Matrix Chain multiplication

→ what is matrix multiplication

Assume a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$\underset{row}{2} \times \underset{c}{3}$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$3 \times 2$$

for multiplicate

$$2 \times \boxed{3 \qquad 3} \times 2$$

must be equal.

(2)

$$C = \begin{bmatrix} a_{11} \times b_{11} + a_{12} \times b_{21} + a_{13} \times b_{31} & a_{11} b_{12} + a_{12} b_{22} + a_{13} b_{32} \\ a_{21} \times b_{12} + a_{22} \times b_{21} + a_{23} \times b_{31} & a_{21} b_{12} + a_{22} \times b_{22} + a_{23} b_{32} \end{bmatrix}$$

Dimention = 2×2

Q So how many multiplicatio done?

Count = 12 multiplicatio.

So

$$2 \times 3 \times 2 = 12.$$

⇒ Now few things

(1) 2 matrix can be multiplied if

$$2 \times \boxed{3 \qquad 3} \times 2$$

↓

Same

(2) Resultant matrix will be extremes

e.g. $\boxed{2} \times 3 \quad 3 \times \boxed{2}$.

(3) No of multiplicln are.

$$2 \quad \boxed{3 \quad 3} \quad 2$$

we lined it as 1

$2 \times 3 \times 2 = 12$.

(4) No of diamentions are.

2 3 3 2

So 4

but 3 is treated same

So 3 diamentions

$$\underbrace{2}_{1} \quad \underbrace{\overbrace{3 \quad 3}^{\text{common.}}}_{2} \quad \underbrace{2}_{3}$$

Now assume we have 3 matrix.

$$A_1 \times A_2 \times A_3$$

$$\underset{d_0}{2} \times \underbrace{3 \quad 3}_{d_1} \times \underbrace{4 \quad 4}_{d_2} \times \underset{d_3}{2}$$

$\Rightarrow$ can be multiplied

We can only multiply 2 matrix at any given time so

we can.

$(A_1 \times A_2) \times A_3$ or $A_1 \times (A_2 \times A_3)$

Both will give same result so associative property holds.

Our focus is not multiplication but order of multiplication.

$(A_1 \times A_2) \times A_3$

2 3   3 4   4 2

↓ As no multiply.

$2 \times 3 \times 4 = 24$ 0

New dimension are

New dimension — $\boxed{2 \times 4}$ 4×2

$2 \times 4 \times 2 = 16$

$\boxed{24 + 0 + 16 = 40}$

for $A_1 \times (A_2 \times A_3)$

3 4 4 2.

$0 + 3 \times 4 \times 2 = 24.$

new = 3 2.

2 3 3 2.

$2 \times 3 \times 2 \Rightarrow 12.$

$\boxed{\text{So } 24 + 0 + 12 = 36}$

~~24×0~~

$24 + 0 + 2 \times 4 \times 2 = 40$ $\quad 0 + 24 + 2 \times 3 \times 3 = 36$ ✓

This is better approach. So at very small level, this is the difference what if the number is large.

So this is matrix chain multiplication

If $A_1 \times A_2 \cdots A_{10}$

So how to multiply.

So we need to minimize no. of multiplication.

hence ~~greedy~~ DP approach.

So we need formula.

$C[1,3] \quad A_1 \times A_2 \times A_3$

$2 \quad 3 \quad 3 \quad 4 \quad 4 \quad 2$

$d_0 \quad d_1 \quad d_1 \quad d_2 \quad d_2 \quad d_3$

$(A_1 \times A_2) \times A_3 \qquad\qquad A_1 \times (A_2 \times A_3)$

$C[1,2] \quad C[3,3] \qquad\qquad C[1,1] \times C[2,3]$

$=24 \quad =0 \qquad\qquad =0 \quad =24$

$2 \; [4 \quad 4] \; 2 \qquad\qquad 2 \; [3 \quad 3] \; 2$

$c[1,2] + c[3,3] + d_0 * d_2 * d_3 = 40 \qquad c[1,1] + c[2,3] + d_0 * d_1 * d_3 = 36$

$i\,k \quad k+1\,j \quad d_{i-1} \; d_k \; d_j \qquad i\,k \quad k+1\,j \quad d_{i-1} \; d_k \; d_j$

$$c[i,j] = \min [c[i,k] + c[k+1,j]] + d_{i-1} \times d_k \times d_j .$$

where $i \le k < j$.

