MT-104
Linear Algebra

National University of Computer and Emerging Sciences

Fall 2020

December 19, 2020

Lecture 24

Orthogonality

Key Ideas

- The **dot product** of two vectors x, y in R^n is

$$x \cdot y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = x^T y.$$

Example

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \\ 5 \end{pmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + 5 \cdot 5 = 57.$$

Key Ideas

- The **length** or **norm** of a vector x in R^n is

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Example

$$\left\| \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \right\| = \sqrt{1^2 + 4^2 + 3^2} = \sqrt{26}.$$

- Two vectors x, y are **orthogonal** or **perpendicular** if $x \cdot y = 0$.

Example

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = 1 \cdot 3 + 2 \cdot 3 + 3 \cdot (-3) = 3 + 6 - 9 = 0$$

Key Ideas

Problem: Find *all* vectors orthogonal to both $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Now we have to solve the system of two homogeneous equations

$$
\begin{aligned}
0 &= x \cdot v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = x_1 + x_2 - x_3 \\
0 &= x \cdot w = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = x_1 + x_2 + x_3.
\end{aligned}
$$

In matrix form:

The rows are v and w $\longrightarrow \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The parametric vector form of the solution is

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.
$$

Key Ideas

- An **orthogonal basis** for a subspace W of R^n is a basis of W that is an orthogonal set.

Example

$\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} \right\}$ is an orthogonal basis W,
where

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 2z = 0 \right\}.$$

- Let W be a subspace of R^n. Its **orthogonal complement** is

$$W^\perp = \{ v \text{ in } \mathrm{R}^n \mid v \cdot w = 0 \text{ for all } w \text{ in } W \} \qquad \text{read "}W\text{ perp".}$$

Important Point: For any matrix A:

$$\mathrm{Row}\, A = \mathrm{Col}\, A^T$$

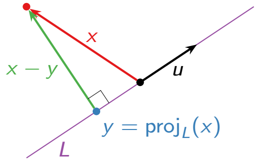
and

$$(\mathrm{Row}\, A)^\perp = \mathrm{Nul}\, A \qquad \mathrm{Row}\, A = (\mathrm{Nul}\, A)^\perp$$

$$(\mathrm{Col}\, A)^\perp = \mathrm{Nul}\, A^T \qquad \mathrm{Col}\, A = (\mathrm{Nul}\, A^T)^\perp$$

Key Ideas

- Let $L = \mathrm{Span}\{u\}$ be a line in $\mathbf{R}^n$, and let x be in $\mathbf{R}^n$. The closest point to x on L is the point
$$\mathrm{proj}_L(x) = \frac{x \cdot u}{u \cdot u}\, u.$$
This point is called the **orthogonal projection of x onto L.**



- Let $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ be an orthogonal set, and let x be a vector in $W = \mathrm{Span}\,\mathcal{B}$. Then

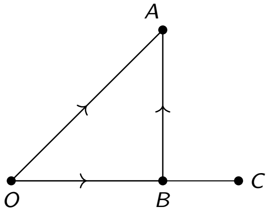
$$x = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \overbrace{\boxed{\frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2}}^{\mathrm{proj}_{L_2}(u_2)} + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

If L_i is the line spanned by u_i, then this says

$$x = \mathrm{proj}_{L_1}(x) + \mathrm{proj}_{L_2}(x) + \cdots + \mathrm{proj}_{L_m}(x).$$

Key Ideas

- By using head to tail rule, we can write



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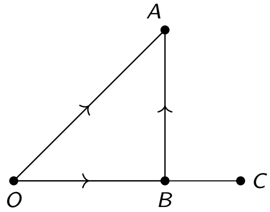

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$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$$
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- Let W be a subspace of R^n, and let x be a vector in R^n. Then $\mathrm{proj}_W(x)$ is the closest point to x in W.

