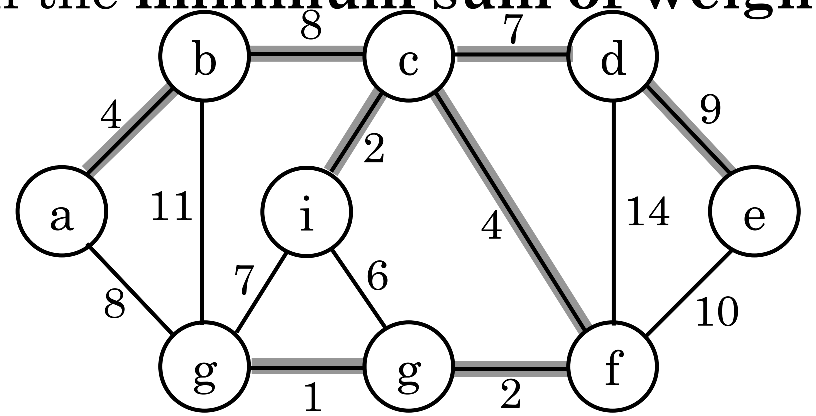
Graph Theory
Minimum Spanning Tree

Design and Analysis of Algorithms

Fall 2022

Minimum Spanning Trees

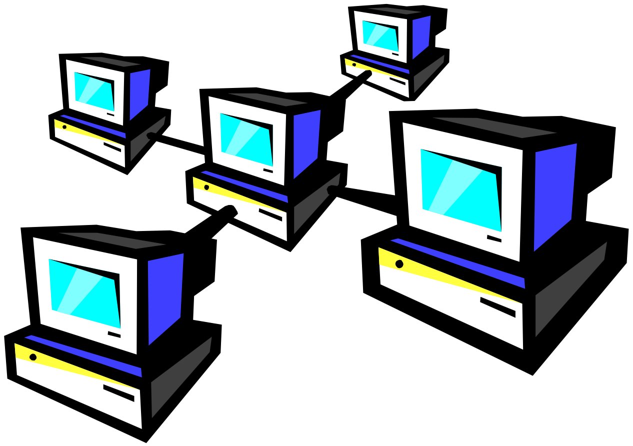
- Spanning Tree
  - A tree (i.e., connected, acyclic graph) which contains all the vertices of the graph
- Minimum Spanning Tree
  - Spanning tree with the **minimum sum of weights**

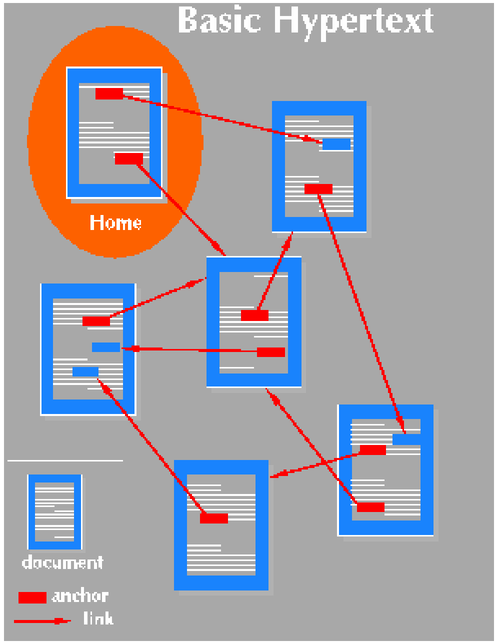


- Spanning forest
  - If a graph is not connected, then there is a spanning tree for each connected component of the graph

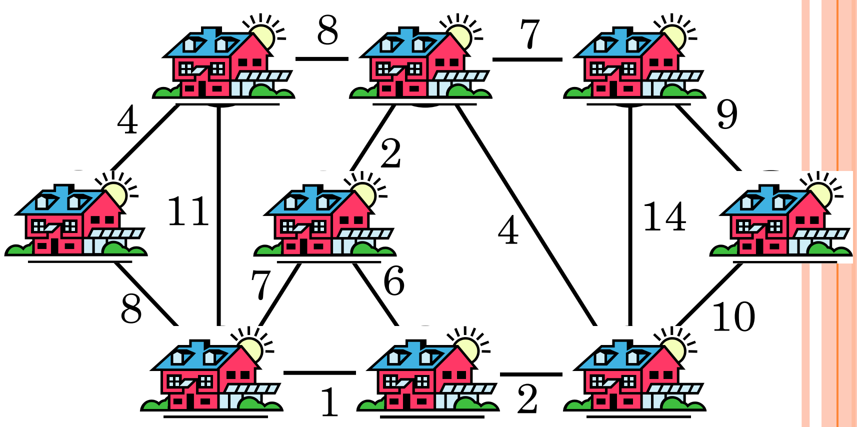
Applications of MST

- Find the least expensive way to connect a set of cities, terminals, computers, etc.





Example



Problem

- A town has a set of houses and a set of roads
- A road connects 2 and only 2 houses
- A road connecting houses u and v has a repair cost w(u, v)

Goal: Repair enough (and no more) roads such that:

1. Everyone stays connected

   i.e., can reach every house from all other houses

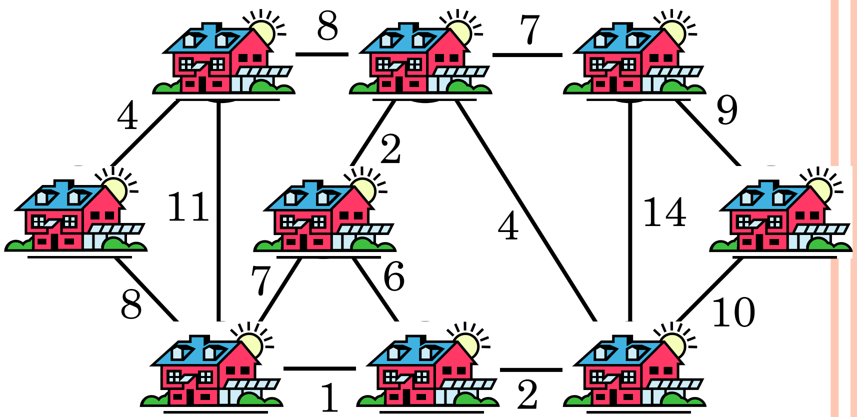
2. Total repair cost is minimum

MINIMUM SPANNING TREES

- A connected, undirected graph:
  - Vertices = houses, Edges = roads
- A **weight** $w(u, v)$ on each edge $(u, v) \in E$

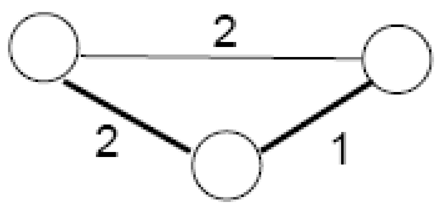
Find $T \subseteq E$ such that:

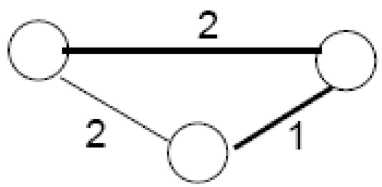
1. T connects all vertices
2. $w(T) = \Sigma_{(u,v)\in T}\, w(u, v)$ is minimized



Properties of Minimum Spanning Trees

- Minimum spanning tree is **not** unique





- MST has no cycles – see why:
  - We can take out an edge of a cycle, and still have the vertices connected while reducing the cost
- # of edges in a MST:
  - |V| - 1

Minimum spanning trees

- A ***spanning tree*** of an undirected connected graph is its connected acyclic subgraph (i.e., a tree) that contains all the vertices of the graph.
- If such a graph has weights assigned to its edges, a ***minimum spanning tree*** is its spanning tree of the smallest weight, where the ***weight*** of a tree is defined as the sum of the
- weights on all its edges.
- The ***minimum spanning tree problem*** is the problem of finding a minimum spanning tree for a given weighted connected graph.

MINIMUM SPANNING TREES

Input: A connected, undirected graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$.

- For simplicity, assume that all edge weights are distinct. (CLRS covers the general case.)

MINIMUM SPANNING TREES

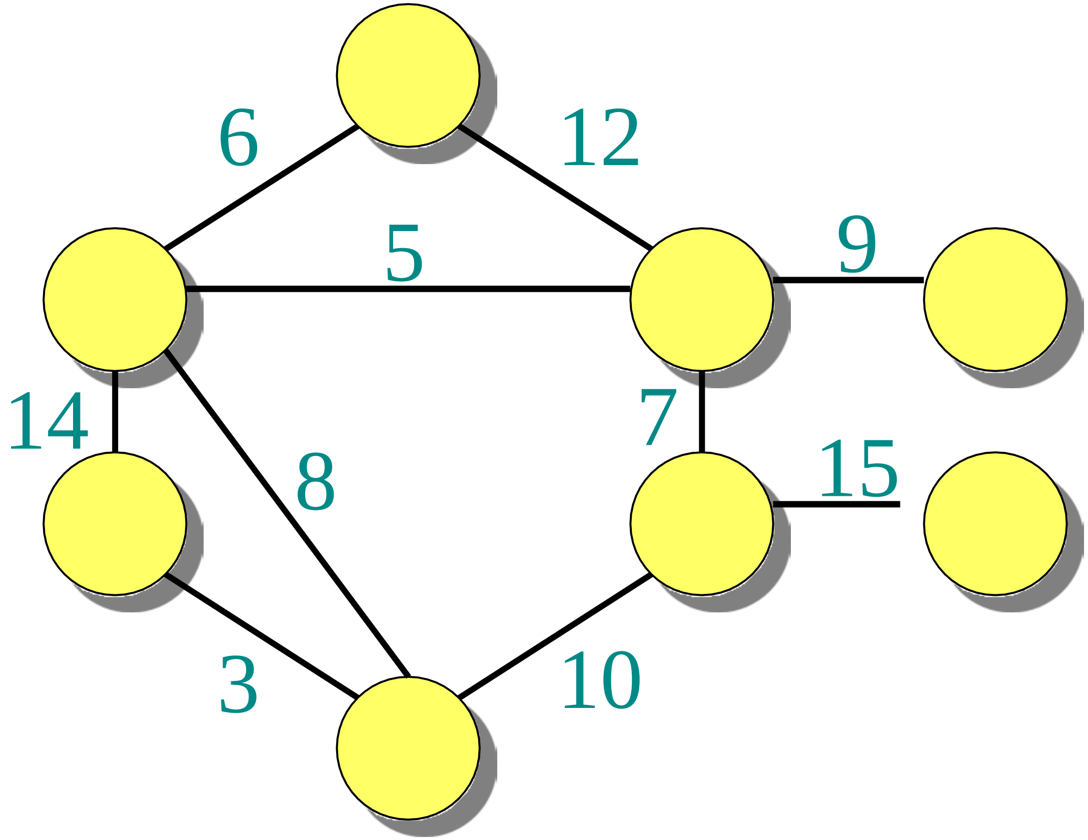
Input: A connected, undirected graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$.

- For simplicity, assume that all edge weights are distinct. (CLRS covers the general case.)

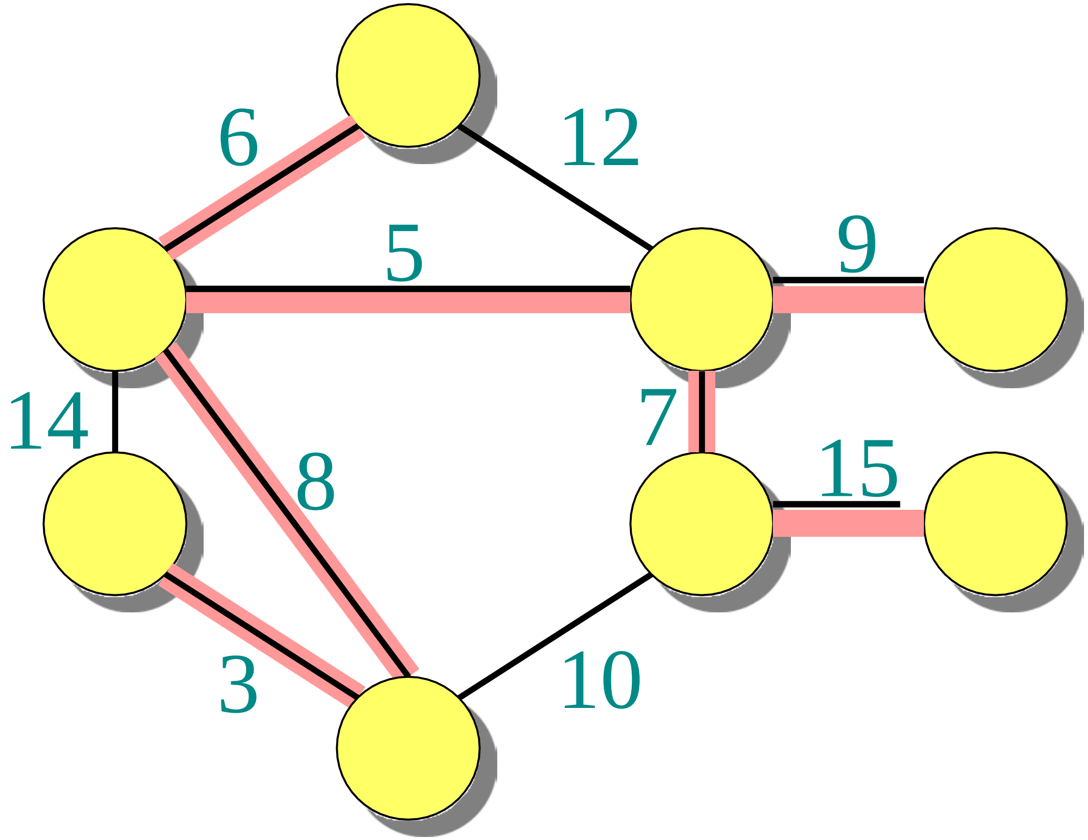
Output: A ***spanning tree*** T — a tree that connects all vertices — of minimum weight:

$$w(T) = \sum_{(u,v) \in T} w(u,v).$$

Example of MST



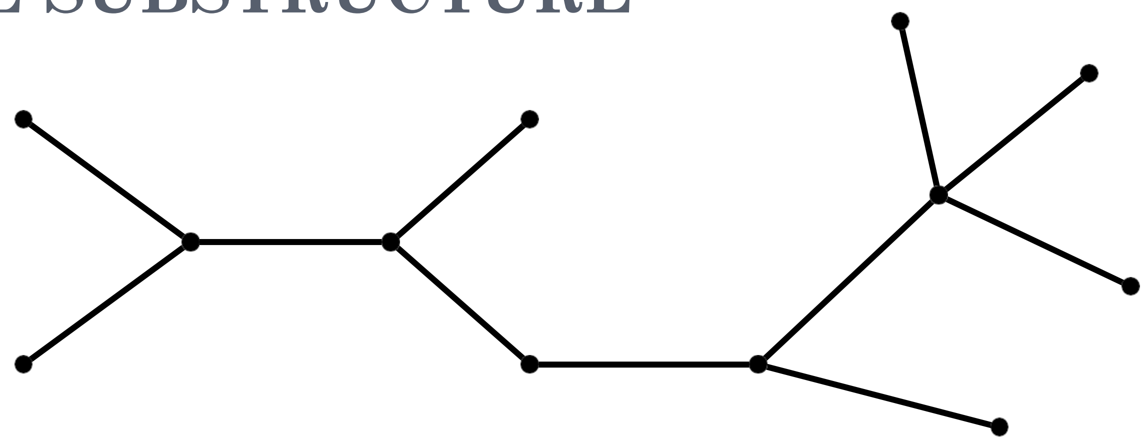
EXAMPLE OF MST



Optimal substructure

MST *T*:

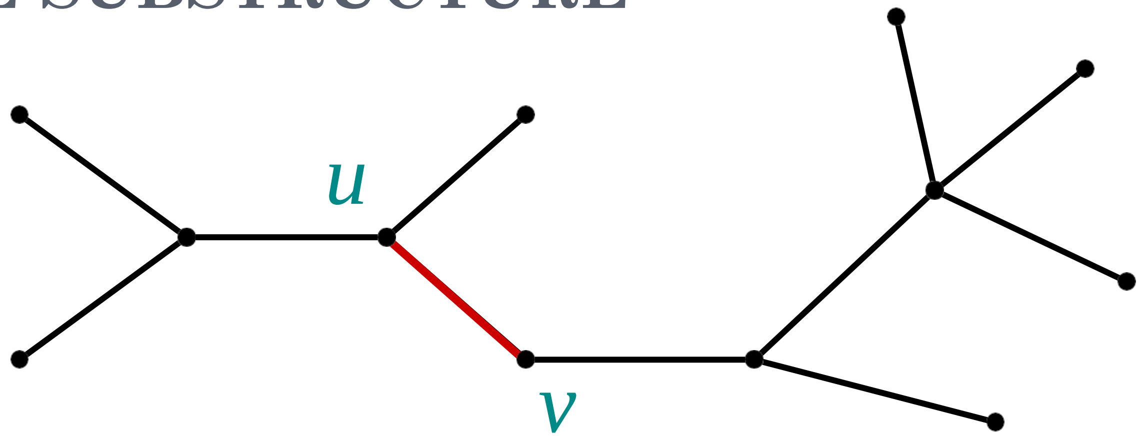
(Other edges of *G* are not shown.)



OPTIMAL SUBSTRUCTURE

MST T:

(Other edges of G are not shown.)



Remove any edge $(u, v) \in T$.

Optimal substructure

MST T:

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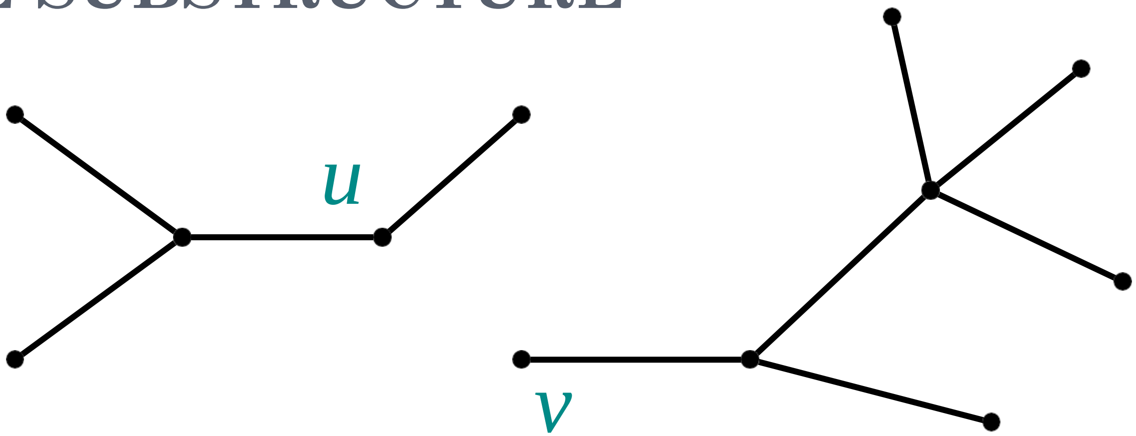

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OPTIMAL SUBSTRUCTURE

MST T:

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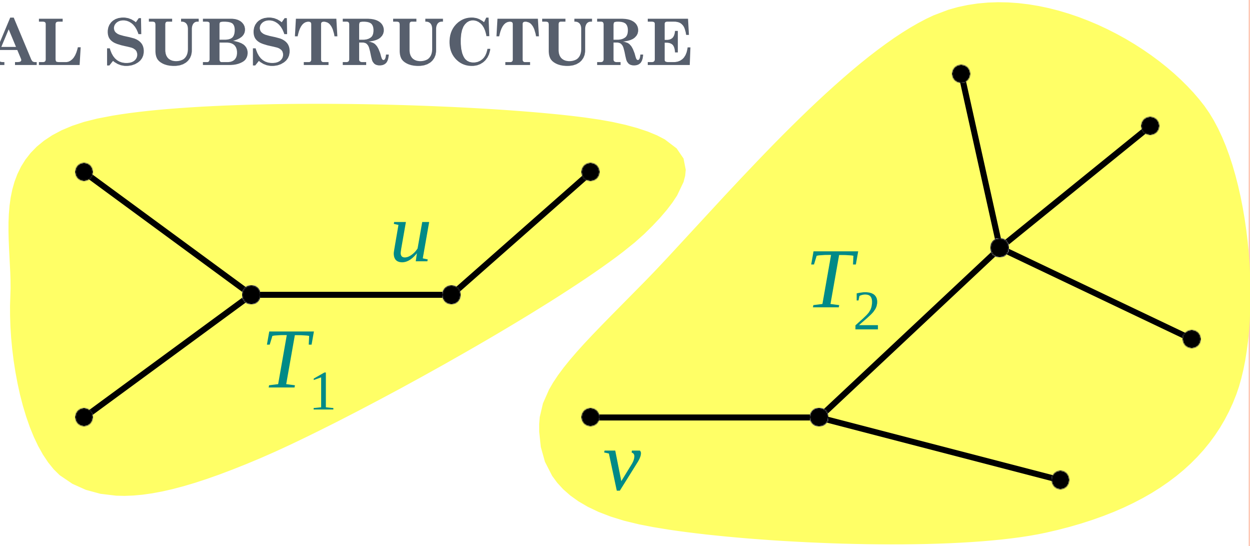


Remove any edge $(u, v) \in T$. Then, T is partitioned into two subtrees T_1 and T_2.

Hallmark for "greedy" algorithms

Greedy-choice property

A locally optimal choice is globally optimal.

```
GENERIC-MST(G, w)
A = ∅
while A does not form a spanning tree
    find an edge (u, v) that is safe for A
    A = A ∪ {(u, v)}
return A
```

- The generic method manages a set of edges A, maintaining the following loop invariant:
- **Prior to each iteration, A is a subset of some minimum spanning tree.**
- At each step, we determine an edge {u, v} that we can add to A without violating this invariant, in the sense that A U {u, v] is also a subset of a minimum spanning tree.
- We call such an edge a ***safe edge*** for A, since we can add it safely to A while maintaining the invariant

Hallmark for "greedy" algorithms

Greedy-choice property

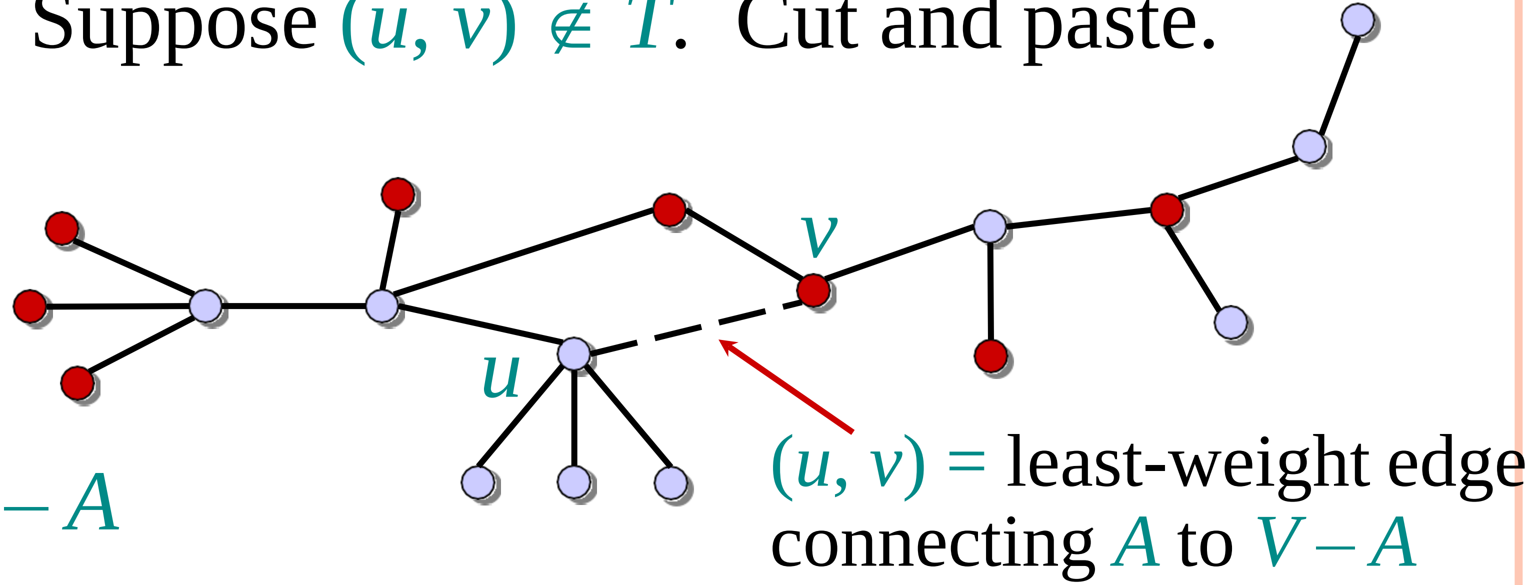
A locally optimal choice is globally optimal.

Theorem. Let T be the MST of $G = (V, E)$, and let $A \subseteq V$. Suppose that $(u, v) \in E$ is the least-weight edge connecting A to $V - A$. Then, $(u, v) \in T$.

Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.

T:



● $\in A$
● $\in V - A$

Proof of theorem

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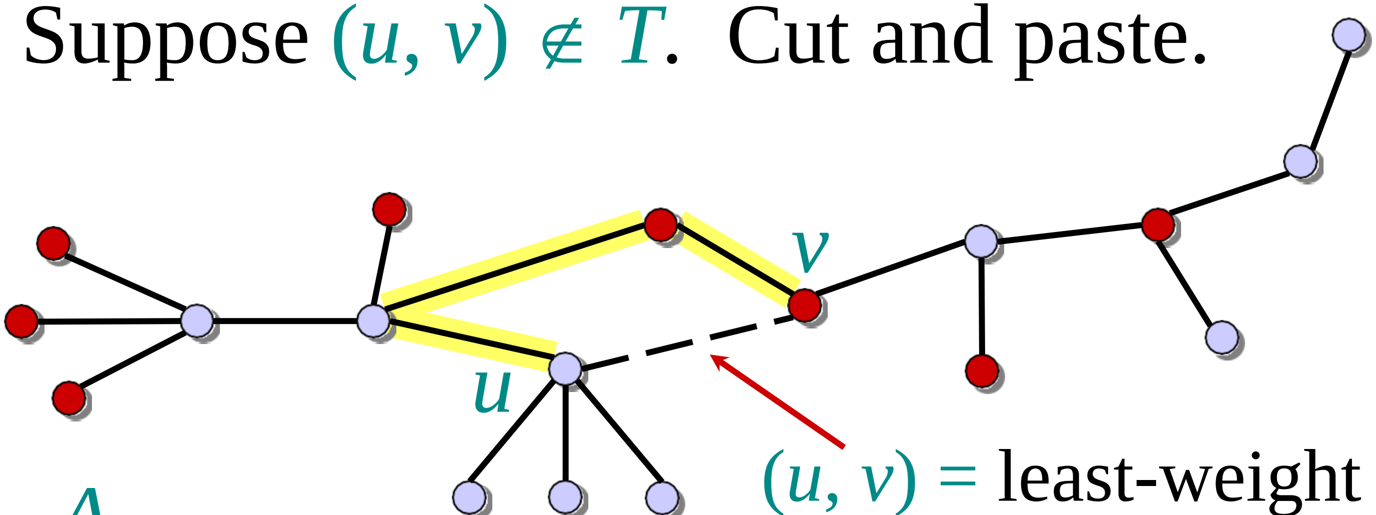

- $\in A$
- $\in V - A$

(u, v) = least-weight edge connecting A to $V - A$

Consider the unique simple path from u to v in T.

Proof of theorem

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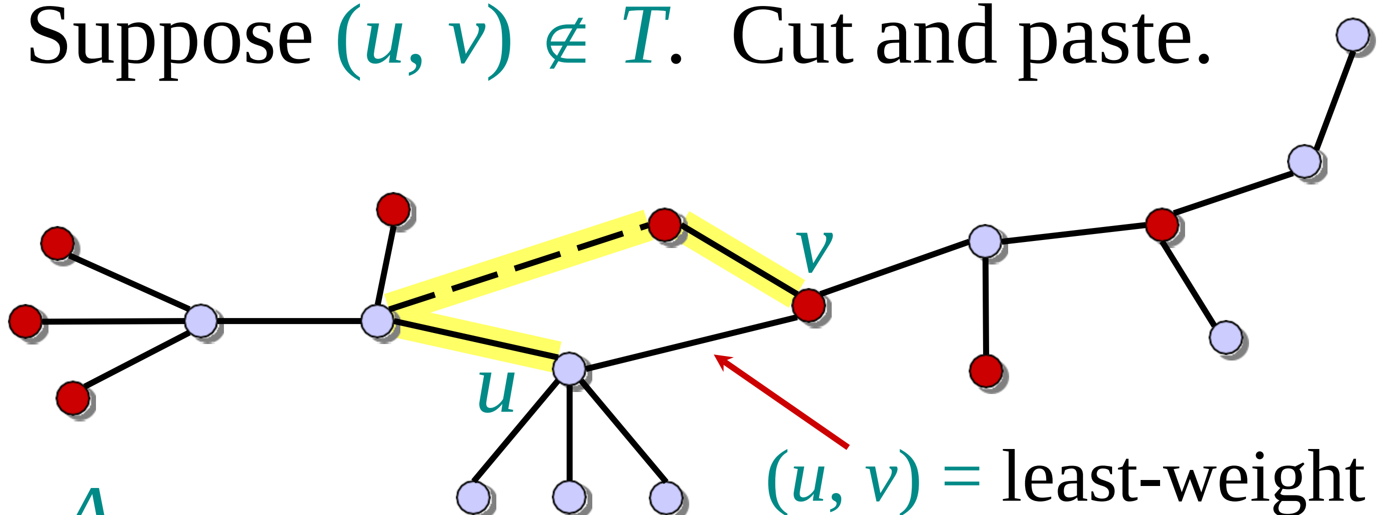

$\in A$

$\in V - A$

(u, v) = least-weight edge connecting A to $V - A$

Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V - A$.

