Course Name:Linear Algebra

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Cramer's rule begins with the clever observation

$$\begin{vmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{vmatrix} = x_1$$

Well, that's great. Now what do we do with this information? Well, note that if $A\mathbf{x} = \mathbf{b}$ then

$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}.$$

If we take determinants of both sides, and note the determinant is multiplicative, we get

$$det(A)x_1 = det(B_1)$$

where B_1 is the matrix we get when we replace column 1 of A by the vector $\mathbf{b}$. So,

$$x_1 = \frac{det(B_1)}{det(A)}.$$

Now, again, there's nothing special here about column 1, or about them being 3×3 matrices. In general if we have the relation $A\mathbf{x} = \mathbf{b}$ then the ith component of $\mathbf{x}$ will be

$$x_i = \frac{det(B_i)}{det(A)},$$

where B_i is the matrix we get by replacing column i of A with $\mathbf{b}$.

Cramer's Rule, Volume, and Linear Transformations

In this lecture, we shall apply the theory discussed in the last two lectures to obtain important theoretical formulae and a geometric interpretation of the determinant.

Cramer's Rule Cramer's rule is needed in a variety of theoretical calculations. For instance, it can be used to study how the solution of $\boldsymbol{Ax} = \boldsymbol{b}$ is affected by changes in the entries of $\boldsymbol{b}$. However, the formula is inefficient for hand calculations, except for 2×2 or perhaps 3×3 matrices.

Theorem 1 (Crammer's Rule) Let $\boldsymbol{A}$ be an invertible $n \times n$ matrix. For any $\boldsymbol{b}$ in $\boldsymbol{R}^n$, the unique solution x of $\boldsymbol{Ax} = \boldsymbol{b}$ has entries given by

$$x_i = \frac{\det A_i(b)}{\det A}, \qquad i = 1, 2, \ldots, n \tag{1}$$

Example 1 Use Cramer's rule to solve the system

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8$$

Solution Write the system in matrix form, $\boldsymbol{Ax} = \boldsymbol{b}$

$$\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \& b = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\det A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix} = 12 - 10 = 2$$

$$A_1(b) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, A_2(b) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

Since det $\boldsymbol{A}$ = 2, the system has a unique solution. By Cramer's rule,

det $A_1(b)$ 24 + 16

Example 2 Consider the following system in which s is an unspecified parameter. Determine the values of s for which the system has a unique solution and use Cramer's rule to describe the solution.

$$\begin{aligned} 3sx_1 - 2x_2 &= 4 \\ -6x_1 + sx_2 &= 1 \end{aligned}$$

Solution Here

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad A_1(b) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix}, \quad A_2(b) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}$$

Since $\det A = 3s^2 - 12 = 3(s+2)(s-2)$

the system has a unique solution when

$\det A \neq 0$

$\Rightarrow 3(s+2)(s-2) \neq 0$

$\Rightarrow s^2 - 4 \neq 0$

$\Rightarrow s \neq \pm 2$

For such an s, the solution is (x_1, x_2), where

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{4s+2}{3(s+2)(s-2)}, \qquad s \neq \pm 2$$

$$x_2 = \frac{\det A_2(b)}{\det A} = \frac{3s+24}{3(s+2)(s-2)} = \frac{s+8}{(s+2)(s-2)}, \qquad s \neq \pm 2$$

Example 3 Solve, by Cramer's Rule, the system of equations

$$2x_1 - x_2 + 3x_3 = 1$$

$$x_1 + 2x_2 - x_3 = 2$$

$$3x_1 + 2x_2 + 2x_3 = 3$$

Solution Here $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 3 & 2 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 2 & -1 \\ 3 & 2 & 2 \end{bmatrix}$

$$A_2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 3 & 3 & 2 \end{bmatrix}, \; A_3 = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 3 & 2 & 3 \end{bmatrix}$$

$$D = \det A = 2\cdot 6 - 1\cdot(-5) + 3(-4) = 5$$

$$D_1 = \det A_1 b = 1\cdot 6 + 2\cdot 8 + 3(-5) = 7$$

$$D_2 = \det A_2 b = 2\cdot(7) - 1\cdot(5) + 3(-3) = 0$$

$$D_3 = \det A_3 b = 2\cdot(2) + 1\cdot(-3) + 1(-4) = -3$$

$$So\ \ x_1 = \frac{D_1}{D} = \frac{7}{5}, \quad x_2 = \frac{D_2}{D} = 0, \quad x_3 = \frac{D_3}{D} = -\frac{3}{5}$$

Use Cramer's Rule to solve

Determinants as Area or Volume

In the next application, we verify the geometric interpretation of determinants and we assume here that the usual Euclidean concepts of length, area, and volume are already understood for $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$.

Theorem 3 If $\boldsymbol{A}$ is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If $\boldsymbol{A}$ is a 3×3 matrix, the volume of the parallelepiped determined by the columns of $\boldsymbol{A}$ is $|\det A|$.

Example 6 Calculate the area of the parallelogram determined by the points (-2, -2), (0, 3), (4, -1) and (6, 4).

