LECTURE: 31:

Section 4.1 : (Continuation)

Non-Homogeneous Equations:-

General Solution - Non Homogeneous Equations:-

Let y_p be any particular solution of the non-homogeneous linear n^{th} order D.E in an interval I and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of homogeneous D.E on I. The General Solution is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p$$
$$= y_c + y_p.$$

Ex:- Let $y_c = c_1 e^{2x} + c_2 e^{5x}$ is Complementary Solution of H.DE $y'' - 7y' + 10y = 0$ and let $y_p = 6e^x$ is particular Solution of Non H.DE $y'' - 7y' + 10y = 24e^x$.

Verify the $y = y_c + y_p = c_1 e^{2x} + c_2 e^{5x} + 6e^x$ is the General Solution of $y'' - 7y' + 10y = 24e^x$.

Sol: If $y = c_1 e^{2x} + c_2 e^{5x} + 6e^x$.

$$y' = 2c_1 e^{2x} + 5c_2 e^{5x} + 6e^x$$

$$y'' = 4c_1 e^{2x} + 25c_2 e^{5x} + 6e^x.$$

Put values in D.E.

$$y'' - 7y' + 10y = 4c_1 e^{2x} + 25c_2 e^{5x} + 6e^x - 7(2c_1 e^{2x} + 5c_2 e^{5x} + 6e^x)$$
$$+ 10(c_1 e^{2x} + c_2 e^{5x} + 6e^x).$$

$= (4c_1e^{2x} - 14c_1e^{2x} + 10c_1e^{2x}) - (25c_2e^{5x} - 35c_2e^{5x} + 10c_2e^{5x})$
$+ (6e^x - 42e^x + 60e^x)$

$= (14c_1 - 14c_1)e^{2x} + (35c_2 - 35c_2)e^{5x} + (66 - 42)e^x$

$= 24e^x$.

Super Position Principle - Non Homogeneous Equ's :-

Let $y_{P_1}, y_{P_2}, \cdots, y_{P_k}$ be k Particular Solutions of non-Homogeneous Linear nth order D.E on I Corresponding to k distinct functions $g_1, g_2, \cdots, g_k$. Suppose y_P be the Particular Solution of Corresponding D.E

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_i(x)$$

Then

$$y_P = y_{P_1}(x) + y_{P_2}(x) + \cdots + y_{P_k}(x).$$

is Particular Solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$

Ex:-

$y_{P_1} = -4x^2$ is a Particular Solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$

$y_{P_2} = e^{2x}$ " " " " " $y'' - 3y' + 4y = 2e^{2x}$

$y_{P_3} = xe^x$ " " " " $y'' - 3y' + 4y = 2xe^x - e^x$.

By superposition Principle

$$y_P = y_{P_1} + y_{P_2} + y_{P_3} = -4x^2 + e^{2x} + xe^x .$$

is a Solution of

$$y'' - 3y' + 4y = \underbrace{-16x^2 + 24x - 8}_{g_1(x)} + \underbrace{2e^{2x}}_{g_2(x)} + \underbrace{2xe^x - e^x}_{g_3(x)}$$

NOTE:- If y_{P_i} are the Particular Solutions of n^{th} order Non Homogeneous D.E's for $i = 1, 2, \dots k$ then the Linear Combination

$$y_P = c_1 y_{P_1} + c_2 y_{P_2} + \dots + c_k y_{P_k}$$

is also a Particular Solution, when R.H.S of Equ is the Linear Combination.

$$c_1 g_1(x) + c_2 g_2(x) + \dots + c_k g_k(x) .$$

Ex:- If $y_{P_1} = 3e^{2x}$ is a P.S of $y'' - 6y' + 5y = -9e^{2x}$

$y_{P_2} = x^2 + 3x$ is a P.S of $y'' - 6y' + 5y = 5x^2 + 3x - 16$.

Then

$$y_P = c_1 y_{P_1} + c_2 y_{P_2} = 3c_1 e^{2x} + c_2(x^2 + 3x)$$

is a Particular Solution of

$$y'' - 6y' + 5y = -9c_1 e^{2x} + c_2(5x^2 + 3x - 16).$$

Practice Problem:-

1) Verify that the given two Parameter family of functions is the general Solution of non Homogeneous D.E on indicated Interval.

$$y'' + y = \sec x \; ;$$

$$y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \ln(\cos x)$$

$$(-\pi/2, \pi/2).$$

Related Questions:-

Ex: 4.1

Q: 31 - 35.

