Course Name:Linear Algebra(MT 104)

Topic: Linear Independence (Exercise 1.7) & Applications of Linear Equations (Exercise 1.6)

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Recall

For System of *nxn* Linear Equations and if No of pivot columns = n, then

- Non Homogeneous system $Ax = b$ has unique solution.
- Columns of A will span $\mathbb{R}^n$.
- Homogeneous system $Ax = 0$ has only trivial solution.
- Columns of A will be linearly independent.

For System of *nxn* Linear Equations and if No of pivot columns $< n$ then

- Non Homogeneous system $Ax = b$ will not have solution for all $b \in \mathbb{R}^n$.
- Columns of A will not span $\mathbb{R}^n$.
- Homogeneous system $Ax = 0$ has non-trivial solution.
- Columns of A will be linearly dependent.

If System of Linear Equations is Under Determined (less equations in more variables) then

- Homogeneous system $Ax = 0$ has non trivial solution.
- Columns of A are linearly dependent.
- If Order of A is e.g 2x3 then columns of A will span $\mathbb{R}^2$ only if pivot columns =2, i.e in this case the non homogeneous system $Ax = b$ will have either no solution or infinite solution.
- If Order of A is e.g 2x3 then columns of A will not span $\mathbb{R}^2$ only if pivot columns < 2, i.e in this case the non homogeneous system $Ax = b$ will not have solution for $b \in \mathbb{R}^2$..

If System of Linear Equations is Over Determined (more equations in less variables) and if pivot columns = number of unknowns, then

- Ax=0 has only trivial solution.
- Columns of A are Linearly Independent
- If Order of A is e.g 3x2 then columns of A will not span $\mathbb{R}^3$. i.e., in this case the non homogeneous system $Ax = b$ is not consistent for all $b \in \mathbb{R}^3$.

Applications of System of Linear Equations

We will discuss here 4 applications of System of Linear Equations

1. A Homogeneous System in Economics (The system of 500 equations in 500 variables, known as a Leontief "input–output" (or "production") model.)
   There exist equilibrium prices that can be assigned to the total outputs of the various sectors in such a way that the income of each sector exactly balances its expenses.
2. Balancing Chemical Equations
3. Network Flow
4. Linear Equations and Electrical Networks

Input-Output Model

- Suppose that an economy is divided into sectors
- Each sector outputs goods and/or services which are then purchased by the other sectors
- There exist *equilibrium prices* that can be assigned to the total outputs of the various sectors in such a way that the income of each sector exactly balances its expenses

A Simplified Example

- Let's consider an economy that has three sectors: Coal, Electric, and Steel
- This table indicates how much of each sector's distribution is purchased by the other sectors:

Distribution of output from...

Coal	Electric	Steel	...purchased by
0%	40%	60%	Coal
60%	10%	20%	Electric
40%	50%	20%	Steel

Example (Cont.)

- Let p_C, p_E, and p_S be the total prices of the outputs of Coal, Electric, and Steel, respectively

- We are assuming that the income from each sector equals its expenses

Distribution of output from...

Coal	Electric	Steel	...purchased by
0%	40%	60%	**Coal**
60%	10%	20%	**Electric**
40%	50%	20%	**Steel**

This gives us three equations:

$$p_C = 0.4p_E + 0.6p_S$$
$$p_E = 0.6p_C + 0.1p_E + 0.2p_S$$
$$p_S = 0.4p_C + 0.5p_E + 0.2p_S$$

Example (Cont.)

Writing these equations in standard form, we have a homogeneous system:

$$
\begin{aligned}
p_C - 0.4p_E - 0.6p_S &= 0 \\
-0.6p_C + 0.9p_E - 0.2p_S &= 0 \\
-0.4p_S - 0.5p_E + 0.8p_S &= 0
\end{aligned}
$$