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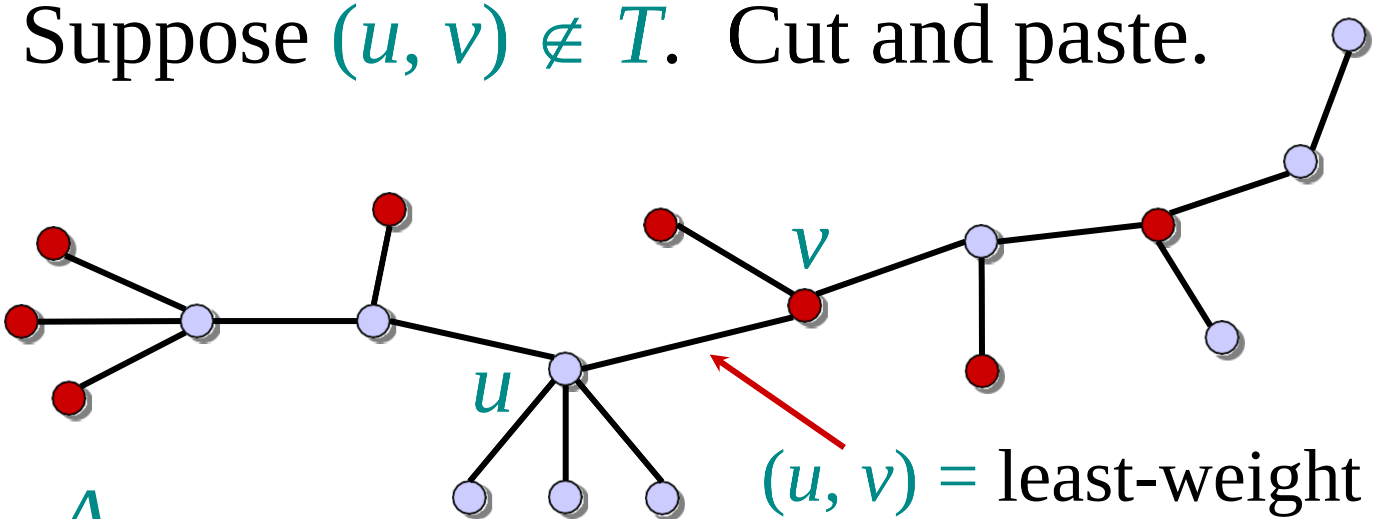


● $\in A$

● $\in V - A$

Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V - A$.

A lighter-weight spanning tree than T results.

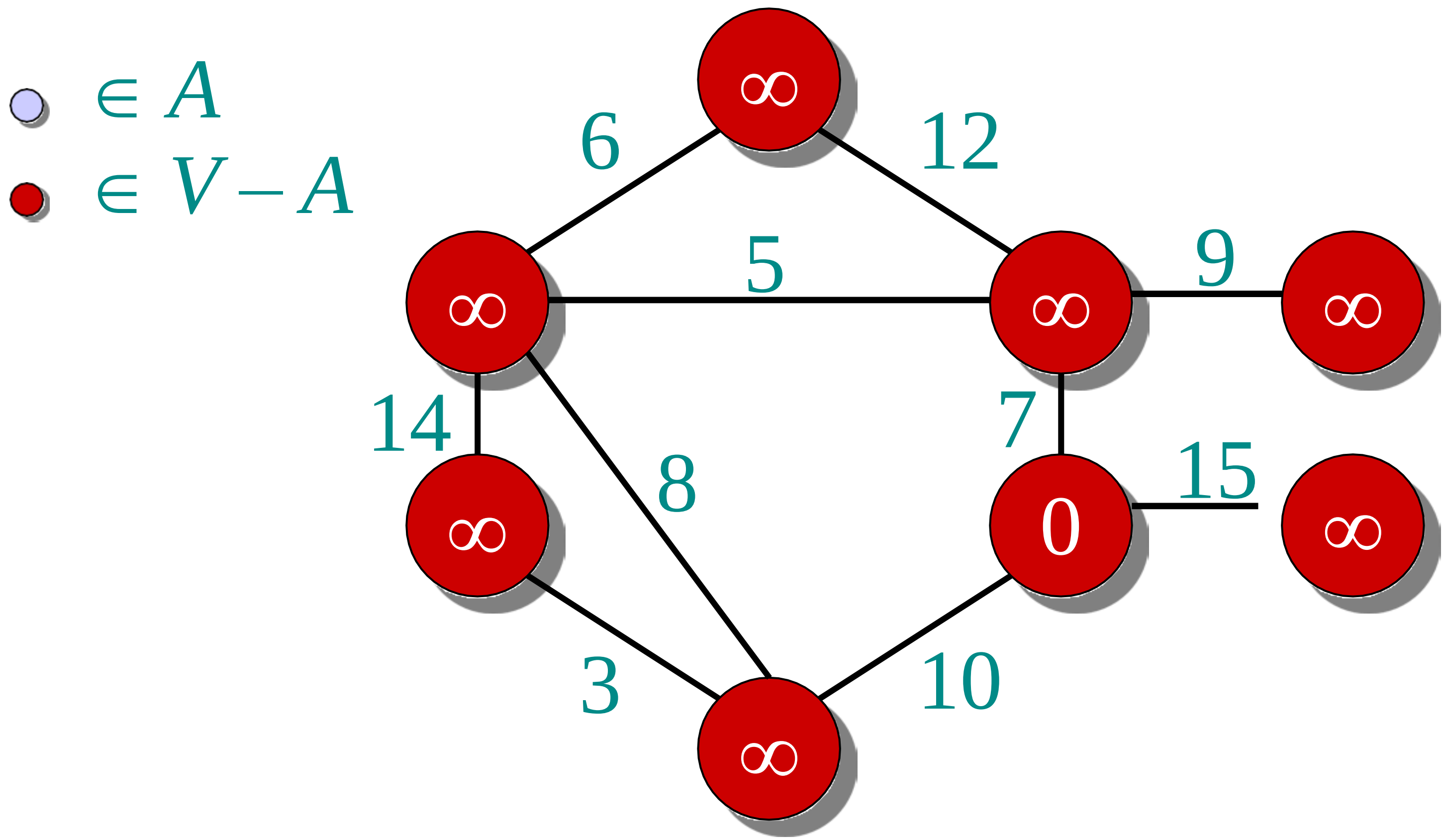
PRIM'S ALGORITHM

IDEA: Maintain $V - A$ as a priority queue Q. Key each vertex in Q with the weight of the least-weight edge connecting it to a vertex in A.

$Q \leftarrow V$
$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$
while $Q \neq \emptyset$
    do $u \leftarrow$ EXTRACT-MIN(Q)
        for each $v \in Adj[u]$
            do if $v \in Q$ and $w(u, v) < key[v]$
                then $key[v] \leftarrow w(u, v)$ ⊳ DECREASE-KEY
                    $\pi[v] \leftarrow u$

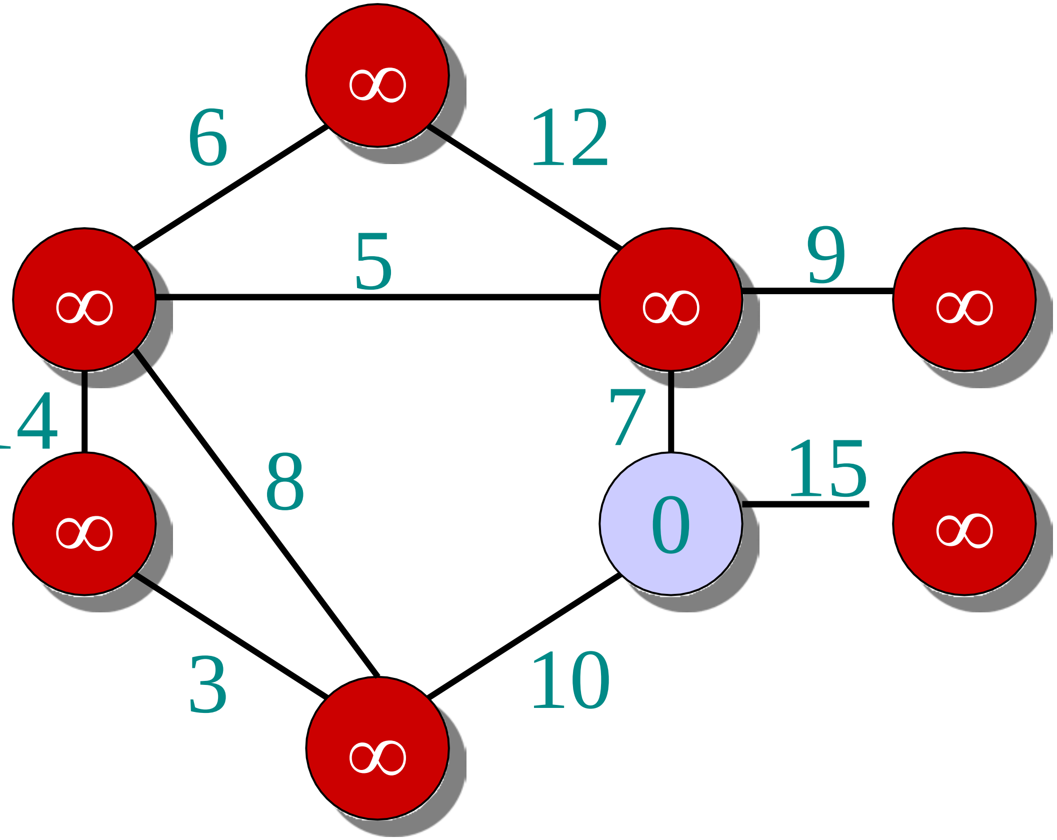
At the end, $\{(v, \pi[v])\}$ forms the MST.

Example of Prim's algorithm



Example of Prim's algorithm

- $\in A$
- $\in V - A$



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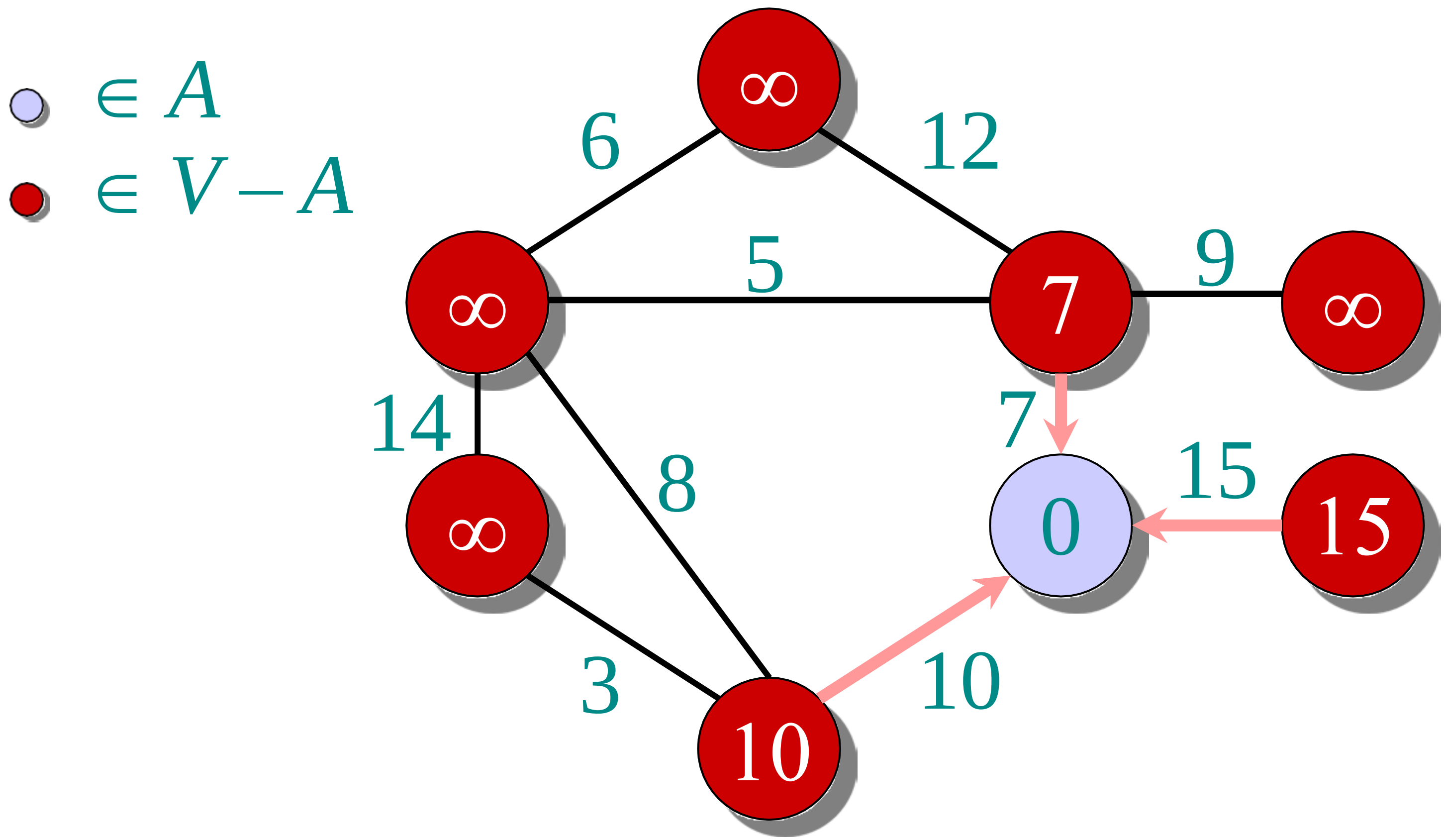

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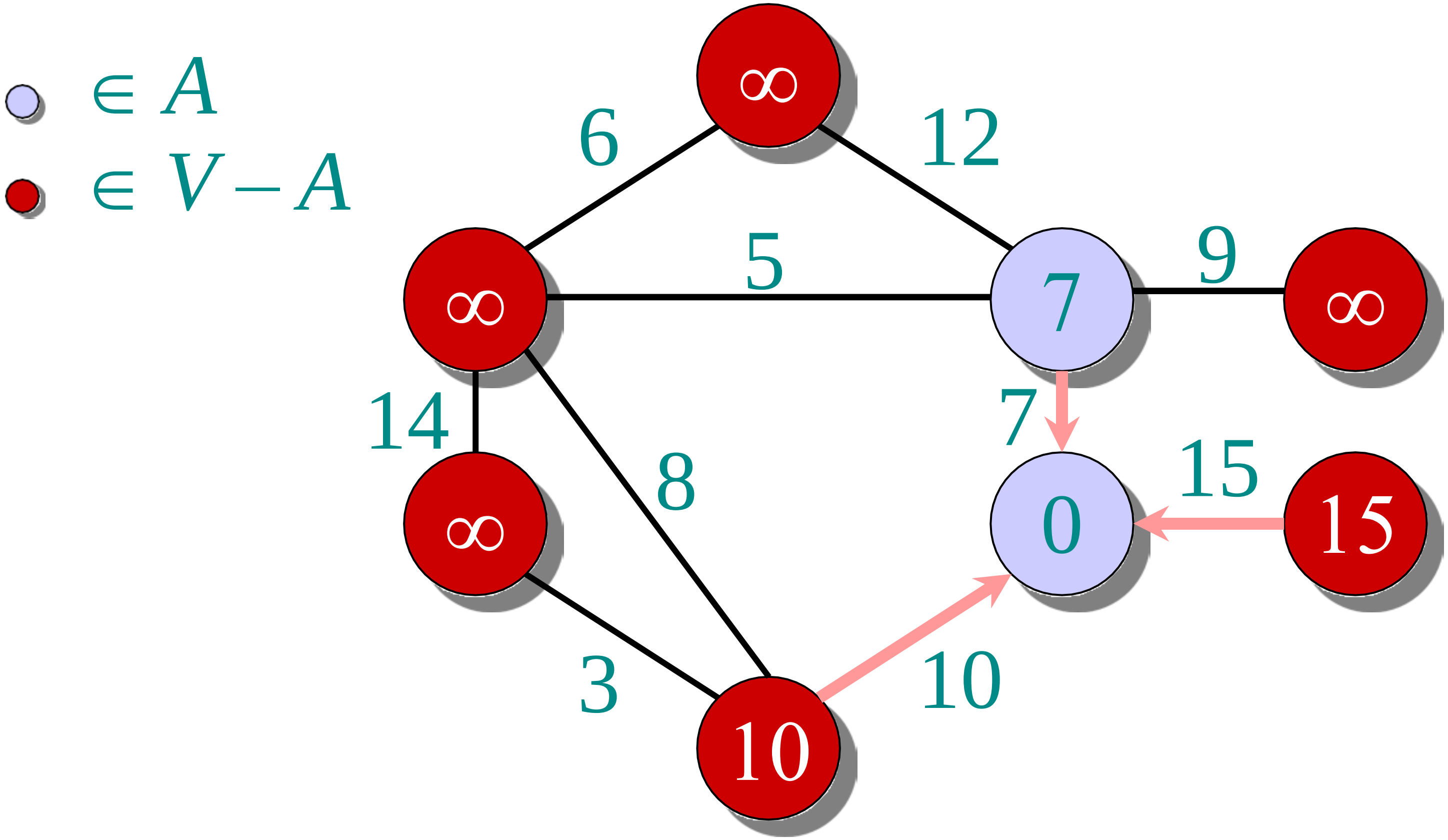

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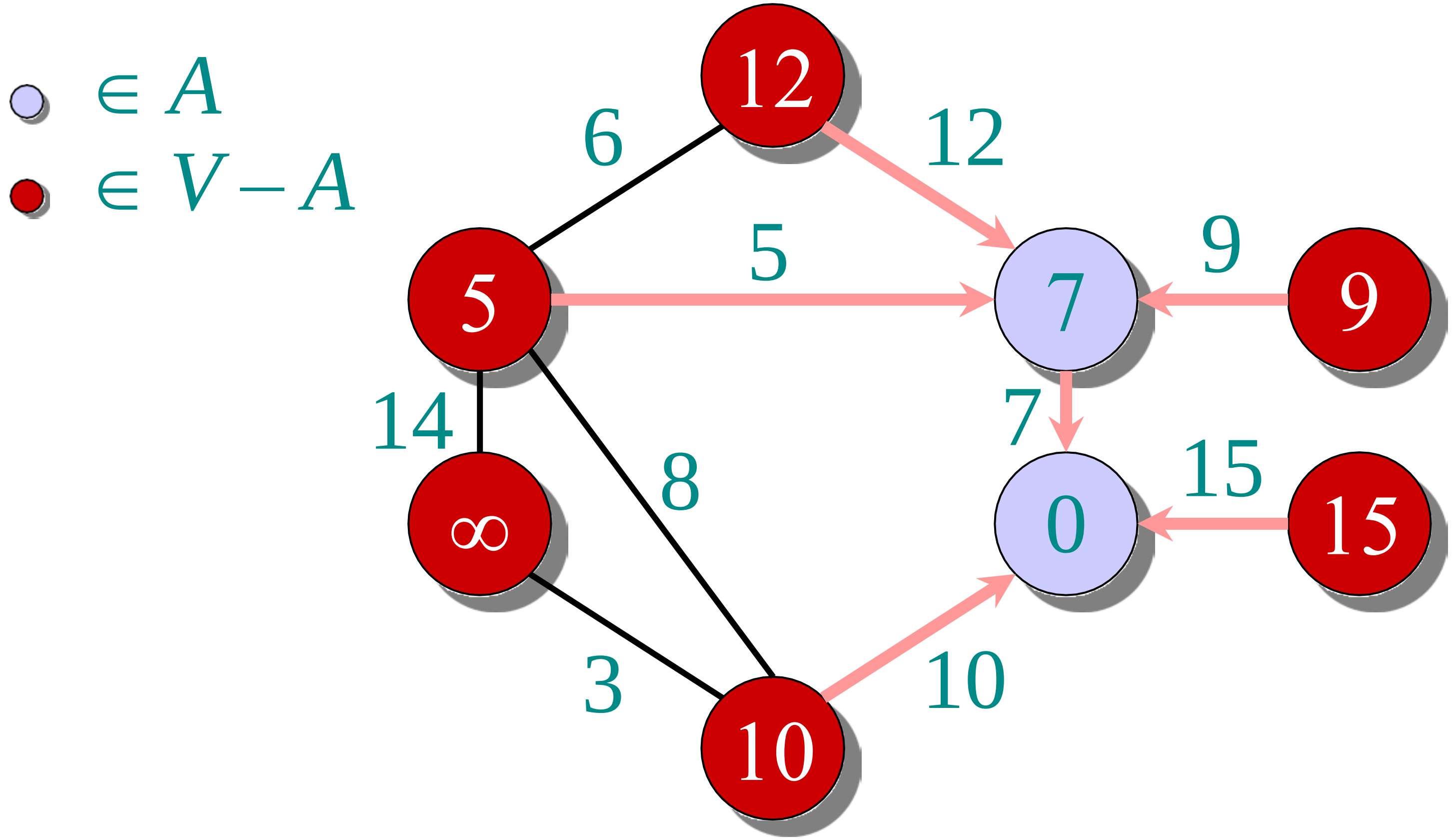

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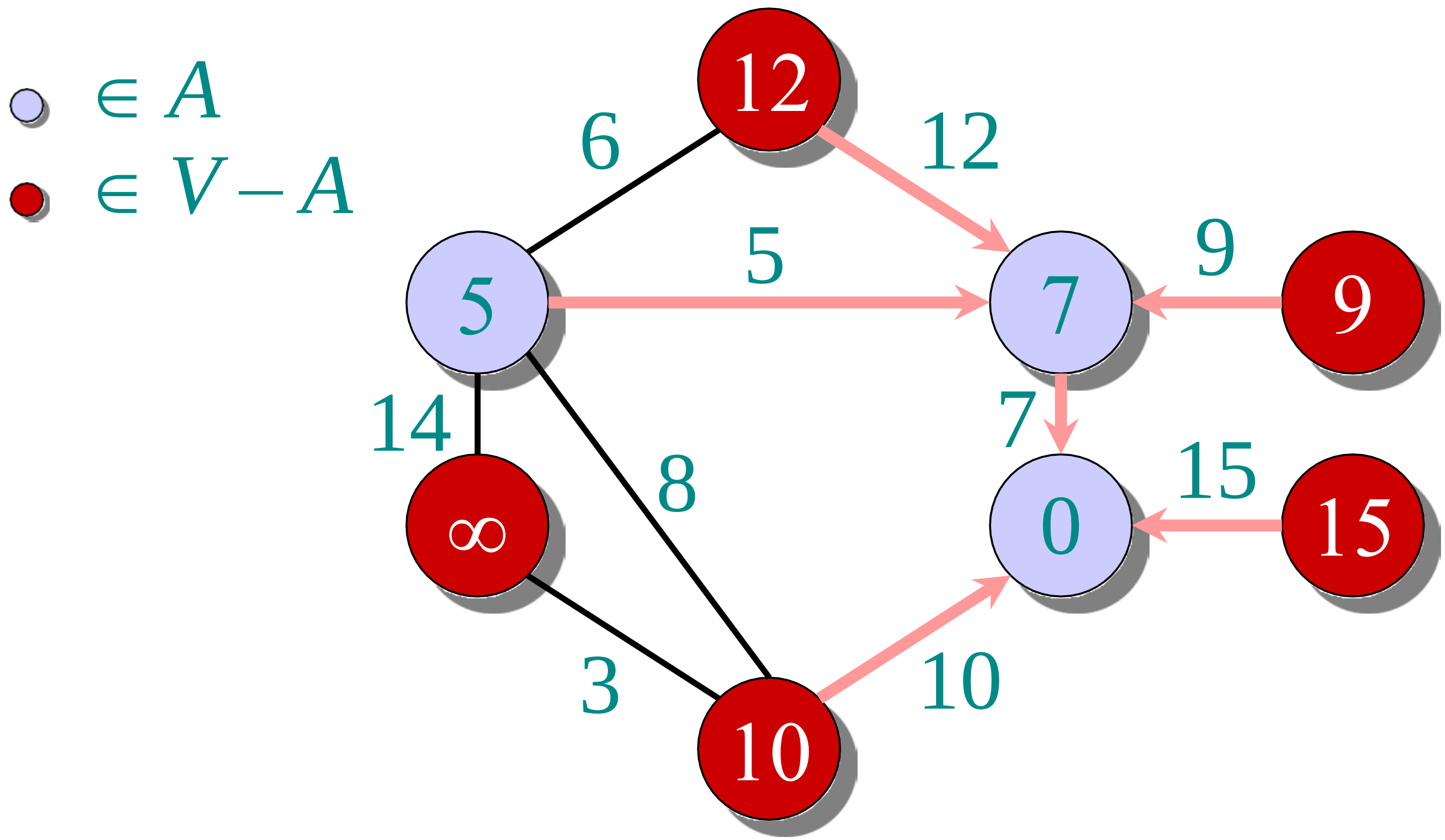

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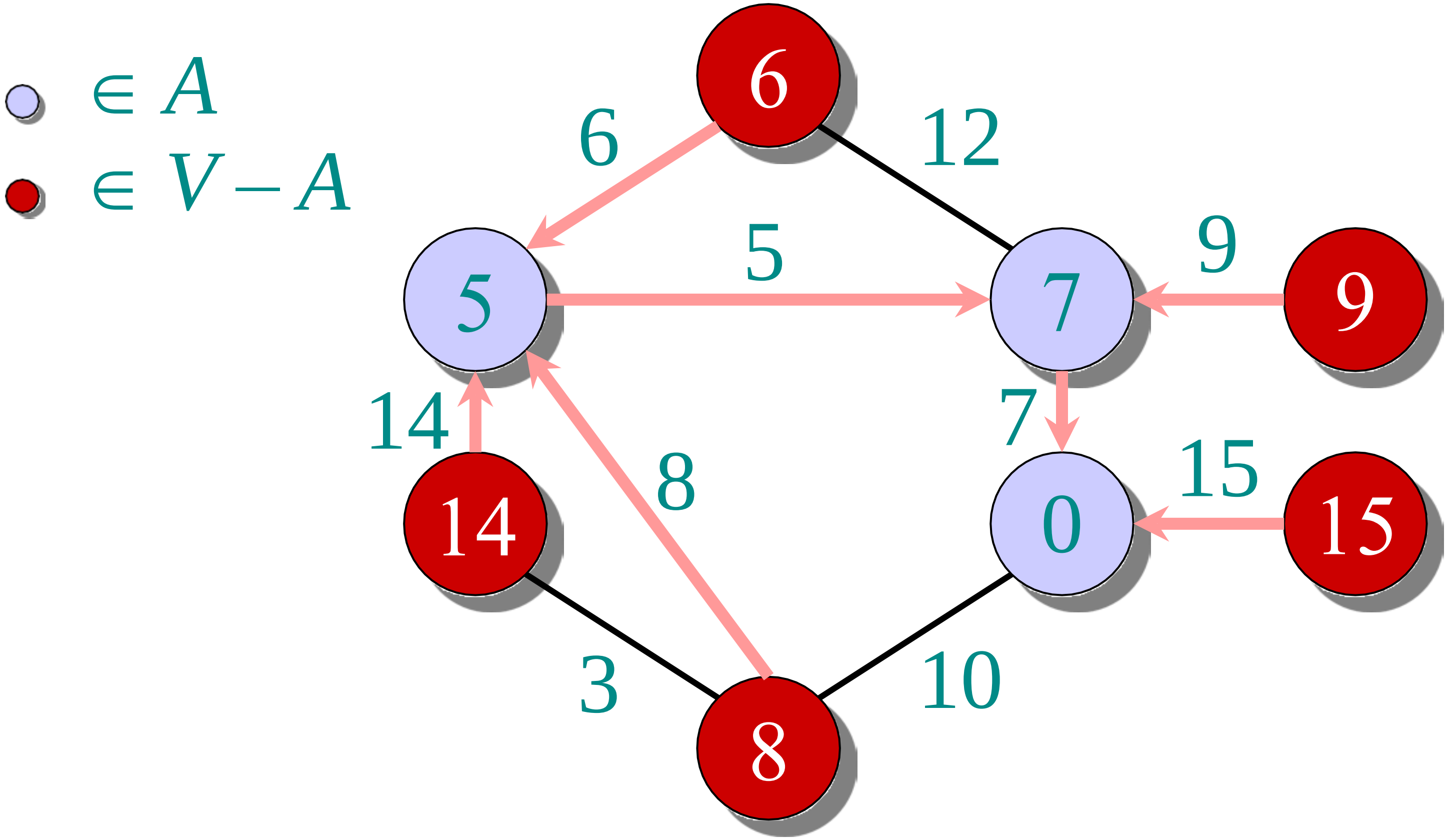

