Matrix Chain Multiplication by Dynamic Programming

Spring 2022

Outline

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- Review of matrix multiplication.
- The chain matrix multiplication problem.
- A dynamic programming algorithm for chain matrix multiplication.

Rules to multiply Two Matrices

The number of ***columns of the 1st matrix*** *must equal the number of* ***rows of the 2nd matrix****.*
And the result will have the same number of ***rows as the 1st matrix****, and the same number of* ***columns as the 2nd matrix****.*

$$(m \times n) \cdot (n \times k) = (m \times k)$$

product is defined

$$\begin{array}{c} \vec{a_1}\rightarrow \\ \vec{a_2}\rightarrow \end{array} \begin{bmatrix}1 & 7\\ 2 & 4\end{bmatrix} \cdot \overset{\begin{array}{cc} \vec{b_1} & \vec{b_2} \\ \downarrow & \downarrow \end{array}}{\begin{bmatrix}3 & 3\\ 5 & 2\end{bmatrix}} = \begin{bmatrix}\overrightarrow{a_1}\cdot\overrightarrow{b_1} & \overrightarrow{a_1}\cdot\overrightarrow{b_2}\\ \overrightarrow{a_2}\cdot\overrightarrow{b_1} & \overrightarrow{a_2}\cdot\overrightarrow{b_2}\end{bmatrix}$$

$$A \qquad\qquad B \qquad\qquad\qquad C$$

Matrix Multiplication

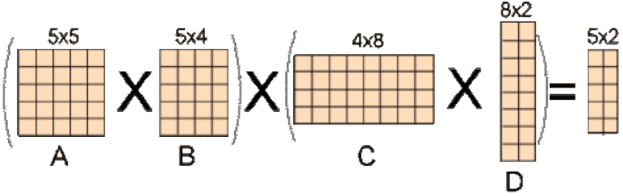
"Dot Product"

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & \\ & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix} \checkmark$$

Matrix Multiplication


5x5
5x4
4x8
8x2
5x2
A
B
C
D

Matrix: An $n \times m$ matrix $A = [a[i,j]]$ is a two-dimensional array

$$A = \begin{bmatrix} a[1,1] & a[1,2] & \cdots & a[1,m-1] & a[1,m] \\ a[2,1] & a[2,2] & \cdots & a[2,m-1] & a[2,m] \\ \vdots & \vdots & & \vdots & \vdots \\ a[n,1] & a[n,2] & \cdots & a[n,m-1] & a[n,m] \end{bmatrix},$$

which has n rows and m columns.

Example

A 4×5 matrix:

$$\begin{matrix} 12 & 8 & 9 & 7 & 6 \\ 7 & 6 & 89 & 56 & 2 \\ 5 & 5 & 6 & 9 & 10 \\ 8 & 6 & 0 & -8 & -1 \end{matrix}.$$

Review of Matrix Multiplication

The product $C = AB$ of a $p \times q$ matrix A and a $q \times r$ matrix B is a $p \times r$ matrix C given by

$$c[i,j] = \sum_{k=1}^{q} a[i,k]b[k,j], \qquad \text{for } 1 \le i \le p \text{ and } 1 \le j \le r$$

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Example

$$A = \begin{bmatrix} 1 & 8 & 9 \\ 7 & 6 & -1 \\ 5 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 8 \\ 7 & 6 \\ 5 & 5 \end{bmatrix}, \quad C = AB = \begin{bmatrix} 102 & 101 \\ 44 & 87 \\ 70 & 100 \end{bmatrix}.$$

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- Matrix multiplication is NOT commutative, e.g.,

  $$A_1A_2 \neq A_2A_1$$

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For $p = 5$, $q = 4$, $r = 6$ and $s = 2$,

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Implication: Multiplication "sequence" (parenthesization) is important!!

Conclusion

Order of operations makes a huge difference. How do we compute the minimum?

Each parenthesization defines a set of **n-1** matrix multiplications. We just need to pick the parenthesization that corresponds to the best ordering.

Trying all possible parenthesizations is a bad idea!!!

- Review of matrix multiplication.
- The chain matrix multiplication problem.
- A dynamic programming algorithm for chain matrix multiplication.

The Chain Matrix Multiplication Problem

Definition (Chain matrix multiplication problem)

Given dimensions $p_0, p_1, \ldots, p_n$, corresponding to matrix sequence A_1, $A_2, \ldots, A_n$ in which A_i has dimension $p_{i-1} \times p_i$, determine the "multiplication sequence" that minimizes the number of scalar multiplications in computing $A_1 A_2 \cdots A_n$.

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$$A_1A_2A_3A_4 = (A_1A_2)(A_3A_4) = A_1(A_2(A_3A_4)) = A_1((A_2A_3)A_4)$$
$$= ((A_1A_2)A_3)(A_4) = (A_1(A_2A_3))(A_4)$$

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Is there a better approach?

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Yes – DP

Developing a Dynamic Programming Algorithm

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  $A_{i..j} = A_i A_{i+1} \cdots A_j$.
  - Note that $A_{i..j}$ is a $p_{i-1} \times p_j$ matrix.

