MT-104
Linear Algebra

National University of Computer and Emerging Sciences

Fall 2020

December 17, 2020

Lecture On Orthogonal Projections

Orthogonality

Orthonormal Basis

Definition

A set of vectors in $\mathbb{R}^n$ is called an **orthonormal set** if it is an orthogonal set of unit vectors.

Theorem

Let $\{x_1, x_2, ..., x_k\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and let w be any vector in W. Then the unique scalars $c_1, c_2, ..., c_k$ such that

$$w = c_1 x_1 + c_2 x_2 + ... + c_k x_k$$

are given by

$$c_i = w \cdot x_i, \quad i = 1, 2, ..., k.$$

Orthogonal Matrices

Theorem

The matrix Q (square or rectangular) has orthonormal columns if and only if $Q^TQ = I$

Proof.

If Q has orthonormal columns then,

$$(Q^TQ)_{ij} = q_i \cdot q_j = I.$$

Conversely,
If $Q^TQ = I$, then

$$q_i \cdot q_j = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{if} i = j. \end{cases}$$

$\square$

Orthogonal Matrices

Theorem

Any square matrix Q whose columns form an orthonormal set is called **Orthogonal Matrix**.

Theorem

Let Q be an $n \times n$ matrix. Then the following statements are equivalent:

1. Q is orthogonal.
2. $Q^T = Q^{-1}$.
3. $\|Q\mathbf{x}\| = \|\mathbf{x}\|$.
4. $Q\mathbf{x} \cdot Q\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.

   where $\mathbf{x}$ and $\mathbf{y}$ are from R^n.

Orthogonal Matrices

Theorem

Let Q be an orthogonal matrix.

1. Q^{-1} is orthogonal.
2. $det(Q) = \pm 1$
3. If λ is an eigenvalue of Q, then $|\lambda| = 1$.
4. Product of orthogonal matrices of same size is another orthogonal matrix.
5. Rows of Q forms an orthonormal set.

Orthogonal Matrices

Examples

- $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & cos\theta \end{bmatrix}.$
- $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$

Orthogonal Complements

Definition

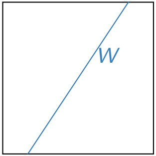
Let W be a subspace of $\mathbb{R}^n$. Its **orthogonal complement** is

$$W^{\perp} = \{ v \text{ in } \mathbb{R}^n \mid v \cdot w = 0 \text{ for all } w \text{ in } W \} \qquad \text{read "}W\text{ perp".}$$

$W^{\perp}$ is orthogonal complement

Pictures:

The orthogonal complement of a line in $\mathbb{R}^2$ is



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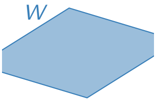

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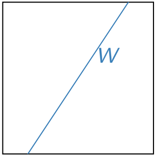

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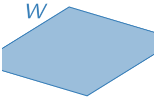

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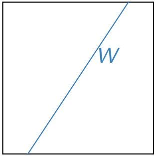

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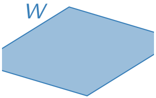

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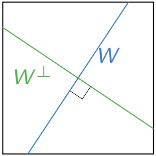

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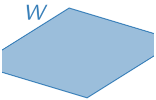

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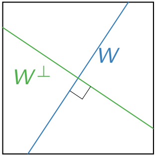

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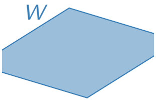

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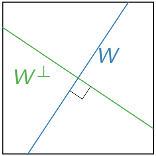

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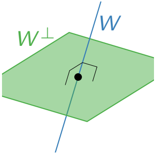

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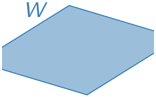

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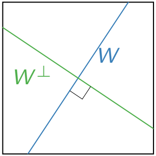

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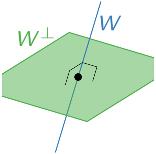

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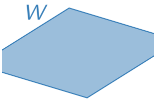

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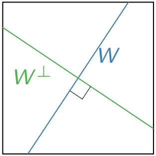

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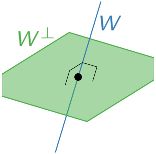

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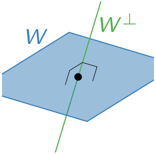

Orthogonal Complement

Let W be a plane in $\mathbf{R}^4$. How would you describe $W^\perp$?

