④

$a_1 = 3b_1 + 2(b_2 + b_3)$

$3b_1 + 2b_2 + 2b_3 = 4$. —— ⑥

Now, Put, $n = 2$ in eqn ④.

$a_2 = 9b_1 + 4b_2 + 8b_3$

$9b_1 + 4b_2 + 8b_3 = 8$ —— ⑦

By solving eqn ⑤, ⑥ & ⑦

$\boxed{b_1 = -4}$, $\boxed{b_2 = 5}$, $\boxed{b_3 = 3}$.

$\therefore a_n = -4 \, 3^n + (5 + 3n) 2^n$.

$$\Rightarrow \boxed{a_n = (5 + 3n) 2^n - 4 \cdot 3^n}$$

Q $f_n = f_{n-1} + f_{n-2} \to$ ① $n \geq 2$, $f_0 = f_1 = 1$.

Put, $f_n = \alpha^n$ in eqn ①

$\alpha^n - \alpha^{n-1} - \alpha^{n-2} = 0$ —— ②

Divide eqn ② by α^{n-2}.

$\alpha^2 - \alpha - 1 = 0$ —— ③

$$\alpha = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \alpha = \frac{1-\sqrt{5}}{2}.$$

Then, soln

$$\boxed{f_n = b_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + b_2 \left(\frac{1-\sqrt{5}}{2}\right)^n} \quad \text{—④}$$

Put, $n = 0$ in eqn ④

$f_0 = b_1 + b_2$.

$b_1 + b_2 = 1$ —— ⑤

Put, $n = 1$ in eqn ④

$$f_1 = \left(\frac{1+\sqrt{5}}{2}\right) b_1 + \left(\frac{1-\sqrt{5}}{2}\right) b_2.$$

$$\frac{1+\sqrt{5}}{2} b_1 + \left(\frac{1-\sqrt{5}}{2}\right) b_2 = 1. \quad \text{—⑥}$$

from eqn ⑤ & ⑥.

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$\cancel{\left(\frac{1+\sqrt5}{2}\right)} b_1 + \left(\frac{1+\sqrt5}{2}\right) b_2 = \left(\frac{1+\sqrt5}{2}\right).$

$+ \cancel{\left(\frac{1+\sqrt5}{2}\right)} b_1 + \left(\frac{1-\sqrt5}{2}\right) b_2 = 1$

$- \qquad - \qquad -$

$\left(\frac{1+\sqrt5}{2} - \frac{1+\sqrt5}{2}\right) b_2 = \left(\frac{1+\sqrt5}{2}\right) - 1.$

$\sqrt5 \; b_2 = \frac{\sqrt5-1}{2}.$

$\boxed{b_2 = + \cancel{\frac{1}{\sqrt5}}} \quad \frac{\sqrt5-1}{2\sqrt5}$

$\boxed{b_1 = -\frac{\sqrt5-1}{2\sqrt5}}$

$\Rightarrow$ ~~fn~~ $\boxed{\cancel{f_n} = -\frac{1}{\cancel{\sqrt5}} \left(\cancel{\frac{\sqrt5-\sqrt5}{2\sqrt5}}\right)^n \cancel{\frac{1}{\sqrt5}} \left(\cancel{\frac{\sqrt5-\sqrt5}{2}}\right)^n}$

$\Rightarrow \quad f_n = \left(\frac{1+\sqrt5}{2\sqrt5}\right)\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2\sqrt5}\right)\left(\frac{1-\sqrt5}{2}\right)^n.$

$\Rightarrow \quad \boxed{f_n = \frac{1}{\sqrt5}\left[\left(\frac{1+\sqrt5}{2}\right)^{n+1} - \left(\frac{1-\sqrt5}{2}\right)^{n+1}\right]}$

Q $a_{r+4} + 8a_{r+2} + 16a_r = 0$ — (1)

Put, $a_r = \alpha^r$ in eq^n (1)

$\alpha^{r+4} + 8\alpha^{r+2} + 16\alpha^r = 0$ — (2)

Divide eq^n (2) by α^r.

$\alpha^4 + 8\alpha^2 + 16 = 0$ — (3)

$\alpha^4 + 4\alpha^2 + 4\alpha^2 + 16 = 0$

$\alpha^2(\alpha^2+4) + 4(\alpha^2+4) = 0$

$(\alpha^2+4)(\alpha^2+4) = 0$

$\alpha = \pm 2i, \pm 2i$

$\sqrt{-4\times4}$
$\frac{-4i}{2}$

Then sol^n

$\boxed{a_r = (b_1 + b_2 r)(2i)^r + (b_3 + b_4 r)(-2i)^r.}$

Q Solve $a_{n+1} - 1.5a_n = 0, n \geq 0$

Put, $a_n = \alpha^n$ in eq^n (1)

$\alpha^{n+1} - 1.5\alpha^n = 0.$ — (2)

Divide eq^n (2) by α^n.

$\alpha - 1.5 = 0.$

$\boxed{\alpha = 1.5}$

Then sol^n

$\boxed{a_n = b_1 (1.5)^n}$

Put, $a_n = \alpha^n$ in eq^n ①

$\alpha^n - 2\alpha^{n-1} + \alpha^{n-2} = 0$ — ②

Divide eq^n ② by α^{n-2}.

$\alpha^2 - 2\alpha + 1 = 0$ — ③

$\alpha^2 - \alpha - \alpha + 1 = 0$

$\alpha(\alpha - 1) - 1(\alpha - 1) = 0$

$(\alpha - 1)(\alpha - 1) = 0$

$\boxed{\alpha = 1, 1}$

Then sol^n.

$\boxed{a_n = (b_1 + b_2 n)}$

Q $a_n = 5a_{n-1} + 6a_{n-2}$. — ① , $n \geq 0$, $a_0 = a_1 = 3$.

Put, $a_n = \alpha^n$ in eq^n ①

$\alpha^n - 5\alpha^{n-1} - 6\alpha^{n-2} = 0$. — ②

Divide eq^n ② by α^{n-2}.

$\alpha^2 - 5\alpha - 6 = 0$. — ③

$\alpha^2 - 6\alpha + \alpha - 6 = 0$

$\alpha(\alpha - 6) + 1(\alpha - 6) = 0$

$(\alpha + 1)(\alpha - 6) = 0$

$\alpha = -1, 6$.

Then sol^n -

$\boxed{a_n = b_1(-1)^n + b_2 6^n}$ — ④

Put, $n = 0$

$a_0 = b_1(-1)^0 + b_2(6)^0$.

$b_1 + b_2 = 3$. — ⑤

Put, $n = 1$.

$a_1 = b_1(-1)^1 + b_2(6)^1$

$-b_1 + 6b_2 = 3$. — ⑥

On solving eq^n ⑤ & ⑥ -

$$\begin{aligned} b_1 + b_2 &= 3 \\ -b_1 + 6b_2 &= 3 \\ \hline 7b_2 &= 6. \end{aligned}$$

$\boxed{b_2 = \frac{6}{7}}$, $\boxed{b_1 = \frac{15}{7}}$.

$\Rightarrow \boxed{a_n = \frac{15}{7}(-1)^n + \left(\frac{6}{7}\right)(6)^n}$

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- Solution of Non-Homogenous Recurrence Relation:-

$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)$ —— ①

1) Put, $f(n) = 0$ and find $a_n^{(h)}$ (Homogenous solution)
2) Find particular solution or trial solution $a_n^{(P)}$ with the help of $f(n)$.
3) Determine coefficients of a_n^P by putting it into eq^n ①
4) complete solution $a_n = a_n^{(h)} + a_n^{(P)}$
5) Determine coefficients with the help of initial condition if given.

S.N	f(n)	Trial function (or) Particular solution ($a_n^{(P)}$)
1.	A (constant)	A_0.
2.	c^n [c is not a characteristic root]	$A_0 c^n$.
3.	c^n [c is a characteristic root with multiplicity s]	$A_0 n^s c^n$.
4.	P(n) [Polynomial of degree m]	$A_0 + A_1 n + A_2 n^2 + \cdots + A_m n^m$.
5.	$c^n \cdot P(n)$ [c is not a characteristic root]	$[A_0 + A_1 n + A_2 n^2 + \cdots + A_m n^m] c^n$.
6	$c^n P(n)$ [c is a characteristic root with multiplicity s]	$[A_0 + A_1 n + A_2 n^2 + \cdots + A_m n^m] n^s c^n$

Q $y_k - y_{k-1} - 6y_{k-2} = -30$. —— ① $y_0 = 20$, $y_1 = -5$.

for Homogenous sol^n make RHS of eq^n ① $= 0$ ie.

$f(n) = 0$.

Now consider the eq^n -

$y_k - y_{k-1} - 6y_{k-2} = 0$ —— ②

Put, $y_k = \alpha^k$ in eq^n ②

$\alpha^k - \alpha^{k-1} - 6\alpha^{k-2} = 0$.

Divide eq^n α^{k-2}

$\alpha^2 - \alpha - 6 = 0$. —— ③

$\Rightarrow \alpha = -2, 3$

$\boxed{y_k^{(h)} = b_1(-2)^k + b_2(3)^k}$ —— ④

Now, $f(k) = -30$ ie. constant.

So, particular sol^n.

$y_k^{(P)} = A$ —— ⑤

Now, substitute eq^n ⑤ in eq ①

$A - A - 6A = -30$

$A = 5$.

So, $\boxed{y_k^{(P)} = 5}$ —— ⑥

Complete sol^n will be -

$$y_k = y_k^{(h)} + y_k^{(p)}$$

$$\boxed{y_k = b_1(-2)^k + b_2 3^k + 5} \quad \text{—(7)}$$

Put, $k = 0$ in eq^n (7) & $k = 1$

$y_0 = b_1 + b_2 + 5$

$20 = b_1 + b_2 + 5$

$\Rightarrow \boxed{b_1 + b_2 = 15}$

$y_1 = -2b_1 + 3b_2 + 5$

$-5 = -2b_1 + 3b_2 + 5$

$\Rightarrow \boxed{2b_1 - 3b_2 = 10}$

$$\begin{aligned} \cancel{2b_1} + 2b_2 &= 30 \\ +\cancel{2b_1} - 3b_2 &= 10 \\ - \quad + \quad &\quad - \\ \hline \cancel{5} b_2 &= \cancel{20}\,4 \end{aligned}$$

$\boxed{b_2 = 4}$ $\qquad$ $b_1 = 15 - 4$, $\boxed{b_1 = 11}$

Sol^n will be –

$$\boxed{y_k = (11)(-2)^k + 4 \cdot 3^k + 5}$$

Q $\quad a_n - 6a_{n-1} + 8a_{n-2} = 3^n$ —(1) $\quad a_0 = 3,\ a_1 = 7$

$\alpha^n - 6\alpha^{n-1} + 8\alpha^{n-2} = 0$

$\alpha^2 - 6\alpha + 8 = 0$

$\alpha^2 - 4\alpha - 2\alpha + 8 = 0$

$\alpha(\alpha - 4) - 2(\alpha - 4) = 0$

$(\alpha - 4)(\alpha - 2) = 0$

$\alpha = 4, 2$

$a_n^{(h)} = b_1(2)^n + b_2(4)^n$.

$f(n) = 3^n$.

Particular sol^n will be –

$a_n^{(p)} = A3^n$.

Put in eq^n (1)

$A \cdot 3^n - 6 \cdot A \cdot 3^{n-1} + 8A \cdot 3^{n-2} = 3^n$.

$A \cdot \cancel{3^n}\left(1 - \frac{6}{3} + \frac{8}{9}\right) = \cancel{3^n}$

$A\left(\frac{9 - 18 + 8}{9}\right) = 1$

$\Rightarrow \quad A = -9$

$a_n^{(p)} = -9 \cdot 3^n$.

Complete sol^n.

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = b_1 \cdot 2^n + b_2 \cdot 4^n - 9 \cdot 3^n$ —(2)

Put, $n = 0$ & $n = 1$ in eq^n —(2)

$a_0 = b_1 + b_2 - 9$

$b_1 + b_2 - 9 = 3$

$b_1 + b_2 = 12$

$a_1 = 2b_1 + 4b_2 - 27$

$7 = 2b_1 + 4b_2 - 27$

$2b_1 + 4b_2 = 34$

$b_1 + 2b_2 = 17$

$b_1 + b_2 = 12$

$\underline{\pm b_1 \pm 2b_2 = \pm 17}$

$-b_2 = -5$

$\boxed{b_2 = 5}$, $\boxed{b_1 = 7}$

Sol^n is

$\boxed{a_n = 7 \cdot 2^n + 5 \cdot 4^n - 9 \cdot 3^n}$

Q $a_{r+2} - 5a_{r+1} + 6a_r = (r+1)^2$.

$\alpha^{r+2} - 5\alpha^{r+1} + 6\alpha^r = 0$

$\alpha^2 - 5\alpha + 6 = 0$

$\alpha^2 - 3\alpha - 2\alpha + 6 = 0$

$\alpha(\alpha - 3) - 2(\alpha - 3) = 0$

$(\alpha - 3)(\alpha - 2) = 0$

$\alpha = 3, 2$.

Sol^n is.

$\boxed{a_r^{(h)} = b_1 2^r + b_2 3^r}$

$f(n) = (r+1)^2 = r^2 + 2r + 1$

$\boxed{a_r^{(P)} = A_0 + A_1 r + A_2 r^2}$

$A_0 + A_1(r+2) + A_2(r+2)^2 - 5(A_0 + A_1(r+1) + A_2(r+1)^2) +$

$6(A_0 + A_1 r + A_2 r^2) = r^2 + 2r + 1$

$\Rightarrow A_0 + A_1(r+2) + A_2(r^2 + 4 + 4r) - 5[A_0 + A_1(r+1) + A_2(r^2 + 1 + 2r)]$

$+ 6(A_0 + A_1 r + A_2 r^2) = r^2 + 2r + 1$

Comparing coefficient of r^2 on both sides.

$A_2 - 5A_2 + 6A_2 = 1$

$\boxed{A_2 = \frac{1}{2}}$

Comparing coefficient of r.

$A_1 + 4A_2 - 5A_1 - 10A_2 + 6A_2 = 2$

$\Rightarrow 2A_1 - 6A_2 = 2$

$\boxed{A_1 = \frac{5}{2}}$

comparing constants -

$A_0 + 2A_1 + 4A_2 - 5A_0 - 5A_1 - 5A_2 + 6A_0 = 1$

$2A_0 - 3A_1 - A_2 = 1$

$\Rightarrow 2A_0 - \frac{15}{2} - \frac{1}{2} = 1$

$\boxed{A_0 = 9/2}$

$\boxed{a_r^{(P)} = \frac{9}{2} + \frac{5}{2} r + \frac{1}{2} r^2}$.

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Q. $a_r - 9a_{r-1} + 20\, a_{r-2} = 2\cdot 5^r$

$\alpha^2 - 9\alpha + 20 = 0$

$\alpha^2 - 4\alpha - 5\alpha + 20 = 0$

$\alpha(\alpha-4) - 5(\alpha-4) = 0$

$(\alpha-4)(\alpha-5) = 0$

$\alpha = 4, 5$

$$\boxed{a_r^{(h)} = b_1 (4)^r + b_2 (5)^r}$$

$a_r^{(P)} = A_0 r (5)^r$ Particular solⁿ.

Put in eqⁿ 1

$\cancel{2(5)^r = 9\cdot 2(5)^{r-1} + 20\cdot 2(5)^{r-2} = 2(5)^r}$

$A_0 \cdot r \cdot 5^r - 9A_0(r-1)5^{r-1} + 20 A_0 (r-2) 5^{r-2} = 2\cdot 5^r$

$\Rightarrow \cancel{5^r}\left[A_0 r - \frac{9}{5}A_0(r-1) + \frac{20}{25}A_0(r-2)\right] = 2\cdot\cancel{5^r}$

$\Rightarrow A_0 r - \frac{9}{5}A_0 r + \frac{9}{5}A_0 + \frac{20}{25}A_0 r - \frac{40}{25}A_0 = 2$

$\Rightarrow \frac{25A_0 r - 45A_0 r + 20A_0 r}{25} + \frac{45A_0 - 40A_0}{25} = 2$

$\Rightarrow +5A_0 = 50$

$\Rightarrow A_0 = +\frac{50}{5}$ 10

$\Rightarrow \boxed{A_0 = +10}$

$\Rightarrow a_r^{(P)} = +10\, r(5)^r$

Complete solⁿ

$a_r = a_r^{(h)} + a_r^{(P)}$

$$\boxed{a_r = b_1 \cdot 4^r + b_2 5^r + 10\, r(5)^r}$$

Q $a_r - 7a_{r-1} + 12 a_{r-2} = r\cdot 4^r$ —— 1

$\alpha^2 - 7\alpha + 12 = 0$

$\Rightarrow \alpha = 3, 4$

Solⁿ is

$a_r^{(h)} = b_1(3)^r + b_2(4)^r$

Particular solⁿ

$a_r^{(P)} = (A_0 + A_1 r)\cdot r \cdot 4^r$

Put in eqⁿ 1

$(A_0 r + A_1 \cdot r^2)4^r - 7(A_0(r-1) + A_1 \cdot (r-1)^2)4^{r-1} + 12(A_0(r-2) + A_1(r-2)^2)4^{r-2} = r\cdot 4^r$

$\Rightarrow \cancel{4^r}\left[A_0 r + A_1 r^2 - \frac{7}{4}\left[A_0 r - A_0 + A_1 r^2 + A_1 - 2A_1 r\right) + \frac{12}{16}\left(A_0 r - 2A_0 + A_1 r^2 + 4A_1 - 4A_1 r\right)\right] = r\cdot\cancel{4^r}$

$\Rightarrow \cancel{A_0} - \frac{7}{4}\cancel{A_0} + \frac{14}{4}A_1 + \frac{3}{4}\cancel{A_0} - \frac{12}{4}A_1 = 1.$

$\frac{1}{2}A_1 = 1$

$\Rightarrow \boxed{A_1 = 2.}$

$\Rightarrow \frac{7}{4}A_0 - \frac{7}{4}A_1 - \frac{6}{4}A_0 + \frac{12}{4}A_1 = 0.$

$\Rightarrow \frac{1}{4}A_0 + \frac{5}{4}A_1 = 0$

$\frac{1}{4}A_0 = -\frac{10}{4}.$

$\boxed{A_0 = -10.}$

Soln will be

$a_r^{(P)} = (2r-10)r.4^r.$

complete soln

$a_r = a_r^{(P)} + a_r^{(h)}$

$\boxed{a_r = b_1(3)^r + b_2(4)^r + (2r-10)r.4^r}$

Q $S(K) - 9S(K-1) + 8S(K-2) = 9K+1$ — ① $S(0) = S(1) = 1$

$\alpha^2 - 9\alpha + 8 = 0.$

$\alpha = 1, 8.$

$S(K) = b_1(1)^K + b_2(8)^K.$

$\boxed{S(K)^{(h)} = b_1 + b_2(8)^K}$ — ②

Particular soln –

$\boxed{S(k)^{(P)} = (A_0 + A_1K)K}.$

Put in eqn ①.

$(A_0K + A_1K^2) - 9(A_0(K-1) + A_1(K-1)^2) + 8(A_0(K-2) + A_1(K-2)^2) = 9K+1$

$A_0K + A_1K^2 - 9A_0K + 9A_0 + 9A_1K^2 + 9A_1 - 18A_1K + 8A_0K - 16A_0 + 8A_1K^2 + 32A_1 - 32A_1K = 9K+1$

Comparing coefficients of K.

$\cancel{A_0} - 9\cancel{A_0} - 18A_1 + 8\cancel{A_0} - 32A_1 = 9$

$-14A_1 = 9$

$\boxed{A_1 = -\frac{9}{14}}$

Comparing constant -

$9A_0 + 9A_1 - 16A_0 + 32A_1 = 1.$

$-7A_0 + 23A_1 = 1.$

$7A_0 = -23\frac{\times 9}{14} - 1$

$\boxed{A_0 = -\frac{221}{98}}$

Put, $k=0$ in eq ②

$s(0) = b_1 + b_2$

$1 = b_1 + b_2$.

Put, $k=1$ in eq^n ②

$s(1) = b_1 + 8b_2$

$b_1 + 8b_2 = 1$.

$$\begin{array}{r} b_1 + b_2 = 1 \\ \pm b_1 \pm 8b_2 = 1 \\ \hline -7b_2 = 0 \end{array}$$

complete $sol^{\underline{n}}$

$s(k) = s_k^{(P)} + s_k^{(h)}$

$s(k) = b_1 + b_2(8)^k + (A_0 + A_1 k) k$.

$= b_1 + b_2(8)^k + \left(-\frac{221}{98} - \frac{9}{14}k\right) k$.

$s(k) = b_1 + b_2(8)^k - \frac{221}{98}k - \frac{9}{14}k^2$. ——— ③

Put $k=0$ in eq^n ③

$s(0) = b_1 + b_2$

$b_1 + b_2 = 1$.

