Course Name:Linear Algebra(MT 104)

Topic: Linear Independence (Exercise 1.7)

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- What would be $Span\{\vec{u}\}$, for $\vec{u} \neq \vec{0}$ and $\vec{u} = \vec{0}$.
- What would be $Span\{\vec{u}, \vec{v}\}$, for $\vec{u}$ isn't multiple of $\vec{v}$ and what for $\vec{u}$ is multiple of $\vec{v}$.

In the Last Lecture, we have solved following Homogenous System

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_2 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

which lead to the following augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

means $x_3 = t(say)$ is free. Hence solution

$$\vec{x} = \begin{bmatrix} 4/3t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} = t\vec{v}, \quad \text{where} \quad \vec{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

Which is parametric equation of line passing through $\vec{0}$ and $\vec{v}$

Now consider the **non Homogeneous System**

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_2 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= -4 \end{aligned}$$

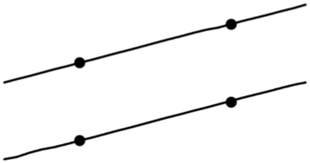
which lead to the following augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

means $x_3 = t(say)$ is free. Hence solution

$$\vec{x} = \begin{bmatrix} -1 + 4/3t \\ 2 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} = \vec{p} + t\vec{v}.$$

Which is parametric equation of line passing through $\vec{p}$ and $\vec{v}$



Parallel solution sets of
$A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$

Theorem

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Recall

- Solution of homogeneous system $Ax = 0$.
- What would be $Span\{\vec{u}\}$, for $\vec{u} \neq \vec{0}$ and $\vec{u} = \vec{0}$.
- What would be $Span\{\vec{u}, \vec{v}\}$, for $\vec{u}$ isn't multiple of $\vec{v}$ and what for $\vec{u}$ is multiple of $\vec{v}$.
- What does mean by $2\vec{u} - 3\vec{v} - \vec{w} = \vec{0}$. $\{Question\#26(Ex1.4)\}$.
- What is dependence in general?

1.7 Linear Independence

- Linear Independence and Homogeneous System
- Linear Independence: Definition
- Linear Independence of Matrix Columns
- Special Cases
  - A Set of One Vector
  - A Set of Two Vectors
  - A Set Containing the $\mathbf{0}$ Vector
  - A Set Containing Too Many Vectors
- Characterization of Linearly Dependent Sets
  - Theorem: Linear Dependence and Linear Combination

Example

A homogeneous system such as

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can be viewed as a vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The vector equation has the trivial solution ($x_1 = 0$, $x_2 = 0$, $x_3 = 0$), but is this the *only solution*?

Linear Independence

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

Linear Dpendence

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights $c_1, \ldots, c_p$,not all 0, such that

$$\underbrace{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.}_{\uparrow}$$

linear dependence relation

(when weights are not all zero)

Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

a. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

b. If possible, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution: (a)

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & -1 & 18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is a free variable $\Rightarrow$ there are nontrivial solutions.

Linear Independence: Example (cont.)

$$\Rightarrow \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \text{ is a linearly dependent set}$$

(b) Reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 33 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{array}{ll} x_1 & = \\ x_2 & = \\ x_3 & \end{array}$$

Let $x_3 =$ _____ (any nonzero number).

Then $x_1 =$ _____ and $x_2 =$ _____.

$$\_\_\_\_ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\_\_\_\_\mathbf{v}_1 + \_\_\_\_\mathbf{v}_2 + \_\_\_\_\mathbf{v}_3 = \mathbf{0}$$

(one possible linear dependence relation)

Example (Linear Dependence Relation)

$$-33\begin{bmatrix}1\\3\\5\end{bmatrix}+18\begin{bmatrix}2\\5\\9\end{bmatrix}+1\begin{bmatrix}-3\\9\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

can be written as the matrix equation:

$$\begin{bmatrix}1&2&-3\\3&5&9\\5&9&3\end{bmatrix}\begin{bmatrix}-33\\18\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution to $A\mathbf{x}=\mathbf{0}$.

The columns of matrix A are linearly independent if and only if the equation $A\mathbf{x}=\mathbf{0}$ has *only* the trivial solution.

