Matt Mazur

Home
About
Archives
Contact
Projects

About

Hey there! I'm the founder of Preceden, a web-based timeline maker, and data wrangler at Help Scout, a company that makes customer support tools. I also built Lean Domain Search and many other software products over the years.

A Step by Step Backpropagation Example

Background

Backpropagation is a common method for training a neural network. There is no shortage of papers online that attempt to explain how backpropagation works, but few that include an example with actual numbers. This post is my attempt to explain how it works with a concrete example that folks can compare their own calculations to in order to ensure they understand backpropagation correctly.

Backpropagation in Python

You can play around with a Python script that I wrote that implements the backpropagation algorithm in this Github repo.

Backpropagation Visualization

For an interactive visualization showing a neural network as it learns, check out my Neural Network visualization.

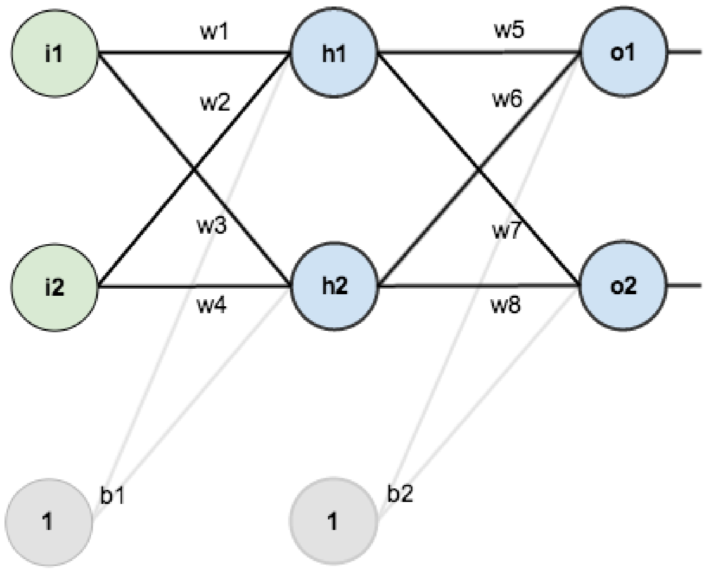
Additional Resources

If you find this tutorial useful and want to continue learning about neural networks, machine learning, and deep learning, I highly recommend checking out Adrian Rosebrock's new book, Deep Learning for Computer Vision with Python. I really enjoyed the book and will have a full review up soon.

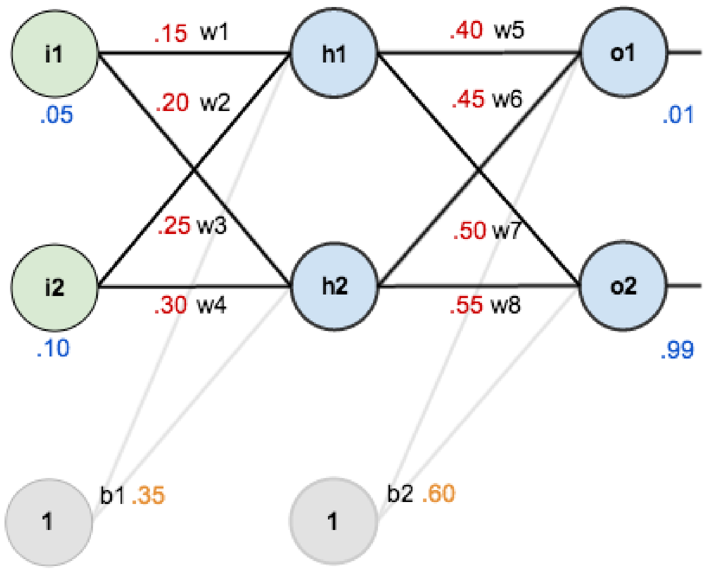
Overview

For this tutorial, we're going to use a neural network with two inputs, two hidden neurons, two output neurons. Additionally, the hidden and output neurons will include a bias.

