Probability and Statistics

Assignment 5

Total Mark:100

Question No.1(8 Marks):

An overseas shipment of 5 foreign automobiles contains 2 that have slight paint blemishes. If an agency receives 3 of these automobiles at random, list the elements of the sample space S, using the letters B and N for blemished and non-blemished, respectively; then to each sample point assign a value x of the random variable X representing the number of automobiles with paint blemishes purchased by the agency.

Solution:

Let X be the number of automobiles purchased with paint blemishes

Automobiles = [N_1, N_2, N_3, B_1, B_2]

Elements for sample space S for X =0, 1, 2

X	Sample Space
0	$N_1\ N_2\ N_3$
1	$N_1\ N_2\ B_1$
1	$N_1\ N_2\ B_2$
1	$N_1\ N_3\ B_1$
1	$N_1\ N_3\ B_2$
1	$N_2\ N_3\ B_1$
1	$N_2\ N_3\ B_2$
2	$N_1\ B_1\ B_2$
2	$N_2\ B_1\ B_2$
2	$N_3\ B_1\ B_2$

Question No.2(10 Marks):

An investment firm offers its customers municipal bonds that mature after varying numbers of years. Given that the cumulative distribution function of T, the number of years to maturity for a randomly selected bond, is

$$F(t) = \begin{cases} 0, & t < 1, \\ \frac{1}{4}, & 1 \le t < 3, \\ \frac{1}{2}, & 3 \le t < 5, \\ \frac{3}{4}, & 5 \le t < 7, \\ 1, & t \ge 7, \end{cases}$$

find

(a) P (T = 5);

(b) P (T > 3);

(c) P (1.4 < T < 6);

Solution:

(a)

P (T = 5) = P (T ≤ 5) – P (T < 5)

$= F(5) - \lim_{t \to 5} P(T \le t)$

$= F(5) - \lim_{t \to 5} F(t)$

$= F(5) - \lim_{t \to 5} (1/2)$

= 3/4 – 1/2

= 1/4

(b)

P (T > 3) = 1 - P (T ≤ 3)

= 1 – F (3)

= 1 – 1/2

= 1/2

(c)

P (1.4<T<6) = P (T < 6) – P (T ≤ 1.4)

$= \lim_{t \to 6} P(T \le t) - P(T \le 1.4)$

$= \lim_{t \to 6} F(t) - F(1.4)$

$= \lim_{t \to 6} (3/4) - (1/4)$

= 3/4 – 1/4

= 1/2

Question No.3(10 Marks):

The discrete random variable 'W' has probability distribution as shown

W	P(W=w)
-3	0.1
-2	0.25
-1	0.3
0	0.15
1	d

Find,

(a) Value of d

(b) P ($-3 \leq W < 0$)

(c) P ($W > -1$)

(d) P ($-1 < W < 1$)

Solution:

(a)

As $\sum (P(W=w)) = 1$

$0.1 + 0.25 + 0.3 + 0.15 + d = 1$

$0.8 + d = 1$

$d = 1 - 0.8$

$d = 0.2$

(b)

$P(-3 \leq W < 0) = P(W = -3) + P(W = -2) + P(W = -1)$

$= 0.1 + 0.25 + 0.3 = 0.65$

(c)

P (W > -1) = P (W ≥ 0)

= P (W = 0) + P (W = 1)

= 0.15 + 0.2 = 0.35

(d)

P (-1 < W < 1) = P (W = 0)

= 0.15

Question No.4(10 Marks):

The total number of hours, measured in units of 100 hours, that a family runs a vacuum cleaner over a period of one year is a continuous random variable X that has the density function

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ 2 - x, & 1 \le x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the probability that over a period of one year, a family runs their vacuum cleaner
(a) less than 120 hours;
(b) between 50 and 100 hours.

Solution:

(a)

P (X < (120/100)) = P (X < 1.2)

P (X < 1.2) = $\int_0^{1.2} f(x)dx$

= $\int_0^1 f(x)dx$ + $\int_1^{1.2} f(x)dx$

= $\int_0^1 x\ dx$ + $\int_1^{1.2} (2 - x)dx$

= $(x^2/2) \Big|_0^1 + (2x - (x^2/2)) \Big|_1^{1.2}$

$= ((1/2) - 0) + (2x1.2 - ((1.2)^2/2) – 2x1 + ((1)^2/2))$

$= 1/2 + 2.4 – 1.44/2 – 2 + 1/2$

$= 0.68$

(b)

$P ((50/100) < X < (100/100)) = P (0.5 < X < 1)$

$P (0.5 < X < 1) = \int_{0.5}^{1} f(x)dx$

$= \int_{0.5}^{1} x\ dx$

$= (x^2/ 2) \Big|_{0.5}^{1}$

$= (1)^2/2 – (0.5)^2/2$

$= 1/2 - 1/8$

$= 0.375$

<u>Question No.5(10 Marks):</u>

Suppose a certain type of small data processing firm is so specialized that some have difficulty making a profit in their first year of operation. The probability density function that characterizes the proportion Y that makes a profit is given by

$$f(y) = \begin{cases} ky^4(1-y)^3, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What is the value of k that renders the above a valid density function?
(b) Find the probability that at most 50% of the firms make a profit in the first year.
(c) Find the probability that at least 80% of the firms make a profit in the first year.

