Course Name:Linear Algebra

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Eigenvectors and Linear Transformations

- Recall the definition of similar matrices:
  Let A and C be $n \times n$ matrices. We say that A **is similar to** C in case $A = PCP^{-1}$ for some invertible matrix P.

- A square matrix A is **diagonalizable** if A is similar to a diagonal matrix D.

- An important idea of this section is to see that the mappings

$$\mathbf{x} \mapsto A\mathbf{x} \text{ and } \mathbf{w} \mapsto D\mathbf{w}$$

  are essentially the same when viewed from the proper perspective. Of course, this is a huge breakthrough since the mapping $\mathbf{w} \mapsto D\mathbf{w}$ is quite simple and easy to understand. In some cases, we may have to settle for a matrix C which is simple, but not diagonal.

Recall

Similarity Invariants for Similar Matrices A and C

Property	*Description*
Determinant	A and C have the same determinant
Invertibility	A is invertible <=> C is invertible
Rank	A and C have the same rank
Nullity	A and C have the same nullity
Trace	A and C have the same trace
Characteristic Polynomial	A and C have the same char. polynomial

Recall

Characteristic Polynomial	A and C have the same char. polynomial
Eigenvalues	A and C have the same eigenvalues
Eigenspace dimension	If λ is an eigenvalue of A and C, then the eigenspace of A corresponding to λ and the eigenspace of C corresponding to λ have the same dimension.

Exercise 5.4

Eigenvectors and Linear transformations.

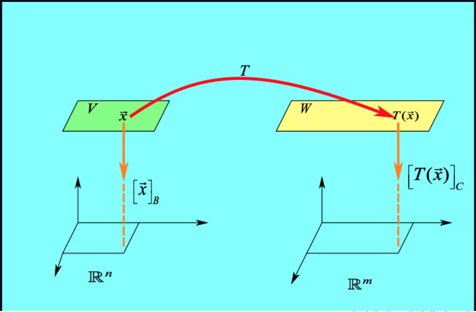
How is the factorization $A = PDP^{-1}$ related to linear transformations?

Any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be implemented by left-multiplication by a matrix A, the standard matrix of T.

We want to extend this kind of representation to any linear transformation between two finite-dimensional vector spaces.

Suppose V is an n-dimensional vector space and W is an m-dimensional vector space. Let $T : V \to W$ be a linear transformation. Suppose further that B and C are ordered bases for V and W respectively.

For any $\vec{x} \in V$, the coordinate vector $[\vec{x}]_B$ is in $\mathbb{R}^n$ and the coordinate vector of its image, $[T(\vec{x})]_C$, is in $\mathbb{R}^m$.



Let $\left[\vec{b}_1, \quad \ldots, \quad \vec{b}_n\right]$ be the basis B for V. If $\vec{x} = r_1\vec{b}_1 + \cdots + r_n\vec{b}_n$, then

$$[\vec{x}]_B = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

and, since T is linear,

$$T(\vec{x}) = T(r_1\vec{b}_1 + \cdots + r_n\vec{b}_n) = r_1T(\vec{b}_1) + \cdots + r_nT(\vec{b}_n) \qquad (*)$$

Now using the basis C in W, we can write (*) in terms of C-coordinate vectors:

$$\left[T(\vec{x})\right]_C = r_1\left[T(\vec{b}_1)\right]_C + \cdots + r_n\left[T(\vec{b}_n)\right]_C \qquad (**)$$

But since C-coordinate vectors are in $\mathbb{R}^m$, the vector equation

$$\left[T(\vec{x})\right]_C = r_1\left[T(\vec{b}_1)\right]_C + \cdots + r_n\left[T(\vec{b}_n)\right]_C$$

can be written as a matrix equation

$$\left[T(\vec{x})\right]_C = M\left[\vec{x}\right]_B$$

with $$M = \left[\left[T(\vec{b}_1)\right]_C \quad \left[T(\vec{b}_2)\right]_C \quad \cdots \quad \left[T(\vec{b}_n)\right]_C\right]$$

