MT-104
Linear Algebra

National University of Computer and Emerging Sciences

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Systems of Linear Differential Equations

First order linear homogenous differential equation

First order differential Equation

$$x' = kx, \quad k \text{ is} \quad \text{a} \quad \text{constant.}$$

Solution: $x(t) = x_0 e^{kt}$.
First order differential system

$$\begin{aligned} x' &= 4x \\ y' &= 9y \end{aligned}$$

In Matrix form of above system can be written as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

- Matrix is diagonal
- System is uncoupled

First order linear system of differential equations

General first order linear system

$$x' = ax + by$$
$$y' = cx + dy$$

In Matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We can write the above system as

$$X' = AX,$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

- A is not a diagonal matrix.
- Can we diagonalize A?

Diagonalization

$\boxed{\textit{Aim}:}$

To solve the system $X' = AX$.

$\boxed{\textit{Challenges}}$

Matrix is not diagonal.

$\boxed{\textit{Possible Solution}}$

Transform the matrix into a diagonal matrix i.e., diagonalize it.

HOW?

We want to transform

$$X' = AX \xrightarrow{to} Y' = DY.$$

Diagonalization

As $PDP^{-1} = A$, so we can write

$$X' = AX = PDP^{-1}X$$

Pre multiplying by P^{-1} we get

$$P^{-1}X' = DP^{-1}X$$

Since, P is a constant matrix, so

$$\left(P^{-1}X\right)' = D\left(P^{-1}X\right).$$

Put $\left(P^{-1}X\right) = Y$ to get

$$\boxed{Y' = DY}.$$

Uncoupling system of differential equations

Summary

Coupled system of differential equation

$$X' = AX$$

can be transformed (uncoupled) to

$$Y' = DY$$

by using the transformation

$$X = PY.$$

Example

Find a solution to the system

$$\begin{aligned} x' &= x + 3y \\ y' &= 2x + 2y \end{aligned}$$

subject to initial conditions $x(0) = 0,\ y(0) = 5.$

Solution: In Matrix form, we can write it as

$$X' = AX,$$

where $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

We can uncoupled the system by using the transformation

$$X = PY.$$

Example

Eigenvalues:
Characteristic Equation:

$$\lambda^2 - 3\lambda - 4 = 0.$$

Eigenvalues are: $-1,\ 4.$
Corresponding eigenvectors are

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix},\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Hence,

$$P = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix},\ D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}.$$

By using the transformation $X = PY$, we get

$$Y' = DY,$$

where $Y = \begin{bmatrix} u \\ v \end{bmatrix}.$

Example

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Above equations can be written as

$$u' = -u$$
$$v' = 4v.$$

Solving, we get

$$u = c_1 e^{-t}, \; v = c_2 e^{4t}.$$

As $X = PY$, so, we can write

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

$$x = 3u + v$$
$$y = -2u + v.$$

Substituting values of u and v, we get

$$x = 3c_1 e^{-t} + c_2 e^{4t}$$
$$y = -2c_1 e^{-t} + c_2 e^{4t}.$$

Example

Since, $x(0) = 0$ and $y(0) = 5$, we get

$$\begin{aligned} 0 &= 3c_1 + c_2 \\ 5 &= -2c_1 + c_2. \end{aligned}$$

Solving, above system we get
$c_1 = -1, \ c_2 = 3.$
In matrix form we can the solution as

$$X = -x_1 e^{-t} + 3x_2 e^{4t}$$

where $x_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are the eigenvectors corresponding to eigenvalues -1 and 4 respectively.

Example

Find a solution to the system

$$r'(t) = w(t) - 12$$
$$w'(t) = -r(t) + 10$$

Solution:

$\boxed{\textit{Issue}:}$

Presence of -12 and 10.