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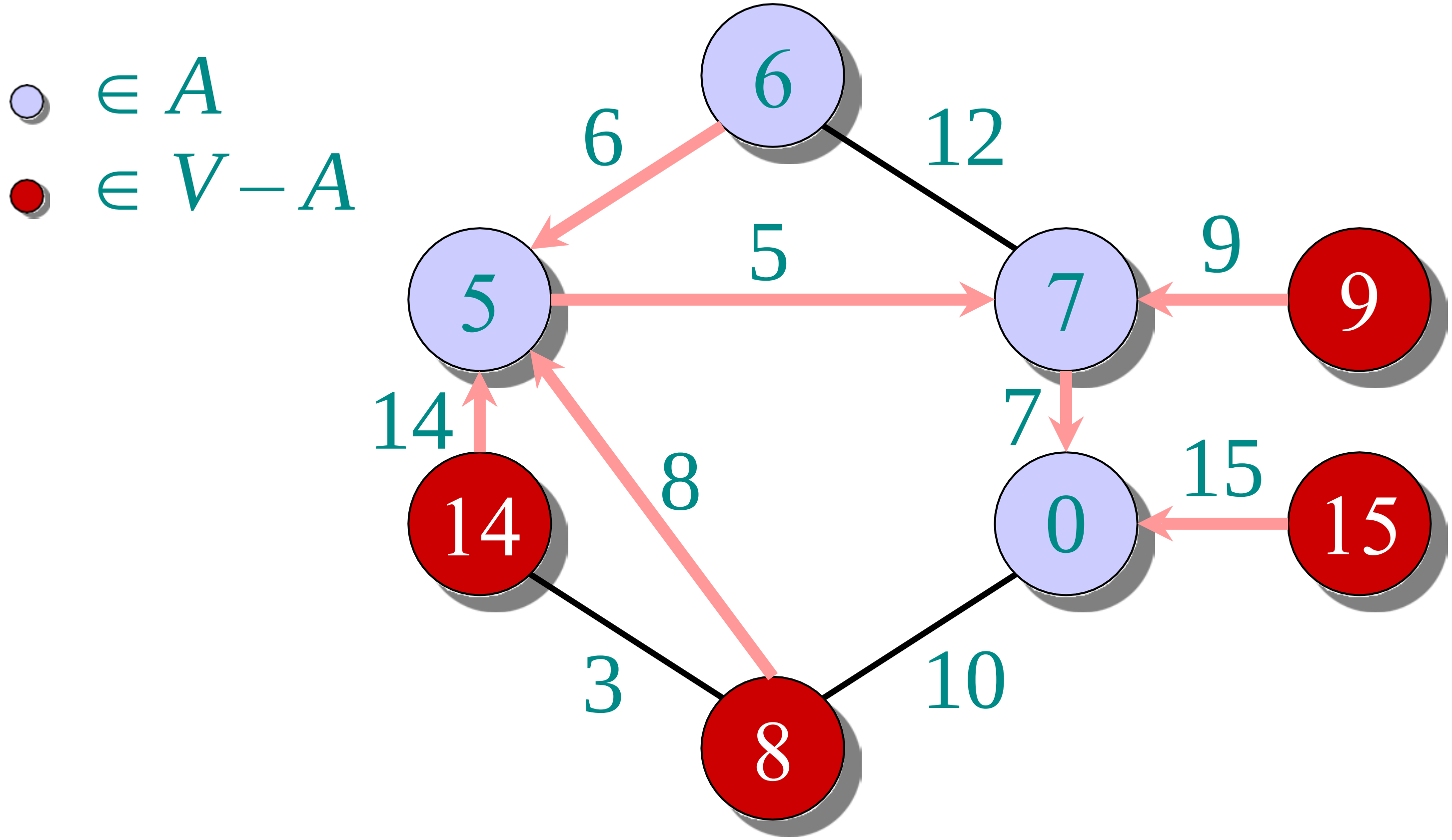

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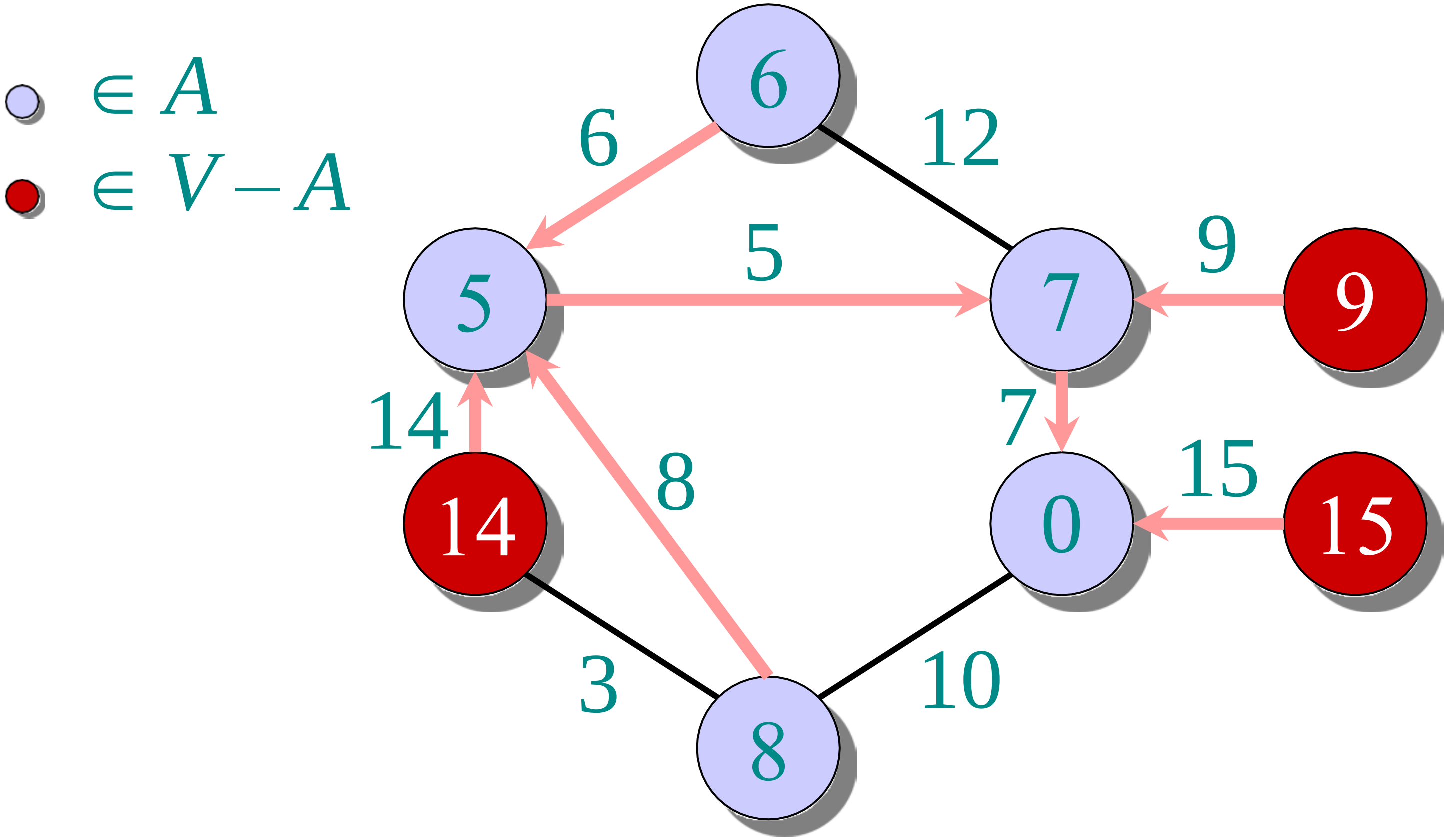

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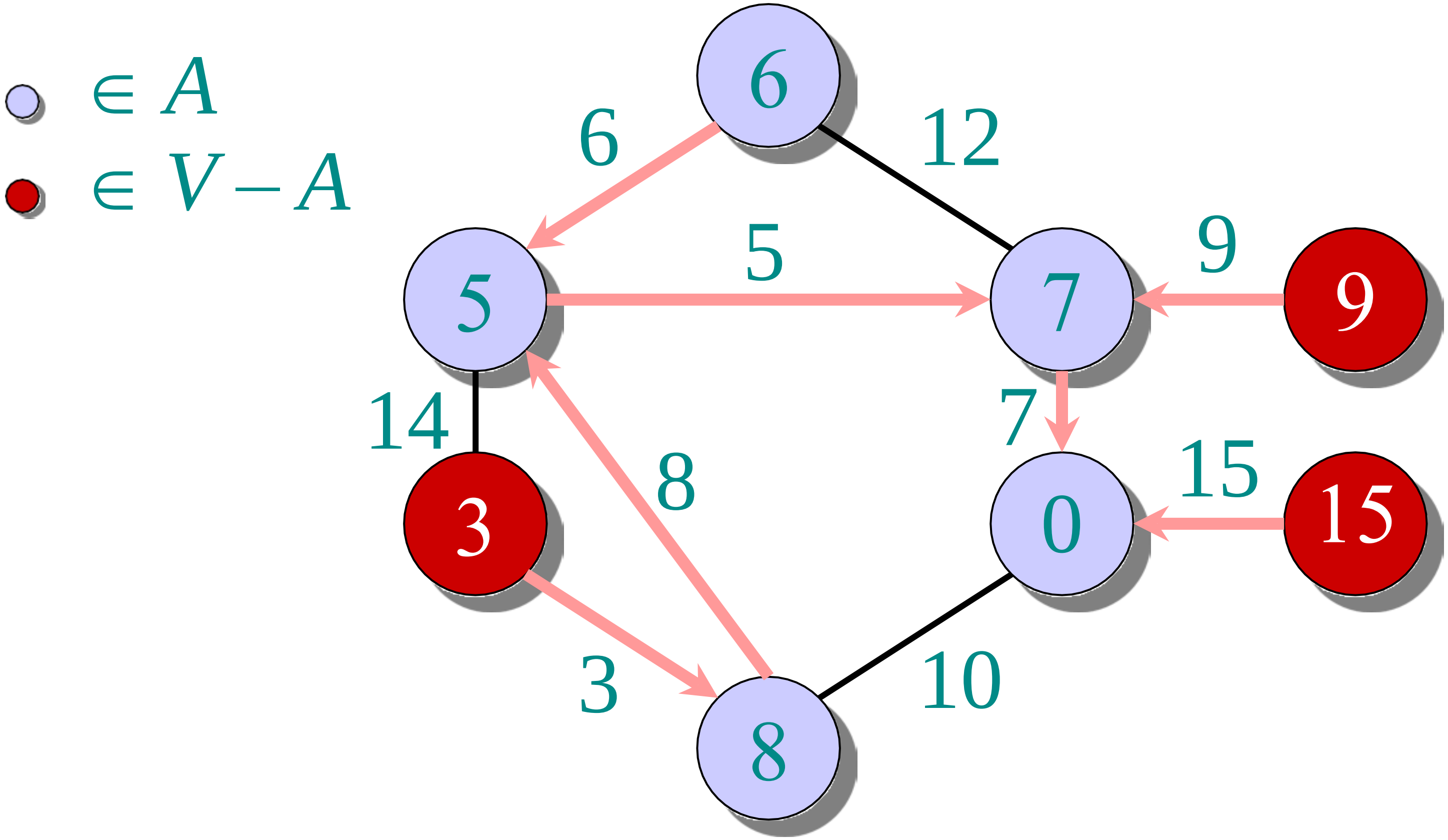

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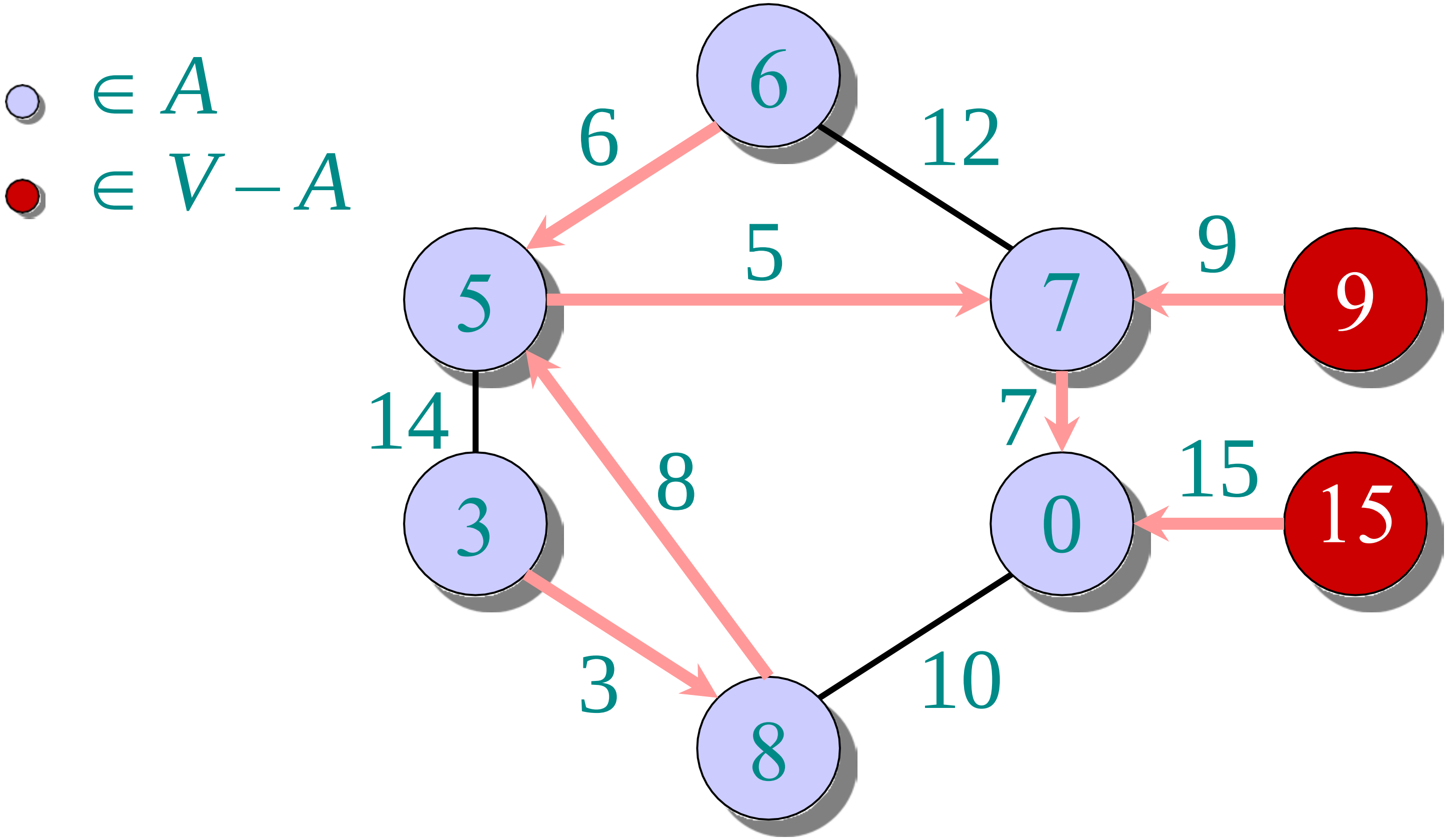

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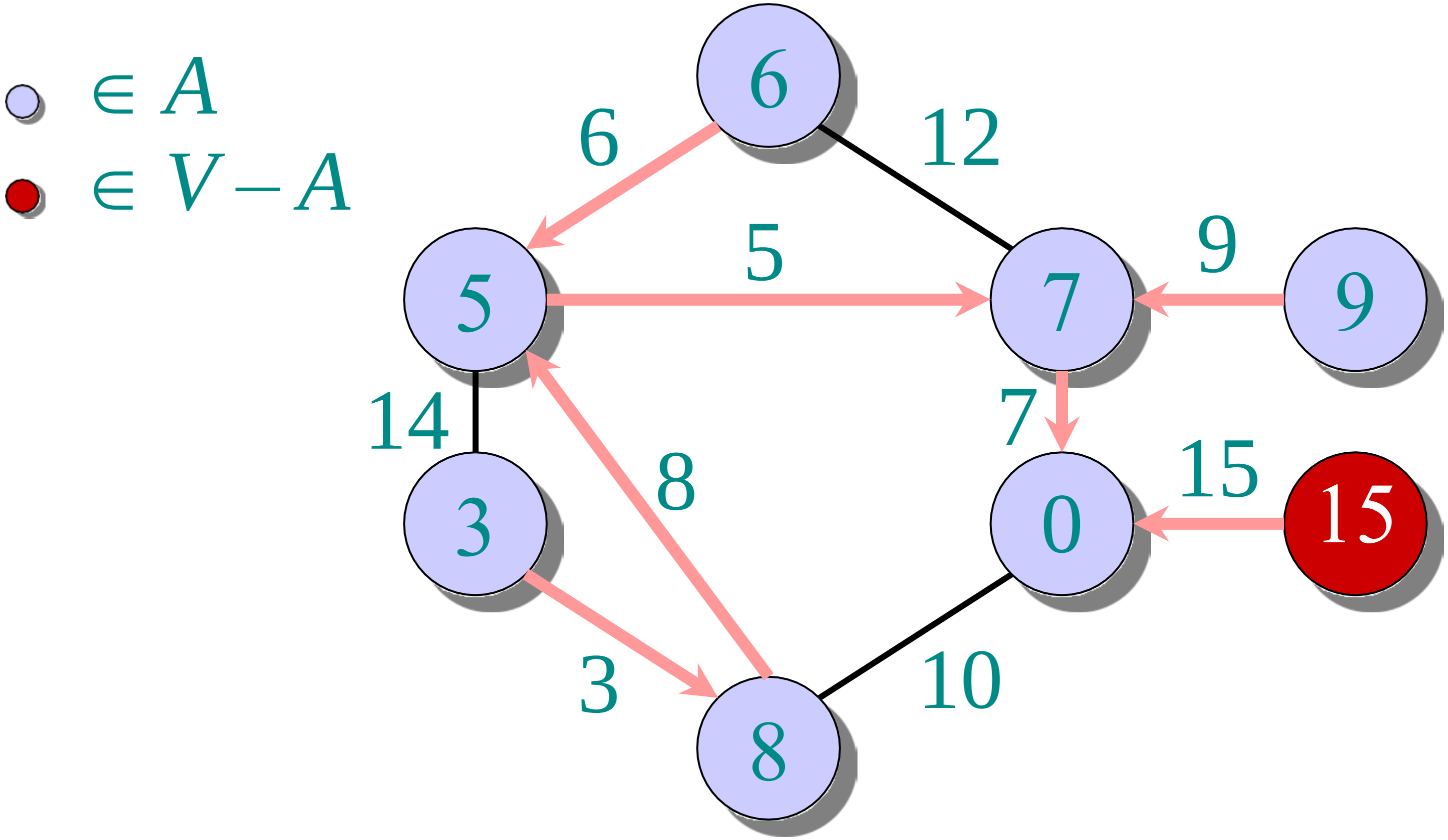

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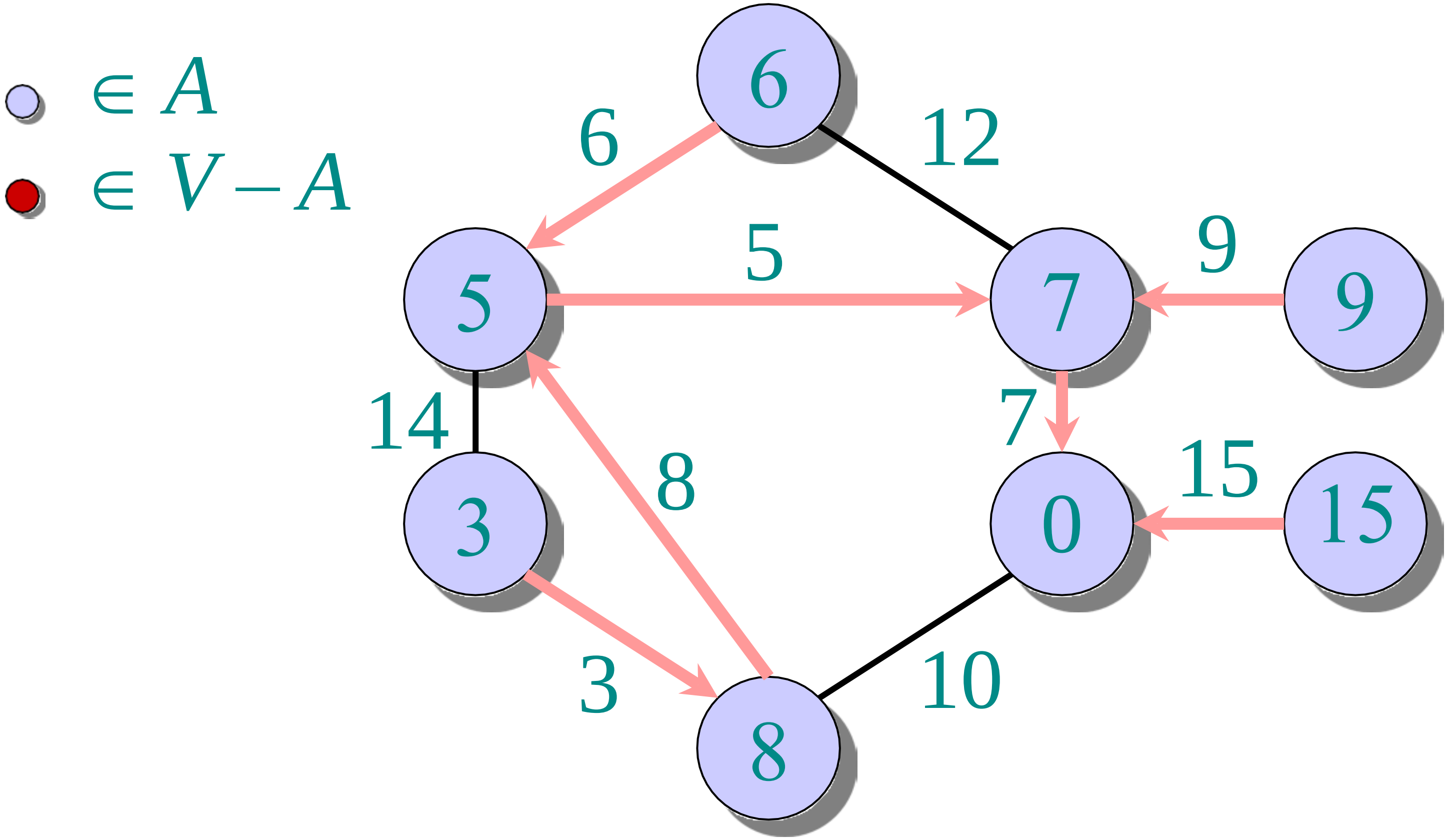

Analysis of Prim

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Analysis of Prim

$\Theta(V)$ total $\left\{\begin{array}{l} Q \leftarrow V \\ key[v] \leftarrow \infty \text{ for all } v \in V \\ key[s] \leftarrow 0 \text{ for some arbitrary } s \in V \end{array}\right.$

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$|V|$ times $\left\{ \begin{array}{l} \textbf{while } Q \neq \emptyset \\ \quad \textbf{do } u \leftarrow \text{EXTRACT-MIN}(Q) \\ \qquad \textbf{for} \text{ each } v \in Adj[u] \\ \qquad\quad \textbf{do if } v \in Q \text{ and } w(u, v) < key[v] \\ \qquad\qquad \textbf{then } key[v] \leftarrow w(u, v) \\ \qquad\qquad\qquad \pi[v] \leftarrow u \end{array} \right.$

ANALYSIS OF PRIM

$\Theta(V)$ total

$Q \leftarrow V$
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$key[s] \leftarrow 0$ for some arbitrary $s \in V$

$|V|$ times

while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)

$degree(u)$ times

    for each $v \in Adj[u]$
      do if $v \in Q$ and $w(u, v) < key[v]$
        then $key[v] \leftarrow w(u, v)$
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ANALYSIS OF PRIM

$\Theta(V)$ total $\left\{ \begin{array}{l} Q \leftarrow V \\ key[v] \leftarrow \infty \text{ for all } v \in V \\ key[s] \leftarrow 0 \text{ for some arbitrary } s \in V \end{array} \right.$

while $Q \neq \varnothing$

$|V|$ times

do $u \leftarrow$ EXTRACT-MIN(Q)

$degree(u)$ times

for each $v \in Adj[u]$

do if $v \in Q$ and $w(u, v) < key[v]$

then $key[v] \leftarrow w(u, v)$

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Handshaking Lemma $\Rightarrow$ $\Theta(E)$ implicit DECREASE-KEY's.

ANALYSIS OF PRIM

$\Theta(V)$ total:

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$|V|$ times:

while $Q \neq \emptyset$
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$degree(u)$ times:

  for each $v \in Adj[u]$
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Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$

Analysis of Prim (continued)

Time $= \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

Analysis of Prim (continued)

Time = $\Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total

Analysis of Prim (continued)

$$\text{Time} = \Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$

ANALYSIS OF PRIM (CONTINUED)

$$\text{Time} = \Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$

Analysis of Prim (continued)

Time = $\Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
Fibonacci heap	$O(\lg V)$ amortized	$O(1)$ amortized	$O(E + V \lg V)$ worst case

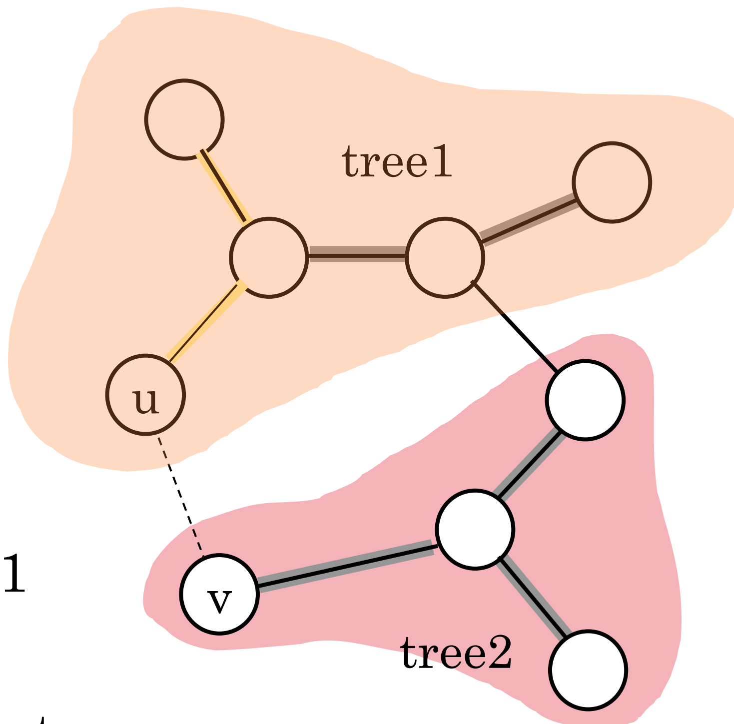
MST ALGORITHMS

Kruskal's algorithm (see CLRS):

- Uses the ***disjoint-set data structure*** (see CLRS, Ch. 21).
- Running time $= O(E \lg V)$.