4.2 Reduction of order:-

Consider Homogeneous Linear Second order D.E

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

It has a General Solution

$$y = c_1 y_1 + c_2 y_2.$$

where y_1, y_2 are Linearly Indep set of Solutions.

⇒ In Reduction of order Method we have given one Solution y_1, we need to find the second Solution y_2, So that y_1 and y_2 is a Linearly independent set on I.

⇒ If we have Linearly Indep Solutions y_1 and y_2, there Quotient must be non Constant, i.e

$$\frac{y_2}{y_1} = \text{Non Constant } (u(x))$$

⇒ Function $u(x)$ can be found by Substituting $y_2 = u(x)\, y_1(x)$ into given D.E. This method is called Reduction of order.

Ex:- Given that $y_1 = e^x$ is a Solution of $y'' - y = 0$ on the interval $(-\infty, +\infty)$, use reduction of order to find a Second Solution.

Solu:- Let $y = u(x)\, y_1(x)$

$y = u(x)\, e^x$.

$y' = u e^x + u' e^x$

$y'' = u e^x + e^x u' + u'' e^x + u' e^x$.

$= u e^x + 2e^x u' + e^x u''$

Given D.E $\Rightarrow$

$y'' - y = 0$

$\cancel{u e^x} + 2e^x u' + e^x u'' - \cancel{u e^x} = 0$

$e^x (u'' + 2u') = 0$

$\because e^x \neq 0$

$u'' + 2u' = 0$ — (a)

Let $w = u'$

$w' = u''$

Equ(a) $\Rightarrow$

$w' + 2w = 0$ is Linear 1st order D.E.

using I.F $e^{2\int dx} = e^{2x}$, we can write.

$\frac{d}{dx}[e^{2x} . w] = 0$

$e^{2x} . w = C_1$

$$w = c_1 e^{-2x}$$

Resubstitute $w = u'$

$$u' = c_1 e^{-2x}$$

$$\int u' \, dx = c_1 \int e^{-2x} \, dx$$

$$u = -\frac{1}{2} c_1 e^{-2x} + c_2$$

$$y = ue^x = \left(-\frac{1}{2} c_1 e^{-2x} + c_2\right) e^x$$

$$= -\frac{1}{2} c_1 e^{-x} + c_2 e^x$$

we get second solution e^{-x}

Because $W(e^x, e^{-x}) \neq 0 \Rightarrow$ Solutions are L.I on $(-\infty, \infty)$.

Standard form of Homogeneous D.E:-

Consider homogeneous Linear Second order D.E

$$a_2(x) y'' + a_1(x) y' + a_0(x) y = 0 \qquad — (1)$$

Divide Equ by $a_2(x)$, we can write

$$y'' + P(x) y' + Q(x) y = 0 \qquad — (2)$$

where $P(x)$ and $Q(x)$ are Continuous on Some Interval.

Formula to find y_2, if y_1 is Given:-

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^2} dx$$

Ex: The Function $y_1 = x^2$ is a Solution of $x^2y'' - 3xy' + 4y = 0$. Find the General Solution of the D.E on the interval $(0, \infty)$.

Solu:- First Convert the D.E into its Standard form; that is

$$y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0$$

we have $P(x) = -\frac{3}{x}$.

By Applying Formula

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^2} dx$$

$$= x^2 \int \frac{e^{3\int 1/x \, dx}}{x^4} dx$$

$$= x^2 \int \frac{e^{3 \ln x}}{x^4} dx$$

$$= x^2 \cdot \int \frac{x^3}{x^4} dx$$

$$= x^2 \cdot \int 1/x \, dx$$

$$= x^2 \ln x$$

General Solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 x^2 + c_2 x^2 \ln x$$

where y_1 and y_2 are Linearly Independent.

Practice Problems:-

Use Reduction of order or Formula to find a Second Solution $y_2(x)$, where y_1 is Given.

1) $y'' + 2y' + y = 0$; $y_1 = xe^{-x}$

2) $y'' - y = 0$; $y = \cosh x$

3) $x^2y'' - 7xy' + 16y = 0$; $y_1 = x^4$.

4) $4x^2y'' + y = 0$; $y_1 = x^{1/2} \ln x$

5) $y'' - 4y = 2$; $y_1 = e^{-2x}$.

Related Questions:-

Ex 4.2

Q: 1-20.