The matrix M is a matrix representation of T called the *matrix for T relative to the bases B and C.*

$$
\begin{tikzcd}
\vec{x} \arrow[r, "T"] \arrow[d] & T(\vec{x}) \arrow[d] \\
{[\vec{x}]_B} \arrow[r, "M\,[\vec{x}]_B"] & {[T(\vec{x})]_C}
\end{tikzcd}
$$

$$[T(\vec{x})]_C = M[\vec{x}]_B$$

$$M = \left[[T(\vec{b}_1)]_C \quad [T(\vec{b}_2)]_C \quad \cdots \quad [T(\vec{b}_n)]_C \right]$$

Example: Suppose $B=\{\vec{b}_1,\vec{b}_2\}$ is a basis for V and $C=\{\vec{c}_1,\vec{c}_2,\vec{c}_3\}$ is a basis for W. Let $T:V\to W$ be a linear transformation with the property that $T(\vec{b}_1)=3\vec{c}_1-2\vec{c}_2+5\vec{c}_3$ and $T(\vec{b}_2)=4\vec{c}_1+7\vec{c}_2-\vec{c}_3$ Find the matrix M for T relative to B and C.

The C-coordinate vectors for the images of $\vec{b}_1$ and $\vec{b}_2$ are

$$\left[T(\vec{b}_1)\right]_C=\begin{bmatrix}3\\-2\\5\end{bmatrix} \text{ and } \left[T(\vec{b}_2)\right]_C=\begin{bmatrix}4\\7\\-1\end{bmatrix}$$

Then the matrix $M=\begin{bmatrix}3&4\\-2&7\\5&-1\end{bmatrix}$

Linear transformations on $\mathbb{R}^n$.

Theorem. Diagonal matrix representation.
Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix.

If B is a basis for $\mathbb{R}^n$ formed from the columns of P,

then D is the B-matrix for the transformation $\vec{x} \mapsto A\vec{x}$.

Linear transformations $T : V \to V$, from a vector space V into itself.

When the domain and codomain of T are the same, that is when $T : V \to W$ and W is the same as V and basis C is the same as B, the matrix

$$M = \left[\left[T(\vec{b}_1)\right]_C \quad \left[T(\vec{b}_2)\right]_C \quad \cdots \quad \left[T(\vec{b}_n)\right]_C \right]$$

is called the B-matrix for T, written $\left[T\right]_B$.

The B-matrix for $T : V \to V$ satisfies the equation

$$\left[T(\vec{x})\right]_B = \left[T\right]_B \left[\vec{x}\right]_B, \quad \forall \vec{x} \in V.$$

Exercise 5.5

Recall that the characteristic equation for an $n \times n$ matrix is a polynomial of degree n. As such it always has exactly n roots.

So far we have been considering 2×2 matrices with 2 real and distinct roots. In this case the characteristic equation is

$$\lambda^2 - T\lambda + D = 0,$$

where T is the trace of the matrix, and D is the determinant. We have the roots

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} \text{ with } T^2 - 4D > 0$$

Now we will consider the case where $T^2 - 4D < 0$.

In this case we'll need complex numbers to determine eigenvalues and eigenvectors.

Complex numbers - quick review

Why do we have complex numbers?

Consider the equation $x^2+1=0$. Solving for x we get

$$x^2=-1 \text{ and so } x=\pm\sqrt{-1}$$

Somehow we need to make sense of $\sqrt{-1}$.

We have a poor mathematical system if it can't handle the simple equation $x^2+1=0$.

Clearly, the symbol $\sqrt{-1}$ does not represent a real number. Long ago $\sqrt{-1}$ was called an "imaginary" number and the name, unfortunately, still sticks today.

We need a number system that will include things like $\sqrt{-1}$. This will turn out to be the *complex number system* $\mathbb{C}^n$.

Euler will help us here; he wanted to make use of these numbers and so he began by setting $i = \sqrt{-1}$.

We then have

$$i^2 = \left(\sqrt{-1}\right)^2 = -1, \qquad i^3 = i \cdot i^2 = (-1) \cdot i = -i,$$

$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1,\ i^5 = i^2 \cdot i^2 \cdot i = (-1)(-1)i = i \text{ and so on.}$$

In general a complex number is written in the form

$z = a + bi$, where a and b are real numbers, $i = \sqrt{-1}$.

a is called the *real part* of z, written $\operatorname{Re}(z)$,

and

b is called the *imaginary part* of z, written $\operatorname{Im}(z)$.