A. The zero space $\{0\}$.

B. A line in $\mathbf{R}^4$.

C. A plane in $\mathbf{R}^4$.

D. A 3-dimensional space in $\mathbf{R}^4$.

E. All of $\mathbf{R}^4$.

Orthogonal Complements

Basic properties

Let W be a subspace of $\mathbf{R}^n$.

Facts:

1. $W^\perp$ is also a subspace of $\mathbf{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $W = \operatorname{Span}\{v_1, v_2, \ldots, v_m\}$, then

$$\begin{aligned} W^\perp &= \text{all vectors orthogonal to each } v_1, v_2, \ldots, v_m \\ &= \left\{x \text{ in } \mathbf{R}^n \mid x \cdot v_i = 0 \text{ for all } i = 1, 2, \ldots, m\right\} \\ &= \operatorname{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}. \end{aligned}$$

Let's check 1.

- Is 0 in $W^\perp$? Yes: $0 \cdot w = 0$ for any w in W.
- Suppose x, y are in $W^\perp$. So $x \cdot w = 0$ and $y \cdot w = 0$ for all w in W. Then $(x + y) \cdot w = x \cdot w + y \cdot w = 0 + 0 = 0$ for all w in W. So $x + y$ is also in $W^\perp$.
- Suppose x is in $W^\perp$. So $x \cdot w = 0$ for all w in W. If c is a scalar, then $(cx) \cdot w = c(x \cdot 0) = c(0) = 0$ for any w in W. So cx is in $W^\perp$.

Problem: Find all vectors orthogonal to some number of vectors $v_1, v_2, \ldots, v_m$ in $\mathbf{R}^n$.

This is the same as finding all vectors x such that

$$0 = v_1^T x = v_2^T x = \cdots = v_m^T x.$$

Putting the *row* vectors $v_1^T, v_2^T, \ldots, v_m^T$ into a matrix, this is the same as finding all x such that

$$\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix} x = \begin{pmatrix} v_1 \cdot x \\ v_2 \cdot x \\ \vdots \\ v_m \cdot x \end{pmatrix} = 0.$$

Important

The set of all vectors orthogonal to some vectors $v_1, v_2, \ldots, v_m$ in $\mathbf{R}^n$ is the *null space* of the $m \times n$ matrix

$$\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}.$$

In particular, this set is a subspace!

Orthogonal Complements

Row space, column space, null space

Definition

The **row space** of an $m \times n$ matrix A is the span of the *rows* of A. It is denoted $\mathrm{Row}\,A$.

Equivalently, it is the column span of A^T:

$$\mathrm{Row}\,A = \mathrm{Col}\,A^T.$$

It is a subspace of R–.

[5]We showed before that if A has rows $v_1^T, v_2^T, \ldots, v_m^T$, then

$$\mathrm{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \mathrm{Nul}\,A.$$

Hence we have shown:

Fact: $(\mathrm{Row}\,A)^\perp = \mathrm{Nul}\,A.$

Replacing A by A^T, and remembering $\mathrm{Row}\,A^T = \mathrm{Col}\,A$:

Fact: $(\mathrm{Col}\,A)^\perp = \mathrm{Nul}\,A^T.$

Using property 2 and taking the orthogonal complements of both sides, we get:

Fact: $(\mathrm{Nul}\,A)^\perp = \mathrm{Row}\,A$ and $\mathrm{Col}\,A = (\mathrm{Nul}\,A^T)^\perp.$

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Orthogonal Complements of Most of the Subspaces We've Seen

For any vectors $v_1, v_2, \ldots, v_m$:

$$\mathrm{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \mathrm{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}$$

For any matrix A:

$$\mathrm{Row}\,A = \mathrm{Col}\,A^T$$

and

$$(\mathrm{Row}\,A)^\perp = \mathrm{Nul}\,A \qquad \mathrm{Row}\,A = (\mathrm{Nul}\,A)^\perp$$

$$(\mathrm{Col}\,A)^\perp = \mathrm{Nul}\,A^T \qquad \mathrm{Col}\,A = (\mathrm{Nul}\,A^T)^\perp$$

Orthogonal Complements

Computation

Problem: if $W = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, compute $W^\perp$.