Put, $k=1$ in eq^n ③

$s(1) = b_1 + 8b_2 - \frac{221}{98} - \frac{9}{14}$

$1 = b_1 + 8b_2 - \frac{284}{98}$.

$b_1 + 8b_2 = 1 + \frac{284}{98} = \frac{98+284}{98} = \frac{382}{98}$.

$b_1 + 8b_2 = \frac{191}{49}$

$$\begin{array}{r} b_1 + b_2 = 1 \\ + b_1 + 8b_2 = \frac{191}{49} \\ - \quad - \quad - \\ \hline -7b_2 = \frac{49-191}{49} \end{array}$$

$b_2 = \frac{142}{49 \times 7}$ $\Rightarrow$ $\boxed{b_2 = \frac{142}{343}}$

$b_2 = 1 - \frac{142}{343}$ $\Rightarrow$ $\boxed{b_2 = \frac{201}{343}}$

$$\boxed{s(k) = \frac{201}{343} + \frac{142}{343}(8)^k - \frac{221}{98}k - \frac{9}{14}k^2}$$

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9 $y_{n+2} - y_{n+1} - y_n = n^2$ —— (1)

$\alpha^2 - \alpha - 1 = 0$

$\alpha = \frac{1+\sqrt{5}i}{2}, \frac{1-\sqrt{5}i}{2}$ $\qquad \frac{1 \pm \sqrt{5}i}{2}$

$$\boxed{y_n^{(h)} = b_1\left(\frac{1}{2} + \frac{\sqrt{5}}{2}i\right)^n + b_2\left(\frac{1}{2} - \frac{\sqrt{5}}{2}i\right)^n}$$

Particular solⁿ will be -

$$\boxed{y_n^{(P)} = (A_0 + A_1 n + A_2 n^2)}$$

Put this in eqⁿ (1)

$[A_0 + A_1(n+2) + A_2(n+2)^2] - [A_0 + A_1(n+1) + A_2(n+1)^2] -$
$[A_0 + A_1 n + A_2 n^2] = n^2$

$A_0 + A_1 n + 2A_1 + A_2 n^2 + 4A_2 + 4A_2 n - A_0 - A_1 n - A_1 - A_2 n^2 - A_2$
$- 2A_2 n - A_0 - A_1 n - A_2 n^2 = n^2$

compare the coefficients of n^2 -

$\cancel{A_2} - \cancel{A_2} - A_2 = 1$

$\Rightarrow \boxed{A_2 = -1}$

compare the coefficients of n.

$\cancel{A_1} + 4A_2 - A_1 - 2A_2 - \cancel{A_1} = 0$

$-A_1 + 2 = 0$

$-A_1 = +2$

$\Rightarrow \boxed{A_1 = -2}$

compare constants -

$\cancel{A_0} + 2A_1 + 4A_2 - A_0 - A_1 - A_2 - \cancel{A_0} = 0$

$A_1 + 3A_2 - A_0 = 0$

$-2 - 3 - A_0 = 0$

$\Rightarrow \boxed{A_0 = -5}$

$$\boxed{y_n^{(P)} = (-5 - 2n - n^2)}$$

complete solⁿ

$y_n = y_n^{(h)} + y_n^{(P)}$

$$\boxed{y_n = b_1\left(\frac{1+\sqrt{5}}{2}\right)^n + b_2\left(\frac{1-\sqrt{5}}{2}\right)^n - 5 - 2n - n^2}$$

Q $a_r - 6a_{r-1} + 8a_{r-2} = r \cdot 4^r$ —— (1) $\quad a_0 = 8, \; a_1 = 22$

$\alpha^2 - 6\alpha + 8 = 0$

$\alpha^2 - 4\alpha - 2\alpha + 8 = 0$

$\alpha(\alpha - 4) - 2(\alpha - 4) = 0$

$(\alpha - 2)(\alpha - 4) = 0$

$\alpha = 2, 4$

$$\boxed{a_r^{(h)} = b_1 2^r + b_2 4^r}$$

Particular solⁿ -

$$\boxed{a_r^{(P)} = (A_0 + A_1 r)\, r \cdot 4^r}$$

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Put in eq^n (1)

$$(A_0 + A_1 r) r \cdot 4^r - 6(A_0(r-1) + A_1(r-1)^2)4^r + 8(A_0(r-2) + A_1(r-2)^2)4^r = r \cdot 4^r$$

$$A_0 r \cdot 4^r + A_1 r^2 4^r - 6A_0 r \cdot 4^r + 6A_0 4^r - 6A_1 r^2 \cdot 4^r - 6A_1 4^r + 12A_1 r \cdot 4^r + 8[A_0 r - 2A_0 + A_1 r^2 + 4A_1 - 4A_1 r)4^r = r4^r$$

$$A_0 r + A_1 r^2 - 6A_0 r + 6A_0 - 6A_1 r^2 - 6A_1 + 12A_1 r + 8A_0 r - 16A_0 + 8A_1 r^2 + 32A_1 - 32A_1 r = r$$

compare coefficients of r.

$$2(A_0 r + A_1 r^2) - 3(A_0(r-1) + A_1(r^2 - 2r + 1)) + (A_0(r-2) + A_1(r^2 - 4r + 4)) = 2 \cdot r$$

$$\cancel{2A_0} - \cancel{3A_0} + 6A_1 + \cancel{A_0} - 4A_1 = 2$$

$$2A_1 = 2$$

$$\Rightarrow A_1 = 1$$

$$3A_0 - 3A_1 - 2A_0 + 4A_1 = 0$$

$$A_0 + A_1 = 0$$

$$\Rightarrow A_0 = -1$$

$$\boxed{a_r^{(P)} = (r-1) \cdot r \cdot 4^r}$$

complete sol^n

$$a_r = a_r^{(h)} + a_r^{(P)}$$

$$\boxed{a_r = b_1 \cdot 2^r + b_2 \cdot 4^r + (r-1) \cdot r \cdot 4^r} \quad \text{— (2)}$$

Put, $r=0$ & $r=1$ in eq^n (2)

$$a_0 = b_1 + b_2$$

$$\Rightarrow b_1 + b_2 = 8$$

$$a_1 = 2b_1 + 4b_2$$

$$\Rightarrow b_1 + 2b_2 = 11$$

$$\boxed{b_1 = 5}$$

$$\cancel{b_1} + b_2 = 8$$

$$-\cancel{b_1} - 2b_2 = -11$$

$$-b_2 = -3$$

$$\boxed{b_2 = 3}$$

Now, sol^n –

$$\boxed{a_r = 5 \cdot 2^r + 3 \cdot 4^r + (r-1) r 4^r}$$

* Generating Function (G.F.) :-

The generating function for the sequence $a_0, a_1, a_2, a_3, a_4, \ldots, a_k, \ldots$ of real number is infinite series given by

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k \cdot x^k$$

Here, 'x' is considered as just as symbol called indeterminant and it is not a variable which is replaced by numbers belonging to same domain.

Q Find the generating function of the given sequence $1, 2, 4, 8, ----$

→ Here, $a_0 = 1$, $a_1 = 2$, $a_2 = 4$, $a_3 = 8$

$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ----$

$G(x) = 1 + 2x + 4x^2 + 8x^3 + ----$

$G(x) = 1 + 2x + 2^2 \cdot x^2 + 2^3 \cdot x^3 + ----$

$\boxed{G(x) = \frac{1}{1-2x}} = \sum_{r=0}^{\infty} 2^r \cdot x^r$

a, r

$T_n = a r^{n-1}$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_\infty = \frac{a}{1-r}$

Q $1, -1, 1, -1, -----$

$a_0 = 1$, $a_1 = -1$, $a_2 = 1$, $a_3 = -1$

$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ---$

$G(x) = 1 + -x + x^2 - x^3 + ----$

$\boxed{G(x) = \frac{1}{1+x}} = \sum_{r=0}^{\infty} (-x)^r$

Q $1, 1, 0, 1, 1, 1, -----$

Here, $a_0 = 1$, $a_1 = 1$, $a_2 = 0$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$.

$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + ---$

$G(x) = (1 + x + x^3 + x^4 + x^5 + --- + x^2) - x^2$

$\boxed{G(x) = \frac{1}{1-x} - x^2}$

Q $0, 0, 1, 1, 1, ----$

Here, $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$

$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 ---$

$G(x) = 0 + 0 + x^2 + x^3 + x^4 + ----$

$\boxed{G(x) = \frac{x^2}{1-x}}$

Q Find the generating function of the sequence whose nth term is $a_n = n$.

Here, $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, ---

$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ----$

$G(x) = 0 + x + 2x^2 + 3x^3 + 4x^4 ---$

$x\, G(x) = \quad + x^2 + 2x^3 + 3x^4 + -----$

$(1-x)\, G(x) = x + x^2 + x^3 + x^4$

$\Rightarrow \boxed{G(x) = \frac{x}{(1-x)^2}}$

Here, it forms AP GP, therefore we multiply the common ratio of GP & shifts the one term.

Q Determine the generating function-

-2^r if r is odd.

$\Rightarrow$ Here, $a_0 = 1$, $a_1 = -2$, $a_2 = 2^2$, $a_3 = -2^3$, $a_4 = 2^4$; - -

$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \text{— —}$

$G(x) = 1 + -2x + 2^2x^2 - 2^3x^3 + 2^4x^4 + \text{— —}$

$$\boxed{G(x) = \frac{1}{1+2x}} = \sum_{r=0}^{\infty} (-2x)^r .$$

Q Find the simple expression for the G.f. of the follow discrete numeric function.

$1, \frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \text{- - - -}, \frac{(r+1)}{3^r}, \text{- - -}$

Here, $a_0 = 1$, $a_1 = \frac{2}{3}$, $a_2 = \frac{3}{9}$, $a_3 = \frac{4}{27}$.

$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \text{— —}$

$G(x) = 1 + \frac{2}{3}x + \frac{3}{9}x^2 + \frac{4}{27}x^3 + \text{— — —}$

$-\frac{x}{3}G(x) = - \frac{1}{3}x + \frac{2}{9}x^2 + \frac{3}{27}x^3 + \text{— —}$

$(1-\frac{x}{3})G(x) = 1 - \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{27}x^3 + \text{— — -}$

$$\boxed{G(x) = \frac{1}{(1-\frac{x}{3})^2} = \frac{9}{(3-x)^2}.}$$

* Solution of recurrence relation using G.F. :-

Q Solve $a_{r+2} - 3a_{r+1} + 2a_r = 0$ —① $a_0 = 2$, $a_1 = 3$.

$\rightarrow$ Let, $G(x) = \sum_{r=0}^{\infty} a_r \cdot x^r$ is the g.f. of this sequence.

Multiplying eq^n ① by x^r and summing from $r=0$ to ∞:

$$\sum_{r=0}^{\infty} a_{r+2} \cdot x^r - 3\sum_{r=0}^{\infty} a_{r+1} \cdot x^r + 2\sum_{r=0}^{\infty} a_r \cdot x^r = 0.$$

Now, Expand all these terms -

$\Rightarrow (a_2 + a_3x + a_4x^2 + \text{- - -}) - 3(a_1 + a_2x + a_3x^2 + \text{- -}) + 2(a_0 + a_1x + a_2x^2 + a_3x^3 + \text{— —}) = 0.$

$\Rightarrow \frac{x^2}{x^2}(a_2 + a_3x + a_4x^2 + \text{- -}) - 3\frac{x}{x}(a_1 + a_2x + a_3x^2 + \text{- -}) + 2(a_0 + a_1x + a_2x^2 + a_3x^3 + \text{— —}) = 0$

$$\Rightarrow \frac{G(x) - a_0 - a_1x}{x^2} - 3\frac{G(x) - a_0}{x} + 2G(x) = 0.$$

$\Rightarrow G(x) - a_0 - a_1x - 3x[G(x) - a_0] + 2x^2G(x) = 0$

$\Rightarrow G(x)[1 - 3x + 2x^2] - a_0 - a_1x + 3xa_0 = 0.$

$\Rightarrow G(x)[1 - 3x + 2x^2] - 2 - 3x + 6x = 0$

$\Rightarrow G(x)[1 - 3x + 2x^2] = 2 - 3x.$

$$\Rightarrow G(x) = \frac{2-3x}{1-3x+2x^2}.$$

$$\Rightarrow G(x) = \frac{2-3x}{(1-2x)(1-x)} = \frac{A}{1-2x} + \frac{B}{1-x}.$$

$1-2x = 0$
$x = \frac{1}{2}$
Jis ko 0 ke equal rakhkar uske dwara [illegible] puri value m x ka value rkho.

$1-x = 0$
$x = 1$
$\frac{(2-3)}{(1-2)}$ = 1

$$G(x) = \frac{1}{1-2x} + \frac{1}{1-x}.$$

$$G(x) = \sum_{r=0}^{\infty} (2x)^r + \sum_{r=0}^{\infty} (x)^r.$$

$$\sum_{r=0}^{\infty} a_r \cdot x^r = \sum_{r=0}^{\infty} (2x)^r + \sum_{r=0}^{\infty} (x)^r$$

$$\boxed{a_r = 2^r + 1.}$$

Q. $a_{r+2} - 2a_{r+1} + a_r = 2^r$ ——(1) $a_0 = 2,\ a_1 = 1.$

Let, $G(x) = \sum_{r=0}^{\infty} a_r \cdot x^r$ is the gf of this function.

Multiply eqn (1) by x^r & summing $r=0$ to $r=\infty$

$$\sum_{r=0}^{\infty} a_{r+2}\, x^r - 2\sum_{r=0}^{\infty} a_{r+1}\, x^r + \sum_{r=0}^{\infty} a_r \cdot x^r = \sum_{r=0}^{\infty} 2^r \cdot x^r$$

$$(a_2 + a_3 x + a_4 x^2 + \cdots) - 2(a_1 + a_2 x + a_3 x^2 + \cdots) + (a_0 + a_1 x + a_2 x^2 + \cdots) = 1 + 2x + 2^2 \cdot x^2 + \cdots$$

$$\frac{x^2}{x^2}(a_2 + a_3 x + a_4 x^2 + \cdots) - \frac{2x}{x}(a_1 + a_2 x + a_3 x^2 + \cdots) + (a_0 + a_1 x + a_2 x^2 + \cdots) = 1 + 2x + 2^2 x^2 + \cdots$$

$$\left(\frac{G(x) - a_0 - a_1 x}{x^2}\right) - \frac{2}{x}(G(x) - a_0) + G(x) = 1 + 2x + 2^2 x^2 + \cdots$$

$$G(x) - a_0 - a_1 x - 2x(G(x) - a_0) + x^2 G(x) = x^2(1 + 2x + 2^2 x^2 + \cdots)$$

$$\Rightarrow G(x)[1 - 2x + x^2] - a_0 - a_1 x + 2a_0 x = x^2 + 2x^3 + 2^2 x^4 + \cdots$$

$$\Rightarrow G(x)[1 - 2x + x^2] - 2 - x + 4x = x^2[1 + 2x + 2^2 x^2 + \cdots]$$

$$\Rightarrow G(x)[x^2 - 2x + 1] = x^2[1 + 2x + 2^2 x^2 + \cdots] + 2 - 3x.$$

$$\Rightarrow G(x)(x^2 - 2x + 1) = \frac{x^2}{1-2x} + 2 - 3x.$$

$$\Rightarrow G(x) = \frac{7x^2 - 7x + 2}{(1-2x)(x^2 - 2x + 1)}$$

$$\Rightarrow G(x) = \frac{7x^2 - 7x + 2}{(1-2x)(1-x)^2}$$

$$\Rightarrow G(x) = \frac{A}{(1-x)^2} + \frac{B}{(1-x)} + \frac{C}{(1-2x)}$$

$$7x^2 - 7x + 2 = A(1-2x) + B(2x^2 - 3x + 1) + C(x^2 - 2x + 1)$$

$$B = 3,\quad A = -2,\quad C = 1.$$

$$\Rightarrow G(x) = \frac{-2}{(1-x)^2} + \frac{3}{(1-x)} + \frac{1}{(1-2x)}$$

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$$\Rightarrow G(x) = -2\left(1+2x+\frac{2\cdot3}{2!}x^2+\frac{2\cdot3\cdot4}{3!}x^3+\cdots\right)+3\sum_{r=0}^{\infty}x^r+\sum_{r=0}^{\infty}2^r\cdot x^r$$

$$\Rightarrow G(x) = -2(1+2x+3x^2+4x^3+\cdots)+3\sum_{r=0}^{\infty}x^r+\sum_{r=0}^{\infty}2^r\cdot x^r$$

$$\Rightarrow G(x) = -2\sum_{r=0}^{\infty}(r+1)\cdot x^r+3\sum_{r=0}^{\infty}x^r+\sum_{r=0}^{\infty}2^r\cdot x^r.$$

$$\Rightarrow a_r = -2r-2+3+2^r.$$

$$\Rightarrow \boxed{a_r = 2^r-2r+1}.$$

Q $F(K) = F(K-1)+F(K-2)$ —— ① $\quad F(0) = F(1) = 1$.

Let, $G(x) = \sum_{K=0}^{\infty} F(K)\cdot x^K$ is the G.F. of this func.

Multiply eq^n ① by x^K & summing $K=2$ to $K=\infty$.

$$\sum_{K=2}^{\infty}F(K)x^K-\sum_{K=2}^{\infty}F(K-1)x^K-\sum_{K=2}^{\infty}F(K-2)x^K = 0.$$

$$(F(2)x^2+F(3)x^3+\cdots)-(F(1)x^2+F(2)x^3+\cdots)-(F(0)x^2+F(1)x^3+\cdots) = 0.$$

$$(G(x)-F(0)-F(1)x)-x(G(x)-F(0))-x^2G(x) = 0$$

$$G(x)[1-x-x^2]-1-\cancel{x}+\cancel{x} = 0.$$

$$G(x)(1-x-x^2) = 1.$$

$$G(x) = \frac{1}{1-x-x^2}$$

$$G(x) = \frac{-1}{(x^2+x-1)}$$

$$G(x) = \frac{-1}{\left(x-\frac{-1+\sqrt5}{2}\right)\left(x-\frac{-1-\sqrt5}{2}\right)}$$

$$G(x) = \frac{A}{\left(x-\frac{-1+\sqrt5}{2}\right)}+\frac{B}{\left(x-\frac{-1-\sqrt5}{2}\right)}$$

$$A = \frac{-1}{\frac{-1+\sqrt5}{2}-\left(\frac{-1-\sqrt5}{2}\right)} = \frac{-1}{\frac{-\cancel{1}+\sqrt5+\cancel{1}+\sqrt5}{2}} = \frac{-1}{\frac{2\sqrt5}{\cancel{2}}} = \frac{-1}{\sqrt5}$$

$$A = \frac{-1}{\sqrt5}, \quad B = \frac{1}{\sqrt5}.$$

$$G(x) = \frac{-\frac{1}{\sqrt5}}{\left(x-\left(\frac{-1+\sqrt5}{2}\right)\right)}+\frac{\frac{1}{\sqrt5}}{\left(x+\frac{1+\sqrt5}{2}\right)}$$

$$G(x) = \frac{1}{\sqrt5\left(\frac{\sqrt5-1}{2}-x\right)}+\frac{1}{\sqrt5\left(\frac{\sqrt5+1}{2}+x\right)}.$$

$$G(x) = \frac{2}{\sqrt5(\sqrt5-1)}\left(\frac{1}{1-\frac{2x}{\sqrt5-1}}\right)+\frac{2}{\sqrt5(\sqrt5+1)}\left(\frac{1}{1+\frac{2x}{\sqrt5+1}}\right)$$

$$\sum_{k=0}^{\infty} F(k)x^k = \frac{1}{\sqrt5}\left[\frac{2}{\sqrt5-1}\sum_{k=0}^{\infty}\left(\frac{2x}{\sqrt5-1}\right)^k + \frac{2}{\sqrt5+1}\sum_{k=0}^{\infty}\left(\frac{-2x}{\sqrt5+1}\right)^k\right]$$

Compare the coefficients of x^k.

$$F(k) = \frac{1}{\sqrt5}\left[\frac{2}{\sqrt5-1}\left(\frac{2}{\sqrt5-1}\right)^k + \frac{2}{\sqrt5+1}\left(\frac{2}{\sqrt5+1}\right)^k(-1)^k\right]$$

$$\boxed{F(k) = \frac{1}{\sqrt5}\left[\left(\frac{2}{\sqrt5-1}\right)^{k+1} + \left(\frac{2}{\sqrt5+1}\right)^{k+1}(-1)^k\right].}$$

Q $a_r - 2a_{r-1} - 3a_{r-2} = 0$, —① $r \geq 2$, $a_0 = 3$, $a_1 = 1$

→ Let, $G(x) = \sum_{r=0}^{\infty} a_r \cdot x^k$ is the GF of the func.

