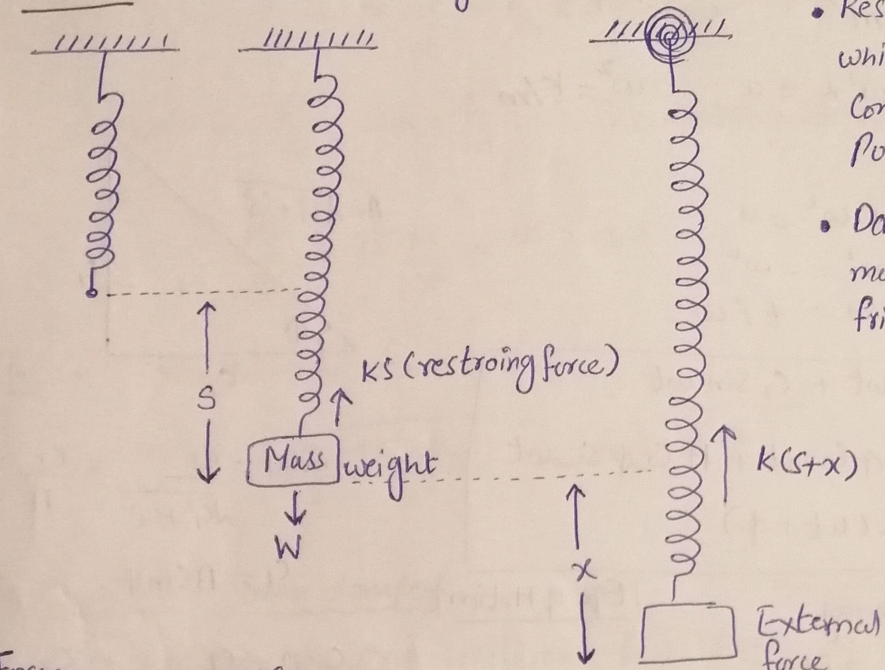
Chapter: 05:- Linear Models: Initial Value Problem.

In this Chapter, we ~~discuss~~ Construct/Consider a Second order differential Equation, with Constant Coefficients.

Problem:- Consider a Spring



- Restoring force is the force which tends the body to come back to its equilibrium Position on either sides.
- Damping force acts against the motion (Air resistive force)/ frictional force of Medium. Air/water

Forces $W \propto S$

- ↓ weight: $W = kS$
- ↑ Restoring force: $F_s \propto (S+x)$

  $F_s = k(S+x)$.
- ↑ Damping force: $F_d \propto dx/dt$

  $F_d \propto \beta \frac{dx}{dt}$.
- ↓ External force: (Applied force) $F(t)$.

According to Supper Position Principal, Sum of all forces equals $F = ma$.

$$F = W - F_s \mp F_d + F(t)$$

$$ma = ks - k(S+x) - \beta\frac{dx}{dt} + F(t)$$

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + F(t).$$

$$\boxed{m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx + F(t) = 0}$$

2nd order differential Equation.

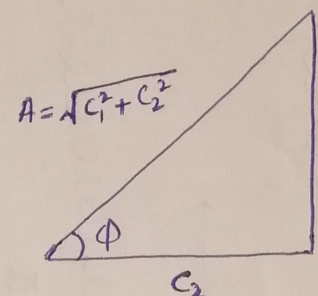
2nd order, Linear, ODE.

Case I:- Free Undamped Motion (Ideal Systems) / Simple Harmonic Motion

$$m\frac{d^2x}{dt^2} + kx = 0.$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0; \quad m \neq 0.$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0; \quad \omega^2 = k/m$$



A.E:- $m^2 + \omega^2 = 0.$

$m^2 = -\omega^2$

$m = \pm i\omega.$

$$\boxed{\begin{aligned} x(t) &= C_1 \cos\omega t + C_2 \sin\omega t \\ &= A\sin\phi\cos\omega t + A\cos\phi\sin\omega t \\ &= A\sin(\omega t + \phi). \end{aligned}}$$

| Equ of Motion |

$\sin\phi = \frac{C_1}{\sqrt{C_1^2 + C_2^2}} = \frac{C_1}{A}$

$C_1 = A\sin\phi.$

$\cos\phi = \frac{C_2}{\sqrt{C_1^2 + C_2^2}} = \frac{C_2}{A}.$

$C_2 = A\cos\phi.$

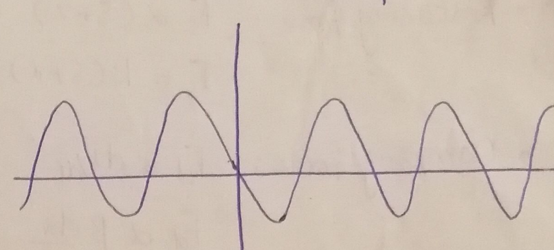
ω = ~~A~~ angular frequency

T = Time Period $= 2\pi/\omega$.

ϕ = Initial Phase / angle at $t = 0$.

A = ~~Amptitude~~ Amplitude

$f = 1/T$ = frequency of Motion, no. of cycles completed each second.