KRUSKAL'S ALGORITHM

- How is it different from Prim's algorithm?
  - Prim's algorithm grows one tree all the time
  - Kruskal's algorithm grows multiple trees (i.e., a forest) at the same time.
  - Trees are merged together using **safe** edges
  - Since an MST has exactly |V| - 1 edges, after |V| - 1 merges, we would have only one component



Kruskal's Algorithm

- Kruskal's algorithm creates a forest of trees
- Initially forest consists of N single node trees (and no edges)
- Sorts the edges by weight and goes through the edges from least weight to greatest weight
- At each step one edge (with least weight) is added so that it joins two trees together
  - As long as the addition does not create a cycle

Kruskal's Algorithm

The halting conditions are as follows:

1. When `|V| - 1` edges have been added
   - In this case we have a minimum spanning tree

2. We have gone through all edges
   - A forest of minimum spanning trees on all connected sub-graphs

Kruskal's Algorithm

1. The forest is constructed – with each node in a separate tree

2. The edges are placed in a priority queue

3. Until we've added `n-1` edges (assumption: connected graph)
    1. Extract the cheapest edge from the queue

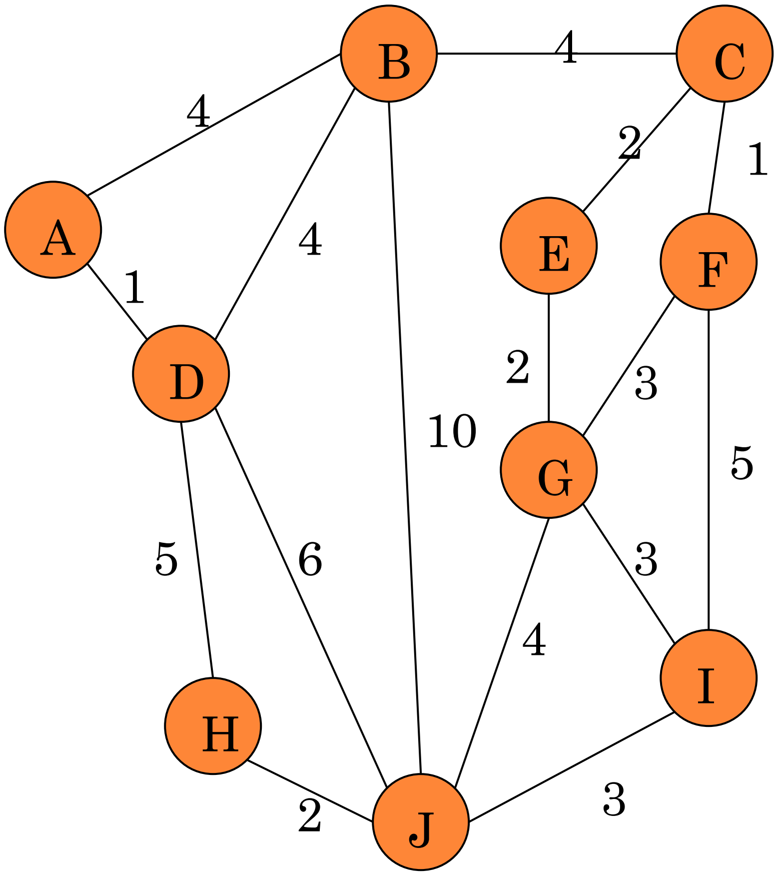
    2. If it forms a cycle, reject it

    3. Else add it to the forest. Adding it to the forest will join two trees

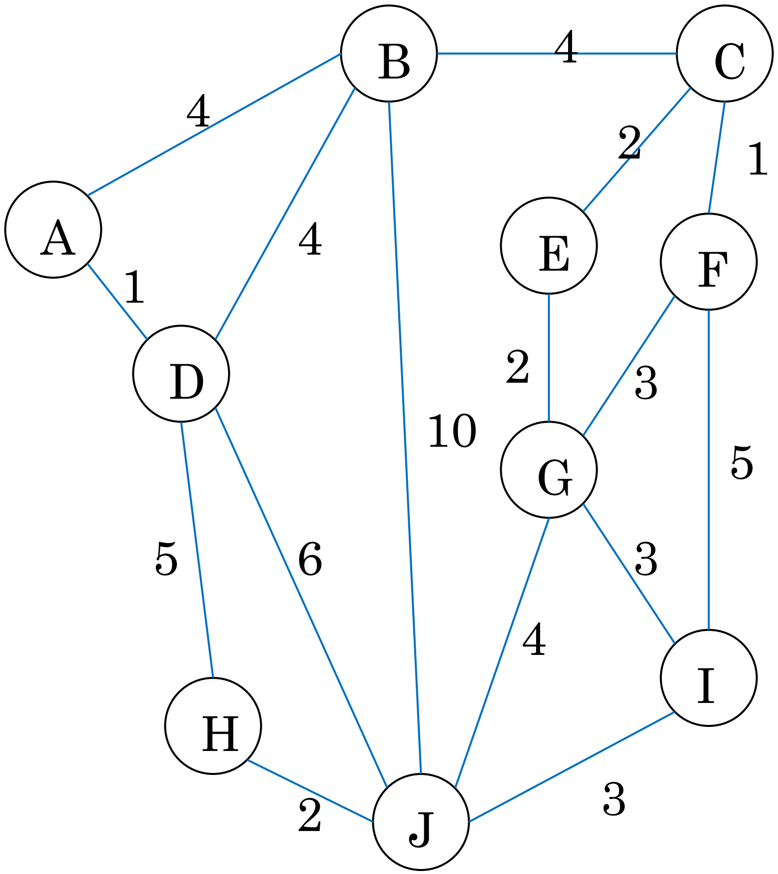
- Every step will have joined two trees in the forest together, so that at the end, there will only be one tree

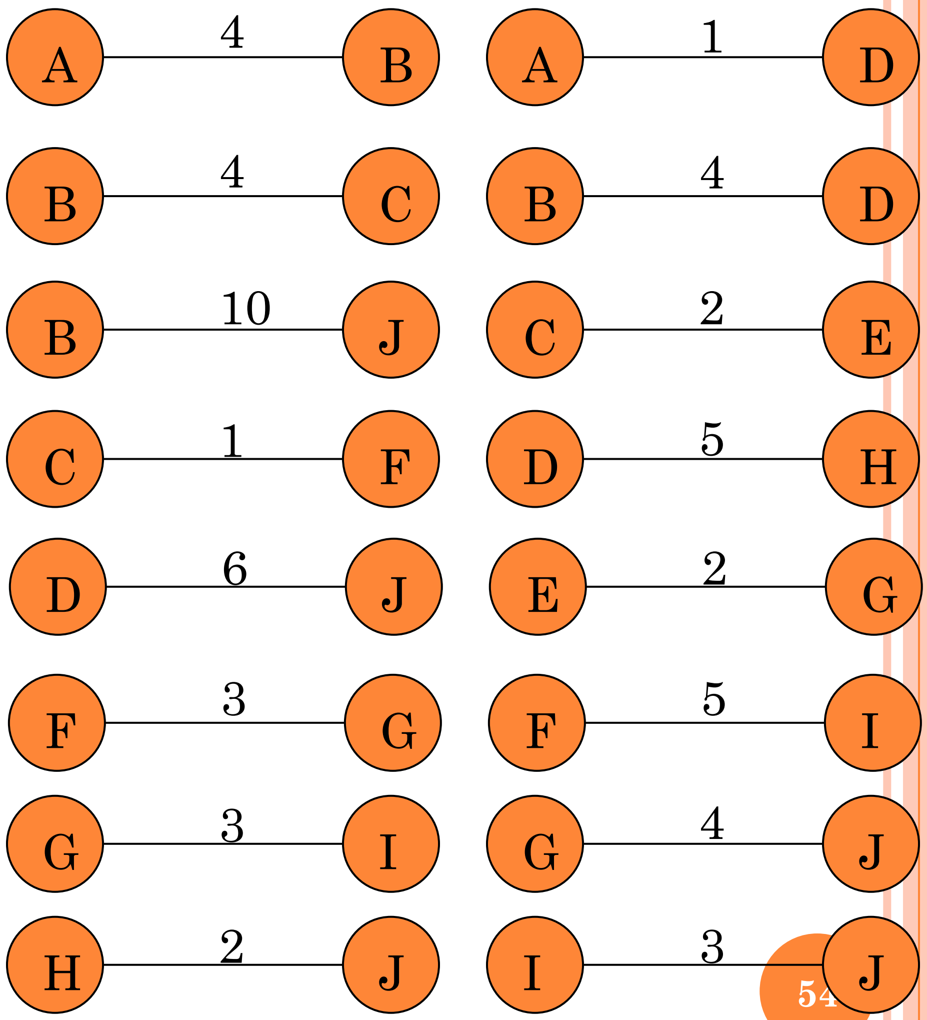
Kruskal's Algorithm – Example

- Complete Graph



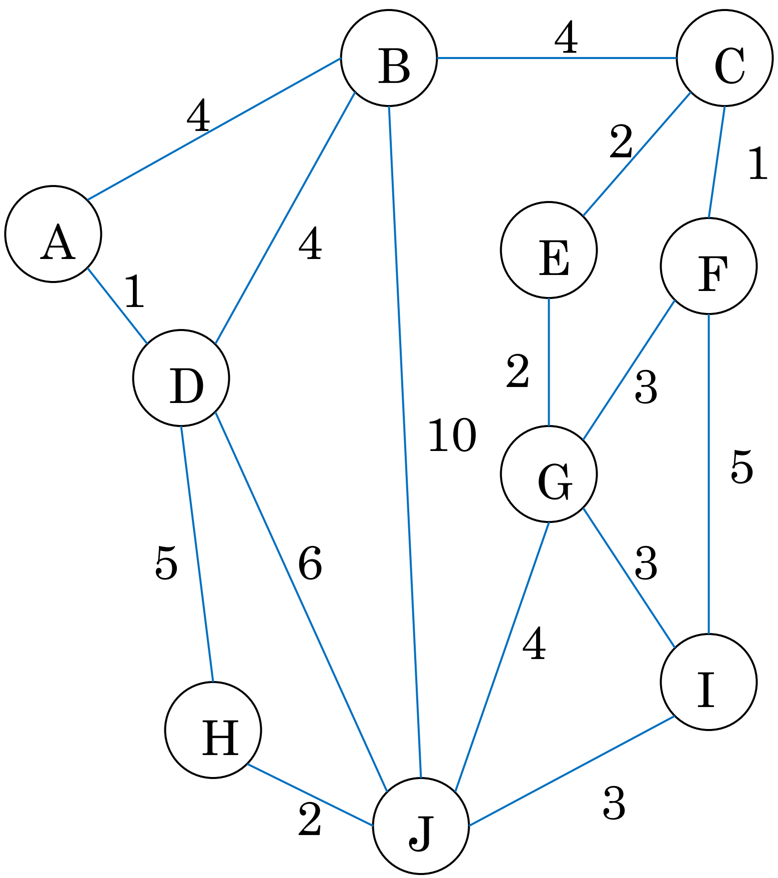
Kruskal's Algorithm – Example

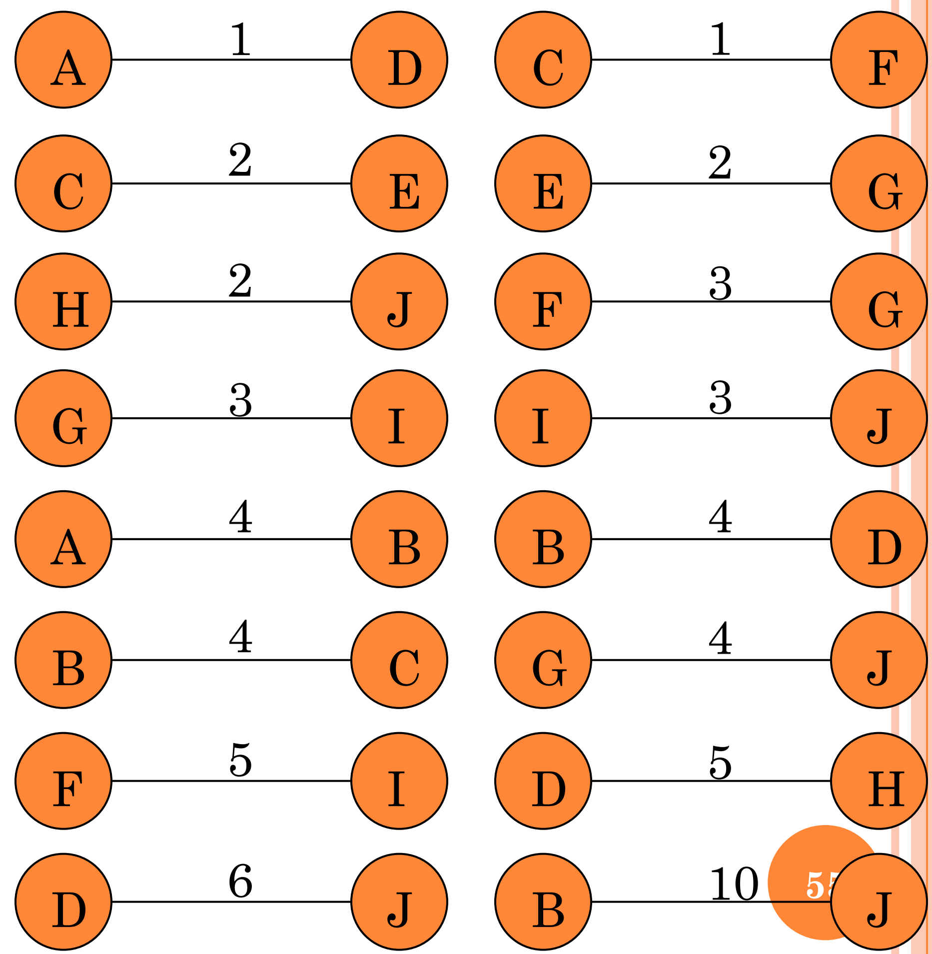




Kruskal's Algorithm – Example

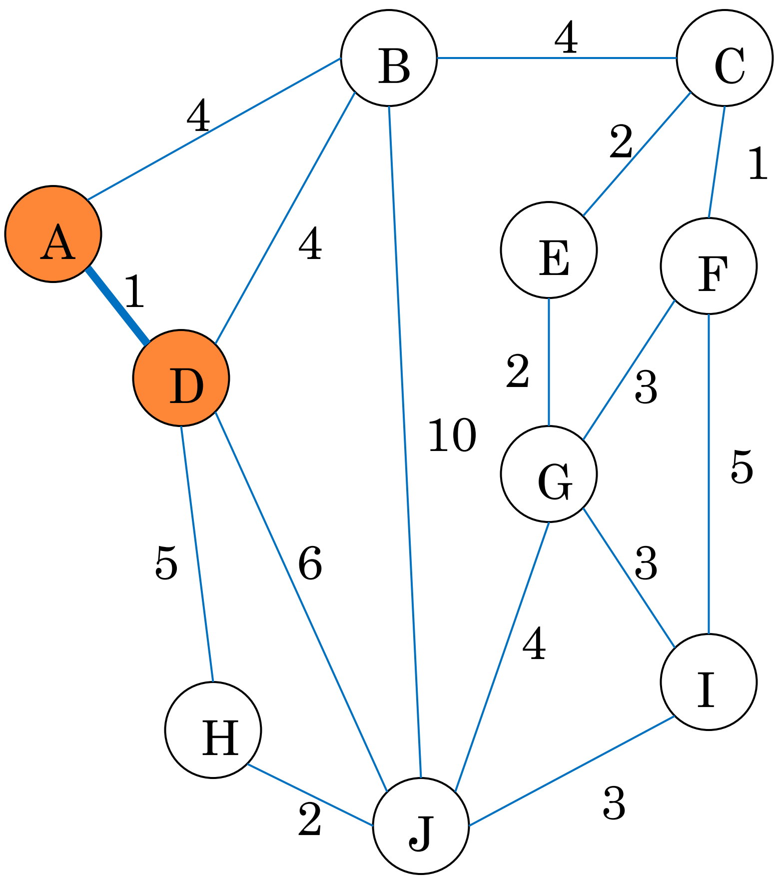
- Sort Edges: In reality priority queue is used