Multiply the eqn-① by x^r & summing $r=2$ to $r=\infty$.

$$\sum_{r=2}^{\infty} a_r x^r - 2\sum_{r=2}^{\infty} a_{r-1} x^r - 3\sum_{r=2}^{\infty} a_{r-2} x^r = 0.$$

$$(a_2x^2 + a_3x^3 + \dots) - 2(a_1x^2 + a_2x^3 + \dots) - 3(a_0x^2 + a_1x^3 + \dots) = 0.$$

$$G(x) - \cancel{G(0)} - a_1 x - 2x(G(x) - a_0) - 3x^2 G(x) = 0.$$

$$G(x)[1 - 2x - 3x^2] - a_0 - a_1x + 2xa_0 = 0.$$

$$G(x)[1 - 2x - 3x^2] - 3 - x + 6x = 0.$$

$$G(x)[-3x^2 - 2x + 1] = 3 - 5x.$$

$$G(x) = \frac{3-5x}{-3x^2-2x+1}$$

$3x^2 + 3x - x - 1$

$3x(x+1) - 1(x+1)$

$$G(x) = \frac{5x-3}{3x^2+2x-1}$$

$$G(x) = \frac{5x-3}{(3x-1)(x+1)}.$$

$$\cancel{(3x-1)(x+1)}\ G(x) = \frac{A}{3x-1} + \frac{B}{x+1}$$

$$5x - 3 \;(\cancel{3x^2+2x-1}) = Ax + A + 3Bx - B.$$

$\cancel{A + 3B = 2}$ $\qquad A - B = \cancel{-1}$.

$\cancel{A - B = -1}$ $\qquad A = -1 + \frac{3}{4}$

$\cancel{4B = 3}$ $\qquad A = -\frac{1}{4}$

$\cancel{B = \frac{3}{4}}$ $\qquad \cancel{A = -1},\ \cancel{B = 2}$

$\cancel{A} + 3B = 5 \qquad A - B = -3.$

$A - B = -3$

$4B = 8/2$

$$G(x) = \frac{1}{3x-1}$$

$\boxed{B = 2}$ $\qquad A = -3 + 2 \Rightarrow \boxed{A = -1}$

$$G(x) = \frac{1}{1-3x} + \frac{2}{x+1}$$

$$G(x) = \sum_{k=0}^{\infty}(3x)^k + 2\sum_{k=0}^{\infty}(-x)^k.$$

Q. $a_n + 3a_{n-1} - 10a_{n-2} = 0$ —— ① $n \geq 2$, $a_0 = 1$, $a_1 = 4$

→ Let, $G(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$ is the GF of the func.

Multiply the eq^n ① by x^n & summing $n=2$ to $n=\infty$.

$$\sum_{n=2}^{\infty} a_n x^n + 3\sum_{n=2}^{\infty} a_{n-1} x^n - 10 \sum_{n=2}^{\infty} a_{n-2} x^n = 0 .$$

$$(a_2 x^2 + a_3 x^3 + \dots) + 3(a_1 x^2 + a_2 x^3 + \dots) - 10(a_0 x^2 + a_1 x^3 + \dots) =$$

$$G(x) - a_0 - a_1 x + 3x(G(x) - a_0) - 10x^2(G(x)) = 0$$

$$G(x)[1 + 3x - 10x^2] - a_0 - a_1 x - 3a_0 x = 0$$

$$G(x)[-10x^2 + 3x + 1] - 1 - 4x - 3x = 0$$

$10x^2 - 5x + 2x - 1$

$5x(2x-1) + 1($

$$G(x) = \frac{7x+1}{-10x^2+3x+1}$$

$$G(x) = \frac{-7x-1}{10x^2-3x-1}$$

$$G(x) = \frac{-(7x+1)}{(5x+1)(2x-1)} = \frac{7x+1}{(1-2x)(5x+1)}$$

$$G(x) = \frac{A}{1-2x} + \frac{B}{5x+1}$$

$$7x+1 = 5Ax + A + B - 2Bx .$$

$5A - 2B = 7 .$

$\cancel{5}A + 5B = 5$

$-7B = 2 .$

$\boxed{B = -\frac{2}{7}}$

$A + B = 1 .$

$A = 1 + \frac{2}{7} = \frac{9}{7} .$

$\boxed{A = \frac{9}{7}}$

$$G(x) = \frac{9}{7(1-2x)} - \frac{2}{7(5x+1)}$$

$$G(x) = \frac{9}{7} \sum_{n=0}^{\infty} (2x)^n - \frac{2}{7} \sum_{n=0}^{\infty} (-5x)^n .$$

$$\boxed{a_n = \frac{9}{7}(2)^n - \frac{2}{7}(-5)^n .}$$

* Pigeon Hole Principle :-

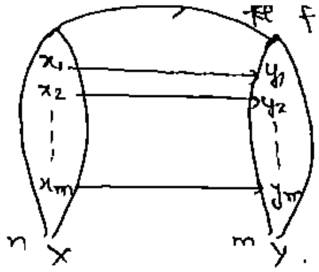
$\lfloor x \rfloor \Rightarrow$ Greatest integer less than or equals to x.

$\lceil x \rceil \Rightarrow$ Smallest integer less than or equals to x.

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 .$$

n = No. of pigeons

m = No. of pigeon holes.



$x_1, x_2 \in X$

$f(x_1) = f(x_2)$

According to pigeon hole principle, if n pigeons are assigned to m pigeons holes and $n > m$. Then, there is atleast one pigeon hole that contains two or more pigeons.

Let, $f: X \to Y$ where, X & Y are finite sets and cardinality of X is 'n' & cardinality of Y is 'm'. & $n > m$.
Then their exist atleast 2 elements x_1 & x_2 such that $f(x_1) = f(x_2)$

Proof:- Let, $X = x_1, x_2, x_3, \ldots, x_m$.
Suppose, f is injective. Then $f(x_1), f(x_2), \ldots, f(x_m)$ are distinct elements in y. So,

As f is one to one. $\boxed{n \le m.}$
But, this is a contradiction to our assumption $\boxed{n > m.}$
Therefore, f is not injective and their must be atleast two distinct elements x_1 & $x_2 \in X$ such that $f(x_1) = f(x_2)$

Q. Find the minimum number of elements to be selected from the set $A = \{3, 4, 5, 6, 7, 8, 9\}$ such that sum of any two elements is 12.

→ The possible sets or cases selecting two elements from set A such that ~~som~~ sum of them is 12 will be as follows -

$\{3,9\}, \{4,8\}, \{5,7\}, \{6,6\}$.

Now, we can observe that each element of A belongs to 1 and only 1 of these sets.
Thus, minimum no. of digits to be selected from set A should be greater than number of sets. So, minimum no. of digits selected is 5.

* Extended or Generalisee Pigeon Hole Principle :-

If 'n' pigeons are accomodated in 'm' pigeon holes. and $n > m$. then one of the pigeon hole must contained atleast $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$.

minimum no. of cars painted with same colour.

→ No. of cars = 100
No. of colour = 9.

Let, cars be the pigeons and colours be the pigeons holes

so, $n = 100$
$m = 9$.

So, Acc. to Pigeon hole principle.
The minimum no. of cars painted with same colour -

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{100-1}{9} \right\rfloor + 1 = \left\lfloor \frac{99}{9} \right\rfloor + 1 = 12.$$

Q How many people must you have to gurantee that atleast 9 of them will have birthdays in the same day of the week

→ $n = ?$
$m =$ Days of the week $= 7$.

$\Rightarrow \left\lfloor \frac{n-1}{m} \right\rfloor + 1 = 9.$

$\Rightarrow \left\lfloor \frac{n-1}{7} \right\rfloor + 1 = 9. \Rightarrow \frac{n-1}{7} + \frac{7}{7} = 9.$

$\Rightarrow n-1 = 9 \times 7 - 6$

$\Rightarrow \boxed{n = 57.}$

Q What is the minimum no. of students required in a class to be sure that atleast 5 of them will receive same grades if there are four grades A, B, C, D.

→ $n = ?$
$m = 4$ (No. of grades).

$\Rightarrow \left\lfloor \frac{n-1}{m} \right\rfloor + 1 = 5 \Rightarrow \frac{n-1}{4} + 1 = 5.$

$\Rightarrow n-1+4 = 20$
$\Rightarrow n = 20-3.$
$\Rightarrow \boxed{n = 17}$

Q In how many ways 6 numbers can be choosen from 1 to 15 so that all the choices have same sum.

→ ${}^{15}C_6 = 5005$ ways.

$1+2+3+4+5+6 = 21$ to $10+11+12+13+14+15 = 75$.
Total sum $= 75 - 21 + 1 = 55$.

No. of pigeons $(n) = 5005$.
No. of pigeon holes $(m) = 55$.

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{5005-1}{55} \right\rfloor + 1 = \left\lfloor \frac{5004}{55} \right\rfloor + 1 = \lfloor 90.98 \rfloor + 1$$

$= 91.$

Atleast ways to select 6 no. = 91
Exactly ways to select 6 no. = 91 − 1 = 90

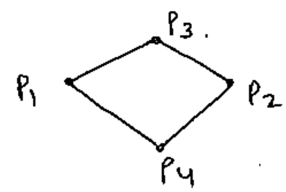
Q.17) Let S = {a, b, c, d, e}. & P = φ be a set of partitions of set S. such that P = { P_1, P_2, P_3, P_4 }.

P_1 = { (a, b, c), (d, e) }
P_2 = { (a, b), (c, d, e) }.
P_3 = { (a, b, c, d, e) }.
P_4 = { (a), (b), (c), (d), (e) }

A POR on P is defined such that $P_i \leq P_j$ or $P_i \lneq = P_j$ where, i, j = 1, 2, 3, 4. If & only if all the elements of P_i are subsets of elements of P_j.

1) Draw the Hasse Diagram of (P, ≤)
2) Is this Poset a lattice.

1)



$P_4 \leq P_1$
$P_4 \leq P_2$
$P_4 \leq P_3$.

P_3 is the power set of all set.

2)

∨	P_1	P_2	P_3	P_4
P_1	P_1	P_3	P_3	P_1.
P_2	P_3	P_2	P_3	P_2.
P_3	P_3	P_3	P_3	P_3.
P_4	P_1	P_2	P_3	P_4.

∧	P_1	P_2	P_3	P_4
P_1	P_1	P_4	P_1	P_4
P_2	P_4	P_2	P_2	P_4.
P_3	P_1	P_2	P_3	P_4.
P_4	P_4	P_4	P_4	P_4.

So, this poset is a lattice.

Q There are two restaurants next to each other one has a sign that says "good food is not cheap" & the other has a sign that says "cheap food is not good": are the signs saying the same thing.

→ p : Food is good.
q : Food is cheap.

$p \to \sim q$.
$q \to \sim p$

p	q	∼p	∼q	p → ∼q	q → ∼p
T	T	F	F	F	F
T	F	F	T	T	~~T~~ F
F	T	T	F	T	T
F	F	T	T	T	T

Yes, they saying the same thing.

UNIT-2.

Algebraic Structure

* n-ary operation :- Let, X be a non-empty set, a mapping function $f: X^n \to X$ is called an n-ary operation on set X where, n is called order of operation.

For n = 1. it is uniary operation.
For n = 2 it is binary operation.
For n = 3 it is ternary operation.
⋮
For n = n it is n-ary operation.

* Algebraic System or Structure :- A system consisting of a non-empty set and 1 or more an n-ary operations on the set is called algebraic system or structure.

An algebric system is represented as -

$(S, f_1, f_2, - - - -)$

Where, S = Non-empty sets.
f_1 & f_2 are n-ary operations on S).

e.g $(R, +)$, $(Z, +)$, $(Z, \cdot)$, $(R, \leq)$

* Properties of Algebraic System :-

Let, $(S, *, +)$ be an algebric system where $*$ and $+$ are binary operations on set S.

S is closed under operation $*$.

Closure Property :- $a * b \in S$, $\forall a, b \in S$.

Associative Property :- $(a*b)*c = a*(b*c)$ $\forall a, b, c \in$

Identity element :- $\exists e \in S$ $a * e = a$ (Right identity) $e * a = a$ (Left identity)

Inverse element :- $\exists a^{-1} \in S$. a^{-1} is called inverse of a under the operation $*$.

$a * a^{-1} = e$.
$a^{-1} * a = e$ $\forall a \in S$.

Commutative property :- $a * b = b * a$, $\forall a, b \in S$.

Where, e is called identity element with respect to operation $*$.

Cancellation Law :- $a * b = a * c \Rightarrow b = c$. (Right cancellation)

$b * a = c * a \Rightarrow b = c$. (Left cancellation)

Distributive law :-

$a * (b+c) = (a*b) + (a*c)$ (Right Distribution)

$(b+c)*a = (b*a) + (c*a)$ (Left Distribution)

Idempotent Property :- An element $a \in S$ is called idempotent element w.r.t operation $*$, if $a * a = a$.

Q The binary operation '$*$' on Q (set of Rational No.'s) defined by $a*b = a+b+2ab$.
Determine where. i) Associative. 2) Commutative
Also find Identity & Inverse element.

→ Let $a, b, c \in Q$.

Associative : $(a*b)*c = a*(b*c)$

LHS $(a*b)*c$.

$\Rightarrow (a+b+2ab)*c$.

$\Rightarrow a+b+2ab+c+2(a+b+2ab)c$.

$\Rightarrow a+b+c+2ab+2ac+2bc+4abc$.

RHS $(a*(b*c))$

$\Rightarrow a*(b+c+2bc)$

$\Rightarrow a+b+c+2bc+2a(b+c+2bc)$

$\Rightarrow a+b+c+2ab+2bc+2ac+4bca$.

Here, LHS = RHS.
So, Associative property follows.

Commutative: $a*b = b*a$.

LHS $a*b$

$\Rightarrow a+b+2ab$.

RHS $b*a$

$\Rightarrow b+a+2ba$.

Here, LHS = RHS
So, Commutative property follows.

Let, $e \in Q$ be the identity element w.r.t '$*$'.

$\forall a \in Q$, $a*e = a$.

$\Rightarrow \cancel{a}+e+2ae = \cancel{a}$.

$\Rightarrow e(1+2a) = 0$.

$\Rightarrow \boxed{e = 0}$ $(\because 1+2a \neq 0)$

Let, $a^{-1} \in Q$ be the inverse of $a \in Q$ w.r.t '$*$'

$a * a^{-1} = e = 0$.

$\Rightarrow a + a^{-1} + 2a \cdot a^{-1} = 0$.

$\Rightarrow a + a^{-1}(1+2a) = 0$

$\frac{\ \ }{1+2a}$ [$\because 1+2a \neq 0$].

* **Semigroup** :- Let, S be a non-empty set and '*' be a binary operation on 'S'. then (S, *) is called a semigroup if it follows -
- closure Property.
- Associative Property.

* **Monoid** :- Let, S be a non-empty set and '*' be binary operation on 'S' then (S, *) is called a Monoid if it follows -
- Closure Property.
- Associative Property.
- Identity Property.

Q Is it a semigroup? (N, +) and (N, •).

closure Property - $2+3 = 5 \in N$. & $2, 3 \in S$.

Associative Property - $(2+3)+4 = 2+(3+4)$

$\Rightarrow 5+4 \qquad 2+7$.

$\Rightarrow 9$ (LHS) $\qquad 9$. (RHS)

LHS = RHS

* Addition Modulo. $a +_m b$ → Remainder. (e.g. $2 +_4 2 = \frac{4}{4} =$

* Multiplication Modulo. $a *_m b$.

Q. Let, $S = 0, 1, 2, 3, 4, 5$ & * be the operation addition modulo 6. show that (S, *) is a semigroup.

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

For Associative -

$1 +_6 (3 +_6 5) = 1 +_6 2 = 3$

$(1 +_6 3) +_6 5 = 4 +_6 5 = 3$

From the above table closure & Associative property follows.

Q Let, * be operation on set $S = Q \times Q$. Given by $(a, b) * (x, y) = (ax, ay+b)$ show that whether (S, *) is a semigroup. also find identity & inverse element on set S if exists.

→ Given, $(a, b) * (x, y) = (ax, ay+b)$

closure Property - (a, b) & $(x, y) \in Q \times Q$.

$a, b, x, y \in Q$.

$\Rightarrow \quad ax \in Q.$

$\Rightarrow \quad ay \in Q.$

$\Rightarrow \quad ay + b \in Q.$

$\Rightarrow \quad (ax, ay+b) \in Q \times Q.$

It follows closure property.

2) <u>Associative</u> <u>Property</u> :-

Let, (a,b), (x,y) & $(u,v) \in Q \times Q$.

$((a,b) * (x,y)) * (u,v) = (a,b) * ((x,y) * (u,v)).$

<u>LHS</u> - $((a,b) * (x,y)) * (u,v)$

$\Rightarrow (ax, ay+b) * (u,v)$

$\Rightarrow (axu, axv + ay+b)$

<u>RHS</u> $(a,b) * ((x,y) * (u,v))$

$\Rightarrow (a,b) * (xu, xv+y)$

$\Rightarrow (axu, a(xv+y)+b)$

$\Rightarrow (axu, axv + ay+b)$

LHS = <u>RHS</u>

Therefore, Associative Property follows.

3) <u>Identity</u> <u>element</u> -

Let. $(e_1, e_2) \in S$.

For, $(a,b) \in S$, $(a,b) * (e_1, e_2) = (a,b)$

$\Rightarrow (ae_1, ae_2 + b) = (a,b)$

$\Rightarrow ae_1 = a \Rightarrow \boxed{e_1 = 1}$

$\Rightarrow ae_2 + \cancel{b} = \cancel{b}$

$\Rightarrow \boxed{e_2 = 0}$

So, $(e_1, e_2) \Rightarrow (1, 0) \in S$.

4) <u>Inverse</u> <u>element</u> -

Let, $(c, d) \in S$ be the inverse of (a,b)

$(a,b) * (c,d) = (1,0)$

$(ac, ad+b) = (1,0)$

$\Rightarrow ac = 1 \Rightarrow c = \frac{1}{a}.$

$\Rightarrow ad+b = 0 \Rightarrow d = -\frac{b}{a}.$

$\left(\frac{1}{a}, -\frac{b}{a}\right).$

$$\left\{ \begin{array}{l} \text{e.g.} \ \left(\frac{1}{2}, \frac{3}{5}\right) \\ \left(2, \frac{-3/5}{1/2}\right) \\ \left(2, -\frac{6}{5}\right). \end{array} \right\}$$

* **Group** :- Let S be a non-empty set with binary operation on 'S' then (S, *) be a is called a Group. if it follows -
  - closure Property
  - Associative Property.
  - Existence of Identity element.
  - Existence of Inverse element.

~~Let (G,*) be a binary algebraic structure~~

e.g. (Z,+), (R,+). are groups.

* **Abelian Group** :- A group (G,*) is called an abelian group or commutative group if binary operation '*' is commutative. i.e.

$$\boxed{a*b = b*a}, \quad \forall a, b \in G.$$

e.g. (Z,+) is an abelian group.

Q (V. Imp) Prove that $G = \{1, \omega, \omega^2\}$ is a group w.r.t multiplicat where, $1, \omega$ & ω^2 are cube roots of unity.

$\omega^3 = 1$

$\omega^4 = \omega^3 \cdot \omega = \omega$.

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

1) Closure Property :- From the table closure property satisfied.

2) Associative Property :- $(1\times\omega)\times\omega^2 = 1\times(\omega\times\omega^2)$

LHS. $\omega\times\omega^2$ $\Rightarrow \omega^3$ $\Rightarrow 1$.

RHS. $1\times\omega^3$ $\Rightarrow 1\times 1$. $\Rightarrow 1$.

3) Existence of Identity Element :-

$1 \in G$ such that

$1\times 1 = 1$.

$\omega\times 1 = \omega$

$\omega^2\times 1 = \omega^2$.

$\therefore$ So, 1 is the identity element.

4) Existence of Inverse Element :-

$1\times 1 = 1$

$\therefore \boxed{1^{-1} = 1}$

$\omega\times\omega^2 = 1$

$\therefore \boxed{\omega^{-1} = \omega^2}$

$\omega^2\times\omega = 1$

$\therefore \boxed{(\omega^2)^{-1} = \omega}$

So, Inverse exist of each element.

So, it is a group.

Q Prove that $G = \{1, 2, 3, 4, 5, 6\}$ is an abelian group under the operation multiplication modulo 7.

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

1) Closure Property :- from the above table closure property satisfied.

2) Associative Property :- from the above table associative property satisfied.

3) Existence of Identity element :-
$1 \in G$ is the identity element.

$1 \times_7 1 = 1$
$2 \times_7 1 = 2$
$3 \times_7 1 = 3$
$4 \times_7 1 = 4$
$5 \times_7 1 = 5$
$6 \times_7 1 = 6$

So, 1 is the identity element.

4) Existence of Inverse element :-

$1 \times_7 1 = 1 \Rightarrow 1^{-1} = 1$
$2 \times_7 4 = 1 \Rightarrow 2^{-1} = 4$
$3 \times_7 5 = 1 \Rightarrow 3^{-1} = 5$
$4 \times_7 2 = 1 \Rightarrow 4^{-1} = 2$
$5 \times_7 3 = 1 \Rightarrow 5^{-1} = 3$
$6 \times_7 6 = 1 \Rightarrow 6^{-1} = 6$

5) Commutative Property :- from the composition table as above it is clear that $\times_7$ is commutative. i.e. $a * b = b * a$, $\forall\, a, b \in G$.

