4.7:- Cauchy Euler Equation:-

LECTURE : 35

A Linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x).$$

is called Cauchy Euler Equation.

Method of Solution:-

Instead of $y = e^{mx}$, we try a Solution of the form $y = x^m$.
Because of this Substitution each term of Cauchy-Euler Equation become a Polynomial in $m$ times $x^m$. Then we get the Auxiliary equation easily.

For example, when we Substitute $y = x^m$, the $2^{nd}$ order D.E becomes.

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = ax^2 m(m-1)x^{m-2} + bx\, mx^{m-1} + c\,x^m = 0$$

$$\Rightarrow [am(m-1) + bm + c]\,x^m = 0.$$

Thus $y = x^m$ is a Solution of D.E whenever $m$ is a Solution of A.E

$$am(m-1) + bm + c = 0.$$

$$am^2 + (b-a)m + c = 0.$$

Then there are three different Cases, depending on whether the roots of the Quadratic Equation are real and distinct, real and Equal or Complex.

Case I:- Distinct Real Roots: (Disc > 0)

If $m_1, m_2 \rightarrow$ Real Roots.

Then $y = x^{m_1}$, $y = x^{m_2}$ $\rightarrow$ forms a fundamental set

General Solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Ex:- $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0.$

Let $y = x^m$. Then $dy/dx = mx^{m-1}$, $d^2y/dx^2 = m(m-1)x^{m-2}$.

we have
$$x^2 m(m-1)x^{m-2} - 2x m x^{m-1} - 4x^m = 0.$$
$$(m(m-1) - 2m - 4)x^m = 0.$$

$$\Rightarrow \quad m(m-1) - 2m - 4 = 0.$$
$$m^2 - 3m - 4 = 0.$$
$$m^2 - 4m + m - 4 = 0$$
$$m(m-4) + 1(m-4) = 0$$
$$(m+1)(m-4) = 0.$$

$$m_1 = -1, \quad m_2 = 4.$$

$$\boxed{y = c_1 x^{-1} + c_2 x^4}$$ General Solution.

Case II:- Repeated Real Roots:- (Disc = 0).

If $m_1, m_2 \rightarrow$ Repeated real roots.

Then $m_1 = m_2$

and we have only one Solution $y = x^{m_1}$.

where $m_1$ can be obtained from the quadratic Equation
$$am_1^2 + (b-a)m_1 + c = 0.$$

i.e
$$m_1 = -\frac{(b-a)}{2a}$$

- Now we Construct a Second Solution $y_2$.

we first write 2nd order Cauchy Equ in Standard form.
$$\frac{d^2y}{dx^2} + \frac{b}{ax}\frac{dy}{dx} + \frac{c}{ax^2}y = 0.$$

where $P(x) = b/ax$
$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx.$$
$$= x^{m_1} \int \frac{e^{-b/a \int 1/x\, dx}}{x^{2m_1}}\,dx.$$
$$= x^{m_1} \int \frac{e^{-b/a \ln x}}{x^{2m_1}}\,dx.$$
$$= x^{m_1} \int x^{-b/a} \cdot x^{-2m_1}\,dx.$$
$$= x^{m_1} \int x^{-b/a} \cdot x^{(b-a)/a}\,dx.$$
$$= x^{m_1} \int x^{-1}\,dx.$$
$$\boxed{y_2 = x^{m_1} \ln x}$$

The General Solution is
$$\boxed{y = c_1 x^{m_1} + c_2 x^{m_1} \ln x}$$

Example:-

$$4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = 0.$$

Let $y = x^m$.

we get

$$4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = x^m(4m(m-1) + 8m + 1)$$
$$= x^m(4m^2 + 4m + 1) = 0.$$

$\Rightarrow 4m^2 + 4m + 1 = 0.$

$\Rightarrow (2m+1)^2 = 0.$

$\Rightarrow m = -\frac{1}{2}, -\frac{1}{2}.$

$$\boxed{y = c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x}$$ General Solution.

Case III:- Conjugate Complex Roots:- (Disc < 0)

If $m_1, m_2 \to$ Conjugate complex Roots.

Then $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ where $\alpha, \beta > 0$.

Then Solution is

$$y = C_1 x^{\alpha + i\beta} + C_2 x^{\alpha - i\beta}.$$
$$= (C_1 x^{i\beta} + C_2 x^{-i\beta})x^{\alpha}$$ where $y_1 = x^{\alpha} x^{i\beta}$, $y_2 = x^{\alpha} x^{-i\beta}$.

we can write $x^{i\beta}$ as

$$x^{i\beta} = (e^{\ln x})^{i\beta} = e^{i\beta \ln x}$$

By Eulers formula.

$$x^{i\beta} = \cos(\beta \ln x) + i\sin(\beta \ln x)$$
$$x^{-i\beta} = \cos(\beta \ln x) - i\sin(\beta \ln x).$$

Adding and Substracting $y_1 + y_2$.