Sometimes we can determine linear independence of a set with minimal effort.

Example (1. A Set of One Vector)

Consider the set containing one nonzero vector: $\{\mathbf{v}_1\}$

The only solution to $x_1\mathbf{v}_1 = 0$ is $x_1 =$ _____.

$$\boxed{\text{So } \{\mathbf{v}_1\} \text{ is linearly independent when } \mathbf{v}_1 \neq \mathbf{0}.}$$

Example (2. A Set of Two Vectors)

Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

a. Determine if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly dependent set or a linearly independent set.

b. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent set or a linearly independent set.

Solution: (a) Notice that $\mathbf{u}_2 = \_\_\_\_\_\mathbf{u}_1$. Therefore

$$\_\_\_\_\_\mathbf{u}_1 + \_\_\_\_\_\mathbf{u}_2 = 0$$

This means that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly ________________ set.

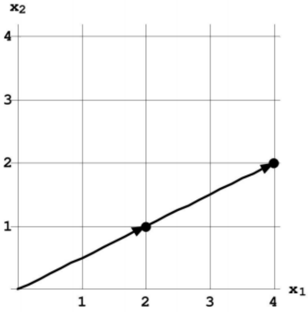
(b) Suppose

$$c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{0}.$$

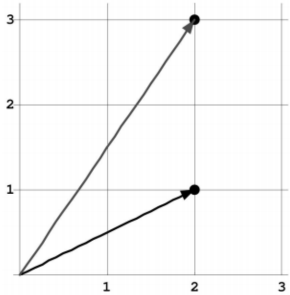
Then $\mathbf{v}_1 = \rule{2cm}{0.4pt}\mathbf{v}_2$ if $c \neq 0$. But this is impossible since $\mathbf{v}_1$ is ______ a multiple of $\mathbf{v}_2$ which means $c =$ _____.
Similarly, $\mathbf{v}_2 = \rule{2cm}{0.4pt}\mathbf{v}_1$ if $d \neq 0$.
But this is impossible since $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ and so $d = 0$.
This means that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly _________________ set.

A set of two vectors is linearly dependent if at least one vector is a multiple of the other.

A set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.



linearly ________________



linearly ________________

Theorem

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ containing the zero vector is linearly dependent.

Proof: Renumber the vectors so that $\mathbf{v}_1 =$ ____. Then

$$\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\mathbf{v}_p = \mathbf{0}$$

which shows that S is linearly ________________.

Theorem

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. I.e. any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ *in* $\mathbf{R}^n$ *is linearly dependent if* $p > n$.

Outline of Proof:

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p \end{bmatrix} \text{ is } n \times p$$

Suppose $p > n$.

$$\Longrightarrow A\mathbf{x} = \mathbf{0} \text{ has more variables than equations}$$

$$\Longrightarrow A\mathbf{x} = \mathbf{0} \text{ has nontrivial solutions}$$

$$\Longrightarrow \text{columns of } A \text{ are linearly dependent}$$

Examples

With the least amount of work possible, decide which of the following sets of vectors are linearly independent and give a reason for each answer.

a. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$

b. Columns of $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \\ 9 & 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 & 8 \end{bmatrix}$

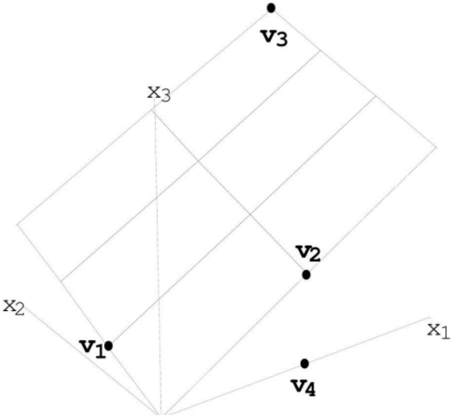
Examples (cont.)

c. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d. $\left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 4 \end{bmatrix} \right\}$

Example

Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ in $\mathbf{R}^3$ in the following diagram. Is the set linearly dependent? Explain



Theorem

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent, and $\mathbf{v}_1 \neq \mathbf{0}$, then some vector $\mathbf{v}_j$ $(j \geq 2)$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.