Example:

dimension $p_{i-1} \times p_i$

i= 1

$$
\begin{array}{lll}
A_1 & 30 \times 35 & = p_0 \times p_1 \\
A_2 & 35 \times 15 & = p_1 \times p_2 \\
A_3 & 15 \times 5 & = p_2 \times p_3 \\
A_4 & 5 \times 10 & = p_3 \times p_4 \\
A_5 & 10 \times 20 & = p_4 \times p_5 \\
A_6 & 20 \times 25 & = p_5 \times p_6
\end{array}
$$

Given a chain $A_1, A_2, \ldots, A_n$ of n matrices, where for i=1, 2, …, n, matrix A_i has dimension $p_{i-1} \times p_i$

Note

In the matrix-chain multiplication problem, we are not actually multiplying matrices.

Our goal is only to ***determine an order for multiplying matrices*** *that has the lowest cost.*

Typically, the time invested in determining this optimal order is more than paid for by the time saved later on when actually performing the matrix multiplications (such as performing only 7500 scalar multiplications instead of 75,000).

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  $A_{i..j} = A_i A_{i+1} \cdots A_j$.
  Note that $A_{i..j}$ is a $p_{i-1} \times p_j$ matrix.
- There are $\Theta(n^2)$ such subproblems. (Why?)
- How can we solve larger problems using subproblem solutions?

Relationships among subproblems

At the last step of *any* optimal multiplication sequence (for a subbroblem), there is some k such that the two matrices $A_{i..k}$ and $A_{k+1..j}$ are multipled together.

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ANS: $A_{i..k}$ and $A_{k+1..j}$ must be computed optimally, so we can apply the same procedure *recursively*.

If the “optimal” solution of $A_{i..j}$ involves splitting into $A_{i..k}$ and $A_{k+1..j}$ at the final step, then parenthesization of $A_{i..k}$ and $A_{k+1..j}$ in the optimal solution must also be optimal

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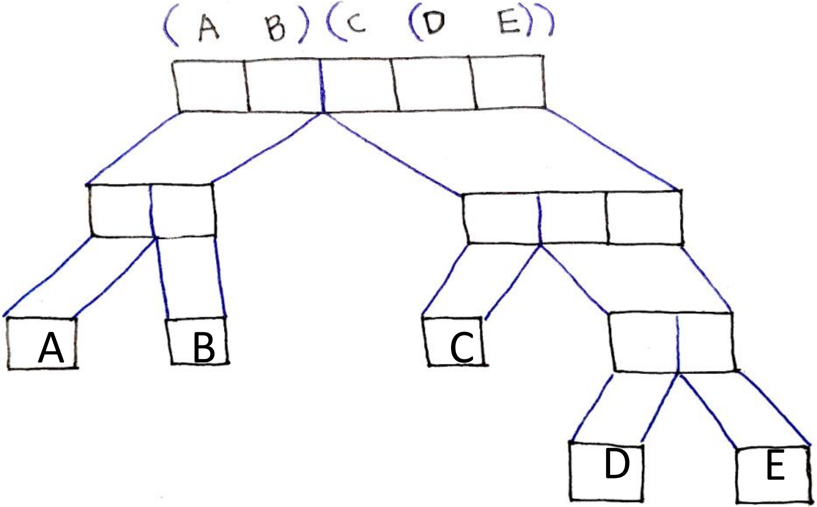
- If parenthesization of $A_{i..k}$ was not optimal, it could be replaced by a cheaper parenthesization, yielding a cheaper final solution, constradicting optimality
- Similarly, if parenthesization of $A_{k+1..j}$ was not optimal, it could be replaced by a cheaper parenthesization, again yielding contradiction of cheaper final solution.

Dynamic Programming Approach



Relationships among subproblems

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$$m[i,j] = \begin{cases} 0 & i = j, \\ \min_{i \leq k < j} (m[i,k] + m[k+1,j] + p_{i-1} p_k p_j) & i < j \end{cases}$$

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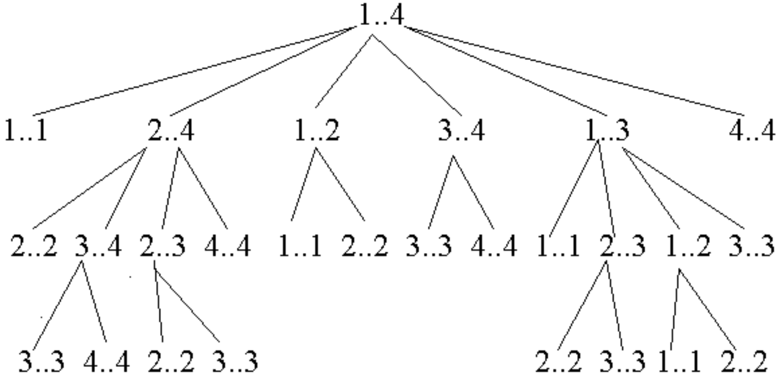
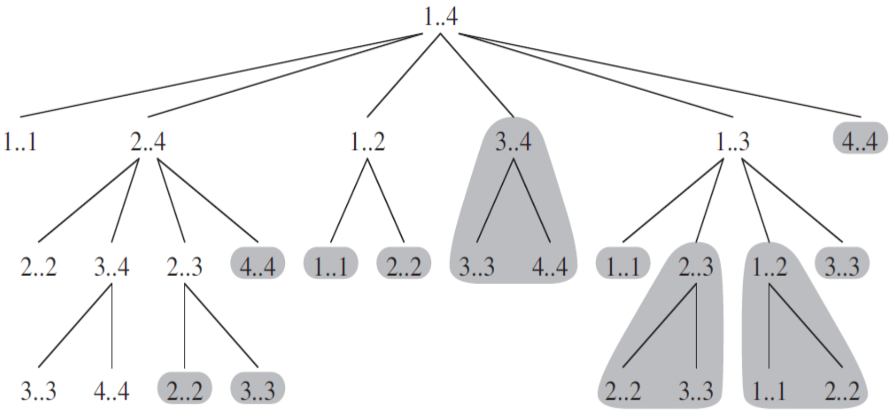
$$A_{i..j} = A_{i..k} A_{k+1..j}$$

