Course Name:Linear Algebra

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4.1 Vector Spaces & Subspaces

- Vector Spaces: Definition
- Vector Spaces: Examples
  - 2×2 matrices
  - Polynomials
- Subspaces: Definition
- Subspaces: Examples
- Determining Subspaces

Many concepts concerning vectors in $\mathbf{R}^n$ can be extended to other mathematical systems.

We can think of a *vector space* in general, as a collection of objects that behave as vectors do in $\mathbf{R}^n$. The objects of such a set are called *vectors*.

Vector Space

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

1. $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

Vector Space (cont.)

3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is vector $-\mathbf{u}$ in V satisfying $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $(cd)\mathbf{u}=c(d\mathbf{u})$.
10. $1\mathbf{u}=\mathbf{u}$.

Example

Let $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,\ b,\ c,\ d \text{ are real} \right\}$

In this context, note that the $\mathbf{0}$ vector is $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$.

Example

Let $n \geq 0$ be an integer and let

$$\mathbf{P}_n = \text{the set of all polynomials of degree at most } n \geq 0.$$

Members of $\mathbf{P}_n$ have the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

where $a_0, a_1, \ldots, a_n$ are real numbers and t is a real variable. The set $\mathbf{P}_n$ is a vector space.

We will just verify 3 out of the 10 axioms here.

Let $\mathbf{p}(t) = a_0 + a_1 t + \cdots + a_n t^n$ and $\mathbf{q}(t) = b_0 + b_1 t + \cdots + b_n t^n$.
Let c be a scalar.

Axiom 1:

The polynomial $\mathbf{p}+\mathbf{q}$ is defined as follows: $(\mathbf{p}+\mathbf{q})(t)=\mathbf{p}(t)+\mathbf{q}(t)$. Therefore,

$$(\mathbf{p}+\mathbf{q})(t)=\mathbf{p}(t)+\mathbf{q}(t)$$

$$=(\_\_\_\_\_\_\_\_\_)+(\_\_\_\_\_\_\_\_\_)\,t+\cdots+(\_\_\_\_\_\_\_\_\_)\,t^n$$

which is also a ____________________ of degree at most ________. So

$\mathbf{p}+\mathbf{q}$ is in $\mathbf{P}_n$.

Axiom 4:

$$\mathbf{0} = 0 + 0t + \cdots + 0t^n$$
(zero vector in $\mathbf{P}_n$)

$$(\mathbf{p} + \mathbf{0})(t) = \mathbf{p}(t) + \mathbf{0} = (a_0 + 0) + (a_1 + 0)t + \cdots + (a_n + 0)t^n$$

$$= a_0 + a_1 t + \cdots + a_n t^n = \mathbf{p}(t)$$
and so $\mathbf{p} + \mathbf{0} = \mathbf{p}$

Axiom 6:

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = (\_\_\_\_\_\_\_\_) + (\_\_\_\_\_\_\_\_)\,t + \cdots + (\_\_\_\_\_\_\_\_)\,t^n$$

which is in $\mathbf{P}_n$.

The other 7 axioms also hold, so $\mathbf{P}_n$ is a vector space.

Vector spaces may be formed from subsets of other vectors spaces. These are called *subspaces*.

Subspaces

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. For each $\mathbf{u}$ and $\mathbf{v}$ are in H, $\mathbf{u} + \mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)

c. For each $\mathbf{u}$ in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

If the subset H satisfies these three properties, then H itself is a vector space.

Example

Let $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : \ a \text{ and } b \text{ are real} \right\}$. Show that H is a subspace of $\mathbf{R}^3$.

Solution: Verify properties a, b and c of the definition of a subspace.

a. The zero vector of $\mathbf{R}^3$ is in H (let $a =$ ______ and $b =$ ______).

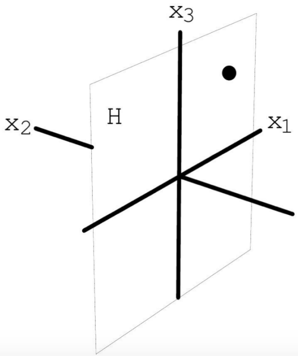
b. Adding two vectors in H always produces another vector whose second entry is ______ and therefore the sum of two vectors in H is also in H. (H is closed under addition)

c. Multiplying a vector in H by a scalar produces another vector in H (H is closed under scalar multiplication).

Since properties a, b, and c hold, V is a subspace of $\mathbf{R}^3$.

Note

Vectors $(a, 0, b)$ in H look and act like the points (a, b) in $\mathbf{R}^2$.