We have to find the null space of the matrix whose rows are $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, which we did before:

$$\operatorname{Nul}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \operatorname{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

$$\operatorname{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \operatorname{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}$$

Orthogonal Complements

Example

Problem Let W be the subspace of $\mathbb{R}^5$ spanned by

$$w_1 = \begin{bmatrix} 1 \\ -3 \\ 5 \\ 0 \\ 5 \end{bmatrix},\ w_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \\ 3 \end{bmatrix},\ w_3 = \begin{bmatrix} 0 \\ -1 \\ 4 \\ -1 \\ 5 \end{bmatrix}.$$

Find a basis for $W^\perp$.

Solution. There are two obvious approaches. We can construct a matrix whose column space is W and can easily construct a matrix whose row space is W.

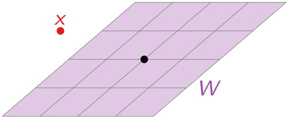
Let $A = \begin{bmatrix} 1 & -1 & 0 \\ -3 & 1 & -1 \\ 5 & 2 & 4 \\ 0 & -2 & -1 \\ 5 & 3 & 5 \end{bmatrix}$

As $W = col(A)$ so, $W^\perp = null(A^T)$.

Solving the homogenous system $A^T x = 0$ will gives us $W^\perp$.

Best Approximation

Suppose you measure a data point x which you know for theoretical reasons must lie on a subspace W.



Due to measurement error, though, the measured x is not actually in W. Best approximation: y is the *closest* point to x on W.

How do you know that y is the closest point?
The vector from y to x is orthogonal to W:
it is in the *orthogonal complement* $W^{\perp}$.

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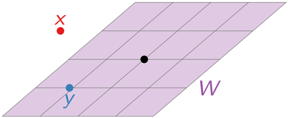

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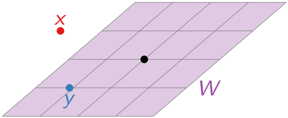

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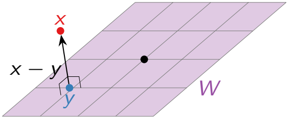

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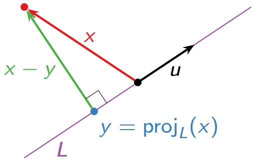
Orthogonal Projection onto a Line

Theorem

Let $L = \text{Span}\{u\}$ be a line in $\mathbf{R}^n$, and let x be in $\mathbf{R}^n$. The closest point to x on L is the point

$$\text{proj}_L(x) = \frac{x \cdot u}{u \cdot u}\, u.$$

This point is called the **orthogonal projection of x onto L.**



Why? Let $y = \text{proj}_L(x)$. We have to verify that $x - y$ is in $L^\perp$. This means proving that $u \cdot (x - y) = 0$.

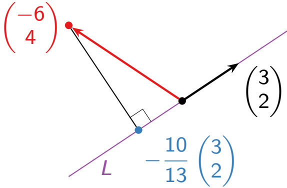
$$u \cdot (x - y) = u \cdot \left(x - \frac{x \cdot u}{u \cdot u} u\right) = u \cdot x - \frac{x \cdot u}{u \cdot u}(u \cdot u) = u \cdot x - x \cdot u = 0.$$

Orthogonal Projection onto a Line

Example

Compute the orthogonal projection of $x = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ onto the line L spanned by $u = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$$y = \mathrm{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u = \frac{-18 + 8}{9 + 4} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = -\frac{10}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$



Orthogonal Bases

Geometric reason

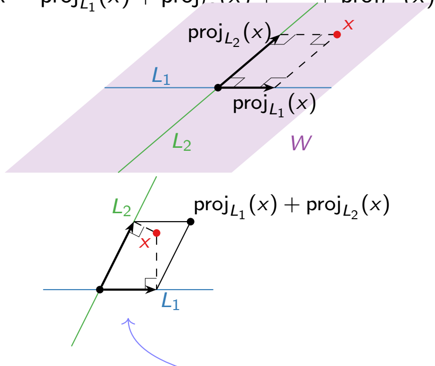
Theorem

Let $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ be an orthogonal set, and let x be a vector in $W = \operatorname{Span}\mathcal{B}$. Then

$$x = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \overset{\text{proj}_{L_2}(u_2)}{\boxed{\frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2}} + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

If L_i is the line spanned by u_i, then this says

$$x = \text{proj}_{L_1}(x) + \text{proj}_{L_2}(x) + \cdots + \text{proj}_{L_m}(x).$$