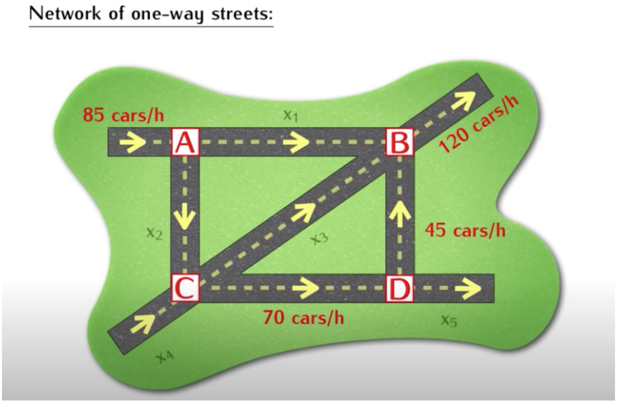
- Solving this system in the normal way gives us this solution set:

$$
\begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = p_S \begin{bmatrix} 0.94 \\ 0.85 \\ 1 \end{bmatrix}
$$

- As usual, our homogeneous system has an infinite number of solutions

- If we knew additional information, say that $p_S = \$100$ million, then we could conclude $p_C = \$84$ million and $p_E = \$85$ million

Network Flow (Traffic Flow): Example



Problem: Find the flow rate of cars on each segment of streets. **Observation:**

- Flow into a junction (node/intersection)= flow out of that junction (node/intersection)
- Total Flow in =Total flow out

Example (Cont.)

	IN	=	OUT
total:	$85 + x_4$	$=$	$120 + x_5$
@ A:	85	$=$	$x_1 + x_2$
@ B:	$x_1 + x_3 + 45$	$=$	120
@ C:	$x_2 + x_4$	$=$	$70 + x_3$
@ D:	70	$=$	$45 + x_5$

Example (Cont.)

$$
\begin{cases}
x_4 - x_5 = 35 \\
x_1 + x_2 = 85 \\
x_1 + x_3 = 75 \\
x_2 - x_3 + x_4 = 70 \\
x_5 = 25
\end{cases}
\qquad
\begin{cases}
x_1 = 75 - x_3 \\
x_2 = 10 + x_3 \\
x_3 = \textit{free} \\
x_4 = 60 \\
x_5 = 25
\end{cases}
$$

$$
\left[\begin{array}{ccccc|c}
0 & 0 & 0 & 1 & -1 & 35 \\
1 & 1 & 0 & 0 & 0 & 85 \\
1 & 0 & 1 & 0 & 0 & 75 \\
0 & 1 & -1 & 1 & 0 & 70 \\
0 & 0 & 0 & 0 & 1 & 25
\end{array}\right]
\xrightarrow{\substack{\textit{row} \\ \textit{reduction}}}
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\left[\begin{array}{ccccc|c}
1 & 0 & 1 & 0 & 0 & 75 \\
0 & 1 & -1 & 0 & 0 & 10 \\
0 & 0 & 0 & 1 & 0 & 60 \\
0 & 0 & 0 & 0 & 1 & 25 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

Balance the following chemical equation (Burning Propane)

$$\boxed{C_3H_8 \ + \ O_2 \ \longrightarrow \ CO_2 \ + \ H_2O}$$

In its complete form this chemical equations says

$$\boxed{x_1 C_3H_8 \ + \ x_2 O_2 \ \longrightarrow \ x_3 CO_2 \ + \ x_4 H_2O}$$

Note.

- The numbers x_1, x_2, x_3, x_4 are positive integers.
- The number of atoms of each element on the left side is the same as the number of atoms of that element on the right side.

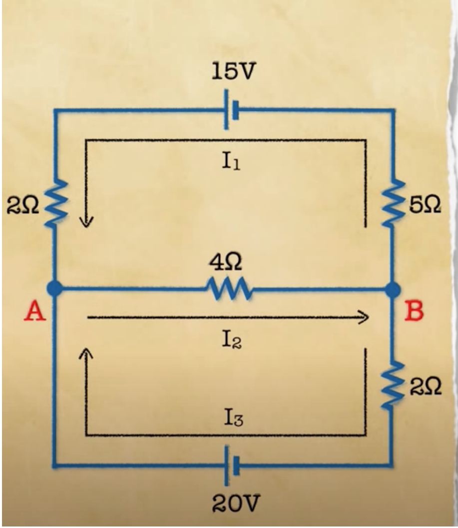
Example (Cont.)

$$x_1 C_3H_8 \; + \; x_2 O_2 \; \longrightarrow \; x_3 CO_2 \; + \; x_4 H_2O$$

Note.