Overlapping subproblems (contd.)



recursion tree for computation of RECURSIVE-MATRIX-CHAIN(p1,4)

Overlapping Sub-Problems



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Fill in the $m[i, j]$ table in an order, such that when it is time to calculate $m[i,j]$, the values of $m[i,k]$ and $m[k+1,j]$ for all k are already available.

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An easy way to ensure this is to compute them in increasing order of the size $(j - i)$ of the matrix-chain $A_{i..j}$:

$m[1,2], m[2,3], m[3,4], \ldots, m[n-3,n-2], m[n-2,n-1], m[n-1,n]$

Set of 2 Matrices

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Set of 3 Matrices

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Set of 4 Matrices

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$\ldots$
$m[1,n-1],\ m[2,n]$

Developing a Dynamic Programming Algorithm

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$\ldots$
$m[1,n-1], \ m[2,n]$
$m[1,n]$

Example for the Bottom-Up Computation

Example

A chain of four matrices A_1, A_2, A_3 and A_4, with $p_0 = 5$, $p_1 = 4$, $p_2 = 6$, $p_3 = 2$ and $p_4 = 7$. Find $m[1, 4]$.

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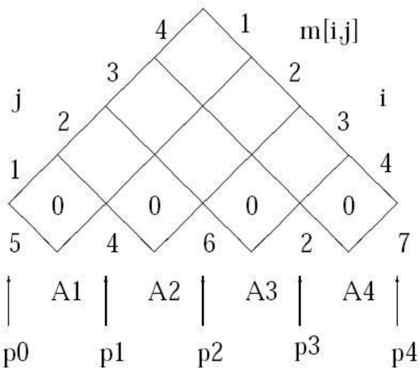
S0: Initialization



Example – Continued

$A1_{5x\mathbf{4}}$, $A2_{\mathbf{4}x6}$, $A3_{6x\mathbf{2}}$, $A4_{\mathbf{2}x7}$

Step 1: Computing $m[1, 2]$

	1	2	3	4
1	0			
2	NA	0		
3	NA	NA	0	
4	NA	NA	NA	0

$A1_{5x\mathbf{4}}$

$A2_{\mathbf{4}x6}$

COST= 5 x **4** x 6=120

M [1,2]= 120

Example – Continued

$A1_{5x4}$, $A2_{4x6}$, $A3_{6x2}$, $A4_{2x7}$

Step 1: Computing $m[1, 2]$

	1	2	3	4
1	0	**120**		
2	NA	0		
3	NA	NA	0	
4	NA	NA	NA	0

$A1_{5x\mathbf{4}}$

$A2_{\mathbf{4}x6}$

COST= 5 x **4** x 6=120

M [1,2]= 120

$A1_{5x4}$, $A2_{4x6}$, $A3_{6x2}$, $A4_{2x7}$

Step 1: Computing *m*[2, 3]

	1	2	3	4
1	0	120		
2	NA	0	**48**	
3	NA	NA	0	
4	NA	NA	NA	0

$A2_{4x6}$
$A3_{6x2}$

COST= 4 x **6** x 2=48
M [2, 3]= 48

Example – Continued

$A1_{5x\mathbf{4}}$, $A2_{\mathbf{4}x6}$, $A3_{6x\mathbf{2}}$, $A4_{\mathbf{2}x7}$

Step 1: Computing $m[3,4]$

	1	2	3	4
1	0	120		
2	NA	0	48	
3	NA	NA	0	**84**
4	NA	NA	NA	0

$A3_{6x\mathbf{2}}$

$A4_{\mathbf{2}x7}$

COST= 6 x **2** x 7=84

M [3,4]= 84

Example – Continued

$A1_{5x\mathbf{4}}$, $A2_{\mathbf{4}x6}$, $A3_{6x\mathbf{2}}$, $A4_{\mathbf{2}x7}$

