Course Name:Linear Algebra(MT 104)

Topic: Guassian Elimination Method (Echelon Form) vs Elimination Method/ Substitution

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Recall if you have a system of linear equations

$$\begin{aligned} 5x + 4y - z &= 0 \\ 10y - 3z &= 11 \\ z &= 1 \end{aligned}$$

It is so easy to solve (why??)

Now if you have the following system

$$\begin{aligned}5x+4y-z &= 4\\ x+2y-z &= 2\\ 3x+2y+z &= 2\end{aligned}$$

what could be first step eqn(1)-5 eqn(2), to get following system

$$\begin{aligned}3y-2z &= 3\\ x+2y-z &= 2\\ 3x+2y+z &= 2\end{aligned}$$

$$-6y+4z=-6$$

Second step eqn(3)-3eqn(2), to get following system

$$\begin{aligned}3y-2z &= 3\\ x+2y-z &= 2\\ y-z &= 1\end{aligned}$$

$$-4y+4z=4$$

Third step eqn(1)-3eqn(3), to get following system

$$\begin{aligned} z &= 0 \\ x + 2y - z &= 2 \\ y - z &= 1 \end{aligned}$$

we can rewrite them as

$$\begin{aligned} x + 2y - z &= 2 \\ y - z &= 1 \\ z &= 0 \end{aligned}$$

Hence, we obtained a triangular form that is easy to solve by using back substitution

$$\begin{aligned} x &= 0 \\ y &= 1 \\ z &= 0 \end{aligned}$$

Another way to look into previous system is

$$\begin{bmatrix} 5x+4y-z=4 \\ x+2y-z=2 \\ 3x+2y+z=2 \end{bmatrix} \xrightarrow{r_1-5r_2} \begin{bmatrix} 3y-2z=3 \\ x+2y-z=2 \\ 3x+2y+z=2 \end{bmatrix} \xrightarrow{r_3-3r_2} \begin{bmatrix} 3y-2z=3 \\ x+2y-z=2 \\ y-z=1 \end{bmatrix}$$

$$\xrightarrow{r_1-3r_3} \begin{bmatrix} z=0 \\ x+2y-z=2 \\ y-z=1 \end{bmatrix}$$

2 Rearranging the equations

$$\begin{bmatrix} x+2y-z=2 \\ y-z=1 \\ z=0 \end{bmatrix}$$

Steps we did while solving system of linear equations

- Multiplying an equation by non-zero constant.
- Adding a multiple of one equation to another equation.
- Interchanging two equations.

In this lecture we will show how these three operations can be performed by using the matrix representation of the linear system and we will show that they give rise to equivalent systems.

In previous Lecture we have discussed that

- A linear system of m equations in n unknowns $x_1, x_2, \dots, x_n$ is a set of equations of the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad . \qquad\quad . \qquad\qquad\quad . \qquad . \\ \quad . \qquad\quad . \qquad\qquad\quad . \qquad . \\ \quad . \qquad\quad . \qquad\qquad\quad . \qquad . \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \dots\dots\dots(1)$$

$a_{11}, \dots, a_{mn}$ are called the coefficients of the system.

Matrix Notation of a Linear System

We can write system (1) in the form of matrices as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$A=$ Matrix of coefficients

$X=$ Matrix of Unknowns

$b=$ Matrix of Constants

Augmented Matrix

Then System (1) can be written as

$$AX = b$$

The matrix

$$[A \quad b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

is called **Augmented Matrix** of the system (1).

Example

Write the augmented matrix for the system of equations:

$$3x_1 + 5x_2 - x_3 = 10$$
$$x_1 + 4x_2 + x_3 = 7$$
$$9x_1 + 2x_3 = 1$$

Solution

There are three variables, and so we will need a column for each. Be careful – notice that the last equation doesn't have an x_2. That will be represented with a 0.

$$\begin{array}{cl} \begin{array}{cccc} x_1 & x_2 & x_3 & \text{the constants} \end{array} & \\ \left[\begin{array}{ccc|c} 3 & 5 & -1 & 10 \\ 1 & 4 & 1 & 7 \\ 9 & 0 & 2 & 1 \end{array}\right] & \begin{array}{l} \text{from first equation} \\ \text{from second equation} \\ \text{from third equation} \end{array} \end{array}$$

What is Row Echelon Form?

