Course Name:Linear Algebra

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5.1 Eigenvectors & Eigenvalues

- Eigenvectors & Eigenvalues
- Eigenspace
- Eigensvalues of Matrix Powers
- Eigensvalues of Triangular Matrix
- Eigenvectors and Linear Independence

The basic concepts presented here - *eigenvectors* and *eigenvalues* - are useful throughout pure and applied mathematics. Eigenvalues are also used to study difference equations and *continuous* dynamical systems. They provide critical information in engineering design, and they arise naturally in such fields as physics and chemistry.

Example

Let $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Examine the images of $\mathbf{u}$ and $\mathbf{v}$ under multiplication by A.

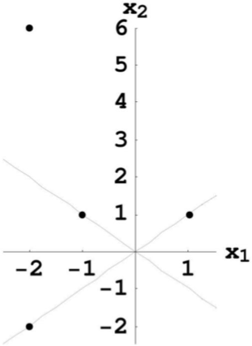
Solution

$$A\mathbf{u} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2\mathbf{u}$$

$\mathbf{u}$ is called an *eigenvector* of A since $A\mathbf{u}$ is a multiple of $\mathbf{u}$.

$$A\mathbf{v} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \neq \lambda \mathbf{v}$$

$\mathbf{v}$ is not an eigenvector of A since $A\mathbf{v}$ is not a multiple of $\mathbf{v}$.



$A\mathbf{u} = -2\mathbf{u}$, but $A\mathbf{v} \neq \lambda\mathbf{v}$

Eigenvectors & Eigenvalues

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

Example

Show that 4 is an eigenvalue of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

Solution: Scalar 4 is an eigenvalue of A if and only if $A\mathbf{x} = 4\mathbf{x}$ has a nontrivial solution.

$$A\mathbf{x} - 4\mathbf{x} = \mathbf{0}$$

$$A\mathbf{x} - 4\,(\_\_\_\_)\,\mathbf{x} = \mathbf{0}$$

$$(A - 4I)\,\mathbf{x} = \mathbf{0}.$$

To solve $(A - 4I)\mathbf{x} = \mathbf{0}$, we need to find $A - 4I$ first:

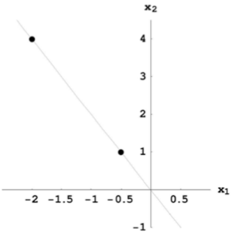
$$A - 4I = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix}$$

Now solve $(A - 4I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -4 & -2 & 0 \\ -4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \qquad \mathbf{x} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Each vector of the form $x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 4$.



Eigenspace for $\lambda = 4$

Warning

The method just used to find eigen*vectors* *cannot* be used to find eigen*values*.

Eigenspace

The set of all solutions to $(A - \lambda I)\mathbf{x} = \mathbf{0}$ is called the **eigenspace** of A corresponding to λ.

Example

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$. An eigenvalue of A is $\lambda = 2$. Find a basis for the corresponding eigenspace.

Solution:

$$A - 2I = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} \_\_\_ & 0 & 0 \\ 0 & \_\_\_ & 0 \\ 0 & 0 & \_\_\_ \end{bmatrix}$$

$$= \begin{bmatrix} 2 - \_\_\_ & 0 & 0 \\ -1 & 3 - \_\_\_ & 1 \\ -1 & 1 & 3 - \_\_\_ \end{bmatrix}$$

$$= \begin{bmatrix} \_\_\_ & 0 & 0 \\ -1 & \_\_\_ & 1 \\ -1 & 1 & \_\_\_ \end{bmatrix}$$

Augmented matrix for $(A-2I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = \_\_\_\_ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

So a basis for the eigenspace corresponding to $\lambda = 2$ is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Example

Suppose λ is eigenvalue of A. Determine an eigenvalue of A^2 and A^3. In general, what is an eigenvalue of A^n?

