LECTURE : 30 :

Section 4.1 : Linear Equations (Continuation)

Homogeneous / Non Homogeneous Equations

A Linear n^{th} order differential Equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \quad -(1)$$

is said to be Homogeneous, whereas as Equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad -(2)$$

with $g(x) \neq 0$ is said to be non-Homogeneous.

Ex:
- $2y'' + 3y' - 5y = 0$ is homogeneous Linear Second order D.E.
- $x^3y''' + 6y' + 10y = e^x$ is non homogeneous Linear third-order D.E.

NOTE:- To Solve non homogeneous Equ, we must Solve associated homogeneous Equ first.

Superposition Principal: Let $y_1, y_2, \dots, y_k$ be the Solutions of the homogeneous n^{th}-order differential Equ (1) on an Interval I. Then the Linear Combination

$$y = c_1y_1(x) + c_2y_2(x) + \dots + c_ky_k(x) \; ; \; c_1, c_2, \dots \text{ are Constants}$$

is also a Solution on the Interval I.

Corollaries:-

⇒ A Constant Multiple $y = c_1 y_1(x)$ of a Solution $y_1(x)$ of a Homogeneous Linear D.E is also a Solution.

⇒ A Homogeneous Linear D.E always Possesses the trivial Solution $y=0$.

Ex:- The functions $y_1 = x^2$, $y_2 = x^2 \ln x$ are the solution of Homogeneous Linear D.E

$$x^2 y''' - 2xy' + 4y = 0 \quad \text{on } (0,\infty)$$

Use Superposition Principal to find the Solution.

Solu:- By Superposition Principal

$$y = c_1 x^2 + c_2 x^2 \ln x$$

is also a Solution of Homogeneous Linear D.E on Interval $(0,\infty)$.

Linear Dependence / Independence:-

A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ is said to be Linearly dependent on an interval I, if there exist $c_1, c_2, \dots, c_n$, not all zero, Such that

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0 \qquad \forall x \in I.$$

If set of functions is not Linearly dependent on I, it is said to be linearly independent.

Ex:- Check Dependence:-

$f_1(x) = \cos^2 x$, $f_2(x) = \sin^2 x$, $f_3(x) = \sec^2 x$, $f_4(x) = \tan^2 x$

on $(-\pi/2, \pi/2)$.

Method: I

$$C_1 \cos^2 x + C_2 \sin^2 x + C_3 \sec^2 x + C_4 \tan^2 x = 0.$$

Let $C_1 = C_2 = 1$, $C_3 = -1$, $C_4 = 1$.

$$\cos^2 x + \sec^2 x - \sec^2 x + \tan^2 x = 0$$

$\because \cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x.$

Implies that functions are Linearly dependent.

Method: II

$$\tan^2 x = -\cos^2 x \mp \sin^2 x + \sec^2 x$$

$$f_4 = (-1) f_1 + (-1) f_2 + f_3.$$

Here f_4 is expressed as a Linear Combination of f_1, f_2, f_3.

$\therefore$ Functions are L. Dependent.

NOTE:- A Set of functions $f_1(x), f_2(x), \dots, f_n(x)$ is L. Dep if at Least one function is expressed as Linear Combination of the remaining functions.

Ex:- $f_1 = \sqrt{x} + 5$, $f_2 = \sqrt{x} + 5x$, $f_3 = x - 1$, $f_4 = x^2$; $(0, \infty)$

Check L. Dep / Indep.

Solu:-

$$\sqrt{x} + 5x = \sqrt{x} + 5 + 5(x-1) + (0)x^2$$

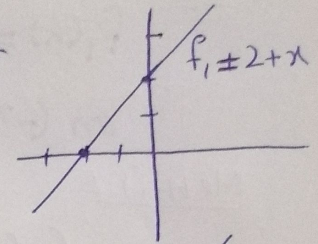
$$f_2 = f_1 + 5 f_3 + 0.f_4$$

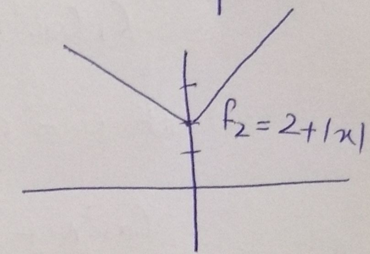
f_2 can be expressed as a Linear Combination of f_1, f_3, f_4.

$\therefore$ function are L. Dependent.

Ex:- $f_1(x) = 2+x$, $f_2(x) = 2+|x|$.