Now 4 matrix.

$A_1 \quad A_2 \quad A_3 \quad A_4.$

$d_0\, d_1 \quad d_1\, d_2 \quad d_2\, d_3 \quad d_3\, d_4.$

$A_1 (A_2 (A_3 A_4))$

$A_1 ((A_2 A_3) A_4)$

$(A_1 A_2)(A_3 A_4)$

$((A_1)(A_2 A_3)) A_4.$

$((A_1 A_2) A_3) A_4.$

~~Catel~~

Catelen number

$$= \frac{{}^{2(n-1)}C_n}{n} = \frac{(2(n-1))!}{n!\, n-1!}$$

$$= \cancel{(2n)!} \quad \cancel{\frac{2(n-1)!}{n!\,(n-1)!}}$$

$$= \cancel{2(3)!}$$

$$= \frac{(6)!}{4!\,3!} = \frac{6 \times 5 \times 4!}{4! \times 3!} = \frac{30}{6} = 5.$$

we need to find the best I. ⑧

$A_1 \times A_2 \times A_3 \times A_4$.

$$C[1,4] = \min_{1 \le k < 4} \begin{cases} 1: & C[1,1] + C[2,4] + d_0 \times d_1 \times d_4 \\ 2: & C[1,2] + C[3,4] + d_0 \times d_2 \times d_4. \\ 3: & C[1,3] + C[4,4] + d_0 \times d_3 \times d_4. \end{cases}$$

($C[1,1] = 0$, $C[2,4] = ?$, $C[4,4] = 0$)

what is 2,4?

$$C[2,4] \quad \min_{2 \le k < 4} \begin{cases} 2: & C[2,2] + C[3,4] + d_1 + d_2 + d_4 \\ 3: & C[2,3] + C[4,4] + d_1 \times d_3 \times d_4. \end{cases}$$

($C[2,2] = 0$, $C[3,4] = ?$, $C[2,3] = ?$, $C[4,4] = 0$)

So we need limitless formula.
good approach is to find smaller value & do it with dp.

we need 2 matrices

c table

i \ j	1	2	3	4
1	0			
2		0		
3			0	
4				0

table

k	1	2	3	4
1				
2				
3				
4				

A_1 A_2 A_3 A_4

3 2 2 4 4 2 2 5.

Matrix Chain Multiplication (2)

So consider

$$A_1 \times A_2 \times A_3 \times A_4$$
$$d_0\ d_1\ d_1\ d_2\ d_2\ d_3\ d_3\ d_4$$

$A_1(A_2(A_3, A_4))$

$A_1((A_2 A_3) A_4)$

$(A_1 A_2)(A_3 A_4)$

$(A_1(A_2 A_3))(A_4)$

$((A_1 A_2) A_3) A_4$

$$c[1-4] = \min_{\substack{1 \le s \\ 1 \\ 2 \\ 3}} \begin{cases} c[1\ 1] + [2 \cdot 4] + d_0 + d_1 + d_4 \\ c[1\ 2] + [3\ 4] + d_0 + d_2 + d_4 \\ c[1, 3] + [4\ 4] + d_0 + d_3 + d_4 \end{cases}$$

Now we need

$A_1(A_2 A_3 A_4)$

$(A_1 A_2)(A_3 A_4)$

$(A_1 A_2 A_3) A_4$

$c(2,4) = k$ $\begin{array}{l} 2 \\ 3 \end{array}\left\{ \begin{array}{l} c[\ \text{—} \\ c[\ \text{— —} \end{array} \right.$

So we need 2^n computations

So we can make 2 table.

i \ j →	1	2	3	4
1	0			
2		0		
3			0	
4				0

Step 1

find 1,1, 2,2, 3,3
4,4

Step 2

find values where diff is one.