Here's the basic structure:



In order to have some numbers to work with, here are the initial weights, the biases, and training inputs/outputs:



The goal of backpropagation is to optimize the weights so that the neural network can learn how to correctly map arbitrary inputs to outputs.

For the rest of this tutorial we're going to work with a single training set: given inputs 0.05 and 0.10, we want the neural network to output 0.01 and 0.99.

The Forward Pass

To begin, lets see what the neural network currently predicts given the weights and biases above and inputs of 0.05 and 0.10. To do this we'll feed those inputs forward though the network.

We figure out the *total net input* to each hidden layer neuron, *squash* the total net input using an *activation function* (here we use the *logistic function*), then repeat the process with the output layer neurons.

> Total net input is also referred to as just *net input* by some sources.

Here's how we calculate the total net input for h_1:

$$net_{h1} = w_1 * i_1 + w_2 * i_2 + b_1 * 1$$

$$net_{h1} = 0.15 * 0.05 + 0.2 * 0.1 + 0.35 * 1 = 0.3775$$

We then squash it using the logistic function to get the output of h_1:

$$out_{h1} = \frac{1}{1+e^{-net_{h1}}} = \frac{1}{1+e^{-0.3775}} = 0.593269992$$

Carrying out the same process for h_2 we get:

$$out_{h2} = 0.596884378$$

We repeat this process for the output layer neurons, using the output from the hidden layer neurons as inputs.

Here's the output for o_1:

$$net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$$

$$net_{o1} = 0.4 * 0.593269992 + 0.45 * 0.596884378 + 0.6 * 1 = 1.105905967$$

$$out_{o1} = \frac{1}{1+e^{-net_{o1}}} = \frac{1}{1+e^{-1.105905967}} = 0.75136507$$

And carrying out the same process for o_2 we get:

$$out_{o2} = 0.772928465$$

Calculating the Total Error

We can now calculate the error for each output neuron using the squared error function and sum them to get the total error:

$$E_{total} = \sum \frac{1}{2}(target - output)^2$$

> Some sources refer to the target as the *ideal* and the output as the *actual*.

> The $\frac{1}{2}$ is included so that exponent is cancelled when we differentiate later on. The result is eventually multiplied by a learning rate anyway so it doesn't matter that we introduce a constant here [1].

For example, the target output for o_1 is 0.01 but the neural network output 0.75136507, therefore its error is:

$$E_{o1} = \frac{1}{2}(target_{o1} - out_{o1})^2 = \frac{1}{2}(0.01 - 0.75136507)^2 = 0.274811083$$

Repeating this process for o_2 (remembering that the target is 0.99) we get:

$$E_{o2} = 0.023560026$$

The total error for the neural network is the sum of these errors:

$$E_{total} = E_{o1} + E_{o2} = 0.274811083 + 0.023560026 = 0.298371109$$

The Backwards Pass

Our goal with backpropagation is to update each of the weights in the network so that they cause the actual output to be closer the target output, thereby minimizing the error for each output neuron and the network as a whole.

Output Layer

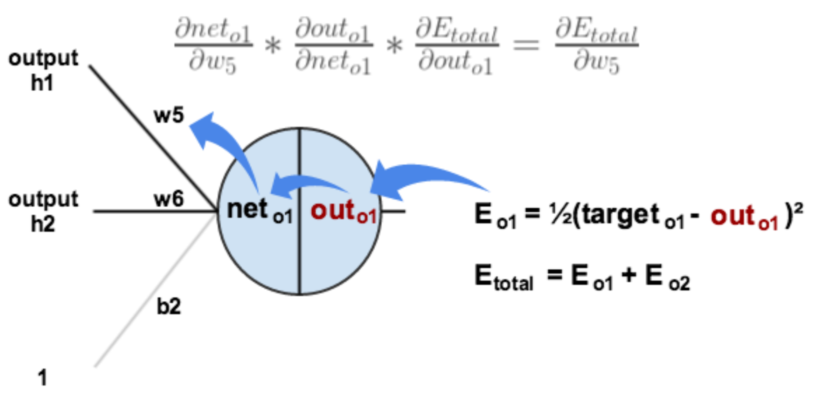
Consider w_5. We want to know how much a change in w_5 affects the total error, aka $\frac{\partial E_{total}}{\partial w_5}$.

