MT-104 Linear Algebra

National University of Computer and Emerging Sciences

Fall 2020

December 8, 2020

Linear Recurrence Relations & Discrete Dynamical Systems

Linear Recurrence Relation

Let $(x_n) = (x_0, x_1, x_2, ...)$ be a sequence of numbers that is defined as follows:

1. $x_0 = a_0, x_1 = a_1, ..., x_{k-l} = a_{k-1}$, where $a_0, a_1, ..., a_{k-l}$ are scalars.
2. For all $n \geq k, x_n = c_1 x_{n-l} + c_2 x_{n-2} + \; + c_k x_{n-k}$ where $c_1, c_2, .., c_k$ are scalars.

If $c_k \neq 0$, the equation in (2) is called a linear recurrence relation of order k.
The equations in (1) are referred to as the initial conditions of the recurrence.

Examples

- $x_{n+2} = x_{n+1} + x_n, \quad x_0 = 1,\ x_1 = 1.$
- $x_{n+1} = 2x_n, \;\; x_0 = 3.$

Linear Recurrence in Matrix Form

I am going to explain it using an example of second order linear recurrence relation
Consider the following linear recurrence relation

$$x_{n+2} = ax_{n+1} + bx_n, \quad x_1 = c_1, \ x_0 = c_0,$$

where c_0 and c_1 are known constants.
We can write it as

$$\begin{aligned} x_{n+2} &= a\, x_{n+1} + b\, x_n \\ x_{n+1} &= x_{n+1}. \end{aligned}$$

In Matrix form, we can write

$$\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$$

$$\boxed{X_{n+1} = A\, X_n}, \ \forall n \geq 0.$$

where $\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$.

Linear Recurrence in Matrix Form

For $n = 0$, we have

$$X_1 = AX_0,$$

where $X_0 = \begin{bmatrix} c_1 \\ c_0 \end{bmatrix}$

For $n = 1$, we can write

$$X_2 = AX_1 = A(AX_0) = A^2X_0.$$

$n = 2$, gives us

$$X_3 = AX_2 = A(A^2X_0) = A^3X_0.$$

Continuing in the same manner, we have

$$X_{n+1} = A^{n+1}X_0.$$

Examples

Suppose each "Gibonacci" number G_{k+2} is the average of the two previous numbers G_{k+1} and G_k. If $G_0 = 0$ and $G_1 = 1$. Find the *kth* term of the sequence only depending upon k.

$\boxed{\textit{Aim}:}$

We want to find the general term of the sequence.

$\boxed{\textit{Steps}:}$

- Matrix Form
- Eigenvalues and Eigenvectors
- Diagonalize

Examples

$$G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k$$
$$G_{k+1} = G_{k+1}.$$

In Matrix Form

$$\boldsymbol{G}_{k+1} = A\boldsymbol{G}_k, \ \ \forall k \geq 0.$$

where $\boldsymbol{G}_{k+1} = \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix}$, $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{G}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\boldsymbol{G}_k = A^k \boldsymbol{G}_0, \ \ \forall k \geq 0.$$

Eigenvalues Characteristic Equation

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0.$$

Eigenvalues are: $1, -\frac{1}{2}$.

Eigenvectors $\lambda = 1$

$$(A - 1I)X = 0$$

Augmented matrix

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

Eigenvector:All non-zero multiples of $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Eigenvalues $\lambda = -\frac{1}{2}$

$$(A + \frac{1}{2}I)X = 0$$

Augmented matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \end{bmatrix}$$

Eigenvector:All non-zero multiples of $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

As $A^k = PD^kP^{-1}$,
so we need to calculate P^{-1}.

$$P^{-1} = \frac{adjP}{detP} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}.$$

So,

$$A^k = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{(-1)^k}{2^k} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}.$$

Simplification gives us

$$A^k = \frac{1}{3} \begin{bmatrix} \frac{(-1)^k}{2^k} + 2 & 1 - \frac{(-1)^k}{2^k} \\ 2 - \frac{2(-1)^k}{2^k} & \frac{2(-1)^k}{2^k} + 1 \end{bmatrix}$$

$$\boldsymbol{G}_k = A^k \boldsymbol{G}_0$$

$$\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} = \frac{1}{3} \begin{bmatrix} \dfrac{(-1)^k}{2^k} + 2 \\ 2 - \dfrac{2(-1)^k}{2^k} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$G_k = \frac{2}{3} - \frac{2}{3}(\frac{-1}{2})^k.$$

Theorem

Let $x_n = ax_{n-1} + bx_{n-2}$ be a recurrence relation. Let λ_1 and λ_2 be the eigenvalues of the associated characteristic equation $\lambda^2 - a\lambda - b = 0$.

1. If $\lambda_1 \neq \lambda_2$, then
$$x_n = c_1\lambda_1^n + c_2\lambda_2^n$$
for some scalars c_1 and c_2.
2. If $\lambda_1 = \lambda_2 = \lambda$, then
$$x_n = c_1\lambda^n + c_2 n\lambda^n$$
for some scalars c_1 and c_2.