$\boxed{\textit{How to resolve it}:}$

Put $w(t) - 12 = y(t)$ and $-r(t) + 10 = x(t)$, we get

$$-x'(t) = y(t)$$
$$y'(t) = x(t).$$

In Matrix form, we can write it as

$$X' = AX,$$

where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

By using the substitution $X = PY$ we get

$$Y' = DY$$

where $P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$, $D = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$.
So, solution is

$$u = c_1 e^{-it}$$
$$v = c_2 e^{it}$$

By using the relation $X = PY$, we get

$$x = c_1 e^{-it} + c_2 e^{it}$$
$$y = c_1 i e^{-it} - i c_2 e^{it}.$$

Solution of the system is

$$r(t) = 10 - c_1 e^{-it} - c_2 e^{it}$$
$$w(t) = 12 + c_1 i e^{-it} - i c_2 e^{it}$$

- In case of single linear differential equation, we have

$$x' = kx, \; k \text{ is a constant.}$$

Solution of the differential equation is

$$x = ce^{kt}.$$

- In case of system of coupled differential equations, we have

$$X' = AX, \; A \text{ is a constant matrix.}$$

Solution of the linear differential system should be

$$X = c\, e^{At}.$$

Exponential of a Matrix

Compute e^{Dt} where $D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$.

Since, $e^x = 1 + x + \frac{x^2}{2!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. so,

$$e^{Dt} = I + Dt + ... = \sum_{n=0}^{\infty} \frac{D^n t^n}{n!}$$

$$e^{Dt} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} 4^n t^n & 0 \\ 0 & t^n \end{bmatrix}$$

$$e^{Dt} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{4^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{1^n}{n!} \end{bmatrix}$$

$$e^{Dt} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{t} \end{bmatrix}$$

Example

Compute e^{At} where $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

For given matrix, we have

$$P = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}, \; D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}.$$

$$\begin{aligned}
e^{At} &= \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} \\
e^{At} &= \sum_{n=0}^{\infty} \frac{PD^n P^{-1}}{n!} \\
e^{At} &= P \sum_{n=0}^{\infty} \frac{D^n t^n}{n!} P^{-1} \\
e^{At} &= Pe^{Dt} P^{-1} \\
e^{At} &= \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{4t} \end{bmatrix} \left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \right)^{-1}.
\end{aligned}$$

Chapter # 6

Motivation

Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$ (Echelon Form, Reduced Echelon Form, Column Space, Null Space)
- Solve the matrix equation $Ax = \lambda x$ (Eigenvalues, eigenvectors and their applications)
- Almost solve the equation $Ax = b$ (We are going to study)

Idea 1: In the real world, data is imperfect.
Suppose you measure a data point x which you know for theoretical reasons must lie on a plane spanned by two vectors u and v. Due to measurement error, though, the measured x is not actually in Span$\{u, v\}$.
In other words, the equation $au + bv = x$ has no solution ($x \in$ Span of u and v).
What do you do?
The real value is probably the *closest* point to x on Span$\{u, v\}$.
Which point is that?

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Idea 1: In the real world, data is imperfect.
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Motivation

Idea 2: The cost of producing t books like this one is nearly linear, $b = C + Dt$, with editing and typesetting in C and then printing and binding in D. C is the set-up cost and D is the cost for each additional book. How to compute C and D? If there is no experimental error, then two measurements of b will determine the line $b = C + Dt$.
But if there is **error**, we must be prepared to "average" the experiments and find an optimal line. Since there are two unknowns C and D to be determined, we now project onto a two-dimensional subspace. A perfect experiment would give a perfect C and D:

$$
\begin{aligned}
C + Dt_1 &= b_1 \\
C + Dt_2 &= b_2 \\
&\cdots\cdots\cdots \\
C + Dt_m &= b_m :
\end{aligned}
$$

This is an overdetermined system, with m equations and only two unknowns. If errors are present, it will have no solution. The best solution $(\widehat{C}, \widehat{D})$ is the $\widehat{x}$ that minimizes the squared error E^2

The Dot Product

Purpose To find *length* and *angle* between two vectors, and in particular, to understand the notion of *orthogonality* (i.e. perpendicularity).