> $\frac{\partial E_{total}}{\partial w_5}$ is read as "the partial derivative of E_{total} with respect to w_5". You can also say "the gradient with respect to w_5".

By applying the chain rule we know that:

$$\frac{\partial E_{total}}{\partial w_5} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial w_5}$$

Visually, here's what we're doing:



$\frac{\partial net_{o1}}{\partial w_5} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial E_{total}}{\partial out_{o1}} = \frac{\partial E_{total}}{\partial w_5}$
output h1
w5
output h2
w6
b2
1
net o1
out o1
E o1 = ½(target o1 - out o1)²
E total = E o1 + E o2

We need to figure out each piece in this equation.

First, how much does the total error change with respect to the output?

$$E_{total} = \frac{1}{2}(target_{o1} - out_{o1})^2 + \frac{1}{2}(target_{o2} - out_{o2})^2$$

$$\frac{\partial E_{total}}{\partial out_{o1}} = 2 * \frac{1}{2}(target_{o1} - out_{o1})^{2-1} * -1 + 0$$

$$\frac{\partial E_{total}}{\partial out_{o1}} = -(target_{o1} - out_{o1}) = -(0.01 - 0.75136507) = 0.74136507$$

> $-(target - out)$ is sometimes expressed as $out - target$

> When we take the partial derivative of the total error with respect to out_{o1}, the quantity $\frac{1}{2}(target_{o2} - out_{o2})^2$ becomes zero because out_{o1} does not affect it which means we're taking the derivative of a constant which is zero.

Next, how much does the output of o_1 change with respect to its total net input?

The partial derivative of the logistic function is the output multiplied by 1 minus the output:

$$out_{o1} = \frac{1}{1+e^{-net_{o1}}}$$

$$\frac{\partial out_{o1}}{\partial net_{o1}} = out_{o1}(1 - out_{o1}) = 0.75136507(1 - 0.75136507) = 0.186815602$$

Finally, how much does the total net input of $o1$ change with respect to w_5?

$$net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$$

$$\frac{\partial net_{o1}}{\partial w_5} = 1 * out_{h1} * w_5^{(1-1)} + 0 + 0 = out_{h1} = 0.593269992$$

Putting it all together:

$$\frac{\partial E_{total}}{\partial w_5} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial w_5}$$

$$\frac{\partial E_{total}}{\partial w_5} = 0.74136507 * 0.186815602 * 0.593269992 = 0.082167041$$

> You'll often see this calculation combined in the form of the delta rule:
>
> $$\frac{\partial E_{total}}{\partial w_5} = -(target_{o1} - out_{o1}) * out_{o1}(1 - out_{o1}) * out_{h1}$$
>
> Alternatively, we have $\frac{\partial E_{total}}{\partial out_{o1}}$ and $\frac{\partial out_{o1}}{\partial net_{o1}}$ which can be written as $\frac{\partial E_{total}}{\partial net_{o1}}$, aka δ_{o1} (the Greek letter delta) aka the *node delta*. We can use this to rewrite the calculation above:
>
> $$\delta_{o1} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} = \frac{\partial E_{total}}{\partial net_{o1}}$$
>
> $$\delta_{o1} = -(target_{o1} - out_{o1}) * out_{o1}(1 - out_{o1})$$

Therefore:

$$\frac{\partial E_{total}}{\partial w_5} = \delta_{o1} out_{h1}$$

Some sources extract the negative sign from δ so it would be written as:

$$\frac{\partial E_{total}}{\partial w_5} = -\delta_{o1} out_{h1}$$

To decrease the error, we then subtract this value from the current weight (optionally multiplied by some learning rate, eta, which we'll set to 0.5):

$$w_5^+ = w_5 - \eta * \frac{\partial E_{total}}{\partial w_5} = 0.4 - 0.5 * 0.082167041 = 0.35891648$$

Some sources use α (alpha) to represent the learning rate, others use η (eta), and others even use ϵ (epsilon).