Orthogonal Bases

Example

Problem: Find the $\mathcal{B}$-coordinates of $x = \binom{0}{3}$, where

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix} \right\}.$$

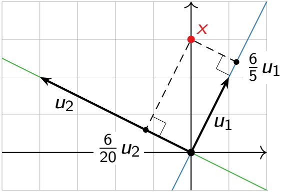
Old way:

$$\left(\begin{array}{cc|c} 1 & -4 & 0 \\ 2 & 2 & 3 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 6/5 \\ 0 & 1 & 6/20 \end{array}\right) \implies [x]_{\mathcal{B}} = \begin{pmatrix} 6/5 \\ 6/20 \end{pmatrix}.$$

New way: note $\mathcal{B}$ is an *orthogonal* basis.

$$x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{3 \cdot 2}{1^2 + 2^2} u_1 + \frac{3 \cdot 2}{(-4)^2 + 2^2} u_2 = \frac{6}{5} u_1 + \frac{6}{20} u_2.$$

So the $\mathcal{B}$-coordinates are $\frac{6}{5}, \frac{6}{20}$.



Orthogonal Bases

Example

Problem: Find the $\mathcal{B}$-coordinates of $x = (6, 1, -8)$ where

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Answer:

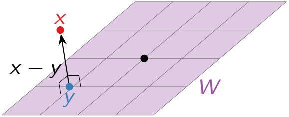
$$\begin{aligned}
[x]_{\mathcal{B}} &= \left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2}, \frac{x \cdot u_3}{u_3 \cdot u_3} \right) \\
&= \left(\frac{6 \cdot 1 + 1 \cdot 1 - 8 \cdot 1}{1^2 + 1^2 + 1^2}, \frac{6 \cdot 1 + 1 \cdot (-2) - 8 \cdot 1}{1^2 + (-2)^2 + 1^2}, \frac{6 \cdot 1 + 1 \cdot 0 + (-8) \cdot (-1)}{1^2 + 0^2 + (-1)^2} \right) \\
&= \left(-\frac{1}{3}, -\frac{2}{3}, 7 \right).
\end{aligned}$$

Check:

$$\begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + 7 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Idea Behind Orthogonal Projections

If x is not in a subspace W, then y in W is the closest to x if $x - y$ is in $W^\perp$:



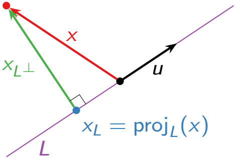
Reformulation: Every vector x can be decompsed uniquely as

$$x = x_W + x_{W^\perp}$$

where $x_W = y$ is the closest vector to x in W, and $x_{W^\perp} = x - y$ is in $W^\perp$.

Example: Let $u = \binom{3}{2}$ and let $L = \mathrm{Span}\{u\}$. Let $x = \binom{-6}{4}$.
Then the closest point to x in L is
$\mathrm{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u$, so

$$x_L = \mathrm{proj}_L(x) = -\frac{10}{13}\begin{pmatrix}3\\2\end{pmatrix} \qquad x_{L^\perp} = x - \mathrm{proj}_L(x) = \begin{pmatrix}-6\\4\end{pmatrix} + \frac{10}{13}\begin{pmatrix}3\\2\end{pmatrix}.$$



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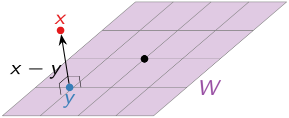

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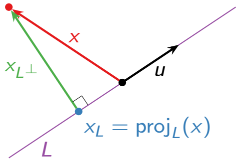

Idea Behind Orthogonal Projections

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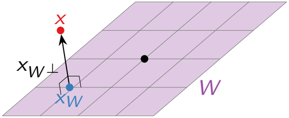

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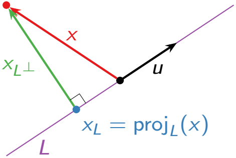


Orthogonal Projections

Definition

Let W be a subspace of $\mathbf{R}^n$, and let $\{u_1, u_2, \ldots, u_m\}$ be an *orthogonal* basis for W. The **orthogonal projection** of a vector x onto W is

$$\mathrm{proj}_W(x) \overset{\text{def}}{=} \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i.$$

Question: What is the difference between this and the formula for $[x]_{\mathcal{B}}$ from before?

Theorem

Let W be a subspace of $\mathbf{R}^n$, and let x be a vector in $\mathbf{R}^n$. Then $\mathrm{proj}_W(x)$ is the closest point to x in W.