The set of functions are Linearly Independent Since they cannot be multiples of each other.





Ex:- $\{\cos 2x, 1, \cos^2 x\}$.

Let $f_1 = \cos 2x$, $f_2 = 1$, $f_3 = \cos^2 x$.

$$\because \cos^2 \theta/2 = \frac{1+\cos\theta}{2}$$

we can write

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$f_3 = \frac{1}{2} f_2 + \frac{1}{2} f_1 \qquad \therefore \text{ functions are L.Dep.}$$

Ex:- $\{e^x, e^{-x}, \sinh x\}$

Let $f_1 = e^x$, $f_2 = e^{-x}$, $f_3 = \sinh x$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

$$f_3 = \frac{1}{2} f_1 - \frac{1}{2} f_2 \qquad \therefore \text{ function are L.Dep.}$$

Ex:- $f_1 = \cos^2 x$, $f_2 = \sin^2 x$, $f_3 = 5$ $\qquad f_3 = 5f_1 + 5f_2$. L.Dep.

Ex:- $\{\ln x, \ln x^5\}$ $\qquad f_2 = 5f_1$ L.Dep.

Ex:- $\{\cos 4x, 2\cos 4x\}$ $\qquad f_2 = 2f_1$ L.Dep.

Wronskian :-

Suppose each function $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ Possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, \ldots f_n) = \begin{vmatrix} f_1, & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

where the Primes denotes derivatives, is Called the wronskian of the functions.

Criteria for L.Ind Solutions:-

Let $y_1, y_2, \ldots, y_n$ be n Solutions of the Homogeneous Linear n^{th} order D.E (1) on interval I. Then the Set of Solutions is Linearly indep on I iff $W(y_1, y_2, \ldots y_n) \neq 0$ $\forall\ x \in I$.

NOTE:- Any set of Linearly Indep Solutions of Homogeneous Linear n^{th} order D.E on I is said to be a fundamental Set of Solutions

Ex:- $y_1 = e^x,\ y_2 = e^{2x},\ y_3 = e^{3x}$

$y''' - 6y'' + 11y' - 6y = 0$

verify that given functions form a fundamental Set of Solutions.

Solu:-

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= e^x \begin{vmatrix} 2e^{2x} & 3e^{3x} \\ 4e^{2x} & 9e^{3x} \end{vmatrix} - e^x \begin{vmatrix} e^{2x} & e^{3x} \\ 4e^{2x} & 9e^{3x} \end{vmatrix} + e^x \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$$

$$= e^x[18e^{5x} - 12e^{5x}] - e^x[9e^{5x} - 4e^{5x}] + e^x[3e^{5x} - 2e^{5x}]$$

$$= 18e^{6x} - 12e^{6x} - 9e^{6x} + 4e^{6x} + 3e^{6x} - 2e^{6x}$$

$$= 2e^{6x} \neq 0.$$

The functions y_1, y_2, y_3 form a fundamental Set of Solutions on $(-\infty, \infty)$.

we Conclude that $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$ is the general Solution of the differential Equation.

Ex:- $\cos(\ln x)$, $\sin(\ln x)$

$x^2 y'' + x y' + y = 0$; $(0, \infty)$

Verify that functions form a fundamental Set of Solution. Also Form the General Solution.

Solu:-

$$W = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ \frac{d}{dx}(\cos(\ln x)) & d/dx(\sin(\ln x)) \end{vmatrix}$$

$$= \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\sin(\ln x)\cdot\frac{1}{x} & \cos(\ln x)\cdot\frac{1}{x} \end{vmatrix}$$

$$= \frac{\cos^2(\ln x)}{x} + \frac{\sin^2(\ln x)}{x} = \frac{1}{x} \neq 0.$$

Therefore functions form a fundamental Set of Solutions on $(0,\infty)$

General Solution is

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x).$$

Practice Problems:-

Determine Linearly Dependence

1) $\{0, x, e^x\}$
2) $\{e^x, e^{-x}, \cosh x\}$
3) $\{x, x^2, 4x-3x^2\}$

Verify that given functions form a fundamental set of Solutions. Form the General Solution.

4) $y^{(4)} + y'' = 0$; $1, x, \cos x, \sin x$. $(0,\infty)$
5) $y'' - 4y = 0$; $\cosh 2x, \sinh 2x$, $(-\infty,\infty)$

Related Questions:-

Ex: 4.1

Q: 15 – 30.