Master Theorm for Dividing function

General form $$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

Assume

$$a \geq 1 \quad b \geq 1 \quad \&$$

$$f(n) = \Theta\left(n^k \log^p n\right)$$

To find complexity, we need two values

1. $\log_b a$

2. k (power of n)

Based on above, we have 3 cases

Case 1: if $\log_b a > k$ then.

$$\Theta\left(n^{\log_b a}\right)$$

Case 2: if $\log_b a = k$

if
$p > -1$ $\quad O(n^k \log^{p+1} n)$
$p = -1$ $\quad O(n^k \log \log n)$
$p < -1$ $\quad O(n^k)$

Case 3

if $\log_b a < k$

$p \geq 0$ $\quad O(n^k \log^p n)$

$p < 0$ $\quad O(n^k)$

Examples

e.g 1 Lets take recurrence relation

$$T(n) = 2T\left(\frac{n}{2}\right) + 1$$

$a = 2 \quad b = 2$

$f(n) = O(1) \Rightarrow (n^0 \log^0 n)$

$k = 0$ & $\log_2 2 = 1$

So case 1

$\Theta(n^1)$

Example

$$T(n) = 4T(n/2) + n$$

1 $\log_4 2 = 2$

So $k = 1$ & $p = 0$

So again case 1

$\Theta(n^2)$

Example = $8T\left(\frac{n}{2}\right) + n$ (Try it)

$\log_2 8 = 3 > k = 1$

Θn^3

Try above example with

$8T\left(\frac{n}{2}\right) + n^2$

Eg

$$T(n) = 9T\left(\frac{n}{3}\right)+1$$

$\log_3 9 = 2 > k = 0$

$O(n^2)$

Now with log.

$$T(n) = 8T\left(\frac{n}{2}\right) + n\log n$$

$\log_b a = 3 > k = 1$.

So $O(n^3)$

Some Case 2 examples

$$T(n) = 2T\left(\frac{n}{2}\right)+n$$

so $\log_2 2 = 1 \quad k = 1$

so equal.

Now Check p.

As no log so $p = 0 > -1$

So

$O(n \log n)$

As p was zero then 1

e.g

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2$$

$\log_2 4 = 2 \quad k = 2$

So

$O(n^2 \log n)$

$4T\left(\frac{n}{2}\right) + n^2 \log n$

$\log_2 4 = 2 \quad k = 2.$

$O(n^2 \log^2 n)$

$4T\left(\frac{n}{2}\right) + n^2 \log(n)$

So

$\cancel{n^2 \log n} \quad n^2 \log^3 n$

So in = case we multiply $\log n$.

$$T(n) = 8T\left(\frac{n}{2}\right) + n^3$$ Do by Shrider

$$= (n^3 \log n)$$

eg

$$T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$$

$\log_2 2 = 1$ $k = 1$ $p = -1$

as

$p = -1$

so

$O(n \log(\log n))$

$$T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log^2 n}$$

so $p = -2 < -1$

log is Too small so

$\boxed{O(n)}$

case 3

eg

$$T(n) = T\left(\frac{n}{2}\right) + n^2$$

$p = 0 \quad \log_2 1 = 0 \quad < \quad k = 2$

$$O(n^2)$$

Try,

$$T(n) = 2T\left(\frac{n}{2}\right) + n^3$$

$$= O(n^3)$$

$$2T\left(\frac{n}{2}\right) + n^2 \log n$$

$\log_2 1 = 0 \quad k = 2$

$p = 1$

So

$$(n^2 \log n)$$

e.g

Try $T(n) = 4T\left(\frac{n}{2}\right) + n^3$

$= n^3$

if

$4T\left(\frac{n}{2}\right) + \frac{n^3}{\log n}$.

As log in denaominale

$p = -1$

we ignore

So $O(n^3)$

Class Task

$4T\left(\frac{n}{2}\right) + n^3 \log^2 n$

case 3

$= \boxed{O(n^3 \log^2 n)}$

Task

3.

$$T(n) = 2T\left(\frac{n}{2}\right) + \frac{n^2}{\log^2 n}$$

case 3

$$\boxed{= O(n^2)}$$

1) $$T(n) = 4T\left(\frac{n}{2}\right) + (n \log n)^2$$

case 2

$$\boxed{= O(n^2 \log^3 n)}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + \frac{n^2}{\log n}$$

case 2

$$\boxed{= n^2 \log \log (n)}$$