LECTURE : 33 :-

4.4 Undetermined Coefficients - Superposition Approach:-

To Solve a nonhomogeneous linear differential Equation:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x) \quad \text{— (1)}$$

i) find the Complementary Solution y_c

ii) find the Particular Solution y_p

iii) we get the General Solution $y = y_c + y_p$.

Method of undetermined Coefficients:-

The Method of obtaining a Particular Solution y_p for a Non homogeneous D.E is Called the Method of undermined Coefficients.
The General method is

- the Coefficients a_i are Constants
- $g(x)$ can be of the forms i.e, a Polynomial function, an exponential function $e^{\alpha x}$, a Sine or Cosine functions $\sin\beta x$ or $\cos\beta x$, or finite Sum or Product of these functions.

i.e $g(x) = 10$, $g(x) = x^2 - 5x$, $g(x) = 15x - 6 + 8e^{-x}$.

$g(x) = \sin 3x - 5x\cos 2x$, $g(x) = xe^{x}\sin x + (3x^2 - 1)e^{-4x}$.

Imp Note:- The Method of U.C is not applicable to equations of the form (1) when

$g(x) = \ln x$, $g(x) = \frac{1}{x}$, $g(x) = \tan x$, $g(x) = \sin^{-1} x$.

and so on.

Table:- Particular Solutions

$g(x)$	Form of y_p
1) 1 (any constant)	A
2) $5x+7$	$Ax+B$
3) $3x^2-2$	Ax^2+Bx+C.
4) x^3-x+1	Ax^3+Bx^2+Cx+E
5) $\sin 4x$	$A\cos 4x + B\sin 4x$.
6) $\cos 4x$	$A\cos 4x + B\sin 4x$.
7) e^{5x}	Ae^{5x}
8) $(9x-2)e^{5x}$	$(Ax+B)e^{5x}$.
9) x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$.
10) $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$.
11) $5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x + (Ex^2+Fx+G)\sin 4x$.
12) $xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x + (Cx+E)e^{3x}\sin 4x$.

Example:- Solve $y''+4y'-2y = 2x^2-3x+6$.

1) First Solve the associated Homogeneous Equation $y''+4y'-2y=0$.

A.E:- $m^2+4m-2=0$.

$m_1=-2-\sqrt{6}$, $m_2=-2+\sqrt{6}$ using Quadratic formula.

Complementary Solution

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}.$$

2) Because $g(x)$ is quadratic Polynomial, y_p is of the form.

$$y_p = Ax^2+Bx+C.$$

To find Coefficients A, B & C.

$$y_p' = 2Ax+B$$

$$y_p'' = 2A$$

Put into given D.E

$y_p'' + 4y_p' - 2y_p = 2x^2 - 3x + 6$.

$2A + 4(2Ax+B) - 2(Ax^2+Bx+C) = 2x^2 - 3x + 6$.

$(-2A)x^2 + (8A - 2B)x + (2A+4B-2C) = 2x^2 - 3x + 6$.

Comparing Coefficients, we get

$-2A = 2$, $\quad 8A - 2B = -3$, $\quad 2A + 4B - 2C = 6$.

$\Rightarrow \boxed{A = -1}$

$8(-1) - 2B = -3$

$-2B = -3 + 8$

$-2B = 5$

$\boxed{B = -\frac{5}{2}}$

$2(-1) + 4\left(-\frac{5}{2}\right) - 2C = 6$.

$-2C = 6 + 2 + 10$.

$-2C = 18$

$\boxed{C = -9}$

$y_p = -x^2 - \frac{5}{2}x - 9$.

3). General Solution

$y = y_c + y_p$

$= C_1 e^{-(2+\sqrt{6})x} + C_2 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9$.

Example:- find a Particular Solution of $y'' - y' + y = 2\sin 3x$.

Let $y_p = A\cos 3x + B\sin 3x$

$y_p' = -3A\sin 3x + 3B\cos 3x$.

$y_p'' = -9A\cos 3x - 9B\sin 3x$

By Putting these values in Given D.E, we get.

$y_p'' - y_p' + y_p = 2\sin 3x$.

$-9A\cos 3x - 9B\sin 3x + 3A\sin 3x - 3B\cos 3x + A\cos 3x + B\sin 3x = 2\sin 3x$.

$(-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 2\sin 3x$.

Comparing Successive terms, we get

$-8A - 3B = 0$ and $3A - 8B = 2$.

Solving both we get $A = \frac{6}{73}$, $B = -\frac{16}{73}$

we get

$\boxed{y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x}$.

Example:- Solve $y''-2y'-3y=4x-5+6xe^{2x}$.