Example

Suppose each "Gibonacci" number G_{k+2} is the average of the two previous numbers G_{k+1} and G_k. If $G_0 = 0$ and $G_1 = 1$. Find the *kth* term of the sequence only depending upon k.

$$G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k$$
$$G_{k+1} = G_{k+1}.$$

Characteristics equation

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0.$$

Eigenvalues are: 1, $-\frac{1}{2}$. So,

$$G_k = c_1(1)^k + c_2(\frac{-1}{2})^k.$$

As $G_0 = 0,\ G_1 = 1$, so we have

$$
\begin{aligned}
0 &= G_0 = c_1 + c_2 \\
1 &= G_1 = c_1 - c_2 \frac{1}{2}.
\end{aligned}
$$

Solving above system, we get

$$c_1 = \frac{2}{3},\ c_2 = -\frac{2}{3}.$$

Hence, G_k is

$$G_k = \frac{2}{3} - \frac{2}{3}\left(\frac{-1}{2}\right)^k.$$

Example

Solve the following recurrence relation with the given initial conditions.

$$y_1 = 1,\ y_2 = 6,\ y_k = 4y_{k-1} - 4y_{k-2},\ \text{k} \geq 3.$$

Characteristics equation $\lambda^2 - 4\lambda + 4 = 0$.
Solution of the quadratic equation is
Eigenvalues: $\lambda_1 = 2,\ \lambda_1 = 2$. So,

$$y_k = c_1(2)^k + c_2 k 2^k.$$

As, $y_1 = 1$, so, $2c_1 + 2c_2 = 1$,
$y_2 = 6$, so, $4c_2 + 8c_2 = 6$.
Solution of above system is

$$c_1 = -\frac{1}{2},\ c_2 = 1.$$

Hence,

$$y_k = -\frac{1}{2}2^k + k2^k.$$

Practice Problems

1. Solve the recurrence relation with the given initial conditions.
   1.1 $a_0 = 4,\ a_1 = 1,\ a_n = a_{n-1} - a_{n-2}/4$, for $n \geq 2$.
   1.2 $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$, subject to $a_0 = 2, a_1 = 2, a_2 = 4$, for $n \geq 3$.
2. Find the limiting values of y_k and z_k, $(k \to \infty)$ if

$$y_{k+1} = .8y_k + .3z_k \quad y_0 = 0$$
$$z_{k+1} = .2y_k + .7z_k, \quad z_0 = 5.$$

3. Suppose there is an epidemic in which every month half of those who are well become sick, and a quarter of those who are sick become dead. Find the steady state for the corresponding Markov process $\begin{bmatrix} d_{k+1} \\ s_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1/4 & 0 \\ 0 & 3/4 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} d_k \\ s_k \\ w_k \end{bmatrix}$.

Dynamical System

A dynamical system is a sequence of vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots$ in $\mathbb{R}^n$ with an associated $n \times n$ matrix A, such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for all k. The idea, as in what we have seen before, is that $\mathbf{x}_k$ represents the state of some system at time k, and multiplying by A moves the system forward in time.

If A is diagonalizable, there is a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for $\mathbb{R}^n$ made up of eigenvectors of A with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. In particular, we can write

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$$

for some unique set of constants $c_1, \ldots, c_n$. Applying A to both sides gives

$$\mathbf{x}_1 = \mathbf{A}x_0 = c_1\lambda_1\mathbf{v}_1 + \ldots + c_n\lambda_n\mathbf{v}_n.$$

Note that the same λ_i may appear more than once if some eigenspace has dimension > 1. In general, applying A multiple times gives

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \ldots + c_n\lambda_n^k\mathbf{v}_n.$$

So a dynamical system with a diagonalizable matrix A is easy to analyze.

Example 1.1. This is Lay's example, so we will go with it, though with some nicer numbers. Consider a forest populated by rats and owls. In the absence of owls, the rats will reproduce off of vegetable matter in the forest, but owls will kill the rats. On the other hand, the owls will starve to death without rats. The populations O_k, R_k of rats and owls in month k are given by the equations

$$\begin{aligned} O_{k+1} &= \frac{1}{2}O_k + \frac{1}{4}R_k \\ R_{k+1} &= -p\,O_k + \frac{3}{2}R_k. \end{aligned}$$

Here p is a parameter that we will vary to see how greater predation affects the system.

Organizing the populations in the form of a vector $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$, we see $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix} 1/2 & 1/4 \\ -p & 3/2 \end{bmatrix}.$$

The characteristic polynomial of this matrix is $\lambda^2 - 2\lambda + \frac{p+3}{4}$, which has real roots if and only if $p \leq 1$. Moreover, in the case that $p < 1$, it is easy to see from the quadratic formula that there will be two distinct eigenvalues, and so A will be diagonalizable. If $p = 1$, there is only one eigenvalue, and a calculation of the eigenspace shows that A is not diagonalizable.