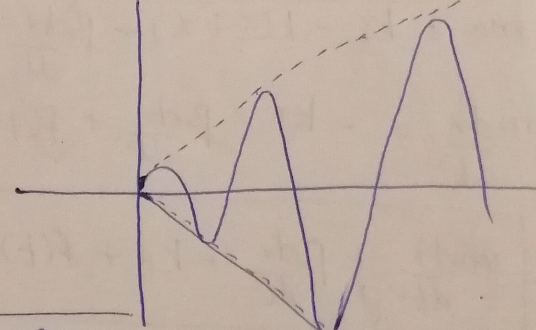
Case: II:- Forced Undamped Motion:-

$$m\frac{d^2x}{dt^2} + kx = F(t).$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{1}{m}F(t). \Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = F(t).$$

$$x(t) = C_1\cos\omega t + C_2\sin\omega t + x_p(t).$$

$$= A\sin(\omega t + \phi) + x_p(t).$$



$t \to \infty$

$x(t) \to \infty$.

{ Pure resonance occurs when external force equals natural resonance of the spring. internal frequency

This Phenomina is Pure Resonance.

Case III :- Free Damped Motion :-

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0.$$

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \,; \qquad m \neq 0.$$

we take $\omega^2 = \frac{k}{m}$, $2\lambda = \frac{\beta}{m}$; λ : damping factor.

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0.$$

A.E :- $m^2 + 2\lambda m + \omega^2 = 0.$

$$m = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

(i) $\lambda^2 - \omega^2 > 0$:- over damped :-

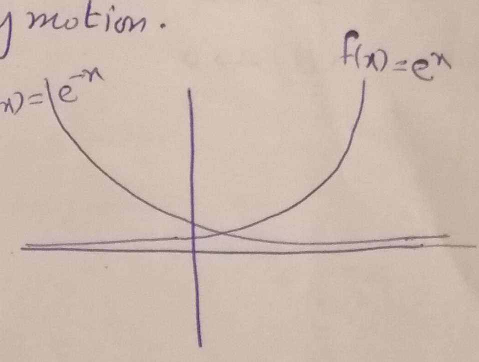
In this case damping coefficient β is large as compared to the spring constant k.

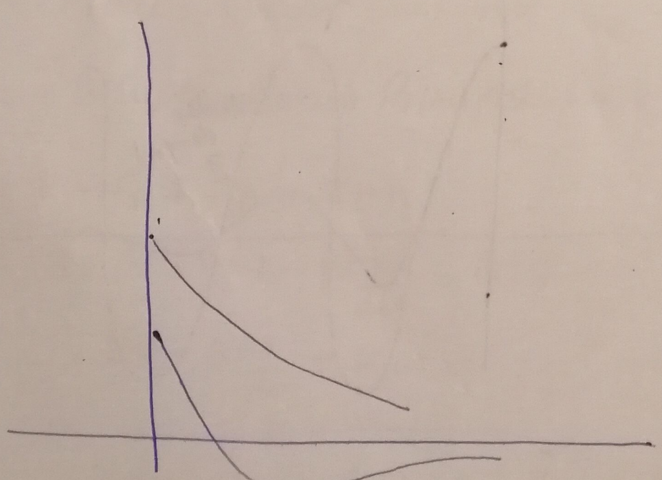
$$\left\{\begin{aligned} &\because\ \lambda^2 - \omega^2 > 0 \\ &\frac{\beta^2}{4m^2} - \frac{k^2}{m^2} > 0. \\ &\beta^2 > 4k^2. \\ &\beta > 2k \end{aligned}\right.$$

$$x(t) = c_1 e^{(-\lambda + \sqrt{\lambda^2 - \omega^2})t} + c_2 e^{(-\lambda - \sqrt{\lambda^2 - \omega^2})t}$$

$$= e^{-\lambda t}\left[c_1 e^{\sqrt{\lambda^2 - \omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2}\,t}\right]$$

This Equation shows a smooth and non oscillatory motion.

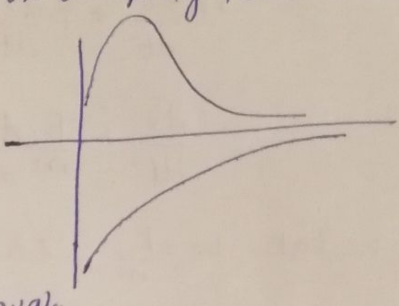




motion of an overdamped system.

ii) $\lambda^2-\omega^2=0$:- Critically damped :-

In this case Slight / Small Change / decrease in damping force would result in oscillatory motion.



$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

$$= e^{\lambda t}(c_1 + c_2 t)$$

It also seems that the mass can pass through the equilibrium position at most one time.

iii) $\lambda^2-\omega^2<0$:- Underdamped :-

In this case damping coefficient is small in comparism to the spring constant.

The roots m_1, m_2 are now complex

$$m_{1,2} = -\lambda \pm \sqrt{\omega^2-\lambda^2}\, i$$

$$x(t) = c_1 e^{-\lambda t} \cos\sqrt{\omega^2-\lambda^2}\, t + c_2 e^{-\lambda t} \sin\sqrt{\omega^2-\lambda^2}\, t.$$

$$= e^{-\lambda t}\left(c_1 \cos\sqrt{\omega^2-\lambda^2}\, t + c_2 \sin\sqrt{\omega^2-\lambda^2}\, t\right).$$

$t \to \infty$.

$x(t) \to 0$.