LECTURE : 26 :

2.5 Solutions By Substitution :-

① Homogeneous Equation:-

A function $f(x,y)$ is Homogeneous of degree α, if it satisfies the Property

$$\boxed{f(tx, ty) = t^{\alpha} f(x,y)}, \quad \alpha \text{ is a real no.}$$

e.g:

$f(x,y) = x^3 + y^3$ is Homogeneous function of degree 3.

$$f(tx, ty) = (tx)^3 + (ty)^3 = t^3x^3 + t^3y^3$$
$$= t^3(x^3+y^3)$$
$$= t^3 f(x,y)$$

whereas

$f(x,y) = x^3 + y^3 + 1$ is not Homogeneous.

$\Rightarrow$ A first order D.E of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be Homogeneous, if both Coefficient functions M and N are Homogeneous function of same degree.

that is

$$M(tx, ty) = t^{\alpha} M(x,y)$$

$$\& \quad N(tx, ty) = t^{\alpha} N(x,y).$$

NOTE: Here first order D.E is Homogeneous due to Homogeneous function.

Substitution: If first order D.E is Homogeneous, then we can use two types of Substitutions, which results Homogeneous D.E into a Seperable D.E.

① $y = ux$

$dy = u\,dx + x\,du$

② $x = vy$

$dx = v\,dy + y\,dv$

By Putting any of these two Substitutions into Homogeneous D.E, Resulting Eqn is Seperable D.E.

Example:- $(x^2+y^2)dx + (x^2-xy)dy = 0$

Solu:-

$M(x,y) = x^2+y^2$, $N(x,y) = x^2 - xy$

$$M(tx, ty) = t^2x^2 + t^2y^2$$
$$= t^2(x^2+y^2)$$
$$= t^2 M(x,y)$$

$$N(tx, ty) = t^2x^2 - (tx)(ty)$$
$$= t^2(x^2 - xy)$$
$$= t^2 N(x,y)$$

Both functions M and N are Homogeneous of same degree 2. Therefore given D.E is Homogeneous.

Let $y = ux$

$dy = u\,dx + x\,du$.

Substitute in Given D.E.

$$(x^2+y^2)\,dx + (x^2 - xy)\,dy = 0$$

$$(x^2 + u^2x^2)\,dx + (x^2 - ux^2)(u\,dx + x\,du) = 0$$

$$x^2dx + \cancel{u^2x^2dx} + ux^2dx + x^3du - \cancel{u^2x^2dx} - ux^3du = 0$$

$$x^2(1+u)\,dx + x^3(1-u)\,du = 0$$

$$x^3(1-u)\,du = -x^2(1+u)\,dx$$

$$\frac{(1-u)}{(1+u)}\,du = -\frac{x^2}{x^3}\,dx.$$

$$\left(\frac{1-u}{1+u}\right)du + \frac{1}{x}\,dx = 0$$

→ Equ is now in Seperable form

$$\frac{du}{dx} = \left(-\frac{1}{x}\right)\left(\frac{1+u}{1-u}\right)$$

Integrating, we get

$$\int\left(\frac{1-u}{1+u}\right)du + \int\frac{1}{x}\,dx = \int 0.$$

$$\int\left(-1+\frac{2}{1+u}\right)du + \int\frac{1}{x}\,dx = \int 0$$

$$\begin{array}{r|l} & -1 \\ \hline 1+u & 1-u \\ & -1-u \\ & +\quad + \\ \hline & 2 \end{array}$$

$$-u + 2\ln(1+u) + \ln x = \ln c$$

Re substitute value of u, which is $u = y/x$

$$-\frac{y}{x} + 2\ln\left(1+\frac{y}{x}\right) + \ln x = \ln c$$

$$2\ln\left(1+\frac{y}{x}\right) + \ln x - \ln c = \frac{y}{x}$$

$$\ln\left(1+\frac{y}{x}\right)^2 + \ln\left(\frac{x}{c}\right) = \frac{y}{x}$$

$$\ln\left(\frac{x+y}{x}\right)^2 + \ln\left(\frac{x}{c}\right) = \frac{y}{x}$$

$$\ln\left[\frac{(x+y)^2}{x^2}\right] + \ln\left(\frac{x}{c}\right) = \frac{y}{x}$$

$$\ln\left[\frac{(x+y)^2}{x^2} \cdot \frac{x}{c}\right] = \frac{y}{x}$$

$$\ln\left(\frac{(x+y)^2}{xc}\right) = \frac{y}{x}$$

Taking exponential

$$\frac{(x+y)^2}{xc} = e^{y/x}$$

$$\boxed{(x+y)^2 = cxe^{y/x}}$$

Implicit Solution of Homogeneous D.E.

Practice Problems:-

1) $(y^2 + yx)\,dx - x^2\,dy = 0$

2) $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$

3) $xy^2 \dfrac{dy}{dx} = y^3 - x^3 \,, \; y(1) = 2$

Related Exercise Questions:-

Ex 2.5

Q: 1 – 14