Solution: Since λ is eigenvalue of A, there is a nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Then

$$\_\_\_A\mathbf{x} = \_\_\_\lambda\mathbf{x}$$

$$A^2\mathbf{x} = \lambda A\mathbf{x}$$

$$A^2\mathbf{x} = \lambda\_\_\_\mathbf{x}$$

$$A^2\mathbf{x} = \lambda^2\mathbf{x}$$

Therefore λ^2 is an eigenvalue of A^2.

Show that λ^3 is an eigenvalue of A^3:

$$\_\_\_A^2\mathbf{x} = \_\_\_\lambda^2\mathbf{x}$$

$$A^3\mathbf{x} = \lambda^2 A\mathbf{x}$$

$$A^3\mathbf{x} = \lambda^3\mathbf{x}$$

Therefore λ^3 is an eigenvalue of A^3.

In general, ______ is an eigenvalue of A^n.

Theorem (1)

The eigenvalues of a triangular matrix are the diagonal entries.

Proof for the 3×3 *Upper Triangular Case*: Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}.$$

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}.$$

By definition, λ is an eigenvalue of A if and only if $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution. This occurs if and only if $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a free variable. When does this occur?

Theorem (2)

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a linearly independent set.

5.2 The Characteristic Equation

- The Characteristic Equation: Definition and Examples
- The Invertible Matrix Theorem (continued)
- Row Reductions and Determinants
- Similarity
- Application to Markov Chains

$$A\mathbf{x} = \lambda\mathbf{x}$$

Find eigenvectors $\mathbf{x}$ by solving $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$.

How Do We Find the Eigenvalues λ?

$\mathbf{x}$ must be nonzero

$$\Downarrow$$

$(A - \lambda I)\,\mathbf{x} = \mathbf{0}$ must have nontrivial solutions

$$\Downarrow$$

$(A - \lambda I)$ is not invertible

$$\Downarrow$$

$$\det(A - \lambda I) = 0$$

(called the *characteristic equation*)

$$\boxed{\text{Solve } \det(A - \lambda I) = 0 \text{ for } \lambda \text{ to find the eigenvalues.}}$$

Characteristic polynomial: $\det(A - \lambda I)$

Characteristic equation: $\det(A - \lambda I) = 0$

Example

Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$.

Solution: Since

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{bmatrix},$$

the equation $\det(A - \lambda I) = 0$ becomes

$$-\lambda(5-\lambda) + 6 = 0 \quad \Longrightarrow \quad \lambda^2 - 5\lambda + 6 = 0$$

Factor:

$$(\lambda - 2)(\lambda - 3) = 0.$$

So the eigenvalues are 2 and 3.

Example

Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.

Solution: For a 3×3 matrix or larger, recall that a determinant can be computed by cofactor expansion.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & -5-\lambda & 0 \\ 1 & 8 & 1-\lambda \end{vmatrix}$$

$$= (-5-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (-5-\lambda)\left[(1-\lambda)^2 - 1\right]$$

$$= (-5-\lambda)\left[1 - 2\lambda + \lambda^2 - 1\right] = -(5+\lambda)\,\lambda\,[-2+\lambda] = 0$$

$$\Rightarrow \lambda = -5, 0, 2$$

Theorem (IMT (cont.))

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

s. *The number 0 is not an eigenvalue of A.*

t. $\det A \neq 0$

Algebraic Multiplicity

The **(algebraic) multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation.

Recall that if B is obtained from A by a sequence of row replacements or interchanges, but without scaling, then $\det A = (-1)^r \det B$, where r is the number of row interchanges.