Q Is $G = \{1, 2, 3, 4\}$ a group under addition modulo 5

$+_5$	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

Here, $0 \notin G$. So, it does not follow closure property.
So, it is not a Group.

$\Rightarrow$ $(Z_m, +_m)$ is always a Group.

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$\times_{10}$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

1) Closure Property :- from above ~~table~~ table closure property satisfied..

2) Associative Property :- from the above table, associative property satisfied.

3) Existence of Identity element :-

$1 \in G$ is an identity element.

$1 \times_{10} 1 = 1$

$3 \times_{10} 1 = 3$

$7 \times_{10} 1 = 7$

$9 \times_{10} 1 = 9$

4) Existence of Inverse element :-

$1 \times_{10} 1 = 1 \Rightarrow 1^{-1} = 1$

$3 \times_{10} 7 = 1 \Rightarrow 3^{-1} = 7$

$7 \times_{10} 3 = 1 \Rightarrow 7^{-1} = 3$

$9 \times_{10} 9 = 1 \Rightarrow 9^{-1} = 9$.

5) Commutative Property :- from the composition table it is clear that $\times_{10}$ is commutative. i.e. $a * b = b * a, \forall a, b \in G$.

Q Show that, $G = \{1, -1, i, -i\}$ where $i = \sqrt{-1}$ is an abelian group under the operation multiplication.

→

$\times$	1	−1	i	$-i$
1	1	−1	i	$-i$
−1	−1	1	$-i$	i
i	i	$-i$	−1	1
$-i$	$-i$	i	1	−1

1) Closure Property :- from the above ~~that~~ table closure property satisfied.

2) Associative Property :- from the above table, associative property satisfied.

3) Existence of Identity element :-

$1 \in G$ is an identity element

$1 \times 1 = 1$

$-1 \times 1 = -1$

$i \times 1 = i$

$-i \times 1 = -i$

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4) <u>Existence of Inverse Element :-</u>

$1 \times 1 = 1 \Rightarrow 1^{-1} = 1$

$-1 \times -1 = 1 \Rightarrow (-1)^{-1} = -1$

$i \times -i = 1 \Rightarrow i^{-1} = -i$

$-i \times i = 1 \Rightarrow (-i)^{-1} = i$

5) <u>Commutative Property</u> :- from the composition table it is clear that, $\times$ is commutative i.e. $a * b = b * a$, $\forall a, b \in G$.

* Properties of a Group :-

- <u>The Identity element of a Group $(G, *)$ is unique.</u>

$\Rightarrow$ Let, e_1 & $e_2 \in G$ are identity elements of $(G, *)$.

$e_1 * e_2 = e_2 * e_1 = e_2$ —— (1)

$e_2 * e_1 = e_1 * e_2 = e_1$ —— (2)

From (1) & (2)

$\boxed{e_1 = e_2}$

- <u>The Inverse of every element of $(G, *)$ is unique.</u>

$\Rightarrow$ Let, if possible b & c are inverse of $a \in G$, and $e \in G$ is the identity element.

$b * a = a * b = e$ —— (1)

$c * a = a * c = e$ —— (2)

Now, $b = e * b$

$b = (c * a) * b$

$b = c * (a * b)$

$b = c * e$

$\boxed{b = c}$

$\boxed{a * a^{-1} = a^{-1} * a \ldots}$

- <u>$(a * b)^{-1} = b^{-1} * a^{-1}$ for $a, b \in G$.</u>

$\Rightarrow (a * b) * (b^{-1} * a^{-1})$	$\Rightarrow (b^{-1} * a^{-1}) * (a * b)$
$\Rightarrow a * (b * b^{-1}) * a^{-1}$	$\Rightarrow b^{-1} * (a^{-1} * a) * b$
$\Rightarrow a * e * a^{-1}$	$\Rightarrow b^{-1} * e * b$
$\Rightarrow a * a^{-1}$	$\Rightarrow b^{-1} * b$
$\Rightarrow e$	$\Rightarrow e$

- <u>Cancellation Laws are true in a Group.</u>

$a * b = a * c \Rightarrow b = c$ (Left cancellation)

$b * a = c * a \Rightarrow b = c$ (Right cancellation)

1) $a * b = a * c$

$\Rightarrow a^{-1} * (a * b) = a^{-1} * (a * c)$

$\Rightarrow (a^{-1} * a) * b = (a^{-1} * a) * c$

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$\Rightarrow \quad e * b = e * c$

$\Rightarrow \quad \boxed{b = c}$

2) $b * a = c * a$

$(b * a) * a^{-1} = (c * a) * a^{-1}$

$\Rightarrow \quad b * (a * a^{-1}) = c * (a * a^{-1})$

$\Rightarrow \quad b * e = c * e$

$\Rightarrow \quad \boxed{b = c}$

- $(G, *)$ can not have an idempotent element except the Identity element e.

$\Rightarrow$ Let if possible $a \in G$ be an idempotent element of $(G, *)$ rather than identity element 'e'.

$a * a = a$ — ①

Now, $e = a * a^{-1}$

$e = (a * a) * a^{-1}$ (From 1)

$e = a * (a * a^{-1})$ (Associative).

$e = a * e$

$\boxed{e = a}$

Q Prove that inverse of the inverse of an element of a group is equal to element itself.

$\rightarrow$ We need to prove that, $(a^{-1})^{-1} = a, \ \forall a \in G$.

Since, a^{-1} is inverse of a.

then, $a * a^{-1} = a^{-1} * a = e$ — ①

Now, Inverse of a^{-1} is $(a^{-1})^{-1}$.

$a^{-1} * (a^{-1})^{-1} = e$

$\Rightarrow \quad a * (a^{-1} * (a^{-1})^{-1}) = a * e$

$\Rightarrow \quad (a * a^{-1}) * (a^{-1})^{-1} = a$

$\Rightarrow \quad e * (a^{-1})^{-1} = a$

$\Rightarrow \quad \boxed{(a^{-1})^{-1} = a}$

Q If a & b are any two elements of a group G then $(a*b)^2 = a^2 * b^2$ if & only if G is abelian.

$\rightarrow$ 1) If G is abelian. i.e. $a * b = b * a$.

Now, $(a*b)^2 = (a * b) * (a*b)$ [Associative].

$= a * (b * a) * b$ [From Given condi

$= a * (a*b) * b$ (Associative)

$= (a*a) * (b*b)$

$\boxed{(a*b)^2 = a^2 * b^2}$

2) Let, $(a*b)^2 = (a^2 * b^2)$

$\Rightarrow (a*b)*(a*b) = (a*a)*(b*b)$

$\Rightarrow e\, a*(b*a)*b = e \cdot a*(a*b)*b.$

$\Rightarrow (b*a)*b = (a*b)*b$ [Left cancellation]

$\Rightarrow \boxed{(b*a) = (a*b)}$ [Right cancellation.

Hence, G is abelian.

* **Finite Group :-** A group $(G, *)$ be a finite group if no. of elements in G is finite.

e.g $G = \{0, 1, 2, 3, 4, 5\}$, under $+_6$ is a finite group.

* **Infinite Group :-** A group $(G, *)$ is called infinite group if no. of elements in G is infinite.

eg $(Z, +)$ is infinite.

* **Order of Group :-** order of group G is the no. of elements in group G it is denoted by $O(G)$ or $|G|$.

Note! A group of order 1 i.e $O(G) = 1$ has only the identity element

Imp* **Order of an Element :-** If the element $a \in G$ then the least positive integer n for which $\boxed{a^n = e}$. w.r.t. binary operation '$*$' where e is an identity element is called order of element a. and it is denoted by $O(a)$.

Q Consider $G = \{1, 2, 4, 7, 8, 11, 13, 14\}$ under $\times_{15}$. find the order of all the elements.

→ Here, $e = 1$.

$o(1) \Rightarrow 1^n = e. \Rightarrow 1^n = 1.$

$O(1) \Rightarrow 1^1 = 1 \Rightarrow O(1) = 1.$

$O(2) \Rightarrow 2^4 = 1 \Rightarrow O(2) = 4.$ or $2^4 = 2\times2\times2\times2 = 16 = 1 \bmod 15$

$O(4) \Rightarrow 4^2 = 1 \Rightarrow O(4) = 2.$ $16 \equiv 1 \bmod 15 \Rightarrow$

$O(7) \Rightarrow 7^4 = 1 \Rightarrow O(7) = 4.$ $2401 \equiv 1 \bmod 15 \Rightarrow O(7) = 4.$

$O(8) \Rightarrow 8^4 = 1 \Rightarrow O(8) = 4$

$O(11) \Rightarrow 11^2 = 1 \Rightarrow O(11) = 2.$

$O(13) \Rightarrow 13^4 = 1 \Rightarrow O(13) = 4.$

$O(14) \Rightarrow 14^2 = 1 \Rightarrow O(14) = 2$

Q $G = \{a, a^2, a^3, a^4, a^5, a^6 = e\}$. Find order of every element?

→ Here, $a^6 = e$, $O(G) = 6$

$O(a) \Rightarrow a^6 = e \Rightarrow O(a) = 6.$

$O(a^2) \Rightarrow (a^2)^3 = e \Rightarrow O(a^2) = 3$

$O(a^3) \Rightarrow (a^3)^2 = e \Rightarrow O(a^3) = 2.$

$o(a^4) \Rightarrow (a^4)^3 = a^{12} \Rightarrow (a^6)^2 = (e)^2 = e \Rightarrow o(a^4) = 3.$

$O(a^5) \Rightarrow (a^5)^6 = a^{30} \Rightarrow (a^6)^5 = (e)^5 = e \Rightarrow o(a^5) = 6.$

$O(a^6) \Rightarrow (a^6)^1 = e. \Rightarrow o(a^6) = 1.$

Q $G = \{0,1,2,3,4,5,6,7,8,9\}$. is a group under $+_{10}$. Find the order of every element:

→ Here, $e = 0$. $O(G) = 10$.

$o(0) \Rightarrow 0^1 = 0 \Rightarrow o(0) = 1.$ $[0 \equiv 0 \bmod 10]$

$o(1) \Rightarrow 1^{10} = \underbrace{1+1+1+\dots+1}_{10 \text{ times}} = 10 \equiv 0 \bmod 10 \Rightarrow o(1) = 10.$

$o(2) \Rightarrow 2^5 = 2+2+2+2+2 = 10 \equiv 0 \bmod 10 \Rightarrow o(2) = 5.$

$o(3) \Rightarrow 3^{10} = \underbrace{3+3+\dots+3}_{10 \text{ times}} = 30 \equiv 0 \bmod 10 \Rightarrow o(3) = 10$

$o(4) \Rightarrow 4^5 = 4+4+4+4+4 = 20 \equiv 0 \bmod 10 \Rightarrow o(4) = 5$

$o(5) \Rightarrow 5^2 = 5+5 = 10 \equiv 0 \bmod 10 \Rightarrow o(5) = 2.$

$o(6) \Rightarrow 6^5 = 6+6+6+6+6 = 30 \equiv 0 \bmod 10 \Rightarrow o(6) = 5$

$o(7) \Rightarrow 7^{10} = \underbrace{7+7+\dots+7}_{10 \text{ times}} = 70 \equiv 0 \bmod 10 \Rightarrow o(7) = 10$

$O(8) \Rightarrow 8^5 = 8+8+8+8+8 = 40 \equiv 0 \bmod 10 \Rightarrow o(8) = 5$

$O(9) \Rightarrow 9^{10} = \underbrace{9+9+\dots+9}_{10 \text{ times}} = 90 \equiv 0 \bmod 10 \Rightarrow o(9) = 10$

Properties of Order of Element:-

- If a group The order of a finite group G is 'n' then order of every element of the group is finite or less than or equal to order of group.

$$\boxed{o(a) \le o(G), \forall a \in G.}$$

- imp. If order of an element $a \in G$ is 'n' then, $\boxed{a^k = e}$ iff n is divisor of k.

Proof:- n is order of a. So, $a^n = e$.

Let, $k \in Z^+$

for some $p, q \in Z^+$

$$\boxed{k = n \cdot p + q} \qquad 0 \le q < n$$

Now, $a^k = a^{np+q} = a^{np} \cdot a^q$

$= (a^n)^p \cdot a^q$

$= e^p \cdot a^q$

$= e \cdot a^q$

$$\Rightarrow \boxed{a^k = a^q} = e.$$

It is not possible. unless $q = 0$.

$$\boxed{k = np.}$$

So, n is the divisor of k.

⇒ Let, n is divisor of k. then $\exists p \in Z^+$.

$k = np.$

Now, $a^k = a^{np} = (a^n)^p = (e)^p = e$.

So, $\boxed{a^k = e}$.

* Cyclic Group :- A group $(G, *)$ is said to be cyclic if there exist an element $a \in G$ such that every element $x \in G$ can be expressed as $x = a^n$

$\boxed{x = a^n}$ $\exists$ integer n. $[n \in z]$

In this case cyclic group G is said to be generated by 'a'. So, 'a' is called Generator of 'G'. 'G' is also denoted by $\{a\}$.

e.g. If $G = \{1, -1, i, -i\}$

then, $(G, \times)$ is a cyclic group.

$(i)^1 = i$

$(i)^2 = -1$.

$(i)^3 = i^2 \cdot i = -i$

$i^4 = i^2 \cdot i^2 = 1$.

So, i is the Generator of that group.

For this cyclic group '$-i$' is also a generator.

Properties :-

- A cyclic group is abelian.

$(G, *)$ & $\{a\}$ is generator.

Let, $x, y \in G$.

$x = a^m$, $y = a^n$, $m, n \in z$.

Now, $x * y = a^m * a^n$
$= a^{m+n}$
$= a^{n+m}$
$= a^n * a^m$

$\boxed{x * y = y * x}$

- If 'a' is generator of cyclic group so, 'a^{-1}' is also Generator of cyclic group.

$x = a^n$, $n \in z$.

$\boxed{x = (a^{-1})^{-n}}$, $-n \in z$.

- If $(G, *)$ is a finite cyclic group ~~an~~ generated by an element $a \in G$. & a is of order 'n' then $a^n = e$ so that $G = \{a, a^2, a^3, \ldots a^n = e\}$.

- If $(G, *)$ is a finite cyclic group of order 'n' $[O(G) = n]$ ~~and~~ generated by 'a' then a^m is also generator of $(G, *)$ if & only if m & n are relative primes.

Show that {1, ω, ω²} is a cyclic group w.r.t multiplication

$(\omega)^1 = \omega$

$(\omega)^2 = \omega^2$

$(\omega)^3 = 1$.

→ 'ω' is the generator of 'G.'

Q Show that G = {1, 2, 4, 5, 7, 8} under X_9 is cyclic. How many generators are there and what are they.

→ G = { (1, 2, 4, 5, 7, 8), X_9 }.

X_9	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

(Z_m, $+_m$)
Z_m = {0, 1, ...}
(Z_p, $\times_m$)
Z_p = set of ...
Z_p = set of ... with ...

1 is the identity element.

$2^1 = 2 = 2 \bmod 9$
$2^2 = 4 = 4 \bmod 9$
$2^3 = 8 = 8 \bmod 9$
$2^4 = 16 = 7 \bmod 9$
$2^5 = 32 = 5 \bmod 9$
$2^6 = 64 = 1 \bmod 9$.

Since, 5 is the inverse of 2, so, 5 is the generator.

There are total 2 generator i.e. 2 & 5.

Q Show that $(Z_5, +_5)$ is a cyclic group. What are its generators

→ Z_5 = { 0, 1, 2, 3, 4 }.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

'0' is the identity element of Z_5 under '$+_5$'.

$1^1 = 1 \equiv 1 \bmod 5$
$1^2 = 1+1 = 2 \equiv 2 \bmod 5$
$1^3 = 1+1+1 = 3 \equiv 3 \bmod 5$
$1^4 = 1+1+1+1 = 4 \equiv 4 \bmod 5$
$1^5 = 1+1+1+1+1 = 5 \equiv 0 \bmod 5$.

$2^1 = 2 \equiv 2 \bmod 5$
$2^2 = 4 \equiv 4 \bmod 5$.
$2^3 = 6 \equiv 1 \bmod 5$
$2^4 = 8 \equiv 3 \bmod 5$
$2^5 = 10 \equiv 0 \bmod 5$

4 is the inverse of 1 & 3 is the inverse of 2.

So, 4 and 3 are also generators.

So, Total no. of generators are. 1, 2, 3, 4.

* Permutation Group :- A bijective function from A to A where A is non-empty is called a permutation of A.

e.g. $A = \{1, 2\}$

$P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ $P_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

* Equality of Permutations :- Let, P_1 and P_2 are two permutations on set A, P_1 and P_2 are said to be equal, if there corresponding images are equal.

$P_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ $P_2 = \begin{pmatrix} 3 & 2 & 1 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

* Identity Permutation :- If the image of element is same as element itself then permutation is said to be identity permutation. It is denoted by 'I or I_A'.

e.g. If $A = \{1, 2, 3\}$ $\Rightarrow$ $I_A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

* Inverse Permutation :- (~~Since permutation is a~~ bijective ~~function~~)

Each permutation 'P' is invertible since it is bijective & it is denoted by P^{-1}

$P = \begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix}$ $\Rightarrow$ $P^{-1} = \begin{pmatrix} a & b & c & d \\ d & a & b & c \end{pmatrix}$

$P^{-1} = \begin{pmatrix} b & c & d & a \\ a & b & c & d \end{pmatrix}$

* Product of Permutation :-

note (The no. of elements is known as degree of permutation)

The product of permutation P_1 & P_2 of same degree is denoted by $P_1 \circ P_2$.

$A = \{1, 2, 3, 4\}$

$P_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$ $P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$

$P_1 \circ P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$

V.Imp * Permutation Group :- A set G of all permutations on a non-empty set A under the binary operation '*' (Product of Permutation) is called Permutation Group. If A has 'n' elements then ~~the~~ given permutation group formed by A is also called symmetric group of degree 'n'. Denoted by 'S_n' and 'S_n' has $n!$ elements.

* Cyclic Permutation :- Let, $A = \{x_1, x_2, ---, x_n\}$ and Let, $t_1, t_2, ---, t_k \in A$. and permutation 'P' : $A \rightarrow A$ defined by

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$P(t_2) = t_3$

$P(t_3) = t_4$

$\vdots$

$P(t_{k-1}) = t_k$

$P(t_k) = t_1$.

is called a cyclic permutation of length 'k'.

Eg. $P = \begin{pmatrix} a & b & c & d & e & f \\ a & d & b & e & c & f \end{pmatrix}$

$P = (b\ d\ e\ c)$

It is cyclic.

It is not necessary that full permutation is cyclic, some elements would be cyclic. This permutation has a cycle of length 4.

* Disjoint cycles :- Two cycles P_1 and P_2 are called the disjoint on set A if there is no element $a \in A$ such that $a \in$ both cycles.

Eg $P_1 = (b\ d\ e)$. $P_2 = (a\ c\ f)$ are disjoint cycles

* Transpositions :- A cycle of length 2 is called a transposition.

Eg. (a, b) . $(1, 2)$ are transpositions.

NOTE: • If there is a cycle of length 'n' then there are $(n-1)$ transpositions.

• Every permutation can be resolved as a product of finite no. of permutations. (transposition).

* Even Permutation :- A permutation is said to be even if it can be expressed as a product of even no. of transpositions.

* odd Permutation :- A permutation A is said to be odd if it can be expressed as a product of odd no. of transpositions.

NOTE: A permutation can't be both even or odd.
Product of two odd permutations is ~~odd~~ even.
Product of two even permutation is even.
Product of even & odd permutation is odd.

Q Express given permutations as product of cycles where -

$P = \begin{pmatrix} a & b & c & d & e & f \\ f & e & b & d & c & a \end{pmatrix}$

$= (a, f) \circ (b, e, c)$ [It is ~~even~~ odd permutation].

$= (a, f) \circ (b, e) \circ (b, c)$

We can verify by this.

$\begin{pmatrix} a & b & c & d & e & f \\ f & b & c & d & e & a \end{pmatrix}$

$\begin{pmatrix} a & b & c & d & e & f \\ a & e & b & d & c & f \end{pmatrix}$

$= \begin{pmatrix} a & b & c & d & e & f \\ f & e & b & d & x & a \end{pmatrix}$

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* Q Determine whether the given permutations are even or odd.

$$P_1 = \begin{pmatrix} a & b & c & d & e & f \\ e & f & b & d & a & c \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$

$P_1 = (a, e) \circ (b, f, c) = (a, e) \circ (b, f) \circ (b, c)$ [odd Per]

$P_2 = (1, 6) \circ (2, 3, 4, 5) = (1, 6) \circ (2, 3), (2, 4), (2, 5)$. [Even Per]

Q $A = (1, 2, 3, 4, 5)$ - Find the product $(1, 3) \circ (2, 4, 5) \circ (2, 3)$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 5 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix}$$

* Imp **Subgroup :-** If $(G, *)$ is a group and H subset of G then $(H, *)$ is said to be subgroup of $(G, *)$ if $(H, *)$ is also a group itself. i.e. H must be satisfy following properties -

- Closure Property :- $\forall a, b \in H$, $a * b \in H$.
- Identity Element :- $\exists e \in H$, $a * e = a$, $\forall a \in H$.
- Inverse Element :- $\exists a^{-1} \in H$, $a * a^{-1} = e$, $\forall a \in H$.

