Course Name:Linear Algebra

Course Code: MT 104

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1.8 Introduction to Linear Transformations

- Matrix Transformations
  - Matrix Acting on Vector
    - Matrix-Vector Multiplication
  - Transformation: Domain and Range
  - Examples
  - Applications
    - Computer Graphics
- Linear Transformation
  - Definition
  - Examples
    - Matrix Transformations

Another Way to Vview $A\mathbf{x} = \mathbf{b}$

Matrix A is an object acting on $\mathbf{x}$ by multiplication to produce a new vector $A\mathbf{x}$ or $\mathbf{b}$.

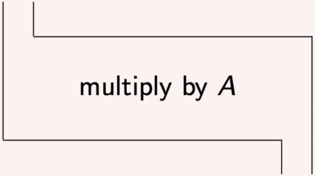
Example

$$\begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ -12 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrix Transformations

Suppose A is $m \times n$. Solving $A\mathbf{x} = \mathbf{b}$ amounts to finding all ____ in $\mathbf{R}^n$ which are transformed into vector $\mathbf{b}$ in $\mathbf{R}^m$ through multiplication by A.



transformation "machine"

Transformation

A **transformation** T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbf{R}^n$ a vector $T(\mathbf{x})$ in $\mathbf{R}^m$.

$$T : \mathbf{R}^n \longrightarrow \mathbf{R}^m$$

Terminology

$\mathbf{R}^n$: **domain** of T

$\mathbf{R}^m$: **codomain** of T

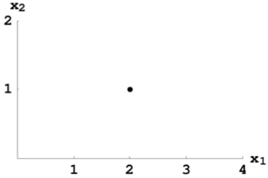
$T(\mathbf{x})$ in $\mathbf{R}^m$ is the **image** of $\mathbf{x}$ under the transformation T

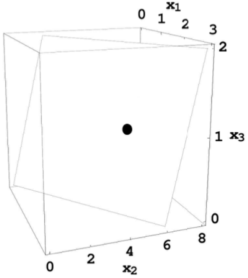
Set of all images $T(\mathbf{x})$ is the **range** of T

Example

Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. Define $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

Then if $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$





Example

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Define a transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

a. Find an $\mathbf{x}$ in $\mathbf{R}^3$ whose image under T is $\mathbf{b}$.

b. Is there more than one $\mathbf{x}$ under T whose image is $\mathbf{b}$. *(uniqueness problem)*

c. Determine if $\mathbf{c}$ is in the range of the transformation T. *(existence problem)*

Solution: (a) Solve ______=____ for $\mathbf{x}$, or

$$\begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$$

Augmented matrix:

$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ -5 & 10 & -15 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} &x_1 = 2x_2 - 3x_3 + 2 \\ &x_2 \text{ is free} \\ &x_3 \text{ is free} \end{aligned}$$

Let $x_2 =$ _____ and $x_3 =$ _____. Then $x_1 =$ _____.

$$\text{So } \mathbf{x} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

(b) Is there an $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{b}$?

$$\text{Free variables exist}$$
$$\Downarrow$$
$$\text{There is more than one } \mathbf{x} \text{ for which } T(\mathbf{x}) = \mathbf{b}$$

(c) Is there an $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{c}$? This is another way of

asking if $A\mathbf{x} = \mathbf{c}$ is _______________.
Augmented matrix:

$$\begin{bmatrix} 1 & -2 & 3 & 3 \\ -5 & 10 & -15 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