Step 1: Computing $m[1, 3]$

	1	2	3	4
1	0	120		
2	NA	0	48	
3	NA	NA	0	84
4	NA	NA	NA	0

A1. (A2 . A3) or (A1. A2). A3

Two Possibilities to multiply 3 matrices

Example – Continued

$A1_{5x4}$, $A2_{4x6}$, $A3_{6x2}$, $A4_{2x7}$

Step 1: Computing $m[1, 3]$

A1. (A2 . A3)   or   (A1. A2). A3

$A1_{5x4}$, ($A2_{4x6}$, $A3_{6x2}$)
=m[1,1]+m[2,3]+5*4*2
=0+ 48 + 40
=88

($A1_{5x4}$. $A2_{4x6}$) . $A3_{6x2}$
= m[1,2]+ m[3,3]+ 5x6x2
=120 + 0 + 60
=180

Pick Minimum

Two Possibilities to multiply 3 matrices

Example – Continued

$A1_{5x\mathbf{4}}$, $A2_{\mathbf{4}x6}$, $A3_{6x\mathbf{2}}$, $A4_{\mathbf{2}x7}$

Step 1: Computing $m[1, 3]$

	1	2	3	4
1	0	120	88	
2	NA	0	48	
3	NA	NA	0	84
4	NA	NA	NA	0

A1. (A2 . A3) or (A1. A2). A3

Two Possibilities to multiply 3 matrices

Example – Continued

$A1_{5x\mathbf{4}}$, $A2_{\mathbf{4}x6}$, $A3_{6x\mathbf{2}}$, $A4_{\mathbf{2}x7}$

Step 1: Computing $m[2, 4]$

	1	2	3	4
1	0	120	88	
2	NA	0	48	
3	NA	NA	0	84
4	NA	NA	NA	0

A2. (A3 . A4) or (A2. A3). A4

Two Possibilities to multiply 3 matrices

Example – Continued

$A1_{5x\mathbf{4}}$, $A2_{\mathbf{4}x6}$, $A3_{6x\mathbf{2}}$, $A4_{\mathbf{2}x7}$

Step 1: Computing $m[2, 4]$

A2. (A3 . A4)   or   (A2. A3). A4

$A2_{4x\mathbf{6}}$. ($A3_{\mathbf{6}x2}$, $A4_{2x7}$)

=m[2,2]+m[3,4]+ 4x6x7

=0+ 84 + 168

=252

($A2_{4x6}$. $A3_{6x\mathbf{2}}$) . $A4_{\mathbf{2}x7}$

= m[1,2]+ m[3,3]+ 4x2x7

=48 + 0 + 56

=104

Pick Minimum

Two Possibilities to multiply 3 matrices

Example – Continued

$A1_{5x4}$, $A2_{4x6}$, $A3_{6x2}$, $A4_{2x7}$

Step 1: Computing $m[2,4]$

	1	2	3	4
1	0	120	88	
2	NA	0	48	**104**
3	NA	NA	0	84
4	NA	NA	NA	0

A1. (A2 . A3) or (A1. A2). A3

Two Possibilities to multiply 3 matrices

Example – Continued

$A1_{5x\mathbf{4}}$, $A2_{\mathbf{4}x6}$, $A3_{6x\mathbf{2}}$, $A4_{\mathbf{2}x7}$

Step 1: Computing $m[2, 4]$

	1	**2**	**3**	**4**
1	0	120	88	
2	NA	0	48	**104**
3	NA	NA	0	84
4	NA	NA	NA	0

A1. (A2 . A3. A4) or (A1. A2). (A3 . A4) or (A1 .A2. A3). A4

Three Possibilities

$A1_{5x4}$, $A2_{4x6}$, $A3_{6x2}$, $A4_{2x7}$

A1. (A2 . A3. A4) or (A1. A2). (A3 . A4) or (A1 .A2. A3). A4

M[1,4]= m[1,1] + m[2,4] + 5 x4 x 7,
m[1,2]+ m[3,4] + 5x6x7,
m[1, 3] + m[4,4] + 5x 2x 7

M[1,4]= 0 + 104+ 140,
120+84+210,
88+0+70

M[1,4]= 244,
414,
158

Three Possibilities

Pick Minimum

Example – Continued

$A1_{5x4}$, $A2_{4x6}$, $A3_{6x2}$, $A4_{2x7}$

Step 1: Computing $m[1, 4]$

	1	2	3	4
1	0	120	88	**158**
2	NA	0	48	104
3	NA	NA	0	84
4	NA	NA	NA	0

A1. (A2 . A3. A4) or (A1. A2). (A3 . A4) or (A1 .A2. A3). A4

Three Possibilities

Example – Continued

$A1_{5x\mathbf{4}}$, $A2_{\mathbf{4}x6}$, $A3_{6x\mathbf{2}}$, $A4_{\mathbf{2}x7}$

Split point Matrix

S	1	2	3	4
1		1	1	**3**
2			2	3
3				3
4				

Example – Continued

Step 1: Computing $m[1, 2]$

By definition

$$m[1,2] = \min_{1 \le k < 2} (m[1,k] + m[k+1,2] + p_0 p_k p_2)$$

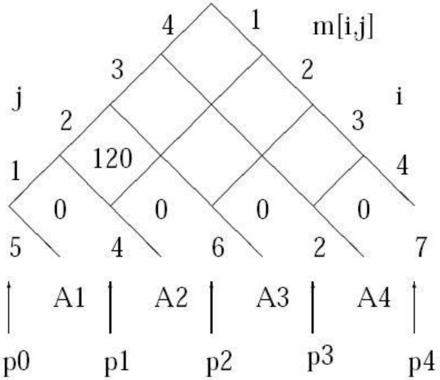
$$= m[1,1] + m[2,2] + p_0 p_1 p_2 = 120 .$$

