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ASSIGNMENT 6

Q1. GIVEN:

$$E[(X-1)^2] = 10$$
$$E[(X-2)^2] = 6$$

Now, Consider, $E[(X-1)^2] = 10$

$$E[X^2-2X+1] = 10$$
$$E(X^2-2X+1) = 10$$
$$E(X^2) - 2E(X) + 1 = 10$$
$$E(X^2) - 2E(X) = 9 \rightarrow (1)$$

Now, Consider, $E[(X-2)^2] = 6$

$$E[X^2-4X+4] = 6$$
$$E(X^2) - 4E[X] + 4 = 6$$
$$E(X^2) - 4E(X) = 2 \rightarrow (2)$$

Subtract (2) from (1), we get

$$E(X^2) - 2E(X) - E(X^2) + 4E(X) = 9-2$$
$$2E(X) = 7$$
$$E(X) = \frac{7}{2}$$

Putting $E(X) = \frac{7}{2}$ in 1, we get

$$E(X^2) - 2\left[\frac{7}{2}\right] = 9$$
$$E(X^2) = 9+7$$

$E[x^2] = 16$

Mean of the random Variable X is, $\mu = E[X]$

$\boxed{= \frac{7}{2}}$

Variance of the random variable X is as follows

$$\sigma^2 = E[x^2] - \{E[x]\}^2$$

$$= 16 - [\tfrac{7}{2}]^2$$

$$= 16 - \frac{49}{4}$$

$$\boxed{= 3.75}$$

Q2.

$$f(y) = \begin{cases} \frac{1}{4} e^{-y/4} & ; \quad 0 \le y < \infty \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

a. The mean time to reflex is,

$$E[y] = \int_{-\infty}^{\infty} y f[y]\, dy$$

$$= \int_{0}^{\infty} y \cdot \frac{1}{4} e^{-y/4}\, dy\ dy = \lim_{t\to\infty} \int_{0}^{t} y \cdot \frac{1}{4} e^{-y/4}\, dy$$

We will solve this integral by using the integration by part method which is:

$$\int_a^b u(y)\, v'(y)\, dy = [u(y)\, v(y)]\Big|_a^b - \int_a^b u'(y)\, v(y)\, dy$$

In our Specific case:

$$u(y) = y, \quad u'(y) = 1$$

$$v'(y) = \frac{1}{4} e^{-\frac{y}{4}} \Rightarrow v(y) = -e^{\frac{y}{4}}$$

$$E[Y] = \lim_{t\to\infty} \int_0^t y \cdot \frac{1}{4} e^{-\frac{y}{4}}\, dy$$

$$= \lim_{t\to\infty} \left[\left[-ye^{-\frac{y}{4}} \right] \Big|_0^t - \int_0^1 -e^{-\frac{y}{4}} dy \right]$$

$$= \lim_{t\to\infty} \left[-te^{-\frac{t}{y}} - \left[4e^{-\frac{y}{4}} \right] \Big|_0^t \right]$$

$$= \lim_{t\to\infty} \left[-te^{-t/y} - 4e^{-t/4} + 4 \right]$$

$$= \lim_{t\to\infty} \left(4 - [t+4] e^{-t/4} \right)$$

$$\boxed{= 4}$$

b. $E[Y^2] = \int_{-\infty}^{\infty} y^2 f(y)\, dy = \int_0^{\infty} y^2 \cdot \frac{1}{4} e^{-\frac{y}{4}} dy = \lim_{t\to\infty} \int_0^t y^2 \cdot \frac{1}{4} e^{-\frac{y}{4}} dy$

Again, we will solve this integral by the use of the integration by parts method. In this case, we take

$$u(y) = y^2 \Rightarrow u'(y) = 2y$$

$$v'(y) = \frac{1}{4} e^{-\frac{y}{4}} \Rightarrow u(y) = -e^{-\frac{y}{4}}$$

$$E[Y^2] = \lim_{t\to\infty} \int_0^t y^2 \cdot \frac{1}{4} e^{-\frac{y}{4}} dy$$

$$= \lim_{t\to\infty} \left[\left[-y^2 e^{-\frac{y}{2}} \right] \Big|_0^t - \int_0^t -2ye^{-\frac{y}{4}} dy \right]$$

$$= \lim_{t\to\infty} \left[-t^2 e^{-\frac{t}{4}} + 8 \int_0^t y \cdot \frac{1}{4} e^{-\frac{y}{4}} dy \right]$$

$$= \lim_{t\to\infty} \left[-t^2 e^{-\frac{t}{4}} \right] + 8 \lim_{t\to\infty} \left(\int_0^1 y \cdot \frac{1}{4} e^{\frac{y}{2}} dy \right)$$

$$= 0 + 8E[X]$$

$$= 8 \cdot 4 \boxed{= 32}$$

$$Var[Y] = E[Y^2] - [E[Y]]^2 = 32 - (4)^2 = 32 - 16$$

$$\boxed{= 16}$$

Q3. $f(x) = \begin{cases} \frac{1}{5} e^{\frac{-x}{5}}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$

a. $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x \cdot \frac{1}{5} e^{-\frac{x}{5}} dx$