<u>Solution:</u>

(a) If f is a valid density function it must satisfy $\int_{-\infty}^{\infty} f(y)dy = 1$. By using that we may find out k:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(y)dy \\
&= \int_0^1 ky^4(1-y)^3 dy \\
&= k\int_0^1 y^4(1-3y+3y^2-y^3)dy \\
&= k\int_0^1 (y^4-3y^5+3y^6-y^7)dy \\
&= k\left(\frac{1}{5}y^5 - \frac{1}{2}y^6 + \frac{3}{7}y^7 - \frac{1}{8}y^8\right)\Big|_0^1 \\
&= k(\frac{1}{5} - \frac{1}{2} + \frac{3}{7} - \frac{1}{8}) \\
&= \mathbf{\frac{k}{280}}
\end{aligned}$$

Therefore,

$$\boxed{k = 280}$$

(b) We need to find the probability $P(Y \leq 0.5)$.

$$\begin{aligned}
P(Y < 0.5) &= P(0 \leq Y \leq 0.5) \\
&= \int_0^{0.5} 280y^4(1-y)^3 dy \\
&= 280\int_0^{0.5} y^4(1-3y+3y^2-y^3)dy \\
&= 280\int_0^{0.5} (y^4-3y^5+3y^6-y^7)dy \\
&= 280\left(\frac{1}{5}y^5 - \frac{1}{2}y^6 + \frac{3}{7}y^7 - \frac{1}{8}y^8\right)\Big|_0^{0.5} \\
&= 280(\frac{1}{5}\cdot 0.5^5 - \frac{1}{2}\cdot 0.5^6 + \frac{3}{7}\cdot 0.5^7 - \frac{1}{8}\cdot 0.5^8) \\
&= \boxed{\frac{93}{256}}
\end{aligned}$$

(c) We need to find the probability $P(Y \geq 0.8)$.

$$\begin{aligned}
P(Y \geq 0.8) &= 1 - P(Y < 0.8) \\
&= 1 - P(0 \leq Y \leq 0.8) \\
&= 1 - \int_0^{0.8} 280y^4(1-y)^3 dy \\
&= 1 - 280\int_0^{0.8} y^4(1 - 3y + 3y^2 - y^3)dy \\
&= 1 - 280\int_0^{0.8} (y^4 - 3y^5 + 3y^6 - y^7)dy \\
&= 1 - 280\left(\frac{1}{5}y^5 - \frac{1}{2}y^6 + \frac{3}{7}y^7 - \frac{1}{8}y^8\right)\Big|_0^{0.8} \\
&= 1 - 280(\frac{1}{5}\cdot 0.8^5 - \frac{1}{2}\cdot 0.8^6 + \frac{3}{7}\cdot 0.8^7 - \frac{1}{8}\cdot 0.8^8) \\
&= \boxed{0.0563}
\end{aligned}$$

Question No.6(15 Marks):

Measurements of scientific systems are always subject to variation, some more than others. There are many strctures for measurement error, and statisticians spend a great deal of time modeling these errors. Suppose the measurement error X of a certain physical quantity is decided by the density function

$$f(x) = \begin{cases} k(3 - x^2), & -1 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Determine k that renders $f(x)$ a valid density function.
(b) Find the probability that a random error in measurement is less than 1/2.
(c) For this particular measurement, it is undesirable if the *magnitude* of the error (i.e., $|x|$) exceeds 0.8. What is the probability that this occurs?

Solution:

Given $f(x) = \begin{cases} k(3-x^2) & -1 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$

a) Given the $f(x)$ is a valid function,

$$\int f(x)\,dx = 1.$$

$$\int_{-1}^{1} k(3-x^2)\,dx = 1$$

$$k\left(3x - x^3/3\right)\Big|_{-1}^{1} = 1$$

$$k\left[\left(3-\frac{1}{3}\right) - \left(-3+\frac{1}{3}\right)\right] = 1.$$

$$k = \frac{3}{16}$$

b) $P(X<0.5) = \int_{-1}^{0.5} \frac{3}{16}(3-x^2)\,dx$

$$P(X<0.5) = \frac{3}{16}\left[3x - \frac{x^3}{3}\right]_{-1}^{0.5} = \frac{99}{128}$$

c) Magnitude of error, $|x| > 0.8$ implies

$P(|X| > 0.8) = 1 - P(-0.8 \le x \le 0.8)$.

$$p(|x| > 0.8) = 1 - \frac{3}{16}\int_{-0.8}^{0.8} (3-x^2)\,dx$$

$$p(|x| > 8) = 1 - \frac{3}{16}\left[3x - \frac{x^3}{3}\right]_{-0.8}^{0.8}$$

$$p(|x| > 8) = 1 - 0.836 = \underline{0.164}$$

Question No.7:

(i)(15 Marks)

Let X denote the amount of space occupied by an article placed in a 1-ft3 packing container. The pdf of X is

$$f(x) = \begin{cases} 90x^8(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Graph the pdf. Then obtain the cdf of X and graph it.