Suppose the echelon form U is obtained from A by a sequence of row replacements or r interchanges, but without scaling.

$$A \sim U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & u_{nn} \end{bmatrix}$$

The **determinant** of A, written $\det A$, is defined as follows:

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix}, & \text{when } A \text{ is invertible} \\ 0, & \text{when } A \text{ is not invertible} \end{cases}$$

Example

Find the eigenvalues of $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution:

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 2 & 3 \\ 0 & 6-\lambda & 10 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

Characteristic equation:

$$(\qquad\qquad)(\qquad\qquad)(\qquad\qquad) = 0.$$

eigenvalues: _____, _____, _____

Example

Find the characteristic polynomial of

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 9 & 1 & 3 & 0 \\ 1 & 2 & 5 & -1 \end{bmatrix}$$

and then find all the eigenvalues and their algebraic multiplicity.

Solution:

$$\det\left(A - \lambda I\right) = \begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 5 & 3-\lambda & 0 & 0 \\ 9 & 1 & 3-\lambda & 0 \\ 1 & 2 & 5 & -1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(3-\lambda)(3-\lambda)(-1-\lambda) = 0$$

eigenvalues: _____, _____, _____

Numerical methods for finding approximating eigenvalues are based upon Theorem 4 to be described shortly.

Similarity

For $n \times n$ matrices A and B, we say the A is **similar** to B if there is an invertible matrix P such that

$$P^{-1}AP = B \qquad \text{or equivalently,} \qquad A = PBP^{-1}.$$

Theorem (4)

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues.

Proof: If $B = P^{-1}AP$, then

$$\begin{aligned} \det(B - \lambda I) &= \det\left[P^{-1}AP - P^{-1}\lambda IP\right] = \det\left[P^{-1}(A - \lambda I)P\right] \\ &= \det P^{-1} \cdot \det(A - \lambda I) \cdot \det P = \det(A - \lambda I). \end{aligned}$$

Example

Consider the migration matrix $M = \begin{bmatrix} .95 & .90 \\ .05 & .10 \end{bmatrix}$ and define $\mathbf{x}_{k+1} = M\mathbf{x}_k$. It can be shown that $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ converges to a steady state vector $\mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. Why?

The answer lies in examining the corresponding eigenvectors. First we find the eigenvalues:

$$\det(M - \lambda I) = \det\left(\begin{bmatrix} .95-\lambda & .90 \\ .05 & .10-\lambda \end{bmatrix}\right) = \lambda^2 - 1.05\lambda + 0.05$$

So solve

$$\lambda^2 - 1.05\lambda + 0.05 = 0$$

By factoring

$$\lambda = 0.05, \ \lambda = 1$$

The eigenvector corresponding to $\lambda = 1$ is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the eigenvector corresponding to $\lambda = 0.05$ is $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Then for a given vector $\mathbf{x}_0$,

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

$$\mathbf{x}_1 = M\mathbf{x}_0 = M\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2\right) = c_1 M\mathbf{v}_1 + c_2 M\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\left(0.05\right)\mathbf{v}_2$$

$$\mathbf{x}_2 = M\mathbf{x}_1 = M\left(c_1\mathbf{v}_1 + c_2\left(0.05\right)\mathbf{v}_2\right) = c_1 M\mathbf{v}_1 + c_2\left(0.05\right)M\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\left(0.05\right)^2\mathbf{v}_2$$

and in general $\quad \mathbf{x}_k = c_1\mathbf{v}_1 + c_2\left(0.05\right)^k\mathbf{v}_2$

and so $\lim_{k\to\infty} \mathbf{x}_k = \lim_{k\to\infty}\left(c_1\mathbf{v}_1 + c_2\left(0.05\right)^k\mathbf{v}_2\right) = c_1\mathbf{v}_1$

and this is the steady state vector $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ when $c_1 = \frac{1}{2}$.

$$\underset{C\leftarrow B}{P} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \qquad \underset{B\leftarrow C}{P} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$\therefore [x]_C = \underset{C\leftarrow B}{P} [x]_B.$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ +1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 \\ +1 - 3/2 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$[x]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$[x]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{s.t.} \quad \underline{x} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 \\ -1/2 \end{bmatrix}$$

$$\Rightarrow x_1 = 5/2$$