- $y_1 + y_2 = 2x^{\alpha}\cos(\beta \ln x)$

$\frac{1}{2}y_1 + \frac{1}{2}y_2 = x^{\alpha}\cos(\beta \ln x)$

- $y_1 - y_2 = 2ix^{\alpha}\sin(\beta \ln x)$

$-\frac{i}{2}y_1 + \frac{i}{2}y_2 = x^{\alpha}\sin(\beta \ln x)$

$\because W(x^{\alpha}\cos(\beta \ln x), x^{\alpha}\sin(\beta \ln x)) \neq 0$

it forms a fundamental set

Therefore our solution is of the form.

$$y = x^{\alpha}[C_1\cos(\beta \ln x) + C_2\sin(\beta \ln x)].$$

Example:- Solve $x^3\frac{d^3y}{dx^3} + 5x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 8y = 0$.

Let $y = x^m$

$\frac{dy}{dx} = mx^{m-1}$, $\frac{d^2y}{dx^2} = m(m-1)x^{m-2}$, $\frac{d^3y}{dx^3} = m(m-1)(m-2)x^{m-3}$.

we have

$$x^3\frac{d^3y}{dx^3} + 5x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 8y = x^3m(m-1)(m-2)x^{m-3} + 5x^2m(m-1)x^{m-2} + 7xmx^{m-1} + 8x^m = 0.$$

$\Rightarrow [m(m-1)(m-2) + 5m(m-1) + 7m + 8]x^m = 0$.

$\Rightarrow [m(m^2-3m+2) + 5m^2 - 5m + 7m + 8]x^m = 0$.

$\Rightarrow [m^3 - 3m^2 + 2m + 5m^2 + 2m + 8]x^m = 0$.

$\Rightarrow [m^3 + 2m^2 + 4m + 8]x^m = 0$.

$\Rightarrow [m^2(m+2) + 4(m+2)]x^m = 0$.

$\because (m+2)(m^2+4) = 0$.

$m = -2$, $m = \pm 2i$.

$\boxed{y = c_1x^{-2} + c_2\cos(2\ln x) + c_3\sin(2\ln x).}$ General Solution.

NOTE:- The U.C Method is not applicable when D.E has a variable coefficients. Therefore we use Variation of parameter method to find the particular Solution.

Ex:- $x^2y'' - 3xy' + 3y = 2x^4e^x$.

Homogeneous: $x^2y'' - 3xy' + 3y = 0$

A.E $\quad m(m-1) - 3m + 3 = 0$

$m^2 - 4m + 3 = 0$.

$(m-1)(m-3) = 0$

$m = 1$, $m = 3$.

$\boxed{y_c = c_1x + c_2x^3}$

where

$y_1 = x$, $y_2 = x^3$

Let $y_p = u_1 y_1 + u_2 y_2$.

~~with $f(x) = 2x^4e^x$~~ First write D.E in standard form

i.e $y'' + P(x)y' + Q(x)y = f(x)$.

we have

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x.$$

~~[illegible]~~ we get $f(x) = 2x^2e^x$, with $y_1 = x$, $y_2 = x^3$.

$$W(y_1, y_2) = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^3 - x^3 = 2x^3.$$

$$u_1' = -\frac{y_2 f(x)}{W} = -\frac{x^3 \cdot 2x^2e^x}{2x^3} = -x^2e^x,$$

$$u_1 = -\int x^2e^x\,dx$$

$$= -\left[x^2\int e^x dx - \int\left(\int e^x dx\right)(2x)dx\right]$$

$$= -x^2e^x + 2\int x e^x dx$$

$$= -x^2e^x + 2x\int e^x dx - 2\int e^x dx$$

$$= -x^2e^x + 2xe^x - 2e^x$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{x \cdot 2x^2e^x}{2x^3} = e^x.$$

$$u_2 = \int e^x dx$$

$$u_2 = e^x$$

$$y_p = u_1y_1 + u_2y_2 = (-x^2e^x + 2xe^x - 2e^x)x + e^x x^3$$

$$= \cancel{-x^3e^x} + 2x^2e^x - 2xe^x + \cancel{x^3e^x} = 2x(x-1)e^x.$$

$$\boxed{y = c_1x + c_2x^3 + 2x(x-1)e^x}$$

## Practice Problems:-

1) Solve Cauchy Euler Equations.

1) $x^2y'' + 3xy' - 4y = 0$

2) $x^3y''' + xy' - y = 0$

3) $3x^2y'' + 6xy' + y = 0$.

4) $x^2y'' - 5xy' + 8y = 8x^6$ ; $y(1/2) = 0$ ; $y'(1/2) = 0$.

## Related Questions:-

Ex 4.7

Q: 1-30.