One Step Subgroup Test :-

The necessary & sufficient condition for a non-empty subset 'H' of $(G, *)$ to be subgroup of 'G' is

$\boxed{a * b^{-1} \in H, \forall a, b \in H}$.

Proof :-

1) Necessary Condition :- Let, $(H, *)$ be a subgroup and $a, b \in H \Rightarrow b^{-1} \in H$.
$\Rightarrow a * b^{-1} \in H$ (closure Property).

2) Sufficient Condition :- If $a * b^{-1} \in H$, $\forall a, b \in H$.
Put, $b = a$
$\Rightarrow a * a^{-1} \in H$.
$\Rightarrow e \in H$ (e is identity element)

$\therefore$ Identity Element exists in H.

$(G, *)$, $H \subseteq G$, $G \subseteq G$, $\{e\} \subseteq G$

Remark: 1) Identity of H will be same as Identity of G.
2) $(G; *)$ & $(e, *)$ are trivial subgroups of G.
3) Other than trivial subgroups all sub groups are called

(2) $a; e \in H$.
$e * a^{-1} \in H$.
$\Rightarrow a^{-1} \in H$.
$\therefore$ Inverse element exists

(3) $a * (b^{-1})^{-1} \in H$.

$\Rightarrow$ $a * b \in H$.

$\therefore$ closure property exists.

so, H is a subgroup.

Q. The intersection of any 2 sub-groups of a group $(G,*)$ is again a subgroup of $(G,*)$.

$\rightarrow$ Let, H_1 & H_2 be two subgroups of a group $(G,*)$ since, identity element exist or belong to both subgroups H_1 & H_2.

Therefore, $(H_1 \cap H_2, *)$ is $H_1 \cap H_2 \neq 0$

Let, $a, b \in H_1 \cap H_2$

$\Rightarrow a, b \in H_1$ & $a, b \in H_2$.

$\Rightarrow (a \in H_1$ and $a \in H_2)$ and $(b \in H_1$ and $b \in H_2)$ —— ①

Since, H_1 & H_2 are subgroups

$b^{-1} \in H_1$ and $b^{-1} \in H_2$ —— ②

from eqn ① & ②.

$a * b^{-1} \in H_1$ and $a * b^{-1} \in H_2$

$\Rightarrow$ $\boxed{a * b^{-1} \in (H_1 \cap H_2)}$

By one step sub-group test 2 subgroup is a group of $(G,*)$

Note: Every subgroup of a cyclic group is cyclic.

Q Prove the following & given an counter example. If $(H_1,*)$ & $(H_2,*)$ are both subgroups of the group $(G,*)$. then - $(H_1 \cup H_2, *)$ is also a subgroup of $(G,*)$.

$\rightarrow$ let, $(Z,+)$ The union of two subgroups may or may not be a subgroup. The counter example is given below -

Let $H_1 = \{ ---, -6, -3, 0, 3, 6 -- \}$.

Let, $H_2 = \{ ---, -10, -5, 0, 5, 10, -- \}$.

$\Rightarrow H_1 \cup H_2 = \{ ----, \pm 10, \pm 6, \pm 5, \pm 3, 0 \}$.

Since, $6 + 5 = 11 \notin H_1 \cup H_2$.

So, $H_1 \cup H_2$ is not a subgroup.

* Coset :- Let, H be a subgroup of group G. and let $a \in G$. then -

$Ha = \{ ha, h \in H$ is called Right coset generated by H & a. also the set

$aH = \{ ah, h \in H$ is called Left coset generated by a & H.

The element a is called representative of aH or Ha.

Eg - $(Z_9, +_9)$. $o(Z_9) = 9$

$Z_9 = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8 \}$.

{0, 3, 6}

$0 +_9 H = \{0, 3, 6\}$
$1 +_9 H = \{1, 4, 7\}$
$2 +_9 H = \{2, 5, 8\}$
$3 +_9 H = \{3, 6, 0\}$
$4 +_9 H = \{4, 7, 1\}$
$5 +_9 H = \{5, 8, 2\}$
$6 +_9 H = \{6, 0, 3\}$
$7 +_9 H = \{7, 1, 4\}$
$8 +_9 H = \{8, 2, 5\}$

$\Rightarrow 0 +_9 H = 3 +_9 H = 6 +_9 H$
$\Rightarrow 1 +_9 H = 4 +_9 H = 7 +_9 H$
$\Rightarrow 2 +_9 H = 5 +_9 H = 8 +_9 H$.

Two subgroups $(e, +m)$ $(Z_m, +_m)$ — If m is prime then 2 subgroups are possible

$(Z_m, +_m) \Rightarrow Z_m = \{0, 1, \ldots, (m-1)\}$

$(U(m), \times_m)$
U(m) is the integers greater than 1. where, m is coprime.
$U(30) = \{1, 7, 11, 13, 17, 19, 23, 29\}$

Distinct coset – $0 +_9 H$, $1 +_9 H$, $2 +_9 H$.

Remark: 1) Both aH & Ha are subset of G.
2) If e is the identity element of G then –
$\boxed{eH = He = H}$
3) If group is abelian then left coset = Right coset.
$aH = Ha$.

Properties of Coset :-
Let 'H' be a subgroup of 'G' & let $a, b \in G$. then –
- $a \in aH$
- $aH = H$ iff $a \in H$.
- $aH = bH$ or $aH \cap bH = \phi$.

Q Let $H = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$ find all the left cosets of H in Z.

$H = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$
$0 \in Z$ is the identity element.
$0 + H = H = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$
$1 + H = \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$
$2 + H = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$
$3 + H = \{\ldots, -6, -3, 0, 3, 6, 9, 12, \ldots\}$.

cosets of H = 0+H, 1+H, 2+H.

Q Find all the left cosets of $H = \{1, 11\}$. $O(H) = 2$.
in U(30) where, U(30) is a set of all +ve integers less than 30 & relatively prime to 30 under. U_{30} is a group under $\times_{30}$.

$U(30) = \{1, 7, 11, 13, 17, 19, 23, 29\}$ $O(U(30)) = 8$
$H = \{1, 11\}$.

$1 \times_{30} H = \{1, 11\}$
$7 \times_{30} H = \{7, 17\}$
$11 \times_{30} H = \{11, 1\}$
$13 \times_{30} H = \{13, 23\}$
$17 \times_{30} H = \{17, 7\}$
$19 \times_{30} H = \{19, 29\}$
$23 \times_{30} H = \{23, 13\}$.
$29 \times_{30} H = \{29, 19\}$

4 Distinct cosets ~~&~~ left cosets.
1H, 7H, 13H, 19H.

group 'G' then the no. of distinct right or left cosets of H in G is called the Index of H in G.
It is denoted by $[G:H]$ or $i_G(H)$ [Index of H in G].

eg $H = (1, 11)$

$[G:H] = i_G(H) = 4$.

Lagrange's Theorem :- (order of subgroup divides order of G)

If G is a finite group and H is a subgroup of G then order of subgroup divides the order of group G [$o(H)$ divides $o(G)$] moreover the number of ~~$o(H)$~~ distinct left or right cosets of H in G equals to $\frac{o(G)}{o(H)}$.

Proof :- Let H be a subgroup of order 'm' of a Group G of order 'n'.

i.e. $o(G) = n$ & $o(H) = m$.

Let, $H = \{h_1, h_2, h_3, \ldots, h_m\}$ & $G = \{a_1, \ldots\}$

Let, $a \in G$ then $aH = \{ah_1, ah_2, ah_3, \ldots, ah_m\}$ is left coset containing m distinct elements.

since, $ah_i = ah_j \Rightarrow \boxed{h_i = h_j}$ By cancellation law of G.

Thus, every left coset of H in G has m distinct elements. since, G is a finite group then the no. of distinct left cosets will also be finite. Let it be 'k'. Then the union of these 'k' distinct left cosets of 'H' in 'G' = G. i.e. if $a_1H, a_2H, a_3H, \ldots a_kH$ are distinct left cosets of H in G then,

$G = a_1H \cup a_2H \cup a_3H \ldots \cup a_kH$

$\Rightarrow o(G) = o(a_1H) + o(a_2H) + o(a_3H) + \ldots + o(a_kH)$

$\Rightarrow n = m + m + m + \ldots + m$, (k times)

$\Rightarrow n = m \cdot k$

$\Rightarrow k = \frac{n}{m}$

$\Rightarrow \boxed{k = \frac{o(G)}{o(H)}}$ where, $k \in Z^+$.

Q Find all the subgroups of-
i) $(Z_{12}, +_{12})$
ii) $(Z_5, +_5)$

i) $Z_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.
Possible order of subgroups = 1, 2, 3, 4, 6, 12.

$2^1 = 2 \qquad \{0, 2, 4, 6, 8, 10\} \Rightarrow o(2) = 6$.
$2^2 = 4$
$2^3 = 6$
$2^4 = 8 \qquad \Rightarrow (0, +_{12})$
$2^5 = 10$
$2^6 = 0 \qquad (Z_{12}, +_{12})$

$3^1 = 3$
$3^2 = 6$
$3^3 = 9$
$3^4 = 0$

$\{0, 3, 6, 9\} \Rightarrow o = 4.$

$4^1 = 4$
$4^2 = 8$
$4^3 = 0$

$\{0, 4, 8\} \Rightarrow o = 3.$

$6^1 = 6$
$6^2 = 0$

$\{0, 6\}. \Rightarrow o = 2.$

in $(Z_5, +_5) \Rightarrow Z_5 = \{0, 1, 2, 3, 4\}.$

Subgroups — $(0, +_5)$, $(Z_5, +_5)$

Q Let, G be a group of order 'p' where 'p' is a prime no. Find all subgroups of G.

→ Since, $O(G) = p$ which is prime.
So, 1 & p are only divisors of $O(G) = p$.
By Lagrange's Theorem-
order of subgroups divides the order of group if H be the subgroup of G then $O(H) = 1$ or p.
Thus, $\{e\}$ and G itself are the subgroups of G

* Product of Two SubGroups :-

Let, H and K be two subgroups of a group 'G' then their product is ~~defi~~ denoted by HK and defined as

$$\boxed{HK = \{hk : h \in H \text{ and } k \in K\}.}$$

If H and K be two subgroups of a group G then HK is a subgroup of G if and only if $\boxed{HK = KH.}$

Q If H subset of K, $(H \subseteq K)$ be two subgroups of finite group G. then show that $[G:H] = [G:K][K:H]$.

Let, G be a finite group and H & K are two subgroups of G such that $H \subseteq K$. then, H is also the subgroup of K.

By Lagrange's Theorem-

$$[G:H] = \frac{O(G)}{O(H)} \quad \text{— (1)}$$

$$[G:K] = \frac{O(G)}{O(K)} \quad \text{— (2)}$$

$$[K:H] = \frac{O(K)}{O(H)} \quad \text{— (3)}$$

{ Every cyclic group is abelian. but not every abelian group is cyclic }.

multiply eqn (2) & (3)

$$[G:K][K:H] = \frac{O(G)}{\cancel{O(K)}} \times \frac{\cancel{O(K)}}{O(H)}$$

$$[G:K][K:H] = \frac{O(G)}{O(H)}$$

$$\boxed{[G:K][K:H] = [G:H]} \quad \text{— From eqn (1) —}$$

Normal Subgroup: ... normal subgroup of G if $\boxed{Ha = aH}$ $\forall\, a \in G$. i.e. the right coset & left coset of H in G generated by 'a' are same.

Remark: • Clearly every subgroup 'H' of ~~and~~ an abelian group 'G' is a normal subgroup of 'G'. for $a \in G$ and $h \in H$

$\boxed{ah = ha}$.

• since a cyclic group is abelian therefore every subgroup of a cyclic group is normal.

Theorem A subgroup H of a group G is normal if & only if

$\boxed{g^{-1} h g \in H \quad \text{for } \forall\, h \in H, \ \forall\, g \in G.}$

Q. Show that ~~such~~ $H = \{(1,b) : b \in R\}$ is a normal subgroup of $G = \{(a,b) : a \neq 0,\ b \in R\}$ under the composition '$*$' defined by $(a,b) * (c,d) = (ac,\ bc+d)$.

$\rightarrow$ Let, $(e_1, e_2) \in G$ be the idendity element.

$\forall\, (a,b) \in G$.

$\Rightarrow (a,b) * (e_1, e_2) = (a,b)$.

$\Rightarrow (ae_1,\ be_1 + e_2) = (a,b)$.

$\Rightarrow ae_1 = a$ & $be_1 + e_2 = b$.

$\Rightarrow e_1 = 1$. $\quad \cancel{b} + e_2 = \cancel{b}$

$\Rightarrow e_2 = 0$.

So, $(1,0)$ is the identity element.

Let, (c,d) be the inverse of $(a,b) \in G$.

$\Rightarrow (a,b) * (c,d) = (1,0)$

$\Rightarrow (ac,\ bc+d) = (1,0)$

$\Rightarrow ac = 1$ & $bc + d = 0$.

$\Rightarrow c = \frac{1}{a}$ $\quad$ $d = -bc$.

$d = -\frac{b}{a} \Rightarrow d = -b \cdot a^{-1}$.

$\Rightarrow c = a^{-1}$

$\left(\frac{1}{a}, -\frac{b}{a}\right)$ be the inverse of (a,b)

Let, $(a,b) \in G$ & $(1,b) \in H$ and, $(a,b)^{-1} = \left(\frac{1}{a}, -\frac{b}{a}\right)$.

Now, $(a,b)^{-1} * (1,b) * (a,b)$

$\left(\left(\frac{1}{a}, -\frac{b}{a}\right) * (1,b)\right) * (a,b)$.

$\Rightarrow \left(\frac{1}{a}, -\frac{b}{a} + b\right) * (a,b) \quad \Rightarrow \left(\frac{1}{a}, \frac{-b+ab}{a}\right) * (a,b)$

$\Rightarrow \left(\frac{1}{\cancel{a}} * \cancel{a},\ \frac{-b+ba}{\cancel{a}} \times \cancel{a} + b\right)$

$\Rightarrow (1,\ -\cancel{b} + ba + \cancel{b})$

$\Rightarrow (1, ab)$, $\quad ab \in R$.

$\Rightarrow (1, ab) \in H$.

Q If N is a Normal subgroup of G. then show that
1) $NaNb = Nab$
2) $aNbN = abN$, $a, b \in G$.

→ 1) $NaNb = N(aN)b$
$= N(Na)b$ ($aN = Na$) (since 'N' is Normal subgroup).
$= NNab$.
$\boxed{NaNb = Nab}$ ($NN = N$).

2) $aNbN = a(Nb)N$
$= a(bN)N$ ($Nb = bN$) (since 'N' is Normal subgroup)
$= abNN$
$\boxed{aNbN = abN}$ ($NN = N$).

* **Homomorphism** :- Let, $(G_1, \cdot)$ and $(G_2, *)$ be two groups then a function or mapping $f : G_1 \to G_2$ is called a Homomorphism if
$\boxed{f(a \cdot b) = f(a) * f(b)}$ $\boxed{\forall\ a, b \in G_1}$.

Thus, 'f' is Homomorphism from $G_1 \to G_2$ then 'f' preserves the composition of G_1 & G_2 i.e. image of composition = composition of images.
$|f(a \cdot b) = f(a) * f(b)|$
The Group G_2 is said to be Homomorphic image of Group G_1.

* **Isomorphism** :- Let, $(G_1, \cdot)$ and $(G_2, *)$ be two groups then a function 'f' : $G_1 \to G_2$ is an isomorphism if
- $\boxed{f \text{ is homomorphism}}$ i.e. $f(a \cdot b) = f(a) * f(b)$ $\forall\ a, b \in G_1$
- $\boxed{f \text{ is Bijective}}$ (one to one & onto).

* **Automorphism** :- An isomorphism of a group 'G' onto itself is called an automorphism of 'G'.

* **Isomorphic Groups** :- A group G_1 is said to be isomorphic to group G_2 if there exist an isomorphism of G_1 on to G_2 and it is denoted by $\boxed{G_1 \cong G_2}$

Q Let, $G_1 = (Z, +)$ and $G_2 = G_1$, and, Let func. $f : G_1 \to G_2$ is defined as $f(x) = 2x$, $x \in Z$. Prove that f is homomorphism. $f(a \cdot b) = f(a) * f(b)$.

→ $G_1 = (Z, +)$, $G_2 = G_1 = (Z, +)$
$f(x) = 2x$, $\forall\ x \in Z$
We have to show that
$f : G_1 \to G_2$
such that, $f(a+b) = f(a) + f(b)$
or $f(x+y) = f(x) + f(y)$, $\forall\ x, y \in Z$.

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Now $f(x+y) = 2(x+y)$

$f(x+y) = 2x + 2y$

$\boxed{f(x+y) = f(x) + f(y)}$

IMP

* Theorems on Homomorphism :-

Theorem 1: Let $f: G_1 \to G_2$ is a homomorphism then-

- $f(e_1) = e_2$. where, e_1 & e_2 are identity elements of G_1 & G_2 resp.
- $f(x^{-1}) = [f(x)]^{-1}$, $\forall\ x \in G_1$.

Proof $f(e_1) = e_2$.

Let, $x \in G_1$ then $f(x) \in G_2$.

$f(x) \cdot e_2 = f(x)$. (e_2 is identity of G_2)

$f(x) \cdot e_2 = f(x \cdot e_1)$. ($e_1$ is identity of G_1)

$\Rightarrow f(x) \cdot e_2 = f(x) \cdot f(e_1)$.

$\Rightarrow \boxed{e_2 = f(e_1).}$ [By left cancellation Law of Group].

Proof $f(x^{-1}) = [f(x)]^{-1}$

Let, $x \in G_1$ then. $f(x) \in G_2$.

$x x^{-1} = x^{-1} \cdot x = e_1$.

$\Rightarrow f(x \cdot x^{-1}) = f(e_1)$

$\Rightarrow f(x) \cdot f(x^{-1}) = (e_2)$. [$f(e_1) = e_2$].

$\Rightarrow \boxed{[f(x)]^{-1} = f(x^{-1})}$

Q Show that a finite cyclic group of order n is isomorphic to Z_n. The group of integers modulo n.

$\to$ $O(G_1) = n$.

$Z_n = \{0, 1, 2, - - - - - (n-1)\}$. $(Z_n, +_n)$

Let, $G_1 = \langle a \rangle = \{a, a^2, a^3, - - - - - - - -, a^n = e\}$ [$\langle a \rangle = a$ is a cyclic group and generator a].

$f: Z_n \to G_1$.

$f(x) = a^x$, $\forall\ x \in Z_n$.

1) f is homomorphism.

2) f is bijective.

1) $f(x+y) = a^{x+y}$

$= a^x \cdot a^y$

$\boxed{f(x+y) = f(x) \cdot f(y)}$

2) Let $f(x_1) = f(x_2)$ $x_1, x_2 \in Z_n$.

$\Rightarrow a^{x_1} = a^{x_2}$.

$\Rightarrow \boxed{x_1 = x_2.}$ So, function is one to one.

Let, $f(x) = y$, $x \in Z_n$ $y \in G_1$.

$\Rightarrow a^x = y$.

$\Rightarrow \boxed{x = \log_a y}$ So, function is onto.

Q. Show that any infinite cyclic group is isomorphic to $(Z, +)$.

→ Let, $G = \langle a \rangle = \{a^n : n \in Z\}$.

$f : Z \to G$.

$f(x) = a^x, \ \forall, x \in Z$.

i) f is homomorphism.
2) f is bijective

i) $f(x+y) = a^{x+y}$
$= a^x \cdot a^y$
$\boxed{f(x+y) = f(x) \cdot f(y)}$ So, it is homomorphism

ii) Let, $f(x_1) = f(x_2)$, $x_1, x_2 \in Z$.
$\Rightarrow a^{x_1} = a^{x_2}$
$\Rightarrow \boxed{x_1 = x_2}$ So, it is one to one

Let, $f(x) = y$
$a^x = y$
$\boxed{x = \log_a y}$. So, it is onto.

Or

$y \in G$
$\Rightarrow a^x \in G \quad y = a^i, \ i \in Z$
$\Rightarrow \boxed{y = f(i)}$ [It is bijective]

So, it is isomorphic to $(Z, +)$.