We can repeat this process to get the new weights w_6, w_7, and w_8:

$$w_6^+ = 0.408666186$$

$$w_7^+ = 0.511301270$$

$$w_8^+ = 0.561370121$$

We perform the actual updates in the neural network *after* we have the new weights leading into the hidden layer neurons (ie, we use the original weights, not the updated weights, when we continue the backpropagation algorithm below).

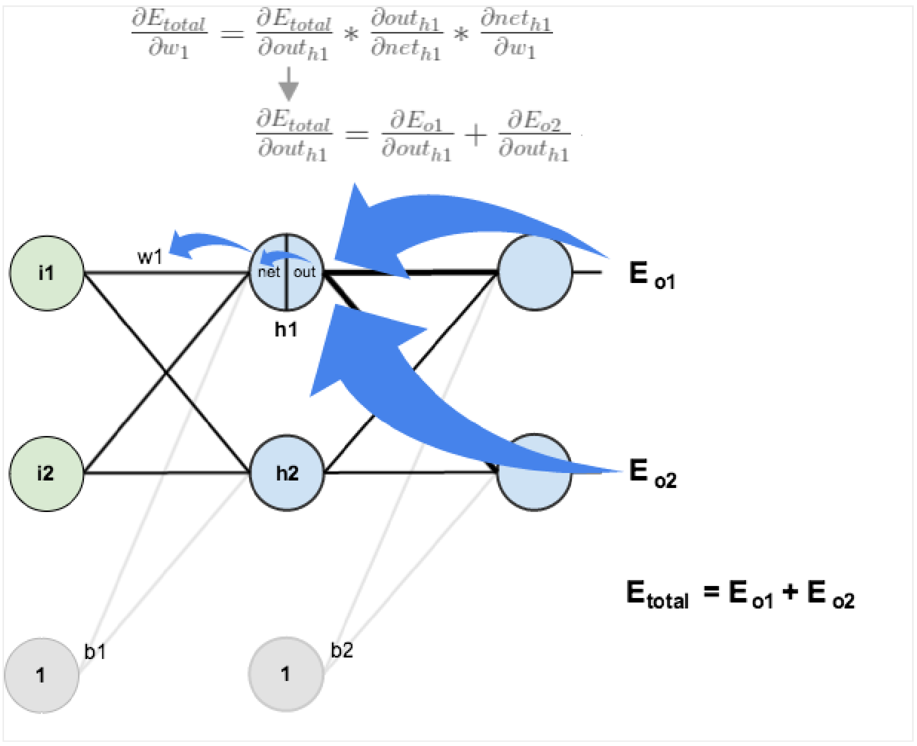
Hidden Layer

Next, we'll continue the backwards pass by calculating new values for w_1, w_2, w_3, and w_4.

Big picture, here's what we need to figure out:

$$\frac{\partial E_{total}}{\partial w_1} = \frac{\partial E_{total}}{\partial out_{h1}} * \frac{\partial out_{h1}}{\partial net_{h1}} * \frac{\partial net_{h1}}{\partial w_1}$$

Visually:



$\frac{\partial E_{total}}{\partial w_1} = \frac{\partial E_{total}}{\partial out_{h1}} * \frac{\partial out_{h1}}{\partial net_{h1}} * \frac{\partial net_{h1}}{\partial w_1}$
$\frac{\partial E_{total}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial out_{h1}} + \frac{\partial E_{o2}}{\partial out_{h1}}$
i1
w1
net
out
h1
i2
h2
E o1
E o2
E total = E o1 + E o2
1
b1
1
b2