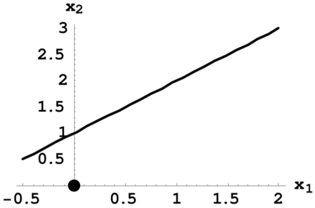


Example

Is $H = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : \ x \text{ is real} \right\}$ a subspace of _______?

I.e., does H satisfy properties a, b and c?

Solution: For *H to be a subspace of $\mathbf{R}^2$, all three properties must hold*



Property (a) fails

Property (a) is not true because ________________________________.

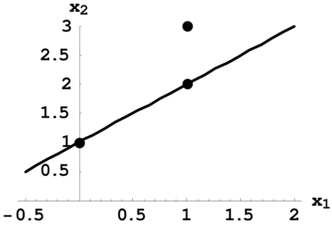
Therefore H is not a subspace of $\mathbf{R}^2$.

Another way to show that H is not a subspace of $\mathbf{R}^2$:

Let

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ then } \mathbf{u}+\mathbf{v} = \begin{bmatrix} \qquad \end{bmatrix}$$

and so $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, which is ____ in H. So property (b) fails and so H is not a subspace of $\mathbf{R}^2$.



Property (b) fails

Theorem (1)

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Proof: In order to verify this, check properties a, b and c of definition of a subspace.

a. $\mathbf{0}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ since

$$\mathbf{0} = \_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\mathbf{v}_p$$

b. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ closed under vector addition, we choose two arbitrary vectors in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p$$

and

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$\mathbf{u} + \mathbf{v} = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p) + (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

$$= (\_\_\_\mathbf{v}_1 + \_\_\_\_\mathbf{v}_1) + (\_\_\_\_\mathbf{v}_2 + \_\_\_\_\mathbf{v}_2) + \cdots + (\_\_\_\_\mathbf{v}_p + \_\_\_\_\mathbf{v}_p)$$

$$= (a_1 + b_1)\,\mathbf{v}_1 + (a_2 + b_2)\,\mathbf{v}_2 + \cdots + (a_p + b_p)\,\mathbf{v}_p.$$

So $\mathbf{u} + \mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

c. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ closed under scalar multiplication, choose an arbitrary number c and an arbitrary vector in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$c\mathbf{v} = c\left(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p\right)$$

$$= \_\_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\_\mathbf{v}_p$$

So $c\mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Since properties a, b and c hold, Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Recap

1. To show that H is a subspace of a vector space, use Theorem 1.
2. To show that a set is not a subspace of a vector space, provide a specific example showing that at least one of the axioms a, b or c (from the definition of a subspace) is violated.

Example

Is $V = \{(a+2b, 2a-3b) : \ a \text{ and } b \text{ are real}\}$ a subspace of $\mathbf{R}^2$? Why or why not?

Solution: Write vectors in V in column form:

$$\begin{bmatrix} a+2b \\ 2a-3b \end{bmatrix} = \begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} 2b \\ -3b \end{bmatrix}$$

$$= \_\_\_\_\_ \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \_\_\_\_\_ \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

So $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and therefore V is a subspace of _____ by Theorem 1.

Example

Is $H = \left\{ \begin{bmatrix} a+2b \\ a+1 \\ a \end{bmatrix} : \ a \text{ and } b \text{ are real} \right\}$ a subspace of $\mathbf{R}^3$?
Why or why not?

Solution: $\mathbf{0}$ is not in H since $a = b = 0$ or any other combination of values for a and b does not produce the zero vector. So property _____ fails to hold and therefore H is not a subspace of $\mathbf{R}^3$.

Example

Is the set H of all matrices of the form $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of $M_{2\times 2}$? Explain.

Solution: Since

$$\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 3a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 3b \end{bmatrix}$$

$$= a\begin{bmatrix} & \\ & \end{bmatrix} + b\begin{bmatrix} & \\ & \end{bmatrix}.$$

Therefore $H = \text{Span}\left\{\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}\right\}$ and so H is a subspace of $M_{2\times 2}$.

4.2 Null Spaces, Column Spaces, & Linear Transformations

- Null Spaces
  - Definition
  - Theorem
  - Examples
- Column Spaces
  - Definition
  - Theorem
  - Examples
- The Contrast Between Nul A and Col A
- Null Spaces & Column Spaces: Review
- Null Spaces & Column Spaces: Examples

Null Space

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbf{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\} \qquad \text{(set notation)}$$

Theorem (2)

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbf{R}^n$.

Proof: Nul A is a subset of $\mathbf{R}^n$ since A has n columns. Must verify properties a, b and c of the definition of a subspace.