A matrix is in row echelon form if it meets the following requirements:

- The first non-zero number from the left (the "leading coefficient") is always to the right of the first non-zero number in the row above.
- Rows consisting of all zeros are at the bottom of the matrix.

$$\begin{bmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 2 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & a_6 \end{bmatrix}$$

Row echelon form. "a" can represent any number.

What is Reduced Row Echelon Form?

Reduced row echelon form is a type of matrix used to solve systems of linear equations. Reduced row echelon form has four requirements:

- The first non-zero number in the first row (**the leading entry**) is the number 1.
- The second row also starts with the number 1, which is further to the right than the leading entry in the first row. For every subsequent row, the number 1 must be further to the right.
- The leading entry in each row must be the only non-zero number in its column.
- Any non-zero rows are placed at the bottom of the matrix.

$$\begin{bmatrix} 1 & 0 & a_1 & 0 & b_1 \\ 0 & 1 & a_2 & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \end{bmatrix}$$

A 3×5 matrix in reduced row echelon form.

Row Reduced Echelon Form (Guassian Elimination Method)

Example(Unique Solution). System:

$$\begin{aligned} x + 2y + z &= 0 \\ x + z &= 4 \\ x + y + 2z &= 1. \end{aligned}$$

We will use this example to illustrate the general method (called *Gaussian elimination*.

General idea: replace the set of equations with an equivalent set of equations (i.e., having the same solutions set) but from which the set of solutions is evident. We find equivalent sets of equations by using the following:

Row operations.

1. multiply an equation by a nonzero scalar
2. swap two equations
3. add a multiple of one equation to another.

The good news is that these operations are all we need to solve any system of linear equations.

We first introduce a convenient way of notating our system:

$$\begin{aligned} x + 2y + z &= 0 \\ x \phantom{{}+2y} + z &= 4 \\ x + y + 2z &= 1. \end{aligned} \quad \rightsquigarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 2 & 1 \end{array}\right).$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 2 & 1 \end{array}\right) \xrightarrow[r_3 \to r_3 - r_1]{r_2 \to r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 0 & 4 \\ 0 & -1 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{r_2 \to -\frac{1}{2} r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow{r_3 \to r_3 + r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{r_1 \to -2r_2 + r_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{r_1 \to -r_3 + r_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

The last augmented matrix is in *reduced echelon form*. We will define this term carefully later.

Translate the lasted augmented matrix back into a system of equations to get a system that is equivalent to the original system, but from which the set of solutions is evident:

$$\begin{aligned} x &= 5 \\ y &= -2 \\ z &= -1. \end{aligned}$$

So there is a unique solution in this case. Now for the most **important step**: check your solution works for the original system:

$$\begin{aligned} 5 + 2(-2) + (-1) &= 0 \\ 5 \qquad\qquad + (-1) &= 4 \\ 5 + (-2) + 2(-1) &= 1. \end{aligned}$$

That works.

Example(No Solution)

The following system is a slight modification of the previous one:

$$\begin{aligned} x + 2y + z &= 0 \\ x \qquad + z &= 4 \\ x + \ \ y + z &= 1. \end{aligned}$$

Converting to the corresponding augmented matrix and performing a sequence of row operations similar to those in the previous example gives

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -1 \end{array}\right).$$

Converting back to equations gives the equivalent system:

$$\begin{aligned} x + z &= 4 \\ y &= -2 \\ 0 &= -1 \end{aligned}$$

which clearly has no solutions. Thus, our original system has no solutions.

Example(Infinite Many Solution)

Consider the System:

$$\begin{aligned} x + 2y + z &= 0 \\ x \qquad + z &= 4 \\ x + \ \ y + z &= 2. \end{aligned}$$

Converting to the corresponding augmented matrix and performing a sequence of row operations similar to those in the previous example gives

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Equivalent system:

$$\begin{aligned} x + z &= 4 \\ y &= -2 \\ 0 &= 0 \end{aligned}$$

which clearly has no solutions. Thus, our original system has no solutions. We now get an infinite set of solutions:

$$\{(x, y, z) : x + z = 4 \text{ and } y = -2\} = \{(x, -2, 4 - x) : x \in \mathbb{R}\}.$$

This is a line in 3-space.