We're going to use a similar process as we did for the output layer, but slightly different to account for the fact that the output of each hidden layer neuron contributes to the output (and therefore error) of multiple output neurons. We know that out_{h1} affects both out_{o1} and out_{o2} therefore the $\frac{\partial E_{total}}{\partial out_{h1}}$ needs to take into consideration its effect on the both output neurons:

$$\frac{\partial E_{total}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial out_{h1}} + \frac{\partial E_{o2}}{\partial out_{h1}}$$

Starting with $\frac{\partial E_{o1}}{\partial out_{h1}}$:

$$\frac{\partial E_{o1}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial out_{h1}}$$

We can calculate $\frac{\partial E_{o1}}{\partial net_{o1}}$ using values we calculated earlier:

$$\frac{\partial E_{o1}}{\partial net_{o1}} = \frac{\partial E_{o1}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} = 0.74136507 * 0.186815602 = 0.138498562$$

And $\frac{\partial net_{o1}}{\partial out_{h1}}$ is equal to w_5:

$$net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$$

$$\frac{\partial net_{o1}}{\partial out_{h1}} = w_5 = 0.40$$

Plugging them in:

$$\frac{\partial E_{o1}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial out_{h1}} = 0.138498562 * 0.40 = 0.055399425$$

Following the same process for $\frac{\partial E_{o2}}{\partial out_{h1}}$, we get:

$$\frac{\partial E_{o2}}{\partial out_{h1}} = -0.019049119$$

Therefore:

$$\frac{\partial E_{total}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial out_{h1}} + \frac{\partial E_{o2}}{\partial out_{h1}} = 0.055399425 + -0.019049119 = 0.036350306$$

Now that we have $\frac{\partial E_{total}}{\partial out_{h1}}$, we need to figure out $\frac{\partial out_{h1}}{\partial net_{h1}}$ and then $\frac{\partial net_{h1}}{\partial w}$ for each weight:

$$out_{h1} = \frac{1}{1+e^{-net_{h1}}}$$

$$\frac{\partial out_{h1}}{\partial net_{h1}} = out_{h1}(1 - out_{h1}) = 0.59326999(1 - 0.59326999) = 0.241300709$$

We calculate the partial derivative of the total net input to h_1 with respect to w_1 the same as we did for the output neuron:

$$net_{h1} = w_1 * i_1 + w_3 * i_2 + b_1 * 1$$

$$\frac{\partial net_{h1}}{\partial w_1} = i_1 = 0.05$$

Putting it all together:

$$\frac{\partial E_{total}}{\partial w_1} = \frac{\partial E_{total}}{\partial out_{h1}} * \frac{\partial out_{h1}}{\partial net_{h1}} * \frac{\partial net_{h1}}{\partial w_1}$$

$$\frac{\partial E_{total}}{\partial w_1} = 0.036350306 * 0.241300709 * 0.05 = 0.000438568$$

You might also see this written as:

$$\frac{\partial E_{total}}{\partial w_1} = (\sum\limits_{o}\frac{\partial E_{total}}{\partial out_{o}} * \frac{\partial out_{o}}{\partial net_{o}} * \frac{\partial net_{o}}{\partial out_{h1}}) * \frac{\partial out_{h1}}{\partial net_{h1}} * \frac{\partial net_{h1}}{\partial w_1}$$

$$\frac{\partial E_{total}}{\partial w_1} = (\sum\limits_{o}\delta_{o} * w_{ho}) * out_{h1}(1 - out_{h1}) * i_1$$

$$\frac{\partial E_{total}}{\partial w_1} = \delta_{h1}i_1$$

We can now update w_1:

$$w_1^{+} = w_1 - \eta * \frac{\partial E_{total}}{\partial w_1} = 0.15 - 0.5 * 0.000438568 = 0.149780716$$