Step 1: Computing $m[1, 2]$

By definition

$$m[1,2] = \min_{1 \le k < 2} (m[1,k] + m[k+1,2] + p_0 p_k p_2)$$

$$= m[1,1] + m[2,2] + p_0 p_1 p_2 = 120.$$



Example – Continued

Step 2: Computing $m[2,3]$

By definition

$$m[2,3] = \min_{2 \le k < 3} (m[2,k] + pm[k+1,3] + p_1 p_k p_3)$$

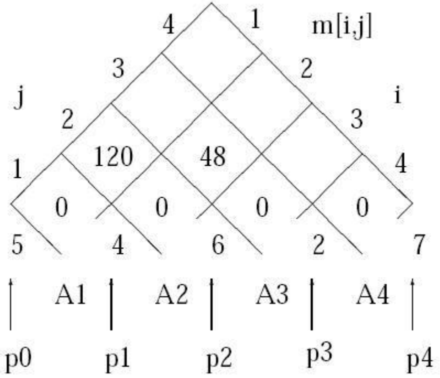
$$= m[2,2] + m[3,3] + p_1 p_2 p_3 = 48 .$$

Step 2: Computing $m[2,3]$

By definition

$$m[2,3] = \min_{2 \le k < 3} (m[2,k] + pm[k+1,3] + p_1 p_k p_3)$$

$$= m[2,2] + m[3,3] + p_1 p_2 p_3 = 48\,.$$



Example – Continued

Step 3: Computing $m[3,4]$

By definition

$$m[3,4] = \min_{3 \le k < 4} (m[3,k] + m[k+1,4] + p_2 p_k p_4)$$

Step 3: Computing $m[3, 4]$

By definition

$$m[3,4] = \min_{3 \le k < 4} (m[3,k] + m[k+1,4] + p_2 p_k p_4)$$

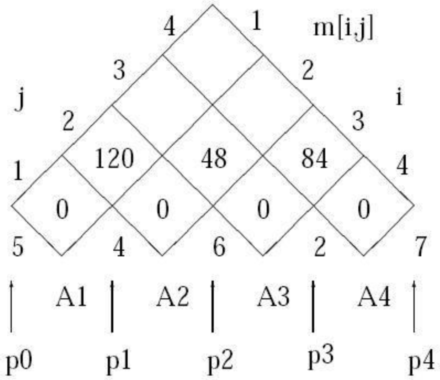
$$= m[3,3] + m[4,4] + p_2 p_3 p_4 = 84.$$

Step 3: Computing $m[3,4]$

By definition

$$m[3,4] = \min_{3 \le k < 4} (m[3,k] + m[k+1,4] + p_2 p_k p_4)$$

$$= m[3,3] + m[4,4] + p_2 p_3 p_4 = 84 .$$



Example – Continued

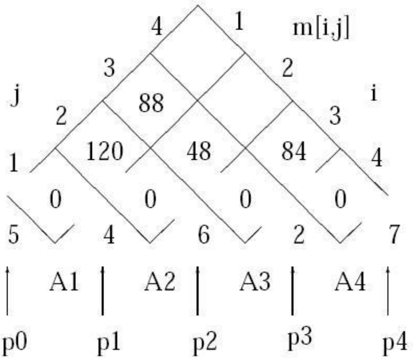
Step 4: Computing $m[1,3]$

By definition

$$m[1,3] = \min_{1 \le k < 3} \left(m[1,k] + m[k+1,3] + p_0 p_k p_3 \right)$$

$$= \min \begin{cases} m[1,1] + m[2,3] + p_0 p_1 p_3 \\ m[1,2] + m[3,3] + p_0 p_2 p_3 \end{cases}$$

$$= 88 \ .$$



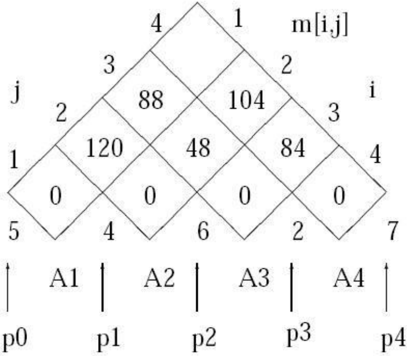
Step 5: Computing $m[2,4]$

By definition

$$m[2,4] = \min_{2 \le k < 4} \left(m[2,k] + m[k+1,4] + p_1 p_k p_4 \right)$$

$$= \min \begin{cases} m[2,2] + m[3,4] + p_1 p_2 p_4 \\ m[2,3] + m[4,4] + p_1 p_3 p_4 \end{cases}$$

$$= 104 \; .$$



Example – Continued

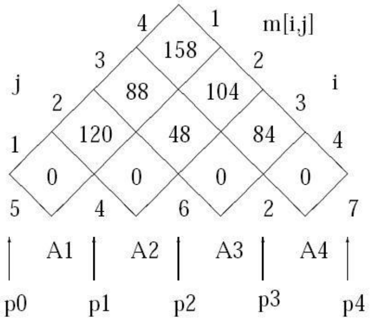
Step 6: Computing $m[1,4]$

By definition

$$m[1,4] = \min_{1 \le k < 4} \left(m[1,k] + m[k+1,4] + p_0 p_k p_4 \right)$$

$$= \min \left\{ \begin{array}{l} m[1,1] + m[2,4] + p_0 p_1 p_4 \\ m[1,2] + m[3,4] + p_0 p_2 p_4 \\ m[1,3] + m[4,4] + p_0 p_3 p_4 \end{array} \right.$$