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  $$x_W = \mathrm{proj}_W(x) \qquad x_{W^\perp} = x - \mathrm{proj}_W(x).$$

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- Let W be a subspace of R^n, and let x be a vector in R^n. Then $\mathrm{proj}_W(x)$ is the closest point to x in W.
  Therefore
  $$x_W = \mathrm{proj}_W(x) \qquad x_{W^\perp} = x - \mathrm{proj}_W(x).$$
  We can decompose any vector
  $$x = \mathrm{proj}_W(x) + x_{W^\perp}$$
- Let W be a subspace of R^n, and let $\{u_1, u_2, \dots, u_m\}$ be an orthogonal basis for W. Let $L_i = \mathrm{Span}\{u_i\}$. Then
  $$\mathrm{proj}_W(x) = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \sum_{i=1}^{m} \mathrm{proj}_{L_i}(x).$$
  So the orthogonal projection is formed by adding orthogonal projections onto perpendicular lines.

Key Ideas

Simple Example Decompose $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in to two perpendicular components one along the $xy-$plane other perpendicular to it.
Solution The xy-plane is $W = \text{Span}\{e_1, e_2\}$, and $\{e_1, e_2\}$ is an orthogonal basis.

$$x_W = \text{proj}_W \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{x \cdot e_1}{e_1 \cdot e_1} e_1 + \frac{x \cdot e_2}{e_2 \cdot e_2} e_2 = \frac{1 \cdot 1}{1^2} e_1 + \frac{1 \cdot 2}{1^2} e_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

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$$x_{W^\perp} = x - \text{proj}_W(x) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

We can write

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

Note that $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ lies along the W and $\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ is perpendicular to W.

The Gram–Schmidt Process

Let $\{v_1, v_2, \dots, v_m\}$ be a basis for a subspace W of $\mathbb{R}^n$. Define:

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m. $u_m = v_m - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2, \ldots, u_{m-1}\}}(v_m) \;= v_m - \displaystyle\sum_{i=1}^{m-1} \frac{v_m \cdot u_i}{u_i \cdot u_i}\, u_i$

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Then $\{u_1, u_2, \ldots, u_m\}$ is an *orthogonal* basis for the same subspace W.

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Remark

In fact, for every i between 1 and n, the set $\{u_1, u_2, \dots, u_i\}$ is an orthogonal basis for $\mathrm{Span}\{v_1, v_2, \dots, v_i\}$.

The Gram–Schmidt Process

Two vectors

Find an orthogonal basis $\{u_1, u_2\}$ for $W = \mathrm{Span}\{v_1, v_2\}$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Run Gram–Schmidt:

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2. $u_2 = v_2 - \dfrac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \dfrac{2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

Why does this work?

- First we take $u_1 = v_1$.
- Now we're sad because $u_1 \cdot v_2 \neq 0$, so we can't take $u_2 = v_2$.
- Fix: let $L_1 = \mathrm{Span}\{u_1\}$, and let $u_2 = (v_2)_{L_1^\perp} = v_2 - \mathrm{proj}_{L_1}(v_2)$.
- By construction, $u_1 \cdot u_2 = 0$, because $L_1 \perp u_2$.

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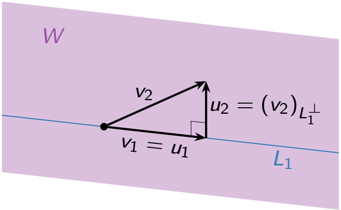

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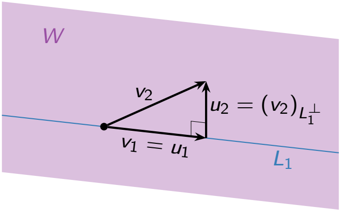

Important: $\mathrm{Span}\{u_1, u_2\} = \mathrm{Span}\{v_1, v_2\} = W$: this is an *orthogonal* basis for the *same* subspace.

The Gram–Schmidt Process

Three vectors

Find an orthogonal basis $\{u_1, u_2, u_3\}$ for $W = \mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{R}^3$, where

$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} \qquad v_2 = \begin{pmatrix}1\\1\\1\end{pmatrix} \qquad v_3 = \begin{pmatrix}3\\1\\1\end{pmatrix}.$$

Run Gram–Schmidt:

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The Gram–Schmidt Process

Three vectors, continued

$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix},\ v_2 = \begin{pmatrix}1\\1\\1\end{pmatrix},\ v_3 = \begin{pmatrix}3\\1\\1\end{pmatrix} \overset{\text{G–S}}{\rightsquigarrow} u_1 = \begin{pmatrix}1\\1\\0\end{pmatrix},\ u_2 = \begin{pmatrix}0\\0\\1\end{pmatrix},\ u_3 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

Why does this work?