Definition

The **dot product** of two vectors x, y in R^n is

$$x \cdot y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = x^T y.$$

Example

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32.$$

Properties of the Dot Product

- $x \cdot x \geq 0$
- $x \cdot x = 0$ if and only if $x = 0$.
- $x \cdot y = y \cdot x$
- $(x + y) \cdot z = x \cdot z + y \cdot z$
- $(cx) \cdot y = c(x \cdot y)$

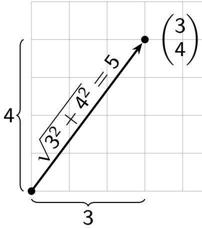
The Dot Product and Length

Definition

The **length** or **norm** of a vector x in R^n is

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

The Pythagorean theorem!



$$\left\| \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\| = \sqrt{3^2 + 4^2} = 5$$

Fact

If x is a vector and c is a scalar, then $\|cx\| = |c| \cdot \|x\|$.

The Dot Product and Distance

Definition

The **distance** between two points x, y in $\mathbb{R}^n$ is

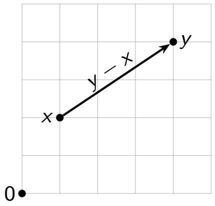
$$\text{dist}(x, y) = \|y - x\|.$$

This is just the length of the vector from x to y.

Example

Let $x = (1, 2)$ and $y = (4, 4)$. Then

$$\text{dist}(x, y) = \|y - x\| = \left\| \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\| = \sqrt{3^2 + 2^2} = \sqrt{13}.$$



Unit Vectors

Definition

A **unit vector** is a vector v with length $\|v\| = 1$.

Example

The unit coordinate vectors are unit vectors:

$$\|e_1\| = \left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\| = \sqrt{1^2 + 0^2 + 0^2} = 1$$

Definition

Let x be a nonzero vector in R^n. The **unit vector in the direction of** x is the vector $\dfrac{x}{\|x\|}$.

This is in fact a unit vector:

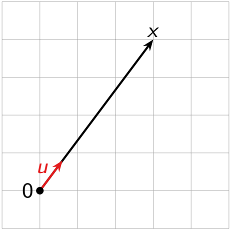
scalar

$$\left\| \frac{x}{\|x\|} \right\| = \frac{1}{\|x\|} \|x\| = 1.$$

Example

What is the unit vector in the direction of $x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$?

$$u = \frac{x}{\|x\|} = \frac{1}{\sqrt{3^2 + 4^2}} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$



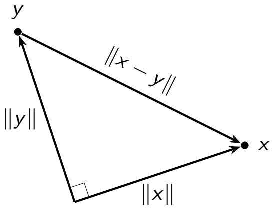
Orthogonality

Definition

Two vectors x, y are **orthogonal** or **perpendicular** if $x \cdot y = 0$.

[3]Why is this a good definition?

The Pythagorean theorem / law of cosines!



$$
\begin{aligned}
\text{x and y are perpendicular} &\iff \|x\|^2 + \|y\|^2 = \|x - y\|^2 \\
&\iff x \cdot x + y \cdot y = (x - y) \cdot (x - y) \\
&\iff x \cdot x + y \cdot y = x \cdot x + y \cdot y - 2x \cdot y \\
&\iff x \cdot y = 0
\end{aligned}
$$

Fact: $x \perp y \iff \|x - y\|^2 = \|x\|^2 + \|y\|^2$

Orthogonality

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Two vectors x, y are **orthogonal** or **perpendicular** if $x \cdot y = 0$.
Notation: $x \perp y$ means $x \cdot y = 0$.

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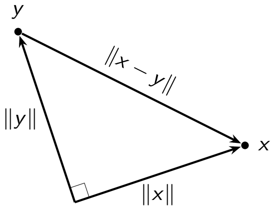

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&\iff x \cdot y = 0
\end{aligned}$$

Fact: $x \perp y \iff \|x-y\|^2 = \|x\|^2 + \|y\|^2$

Orthogonality

Definition

A set of vectors $\{x_1, x_2, ..., x_k\}$ in R^n is called an **orthogonal set** if $x_i \cdot x_j = 0$ whenever $i \neq j$ for $i, j = 1, 2, ..., k$.