$$= 158 \,.$$



Constructing a Solution

- $m[i,j]$ only keeps costs but not actual multiplication sequence.
- To solve problem, need to reconstruct multiplication sequence that yields $m[1,n]$.
- Solution: similar to previous DP algorithm(s) keep an auxillary array $s[*,*]$.
- $s[i,j] = k$ where k is the index that achieves minimum in

$$m[i,j] = \min_{i \leq k < j} (m[i,k] + m[k+1,j] + p_{i-1}p_kp_j).$$

Step 4: Constructing optimal solution

Idea: Maintain an array $s[1..n, 1..n]$, where $s[i,j]$ denotes k for the optimal splitting in computing $A_{i..j} = A_{i..k}A_{k+1..j}$.

Step 4: Constructing optimal solution

Idea: Maintain an array $s[1..n, 1..n]$, where $s[i, j]$ denotes k for the optimal splitting in computing $A_{i..j} = A_{i..k}A_{k+1..j}$.

Question

How to Recover the Multiplication Sequence using $s[i, j]$?

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Question

How to Recover the Multiplication Sequence using $s[i,j]$?

$s[1,n]$ $\quad$ $(A_1 \cdots A_{s[1,n]})(A_{s[1,n]+1} \cdots A_n)$

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How to Recover the Multiplication Sequence using $s[i,j]$?

$s[1,n]$ $\qquad (A_1 \cdots A_{s[1,n]})(A_{s[1,n]+1} \cdots A_n)$

$s[1,s[1,n]]$ $\qquad (A_1 \cdots A_{s[1,s[1,n]]})(A_{s[1,s[1,n]]+1} \cdots A_{s[1,n]})$

Step 4: Constructing optimal solution

Idea: Maintain an array $s[1..n, 1..n]$, where $s[i,j]$ denotes k for the optimal splitting in computing $A_{i..j} = A_{i..k}A_{k+1..j}$.

Question

How to Recover the Multiplication Sequence using $s[i,j]$?

$s[1,n]$	$(A_1 \cdots A_{s[1,n]})(A_{s[1,n]+1} \cdots A_n)$
$s[1,s[1,n]]$	$(A_1 \cdots A_{s[1,s[1,n]]})(A_{s[1,s[1,n]]+1} \cdots A_{s[1,n]})$
$s[s[1,n]+1,n]$	$(A_{s[1,n]+1} \cdots A_{s[s[1,n]+1,n]})(A_{s[s[1,n]+1,n]+1} \cdots A_n)$

Step 4: Constructing optimal solution

Idea: Maintain an array $s[1..n, 1..n]$, where $s[i,j]$ denotes k for the optimal splitting in computing $A_{i..j} = A_{i..k}A_{k+1..j}$.

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How to Recover the Multiplication Sequence using $s[i,j]$?

$s[1,n]$	$(A_1 \cdots A_{s[1,n]})(A_{s[1,n]+1} \cdots A_n)$
$s[1,s[1,n]]$	$(A_1 \cdots A_{s[1,s[1,n]]})(A_{s[1,s[1,n]]+1} \cdots A_{s[1,n]})$
$s[s[1,n]+1,n]$	$(A_{s[1,n]+1} \cdots A_{s[s[1,n]+1,n]})(A_{s[s[1,n]+1,n]+1} \cdots A_n)$
:	:

Step 4: Constructing optimal solution

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Question

How to Recover the Multiplication Sequence using $s[i, j]$?

$$
\begin{array}{ll}
s[1,n] & (A_1 \cdots A_{s[1,n]})\,(A_{s[1,n]+1} \cdots A_n) \\
s[1,s[1,n]] & (A_1 \cdots A_{s[1,s[1,n]]})\,(A_{s[1,s[1,n]]+1} \cdots A_{s[1,n]}) \\
s[s[1,n]+1,n] & (A_{s[1,n]+1} \cdots A_{s[s[1,n]+1,n]})\,(A_{s[s[1,n]+1,n]+1} \cdots A_n) \\
\vdots & \vdots
\end{array}
$$

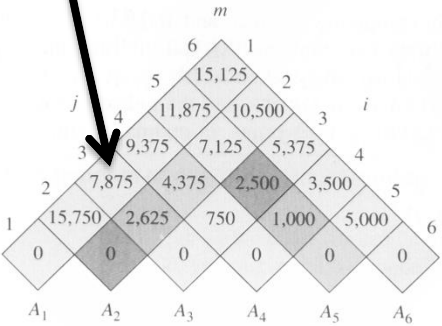
Apply recursively until multiplication sequence is completely determined.

