MT-104
Linear Algebra

National University of Computer and Emerging Sciences

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Lecture

Applications of Orthogonal Diagonalization

Quadratic forms

- An expression of the form

$$ax^2 + by^2 + cxy$$

is called a quadratic form in x and y.

- Quadratic form in two variables $ax^2 + by^2 + cxy$ can be written in following form

$$ax^2 + by^2 + cxy = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- A quadratic form in n variables is a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form where A is a symmetric $n \times n$ matrix and x is in $\mathbb{R}^n$. We refer to A as the matrix associated with f

Example

Remove the cross product term from the following Quadratic form

$$5x^2 - 4xy + 5y^2 = 48.$$

Solution
Given quadratic form can be written as

$$X^T AX = 48$$

where $A = \begin{pmatrix} 5 & -2 \\ -2 & 4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$.
By using the transformation $X = QY$, we get

$$(QY)^T A(QY) = 48.$$

As A is a symmetric matrix so we can find an orthogonal matrix Q such that

$$Q^T AQ = D.$$

So, we have

$$Y^T DY = 48.$$

where $Y = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and D is a diagonal matrix whose diagonal entries are the eigenvalues of A.

Example

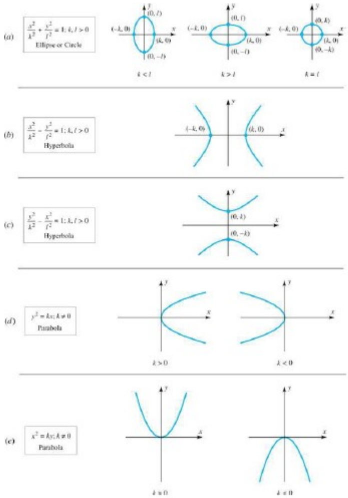
Eigenvalues of A are 3 and 7

Corresponding Unit Eigenvectors are: $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$

$$D = \begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Hence, we have

$$3x_1^2 + 7y_1^2 = 48.$$



(a)
$\frac{x^2}{k^2}+\frac{y^2}{l^2}=1;\ k,l>0$
Ellipse or Circle
$k<l$
$k>l$
$k=l$
(b)
$\frac{x^2}{k^2}-\frac{y^2}{l^2}=1;\ k,l>0$
Hyperbola
(c)
$\frac{y^2}{k^2}-\frac{x^2}{l^2}=1;\ k,l>0$
Hyperbola
(d)
$y^2=kx;\ k\neq 0$
Parabola
$k>0$
$k<0$
(e)
$x^2=ky;\ k\neq 0$
Parabola

Example

Describe the conic C whose equation is

$$5x^2 - 4xy + 8y^2 + 4\sqrt{5}x - 16\sqrt{5}y + 4 = 0.$$

Solution
In matrix form we can write

$$x^T AX + KX + 4 = 0,$$

where $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}$, $K = (4\sqrt{5} \quad -16\sqrt{5})$.
Put $X = QY$ to get

$$Y^T DY + KQY + 4 = 0,$$

where

$$Y = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad Q = \begin{pmatrix} 2\sqrt{5} & -1\sqrt{5} \\ 1\sqrt{5} & 2\sqrt{5} \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$$

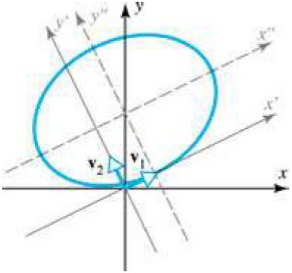
$$4x_1^2 + 9y_1^2 - 8x_1 - 36y_1 + 4 = 0.$$

Example

$$4(x_1 - 1)^2 + 9(y_1 - 2)^2 = 36.$$

Put $x'' = x_1 - 1, \quad y'' = y_1 - 2$, we get

$$\frac{x''^2}{9} + \frac{y''^2}{4} = 1.$$



Definition

A quadratic form $f(x) = x^T A x$ is classified as one of the following:

1. positive definite if $f(x) > 0$ for all $x \neq 0$
2. positive semidefinite if $f(x) \geq 0$ for all x
3. negative definite if $f(x) < 0$ for all $x \neq 0$
4. negative semidefinite if $f(x) \leq 0$ for all x
5. indefinite if $f(x)$ takes on both positive and negative values

Theorem

Let A be an $n \times n$ symmetric matrix. The quadratic form $f(X) = X^T AX$ is

- positive definite if and only if all of the eigenvalues of A are positive
- positive semidefinite if and only if all of the eigenvalues of A are nonnegative
- negative definite if and only if all of the eigenvalues of A are negative
- negative semidefinite if and only if all of the eigenvalues of A are non positive
- indefinite if and only if A has both positive and negative eigenvalues.

CONSTRAINED OPTIMIZATION

Find the maximum and minimum values of $Q(\mathbf{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$ subject to the constraint $x^T x = 1$.

Solution

$$
\begin{aligned}
Q(\mathbf{x}) &= 9x_1^2 + 4x_2^2 + 3x_3^2 \\
&\leq 9x_1^2 + 9x_2^2 + 9x_3^2 \\
&= 9(x_1^2 + x_2^2 + x_3^2) \\
&= 1
\end{aligned}
$$

So the maximum value of $Q(\mathbf{x})$ cannot exceed 9 when $\mathbf{x}$ is a unit vector. Thus 9 is the maximum value of $Q(\mathbf{x})$.

To find the minimum value of $Q(\mathbf{x})$, observe that

$$Q(\mathbf{x}) \geq 3x_1^2 + 3x_2^2 + 3x_3^2 = 3$$

$Q(\mathbf{x}) = 3$ is the minimum value

Theorem

Let $f(\mathbf{x}) = x^T Ax$ be a quadratic form with associated $n \times n$ symmetric matrix A. Let the eigenvalues of A be $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. Then the following are true, subject to the constraint $\|\mathbf{x}\| = 1$

- $\lambda_1 \geq f(\mathbf{x}) \geq \lambda_n$
- The maximum value of $f(\mathbf{x})$ is λ_1, and it occurs when x is a unit eigenvector corresponding to λ_1.
- The minimum value of $f(\mathbf{x})$ is λ_n, and it occurs when x is a unit eigenvector corresponding to λ_n.

Example

Find the maximum and minimum values of $Q(\mathbf{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$ subject to the constraint $x^T x = 1$.

Solution
Matrix corresponding to given quadratic form is

$$A = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Eigenvalues of A are $9,\ 4,\ 3$.

Eigenvector corresponding to 9 is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Eigenvector corresponding to 3 is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$