Course Name:Linear Algebra

Course Code: MT 104

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Figure 6.4 shows two different coordinate systems for $\mathbb{R}^2$, each arising from a different basis. Figure 6.4(a) shows the coordinate system related to the basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$, while Figure 6.4(b) arises from the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$, where

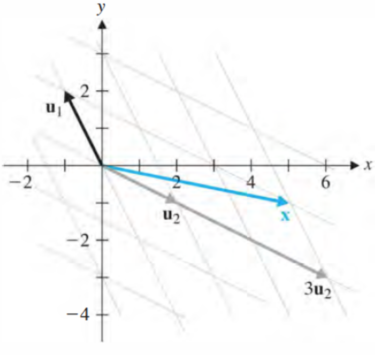
$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The same vector $\mathbf{x}$ is shown relative to each coordinate system. It is clear from the diagrams that the coordinate vectors of $\mathbf{x}$ with respect to $\mathcal{B}$ and $\mathcal{C}$ are

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

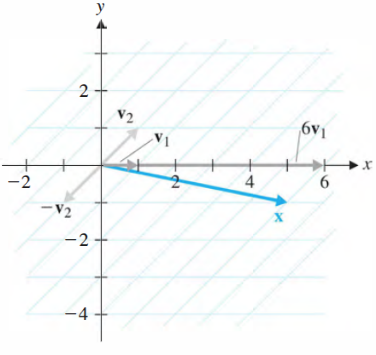
respectively. It turns out that there is a direct connection between the two coordinate vectors. One way to find the relationship is to use $[\mathbf{x}]_{\mathcal{B}}$ to calculate

$$\mathbf{x} = \mathbf{u}_1 + 3\mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$



y
x
2
−2
−4
−2
2
4
6
$\mathbf{u}_1$
$\mathbf{u}_2$
$3\mathbf{u}_2$
$\mathbf{x}$

(a)



y
x
2
−2
−4
−2
2
4
6
$\mathbf{v}_2$
$\mathbf{v}_1$
$6\mathbf{v}_1$
$-\mathbf{v}_2$
$\mathbf{x}$

(b)

. From $\mathbf{x} = \mathbf{u}_1 + 3\mathbf{u}_2$, we have

$$[\mathbf{x}]_C = [\mathbf{u}_1 + 3\mathbf{u}_2]_C = [\mathbf{u}_1]_C + 3[\mathbf{u}_2]_C$$

,

$$\begin{aligned}[\mathbf{x}]_C &= [[\mathbf{u}_1]_C [\mathbf{u}_2]_C] \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= P[\mathbf{x}]_B \end{aligned}$$

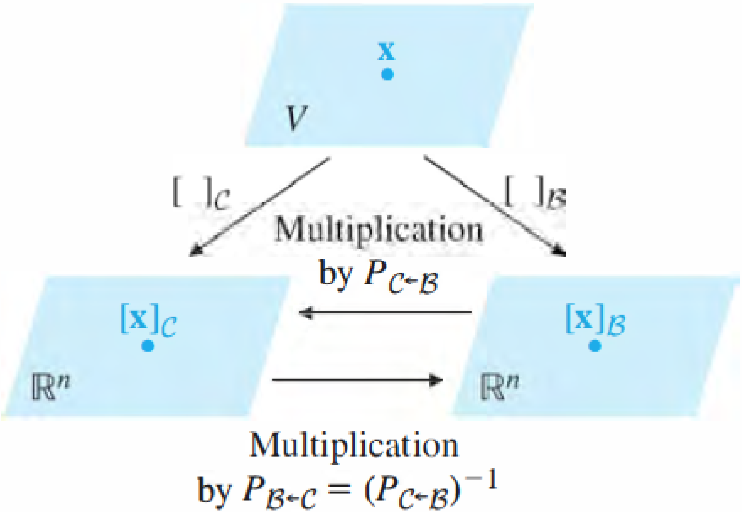
Definition Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V. The $n \times n$ matrix whose columns are the coordinate vectors $[\mathbf{u}_1]_{\mathcal{C}}, \ldots, [\mathbf{u}_n]_{\mathcal{C}}$ of the vectors in $\mathcal{B}$ with respect to $\mathcal{C}$ is denoted by $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and is called the ***change-of-basis matrix*** from $\mathcal{B}$ to $\mathcal{C}$. That is,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [\,[\mathbf{u}_1]_{\mathcal{C}}\,[\mathbf{u}_2]_{\mathcal{C}} \;\cdots\; [\mathbf{u}_n]_{\mathcal{C}}\,]$$

Think of $\mathcal{B}$ as the "old" basis and $\mathcal{C}$ as the "new" basis. Then the columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are just the coordinate vectors obtained by writing the old basis vectors in terms of the new ones. Theorem 6.12 shows that Example 6.45 is a special case of a general result.

Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V and let $P_{\mathcal{C}\leftarrow\mathcal{B}}$ be the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Then

a. $P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.
b. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the unique matrix P with the property that $P[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.
c. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible and $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = P_{\mathcal{B}\leftarrow\mathcal{C}}$.