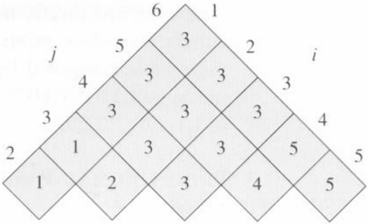
Example 2:

$$
\begin{array}{lll}
A_1 & 30 \times 35 & = p_0 \times p_1 \\
A_2 & 35 \times 15 & = p_1 \times p_2 \\
A_3 & 15 \times 5 & = p_2 \times p_3 \\
A_4 & 5 \times 10 & = p_3 \times p_4 \\
A_5 & 10 \times 20 & = p_4 \times p_5 \\
A_6 & 20 \times 25 & = p_5 \times p_6
\end{array}
$$

Order Products	Matrix Size	
A2 x A3 =	35 x 5	2,625
A1 (A2 x A3) =	30 X 5	5,250 + 2,625

the m and s table computed by MATRIX-CHAIN-ORDER for n=6





$m[2,5]=$
$min\{$
$m[2,2]+m[3,5]+p_1p_2p_5=0+2500+35\times15\times20=13000,$
$m[2,3]+m[4,5]+p_1p_3p_5=2625+1000+35\times5\times20=7125,$
$m[2,4]+m[5,5]+p_1p_4p_5=4375+0+35\times10\times20=11374$
$\}$
$=7125$

Constructing the optimal solution

```
PRINT-OPTIMAL-PARENS(s, i, j)
if i == j
        print "A"_i
else print "("
        PRINT-OPTIMAL-PARENS(s, i, s[i, j])
        PRINT-OPTIMAL-PARENS(s, s[i, j] + 1, j)
        print ")"
```

- **example:** $((A_1(A_2A_3))((A_4A_5)A_6))$

Constructing the optimal solution

```
PRINT-OPTIMAL-PARENS(s, i, j)
if i == j
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    PRINT-OPTIMAL-PARENS(s, i, s[i, j])
    PRINT-OPTIMAL-PARENS(s, s[i, j] + 1, j)
    print ")"
```

- **example:** $((A_1(A_2A_3))((A_4A_5)A_6))$

Example (Finding the Multiplication Sequence)

Consider $n = 6$. Assume array $s[1..6, 1..6]$ has been properly constructed.

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$$s[1,6] = 3$$

Step 4...

Example (Finding the Multiplication Sequence)

Consider $n = 6$. Assume array $s[1..6, 1..6]$ has been properly constructed. The multiplication sequence is recovered as follows.

$$s[1, 6] = 3 \qquad (A_1A_2A_3)(A_4A_5A_6)$$

Step 4...

Example (Finding the Multiplication Sequence)

Consider $n = 6$. Assume array $s[1..6, 1..6]$ has been properly constructed. The multiplication sequence is recovered as follows.

$$s[1,6] = 3 \qquad (A_1A_2A_3)(A_4A_5A_6)$$
$$s[1,3] = 1$$

Step 4...

Example (Finding the Multiplication Sequence)

Consider $n = 6$. Assume array $s[1..6, 1..6]$ has been properly constructed. The multiplication sequence is recovered as follows.

$$\begin{array}{ll} s[1,6] = 3 & (A_1A_2A_3)(A_4A_5A_6) \\ s[1,3] = 1 & (A_1(A_2A_3)) \end{array}$$

Example (Finding the Multiplication Sequence)

Consider $n = 6$. Assume array $s[1..6,\ 1..6]$ has been properly constructed. The multiplication sequence is recovered as follows.

$$
\begin{array}{ll}
s[1,6] = 3 & (A_1A_2A_3)(A_4A_5A_6) \\
s[1,3] = 1 & (A_1(A_2A_3)) \\
s[4,6] = 5 &
\end{array}
$$

Example (Finding the Multiplication Sequence)

Consider $n = 6$. Assume array $s[1..6,\ 1..6]$ has been properly constructed. The multiplication sequence is recovered as follows.

$$\begin{array}{ll} s[1,6]=3 & (A_1A_2A_3)(A_4A_5A_6) \\ s[1,3]=1 & (A_1(A_2A_3)) \\ s[4,6]=5 & ((A_4A_5)A_6) \end{array}$$

Step 4...

Example (Finding the Multiplication Sequence)

Consider $n = 6$. Assume array $s[1..6, 1..6]$ has been properly constructed. The multiplication sequence is recovered as follows.

$$\begin{array}{ll} s[1,6]=3 & (A_1A_2A_3)(A_4A_5A_6) \\ s[1,3]=1 & (A_1(A_2A_3)) \\ s[4,6]=5 & ((A_4A_5)A_6) \end{array}$$

Hence the final multiplication sequence is

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$$\begin{array}{ll} s[1,6]=3 & (A_1A_2A_3)(A_4A_5A_6) \\ s[1,3]=1 & (A_1(A_2A_3)) \\ s[4,6]=5 & ((A_4A_5)A_6) \end{array}$$

Hence the final multiplication sequence is

$$(A_1(A_2A_3))((A_4A_5)A_6).$$

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

for i = 1 **to** n **do** $m[i, i]$ =

The Dynamic Programming Algorithm

Matrix-Chain(*p*, *n*):// l is length ofsub-chain

for $i = 1$ **to** n **do** $m[i, i] = 0$;