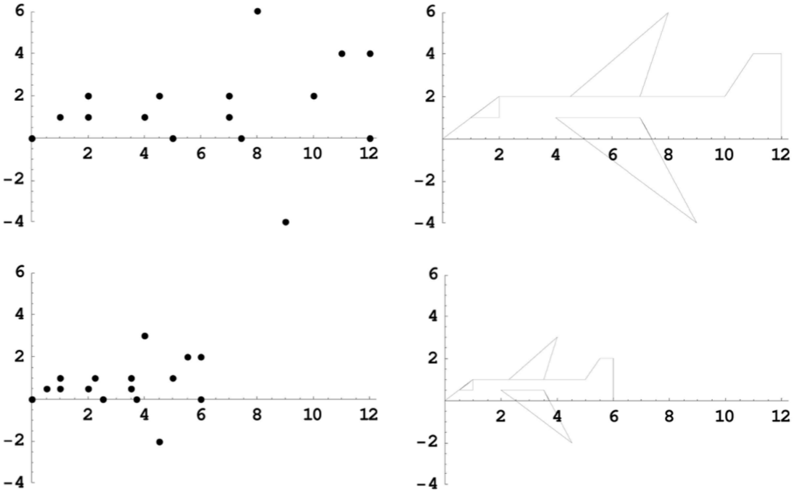
$\mathbf{c}$ is not in the _______________ of T.

Matrix transformations have many applications - including *computer graphics*

Example

Let $A = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}$. The transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is an example of a **contraction** transformation. The transformation $T(\mathbf{x}) = A\mathbf{x}$ can be used to move a point $\mathbf{x}$.

$$\mathbf{u} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \qquad T(\mathbf{u}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$


6
4
2
−2
−4
2
4
6
8
10
12

If A is $m \times n$, then the transformation $T(\mathbf{x}) = A\mathbf{x}$ has the following properties:

$$T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = \_\_\_\_\_\_ + \_\_\_\_\_\_$$

$$= \_\_\_\_\_\_ + \_\_\_\_\_\_$$

and

$$T(c\mathbf{u}) = A(c\mathbf{u}) = \_\_\_\_A\mathbf{u} = \_\_\_\_T(\mathbf{u})$$

for all $\mathbf{u}$,$\mathbf{v}$ in $\mathbf{R}^n$ and all scalars c.

Linear Transformation

A transformation T is **linear** if:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T.
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in the domain of T and all scalars c.

Every matrix transformation is a **linear** transformation.

RESULT

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0} \qquad \text{and} \qquad T(c\mathbf{u} + d\mathbf{v}) = c\mathbf{T}(\mathbf{u}) + d\mathbf{T}(\mathbf{v}).$$

Proof:

$$T(\mathbf{0}) = T(0\mathbf{u}) = \_\_\_\_ T(\mathbf{u}) = \_\_\_\_\_.$$

$$T(c\mathbf{u} + d\mathbf{v}) = T(\qquad\qquad) + T(\qquad\qquad)$$

$$= \_\_\_\_\_ T(\qquad) + \_\_\_\_\_ T(\qquad)$$

Example

Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Suppose $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation which maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: First, note that

$$T(\mathbf{e}_1) = \_\_\_\_\_\_ \qquad \text{and} \qquad T(\mathbf{e}_2) = \_\_\_\_\_\_.$$

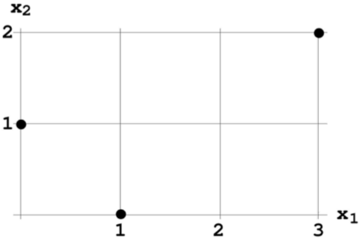
Also

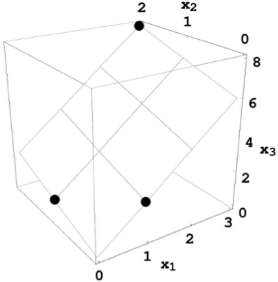
$$\_\_\_\mathbf{e}_1 + \_\_\_\mathbf{e}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Then

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = T\left(\_\_\_\mathbf{e}_1 + \_\_\_\mathbf{e}_2\right) =$$

$$\_\_\_T\left(\mathbf{e}_1\right) + \_\_\_T\left(\mathbf{e}_2\right) =$$





$$T\left(3\mathbf{e}_1 + 2\mathbf{e}_2\right) = 3T\left(\mathbf{e}_1\right) + 2T\left(\mathbf{e}_2\right)$$

Also

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\_\_\_\_\_\mathbf{e}_1 + \_\_\_\_\_\mathbf{e}_2\right) =$$

$$\_\_\_\_T\left(\mathbf{e}_1\right) + \_\_\_\_T\left(\mathbf{e}_2\right) =$$

Example

Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(x_1, x_2, x_3) = (|x_1 + x_3|, 2 + 5x_2)$. Show that T is a not a linear transformation.