Let's choose $p = 1/9$. In this case, the eigenvalues are $\lambda_1 = 1 + \sqrt{2}/3$ and $\lambda_2 = 1 - \sqrt{2}/3$. Corresponding eigenvectors are $\mathbf{v}_1 = (3 - 2\sqrt{2}, 2/3)$ and $\mathbf{v}_2 = (3 + 2\sqrt{2}, 2/3)$.

Let's say we start with some population vector $\mathbf{x}_0$ with nonzero entries (we start with more than zero owls and rats). Writing $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, we see that

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2.$$

Now, notice that $0 < \lambda_2 < 1$ (in fact, $\lambda_2 \approx 0.057$). Therefore, $\lambda_2^k \to 0$ as $k \to \infty$ and so

$$\mathbf{x}_k \approx c_1 \lambda_1^k \mathbf{v}_1$$

as $k \to \infty$. Now, $\lambda_1 \approx 1.94$ and so both the owls and rats grow in population. In fact, the population of each almost doubles month-by-month.

Graphical Description of Solutions

When A is 2×2, algebraic calculations can be supplemented by a geometric description of a system's evolution. We can view the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ as a description of what happens to an initial point $\mathbf{x}_0$ in $\mathbb{R}^2$ as it is transformed repeatedly by the mapping $\mathbf{x} \mapsto A\mathbf{x}$. The graph of $\mathbf{x}_0, \mathbf{x}_1, \ldots$ is called a **trajectory** of the dynamical system.

EXAMPLE 2 Plot several trajectories of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, when

$$A = \begin{bmatrix} .80 & 0 \\ 0 & .64 \end{bmatrix}$$

SOLUTION The eigenvalues of A are .8 and .64, with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. If $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then

$$\mathbf{x}_k = c_1(.8)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(.64)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Of course, $\mathbf{x}_k$ tends to $\mathbf{0}$ because $(.8)^k$ and $(.64)^k$ both approach 0 as $k \to \infty$. But *the way* $\mathbf{x}_k$ goes toward $\mathbf{0}$ is interesting. Figure 1 (on page 304) shows the first few terms of several trajectories that begin at points on the boundary of the box with corners at $(\pm 3, \pm 3)$. The points on each trajectory are connected by a thin curve, to make the trajectory easier to see. ■

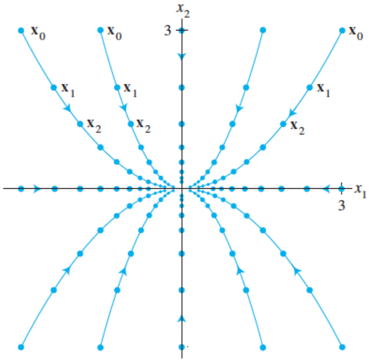

FIGURE 1 The origin as an attractor.

In Example 2, the origin is called an **attractor** of the dynamical system because all trajectories tend toward $\mathbf{0}$. This occurs whenever both eigenvalues are less than 1 in magnitude. The direction of greatest attraction is along the line through $\mathbf{0}$ and the eigenvector $\mathbf{v}_2$ for the eigenvalue of smaller magnitude.

In the next example, both eigenvalues of A are larger than 1 in magnitude, and $\mathbf{0}$ is called a **repeller** of the dynamical system. All solutions of $\mathbf{x}_{k+1} = A\mathbf{x}_k$ except the (constant) zero solution are unbounded and tend away from the origin.[2]

EXAMPLE 3 Plot several typical solutions of the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix} 1.44 & 0 \\ 0 & 1.2 \end{bmatrix}$$

SOLUTION The eigenvalues of A are 1.44 and 1.2. If $\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then

$$\mathbf{x}_k = c_1(1.44)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(1.2)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Both terms grow in size, but the first term grows faster. So the direction of greatest repulsion is the line through $\mathbf{0}$ and the eigenvector for the eigenvalue of larger magnitude. Figure 2 shows several trajectories that begin at points quite close to $\mathbf{0}$. ∎

In the next example, $\mathbf{0}$ is called a **saddle point** because the origin attracts solutions from some directions and repels them in other directions. This occurs whenever one eigenvalue is greater than 1 in magnitude and the other is less than 1 in magnitude. The direction of greatest attraction is determined by an eigenvector for the eigenvalue of smaller magnitude. The direction of greatest repulsion is determined by an eigenvector for the eigenvalue of greater magnitude.

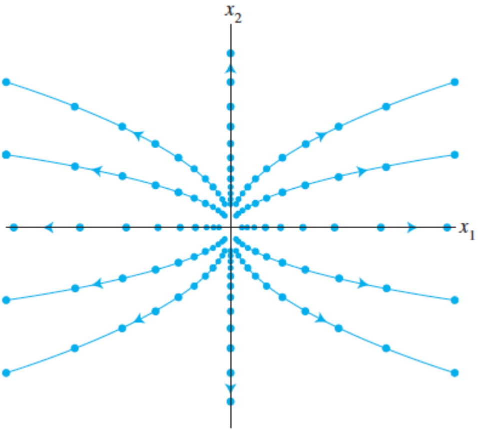

FIGURE 2 The origin as a repeller.