Course Name:Linear Algebra(MT 104)

Topic:Matrix Equation (Exercise 1.4)

Instructor: Dr. Sara Aziz

saraazizpk@gmail.com

September 19, 2020

Discuss **Exercise 1.3**

1.4 The Matrix Equation $\mathbf{Ax} = \mathbf{b}$

- Matrix-Vector Multiplication
  - Linear Combination of the Columns
- Matrix Equation
  - Three Equivalent Ways of Viewing a Linear System
- Existence of Solution
  - Matrix Equation Equivalent Theorem
- Another method for computing $A\mathbf{x}$
  - Row-Vector Rule

Matrix-Vector Multiplication

Key Concepts to Master

Linear combinations can be viewed as
a matrix-vector multiplication.

Matrix-Vector Multiplication

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbf{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is the **linear combination of the columns of A using the corresponding entries in x as weights**. i.e.,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\ \mathbf{a}_1 + x_2\ \mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Matrix-Vector Multiplication: Examples

Example

$$\begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + -6 \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} =$$

$$\begin{bmatrix} 7 \\ 21 \\ 0 \end{bmatrix} + \begin{bmatrix} 24 \\ -12 \\ -30 \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}$$

Matrix-Vector Multiplication: Examples

Example

Write down the system of equations corresponding to the augmented matrix below and then express the system of equations in vector form and finally in the form $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ is a 3×1 vector.

$$\begin{bmatrix} 2 & 3 & 4 & 9 \\ -3 & 1 & 0 & -2 \end{bmatrix}$$

Solution: Corresponding system of equations (fill-in)

Vector Equation:

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \qquad \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \qquad \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}.$$

Matrix equation (fill-in):

Matrix Equation

Three Equivalent Ways of Viewing a Linear System

1. as a system of linear equations;
2. as a vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$; or
3. as a matrix equation $A\mathbf{x} = \mathbf{b}$.

Useful Fact

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a

__________ ________________ of the columns of A.

Matrix Equation: Theorem

Theorem

If A is a $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbf{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

Matrix Equation: Example

Example

Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$?

Solution: Augmented matrix corresponding to $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & b_1 \\ 0 & 1 & 1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ is _______ consistent for all $\mathbf{b}$ since some choices of $\mathbf{b}$ make $-2b_1 + b_3$ nonzero.

Matrix Equation: Example (cont)

$$A = \begin{array}{c} \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{array} \end{array}$$

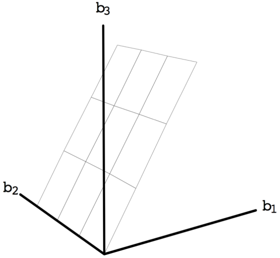
The equation $A\mathbf{x} = \mathbf{b}$ is consistent if

$$-2b_1 + b_3 = 0.$$

(equation of a plane in $\mathbf{R}^3$)

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_3 + x_3\mathbf{a}_3 = \mathbf{b}$$

if and only if $b_3 - 2b_1 = 0$.



Columns of A span a plane in $\mathbf{R}^3$ through $\mathbf{0}$

Instead, if *any* $\mathbf{b}$ in $\mathbf{R}^3$ (not just those lying on a particular line or in a plane) can be expressed as a linear combination of the columns of A, then we say that the columns of A span $\mathbf{R}^3$.

Matrix Equation: Span R^n

Definition

We say that **the columns of** $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p \end{bmatrix}$ **span** $\mathbf{R}^m$ if *every* vector $\mathbf{b}$ in $\mathbf{R}^m$ is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_p$ (i.e. $\text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_p\} = \mathbf{R}^m$).

Theorem (4)

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. *For each $\mathbf{b}$ in $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.*
2. *Each $\mathbf{b}$ in $\mathbf{R}^m$ is a linear combination of the columns of A.*
3. *The columns of A span $\mathbf{R}^m$.*
4. *A has a pivot position in every row.*

Matrix Equation: Example

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible $\mathbf{b}$?

Solution: A has only ____ columns and therefore has at most ____ pivots.

Since A does not have a pivot in every ______, $A\mathbf{x} = \mathbf{b}$

is _______ ________________ for all possible $\mathbf{b}$, according to Theorem 4.

Matrix Equation: Example

Example

Do the columns of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix}$ span $\mathbf{R}^3$?

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix} \sim$$

(no pivot in row 2)

By Theorem 4, the columns of A