LECTURE: 28:

2.5: Solutions by Substitution: Continuation

③ Reduction to Seperation of Variables:-

A differential Equation of the form

$$\frac{dy}{dx} = f(Ax + By + C) \quad - (1)$$

Can always be reduced to an equation with seperable Variables by means of Substitution.

Substitution:

$$\left.\begin{aligned} u &= Ax + By + C, \quad B \neq 0 \\ \frac{du}{dx} &= A + B\frac{dy}{dx}. \end{aligned}\right\}$$

By this Substitution Equation of form (1) reduced into Seperable D.E.

Example: Solve $\frac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0. \quad - (2)$

Solu:- Let $u = -2x + y$

$$\frac{du}{dx} = -2 + \frac{dy}{dx}$$

we get

$$\frac{dy}{dx} = \frac{du}{dx} + 2$$

Equ (2) implies

$$\frac{dy}{dx} = (-2x+7)^2 - 7$$

$$\frac{du}{dx} + 2 = u^2 - 7$$

$$\frac{du}{dx} = u^2 - 9$$

$$\frac{1}{u^2-9}\, du = dx \quad \rightarrow \text{ we get Seperable D.E.}$$

$$\frac{1}{(u-3)(u+3)}\, du = dx$$

Integrating Both sides

$$\int \frac{1}{(u-3)(u+3)}\, du = \int dx .$$

By Partial fraction, we can write.

$$\frac{1}{6}\int \left[\frac{1}{u-3} - \frac{1}{u+3}\right] du = \int dx .$$

$$\frac{1}{6}\left[\ln(u-3) - \ln(u+3)\right] = x + C_1$$

$$\frac{1}{6}\ln\left|\frac{u-3}{u+3}\right| = x + C_1$$

$$\ln\left|\frac{u-3}{u+3}\right| = 6x + 6C_1$$

$$= 6x + C_2$$

By Partial fraction

$$\frac{1}{(u-3)(u+3)} = \frac{A}{u-3} + \frac{B}{u+3}$$

$$1 = A(u+3) + B(u-3)$$

For $u = 3$,

$$1 = 6A \Rightarrow A = 1/6$$

For $u = -3$,

$$1 = -6B \Rightarrow B = -1/6$$

$$\frac{1}{(u-3)(u+3)} = \frac{1/6}{u-3} + \frac{-1/6}{u+3}$$

$$= \frac{1}{6}\left[\frac{1}{u-3} - \frac{1}{u+3}\right]$$

Taking exponential on Both sides, we get

$$\frac{u-3}{u+3} = e^{6x+c_2}$$

$$= e^{6x} \cdot e^{c_2}$$

$$= ce^{6x}$$

Solving in terms of u

$$u-3 = (u+3)ce^{6x}$$

$$u = (u+3)ce^{6x} + 3$$

$$u = uce^{6x} + 3ce^{6x} + 3$$

$$u(1-ce^{6x}) = 3(1+ce^{6x})$$

$$u = \frac{3(1+ce^{6x})}{1-ce^{6x}}$$

Resubstitute value of u,

$$-2x+y = \frac{3(1+ce^{6x})}{1-ce^{6x}}.$$

$$y = 2x + \frac{3(1+ce^{6x})}{1-ce^{6x}}.$$

Apply I.C $y(0) = 0$

$$y = 2x + \frac{3(1+ce^{6x})}{1-ce^{6x}}$$

$$y(0) = 2(0) + \frac{3(1+ce^{0})}{1-ce^{0}}$$

$$0 = \frac{3(1+c)}{1-c}$$

$$1+c = 0$$

$$\boxed{c = -1}$$

we get

$$\boxed{y = -2x + \frac{3(1-e^{6x})}{1+e^{6x}}}$$

we get Explicit Solution of D.E (2).

Practice Problems:-

1) $\frac{dy}{dx} = \frac{1-x-y}{x+y}$ (Hint: Let $u = x+y$)

2) $\frac{dy}{dx} = 1 + e^{y-x+5}$ (Hint: Let $u = y-x+5$)

Related Exercise Questions:- Ex : 2.5

Q : 23 – 30 .