Therefore

$$\boxed{x_W = \mathrm{proj}_W(x) \qquad x_{W^\perp} = x - \mathrm{proj}_W(x).}$$

Why? Let $y = \mathrm{proj}_W(x)$. We need to show that $x - y$ is in $W^\perp$. In other words, $u_i \cdot (x - y) = 0$ for each i. Let's do u_1:

$$u_1 \cdot (x - y) = u_1 \cdot \left(x - \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i \right) = u_1 \cdot x - \frac{x \cdot u_1}{u_1 \cdot u_1}(u_1 \cdot u_1) - 0 - \cdots = 0.$$

Orthogonal Projections

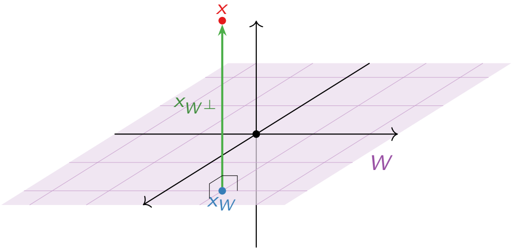
Easy example

What is the projection of $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ onto the xy-plane?

Answer: The xy-plane is $W = \mathrm{Span}\{e_1, e_2\}$, and $\{e_1, e_2\}$ is an orthogonal basis.

$$x_W = \mathrm{proj}_W \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{x \cdot e_1}{e_1 \cdot e_1} e_1 + \frac{x \cdot e_2}{e_2 \cdot e_2} e_2 = \frac{1 \cdot 1}{1^2} e_1 + \frac{1 \cdot 2}{1^2} e_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

So this is the same projection as before.



Orthogonal Projections

More complicated example

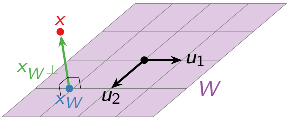
What is the projection of $x = \begin{pmatrix} -1.1 \\ 1.4 \\ 1.45 \end{pmatrix}$ onto $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1.1 \\ -.2 \end{pmatrix} \right\}$?

Answer: The basis is orthogonal, so

$$x_W = \text{proj}_W \begin{pmatrix} -1.1 \\ 1.4 \\ 1.45 \end{pmatrix} = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \frac{(-1.1)(1)}{1^2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{(1.4)(1.1) + (1.45)(-.2)}{1.1^2 + (-.2)^2} \begin{pmatrix} 0 \\ 1.1 \\ -.2 \end{pmatrix}$$

This turns out to be equal to $u_2 - 1.1 u_1$.



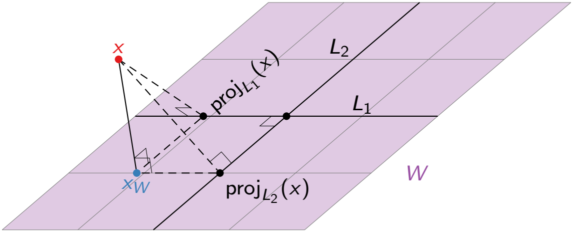
Orthogonal Projections

Picture

Let W be a subspace of $\mathbb{R}^n$, and let $\{u_1, u_2, \ldots, u_m\}$ be an orthogonal basis for W. Let $L_i = \mathrm{Span}\{u_i\}$. Then

$$\mathrm{proj}_W(x) = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \sum_{i=1}^{m} \mathrm{proj}_{L_i}(x).$$

So the orthogonal projection is formed by adding orthogonal projections onto perpendicular lines.



Orthogonal Projections

Properties

First we restate the property we've been using all along.

Best Approximation Theorem

Let W be a subspace of $\mathbf{R}^n$, and let x be a vector in $\mathbf{R}^n$. Then $y = \mathrm{proj}_W(x)$ is the closest point in W to x, in the sense that

$$\mathrm{dist}(x, y') \geq \mathrm{dist}(x, y) \quad \text{for all} \quad y' \text{ in } W.$$

We can think of orthogonal projection as a *transformation*:

$$\mathrm{proj}_W \colon \mathbf{R}^n \longrightarrow \mathbf{R}^n \qquad x \mapsto \mathrm{proj}_W(x).$$

Theorem

Let W be a subspace of $\mathbf{R}^n$.

1. proj_W is a *linear* transformation.
2. For every x in W, we have $\mathrm{proj}_W(x) = x$.
3. For every x in $W^\perp$, we have $\mathrm{proj}_W(x) = 0$.
4. The range of proj_W is
   W.