$= \lim_{t \to \infty} \int_0^t x \cdot \frac{1}{5} e^{\frac{-x}{5}} dx$

We will solve the integral by using the integration by parts method

In this case, we have

$u(x) = x, \quad u'(x) = 1$

$v'(x) = \frac{1}{5} e^{-x/5} \Rightarrow v(x) = -e^{-x/5}$

$E[x] = \lim_{t \to \infty} \left[(-xe^{-\frac{x}{5}}) \Big|_0^x - \int_0^t -e^{-\frac{x}{5}} dx \right]$

$= \lim_{t \to \infty} \left[-te^{-\frac{t}{5}} - [5e^{-x/5}] \Big|_0^t \right]$

$= \lim_{t \to \infty} [-te^{-\frac{t}{5}} - 5e^{-t/5} + 5]$

$= \lim_{t \to \infty} [5 - (t+5) e^{-t/5}]$

$\boxed{= 5}$

b. $E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} x^2 \cdot \frac{1}{5} e^{-\frac{x}{5}} dx$

$= \lim_{t \to \infty} \int_0^1 x^2 \cdot \frac{1}{5} e^{-\frac{x}{5}} dx$

Again, we will solve the integral by using the integration by parts method. In this case, we have

$u(x) = x^2 \Rightarrow u'(x) = 2x$

$v'(x) = \frac{1}{5} e^{-\frac{x}{2}} \Rightarrow v(x) = -e^{-\frac{x}{5}}$

$E[X^2] = \lim_{t \to \infty} \left[(-x^2 e^{-\frac{x}{5}}) \Big|_0^t - \int_0^t -2xe^{-\frac{x}{5}} dx \right]$

$= \lim_{t \to \infty} [-t^2 e^{-\frac{t}{5}}] + 10 \lim_{t \to \infty} \int_0^t \frac{1}{5} x e^{-\frac{x}{5}} dx$

$= 0 + 10\, E(x) \boxed{= 50}$

Using the obtained values, we get the Variance

$\sigma^2_x = E[x^2] - [E(x)]^2 = 50 - (5)^2 = 50 - 25$
$\boxed{= 25}$

and the standard deviation

$\sigma_x = \sqrt{25} \boxed{= 5}$

c. $E[(x+5)^2] = E[x^2 + 10x + 25]$
$= E(x^2) + 10 E[x] + 25$
$= 50 + 10(5) + 25$
$\boxed{= 125}$

Q4. $f(x) \begin{cases} \frac{1}{900} e^{-\frac{x}{900}}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$

a. $E(x) = \int_{-\infty}^{\infty} x f(x) = \int_0^{\infty} x \cdot \frac{1}{900} e^{-\frac{x}{900}} dx$

$= \lim_{t \to \infty} \int_0^t x \cdot \frac{1}{900} e^{-\frac{x}{900}} dx$

We will solve this integral by using integration by part method
In this case, we have,

$u(x) = x \Rightarrow u'(x) = 1$

$v'(x) = \frac{1}{900} e^{-\frac{x}{900}} \Rightarrow v(x) = -e^{-\frac{x}{900}}$

$$E(x) = \lim_{t\to\infty} \left[\left[-x e^{-\frac{x}{900}} \right]\Big|_0^t - \int_0^t -e^{-\frac{x}{900}}\,dx \right]$$

$$= \lim_{t\to\infty} \left[-t e^{-\frac{t}{900}} - \left[900 e^{-\frac{x}{900}} \right]\Big|_0^t \right]$$

$$= \lim_{t\to\infty} \left[-t e^{-\frac{t}{900}} - 900 e^{-\frac{t}{900}} + 900 \right]$$

$$= \lim_{t\to\infty} \left(900 - (t+900) e^{-\frac{t}{900}} \right) \boxed{= 900}$$

b. $$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^\infty x^2 \cdot \frac{1}{900} e^{-\frac{x}{900}}\,dx$$

$$= \lim_{t\to\infty} \int_0^t x^2 \cdot \frac{1}{900} e^{-\frac{x}{900}}\,dx$$

Again, we will solve the integral by using integration by parts method. In this cases we have