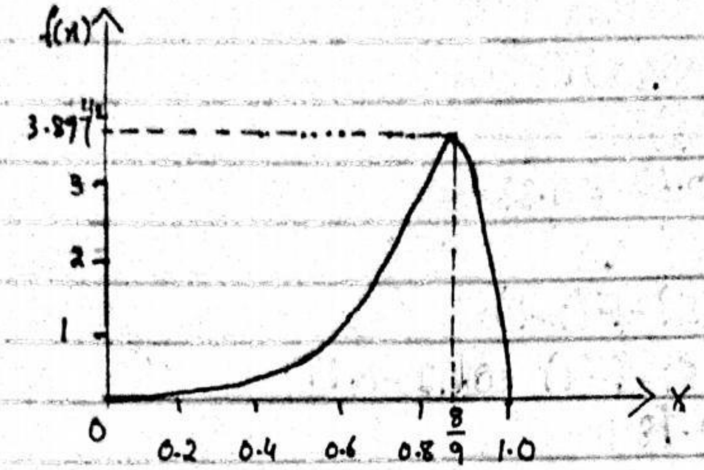
(b) What is P (X ≤ 0.5)?

(c) Using part (a), What is P (0.25 < X ≤ 0.5)? What is P (0.25 ≤ X ≤ 0.5)?

Solution:

(a) pdf $f(x)$ is given to us as

$$f(x) = \begin{cases} 90 x^8 (1-x) & 0 < x < 1 \\ 0 & \text{otherwis.} \end{cases}$$



$$F(x) = \int_0^x 90 y^8 (1-y) \cdot dy$$

$$= 90 \int_0^x y^8 \, dy - 90 \int_0^x y^9 \, dy$$

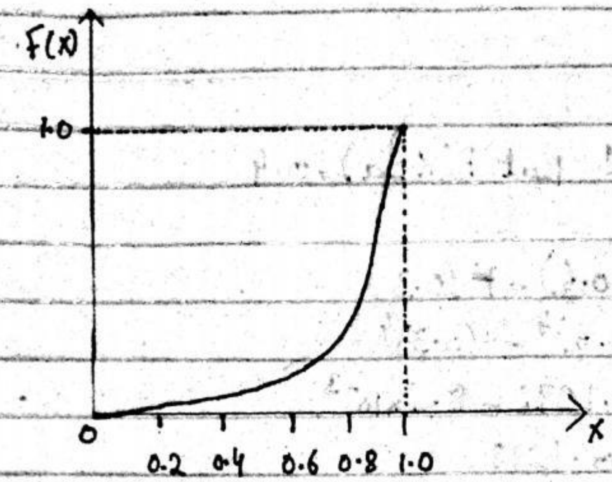
$$= 90 \left[\frac{y^9}{9} \right]_0^x - 90 \left[\frac{y^{10}}{10} \right]_0^x$$

$$= \frac{\cancel{90}^{10}}{\cancel{9}} \left[(x)^9 - (0)^9 \right] - \frac{\cancel{90}^{9}}{\cancel{10}} \left[(x)^{10} - (0)^{10} \right]$$

$$= 10x^9 - 9x^{10}$$

$$= x^9 (10 - 9x)$$

$$F(x) = \begin{cases} 0 & x \leq 0 \\ x^9 (10 - 9x) & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$



(b) Using the given cdf from part (a), we can write:

$$P(X \le 0.5) = F(0.5)$$
$$= (0.5)^9(10 - 9(0.5))$$
$$= (0.5)^9(5.5)$$
$$= 0.01074$$

(c) $P(0.25 < X < 0.5) = F(0.5) - F(0.25)$

$$= [(0.5)^9(10 - 9(0.5))] - [(0.25)^9\{10 - 9(0.25)]$$
$$= (0.01074) - (0.00003)$$
$$= 0.01071$$

$$P(0.25 \le X \le 0.5) = P(0.25 < X < 0.5)$$
$$= 0.01071$$

(ii)(10 Marks)

The random variable X has cumulative distributive function

$$F(x)=\begin{cases}0 & x<0\\ x^4 & 0\leqslant x\leqslant 1\\ 1 & x\geqslant 1\end{cases}$$

Find

(a) P (0.3 < X < 0.6),

(c) the value of a such that P (X > a) = 0.4

Solution:

(a) $P(0.3 < X < 0.6) = F(0.6) - F(0.3)$

$= (0.6)^4 - (0.3)^4$

$= 0.1296 - 8.1\times10^{-3}$

$= 0.1215$

(b)

$P(X > a) = 0.4$

$P(X > a) = 1 - P(X \leq a)$

$1 - P(X \leq a) = 0.4$

$P(X \leq a) = 1 - 0.4$

$P(X \leq a) = F(a)$

$F(a) = 1 - 0.4$

$a^4 = 0.6$

$a = 0.88$