Choose minimum cost (we need k)

k	1	2	3	4
1				
2				
3				
4				

A_1 A_2 A_3 A_4

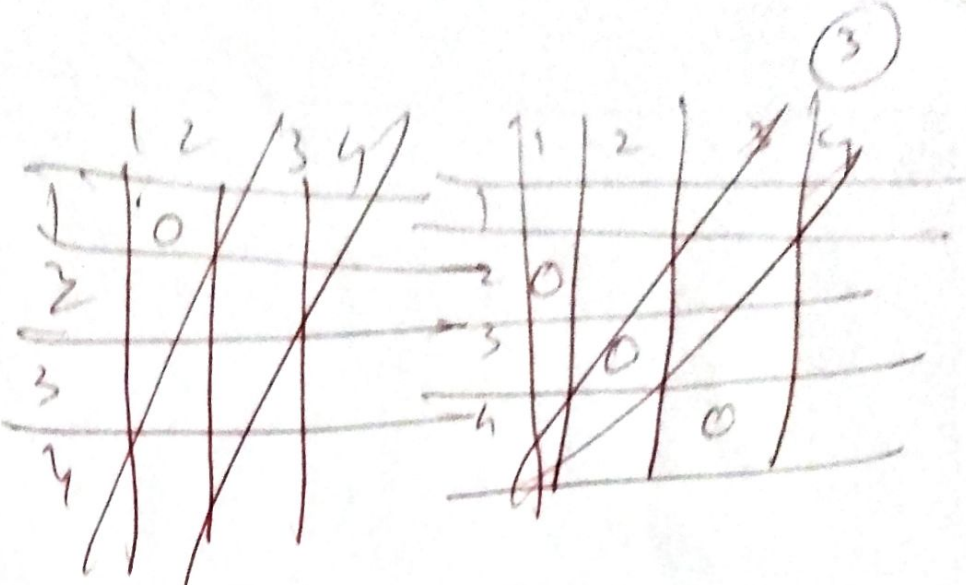
3 2 2 4 4 2 2 5

k table

S table

c[1,2]

k 12



0	24	28	58
	0	16	36
		0	40
			0

k	1	2	3	4
1		1	1	3
2			2	3
3				3
4				

c[1,2] = 24

c[2,3] = 16

c[3,4] = 40

then.

c[1,3] = 28 (k=1)

c{2,4} = 36 (k=3)

then

c{1,4} = 58 (k=3)

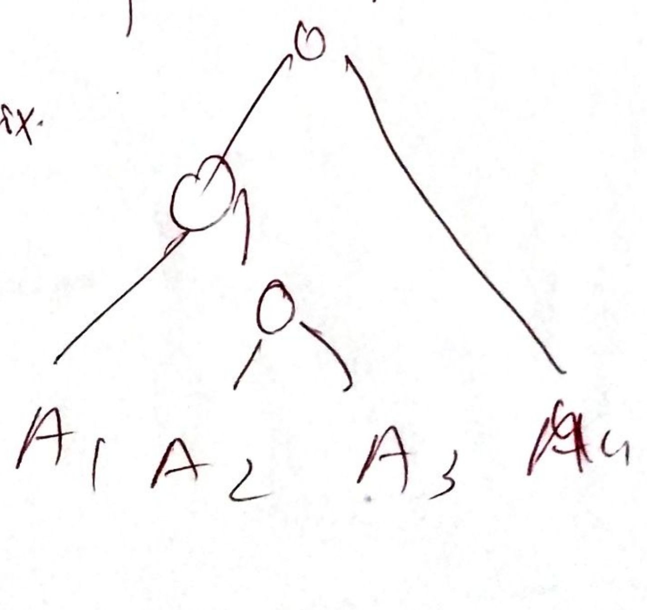
$A_1 \quad A_2 \quad A_3 \quad A_4$

3 2 2 4 4 2 2 5

$((A_1 \; A_2 \; A_3)(A_4))$

$((A_1)(A_2 \; A_3))(A_4)$

k	1	2	3	4
1		1	1	3
2			2	3
3				3
4				



n^2 for matrix.

but for k.

so

$O(n^3)$.