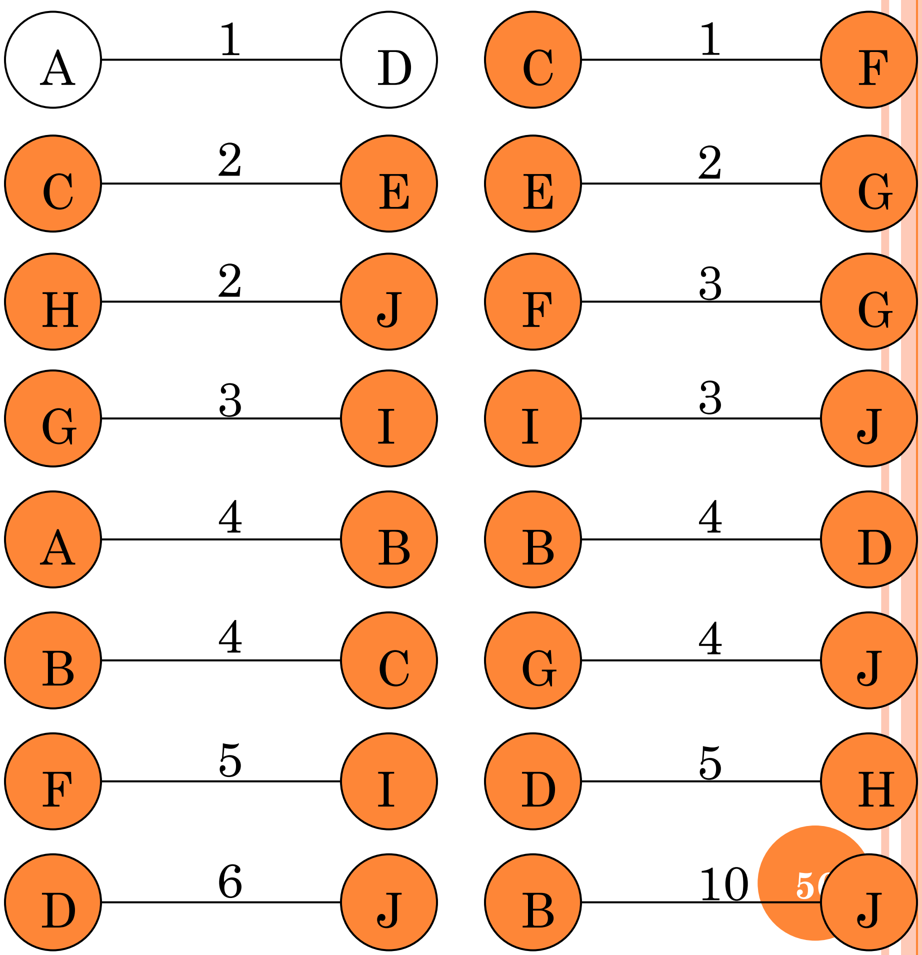




Kruskal's Algorithm – Example

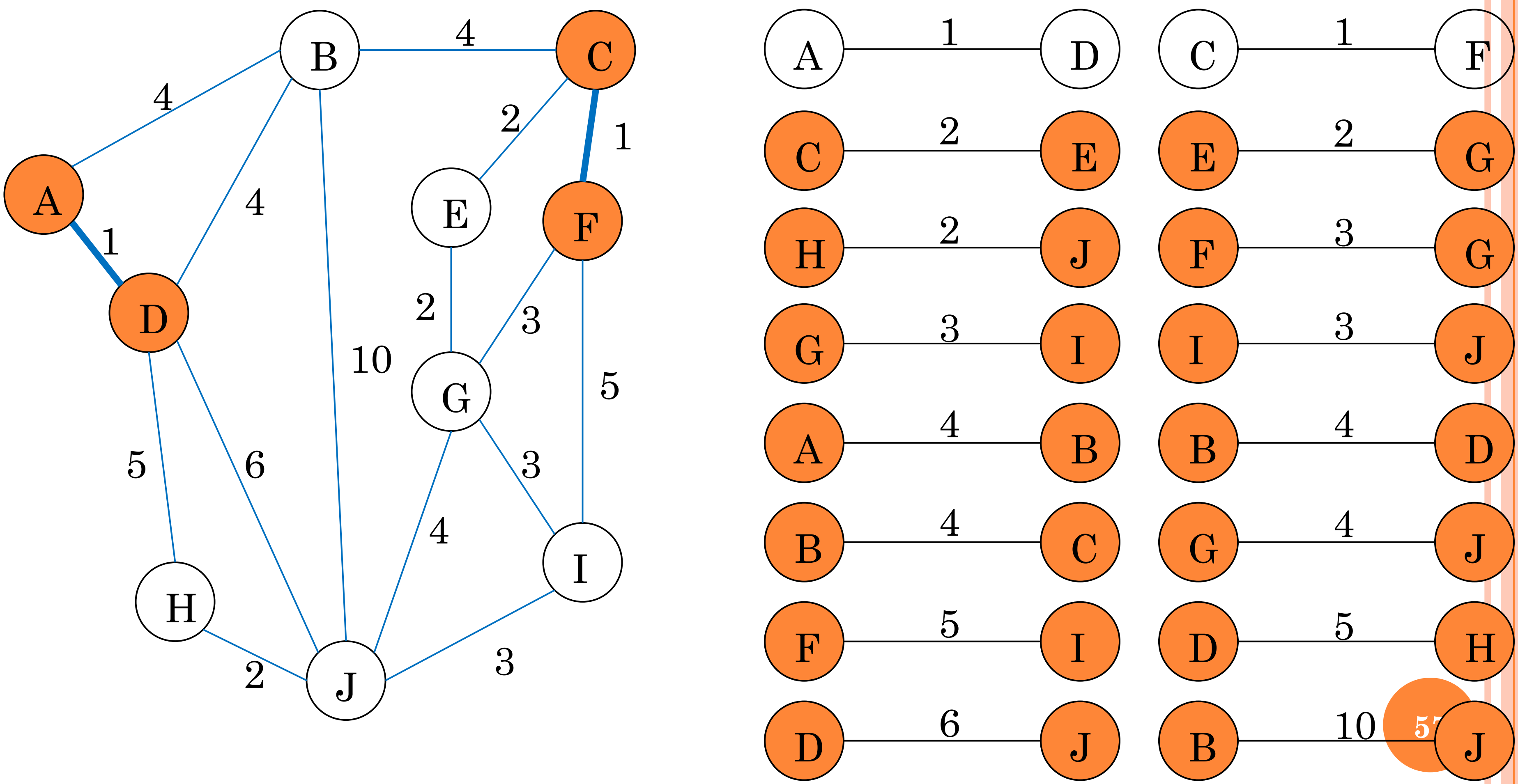
- Add Edge





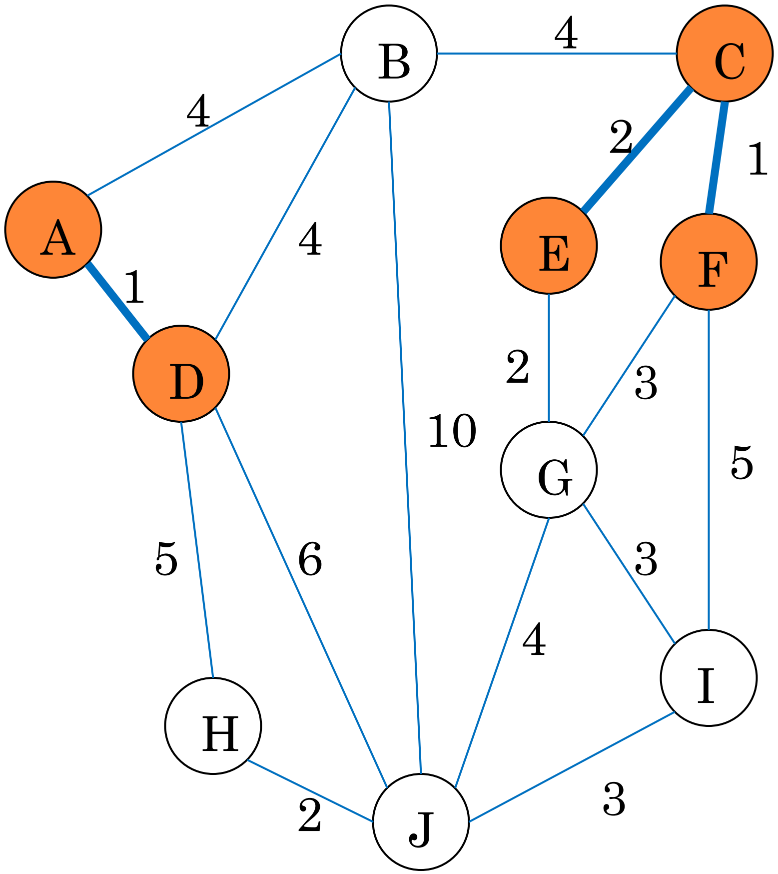
Kruskal's Algorithm – Example

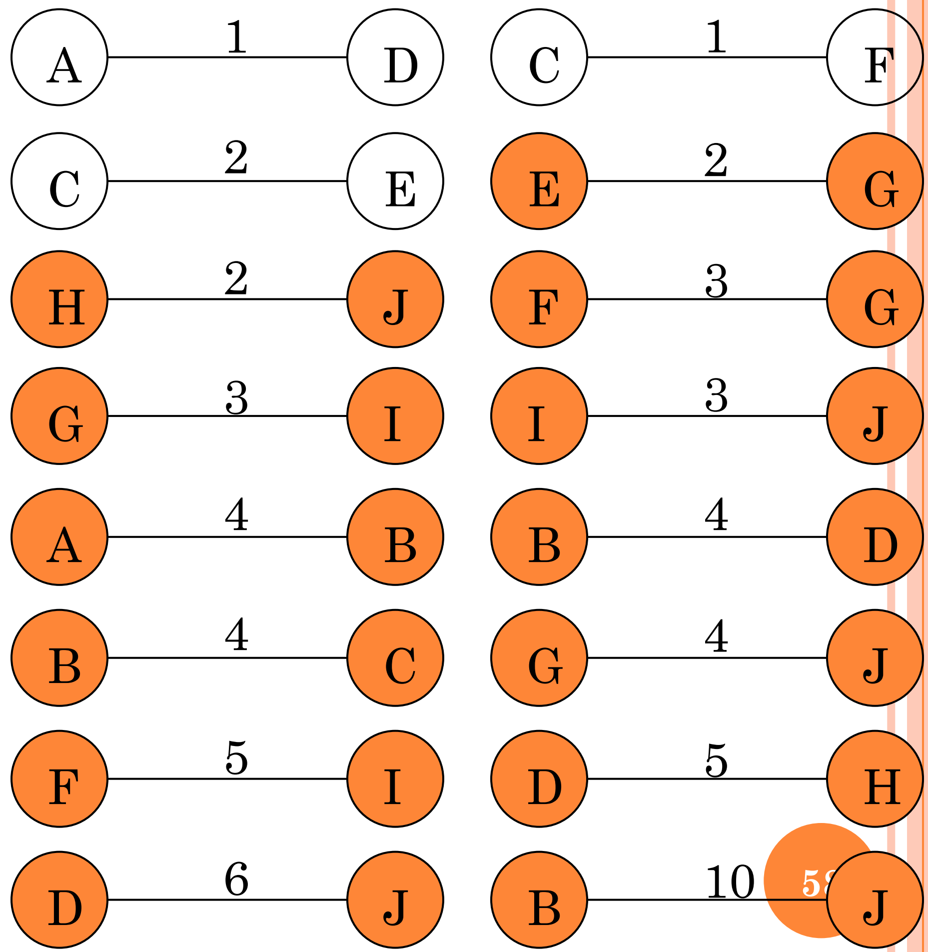
- Add Edge



Kruskal's Algorithm – Example

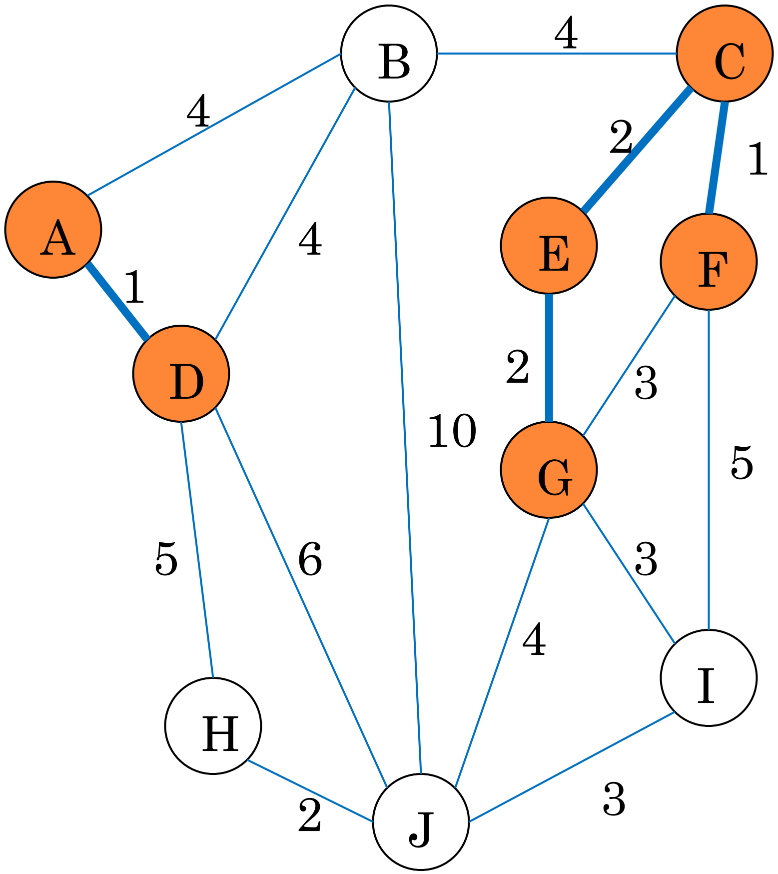
- Add Edge

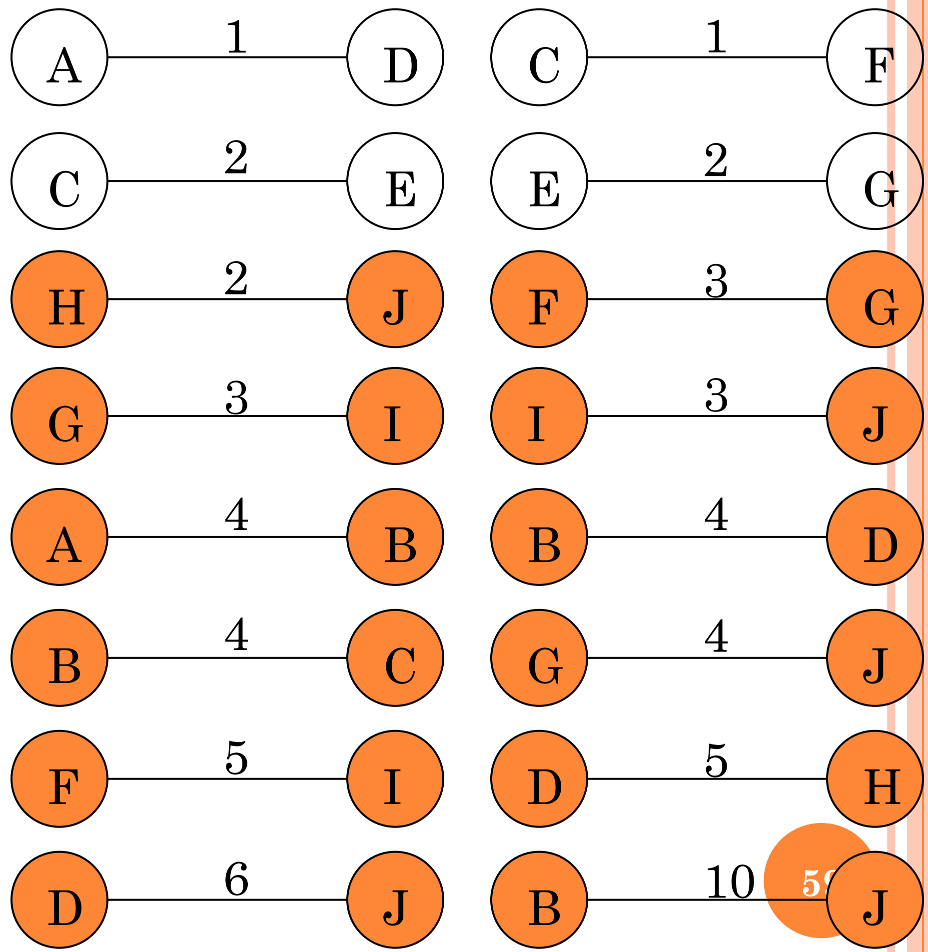




Kruskal's Algorithm – Example

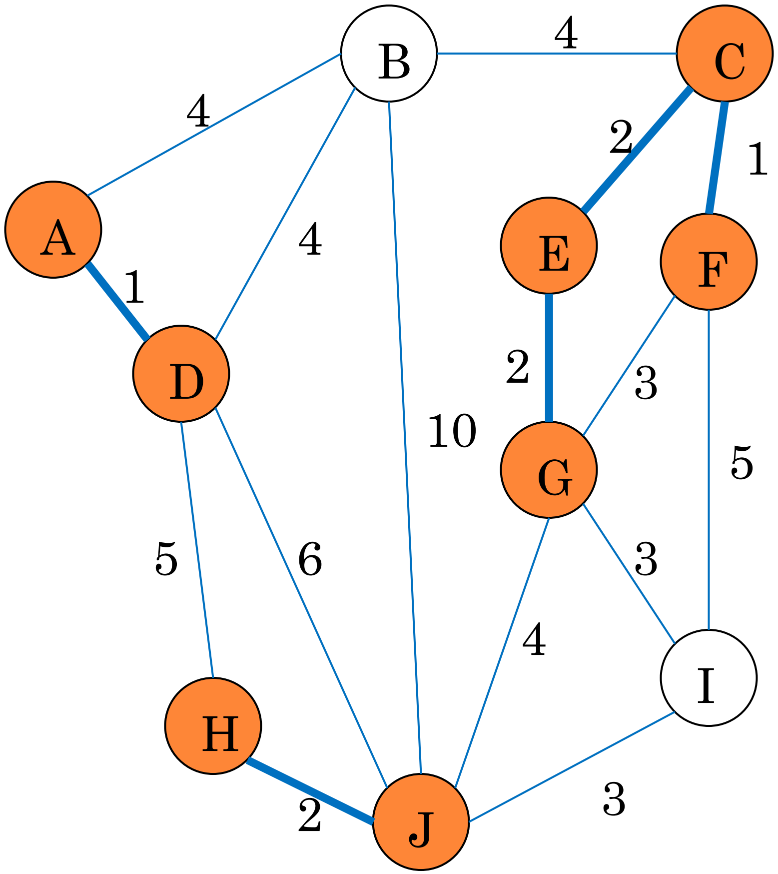
- Add Edge

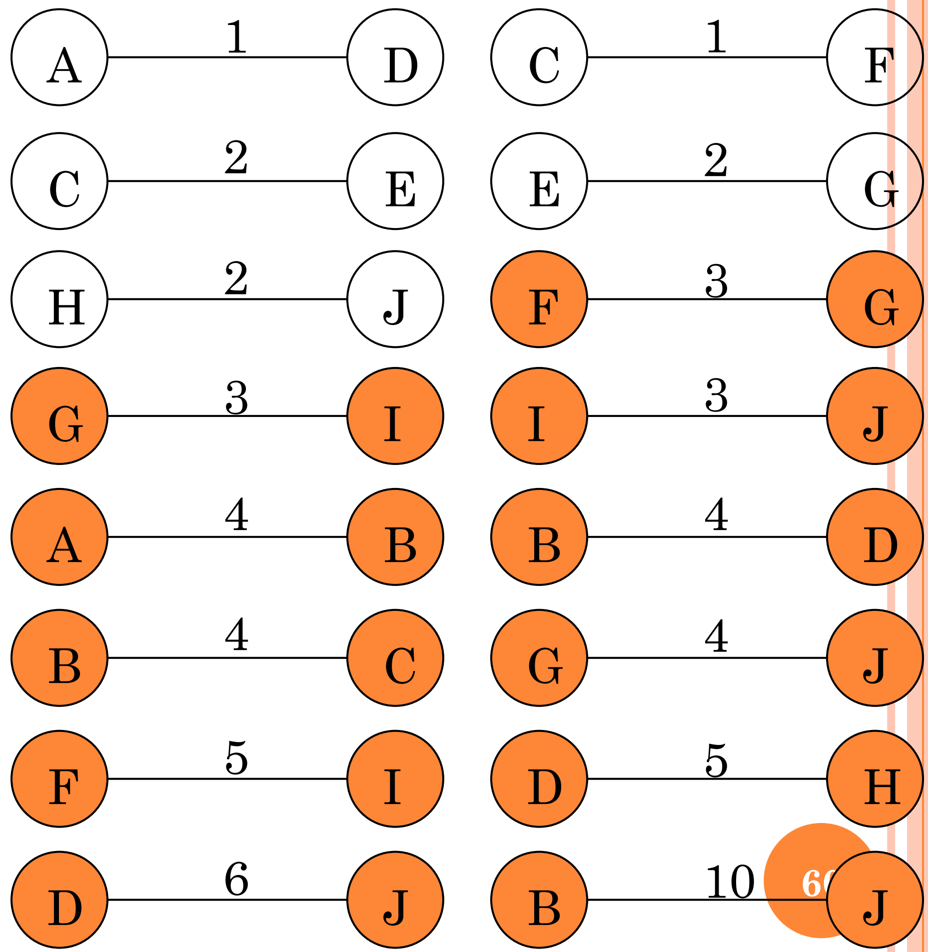




Kruskal's Algorithm – Example

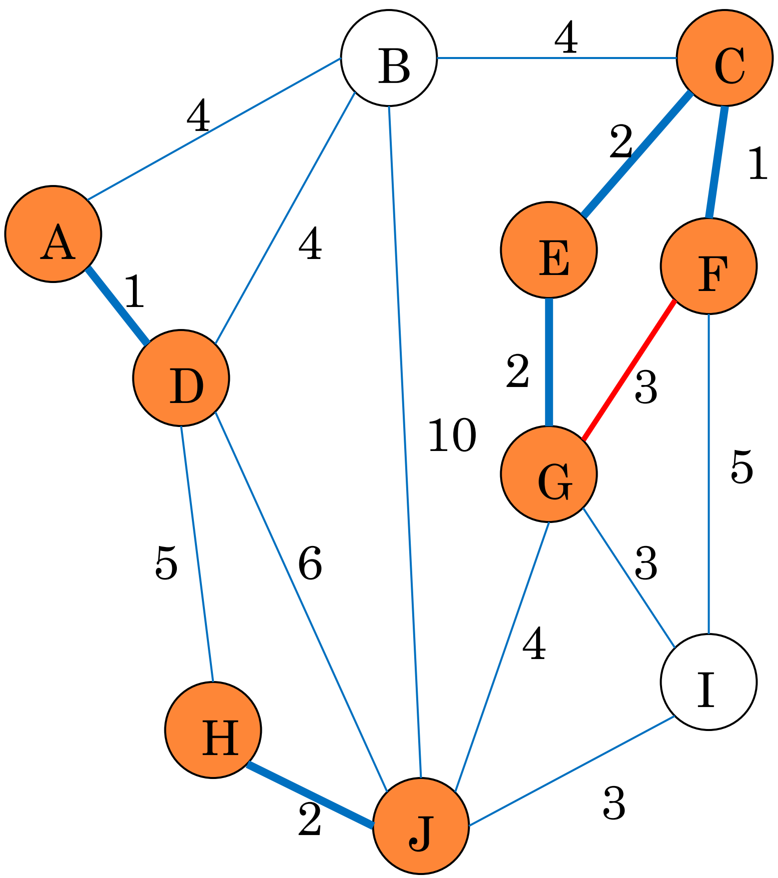
- Add Edge

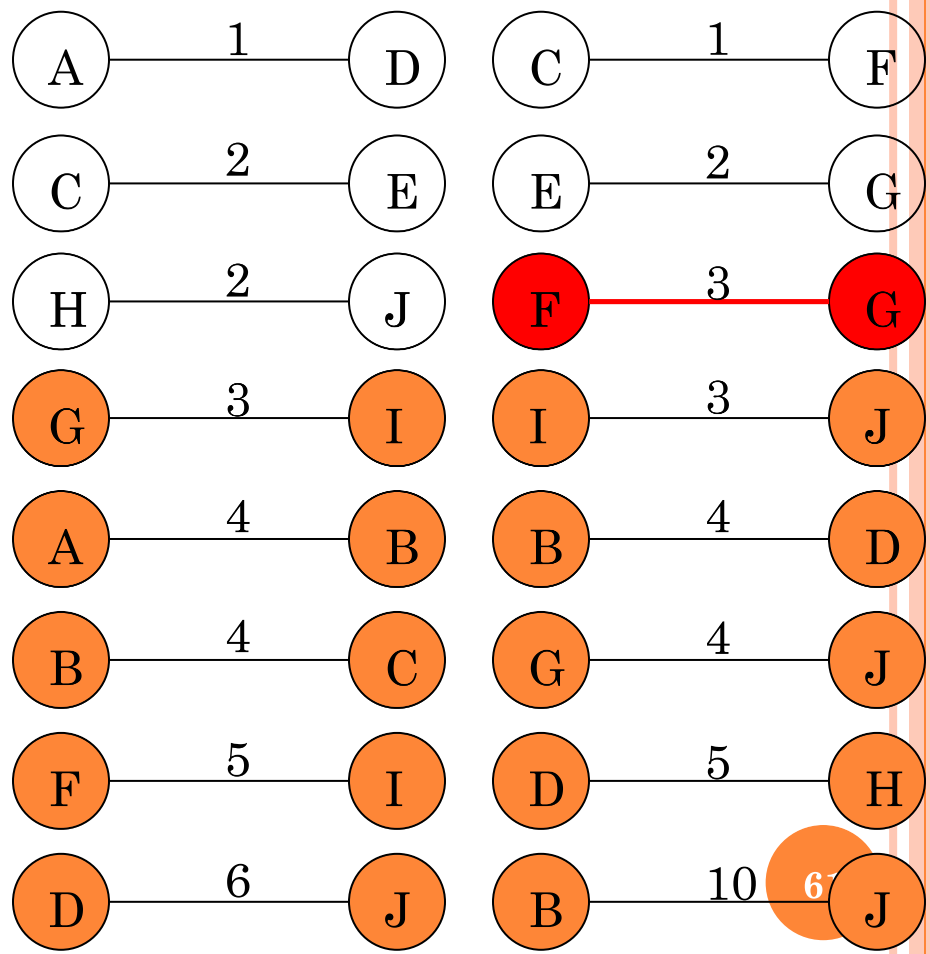




Kruskal's Algorithm – Example

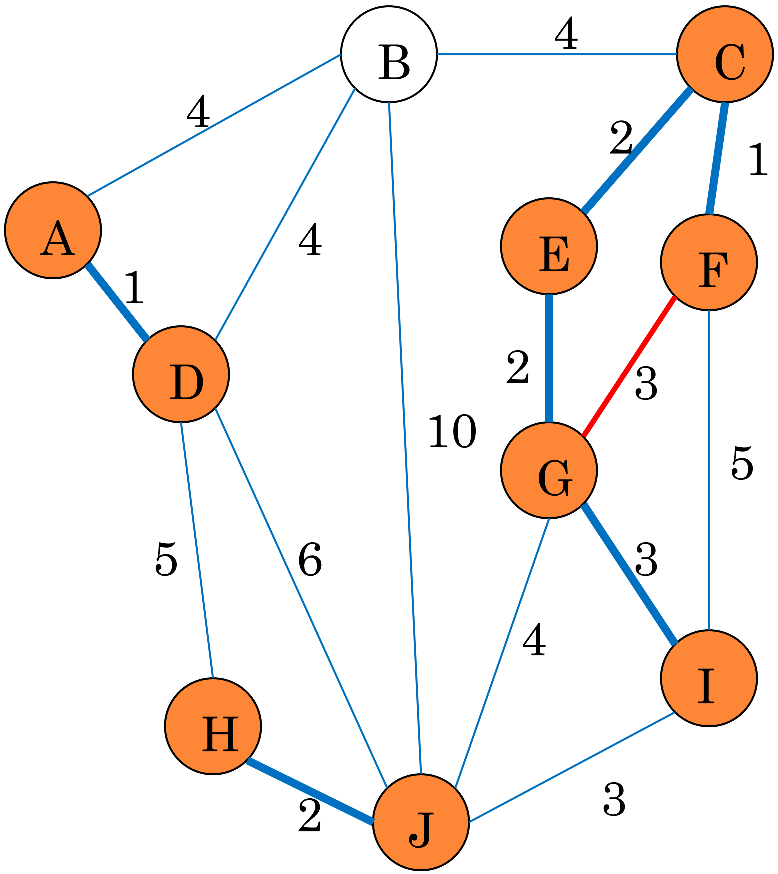
- Add Edge

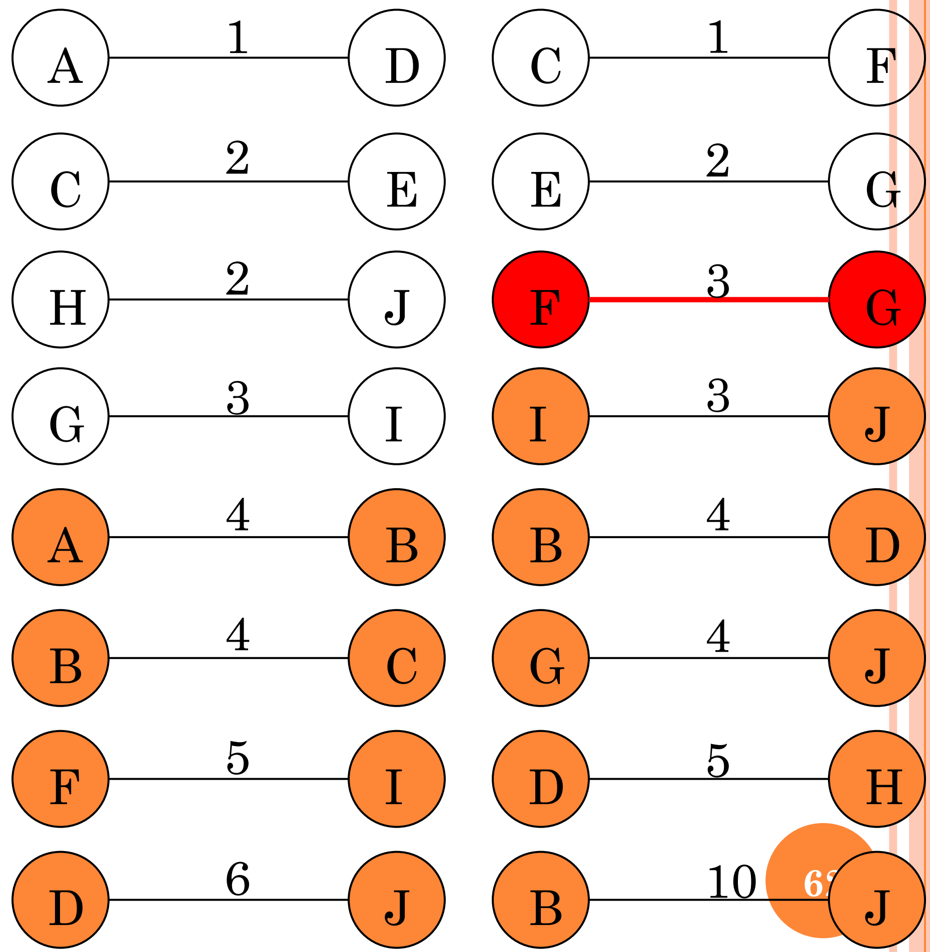




Kruskal's Algorithm – Example

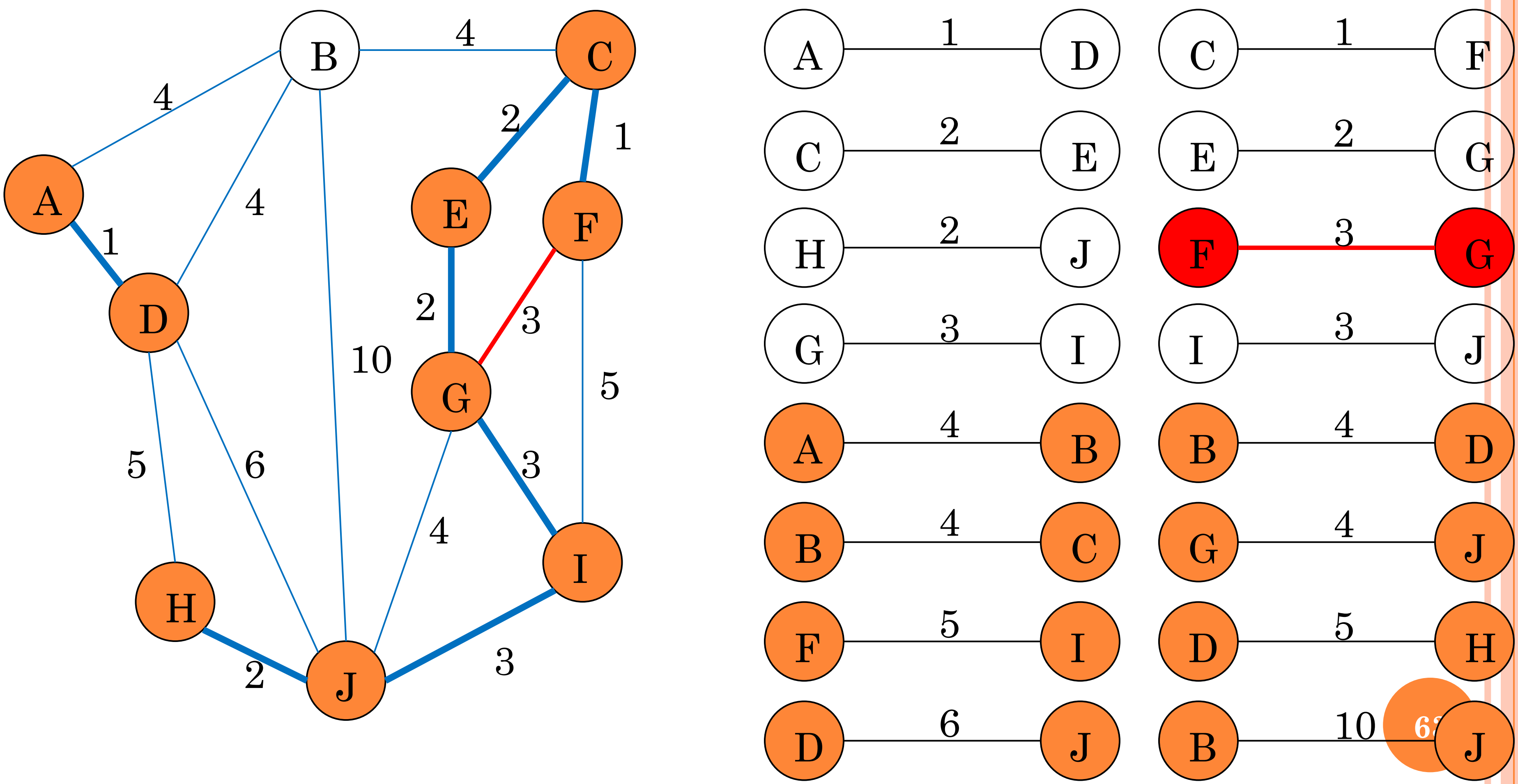
- Add Edge





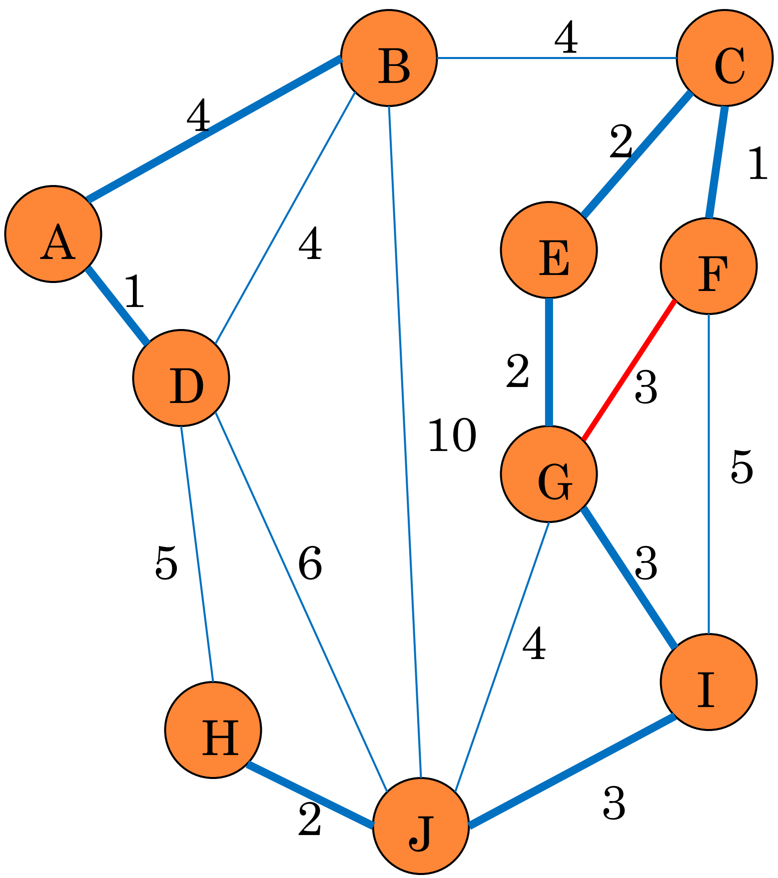
Kruskal's Algorithm – Example

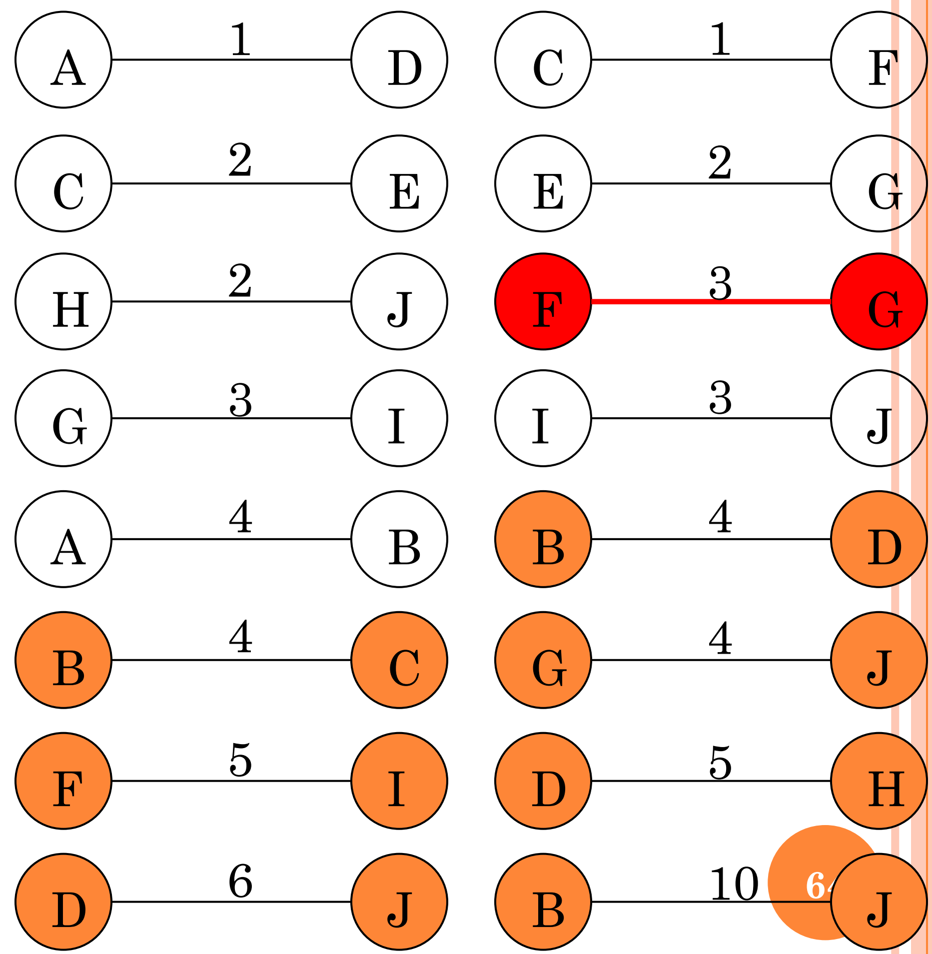
- Add Edge



Kruskal's Algorithm – Example

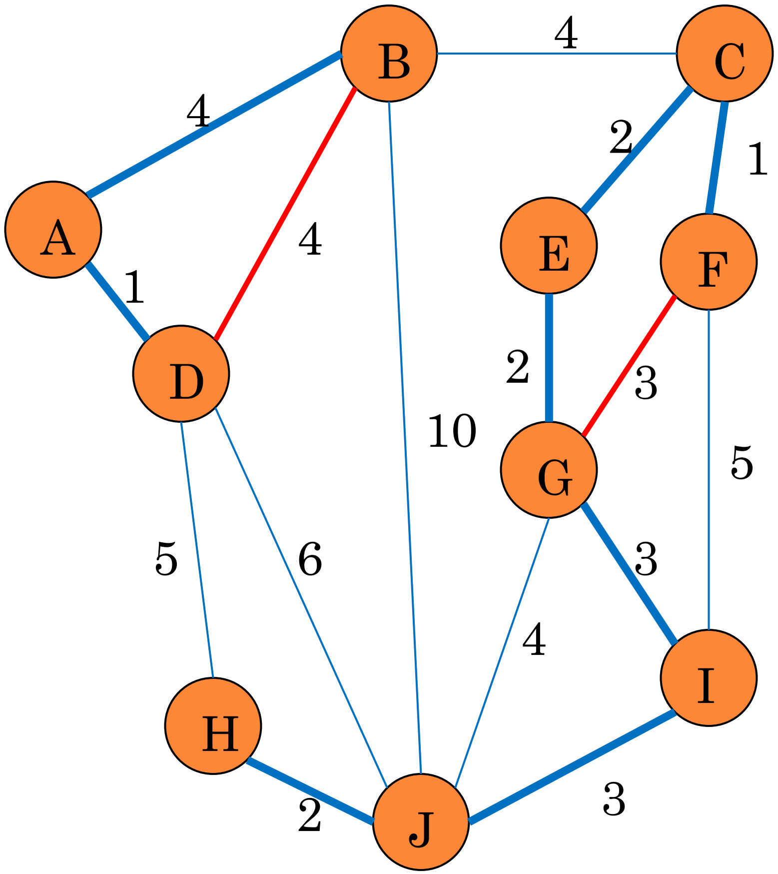
- Add Edge

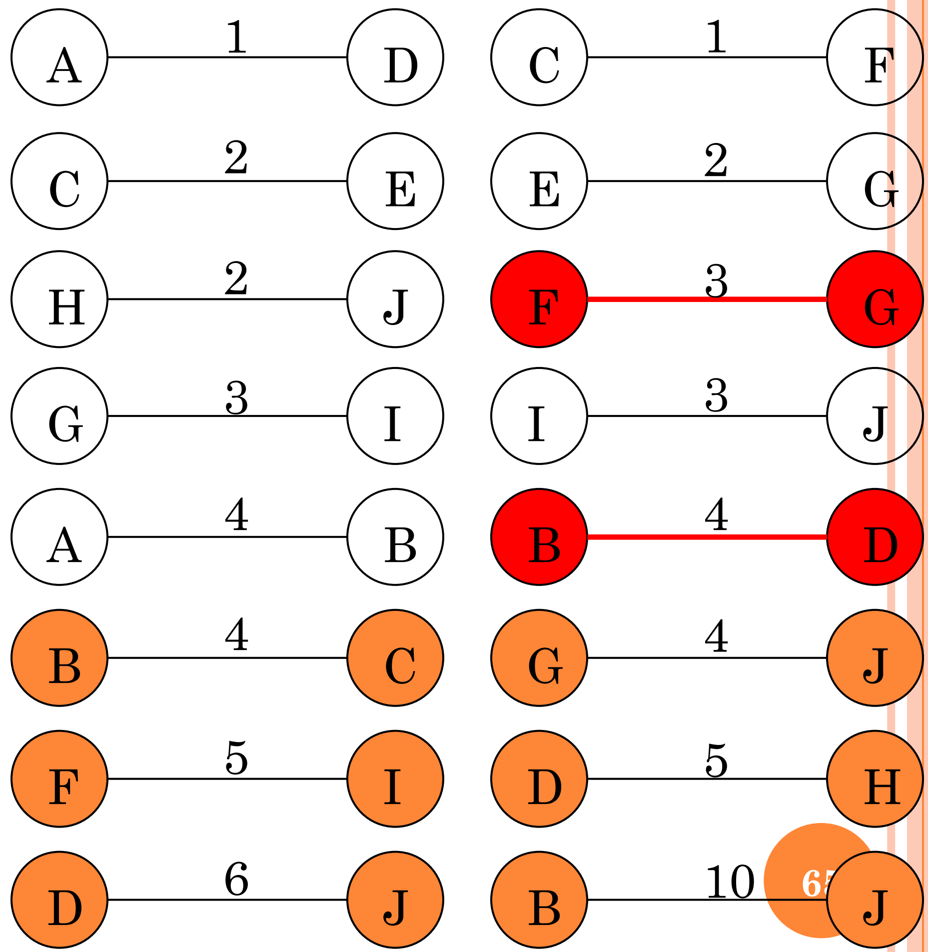




Kruskal's Algorithm – Example

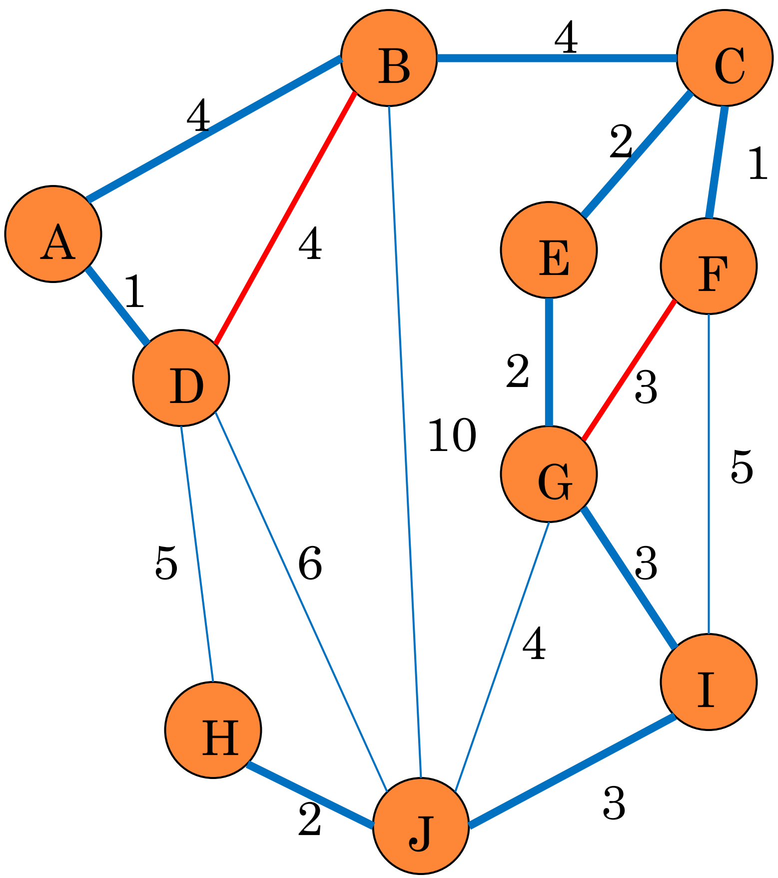
- Add Edge

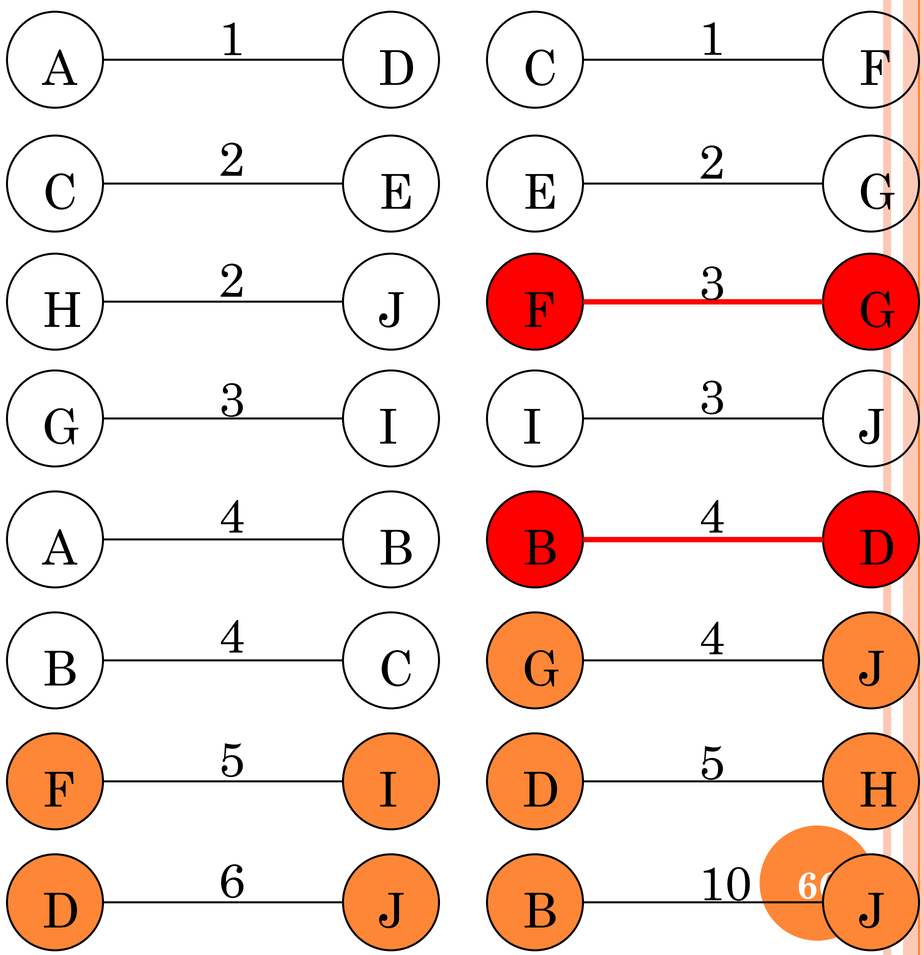




Kruskal's Algorithm – Example

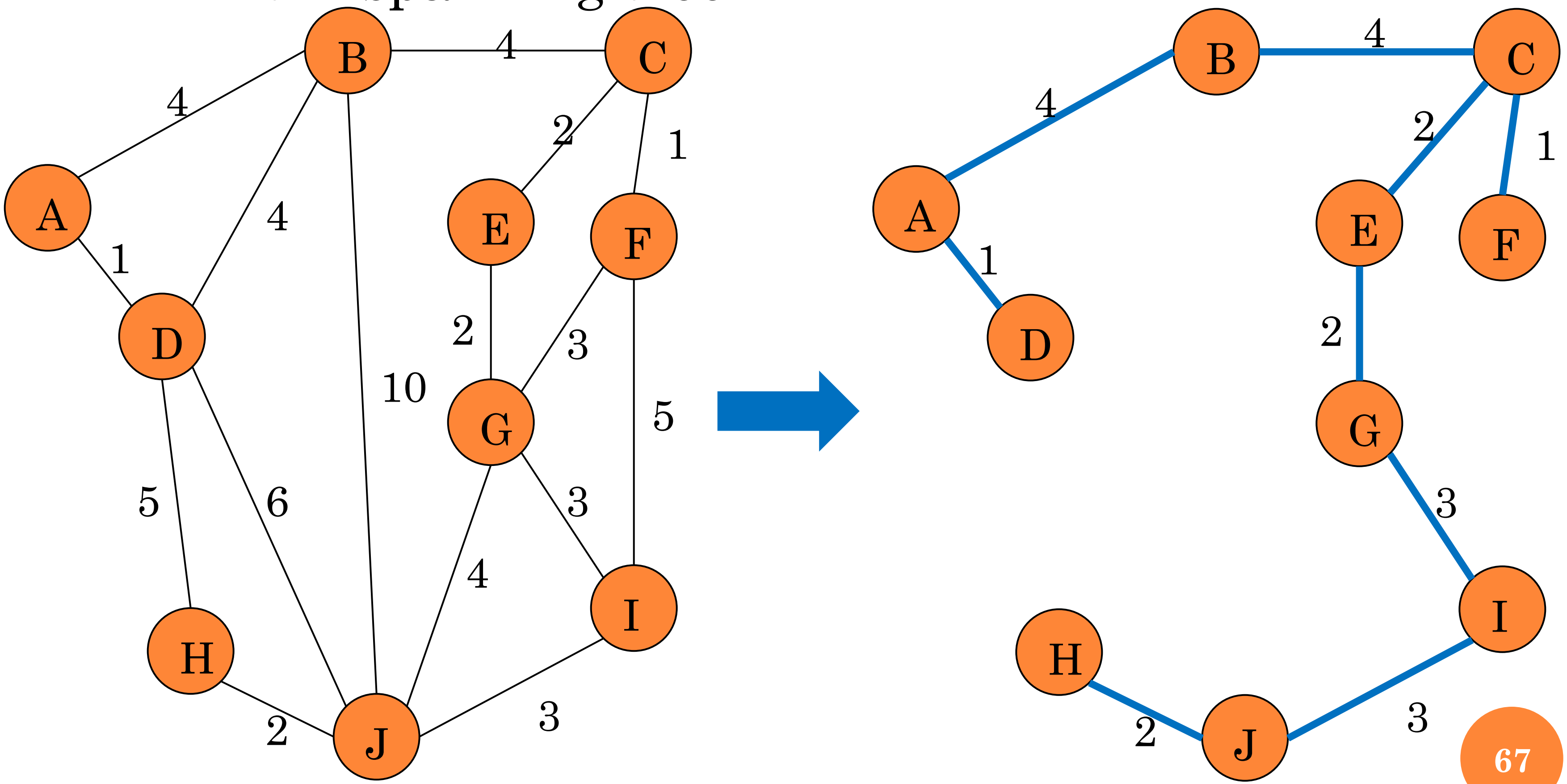
- Add Edge





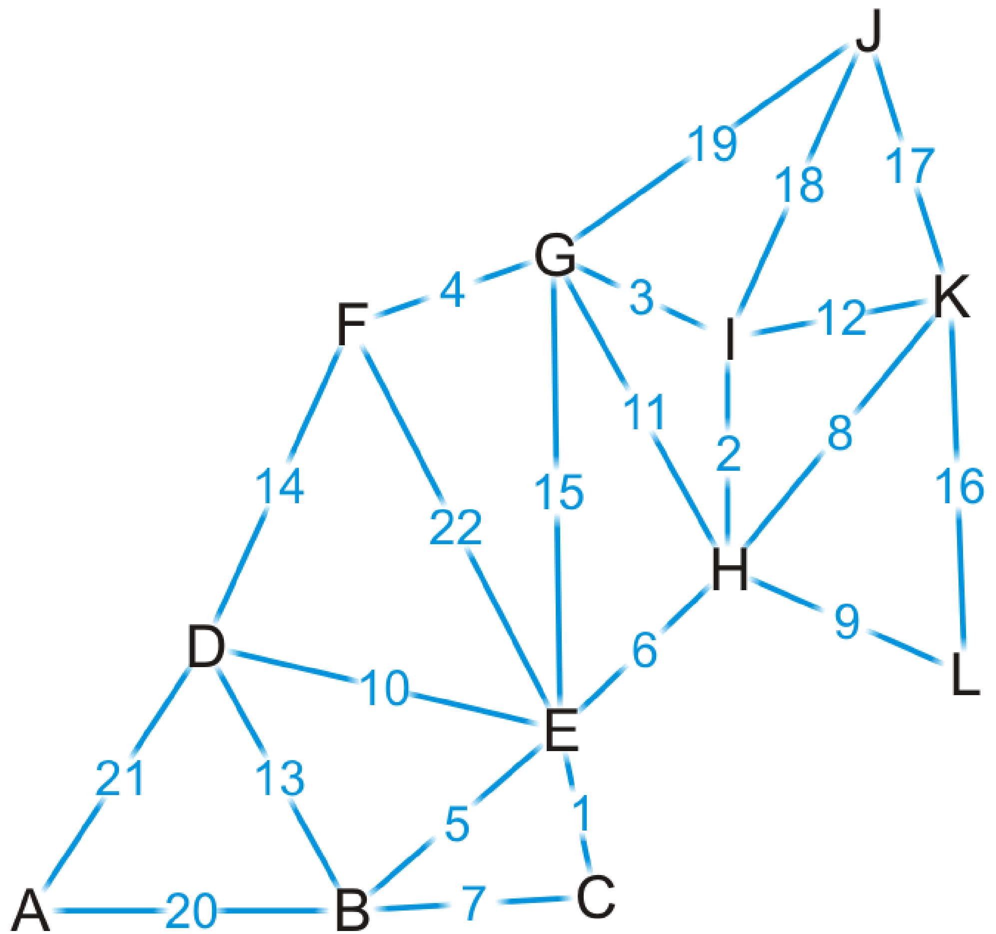
Kruskal's Algorithm – Example