Solution: Another way to write the transformation:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} |x_1 + x_3| \\ 2 + 5x_2 \end{bmatrix}$$

Provide a **counterexample** - example where $T(\mathbf{0}) = \mathbf{0}$, $T(c\mathbf{u}) = c\mathbf{T}(\mathbf{u})$ or $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ is violated.

A counterexample:

$$T(\mathbf{0}) = T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \quad \end{bmatrix} \neq \_\_\_\_\_$$

which means that T is not linear.

Another counterexample: Let $c = -1$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then

$$T(c\mathbf{u}) = T\left(\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} |-1 + -1| \\ 2 + 5(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

and

$$cT(\mathbf{u}) = -1T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = -1\begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}.$$

Therefore $T(c\mathbf{u}) \neq$ ___ $T(\mathbf{u})$ and therefore T is

not __________________.

1.9 The Matrix of a Linear Transformation

- Matrix Transformation: Identity Matrix
- Linear Transformation: Generalized Result
- Matrix of a Linear Transformation
    - Theorem
    - Examples
        - Geometric Linear Transformations of $\mathbf{R}^2$

Identity Matrix

I_n is an $n \times n$ matrix with 1's on the main left to right diagonal and 0's elsewhere. The ith column of I_n is labeled $\mathbf{e}_i$.

Example

$$I_3 = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that

$$I_3\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \_\_\_\_ \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} + \_\_\_\_ \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} + \_\_\_\_ \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \_\_\_\_.$$

Identity Matrix

In general, for $\mathbf{x}$ in $\mathbf{R}^n$, $\quad I_n\mathbf{x} = \_\_\_$

Linear Transformation

From Section 1.8, if $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then $T(c\mathbf{u} + d\mathbf{v}) = c\mathbf{T}(\mathbf{u}) + d\mathbf{T}(\mathbf{v})$.

Generalized Result

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1 T(\mathbf{v}_1) + \cdots + c_p T(\mathbf{v}_p).$$

Example

The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Suppose T is a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^3$ where

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \text{ and } T(\mathbf{e}_2) = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

Compute $T(\mathbf{x})$ for any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: A vector $\mathbf{x}$ in $\mathbf{R}^2$ can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \_\_\_\_ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \_\_\_\_\mathbf{e}_1 + \_\_\_\_\mathbf{e}_2$$

Then

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = \_\_\_\_\_ T(\mathbf{e}_1) + \_\_\_\_\_ T(\mathbf{e}_2)$$

$$= \_\_\_\_\_ \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} + \_\_\_\_\_ \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \qquad\qquad\qquad\qquad \\ \\ \\ \end{bmatrix}.$$

Note that

$$T(\mathbf{x}) = \begin{bmatrix} \qquad \\ \\ \\ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

So

$$T(\mathbf{x}) = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} \mathbf{x} = A\mathbf{x}$$

To get A, replace the identity matrix $[\mathbf{e}_1\ \mathbf{e}_2]$ *with* $[T(\mathbf{e}_2)\ T(\mathbf{e}_2)]$.

Theorem

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbf{R}^n$.

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]$$

$$\uparrow$$

(standard matrix for the linear transformation) T

Example

$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ 4x_1 \\ 3x_1 + 2x_2 \end{bmatrix}$$

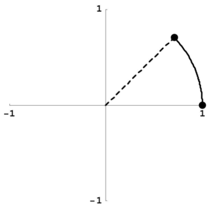
Solution:

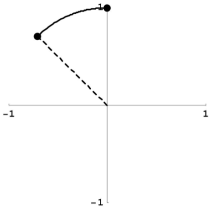
$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} = \text{standard matrix of the linear transformation } T$$

$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \qquad \text{(fill-in)}$$

Example

Find the standard matrix of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ which rotates a point about the origin through an angle of $\frac{\pi}{4}$ radians (counterclockwise).





$$\downarrow$$

$$A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$T(\mathbf{e}_1) = \begin{bmatrix} \quad \\ \quad \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$