Repeating this for w_2, w_3, and w_4

$$w_2^{+} = 0.19956143$$

$$w_3^{+} = 0.24975114$$

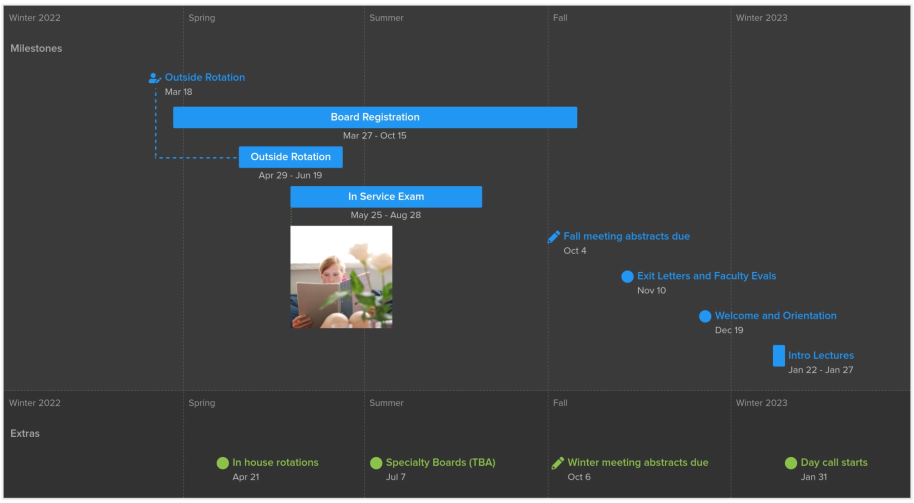
$$w_4^{+} = 0.29950229$$

Finally, we've updated all of our weights! When we fed forward the 0.05 and 0.1 inputs originally, the error on the network was 0.298371109. After this first round of backpropagation, the total error is now down to 0.291027924. It might not seem like much, but after repeating this process 10,000 times, for example, the error plummets to 0.0000351085. At this point, when we feed forward 0.05 and 0.1, the two outputs neurons generate 0.015912196 (vs 0.01 target) and 0.984065734 (vs 0.99 target).

If you've made it this far and found any errors in any of the above or can think of any ways to make it clearer for future readers, don't hesitate to drop me a note. Thanks!

And while I have you…

In addition to dabbling in data science, I run Preceden timeline maker, the best timeline maker software on the web. If you ever need to create a high level timeline or roadmap to get organized or align your team, Preceden is a great option. For example, here's an example graduate education timeline:



Posted on March 17, 2015 by Mazur. This entry was posted in Machine Learning and tagged ai, backpropagation, machine learning, neural networks. Bookmark the permalink.

← Introducing ABTestCalculator.com, an Open Source A/B Test Significance Calculator

TetriNET Bot Source Code Published on Github →

1,019 thoughts on "A Step by Step Backpropagation Example"

← Older Comments

Philippe Perez

— September 24, 2021 at 12:47 pm

I think the formuale below is wrong. You should apply the multivariable version of the chaine rule instead.

\frac{\partial E_{total}}{\partial w_{5}} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial w_{5}}

Reply

Ingo Dahn
— October 5, 2021 at 1:29 pm

Hi, it seems your network visualization is giving an error. numInputs not defined.
Please, let me know when it is working, I'd like to check whether I can reference or include it into my materials.
Best
Ingo Dahn

Reply

com
— December 4, 2021 at 5:25 pm

I'm assuming only one weight on the bias is just for simplicity?

Reply

NotSoUsefulUser
— February 19, 2022 at 8:18 am

Although this is a good post. I don't think the writer understand about bias, that's why there's no concrete explanation about how to update bias in this post.

Reply

almstrand
— March 15, 2022 at 11:45 am

The author likely assigned the same bias for the hidden and output neurons for simplicity. The article does not show the step of updating the bias weights, presumably as it is the exact same process as is used to update other weights (it is actually simpler, since the bias node activation is always 1.)