- Minimum spanning tree

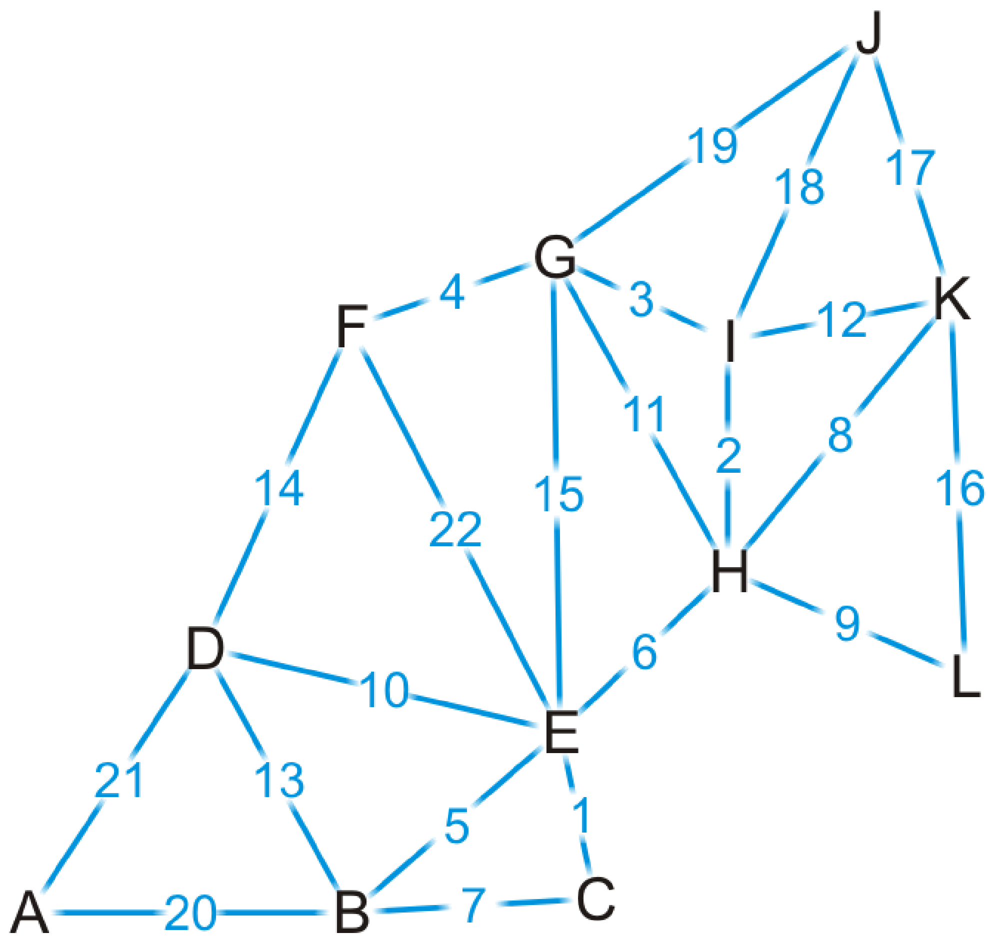


Kruskal's Algorithm – Example

- Complete graph



Kruskal's Algorithm – Example



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
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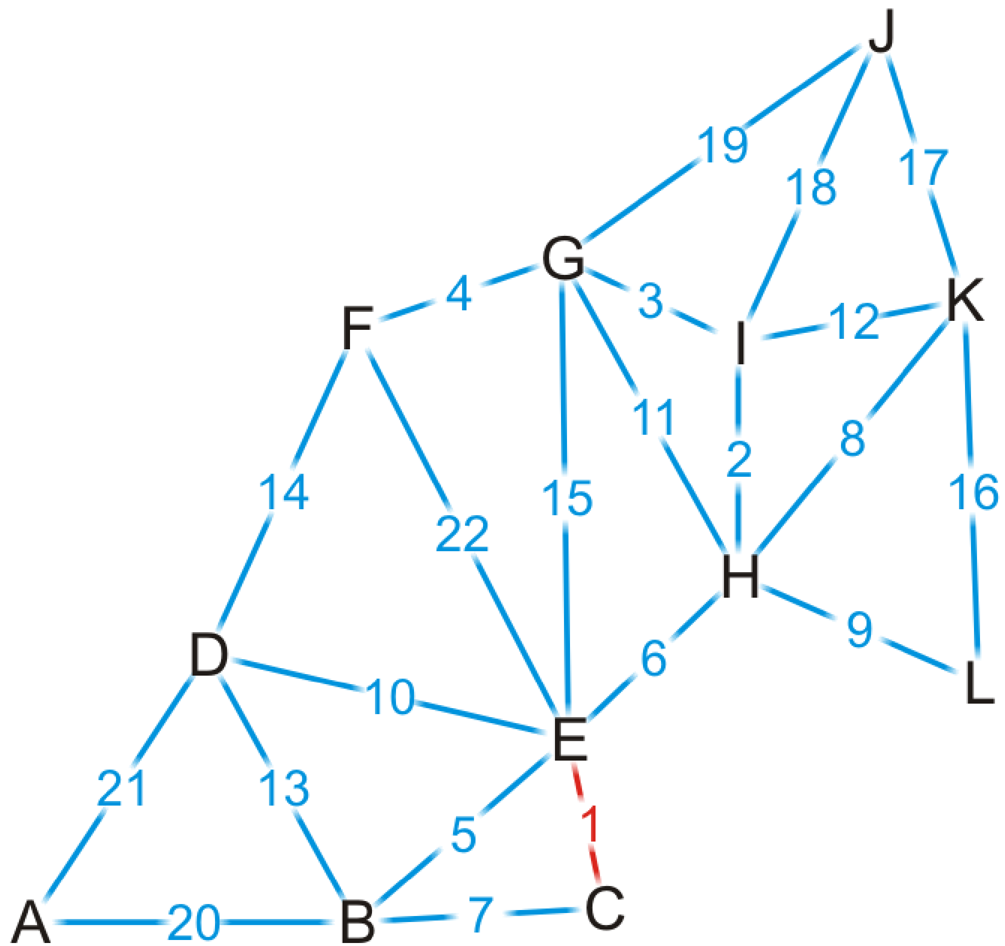
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example



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{B, C}
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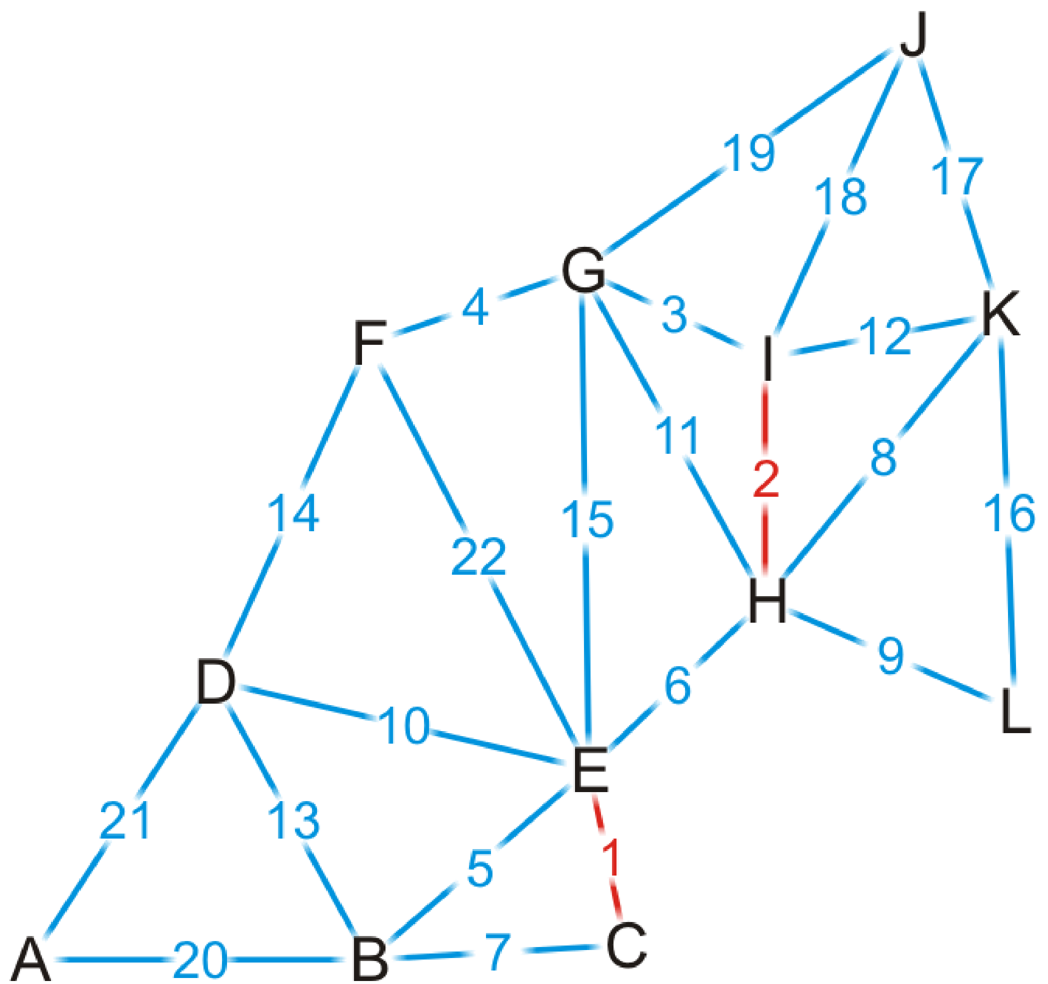
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

KRUSKAL'S ALGORITHM – EXAMPLE



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{H, K}
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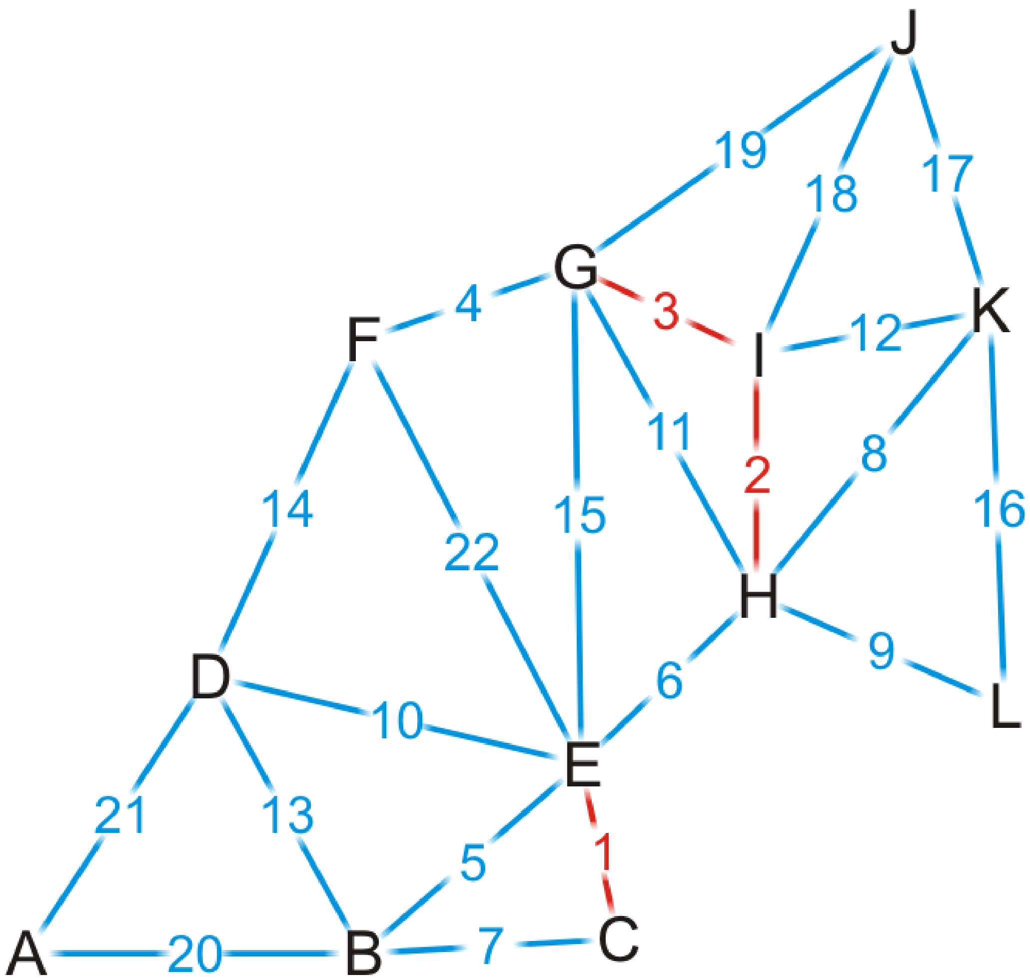
{A, D}
{E, F}

KRUSKAL'S ALGORITHM – EXAMPLE



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{B, C}
{H, K}
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{B, D}
{D, F}
{E, G}
{K, L}
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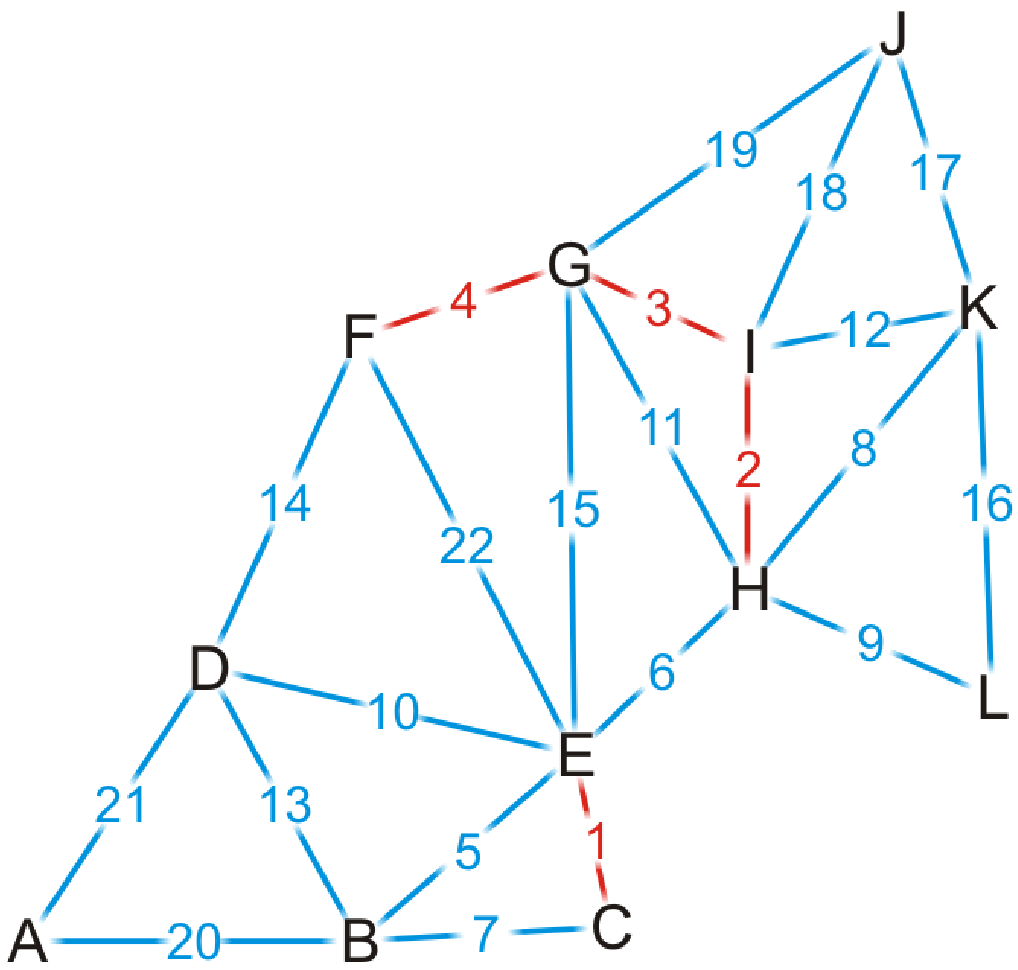
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example



{C, E}
{H, I}
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{E, H}
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{H, K}
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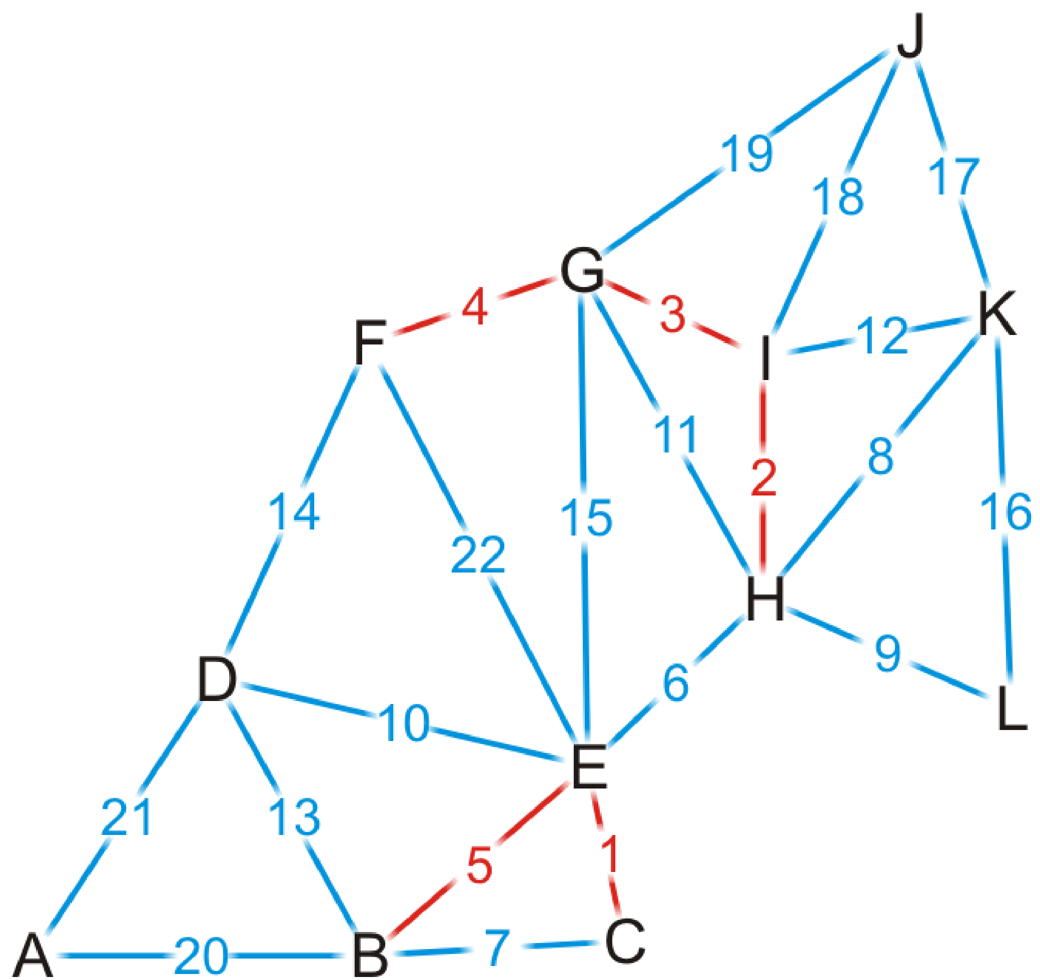
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example