If $b = 0$, we have no imaginary part and the resulting z is real. If $a = 0$, there is no real part and so z is a purely imaginary number.

Eample: Find the eigenvalues of the matrix

$A = \begin{bmatrix} 0.50 & -0.60 \\ 0.75 & 1.10 \end{bmatrix}$ and find a basis for each eigenspace.

The characteristic equation of A is $Det\left(\begin{bmatrix} 0.50 & -0.60 \\ 0.75 & 1.10 \end{bmatrix}\right) = 0$

$$= (0.50 - \lambda)(1.10 - \lambda) - (-0.60 - \lambda)(0.75)$$

$$= \lambda^2 - 1.60\lambda + 1.00 = 0$$

So $\lambda = \dfrac{1.60 \pm \sqrt{(-1.60)^2 - 4}}{2} = \dfrac{1.60 \pm i\sqrt{4 - 2.56}}{2} = 0.80 \pm 0.60i$

Consider the eigenvalue $\lambda = 0.80 - 0.60i$

For $\lambda_1 = 0.80 - 0.60i$, $A - \lambda_1 I = A - (0.80 - 0.60i)I$

$$= \begin{bmatrix} 0.50 & -0.60 \\ 0.75 & 1.10 \end{bmatrix} - \begin{bmatrix} 0.80 - 0.60i & 0.00 \\ 0.00 & 0.80 - 0.60i \end{bmatrix}$$

$$= \begin{bmatrix} -0.30 + 0.60i & -0.60 \\ 0.75 & 0.30 + 0.60i \end{bmatrix}$$

Row reduction by hand is unpleasant however, since $0.80-0.60i$ is an eigenvalue, the system

$$\begin{aligned} (-0.30+0.60i)x_1 - 0.60x_2 &= 0.00 \\ 0.75x_1 + (0.30+0.60i)x_2 &= 0.00 \end{aligned} \qquad (*)$$

has a nontrivial solution. So both of the equations in (*) must determine the *same relationship* between x_1 and x_2. So either equation can be used to express one variable in terms of the other. *The equations are identical.*

Consider the second equation in (*):

$$0.75x_1 + (0.30 + 0.60i)x_2 = 0$$

$$0.75x_1 = -(0.30 + 0.60i)x_2 \Rightarrow x_1 = (-0.40 - 0.80i)x_2$$

To eliminate the decimals, pick $x_2 = 5$. Then $x_1 = -2 - 4i$

A basis for the eigenspace corresponding to $\lambda = 0.80 - 0.60i$

is $\vec{v}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} i$

Similar calculations for the eigenvalue $\lambda_2 = 0.80 + 0.60i$ produce the eigenvector

$$\vec{v}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} i.$$

Notice that the eigenvalues λ_1 and λ_2 are a conjugate pair;

$$\lambda_1 = 0.80 - 0.60i \qquad \lambda_2 = 0.80 + 0.60i$$

They're corresponding eigenvectors are a conjugate pair as well:

$\vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}$. This will always be the case.

A basis for the eigenspace corresponding to $\lambda = 0.80 + 0.60i$

is $\vec{v}_2 = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} i$.

Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

and find a basis for each eigenspace.

Solution

Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ and find a basis for each eigenspace.

The characteristic polynomial of A is $\lambda^2 - 4\lambda + 5 = 0$

So the eigenvalues of A are $\lambda = \dfrac{4 \pm \sqrt{-4}}{2} = \dfrac{4 \pm i\sqrt{4}}{2} = 2 \pm i.$

Set $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i.$

For $\lambda_1 = 2+i$, $A-\lambda_1 I = A-(2+i)I = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix}$.

The equation $\left(A-(2+i)I\right)\vec{x} = \vec{0}$, where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, amounts to

$$x_1 + (1-i)x_2 = 0 \Rightarrow x_1 = -(1-i)x_2 \text{ with } x_2 \text{ free.}$$

A basis vector for the eigenspace is thus $\vec{v}_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$

The eigenvalue $\lambda_2 = 2 - i$, which is the conjugate of λ_1, has the basis vector $\vec{v}_2 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ for the eigenspace.

Conjugate eigenvalues have conjugate eigenvectors.