$$u(x) = x^2 \Rightarrow u'(x) = 2x$$

$$v'(x) = \frac{1}{900} e^{-\frac{x}{900}} \Rightarrow v(x) = -e^{-\frac{x}{900}}$$

$$E(x^2) = \lim_{t\to\infty} \left[\left[-x^2 e^{-\frac{x}{900}} \right]\Big|_0^t - \int_0^t -2x e^{-\frac{x}{900}}\,dx \right]$$

$$= \lim_{t\to\infty} \left[-t^2 e^{-\frac{t}{900}} \right] + 1800 \lim_{t\to\infty} \int_0^t \frac{1}{900} x e^{-\frac{x}{900}}\,dx$$

$$= 0 + 1800\, E(x)$$

$$\boxed{= 1620000}$$

c. Variance:

$$\sigma_x^2 = E(x^2) - \{E(X)\}^2 = 1620000 - (900)^2$$

$$\boxed{= 810000}$$

Standard Deviation:

$$\sigma_x = \sqrt{810000} = \boxed{900}$$

Q5. Let X be the discrete random variable with the given probability table, which represent the firm's profit during the next 6 months.

$E[X] = \sum_x x f(x) = (-5000)0.2 + (1008)0.5 + (30000)0.3$

$= -1000 + 5000 + 9000$

$\boxed{= \$\ 13000}$

Q6.

a. $E[X] = \sum_x x f(x) = (-15000)0.05 + (0)0.15 + (15000)0.15 + (25000)0.30 + (40000)(0.15) + (50000)0.10 + (100000)0.05 + (150000)0.03 + (200000)(0.02)$

$= -750 + 0 + 2250 + 7500 + 6000 + 5000 + 5000 + 4500 + 4000$

$= \$\ 33500$

b. $E(X^2) = \sum_x x^2 f(x) = (-15000)^2 0.05 + (0)^2 0.15 + (15000)^2 0.15 + (50000)^2 0.10 + (100000)^2 0.05 + (25000)^2 0.30 + (40000)^2 0.15 + (150000)^2 0.03 + (200000)^2 0.02$

$= 1.125 \times 10^7 + 0 + 3.375 \times 10^7 + 1.875 \times 10^8 + 2.4 \times 10^8 + 2.5 \times 10^8 + 5 \times 10^8 + 6.75 \times 10^8 + 8 \times 10^8$

$\boxed{= 2.6975 \times 10^9}$

Standard Derivation:

$\sigma_x = \sqrt{E(x^2) - (E(x))^2} = \sqrt{2.6975 \times 10^9 - (33500)^2}$

$\boxed{= \$\ 39689}$

Q7.

$$f(x) = \begin{cases} \frac{3}{4\times 50^3}(50^2 - x^2), & -50 \leq x \leq 50 \\ 0, & \text{elsewhere} \end{cases}$$

a. $E(x) = \int_{-50}^{50} x \cdot \frac{3}{4\times 50^3}(50^2 - x^2)\,dx$

$= \frac{3}{4\times 50^3}\left[50^2 \int_{-50}^{50} x\,dx - \int_{-50}^{50} x^3\,dx\right]$

$= \frac{3}{4\times 50^3}\left[50^2 \cdot \frac{x^2}{2}\Big|_{-50}^{50} - \frac{x^4}{4}\Big|_{-50}^{50}\right]$

$= \frac{3}{4\times 50^3}\left[\frac{50^2}{2}\{50^2 - 50^2\} - \frac{1}{4}\{50^4 - 50^4\}\right]$

$= \frac{3}{4\times 50^3} \times 0$

$\boxed{= 0}$

b. $E(x^2) = \int_{-50}^{50} x^2 \cdot \frac{3}{4\times 50^3}(50^2 - x^2)\,dx$

$= 2 \cdot \frac{3}{4\times 50^3} \int_{-50}^{50} x^2(50^2 - x^2)\,dx$

$= \frac{3}{4\times 50^3}\left[50^2 \int_0^{50} x^2\,dx - \int_0^{50} x^4\,dx\right]$

$= \frac{3}{4\times 50^3}\left[50^2 \cdot \frac{x^3}{3}\Big|_0^{50} - \frac{x^5}{5}\Big|_0^{50}\right]$

$= \frac{3}{4\times 50^3}\left[\frac{50^5}{3} - \frac{50^5}{5}\right] = \frac{3}{2\times 50^3} \times 50^5\left[\frac{1}{3} - \frac{1}{5}\right]$

$= \frac{3}{2} \times 50^2 \times \frac{2}{15}$ $\boxed{= 500}$

c. $\sigma_x = \sqrt{E(x^2) - \{E(x)\}^2}$

$= \sqrt{500 - 0^2}$

$= \sqrt{500}$ $\boxed{= 22.361}$