* Ring :-

An algebraic system $(R, +, \cdot)$ where R is a non-empty set and '+' and '·' are any two binary operations is called a ring if following conditions are satisfied.

- $(R, +)$ is a abelian group.
- $(R, \cdot)$ is a semigroup.
  i.e. $a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in R$.
- operation '·' is distributed over '+'.
  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
  $(b + c) \cdot a = (b \cdot a) + (c \cdot a) \quad , \forall a, b, c \in R$.

e.g Set of Integers (Z) / Set of Real No. (R) under addition and Multiplication is a ring. (with unity 1).

Remark:

- Ring 'R' is commutative iff $\boxed{a \cdot b = b \cdot a}$ $\forall a, b \in R$. i.e. $(R, \cdot)$ is commutative
- An unit or identity element of ring are (if it exists) is an element of semigroup $(R, \cdot)$.
- If $(R, \cdot)$ is monoid then the ring $(R, +, \cdot)$ is called a ring with identity or unit.
- Additive identity of $(R, +)$ is called zero element, or zero identity of ring R

$\boxed{a \cdot b = 0}$ then a & b are divisors of zero or zero divisor

Q. show that $(Z_5, +_5, \times_5)$ is a commutative ring with unity

→ $Z_5 = (0, 1, 2, 3, 4)$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

1) $(Z_5, +_5)$ is abelian :-

1) closure Property : From above table it is satisfied

for Ring

1) $(Z_5, +_5)$ is abelian.
2) $(Z_5, \times_5)$ is semigroup.
3) '$\times_5$' is distributed over '$+_5$'

2) Associative Property : From above table it is satisfied

3) Existence of Identity element :- 0 is the identity element

4) Inverse element :-

$0 +_5 0 = 0 \Rightarrow 0^{-1} = 0$
$1 +_5 4 = 0 \Rightarrow 1^{-1} = 4$
$2 +_5 3 = 0 \Rightarrow 2^{-1} = 3$
$3 +_5 2 = 0 \Rightarrow 3^{-1} = 2$
$4 +_5 1 = 0 \Rightarrow 4^{-1} = 1$

5) commutative Property :- From above table, it is satisfied

$\Rightarrow (Z_5, +_5)$ is abelian.

(2) $(Z_5, \times_5)$ is semigroup :-

1) Closure Property :- from above table, it is satisfied.
2) Associative Property :- from above table, it is satisfied.

$\Rightarrow (Z_5, \times_5)$ is semigroup.

3) '$\times_5$' is distributed over '$+_5$'.

$1 \times_5 (2 +_5 3) = (1 \times_5 2) +_5 (1 \times_5 3) = 2 +_5 3 = 5$

$\Rightarrow 1 \times_5 0 = 2 +_5 3$

$\Rightarrow 0 = 0 \quad \Rightarrow LHS = RHS$

$(2 +_5 3) \times_5 1 = (2 \times_5 1) +_5 (3 \times_5 1)$

$\Rightarrow 0 \times_5 1 = 2 +_5 3$

$\Rightarrow 0 = 0$

$\Rightarrow LHS = RHS$

So, it is a Ring.

4) Ring is commutative. $(Z_5, \times_5)$. $[a \times_5 b = b \times_5 a]$ $\forall a, b \in$

$1 \times_5 2 = 2 \times_5 1 \Rightarrow 2 = 2$.

From above table commutative Property follows.

5) $(Z_r, \times_r)$ Identity element.

$1 \in Z_r$ is a identity element.

$0 \times_r 1 = 0$

$1 \times_r 1 = 1$

$2 \times_r 1 = 2$

$3 \times_r 1 = 3$

$4 \times_r 1 = 4$.

So, 1 is identity element of $(Z_5, \times_r)$.

6) $(Z_r, \times_r)$ is a monoid.

1) Closure Property : From table, it is satisfied

2) Associative Property :- From table, it is satisfied.

3) Identity Element :- from above it is clear 1 is identity element.

So, it is monoid.

7) $(Z_r, +_r)$ is called zero element

So, it is a commutative Ring with unity.

Q Show that the system $(E, +, \cdot)$ of Even integers is a commutative ring without unity w.r.t usual addition and multiplication.

→ E = Set of Even Integers.

1) $(E, +)$ is an abelian. :-

$a+b \in E$, $\forall$ $a, b \in E$.

1) Closure :-

2) Associative :- $\forall$ $a, b, c \in E$.

$a + (b+c) = (a+b) + c$.

3) Existence of Identity element :-

$0 \in E$. is an identity element

Since, $a+0 = a$ $\forall$, $a \in E$

4) Existence of Inverse Element :-

$a^{-1} = -a$.

Since, $a + (-a) = 0$. , $\forall$ $a \in E$

5) Commutative Property :-

$a+b = b+a$, $\forall$ $a, b \in E$.

2) $(E, \cdot)$ is a semigroup :-

1) Closure Property :- $a \cdot b \in E$, $\forall a, b \in E$

2) Associative Property :- $\forall$, $a, b, c \in E$.

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

3) '$\cdot$' is distributed over '+'. $\forall$, $a, b, c \in E$.

$a \cdot (b+c) = (a \cdot b) + (a \cdot c)$

4) commutative Ring $(E, \cdot)$

$\forall$ $a, b \in E$.

$1 \notin E$ so, identity element does not exist.

so. it is a commutative ring without unity.

Q If R is a commutative ring then show that $(a+b)^2 = a^2 + 2ab + b^2$.

$\Rightarrow$ $(a+b)^2 = (a+b)\cdot(a+b)$

$= (a+b)\cdot a + (a+b)\cdot b$ (Right Distribution)

$= a\cdot a + b\cdot a + a\cdot b + b\cdot b$ (Left Distribution)

$= a\cdot a + a\cdot b + a\cdot b + b\cdot b$. (R is commutative. $(a\cdot b = b\cdot a)$).

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

* Properties of Ring :-

- The additive identity or zero element of a ring $(R, +, \cdot)$ is unique.

Proof Let, if possible 0 and $0'$ be additive identity of Ring R.

Let 0 is additive identity and $0' \in R$.

$\Rightarrow$ $0'+0 = 0+0' = 0'$ — ①

Let $0'$ is additive identity and $0 \in R$.

$\Rightarrow$ $0+0' = 0'+0 = 0$ — ②

From eq^n ① & ②.

$$\boxed{0 = 0'}$$

- The additive Inverse of every element of the Ring is unique.

Proof Let, if possible b & c are additive inverse of $a \in R$.

since $0 \in R$ is additive identity

Then, $a+b = b+a = 0$ — ①

& $a+c = c+a = 0$ — ②

Now, $b = b+0$.

$b = b+(a+c)$ From eq^n ②

$b = (b+a)+c$. From eq^n ① [Associative Prop.]

$b = 0+c$.

$$\boxed{b = c}$$

- Cancellation Laws of Addition.

Proof $\forall\ a, b, c, \in R$.

i) $a+b = a+c \Rightarrow b=c$ [Left cancellation]

ii) $b+a = c+a \Rightarrow b=c$ [Right cancellation]

$\Rightarrow$ Let, $a+b = a+c$. [where $-a$ is inverse of $a \in R$]

$\Rightarrow$ $(-a)+(a+b) = (-a)+a+c$

$\Rightarrow$ $(-a+a)+b = (-a+a)+c$.

$\Rightarrow$ $0+b = 0+c$.

$\Rightarrow$ $\boxed{b = c}$

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$\Rightarrow \quad b+a = c+a$

$\Rightarrow \quad b+a+(-a) = c+a+(-a)$

$\Rightarrow \quad b+(a-a) = c+(a-a)$

$\Rightarrow \quad b+0 = c+0$

$\Rightarrow \quad \boxed{b = c}$

- If $(R,+,\cdot)$ is a ring and $a \in R$ then $\boxed{a\cdot 0 = 0\cdot a = 0}$ where, 0 is the Additive Identity or zero element of R.

Proof: consider, $a\cdot 0 + a\cdot a = a\cdot(0+a)$ (Distributive law)

$= a\cdot a. \quad (\because 0+a=a)$

$a\cdot 0 + a\cdot a = 0 + a\cdot a$ [By Right Cancellation Law of Addition]

$\boxed{a\cdot 0 = 0}$

consider. $a\cdot a + 0\cdot a = (a+0)\cdot a$ (Distributive)

$a\cdot a + 0\cdot a = a\cdot a \quad [\because 0+a=0]$

$a\cdot a + 0\cdot a = a\cdot a + 0$ [By Left Cancellation]

$\boxed{0\cdot a = 0}$

- 1) If $(R,+,\cdot)$ is a Ring then $\boxed{-(-a) = a}$

Proof:- $0 \in R$ is additive identity of Ring R

So, $a+a^{-1} = 0, \quad \forall a \in R$

$\Rightarrow \quad a^{-1} = -a$.

ie. $(-a)$ is additive inverse of $a \in R$.

So, $a+(-a) = (-a)+a = 0$.

Now, $(-a)^{-1} = \boxed{-(-a) = a}$.

Since, additive inverse is unique.

2) $\boxed{a\cdot(-b) = (-a)\cdot b = -(a\cdot b)}, \quad \forall a,b \in R$.

Proof: If $0 \in R$ is zero element.

$a\cdot 0 = 0 \quad \forall a \in R$

$\Rightarrow a\cdot(b+(-b)) = 0$

$\Rightarrow a\cdot b + a\cdot(-b) = 0$ [Distributive]

We know that $-a$ is additive inverse of $a \in R$ and it is unique.

So, $\boxed{-(a\cdot b) = a\cdot(-b)}$.

$0\cdot b = 0. \quad \forall a \in R.$

$\Rightarrow \quad (a+(-a))\cdot b = 0$

$\Rightarrow \quad a\cdot b + (-a)\cdot b = 0.$ (Distributive)

$\Rightarrow \quad a\cdot b + (-a)\cdot b = 0.$

so, $\boxed{-(a\cdot b) = (-a)\cdot b}$

3) $\boxed{(-a)\cdot(-b) = a\cdot b}, \quad \forall a,b \in R.$

Proof: consider $(-a)\cdot(-b) = -a\cdot(-b)$

$= -[-(a\cdot b)]$

$\boxed{a\cdot(b-c) = a\cdot b - a\cdot c}$

Proof :- consider, $a\cdot(b-c) = a\cdot(b+(-c))$
$= a\cdot b + a\cdot(-c)$ [Distributive
$$\boxed{a\cdot(b-c) = a\cdot b - a\cdot c}$$

Proof $\boxed{(a-b)\cdot c = a\cdot c - b\cdot c}$

consider, $(a-b)\cdot c = (a+(-b))\cdot c$
$= a\cdot c + (-b)\cdot c$
$$\boxed{(a-b)\cdot c = a\cdot c - b\cdot c}$$

* **Boolean Ring** :- An element $a \in R$ is said to be idempotent if $\boxed{a^2 = a}$

A ring R is called a Boolean Ring if all its elements are idempotent.

i.e. $\boxed{\forall\ a \in R\ ;\quad a^2 = a}$

* **Integral Domain** :- A ring is called integral domain if.

- It is commutative. $[a\cdot b = b\cdot a]$
- It has unit element. $[1 \in R$ Ring with unity$]$
- It is without zero divisors $[a\cdot b = 0$ & $a, b \neq 0]$

eg. $(C,+,\cdot)$, $(Q,+,\cdot)$, $(R,+,\cdot)$ are integral domains.

* **Field** :- A ring 'R' with atleast two elements is called a field if it has following property -

- Ring R is commutative.
- R has unity or unit element.
- R is such that each non-zero element has multiplicative inverse.

e.g Set of Real Numbers is a field, set of complex No. is field

Remark: Every field is an Integral Domain

For complex

$(x+iy) + (e_1+ie_2) = (x+iy)$
$(x+e_1) + i(y+e_2) = x+iy$
$\Rightarrow x+e_1 = x$ & $y+e_2 = y$.
$\Rightarrow e_1 = 0$ & $e_2 = 0$.

Identity element of $(x+iy) \rightarrow (0+i0)$

$(x+iy) + (a+ib) = 0+i0$
$\Rightarrow a = -x$. & $b = -y$.

$(x+iy)^{-1} = -(x+iy)$

Identity element of multiplicative inverse $= (1+0\cdot i)$

$\Rightarrow (x+iy)\cdot(a+ib) = (1+i\cdot 0)$
$\Rightarrow (x+iy)\cdot a + (x+iy)\cdot ib = (1+i\cdot 0)$
$\Rightarrow (x+iy)\cdot a + ix\cdot b - y\cdot b = (1+i\cdot 0)$
$\Rightarrow x\cdot a + i\cdot y\cdot a + ix\cdot b - y\cdot b = (1+i\cdot 0)$
$\Rightarrow (x\cdot a - y\cdot b) + i(y\cdot a + x\cdot b) = (1+i\cdot 0)$
$\Rightarrow x\cdot a - y\cdot b = 1$ & $y\cdot a + x\cdot b = 0$.
$\Rightarrow xb = -ay$.
$b = -\frac{ay}{x}$.

$xa + \frac{ay^2}{x} = 1$
$\Rightarrow x^2a + y^2a = x$
$\Rightarrow \boxed{a = \frac{x}{x^2+y^2}}$

$b = -\frac{x}{(x^2+y^2)}\cdot\frac{y}{x}$

$\boxed{b = -\frac{y}{x^2+y^2}}$

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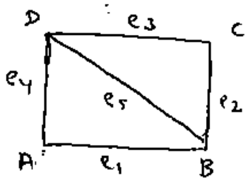
GRAPH THEORY

$V(G) = \{A, B, C, D\}$

$E(G) = \{e_1, e_2, e_3, e_4, e_5\}$

$E(G) = \{(A,B), (B,C), (C,D), (D,A), (B,D)\}$.



The degree of self Loop is 2.

Degree of vertex : $deg(v_i)$ or $de(v_i)$. [No. of Edges associated with it]

$d(A) = 2$ vertex A is even.
$d(B) = 3$ vertex B is odd.
$d(C) = 2$ vertex C is even
$d(D) = 3$. vertex D is odd.

{A tree is always a Graph. but a graph is not always a tree}

'e_1' Edge is incident on A & B

A & B are adjacent vertices.

$\Sigma \deg(v_i) = 10$.

$\boxed{\Sigma \deg(v_i) = 2e} =$ even $\boxed{\text{Hand shaking Theorem}}$.

⇒ Sum of degrees of vertices is also even.

⇒ e = Distinct edges of graph.

Handshaking Theorem Improved :-
The no. of odd vertices in a graph is also even.

* Isolated vertex :- Vertex having zero degree.
* Pendant vertex :- vertex having degree 1.

* Hand Shaking Theorem :- The sum

$$\boxed{\sum_{i=1}^{n} \deg(V_i) = 2e.}$$

Where, e is No. of Distinct edges.

Statement: The sum of degrees of all vertices in a graph is always even. and it is equal to 2× Total No. of edges in the graph.

* **Theorem :** The number of vertices of odd degree in an undirected graph is always even.

Proof Let. G (V, E) be a undirected graph then by Handshaking Theorem

$$\boxed{\sum_{i=1}^{n} \deg(v_i) = 2e} = \text{even} \quad \text{——(1)}$$

Let, V_1 and V_2 be the set of even and odd vertices

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$\Sigma \deg(V_1) + \Sigma \deg(V_2) = \Sigma \deg(V_i)$.

$= \text{even}$ [from eqⁿ ①]

$\Rightarrow \text{even} + \Sigma \deg(V_2) = \text{even}.$

$\Rightarrow \Sigma \deg(V_2) = \text{even} - \text{even}$

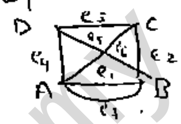
$\Rightarrow \boxed{\Sigma \deg(V_2) = \text{even.}}$

{subtraction of even no. is always even}

since sum of odd degree vertices is even. Then, no. of odd vertices degree will be even.

* Simple Graph :- A graph which has niether self loop nor parallel edges.

Parallel edges. There is more than 1 edge b/w two vertices.

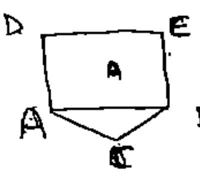


eg e_1 & e_7 are parallel edges.

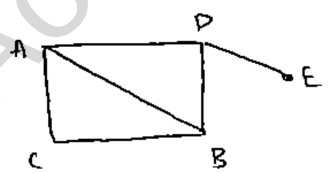
* Multi Graph :- A graph which contain some parallel edges or self-loop is called multi-graph

Q Draw a graph with vertices A, B, C, D, E such that d(A) = d(C) = 2, B is an odd vertex and D and E are adjacent vertices.

→



or



Q Does there exist a simple graph with 5 vertices of given degrees. If so, draw such graph.

1) 1, 2, 3, 4, 5

By Handshaking theorem.

$\Sigma \deg(V) = 15$. which is odd.

But, By HST the sum of degrees of all vertices in a graph is even.

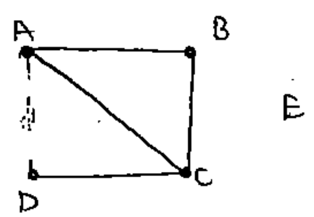
2) 1, 2, 3, 4, 4

$\Sigma \deg(V) = 1+2+3+4+4 = 14$.

Here, Two vertices of degree 4 exists. Each remaining 3 vertices will be of atleast degree 2. But the given sequence of degree contains 1. So, no such graph is possible of given sequence.

3) 0, 1, 2, 2, 3.

$\Sigma \deg(V) = 1+2+2+3+0 = 8$.



Q. Show that the maximum no. of edges in a simple graph with n vertices is $\frac{n \times (n-1)}{2}$.

$\Rightarrow$ i.e. $e_{max} = \frac{n(n-1)}{2}$.

$G(V, E)$.

$\because \sum_{i=1}^{n} \deg(v_i) = 2e$.

$\Rightarrow e = \frac{1}{2} \sum_{i=1}^{n} \deg(v_i)$. —— (1)

If there are 'n' vertices then 'n' can be adjacent to maximum (n−1) vertices.
So, No. of maximum edges associated with one vertex = (n−1)
So, No. of Total maximum edges of graph = n.(n−1)

So, $\sum_{i=1}^{n} \deg(v_i) = n(n-1)$] → Put in eqn (1).

$\Rightarrow \boxed{e_{max} = \frac{1}{2} n(n-1).}$

Q. Show that if all the vertices of an undirected graph are each of odd degree 'K'. show that no. of edges of the graph is a 'multiple of K'.

→ since, each vertex is of a odd degree 'K'.
So, vertices must be in even number say it is 2n,
$n \in Z^+$.
If No. of edges = e, then acc. to Handshaking Theorem –
$\sum_{i=1}^{2n} \deg(v_i) = 2e$.

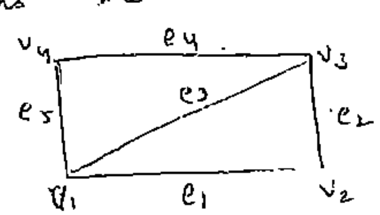
$\Rightarrow \cancel{2}n \times k = \cancel{2}e$

$\Rightarrow \boxed{e = nK}$

Walk
- open → (No edge repeated) Trail → Path (No vertex & edge repeated)
- closed → circuit → cycle

* Terminologies :-

- **Walk** :- It starts from a point (vertex) and ends at a point such that each edge preceed or succeeds its vertices.



eg. $v_1 e_1 v_2 e_2 v_3 e_3 v_1$ is a closed walk.
$v_1 e_1 v_2 e_2 v_3 e_3$ is an open walk.

There are, two types of walk :-

open walk :- $v_1 \neq v_n$.

closed walk :- $v_1 = v_n$.

- **Trail** :- Trail is an open walk with no edge repeated.
- **Path** :- Path is an open walk with neither vertex nor repeated.

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repeated.

- Cycle :- cycle is a closed walk with neither edge nor vertex repeated.

* Some Special types of Graph :-

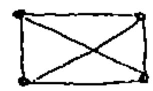
- Complete Graph :- A simple graph is called complete graph in which exactly one edge present b/w every pair of vertices. It is denoted by K_n, where 'n' is no. of vertices.

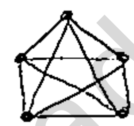
$$\boxed{{}^nC_2 = \frac{n(n-1)}{2}}$$

eg K_1

K_2

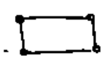
K_3 

K_4 

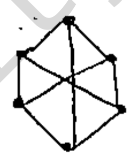
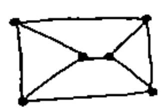
K_5 

- n - Regular Graph :- It is a graph where degree of every vertex is same. If deg is n then it is called n- regular graph.

eg 1- regular graph

2 - regular graph  or 

0- regular graph

3- regular graph  or 

- Null graph (or) void (or) Empty Graph :- The graph with isolated vertices called Null Graph.
A null graph with 'k' vertices is denoted by N_k.

e.g v_1 v_2 $\quad$ v_1 v_2 v_3.