{C, E}
{H, I}
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{E, H}
{B, C}
{H, K}
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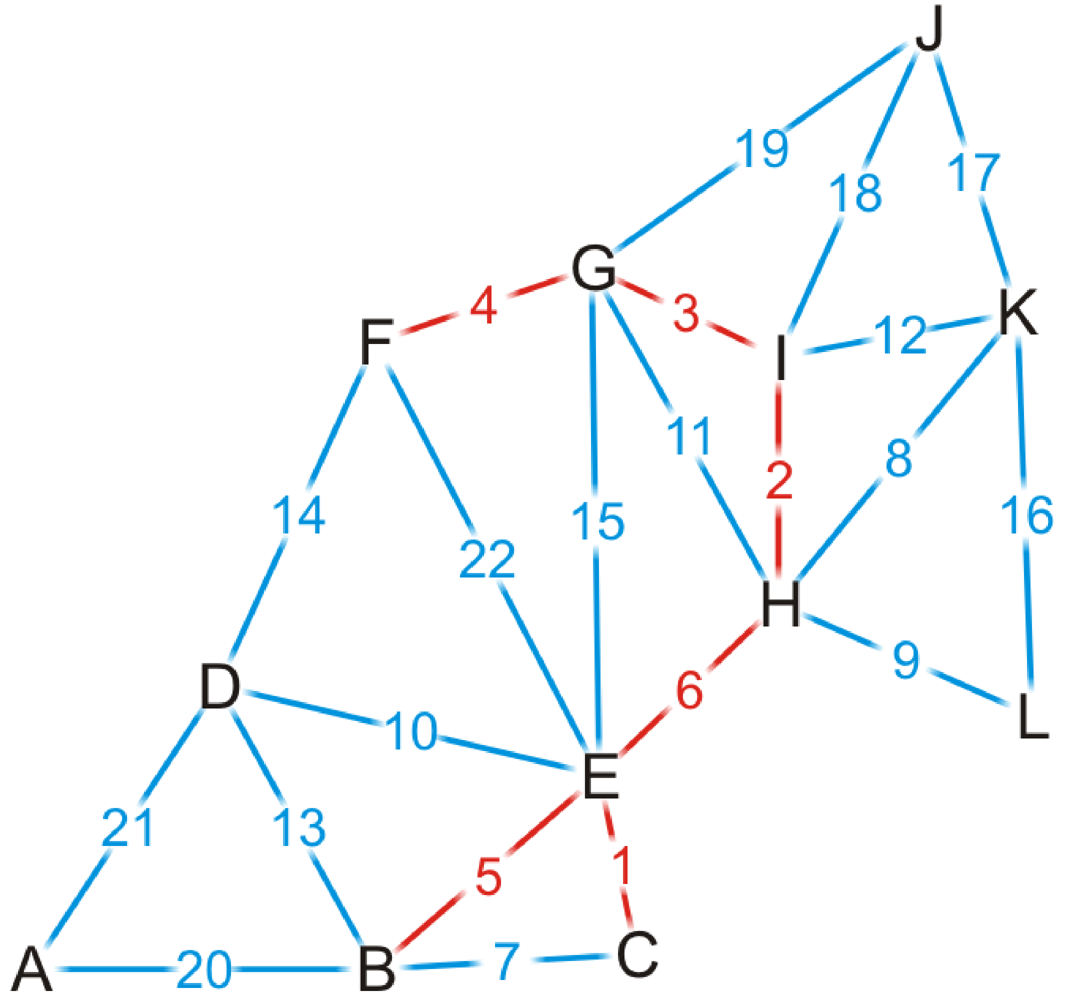
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

- We add edge {E, H}
  - This coalesces the two spanning sub-trees into one



{C, E}
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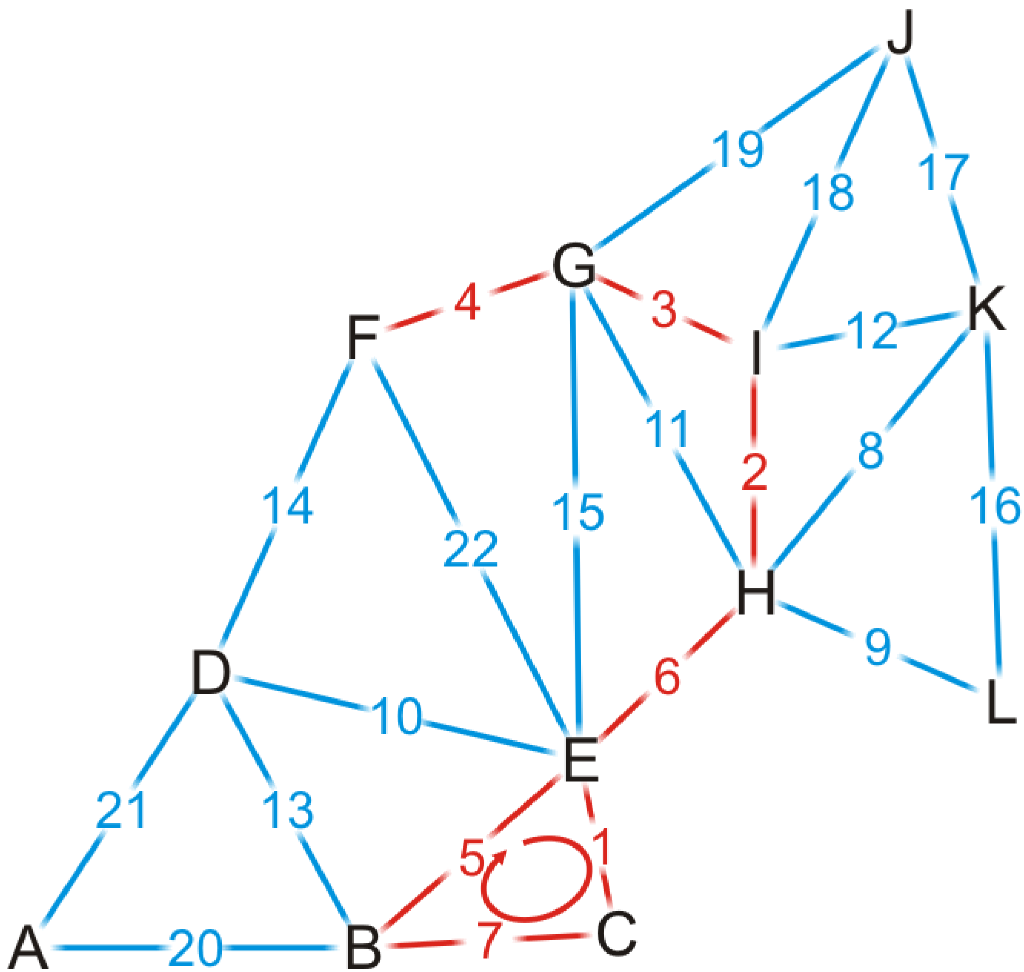
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

- We try adding {B, C}, but it creates a cycle



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{D, E}
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{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
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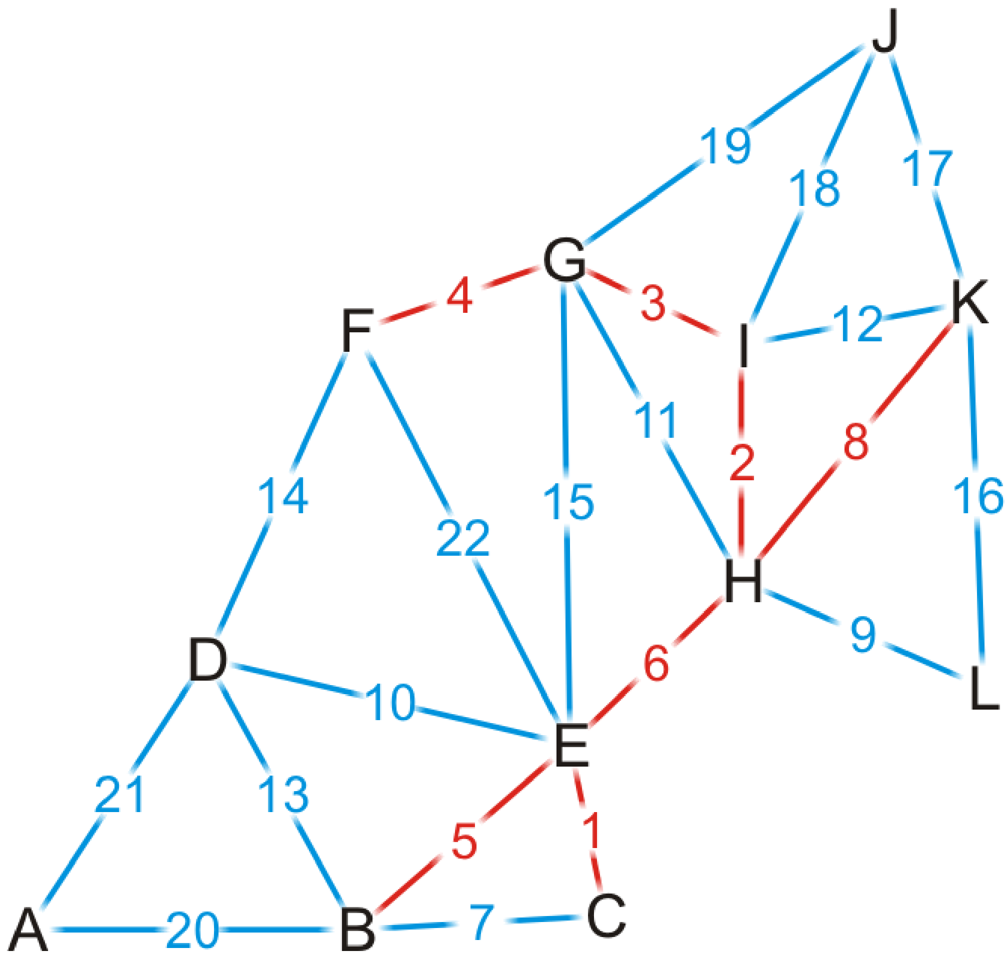
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example



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{B, D}
{D, F}
{E, G}
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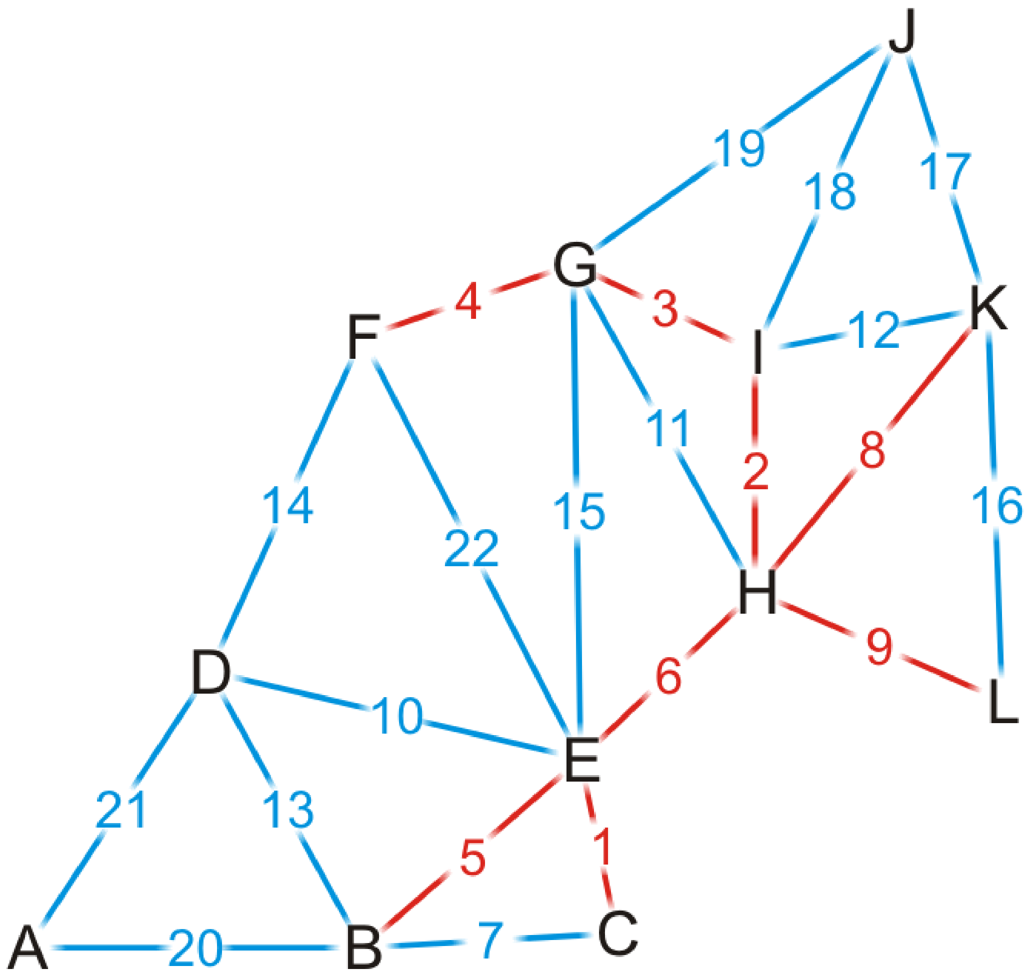
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example



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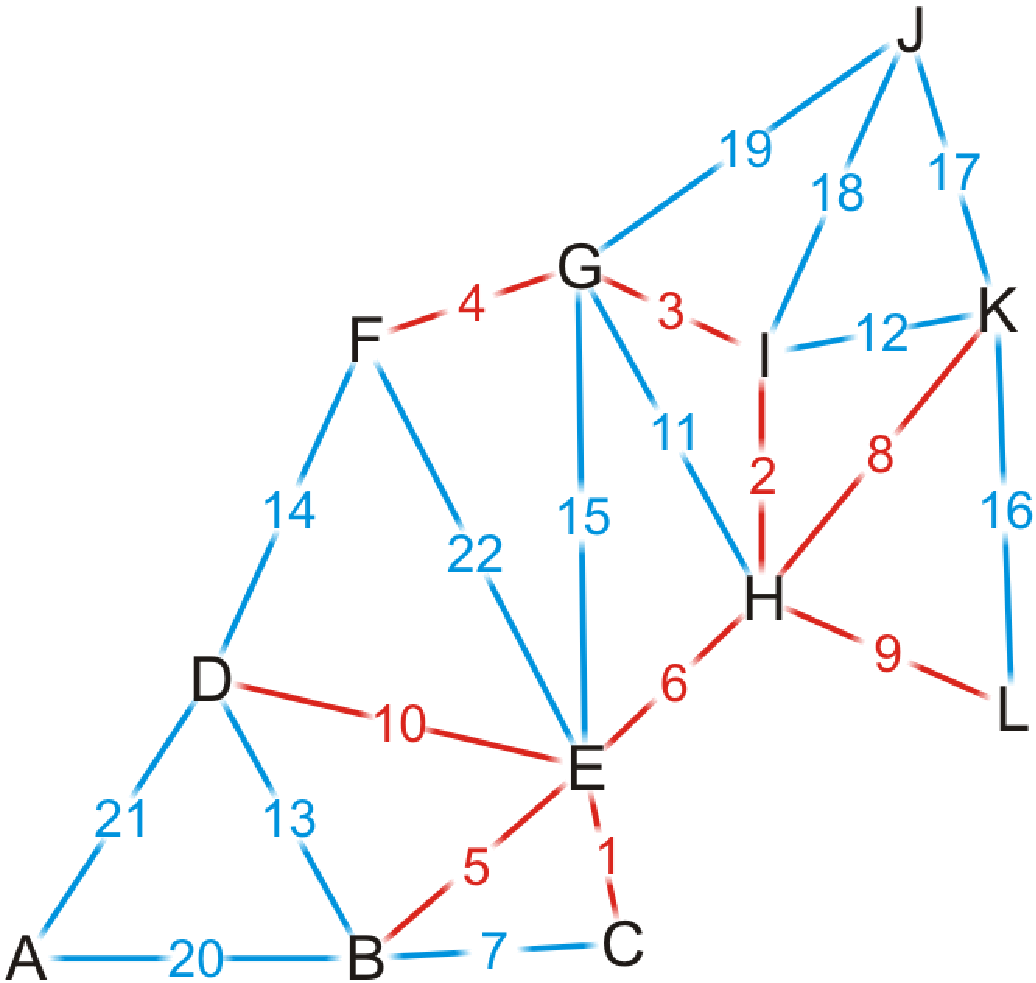
{A, D}
{E, F}

Kruskal's Algorithm – Example



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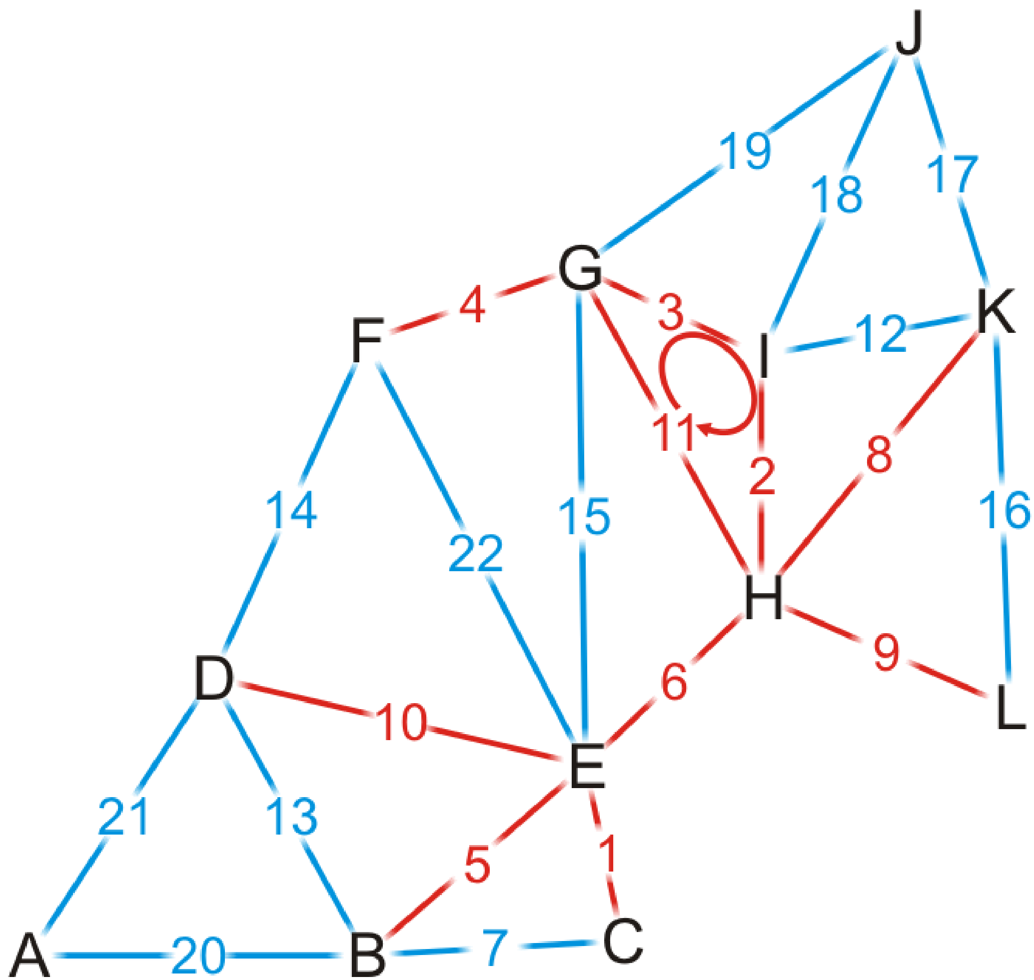
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

- We try adding {G, H}, but it creates a cycle



{C, E}
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{I, K}
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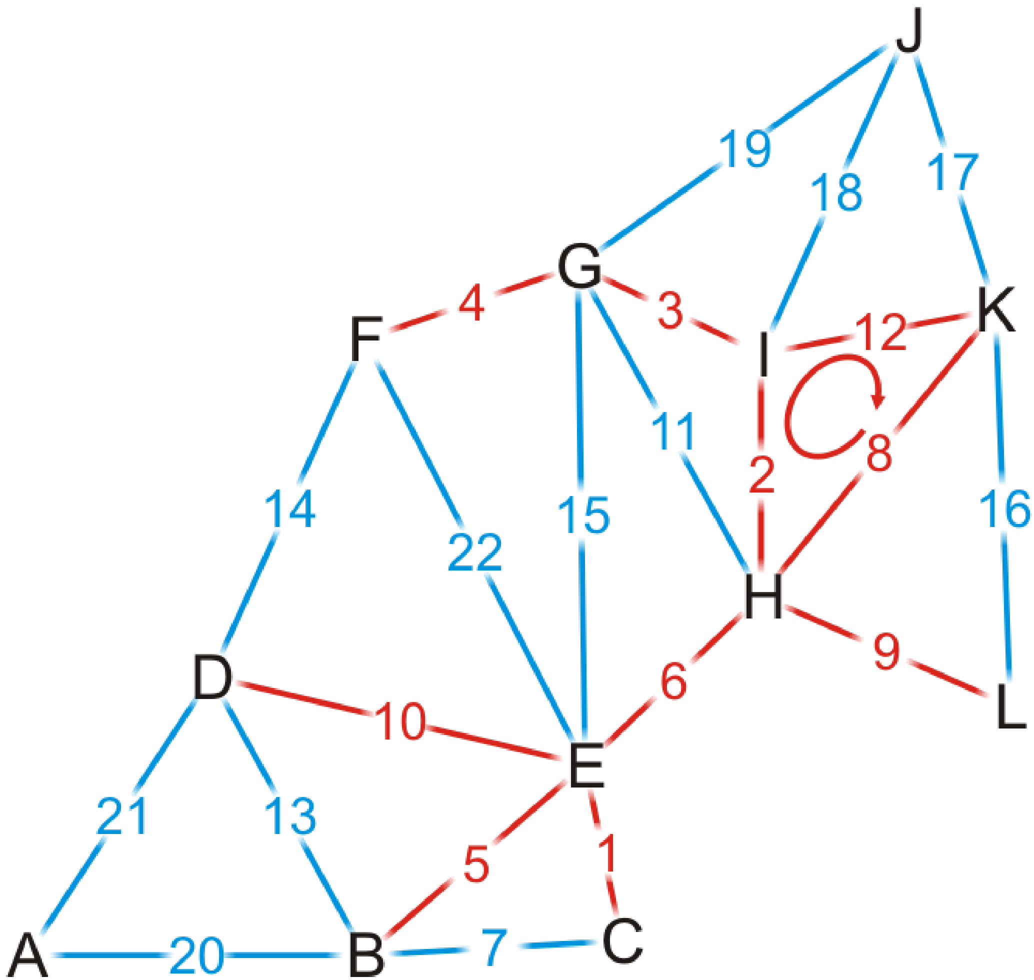
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

- We try adding {I, K}, but it creates a cycle



{C, E}
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{H, K}
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{D, E}
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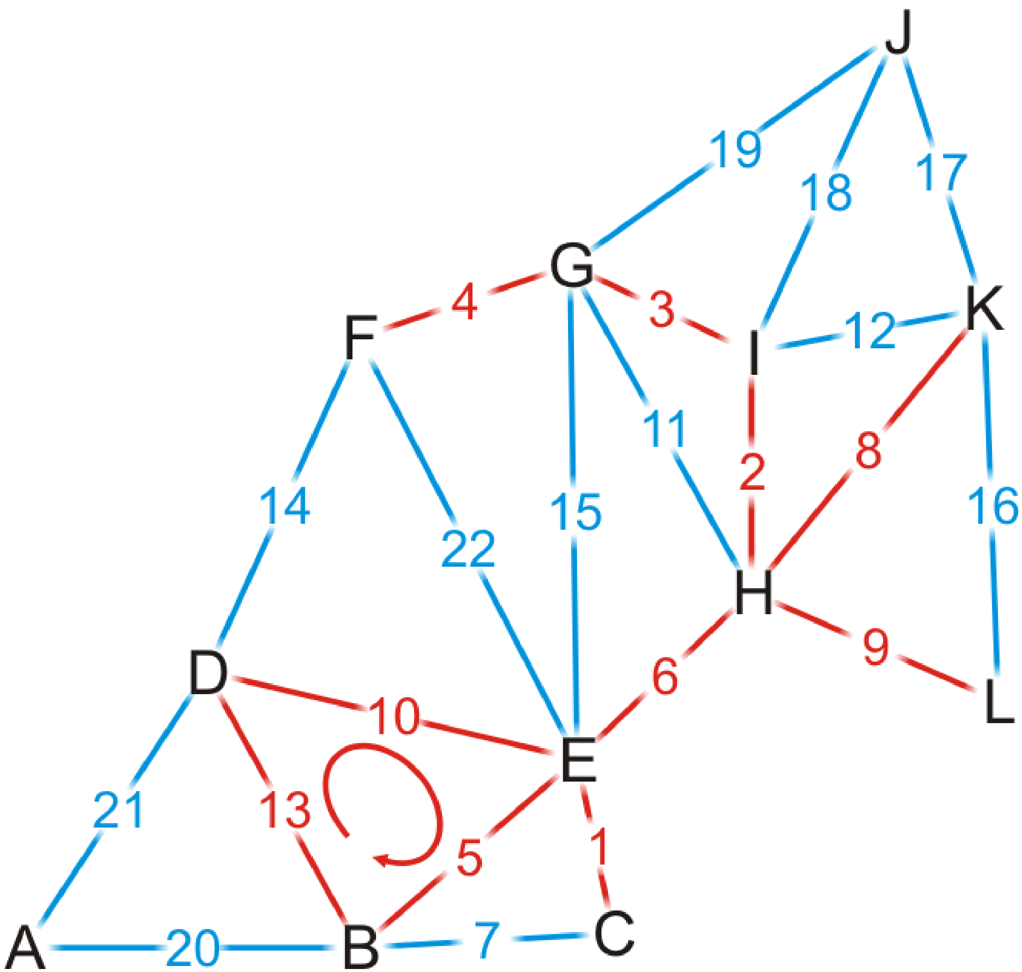
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

- We try adding {B, D}, but it creates a cycle



{C, E}
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{F, G}
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{E, H}
{B, C}
{H, K}
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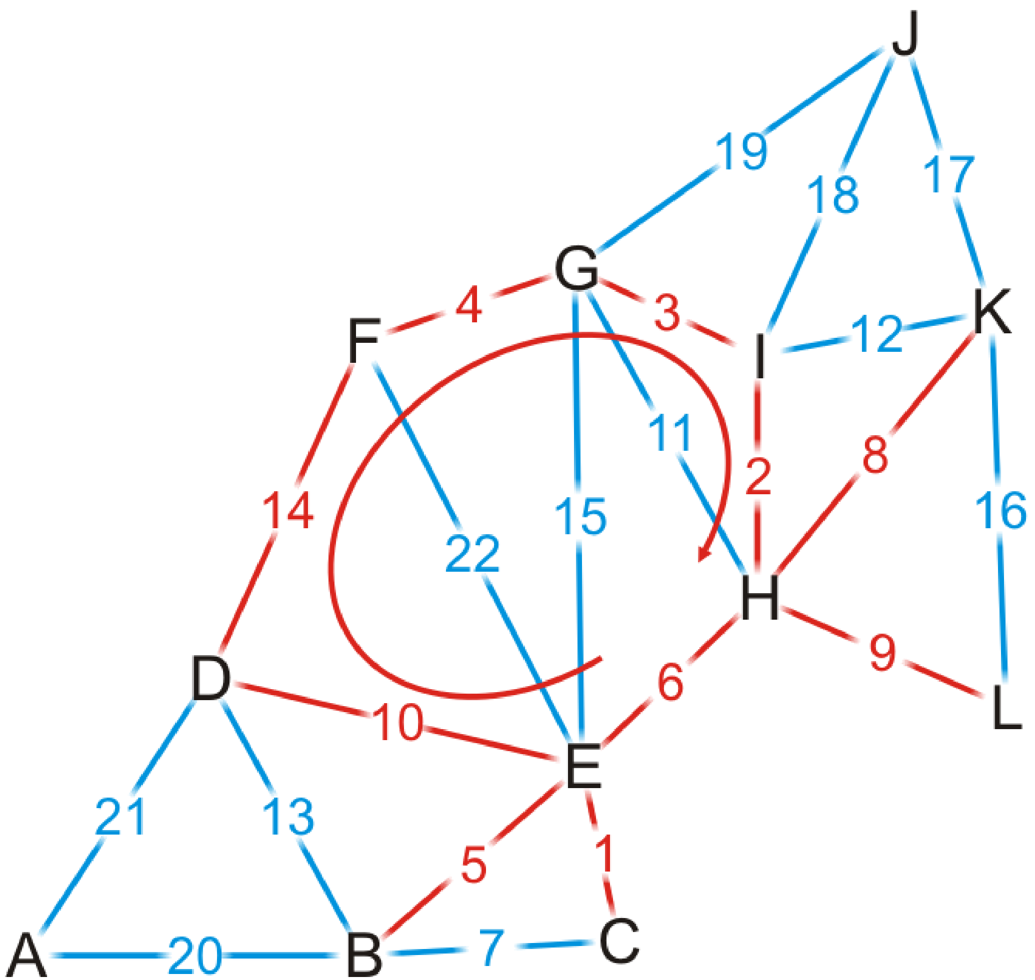
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

- We try adding {D, F}, but it creates a cycle



{C, E}
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{B, D}
{D, F}
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{K, L}
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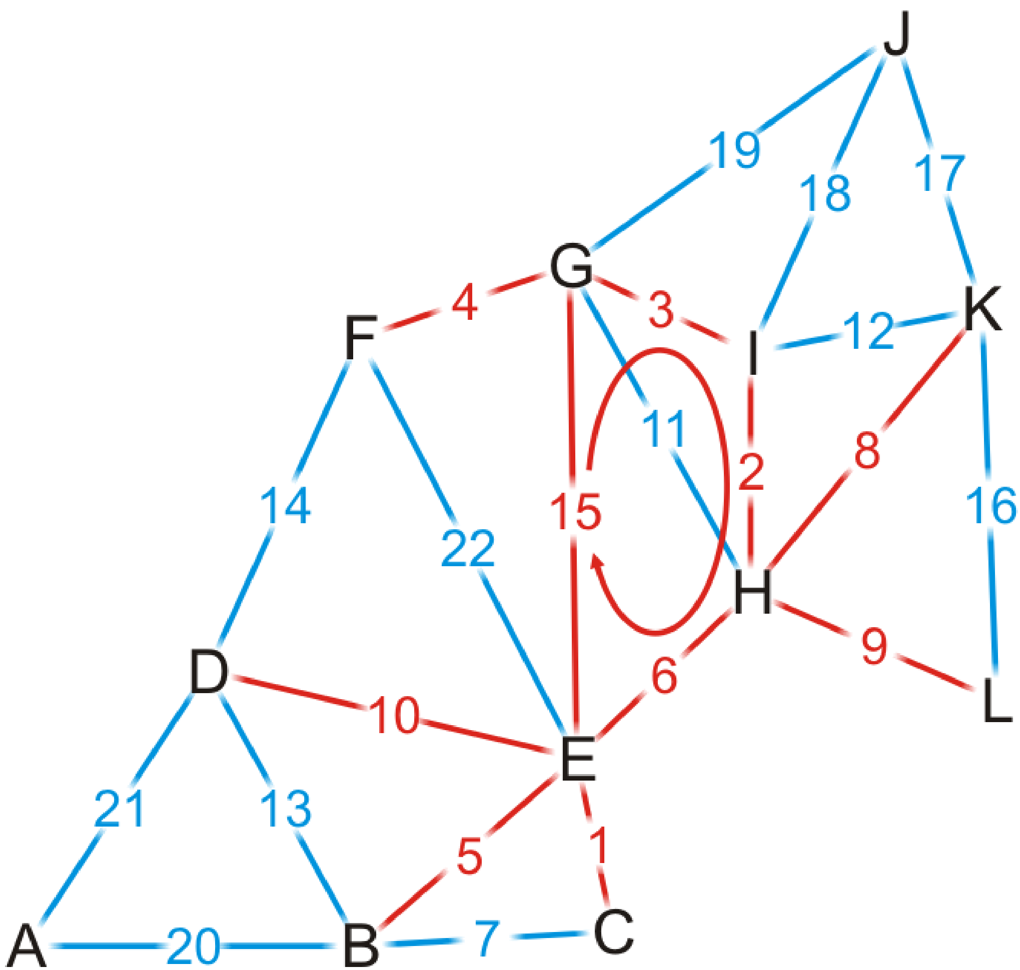
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