- The numbers x_1, x_2, x_3, x_4 are positive integers.
- The number of atoms of each element on the left side is the same as the number of atoms of that element on the right side.

$$\begin{cases} 3x_1 - x_3 = 0 \\ 8x_1 - 2x_4 = 0 \\ 2x_2 - 2x_3 - x_4 = 0 \end{cases} \qquad \begin{cases} x_1 = \frac{1}{4}x_4 \\ x_2 = \frac{5}{4}x_4 \\ x_3 = \frac{3}{4}x_4 \\ x_4 = \mathit{free} \end{cases}$$

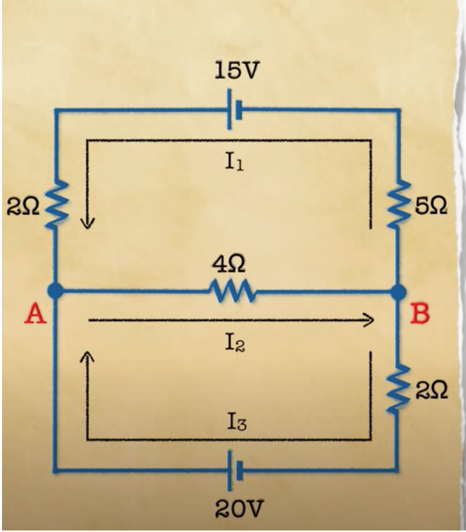


current =
flow rate of electrical charge

Current is measured in amperes:

$$1\text{A} = \begin{pmatrix} \text{the electrical charge} \\ \text{of } 6.2 \cdot 10^{18} \text{ electrons} \\ \text{per second} \end{pmatrix}$$

Goal.
Compute the currents I_1, I_2, I_3.



@A:

$$I_1 + I_3 = I_2$$

Ohm's law:
voltage drop $= I \cdot R$

Kirchoff's second law:
In any loop:
(v. gain) − (v. drop) = 0

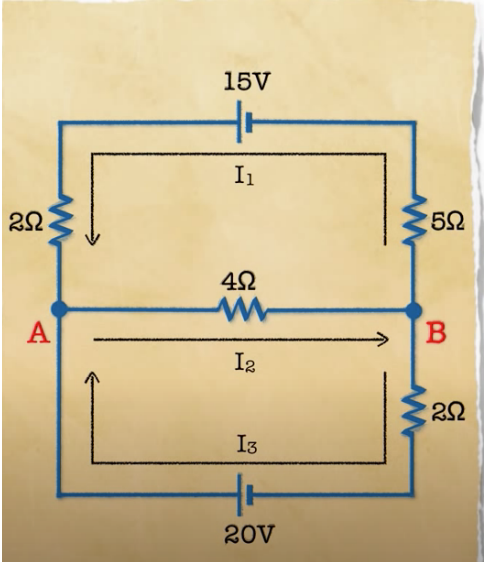
upper loop:

$$15 - 2I_1 - 4I_2 - 5I_1 = 0$$

lower loop:

$$20 - 4I_2 - 2I_3 = 0$$

Example (cont.)



$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 7 & 4 & 0 & 15 \\ 0 & 4 & 2 & 20 \end{array}\right]$$

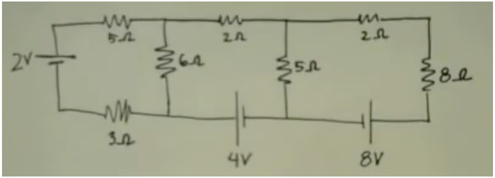
row reduction

$$\begin{array}{cccc} I_1 & I_2 & I_3 & \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.2 \\ 0 & 1 & 0 & 3.4 \\ 0 & 0 & 1 & 3.2 \end{array}\right]} \end{array}$$

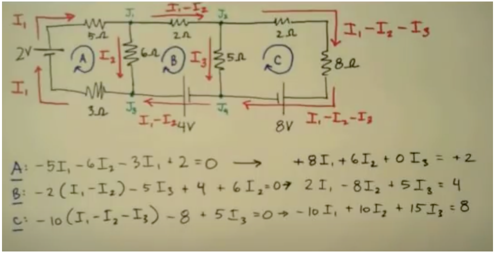
$$\begin{cases} I_1 = 0.2 \\ I_2 = 3.4 \\ I_3 = 3.2 \end{cases}$$

Example

Consider the following Electrical Flow. Determine the flow of current.



Label the flow as



Example (Cont.)