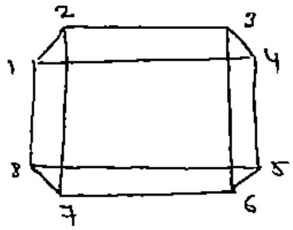
- Bipartite Graph :- A Graph $G(V,E)$ is said to be bipartite if vertex set V can be partition into two disjoint subsets V_1 and V_2. such that each edge in 'E' connects a vertex of V_1 to a vertex of V_2 So that no edge in 'E' connects other two vertices in V_1 or two vertices in V_2. So, (V_1, V_2) is called Bipartite of G.
If V_1 has 'm' vertices and V_2 has 'n' vertices then complete bipartite graph is denoted by $K_{m,n}$.

NOTE: A bipartite Graph has no self loop.

eg K_{23}

K_{33}

Q



$V_1 = \{1, 3, 5, 7\}$
$V_2 = \{2, 4, 6, 8\}$.

Here, vertex 1 is connected to 2, 4, 8 not connected to 6.
vertex 3 is connected to 2, 4, 6 not connected to 8.
vertex 5 is connected to 4, 6, 8 not connected to 2
vertex 7 is connected to 2, 6, 8 not connected to 4

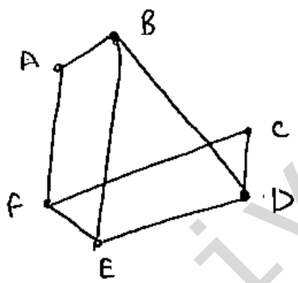
So, it is a bipartite graph.
since, there is no edge b/w 1&6, 3&8, 5&2, 4&7
so, it is not a complete bipartite graph.

Q

$V_1 = \{A, B, C, D\}$
$V_2 = \{E\}$.

Here. vertex A, B, C, D are not connected to each other they all are connected to E.
So, it is a complete bipartite graph $K_{1,4}$

Q

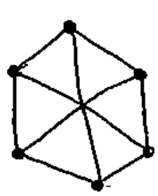
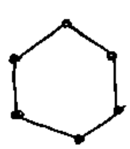


It is not a bipartite graph.
Because disjoint sets are obtained but vertex of set V_2 are connected to each other.

$V_1 = \{B, C\}$
$V_2 = \{A, D, E, F\}$.

• Sub Graph:- A graph $H(V', E')$ is called sub-graph of $G(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.
or $G(V, E)$ is called <u>super graph</u> of $H(V', E')$.

eg

 $\Rightarrow$ 

<u>Sub graph</u>.

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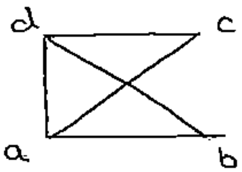
isomorphic to graph $G_2(V_2, E_2)$
if there exits a function $\boxed{f : V_1 \rightarrow V_2}$. such that
- f is $\boxed{\text{bijective}}$. (one to one, onto).
- for all $\boxed{\{a, b\} \in E_1 \quad \{f(a), f(b)\} \in E_2}$ $\forall$ $a, b \in V_1$

such a correspondence is called isomorphism.

e.g.



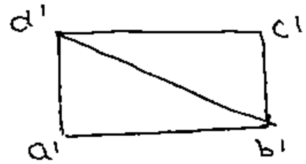
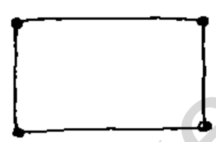


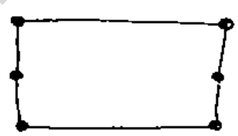
Image of v is that the same circle of degree. Eg f(a) is a d' and we take any one it is that nece that we to b'.

$f(a) = b'$
$f(b) = a'$
$f(c) = c'$
$f(d) = d'$

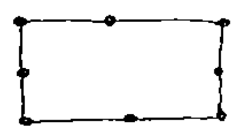
- Homomorphic Graph :- Two graph $G(V, E)$ and $G^*(V^*, E$ are said to be Homeo-morphic if G^* can be obtained from G by dividing edges of G by additional edges.



G.

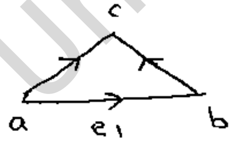


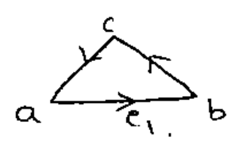
This is a homomorphic graph of G.



This is also a homomorphic gra

* Directed Graph :-





e_1 incident from a.
e_1 is incident to b.
a is adjacent to b.
b is adjacent from a.
a is the Initial vertex.
b is the terminal vertex.

$\boxed{\text{Indegree}}$

Indeg (v) $\quad d^-(v)$

$\boxed{\text{Outdegree}}$

outdeg (v) $\quad d^+(v)$

$d^-(a) = 1 \quad d^-(c) = 1$
$d^+(a) = 1. \quad d^+(c) =$
$d^-(b) = 1$
$d^+(b) = 1$

Total Degree :-

$deg(v) = d(v) = d^-(v) + d^+(v)$

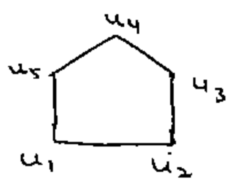
$d(a) = 2$.
$d(b) = 2$
$d(c) = 2$

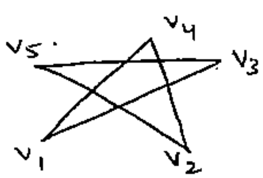
- Source :- vertex whose Indegree is zero. $\boxed{d^-(v) = 0}$ source.
- Sink :- Vertex whose outdegree is zero. $\boxed{d^+(v) = 0}$ sink.
- Self Loop :- Indegree = 1
  outdegree = 1
  Total Degree = 2.

NOTE: Sum of Indegree = Sum of outdegree = e.

$\Sigma d^-(v_i) = \Sigma d^+(v_i) = e$ (No. of edges).

Q.





$f : v_1 \to v_2$.
There are equal no. of edges so, it is bijective.

order of degree of all vertices of Graph G_1 & G_2 are 2 and are same.

$f(u_1) = v_1$.
$f(u_2) = v_3$.
$f(u_3) = v_5$.
$f(u_4) = v_2$.
$f(u_5) = v_4$.

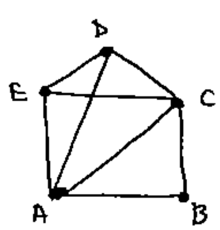
$\{u_1, u_2\} \in E_1 \Rightarrow \{f(u_1), f(u_2)\} = (v_1, v_3) \in E_2$.

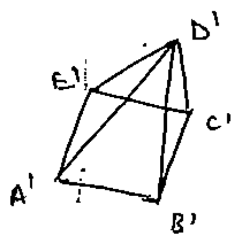
$\{u_1, u_5\} \in E_1 \Rightarrow \{f(u_1), f(u_5)\} = (v_1, v_4) \in E_2$

$\{u_2, u_3\} \in E_1 \Rightarrow \{f(u_2), f(u_3)\} = \{v_3, v_5\} \in E_2$.

$(u_3, u_4) \in E_1 \Rightarrow \{f(u_3), f(u_4)\} = (v_5, v_2) \in E_2$.

Q.





$d(A) = 4$, $d(B) = 2$, $d(C) = 4$, $d(D) = 3$, $d(E) = 3$.

$d(A') = 3$, $d(B') = 3$, $d(C') = 3$, $d(D') = 4$, $d(E') = 3$.

The order of degree of Graph G_1 & G_2 are not same. Therefore it is not isomorphic.

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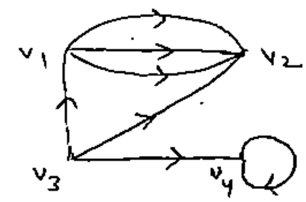
Q



$d^-(v_1) = 1 \qquad d^+(v_1) = 3$
$d^-(v_2) = 4 \qquad d^+(v_2) = 0$
$d^-(v_3) = 0 \qquad d^+(v_3) = 3$
$d^-(v_4) = 2 \qquad d^+(v_4) = 1$

Total degree.

$\Sigma\, d^-(v_i) + \Sigma\, d^+(v_i)$

$7 + 7 = \underline{14}$.

$d(v_1) = d^-(v_1) + d^+(v_1) = 1+3 = 4.$
$d(v_2) = d^-(v_2) + d^+(v_2) = 4+0 = 4$
$d(v_3) = d^-(v_3) + d^+(v_3) = 0+3 = 3$
$d(v_4) = d^-(v_4) + d^+(v_4) = 2+1 = 3.$

* Pendant vertex in directed graph:- vertex whose total degree is 1 (by adding indegree and outdegree).
  - Indegree may be zero then outdegree is 1.
  - outdegree may be zero then Indegree is 1).

But both Indegree & outdegree at same time not be equal to 1.

* Representation of Graph:-

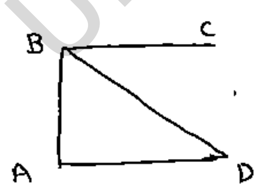
- Undirected Graph:- There are two methods to represent undirected graph.

◉ Adjacency (Vertex-Vertex adjacency Matrix)
◉ Incident (Vertex- Edge Incident Matrix).

Adjacency Matrix :-

$\boxed{A_{ij} = [a_{ij}]}$ where, $\boxed{a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}}$

eg.



$$A = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

If a Graph has M vertices then the order of adjacency matrix will be m×n. and adjacency matrix is represented as- $\boxed{A_{ij} = a_{ij}}$ Where, $a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to} \\ 0 & \text{otherwise.} \end{cases}$

Properties :-

Adjacency Matrix is always symmetric.
If we add rows of matrix then we can get degree of corresponding vertex.

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If we add all non-zero elements of matrix then we get $2e$. where e is No. of edges of graph.

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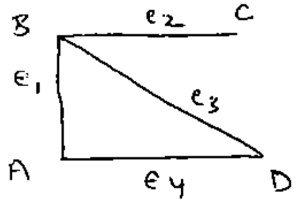
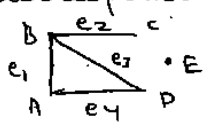
Incidence Matrix :-

$\boxed{B_{ij} = [b_{ij}]}$, where, $b_{ij} = \begin{cases} 1, & \text{if } e_j \text{ is incident on } v_i \\ 0, & \text{otherwise} \end{cases}$



$$I = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

If we add row of matrix then we can get degree of matrix.
on every column there is only or exactly two one's. because edge is incident on two vertices.
If a vertex is isolated then it has all zeroes in its corresponding row/column.

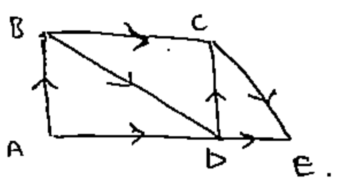


$$A = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Directed Graph:-

Adjacency Matrix :-

$\boxed{A_{ij} = [a_{ij}]}$, where, $a_{ij} = \begin{cases} 1, & \text{if there is an edge directed from } v_i \text{ to } v_j \\ 0, & \text{otherwise} \end{cases}$



$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

It is not symmetric.
The sum of elements of row gives the outdegree of the corresponding vertex.
The sum of elements of column gives the indegree of the corresponding vertex.

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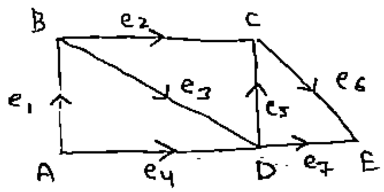
Incidence Matrix :-

$\boxed{B_{ij} = [b_{ij}]}$, where, $b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is initial vertex of edge } e_j \\ -1, & \text{if } v_i \text{ is terminal vertex of edge } e_j \\ 0, & \text{otherwise.} \end{cases}$

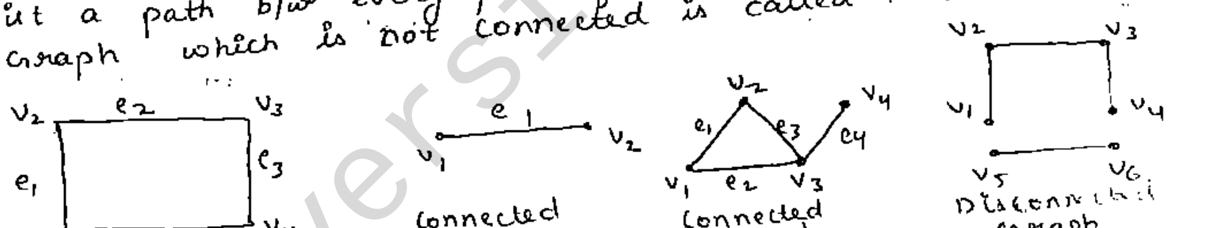


$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

* ~~Distance and Diameter :- consider a connected graph 'G'~~

* Connected Graph (or) Connectivity in Undirected Graph :-

An undirected graph is said to be connected if there exist a path b/w every pair of distinct vertices. A Graph which is not connected is called disconnected graph.



Connected Graph — Connected Graph — Connected Graph — Disconnected Graph

⇒ Every disconnected graph contains two or more connected graph (sub graph) and each of the connected graph of disconnected graph is called $\boxed{\text{component}}$.

Components :-

* Theorem :- A graph 'G' without parallel edges or self loop with 'n' nodes and 'k' components can have almost $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof Let, No. of vertices in i^{th} component be n_i

$n_1 + n_2 + n_3 + \cdots + n_k = n.$

$$\Rightarrow \sum_{i=1}^{k} n_i = n.$$

$$\Rightarrow \sum_{i=1}^{k} n_i - k = n-k.$$

$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2 \geq 2\sum_{i=1}^{k}(n_i-1)+\sum_{i=1}^{k}1 \geq n^2+k^2-2nk.$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2n + k = n^2+k^2-2nk.$$

$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2 + 2\sum_{\substack{i=1 \\ j=1,\, i\neq j}}^{k}(n_i-1)(n_j-k) = n^2+k^2-2nk.$$

less than equals to and

$$\Rightarrow \sum_{i=1}^{k} n_i - \underbrace{(1+1+1+\cdots)}_{k \text{ times}} =$$

$$\Rightarrow \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n-k$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1) = n-k.$$

$$\Rightarrow \left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2.$$

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leave the one digit.

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 \le n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} 1 - 2\sum_{i=1}^{k} n_i \le n^2 + k^2 - 2nk.$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + k - 2n \le n^2 + k^2 - 2nk.$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \le n^2 + k^2 - 2nk - k + 2n.$$

We know that the maximum no. of edges in the i^{th} component

$$= \frac{1}{2} n_i (n_i - 1)$$

so, Total no. of edges of the graph

$$= \frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1)$$

$$= \frac{1}{2} \sum_{i=1}^{k} (n_i^2 - n_i)$$

$$= \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} \sum_{i=1}^{k} n_i.$$

$$= \frac{1}{2} [n^2 + k^2 - 2nk + 2n - k - n]$$

$$= \frac{1}{2} [n^2 + k^2 - 2nk + n - k].$$

$$= \frac{1}{2} [(n-k)^2 + (n-k)]$$

$$\boxed{= \frac{1}{2} (n-k)(n-k+1)}.$$

* Distance and Diameter :-

Consider a connected graph 'G' the distance b/w vertices u & v denoted by $d(u,v)$ is minimum No. of edges b/w vertices u and v.

The diameter of Graph 'G' denoted by 'dia(G)' or '$\Delta(G)$' or 'Δ' is maximum distance b/w any two vertices.

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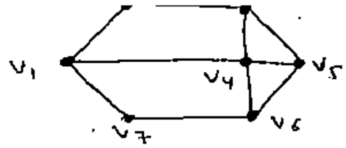


$d(v_1, v_2) = 1$
$d(v_1, v_3) = 2$
$d(v_1, v_4) = 1$
$d(v_1, v_6) = 2$
$d(v_1, v_7) = 1$

$d(v_2, v_3) = 1$
$d(v_2, v_4) = 2$
$d(v_2, v_5) = 2$
$d(v_2, v_6) = 3$
$d(v_2, v_7) = 2$.

$d(v_3, v_4) = 1$
$d(v_3, v_5) = 1$
$d(v_3, v_6) = 2$
$d(v_3, v_7) = 3$

$d(v_4, v_5) = 1$
$d(v_4, v_6) = 1$
$d(v_4, v_7) = 2$.

$d(v_5, v_6) = 1$
$d(v_5, v_7) = 2$

$d(v_6, v_7) = 1$.

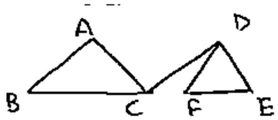
$\boxed{\Delta = 3}$

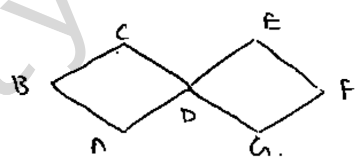
$\boxed{\Delta = \max d(u, v)}$

* Cut-Points and Bridge :-

Let 'G' be a connected graph a vertex 'v' in G is called <u>a cut-point</u> if G − v is disconnected. And edge 'e' of 'G' is called a <u>Bridge</u> if 'G − e' is disconnected.

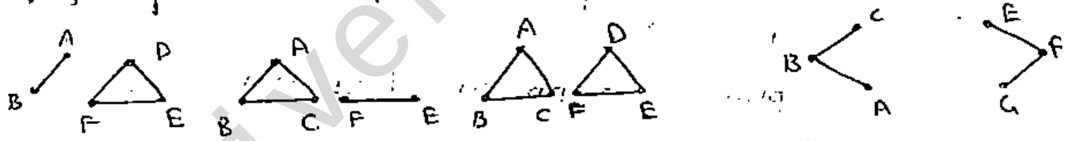
<u>eg</u>





C and D are cut points.
{C, D} edge is Bridge.

D is a cut point.
There is not Bridge.



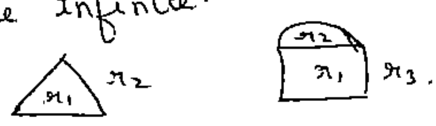
* Planar Graph :-

A graph or a multigraph which can be drawn in the plane so that its edges don't cross each other is said to be planar.

<u>eg</u>

⇒ The planar graph divides the ~~regi~~ plane into two or more parts or regions where one of the part will be infinite.

<u>e.g</u>



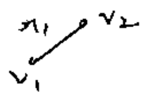
has only a graph which has region which is infinite & is planar

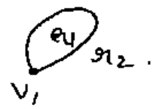
* Euler's Theorem :- Consider any connected planar Graph '$G(V, E)$' with 'R' regions, 'V' vertices & 'E' edges then -

$$\boxed{V + R - E = 2}$$

Proof By Mathematical Induction

1) Induction Based : $E = 1$





$V + R - E = 2$

$\Rightarrow \underline{2 + 1 - 1 = 2}$

$V + R - E = 2$

$\Rightarrow \underline{1 + 2 - 1 = 2}$

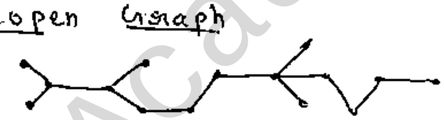
2) For k it is true

$\boxed{V + R - E = 2}$ is true. (or) $V + R - K = 2$.

3) For $(K+1)$. (Inductive Hypothesis).

Case 1 $V = V_1 + 1$. & $E = E_1 + 1$.

For open Graph

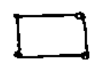


$V + R - E = 2$.

$V_1 + \cancel{1} + R - E_1 - \cancel{1} = 2$

$\boxed{V_1 + R - E_1 = 2}$ It is true.

Case 2 For Closed Graph



$R = R_1 + 1$

$E = E_1 + 1$

$V + R - E = 2$

$V + R_1 + \cancel{1} - E_1 - \cancel{1} = 2$

$\boxed{V + R_1 - E_1 = 2}$

- If a connected planar graph 'G' has E edges & R regions then, $\boxed{R \le \frac{2}{3} E}$
- If a connected planar graph 'G' has V vertices & E edges then, $\boxed{3V - E \ge 6}$
- A complete graph K_n is planar iff $\boxed{n < 5}$
- A complete bipartite Graph $K_{m,n}$ is planar iff $\boxed{m < 3 \ \&}$ $\boxed{n < 3}$.

Q. A planar Graph has 30 vertices each of degree 3. Determine the no. of regions into which this planar Graph can be splitted.

$\rightarrow$ $V = 30$. $d(v_i) = 3$.

$d(v_i) = 3$

$\sum d(v_i) = 2e$

$\overset{15}{\cancel{30}} \times 3 = \cancel{2}e$

$\boxed{e = 45}$

OR sum of degrees $= 3V$
$= 3 \times 30$
$= 90$.

By Hand shaking Th.
$\sum d(V) = 2e$
$90 = 2e \Rightarrow \boxed{e = 45}$

$V - E + R = 2 \Rightarrow 30 - 45 + R = 2$

$\boxed{R = 17}$

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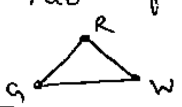
⇒ If every region of a simple planar graph with n vertices & E edges is bounded by k edges then prove that $E = \frac{k(n-2)}{(k-2)}$. ⇒ Solution next page

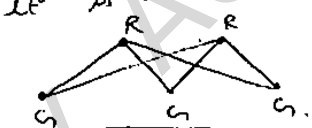
* **Non-Planar Graph** :- A graph 'G' is said to be non planar graph if it can not be drawn in a plane so that no edges cross.