- Once we have u_1 and u_2, then we're sad because v_3 is not orthogonal to u_1 and u_2.
- Fix: let $W_2 = \operatorname{Span}\{u_1, u_2\}$, and let $u_3 = (v_3)_{W_2^\perp} = v_3 - \operatorname{proj}_{W_2}(u_3)$.
- By construction, $u_1 \cdot u_3 = 0 = u_2 \cdot u_3$ because $W_2 \perp u_3$.

Check:

$$u_1 \cdot u_2 = 0 \ \checkmark$$
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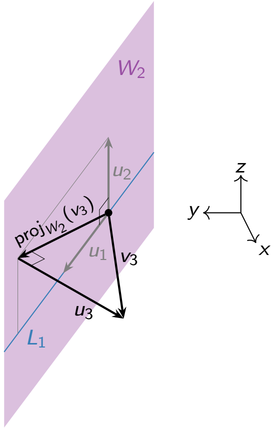

The Gram–Schmidt Process

Three vectors in $\mathbf{R}^4$

Find an orthogonal basis $\{u_1, u_2, u_3\}$ for $W = \mathrm{Span}\{v_1, v_2, v_3\}$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 4 \\ -2 \\ -2 \\ 0 \end{pmatrix}.$$

Run Gram–Schmidt:

1. $u_1 = v_1$
2. $u_2 = v_2 - \dfrac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1 = \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix} - \dfrac{6}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{pmatrix}$
3. $u_3 = v_3 - \dfrac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \dfrac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2$

$$= \begin{pmatrix} 4 \\ -2 \\ -2 \\ 0 \end{pmatrix} - \frac{0}{24} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-20}{25} \begin{pmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ -2 \end{pmatrix}$$

Quiz

What happens if you try to run Gram–Schmidt on a linearly *dependent* set of vectors $\{v_1, v_2, \ldots, v_m\}$?

A. You get an inconsistent equation.

B. For some i you get $u_i = u_{i-1}$.

C. For some i you get $u_i = 0$.

D. You create a rift in the space-time continuum.

Quiz

What happens if you try to run Gram–Schmidt on a linearly *de*pendent set of vectors $\{v_1, v_2, \ldots, v_m\}$?

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If $\{v_1, v_2, \ldots, v_m\}$ is linearly dependent, then some v_i is in $\mathrm{Span}\{v_1, v_2, \ldots, v_{i-1}\} = \mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}$.

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If $\{v_1, v_2, \ldots, v_m\}$ is linearly dependent, then some v_i is in $\mathrm{Span}\{v_1, v_2, \ldots, v_{i-1}\} = \mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}$.

This means

$$\begin{aligned} v_i &= \mathrm{proj}_{\mathrm{Span}\{u_1,u_2,\ldots,u_{i-1}\}}(v_i) \\ &\implies u_i = v_i - \mathrm{proj}_{\mathrm{Span}\{u_1,u_2,\ldots,u_{i-1}\}}(v_i) = 0. \end{aligned}$$

Quiz

What happens if you try to run Gram–Schmidt on a linearly *dependent* set of vectors $\{v_1, v_2, \ldots, v_m\}$?

A. You get an inconsistent equation.

B. For some i you get $u_i = u_{i-1}$.

C. For some i you get $u_i = 0$.

D. You create a rift in the space-time continuum.

If $\{v_1, v_2, \ldots, v_m\}$ is linearly dependent, then some v_i is in $\mathrm{Span}\{v_1, v_2, \ldots, v_{i-1}\} = \mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}$.

This means

$$v_i = \mathrm{proj}_{\mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}}(v_i)$$

$$\implies u_i = v_i - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}}(v_i) = 0.$$

In this case, you can simply discard u_i and v_i and continue: so Gram–Schmidt produces an orthogonal basis from any spanning set!