Lecture 31

Singular Value Decomposition

Matrix Factorization

- Digitalization of a square matrix A

$$P^{-1}AP = D.$$

- Orthogonal Digitalization of symmetric matrix A

$$Q^TAQ = D.$$

- What will if matrix is not symmetric?

- What if even matrix is not square?

Motivation

- The absolute values of the eigenvalues of a symmetric matrix A measure the amounts that A stretches or shrinks certain vectors (the eigenvectors). If $Ax = \lambda x$ and $\|x\| = 1$, then

$$\|Ax\| = \|x\| = |\lambda|\|x\| = |\lambda|.$$

- If

$$A = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix},$$

then the linear transformation $x \to Ax$ maps the unit sphere $\{x : \|x| = 1\}$ in $\mathbb{R}^3$ onto an ellipse in $\mathbb{R}^2$. Find a unit vector x at which the length $\|Ax\|$ is maximized, and compute this maximum length.

$$\|Ax\|^2 = Ax \cdot Ax = (Ax)^T(Ax) = x^T(A^TA)x.$$

Problem is to maximize the quadratic form $x^T(A^TA)x$ subject to the constraint $\|x\| = 1$.

How to solve this problem?

Motivation

- The maximum value is attained at a unit eigenvector of $A^T A$ corresponding to λ_1.

$$A^T A = \begin{pmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{pmatrix}.$$

The eigenvalues are $360, 90, 0$. Maximum eigenvalue is 360 and corresponding eigenvector is

$$v_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}.$$

Hence, the maximum value of $\|Ax\|^2$ is 360, attained when x is the unit vector v_1.

For $\|x\| = 1$, the maximum value of $\|Ax\|$ is

$$\|Av_1\| = \sqrt{360}$$

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Singular Values

- For any matrix A of size $m \times n$

$$A^T A \text{ is symmetric.}$$

- Let $\{v_1, v_2, ..., v_n\}$ be an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$ and let $\lambda_1, \ \lambda_2, ..., \lambda_n$ be the associated eigenvalues.

- $\|Av_1\|^2 = Av_1 \cdot Av_1 = (Av_1)^T (Av_1) = v_1^T (A^T A) v_1 = v_1^T (\lambda_1 v_1) = \lambda_1.$

- All the eigenvalues of $A^T A$ are all nonnegative.

- The singular values of A are the square roots of the eigenvalues of $A^T A$.

Singular Values

Find the singular values of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Solution

$$A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

has eigenvalues 3 and 1.

So, singular values of A are

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3},$$

and

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$$

Singular Value Decomposition

Theorem

Suppose $\{v_1, ..., v_n\}$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A^TA, arranged so that the corresponding eigenvalues of A^TA satisfy $\lambda_1 \geq ... \geq \lambda_n$, and suppose A has r nonzero singular values. Then $\{Av_1, ..., Av_r\}$ is an orthogonal basis for $\mathrm{Col}\ A$, and $\mathrm{rank}\ A = r$.

Proof

$$Av_i \cdot Av_j = (Av_j)^T(Av_i) = v_j^TA^TAv_i = v_j^T\lambda_i v_i = 0.$$

Hence, $\{Av_1, ..., Av_r\}$ is an orthogonal set.

Let $y \in \mathrm{Col}\ A$ then $y = Ax = A(c_1v_1 + ...c_rv_r + c_{r+1}vr + 1 + ... + c_nv_n)$.

Above, equation can be written as

$$y = Ax = c_1Av_1 + ...c_rAv_r + 0 + 0... + 0.$$

So,

$$y \in \mathrm{Span}\ \{Av_1, ..., Av_r\}.$$

Singular Value Decomposition

Theorem

Let A be an $m \times n$ matrix with $\mathrm{rank} r$. Then there exists an $m \times n$ matrix Σ of the form

$$\Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

for which the diagonal entries in D are the first r singular values of A, $\sigma_1 \geq ... \geq \sigma_r > 0$ and there exist an $m \times n$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that

$$A = U\Sigma V^T$$

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Singular Value Decomposition

- For positive definite matrices, *Sigma* is D and $U\Sigma V^T$ is identical to QDQ^T.

- For other symmetric matrices, any negative eigenvalues in D become positive in Σ.

- U and V give orthonormal bases for all four fundamental subspaces: first r columns of U: column space of A

  last $m - r$ columns of U: left nullspace of A

  first r columns of V: row space of A

  last $n - r$ columns of V: nullspace of A

- $AV = U\Sigma$

Singular Value Decomposition

- Eigenvectors of AA^T and A^TA must go into the columns of U and V: $AA^T = (U\Sigma V^T)(V\Sigma^T U^T) = U\Sigma\Sigma^T U^T$ and similarly, $A^TA = V\Sigma^T\Sigma V^T$.

- $A^TAv_j = \sigma_j^2 v_j$. Multiply by A, we get

$$AA^TAv_j = \sigma_j^2 Av_j$$

This is eigenvalue equation

$$AA^T(Av_j) = \sigma_j^2(Av_j).$$

Hence, Av_j is the eigenvector of A^TA and σ_j^2 is the eigenvalue.

So, the unit eigenvector is $Av_j/\sigma_j = u_j$.

In other words,

$$AV = U\Sigma.$$

Example

Find a singular value decomposition of

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Eigenvalues are $2, 1, 0$ and corresponding normalized eigenvectors are

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; v_3 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}.$$

Hence,

$$V = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In order to find U, we compute $u_1 = \frac{1}{\sigma_1} A v_1, \; u_2 = \frac{1}{\sigma_2} A v_2$.
So,

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example

Find a singular value decomposition of

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}.$$

$$A^T A = \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}.$$

The eigenvalues of $A^T A$ are 18 and 0 with corresponding unit eigenvectors

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \; v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

Hence,

$$V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

and

$$\Sigma = \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example

To construct U, first construct Av_1 and Av_2 :

$$Av_1 = \begin{pmatrix} 2/\sqrt{2} \\ -4/\sqrt{2} \\ 4/\sqrt{2} \end{pmatrix}, \quad Av_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$u_1 = \frac{1}{3\sqrt{2}} Av_1$$

In order to write U, we need to extend the set $\{u_1\}$ to orthonormal basis for $\mathbb{R}^3$.

$$U = \begin{pmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{pmatrix}.$$

Singular Value Decomposition

Theorem

Let A be an $m \times n$rix with singular vales $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$. Let $u_1,\ u_2,\ ,...,u_r$ be left singular vectors and let $v_1,\ v_2,\ ,...,v_r$ be right singular vectors of A corresponding to these singular vales. The,

$$A = \sigma_1 u_1 v_1^T + ... + \sigma_r u_r v_r^T.$$