;

for l =

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

for i = 1 **to** n **do** $m[i, i] = 0$;
;
for l = 2 **to**

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

for i = 1 **to** n **do** $m[i, i] = 0$;
;
for l = 2 **to** n **do**
    for i = 1 **to**

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

for $i = 1$ **to** n **do** $m[i, i] = 0$;
;
for $l = 2$ **to** n **do**
    for $i = 1$ **to** $n - l + 1$ **do**
        $j =$

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

for $i = 1$ **to** n **do** $m[i, i] = 0$;

;

for $l = 2$ **to** n **do**

  for $i = 1$ **to** $n - l + 1$ **do**

    $j = i + l - 1$;

    $m[i, j] =$

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

for $i = 1$ **to** n **do** $m[i, i] = 0$;
;
for $l = 2$ **to** n **do**
  for $i = 1$ **to** $n - l + 1$ **do**
    $j = i + l - 1$;
    $m[i, j] = \infty$;
    for k =

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

```
for i = 1 to n do m[i,i] = 0;
;
for l = 2 to n do
    for i = 1 to n − l + 1 do
        j = i + l − 1;
        m[i,j] = ∞;
        for k = i to
```

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

for $i = 1$ **to** n **do** $m[i, i] = 0$;

;

for $l = 2$ **to** n **do**

for $i = 1$ **to** $n - l + 1$ **do**

$j = i + l - 1$;

$m[i, j] = \infty$;

for $k = i$ **to** $j - 1$ **do**

$q =$

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

for $i = 1$ **to** n **do** $m[i, i] = 0$;

;

for $l = 2$ **to** n **do**

for $i = 1$ **to** $n - l + 1$ **do**

$j = i + l - 1$;

$m[i, j] = \infty$;

for $k = i$ **to** $j - 1$ **do**

$q = m[i, k] +$

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

```
for i = 1 to n do m[i,i] = 0;
;
for l = 2 to n do
        for i = 1 to n − l + 1 do
            j = i + l − 1;
            m[i,j] = ∞;
                for k = i to j − 1 do
                    q = m[i,k] + m[k + 1,j] +
```

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

```
for i = 1 to n do m[i,i] = 0;
;
for l = 2 to n do
    for i = 1 to n − l + 1 do
        j = i + l − 1;
        m[i,j] = ∞;
        for k = i to j − 1 do
            q = m[i,k] + m[k+1,j] + p[i − 1] *
```

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

for $i = 1$ **to** n **do** $m[i, i] = 0$;

;

for $l = 2$ **to** n **do**

    for $i = 1$ **to** $n - l + 1$ **do**

        $j = i + l - 1$;

        $m[i, j] = \infty$;

        for $k = i$ **to** $j - 1$ **do**

            $q = m[i, k] + m[k + 1, j] + p[i - 1] * p[k] *$

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;
for $l = 2$ **to** n **do**
    for $i = 1$ **to** $n - l + 1$ **do**
        $j = i + l - 1$;
        $m[i, j] = \infty$;
        for $k = i$ **to** $j - 1$ **do**
            $q = m[i, k] + m[k + 1, j] + p[i - 1] * p[k] * p[j]$;

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            q = m[i,k] + m[k+1,j] + p[i − 1] * p[k] * p[j];
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                m[i,j] =
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            q = m[i,k] + m[k+1,j] + p[i − 1] * p[k] * p[j];
            if q < m[i,j] then
                m[i,j] = q;
                s[i,j] =
```

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

```
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                    if q < m[i,j] then
                        m[i,j] = q;
                        s[i,j] = k;
                    end
                end
        end
end  return
```

The Dynamic Programming Algorithm

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            end
        end
    end
end
return m and s;(Optimum in
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                    if q < m[i,j] then
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                        s[i,j] = k;
                    end
            end
    end
end
return m and s;(Optimum in m[1,n])
```

Complexity: The loops are nested three levels deep.

The Dynamic Programming Algorithm

Matrix-Chain(p, n):// l is length ofsub-chain

```
for i = 1 to n do m[i,i] = 0;
;
for l = 2 to n do     / l is the chain length
        for i = 1 to n − l + 1 do
            j = i + l − 1;
            m[i,j] = ∞ ;
                for k = i to j − 1 do
                    q = m[i,k] + m[k + 1,j] + p[i − 1] * p[k] * p[j];
                    if q < m[i,j] then
                        m[i,j] = q;
                        s[i,j] = k;
                    end
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The Dynamic Programming Algorithm

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                    end
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end
return m and s;(Optimum in m[1,n])
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Complexity: The loops are nested three levels deep. Each loop index takes on

$\leq n$ values. Hence the time complexity is $O(n^3$.

The Dynamic Programming Algorithm

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              for k = i to j − 1 do
                q = m[i,k] + m[k+1,j] + p[i − 1] * p[k] * p[j];
                if q < m[i,j] then
                    m[i,j] = q;
                    s[i,j] = k;
                end
            end
        end
end
return m and s;(Optimum in m[1,n])
```

Complexity: The loops are nested three levels deep. Each loop index takes on $\leq n$ values. Hence thetime complexityis $O(n^3)$. Space complexityis $\Theta(n^2)$.

Reference

Introduction to Algorithms

Thomas H. Cormen

Chapter # 15