* **Graph Coloring** :- Suppose that G(V,E) with no multiple edges, a vertex coloring of G is an assignment of colours to vertices of G such that adjacent vertices have different colours.
A Graph 'G' is M-colourable if there exist a colouring 'G' which uses M colours.
colouring the vertices such a way such that no two adjacent vertices have same colour is called proper colouring otherwise it is called Improper Colouring.

* **Chromatic Number of Graph** :- The minimum number of colours needed to produce a proper colouring of a Graph 'G' is called chromatic No. of 'G' and it is denoted by $\boxed{\chi(G)}$.

e.g





$\boxed{\chi(G) = 3}$ $\boxed{\chi(G) = 2}$

* **Algorithm for Graph Colouring** :- (Welch and Powell)
Wlech and Powell gave algorithm for the colouring of graph 'G'.

step 1 : order the vertices of the Graph 'G' acc. to decreasing degree.

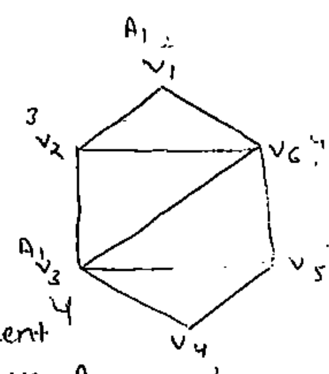
step 2 : Assign first colour A_1 to the first vertex and then assign A_1 to each vertex which is not adjacent to previous vertex which was assign colour A_1 but in sequential order.

step 3 : Repeat step 2 with second colour A_2 and subsequent of non-coloured vertices.

step 4 : Reapeat step 3 with a third colour A_3 then fourth colour A_4 and so on until all vertices are coloured.

e.g

vertices	V_3	V_6	V_2	V_5	V_1	V_4
Degrees	4	4	3	3	2	2



① Now Assign colour A_1 to the vertex V_3. Now, the vertex V_1 is not adjacent to ~~colo~~ vertex V_3. So we assign colour A_1 also to vertex V_1.

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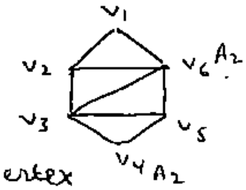
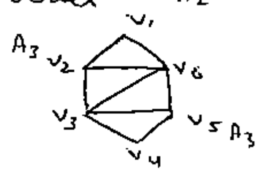
2) Now assign second colour A_2 to vertex v_6. and Now v_4 is not adjacent to vertex v_4. so we assign colour A_2 to vertex v_4.



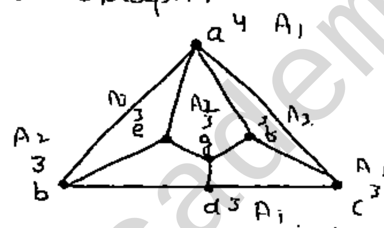
3) Now assign third colour A_3 to vertex v_2. and Now v_5 is not adjacent to vertex v_5. so we assign colour A_3 to vertex v_5.



Now all the vertices are coloured
The chromatic no. of this graph is
$\boxed{\chi(G) = 3}$

Q find the chromatic no. of this Graph.

vertices	a	b	c	d	e	f	g
Degree	4	3	3	3	3	3	3.



1) Now assign colour. A_1 to the vertex 'a'. Now, 'd' is not adjacent to vertex 'a'. so, we assign colour A_1 to vertex 'd'.

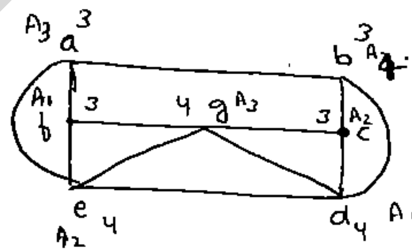
2) Now assign second colour A_2 to the vertex 'b'. Now vertex 'g' and 'c' are not adjacent to vertex 'b'. so we assign colour A_2 to vertex 'g and 'c'.

3) We assign third colour A_3 to vertex 'e'. Now vertex 'f' are not adjacent to vertex e. so, we assign colour A_3 to vertex 'f'.

* Bichromati

Vertices	d	e	g	a	b	c	f
Degree	4	4	4	3	3	3	3.

$\boxed{\chi(G) = 4}$

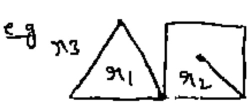


* Degree of Regions in a planar graph:- The no. of edges ~~traversed~~ in a closed walk of the region such that every vertex of the region is traversed is called degree of the region.

eg

$d(r_1) = 3$
$d(r_2) = 3$
$d(r_3) = 4$
$10 = 2e$
$10 = 2\times 5$

eg



$d(r_1) = 3$
$d(r_2) = 6$
$d(r_3) = 7$.
$16 = 2e$
$16 = 2\times 8$

$\boxed{\Sigma\, \deg(r) = 2e}$

r = regions.

If such no. of edges let be 'k' then that region is said to be bounded by k ~~regions~~ edges.

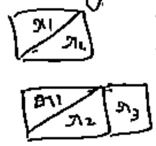
The no. of outer edges in a ~~closed~~ closed walk of the graph is said to be degree of Infinite region.

$$E = \frac{K(n-2)}{(K-2)}$$

$$KR = 2E$$
$$\Rightarrow R = \frac{2E}{K}$$

$$v + R - E = 2$$
$$\Rightarrow n + \frac{2E}{K} - E = 2$$
$$\Rightarrow nK + 2E - EK = 2K$$
$$\Rightarrow E(2-K) = K(2-n)$$
$$\Rightarrow E = \frac{K(2-n)}{(2-K)} \Rightarrow \boxed{E = \frac{K(n-2)}{(K-2)}}$$

In planar Graph region colouring is also done



$r_1 \rightarrow A_1$ colour
$r_2 \rightarrow A_2$ colour

r_1 & $r_3 \rightarrow A_1$ colour
$r_2 \rightarrow A_2$ colour

adjacent region does not have same colour

* Bichromatic No. of Graph :- If a Graph 'G' requires 2 colo for colouring then it is called bichromatic graph.

eg 

Every bipartite graph is bichromatic.

NOTE: • chromatic no. of G will always be less than equal to No. of vertices in Graph 'G'. ($\leq n$).

• If chromatic no. of a subgraph of 'G' is 'm' then $\boxed{\chi(G) \geq m}$

• Chromatic no. of complete graph 'K_n' is equal to no. of vertices $\boxed{\text{in } K_n}$

• The chromatic no. of a Null Graph is "1"

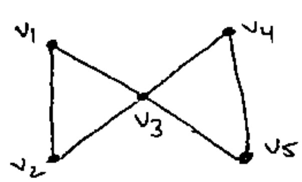
* Euler Graph :-

Euler Path :- A path of graph 'G' which includes each edge 'G' exactly once is called Eulerian path

Eulerian Circuit :- A circuit of graph 'G' which includes each edge of 'G' exactly once.

Eulerian Graph :- A Graph containing Eulerian circuit is called Eulerian Graph.

eg $\boxed{V_3\ V_1\ V_2\ V_3\ V_4\ V_5\ V_3}$



* Theorem :- A connected graph contain an Euler circuit if and only if each of its vertices of even degree.

* Theorem :- A connected graph contain an Euler path ~~iff~~ it has ~~only~~ exactly two vertices of odd degree.

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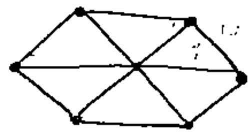
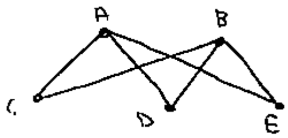
(i) 

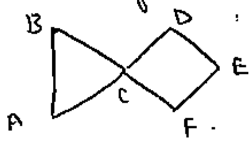
There is no Euler path & Euler circuit - because each vertex is of order 3 which is odd and also there is more than two vertices of odd degree.

(ii) 

There is Euler path because A and B vertex is of oder 3 and there is only two vertex which has odd degree.

A → C → B → E → A → D → B.

There is no Euler circuit because two vertices A & B is of order 3. i.e. odd.

(iii) 

There is no Euler path because there is no vertex which is of odd degree.

There is Euler circuit.

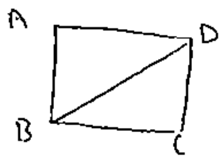
C – A – B – C – D – E – F. So, it is Euler graph.

* Hamiltonian Graph :-

Hamiltonian Graph is named after Irish mathematican Sir William Hamiltonian

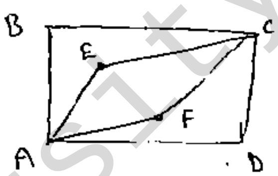
A Hamiltonian circuit in a graph 'G' is a closed path that visit every vertex in 'G' exactly once except the end vertices.

A Graph 'G' is called Hamiltonian Graph if it contains a Hamiltonian circuit.



A – B – C – D – A

Hamiltonian graph.



Not a Hamiltonian Graph.

Hamiltonian Path :- The path obtained by removing any one edge from a hamiltonian circuit is called Hamiltonian path.

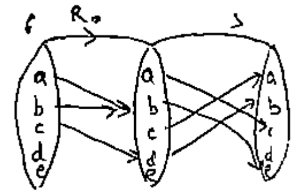
Hamiltonian Path is subgraph of Hamiltonian circuit. but converse is not true.

The length of a Hamiltonian path in a connected graph of 'n' vertices is (n-1) if it exists.

Q $R = \{(a,b), (c,d), (b,b)\}$ $S = \{(d,b), (b,e), (c,a), (a,c)\}$

$A = \{a,b,c,d,e\}$

Compute $R \circ S$ & $S \circ R$.



→ $R_0 S = \{(a,e), (b,e), (c,b)\}$

$S_0 R = \{(d,b), (c,b), (a,d)\}$.

$(R_0 S)_0 R = \{(c,b)\}$

$(S_0 R)_0 S = \{(d,e), (c,e), (a,b)\}$.

Q $R = \{(a, 3a) : a \in Z^+\}$. Z^+.

$S = \{(a, a+1) : a \in Z^+\}$.

$R_0 S = \{(a, 3a+1) : a \in Z^+\}$

$R_0 R = \{(a, 9a)\}$

$R_0 R_0 R = \{(a, 27a)\}$.

$R_0 S_0 R = \{(a, 9a+3)\}$

Q Show that if set A has 3 elements then we can have 2^6 symmetric relations of on A.

Total symmetric relations. $= 2^{n(n+1)/2}$

$= 2^{3(3+1)/2} = 2^{\frac{3\times 4}{2}} = 2^6$.

Q $A = \{1,2\}$. Find $P(A)$

$P(A) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$.

Q Define multiset.

Q Prove n^3+n by MI. is divisible by 3 by MI.

Q $A\{1,2,3,4\}$. i) $R = \{(1,3), (3,1), (1,1), (1,2), (3,3), (4,4)\}$. No Eq

ii) $R = A \times A$. Equivalence

~~show~~ It is equivalence or not.

⇒ 1) Reflexive : $(a,a) \in R$, $\forall a \in A$.

2) Symmetric : $(a,b) \in R \Rightarrow (b,a) \in R$, $\forall (a,b) \in A$.

3) Transitive: $(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$, $\forall a,b,c \in A$.

1) It is not reflexive.

Q 2, 4, 6, 8. How many 4 digit no. we make when repetion is allowed.

→ $4 \times 4 \times 4 \times 4 = 256$.

Q : DNF $(x+y)(x'+y')$. i.e. we find SOP (OR) (Disjunctive Normal Form)

CNF (conjunctive Normal ~~No~~ Form) (POS) form. AND

$\Rightarrow xx' + xy' + yx' + yy'$

$\Rightarrow xy' + x'y$.

(1) ~~Not~~ All books have bibliographies. Quantifiers

$P(x)$: ~~All books~~ x is a book having bibliograp

is $(\forall x\ P(x))$

Q $Y = ((AB)' + A' + AB)'$

$A'B'C'D' + A'B'C'D + A'B'CD + A'B'CD' = A'B'$

Q $A = \{1, 2, 3, 4\}$ aRb iff a is a factor of b.

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$.

$A = \{1, 2, 3, 4, 6\}$.

$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6),$
$(4,4), (6,6)\}$.

R(3) = The set of those elements which are related to 3 in R.

$= \{\cancel{(3,6)}, \cancel{(3,3)}\} \cdot \{3, 6\}$.

$R(6) = \{6\}$.

$R(2,4,6) = R(2) \cup R(4) \cup R(6)$.

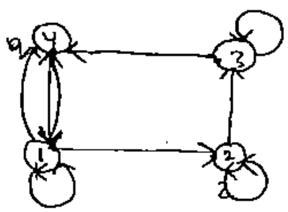
$= \{2,4,6\} \cdot \cup \{4\} \cup \{6\}$

$= \{2, 4, 6\}$.

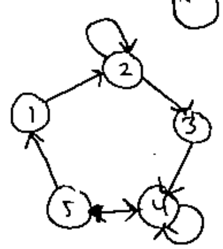
Q $A = \{1, 2, 3, 4\}$. Find relation R & draw the diagraph (directed graph) represented by relation R.

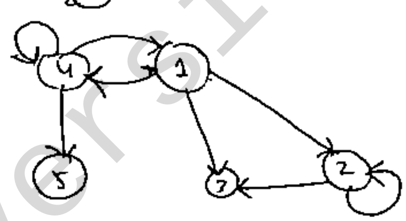
$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$R = \{(1,1), (1,2), (1,4), (2,2), (2,3), (3,3), (3,4),$
$(4,1)\}$.



Q





$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$R = \{(1,2), (2,2), (2,3), (3,4),$
$(5,1), (4,4), (5,4)\}$.

$R = \{(1,2), (1,3), (1,4), (2,2), (2,3),$
$(4,1), (4,4), (4,5)\}$.

Q $A = \{1, 2, 3, 4\}$.

$R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$.

1) It is reflexive $(a,a) \in R$. (Every element related to itself.

since, $(1,1) \in R$, $(2,2) \in R$, $(3,3) \in R$, $(4,4) \in R$.

2) Not irreflexive.

3) It is symmetric. Since, $(1,1) \in R$, $(1,2) \in R \Rightarrow (2,1) \in R$

$(2,2) \in R$, $(3,4) \in R \Rightarrow (4,3) \in R$.

$(4,4) \in R$, $(3,3) \in R$

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5) Transitive Property.

Q $R = \{(1,3), (1,1), (3,1), (1,2), (3,3), (4,4)\}$

1) Not reflexive $(2,2) \notin R$.
2) It is not irreflexive $(1,1) \in R$, $(3,3) \in R$, $(4,4) \in R$
3) Not symmetric $(1,2) \in R \Rightarrow (2,1) \notin R$
4) Transitive Not $(3,1) \in R$ & $(1,2) \in R \Rightarrow (3,2) \notin R$.
5) Not Antisymmetric

Q Let R be a relation on the set of real No's such that $R = \{<x,y> : |x-y| < 1\}$. is it an equivalence relation.

1) Reflexive:- Let $(x,x) \in R \Rightarrow |x-x| \Rightarrow 0 < 1$. Yes. $\forall x \in R$.
2) Symmetric: Let $(x,y) \in R \Rightarrow (y,x) \in R$.

$$|x-y| < 1$$
$$\Rightarrow |-(y-x)| < 1$$
$$\Rightarrow |y-x| < 1. \Rightarrow (y,x) \in R$$

3) Transitive:- Let $(x,y) \in R$ & $(y,z) \in R$.

$|x-y| < 1$ $|y-z| < 1$.

It is not a transitive property.

eg. $x \quad 1 \quad y \quad 1 \quad z$

$|x-y| < 1$. $|x-z| \not< 1$

$|y-z| < 1$.

Not an equivalence relation.

Q The bit strings for the sets $\{1,2,3,4,5\}$ & $\{1,3,5,7,9\}$ are 1111100000 & 1010101010 resp. use bit strings to find the union & intersection of sets.

$\Rightarrow$

$$\left.\begin{array}{r} 1111100000 \\ 1010101010 \\ \hline 1111101010 \end{array}\right] OR \qquad \left.\begin{array}{r} 1111100000 \\ 1010101010 \\ \hline 1010100000 \end{array}\right] AND$$

$A \cup B = \{1,2,3,4,5,7,9\}$. $A \cap B = \{1,3,5\}$.

Q Let $U = \{1,2,3,4,5,6,7,8,9,10\}$ & the ordering of elements of U is in increasing order i.e. $a_i = i$. what bit strings represent the subset of all odd integers in U, subset of all integers not exceeding 5 in U & the subset of integers in U & the subset of even integers in U

$\Rightarrow$ $S_1 = \{1,3,5,7,9\}$. = 1010101010

$S_2 = \{2,4,6,8,10\}$. = 0101010101

$S_3 = \{1,2,3,4,5\}$. = 1111100000.

15/12/14

Q If $X = \{1,2,3,4,5\}$ and the fun. f.g.h are defined from $X \to X$ given by $f = \{(1,2), (2,1), (3,4), (4,5), (5,3)\}$. $h = \{(1,2), (2,2), (3,4), (4,3), (5,$ $g = \{(1,3), (2,5), (3,1), (4,2), (5,4)\}$.

1) verify whether $f \circ g = g \circ f$
2) Explain why f and g have inverse but h does not.
3) Find f^{-1} & g^{-1}
4) show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

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1) $f \circ g(1) = f[g(1)] = f(3) = 4$
$f \circ g(2) = f[g(2)] = f(5) = 3$
$f \circ g(3) = f[g(3)] = f(1) = 2$
$f \circ g(4) = f(g(4)] = f(2) = 1$
$f \circ g(5) = f(g(5)) = f(4) = 5$

$f \circ g = \{(1,4), (2,3), (3,2), (4,1), (5,5)\}$

$g \circ f(1) = g[f(1)] = g(2) = 5$
$g \circ f(2) = g(f(2)] = g(1) = 3$
$g \circ f(3) = g(f(3)] = g(4) = 2$
$g \circ f(4) = g f(4)) = g(5) = 4$
$g \circ f(5) = g(f(5)) = g(3) = 1.$

$g \circ f = \{(1,5), (2,3), (3,2), (4,4), (5,1)\}.$

$\Rightarrow \boxed{f \circ g \neq g \circ f}$

2) f and g are bijective functions therefore inverse of f & g exists but h is not bijective fun. therefore inverse of h does not exist.
$h(1) = h(2) \Rightarrow 1 \neq 2.$

3) $f^{-1} = \{(2,1), (1,2), (4,3), (5,4), (3,5)\}. = \{(1,2), (2,1), (3,5), (4,3), (5,4)\}$
$g^{-1} = \{(3,1), (5,2), (1,3), (2,4), (4,5)\} = \{(1,3), (2,4), (3,1), (4,5), (5,2)\}$

4) $(f \circ g)^{-1} = \{(4,1), (3,2), (2,3), (1,4), (5,5)\} = \{(1,4), (2,3), (3,2), (4,1), (5,5)\}$

$g^{-1} \circ f^{-1}(1) = 4$
$g^{-1} \circ f^{-1}(2) = 3$
$g^{-1} \circ f^{-1}(3) = 2$
$g^{-1} \circ f^{-1}(4) = 1$
$g^{-1} \circ f^{-1}(5) = 5$

$\therefore g^{-1} \circ f^{-1} = \{(1,4), (2,3), (3,2), (4,1), (5,5)\}.$

$\Rightarrow \boxed{(f \circ g)^{-1} = g^{-1} \circ f^{-1}}$ H.P

Uni Imp & 10M

Consider the set Z of integers defined aRb by $b = a^r$ for some +ve integer 'r'. Show that 'R' is a partial order on Z.

$\Rightarrow$ For POR

1) Reflexive :- $b = a^1$ for $1 \in Z^+$
$\Rightarrow aRa, \ \forall a \in Z$

2) Antisymmetric: Let aRb & bRa
$\Rightarrow b = a^r$ & $a = b^s$; $r, s \in Z^+$.
$a = (a^r)^s$
$\Rightarrow a = a^{rs}$

Three conditions 1) $rs = 1$, $r = 1$ & $s = 1 \Rightarrow b = a$ & $a = b$

2) $a = 1$, $\Rightarrow b = 1^r = 1 = a$ $\Rightarrow \boxed{b = a}$

3) $a = -1$ $\Rightarrow b = (-1)^r = -1 = a$. $\Rightarrow \boxed{b = a}$ $\boxed{b \neq 1}$ $a = b^s$ $(-1) = b^s$. $\Rightarrow \underline{b \neq 1}$

3) Transitive :- $aRb \Rightarrow b = a^r$
$bRc \Rightarrow c = b^s$.
$\Rightarrow c = a^{rs}. \Rightarrow c = a^p$, $p \in Z^+$.
$\underline{aRc}$