KRUSKAL'S ALGORITHM – EXAMPLE

- We try adding {E, G}, but it creates a cycle



{C, E}
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{B, E}
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{B, C}
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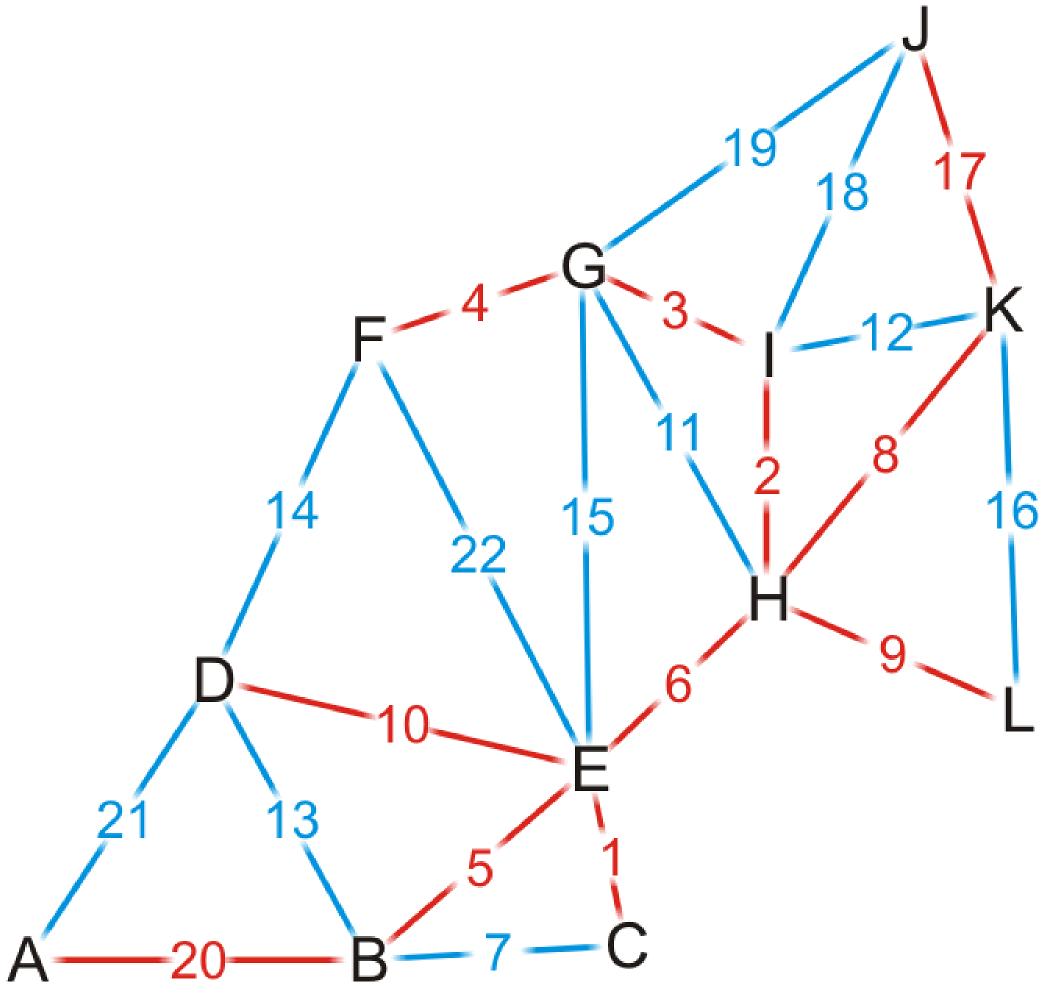
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

- By observation, we can still add edges {J, K} and {A, B}



{C, E}
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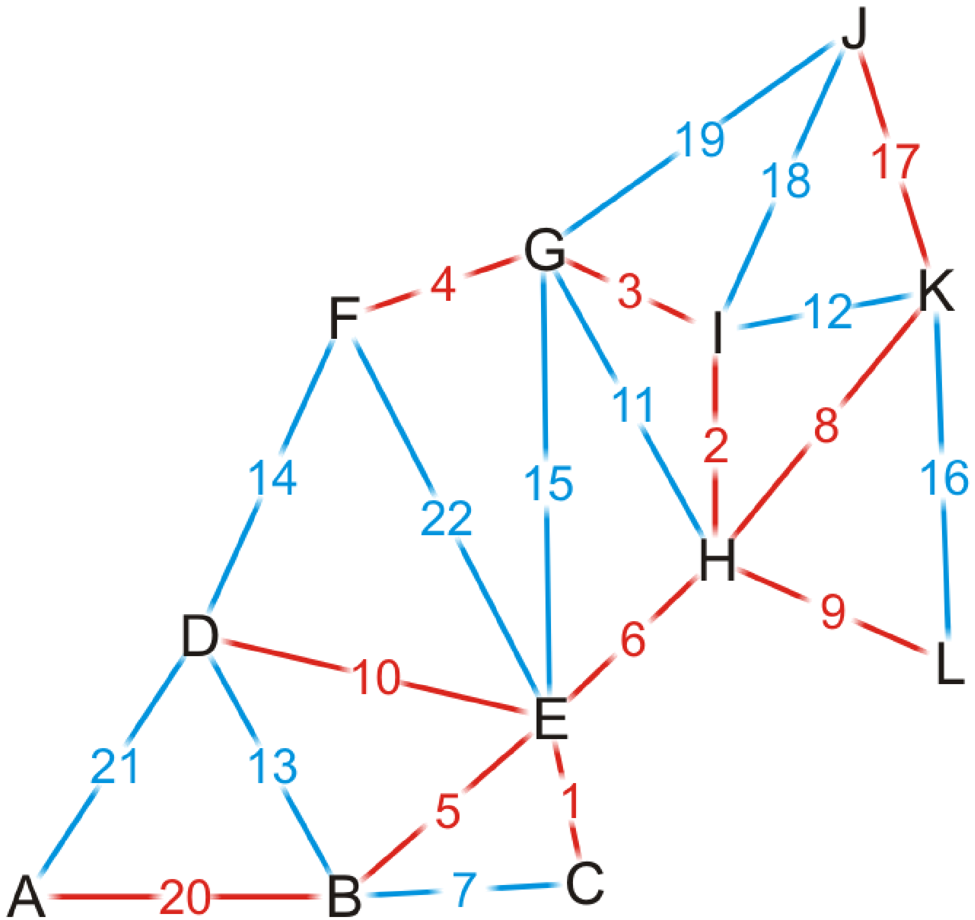
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

- Having added {A, B}, we now have 11 edges
  - We terminate the loop
  - We have our minimum spanning tree



{C, E}
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{E, H}
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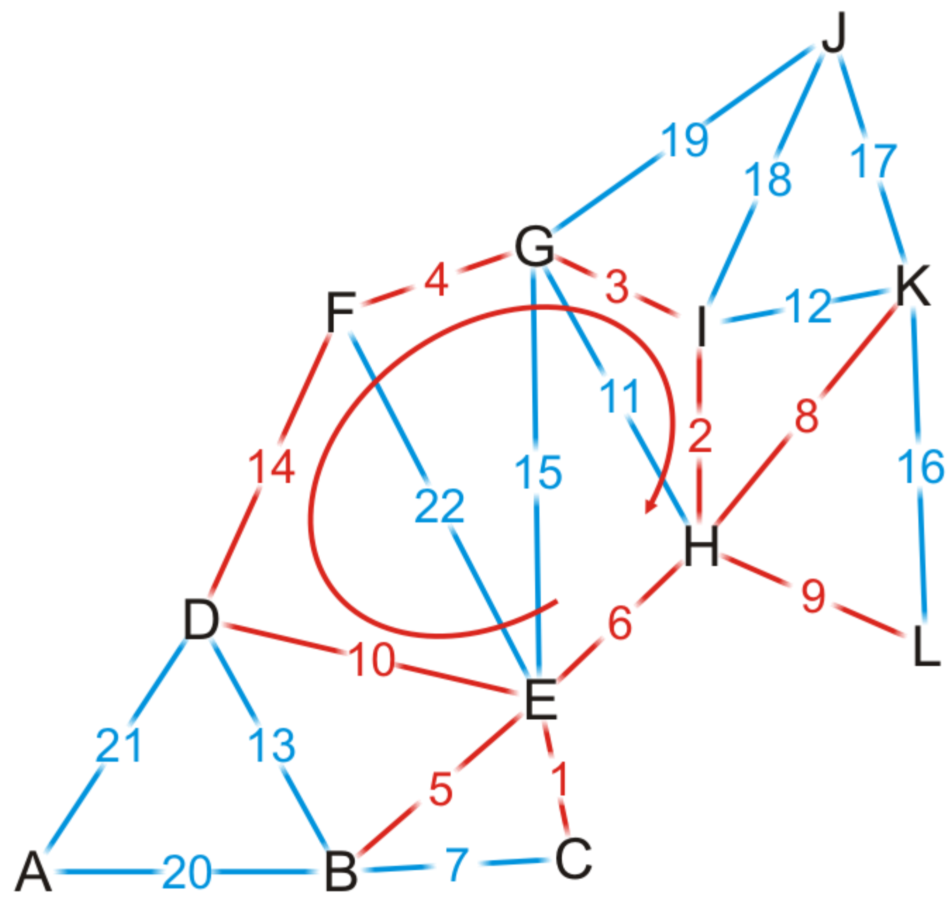
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Detecting a Cycle

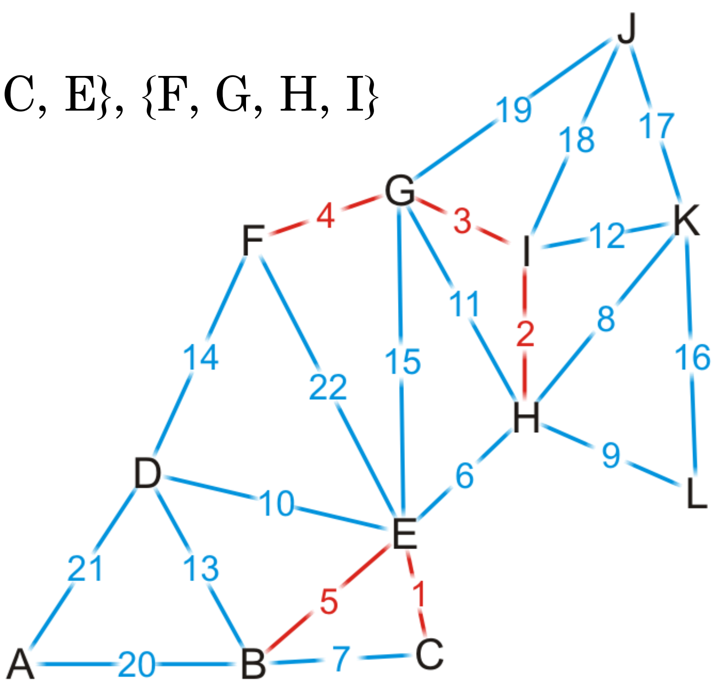
- To determine if a cycle is created, we could perform a traversal
  - A run-time of `O(|V|)`



Detecting a Cycle – Disjoint Sets

- Consider edges in the same connected sub-graph as forming a set

{B, C, E}, {F, G, H, I}

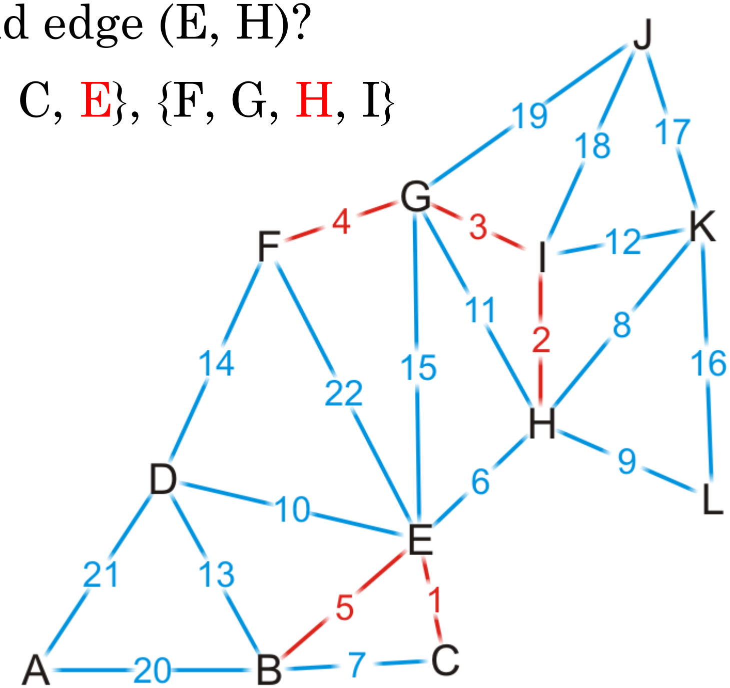


Detecting a Cycle – Disjoint Sets

- Consider edges in the same connected sub-graph as forming a set
- If the vertices of the next edge are in different sets
  - Take the union of the two sets

Add edge (E, H)?

{B, C, E}, {F, G, H, I}

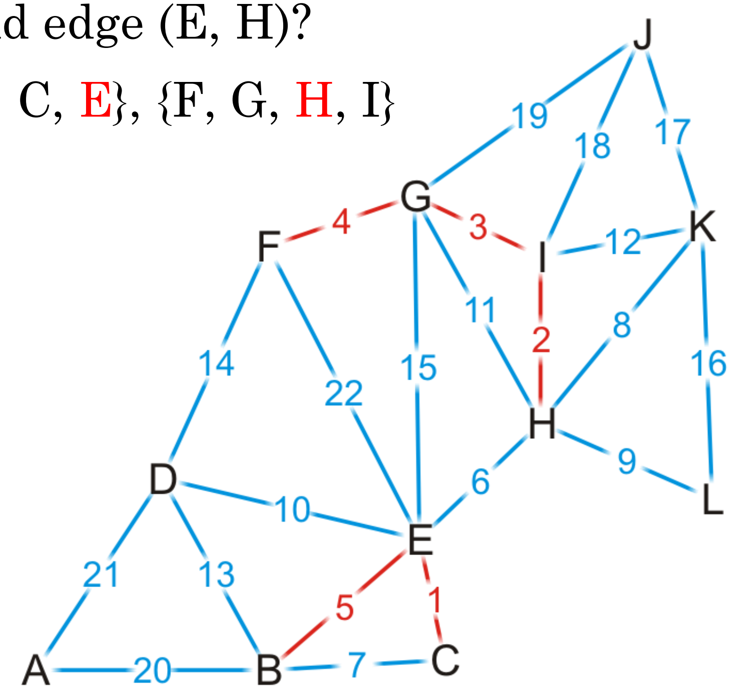


Detecting a Cycle – Disjoint Sets

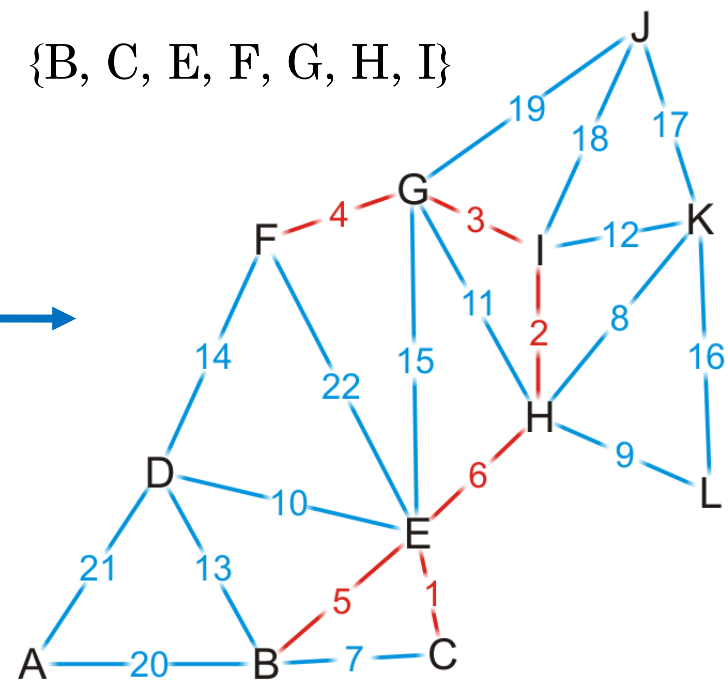
- Consider edges in the same connected sub-graph as forming a set
- If the vertices of the next edge are in different sets
  - Take the union of the two sets

Add edge (E, H)?

{B, C, E}, {F, G, H, I}



{B, C, E, F, G, H, I}

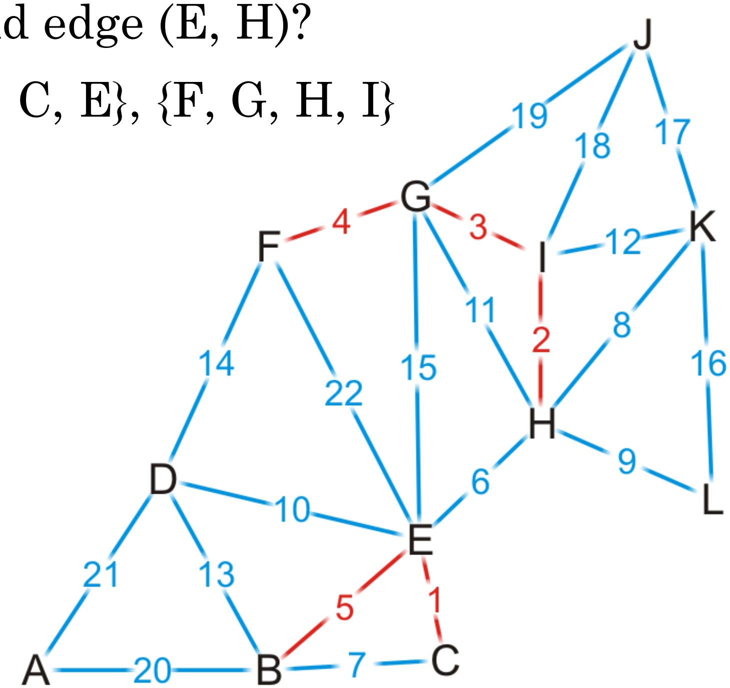


Detecting a Cycle – Disjoint Sets

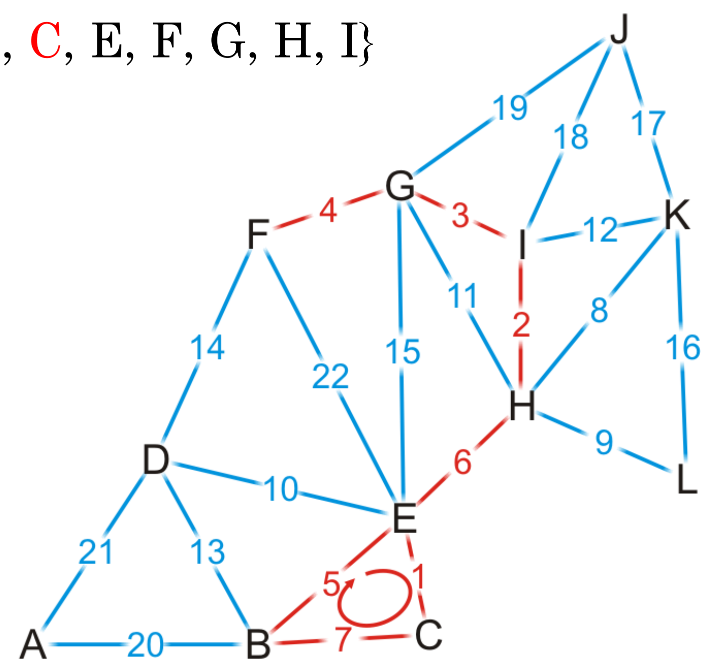
- Consider edges in the same connected sub-graph as forming a set
- If the vertices of the next edge are in different sets
  - Take the union of the two sets
- Do not add an edge if both vertices are in the same set

Add edge (E, H)?

{B, C, E}, {F, G, H, I}



{B, C, E, F, G, H, I}



KRUSKAL(V, E, w)

```
A ← ∅
for each vertex v ∈ V                                   O(V)
        do MAKE-SET(v)
sort E into non-decreasing order by w                   O(ElgE)
for each (u, v) taken from the sorted list              O(E)
        do if FIND-SET(u) ≠ FIND-SET(v)
                then A ← A ∪ {(u, v)}
                        UNION(u, v)                     O(lgV)
return A
```

Running time: O(V+ElgE+ElgV)=O(ElgE) – dependent on the implementation of the disjoint-set data structure

KRUSKAL(V, E, w) (CONT.)

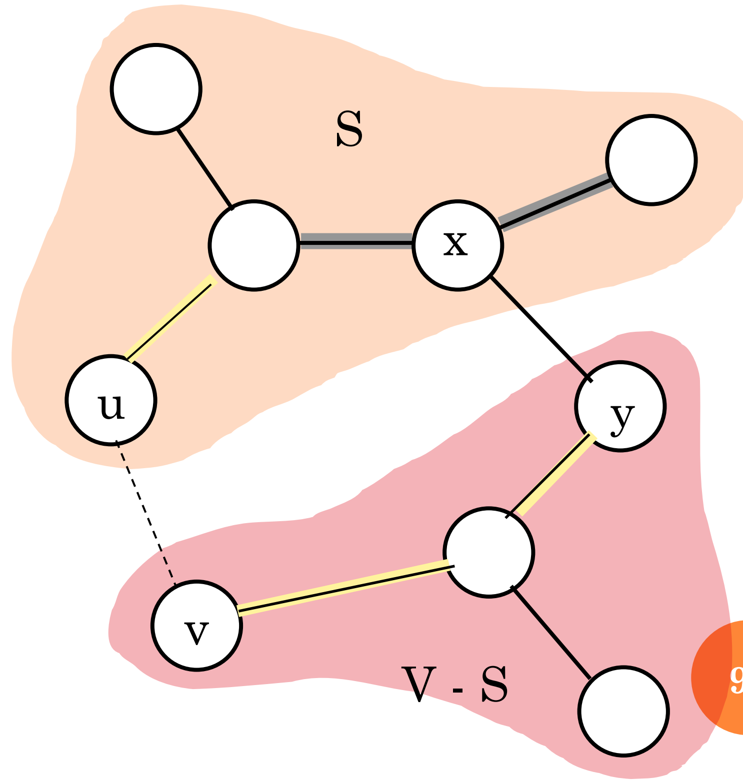
```
A ← ∅
for each vertex v ∈ V                                  O(V)
        do MAKE-SET(v)
sort E into non-decreasing order by w                  O(ElgE)
for each (u, v) taken from the sorted list             O(E)
        do if FIND-SET(u) ≠ FIND-SET(v)
                then A ← A ∪ {(u, v)}
                        UNION(u, v)                    O(lgV)
return A
```

- Running time: $O(V+ElgE+ElgV)=O(ElgE)$
- Since $E=O(V^2)$, we have $lgE=O(2lgV)=O(lgV)$

O(ElgV)

Kruskal's Algorithm

- Kruskal's algorithm is a **"greedy"** algorithm
- Kruskal's greedy strategy produces a globally optimum solution
- Proof for generic approach applies to Kruskal's algorithm too



Kruskal's Algorithm vs Prim's Algorithm

- In **prim's algorithm**, graph must be a **connected**
- **Kruskal's** algorithm can function on **disconnected** graphs too
- **Prim's algorithm** is significantly faster for **dense graphs** with more number of edges than vertices
- **Kruskal's algorithm** runs faster in the case of **sparse graphs**

Problem 1

- **(Exercise 23.2-3, page 637**) Compare Prim's algorithm with and Kruskal's algorithm assuming:

(a) sparse graphs:

In this case, E=O(V)

Kruskal:

O(ElgE)=O(VlgV)

Prim:

- binary heap: O(ElgV)=O(VlgV)
- Fibonacci heap: O(VlgV+E)=O(VlgV)

Problem 1 (cont.)

(b) dense graphs

In this case, $E=O(V^2)$

Kruskal:

$O(ElgE)=O(V^2lgV^2)=O(2V^2lgV)=O(V^2lgV)$

Prim:

- binary heap: $O(ElgV)=O(V^2lgV)$
- Fibonacci heap: $O(VlgV+E)=O(VlgV+V^2)=O(V^2)$

Problem 2

- Suppose that some of the weights in a connected graph G are negative. Will Prim's algorithm still work? What about Kruskal's algorithm? Justify your answers.
  - Yes, both algorithms will work with negative weights. Review the proof of the generic approach; there is no assumption in the proof about the weights being positive.

Problem 3

- Find an algorithm for the "maximum" spanning tree. That is, given an undirected weighted graph G, find a spanning tree of G of maximum cost. Prove the correctness of your algorithm.
  - Consider choosing the "heaviest" edge (i.e., the edge associated with the largest weight) in a cut. The generic proof can be modified easily to show that this approach will work.
  - Alternatively, multiply the weights by -1 and apply either Prim's or Kruskal's algorithms without any modification at all!

MST ALGORITHMS

Kruskal's algorithm (see CLRS):

- Uses the ***disjoint-set data structure*** (see CLRS, Ch. 21).
- Running time $= O(E \lg V)$.

Best to date:

- Karger, Klein, and Tarjan [1993].
- Randomized algorithm.
- $O(V + E)$ expected time.

Reference

- **Introduction to Algorithms**
  - Minimum Spanning Tree
  - Chapter # 23
  - Thomas H. Cormen

  - Disjoint Set
  - http://www.csl.mtu.edu/cs4321/www/Lectures/Lecture%2019%20-%20Kruskal%20Algorithm%20and%20Dis-joint%20Sets.htm

Summary of Graph Algortihms

Breadth-first search

BFS(G, s)
for each vertex $u \; V[G] - \{s\}$
  do $color[u] \leftarrow$ WHITE
    $d[u] \leftarrow \infty$
    $\pi[u] \leftarrow$ NIL
$color[s] \leftarrow$ GRAY
$d[s] \leftarrow 0$
$\pi[s] \leftarrow$ NIL
$Q \leftarrow \varnothing$
ENQUEUE(Q, s)
while $Q \neq \varnothing$
  do $u \leftarrow$ DEQUEUE(Q)
    for each $v \in Adj[u]$
      do if $color[v]$ = WHITE
        then $color[v] \leftarrow$ GRAY
          $d[v] \leftarrow d[u] + 1$
          $\pi[v] \leftarrow u$
          ENQUEUE(Q, v)
$color[u] \leftarrow$ BLACK

Depth-first search

DFS(G)
for each vertex $u \in G.V$
  $u.color$ = WHITE
  $u.\pi$ = NIL
$time = 0$
for each vertex $u \in G.V$
  if $u.color$ == WHITE
    DFS-VISIT(G, u)

DFS-VISIT(G, u)
$time = time + 1$ // whi
$u.d = time$
$u.color$ = GRAY
for each $v \in G.Adj[u]$ // exp
  if $v.color$ == WHITE
    $v.\pi = u$
    DFS-VISIT(G, v)
$u.color$ = BLACK // blac
$time = time + 1$
$u.f = time$

Analysis of Prim

$$\left.\begin{array}{c}\Theta(V)\\ \text{total}\end{array}\right\{\begin{array}{l} Q \leftarrow V \\ key[v] \leftarrow \infty \text{ for all } v \in V \\ key[s] \leftarrow 0 \text{ for some arbitrary } s \in V \end{array}$$

$$\left.\begin{array}{c}|V|\\ \text{times}\end{array}\right\{\begin{array}{l} \textbf{while } Q \neq \emptyset \\ \quad \textbf{do } u \leftarrow \text{EXTRACT-MIN}(Q) \\ \quad \left.\begin{array}{c} degree(u) \\ \text{times}\end{array}\right\{\begin{array}{l} \textbf{for} \text{ each } v \in Adj[u] \\ \quad \textbf{do if } v \in Q \text{ and } w(u,v) < key[v] \\ \qquad \textbf{then } key[v] \leftarrow w(u,v) \\ \qquad\qquad \pi[v] \leftarrow u \end{array}\end{array}$$

Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

Time $= \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

KRUSKAL(V, E, w) (CONT.)

```
A ← ∅                                                  } O(V)
for each vertex v ∈ V
        do MAKE-SET(v)                                 } O(ElgE)
sort E into non-decreasing order by w
for each (u, v) taken from the sorted list              ← O(E)
      do if FIND-SET(u) ≠ FIND-SET(v)
              then A ← A ∪ {(u, v)}                     } ← O(lgV)
                      UNION(u, v)
return A
```

- Running time: O(V+ElgE+ElgV)=O(ElgE)
- Since $E=O(V^2)$, we have lgE=O(2lgV)=O(lgV) } O(ElgV)

Shortest Path