Problem: Show that $\{x_1,\ x_2,\ x_3\}$ is an orthogonal set in R^3 if

$$x_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix},\ x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ x_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

$$\begin{aligned} x_1 \cdot x_2 &= 2(0) + 1(1) + (-1)(1) = 0 \\ x_2 \cdot x_3 &= 0(1) + 1(-1) + (1)(1) = 0 \\ x_1 \cdot x_3 &= 2(1) + 1(-1) + (-1)(1) = 0 \end{aligned}$$

Problem: Find *all* vectors orthogonal to $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

We have to find all vectors x such that $x \cdot v = 0$. This means solving the equation

$$0 = x \cdot v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = x_1 + x_2 - x_3.$$

The parametric form for the solution is $x_1 = -x_2 + x_3$, so the parametric vector form of the general solution is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

For instance, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \perp \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ because $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0.$

Orthogonality

Example

Problem: Find *all* vectors orthogonal to both $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Now we have to solve the system of two homogeneous equations

$$\begin{aligned} 0 &= x \cdot v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = x_1 + x_2 - x_3 \\ 0 &= x \cdot w = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = x_1 + x_2 + x_3. \end{aligned}$$

In matrix form:

The rows are v and w $\longrightarrow \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

The parametric vector form of the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Orthogonal Basis

Theorem

If $\{x_1, x_2, ..., x_k\}$ is an orthogonal set of nonzero vectors in R^n, then these vectors are linearly independent.

Suppose $\{u_1, u_2, \ldots, u_m\}$ is orthogonal. We need to show that the equation

$$c_1 u_1 + c_2 u_2 + \cdots + c_m u_m = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = c_m = 0$.

$$0 = u_1 \cdot (c_1 u_1 + c_2 u_2 + \cdots + c_m u_m) = c_1(u_1 \cdot u_1) + 0 + 0 + \cdots + 0.$$

Hence $c_1 = 0$. Similarly for the other c_i.

Definition

An **orthogonal basis** for a subspace W of R^n is a basis of W that is an orthogonal set.

Problem Show that $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ forms orthogonal basis of R^2.

Orthogonal Basis

Example

Problem Find an orthogonal basis for the subspace W of R^3 given by

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 2z = 0 \right\}.$$

We have already studied how to calculate the basis of W.

$$\begin{bmatrix} y - 2z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},$$

so, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ forms basis of W

They are not orthogonal.

Orthogonal Basis

We want to find a vector of W that is orthogonal to either $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

Let $w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a vector of W and is orthogonal to $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

As $w \in W$, so we can write

$$x - y + 2z = 0.$$

Orthogonality condition implies

$$-2x + z = 0.$$

Solving above two equations, we get

$$y = 5x, \ z = 2x.$$

So,

$$w = x \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}.$$

So, $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$ forms orthogonal basis of W.

Orthogonal Basis

Theorem

Let $\{x_1, x_2, ..., x_k\}$ be an orthogonal basis for a subspace W of R^n and let w be any vector in W. Then the unique scalars $c_1, c_2, ..., c_k$ such that

$$w = c_1 x_1 + c_2 x_2 + ... + c_k x_k$$

are given by

$$c_i = \frac{w \cdot v_i}{v_i \cdot v_i}, \quad i = 1, 2, ..., k.$$

Orthogonal Basis

Example

Problem Show that $B = \left\{ v_1 = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ forms basis of R^2 and write coordinate vector of $w = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ w.r.t B.

As $v_1 \cdot v_2 = (4).(1) + (2)(-2) = 0$, so, v_1 and v_2 are orthogonal and hence forms basis of R^2.
Now, for the coordinate vector

$$w = c_1 v_1 + c_2 v_2.$$

Taking dot product with v_1, we get

$$w \cdot v_1 = c_1 v_1 \cdot v_1$$
$$c_1 = \frac{w \cdot v_1}{v_1 \cdot v_1}$$

Similarly, $c_2 = \dfrac{w \cdot v_2}{v_2 \cdot v_2}$.
As $v_1 \cdot v_1 = 12,\ v_2 \cdot v_2 = 5,\ w \cdot v_1 = 10,\ w \cdot v_2 = -5$. So, $c_1 = \frac{5}{6},\ c_2 = -1$.
Hence, coordinate vector of w is

$$\begin{bmatrix} \frac{5}{6} \\ -1 \end{bmatrix}.$$