Property (a) Show that $\mathbf{0}$ is in Nul A. Since _________, $\mathbf{0}$ is in

__________.

Property (b) If **u** and **v** are in Nul A, show that $\mathbf{u}+\mathbf{v}$ is in Nul A.
Since **u** and **v** are in Nul A,

$$\_\_\_\_\_\_\_\_\_\_ \text{ and } \_\_\_\_\_\_\_\_\_\_.$$

Therefore

$$A(\mathbf{u}+\mathbf{v}) = \_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_ = \_\_\_\_.$$

Property (c) If **u** is in Nul A and c is a scalar, show that $c\mathbf{u}$ in Nul A:

$$A(c\mathbf{u}) = \_\_\_A(\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$

Since properties a, b and c hold, A is a subspace of $\mathbf{R}^n$.

Example

Find an explicit description of Nul A where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$$

Solution: Row reduce augmented matrix corresponding to $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \quad x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + \quad x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

Then

$$\text{Nul } A = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$$

Observations:

1. Spanning set of Nul A, found using the method in the last example, is automatically linearly independent:

$$c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$

$$c_1 = \_\_\_\_\_ \qquad c_2 = \_\_\_\_\_ \qquad c_3 = \_\_\_\_\_$$

2. If Nul A$\neq \{\mathbf{0}\}$, the the number of vectors in the spanning set for Nul A equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.

Column Space

The **column space** of an $m \times n$ matrix A (Col A) is the set of all linear combinations of the columns of A.

If $A = [\mathbf{a}_1 \ \ \ldots \ \ \mathbf{a}_n]$, then

$$\boxed{\text{Col } A = \text{Span}\{\mathbf{a}_1, \ \ldots, \ \mathbf{a}_n\}}$$

Theorem (3)

The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

Why? (Theorem 1, page 194)

Recall that if $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ is a linear combination of the columns of A. Therefore

$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbf{R}^n\}}$$

Example

Find a matrix A such that $W = Col\ A$ where

$$W = \left\{ \begin{bmatrix} x-2y \\ 3y \\ x+y \end{bmatrix} : x, y \text{ in } \mathbf{R} \right\}.$$

Solution:

$$\begin{bmatrix} x-2y \\ 3y \\ x+y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore

$$A = \begin{bmatrix} \quad \end{bmatrix}.$$

By Theorem 4 (Chapter 1),

> The column space of an $m \times n$ matrix A is all of $\mathbf{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^m$.

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) The column space of A is a subspace of $\mathbf{R}^k$ where $k =$_____.

(b) The null space of A is a subspace of $\mathbf{R}^k$ where $k =$_____.

(c) Find a nonzero vector in Col A. (There are infinitely many possibilities.)

$$\_\_\_ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \_\_\_ \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix} + \_\_\_ \begin{bmatrix} 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Example (cont.)

(d) Find a nonzero vector in Nul A. Solve $A\mathbf{x} = \mathbf{0}$ and pick one solution.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 0 \\ 3 & 6 & 10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} x_1 = -2x_2 \\ x_2 \text{ is free} \\ x_3 = 0 \end{array} \quad \Longrightarrow \quad \text{let } x_2 = \_\_\_ \quad \Longrightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Contrast Between Nul A and Col A where A is $m \times n$
(see page 204)

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Review

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, $\mathbf{u} + \mathbf{v}$ is in H.
(In this case we say H is closed under vector addition.)

c. For each $\mathbf{u}$ in H and each scalar c, $c\mathbf{u}$ is in H.
(In this case we say H is closed under scalar multiplication.)

If the subset H satisfies these three properties, then H itself is a vector space.

Theorem (1, 2 and 3 in Sections 4.1 & 4.2)

- *If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.*
- *The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$.*
- *The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.*

Example

Determine whether each of the following sets is a vector space or provide a counterexample.

(a) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x - y = 4 \right\}$

Solution: Since

$$\_\_\_\_ = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

is not in H, H is not a vector space.

Example

(b) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{l} x - y = 0 \\ y + z = 0 \end{array} \right\}$

Solution: Rewrite

$$x - y = 0$$

$$y + z = 0$$

as

$$\begin{bmatrix} & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $V =$Nul A where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Since Nul A is a subspace of $\mathbf{R}^2$, V is a vector space.

Example

(c) $S = \left\{ \begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} : x, y, z \text{ are real} \right\}$

One *Solution:* Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

$$S = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\};$$

therefore S is a vector space by Theorem 1.

Another Solution: Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

$$S = \text{Col } A \qquad \text{where } A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 3 \end{bmatrix};$$

therefore S is a vector space, since a column space is a vector space.

Linear Transformation

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

Kernal and Range

The *kernel* (or **null space**) of T is the set of all vectors $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$. The *range* of T is the set of all vectors in W of the form $T(\mathbf{u})$ where $\mathbf{u}$ is in V.

So if $T(\mathbf{x}) = A\mathbf{x}$, col A = range of T.