$\mathbf{x}$
V
$[\ ]_{\mathcal{C}}$
$[\ ]_{\mathcal{B}}$
Multiplication
by $P_{\mathcal{C}\leftarrow\mathcal{B}}$
$[\mathbf{x}]_{\mathcal{C}}$
$\mathbb{R}^n$
$[\mathbf{x}]_{\mathcal{B}}$
$\mathbb{R}^n$
Multiplication
by $P_{\mathcal{B}\leftarrow\mathcal{C}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$

Find the change-of-basis matrices $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow\mathcal{C}}$ for the bases $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1 + x, x + x^2, 1 + x^2\}$ of $\mathcal{P}_2$. Then find the coordinate vector of $p(x) = 1 + 2x - x^2$ with respect to $\mathcal{C}$.

Solution Changing *to* a standard basis is easy, so we find $P_{\mathcal{B}\leftarrow\mathcal{C}}$ first. Observe that the coordinate vectors for $\mathcal{C}$ in terms of $\mathcal{B}$ are

$$[1 + x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad [x + x^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad [1 + x^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

To find $P_{\mathcal{C}\leftarrow\mathcal{B}}$, we could express each vector in $\mathcal{B}$ as a linear combination of the vectors in $\mathcal{C}$ (do this), but it is much easier to use the fact that $P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1}$, by Theorem 6.12(c). We find that

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

It now follows that

$$\begin{aligned} [p(x)]_{\mathcal{C}} &= P_{\mathcal{C}\leftarrow\mathcal{B}}[p(x)]_{\mathcal{B}} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \end{aligned}$$

Find the coordinate vector $[A]_{\mathcal{B}}$ of $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ with respect to the standard basis $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ of M_{22}.

Solution Since

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= 2E_{11} - E_{12} + 4E_{21} + 3E_{22}$$

we have

$$[A]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 3 \end{bmatrix}$$

In M_{22}, let $\mathcal{B}$ be the basis $\{E_{11}, E_{21}, E_{12}, E_{22}\}$ and let $\mathcal{C}$ be the basis $\{A, B, C, D\}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Find the change-of-basis matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ and verify that $[X]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}}[X]_{\mathcal{B}}$ for $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution 1 To solve this problem directly, we must find the coordinate vectors of $\mathcal{B}$ with respect to $\mathcal{C}$. This involves solving four linear combination problems of the form $X = aA + bB + cC + dD$, where X is in $\mathcal{B}$ and we must find a, b, c, and d. However, here we are lucky, since we can find the required coefficients by inspection.

Clearly, $E_{11} = A$, $E_{21} = -B + C$, $E_{12} = -A + B$, and $E_{22} = -C + D$. Thus,

$$[E_{11}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [E_{21}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad [E_{12}]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [E_{22}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

so

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[E_{11}]_{\mathcal{C}} \; [E_{21}]_{\mathcal{C}} \; [E_{12}]_{\mathcal{C}} \; [E_{22}]_{\mathcal{C}}] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then

$$[X]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}$$

and $$P_{\mathcal{C}\leftarrow\mathcal{B}}[X]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

This is the coordinate vector with respect to $\mathcal{C}$ of the matrix

$$\begin{aligned} -A - B - C + 4D &= -\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = X \end{aligned}$$

as it should be.

Theorem

Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V. Let $B = [[\mathbf{u}_1]_\mathcal{E} \ \cdots \ [\mathbf{u}_n]_\mathcal{E}]$ and $C = [[\mathbf{v}_1]_\mathcal{E} \ldots [\mathbf{v}_n]_\mathcal{E}]$, where $\mathcal{E}$ is any basis for V. Then row reduction applied to the $n \times 2n$ augmented matrix $[C \mid B]$ produces

$$[C \mid B] \to [I \mid P_{\mathcal{C} \leftarrow \mathcal{B}}]$$

(a) Find the coordinate vectors $[\mathbf{x}]_{\mathcal{B}}$ *and* $[\mathbf{x}]_{\mathcal{C}}$ *of* $\mathbf{x}$ *with respect to the bases* $\mathcal{B}$ *and* $\mathcal{C}$*, respectively.*
(b) Find the change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ *from* $\mathcal{B}$ *to* $\mathcal{C}$*.*
(c) Use your answer to part (b) to compute $[\mathbf{x}]_{\mathcal{C}}$*, and compare your answer with the one found in part (a).*
(d) Find the change-of-basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ *from* $\mathcal{C}$ *to* $\mathcal{B}$*.*
(e) Use your answers to parts (c) and (d) to compute $[\mathbf{x}]_{\mathcal{B}}$*, and compare your answer with the one found in part (a).*

1. $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},$

$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ in $\mathbb{R}^2$

$$x = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$[x]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$[x]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{s.t.} \quad x = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

$$\Rightarrow x_1 = 5/2$$

$$x_2 = -1/2 .$$

$$= \begin{bmatrix} 5/2 \\ -1/2 \end{bmatrix}$$

$$[B \mid C]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\underset{B \leftarrow C}{P}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ +1/2 & -1/2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1/2 & +1/2 & 1 & 0 \\ +1/2 & -1/2 & 0 & 1 \end{array}\right]$$

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \qquad \underset{B \leftarrow C}{P} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$\therefore [x]_C = \underset{C \leftarrow B}{P} [x]_B$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ +1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 \\ +1 - 3/2 \end{bmatrix}$$

$$= \begin{bmatrix} +5/2 \\ -1/2 \end{bmatrix} = \text{Verified}$$