Reply

Ismail Atan
— April 8, 2022 at 10:30 am

Can give someone one example for updating one bias? I don't know, how i can update one bias-

Reply

demastersband
— April 20, 2022 at 8:07 pm

I like how you put this, will help for my exam tomorrow

Reply

Ping! - xmioviettel.net

Yoshiaki Fujita

— December 12, 2021 at 11:51 pm

Thanks to such the nice explanation!
I implement my first neural network by referring the post, thanks.

Reply

Justin De Witt

— December 16, 2021 at 1:06 pm

hello!

If there are multiple hidden layers, say two, three, or even more; How do you continue the back-propagation? Do you always take it two layers at a time?

If we are dealing with the leftmost hidden layer, do we need to track how changes to weights in this layer effect the following multiple hidden layers and then finally the output layer (working out error)

Tracing all routes that changes in a far left weight effect error seems to “blow up” very quickly so to speak.

Can anyone point me towards a worked example with at least 1 more hidden layer?

Reply

Albrecht Ehlert

— December 20, 2021 at 11:48 am

Hello Justin,
the back-propagation works also with multiple hidden layers.
You will find a good visualization for neural networks:
http://playground.tensorflow.org/
Sincerly
Albrecht

Reply

Ping! Machine Learning Roadmap 2022 - Codelivly

DBM

— December 29, 2021 at 6:49 am

Thank you for your detailed explanation.
I wonder how you set the outputs to be 0.01 and 0.99 ?
It is arbitrary chosen? Does the amount of the inputs and their vaules impact of the decesion how to set the outputs?

Reply

LL

— January 25, 2022 at 5:12 am

This is multi-label classification, correct?
Otherwise I would be wondering why we don't either use a softmax in the last layer? Also our outputs don't sum to 1.

Thank you and kind regards

Reply

Ping! How to calculate the total error from an arbitrary hidden layer in a neural network back propogation? - PhotoLens

Ager

— February 8, 2022 at 3:00 am

Hmm, in my case, after first round of backpropagation the total error is 0.291027774 instead 0.291027924. And after repeating 10,000 times, the total error is 0.0000351019 instead 0.0000351085. The rest of the calculations matched perfectly

Reply

Ping! Comprendre le DeepLearning et les réseaux de neurones en 10 minutes ! – TUTO3D

Ping! 机器学习整理（神经网络） - 算法网

Eleven

— April 12, 2022 at 4:21 am

I'm totally stumped, I've been over this article like 100 times and scrolled through as many comments and I cannot figure out how back propagation would work with multiple hidden layers. Using the example (I'm using d instead of the swirly thing because I've seen other comments do that and idk how to type the other character):

dEtotal / dW1 = dEtotal / dOutputH1 * stuff I get
dEtotal / dOutputH1 = dEo1/dNeto1 + same for O2
dEo1/dNeto1 = dEo1 / dOutputo1 * dOutputo1 / dNeto1
dEo1 / dOutputo1 = -(targeto1 – outputo1)

I understand that for the output layer, but if you have a second hidden layer in place of the output layer then for some hn

dEhn / dOutputhn = -(targethn – outputhn)
What the hell is the target value? I've been pulling my hair out for like 4 hours and I cannot figure it out.

Reply

kunde

— May 25, 2022 at 1:41 pm

Hi, i noticed that the value of out_h2 is incorrect. since its a logistic function, the sum of the two output can never exceed one but should approximate one in the nearest whole number. i go the value as 0.34468163

Reply

Dancer/Programmer Helen
— June 1, 2022 at 2:39 am

OMG I'm a postgrad student and have learnt about CV,NLP which all covers NN but I'm still confusing with the math behind the learning. Thanks for your detailed formulas and calculations!!
Cheers,
Jiayu

Reply

test
— June 6, 2022 at 9:44 am

This is such a good, detailed exaplanation. Thank you very much, it helped a lot.

Reply

← Older Comments

Leave a Reply

Enter your comment here...

Blog at WordPress.com.