MT-104
Linear Algebra

National University of Computer and Emerging Sciences

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Lecture 26

Orthogonal Diagonalization of Symmetric Matrices

Properties of Symmetric Matrices

Any matrix A is said to be symmetric if

$$A^T = A.$$

Theorem

If A is a real symmetric matrix, then the eigenvalues of A are real.

Theorem

If A is a symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Main Results

Theorem

A square matrix A is **orthogonally diagonalizable** if there exists an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

Theorem (The Spectral Theorem)

Let A be an $n \times n$ real matrix. Then A is symmetric if and only if it is orthogonally diagonalizable.

Examples

Orthogonally diagonalize the following matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Solution

Calculate

Step-I eigenvalues (verify they are real)

Step-II eigenvectors (must be orthogonal corresponding to distinct eigenvalues)

Step-III Q(Previously we were using P)

Characteristic equation: $\lambda^2 - 5\lambda = 0$.

Eigenvalues: $\lambda = 0, 5$.

Eigenvectors: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Example

$$P = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

Columns are not orthonormal

$$Q = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}.$$

Hence,

$$Q^{-1}AQ = D.$$

Example

Orthogonally diagonalize the following matrix

$$A = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}.$$

Solution

Eigenvalues of given matrix are: $7, 7, -2$

Eigenvector corresponding to 7 are $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix}.$

Eigenvector corresponding to -2 is $\begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix}.$

Eigenvectors corresponding to 7 are not orthogonal.

How can we make them orthogonal?

Example

Gram-Schmidt Process

$$\left\{ v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

We want to construct an orthogonal set from $\{v_1,\ v_2\}$.

$$\begin{aligned} w_1 &= v_1 \\ w_2 &= v_2 - proj_{w_1} v_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1}. \end{aligned}$$

$$\left\{ w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ w_2 = \begin{pmatrix} -1/4 \\ 1 \\ 1/4 \end{pmatrix} \right\}$$

.

Hence, set of eigenvectors are

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/4 \\ 1 \\ 1/4 \end{pmatrix}, \begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix} \right\}$$

.

Example

Set of eigenvectors can be written as

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \right\}$$

.

$$\left\| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\| = \sqrt{2}, \quad \left\| \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \right\| = \sqrt{18}, \quad \left\| \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \right\| = 3.$$

Hence,

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & \frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & -\frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{pmatrix}.$$

$$D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Spectral Decomposition

If A is symmetric, then we can find an orthogonal matrix Q and diagonal matrix D such that

$$A = QDQ^T.$$

Then, symmetric matrix A can be written as

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \ldots + \lambda_n u_n u_n^T.$$

Example Construct a spectral decomposition of the following matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Solution

$$A = 0 \begin{pmatrix} -2 \\ 1 \end{pmatrix} (-2 \;\; 1) + 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} (1 \;\; 2).$$