Solution

Let A(-2,-2), B(0,3), C(4,-1) and D(6,4). Fixing one point say A(-2,-2) and find the adjacent lengths of parallelogram which are given by the column vectors as follows;

$$AB = \begin{bmatrix} 0-(-2) \\ 3-(-2) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$AC = \begin{bmatrix} 4-(-2) \\ -1-(-2) \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

So the area of parallelogram ABCD determined by above column vectors

$$= \left| \det \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix} \right| = |2-30| = |-28| = 28$$

Linear Transformations

Determinants can be used to describe an important geometric property of linear transformations in the plane and in $\boldsymbol{R}^3$. If $\boldsymbol{T}$ is a linear transformation and $\boldsymbol{S}$ is a set in the domain of $\boldsymbol{T}$, let $\boldsymbol{T}(\boldsymbol{S})$ denote the set of images of points in $\boldsymbol{S}$. We are interested in how the area (or volume) of $\boldsymbol{T}(\boldsymbol{S})$ compares with the area (or volume) of the original set $\boldsymbol{S}$. For convenience, when $\boldsymbol{S}$ is a region bounded by a parallelogram, we also refer to $\boldsymbol{S}$ as a parallelogram.

Theorem 4 Let $T: R^2 \rightarrow R^2$ be the linear transformation determined by a 2×2 matrix $\boldsymbol{A}$. If $\boldsymbol{S}$ is a parallelogram in $\boldsymbol{R}^2$, then

$$\{\text{area of } \boldsymbol{T}(\boldsymbol{S})\} = |\det \boldsymbol{A}|.\ \{\text{area of } \boldsymbol{S}\}$$

If $\boldsymbol{T}$ is determined by a 3 x 3 matrix $\boldsymbol{A}$, and if $\boldsymbol{S}$ is a parallelepiped in $\boldsymbol{R}^3$, then

$$\{\text{volume of } \boldsymbol{T}(\boldsymbol{S})\} = |\det \boldsymbol{A}|.\ \{\text{volume of } \boldsymbol{S}\}$$

Example 7 Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$.

<u>Solution</u> We claim that E is the image of the unit disk D under the linear transformation **A:D→E** determined by the matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, given as

$\boldsymbol{Au} = \boldsymbol{x}$ where $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in D$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in E$.

$$\Rightarrow \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now $\boldsymbol{Au} = \boldsymbol{x} \Rightarrow \begin{bmatrix} au_1 \\ bu_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then

$$\Rightarrow au_1 = x_1 \quad and \quad bu_2 = x_2$$

$$\Rightarrow u_1 = \frac{x_1}{a} \text{ and } u_2 = \frac{x_2}{b}$$

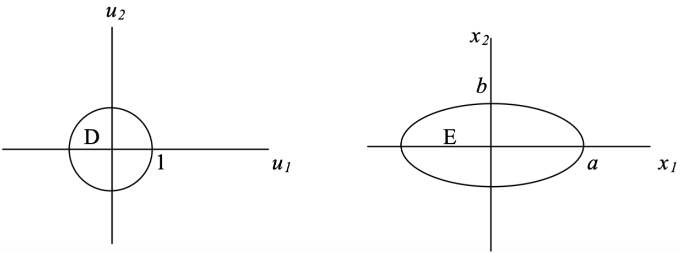
Since $u \in D$ (in the circular disk),it follows that the distance of $\boldsymbol{u}$ from origin will be less than unity i-e

$$\left(u_1^2 - 0\right) + \left(u_2^2 - 0\right) \leq 1$$

$$\Rightarrow \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 \leq 1 \qquad \because u_1 = \frac{x_1}{a}, u_2 = \frac{x_2}{b}$$

Hence by the generalization of theorem 4,

$$\{\text{area of ellipse}\} = \{\text{area of } \boldsymbol{A(D)}\} \qquad (here\ \ T \equiv A)$$
$$= |\det \boldsymbol{A}|.\ \{\text{area of } \boldsymbol{D}\}$$
$$= ab.\pi\,(1)^2 = \pi\,\text{ab}$$



Example 8 Let $\boldsymbol{S}$ be the parallelogram determined by the vectors $b_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, and let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Compute the area of image of $\boldsymbol{S}$ under the mapping $x \to Ax$.

Solution The area of $\boldsymbol{S}$ is $\left|\det\begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}\right| = 14$, and det $\boldsymbol{A}$ = 2. By theorem 4, the area of image of $\boldsymbol{S}$ under the mapping $x \to Ax$ is |det A|. {area of $\boldsymbol{S}$} = 2.14 = 28

15. Let $\boldsymbol{T}: \boldsymbol{R}^3 \to \boldsymbol{R}^3$ be the linear transformation determined by the matrix

$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where a, b, c are positive numbers. Let $\boldsymbol{S}$ be the unit ball, whose bounding surface has the equation $x_1^2 + x_2^2 + x_3^2 = 1$.

a. Show that $\boldsymbol{T}(\boldsymbol{S})$ is bounded by the ellipsoid with the equation $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} + \dfrac{x_3^2}{c^2} = 1$.

b. Use the fact that the volume of the unit ball is $4\pi/3$ to determine the volume of the region bounded by the ellipsoid in part (a).