Step1:- The Solution of associated homogeneous Equ $y''-2y'-3y=0$ is found to be

$$y_c = c_1e^{-x}+c_2e^{3x}.$$

Step2:- To find Particular Solution y_p; we see that $g(x)$ is a sum of two kinds of functions.

$$g(x) = g_1(x) + g_2(x) = \text{Polynomial} + \text{exponentials}.$$

Therefore by Superposition principle.

$$y_p = y_{p_1} + y_{p_2}.$$

where $y_{p_1} = Ax+B$, $y_{p_2} = (Cx+D)e^{2x}$

$$y_p = Ax+B+Cxe^{2x}+De^{2x}.$$

Step3:- $y = y_c + y_p$ General Solution

$$= c_1e^{-x}+c_2e^{3x}-\frac{4}{3}x+\frac{23}{9}-\left(2x+\frac{4}{3}\right)e^{2x}.$$

Example:- A Glitch/fault in the Method:-

find y_p of $y''-5y'+4y=8e^x$.

$$y_c = c_1e^{x}+c_2e^{4x}.$$

$y_p = 4e^x$. the Solution e^x repeates, Also we found that by putting y_p into D.E we get zero.

$$\left.\begin{aligned} y_p''-5y_p'+4y_p &= 8e^x \\ 4e^x-20e^x+16e^x &= 8e^x \\ 0 &= 8e^x \end{aligned}\right\}$$

$\because$ It will not give the values of constants.

To Over come this difficulty.

$$y_p = Axe^x.$$

Therefore

$$y = y_c + y_p$$

$$= c_1e^{x}+c_2e^{4x}-\frac{8}{3}xe^{x}.$$

Ex:- $y'' - 8y' + 25y = 5x^3e^{-x} - 7e^{-x}$. find y_p.

$= (5x^3 - 7)e^{-x}$

$= g(x)$ (Product of Polynomial and exponential).

$y_c = e^{4x}(c_1 \cos 3x + c_2 \sin 3x)$.

$y_p = (Ax^3 + Bx^2 + Cx + D)e^{-x}$.

Ex:- $y'' + 4y = x \cos x$.

$= g(x)$ (Product of Polynomial & Trignometric).

$y_c = C_1 \cos 2x + C_2 \sin 2x$.

$y_p = (Ax + B)\cos x + (Cx + D)\sin x$

Ex:- $y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}$.

$= g(x)$

$= g_1(x) + g_2(x) + g_3(x)$.

$y_c = C_1e^{2x} + C_2 e^{7x}$.

$y_p = y_{p_1} + y_{p_2} + y_{p_3}$ where $y_{p_1} = Ax^2 + Bx + C$.

$y_{p_2} = D\cos 2x + E\sin 2x$.

$y_{p_3} = (Fx + G)e^{6x}$.

$y_p = Ax^2 + Bx + C + D\cos 2x + E\sin 2x + (Fx + G)e^{6x}$.

Ex:- $y'' - 2y' + y = e^x$.

$y_c = C_1 e^x + C_2 x e^x$.

$y_p = Ax^2 e^x$.

$\left\{ \begin{array}{l} y_p = Ae^x \\ \quad = Axe^x \\ \quad = Ax^2e^x \end{array} \right.$

$y = y_c + y_p = C_1e^x + C_2xe^x + \frac{1}{2}x^2e^x$.

Ex:- $y''+y=4x+10\sin x$, $y(\pi)=0$, $y'(\pi)=2$.

$y_c = C_1\cos x + C_2 \sin x$.

$y_p = (Ax+B) + (C\cos x + D\sin x)x$.

$= Ax+B+Cx\cos x + Dx\sin x$.

$y = y_c + y_p$

$= C_1\cos x + C_2\sin x + 4x - 5x\cos x$.

$y(\pi)=0 \Rightarrow C_1 = 9\pi$.

$y'(\pi)=0 \Rightarrow C_2 = 7$.

The General Solution of initial value is then

$y = 9\pi\cos x + 7\sin x + 4x - 5x\cos x$.

Ex:- $y''-6y'+9y = \underbrace{6x^2+2}-\underbrace{12e^{3x}}$.

$y_c = C_1e^{3x} + C_2xe^{3x}$.

$y_p = (Ax^2+Bx+C) + (De^{3x})x^2$

$= \underbrace{Ax^2+Bx+C}_{y_{p_1}} + \underbrace{Dx^2e^{3x}}_{y_{p_2}}$.

$y = y_c + y_p = C_1e^{3x} + C_2xe^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2e^{3x}$.

Ex:- $y''' + y'' = e^x\cos x$.

$\Rightarrow$ Characteristics Equ

$$\left.\begin{array}{ll} m^3+m^2=0. & \\ m^2(m+1)=0. & \\ m^2=0 & ,\ m+1=0. \\ m=0,0 & m=-1 \end{array}\right\} \quad y_c = C_1 + C_2x + C_3e^{-x}$$

$\Rightarrow y_p = e^x(A\cos x + B\sin x) = Ae^x\cos x + Be^x\sin x$.

$\Rightarrow \quad y = y_c + y_p = C_1 + C_2x + C_3e^{-x} - \frac{1}{10}e^x\cos x + \frac{1}{5}e^x\sin x$.

(6) $y'' - 8y' + 20y = 100x^2 - 26xe^x$.

$\Rightarrow$ Homogeneous Equ: $y'' - 8y' + 20y = 0$.

A.E: $m^2 - 8m + 20 = 0$.

$$m = \frac{8 \pm \sqrt{64-80}}{2} = \frac{8 \pm \sqrt{-16}}{2} = \frac{8 \pm 4i}{2} = 4 \pm 2i$$

$y_c = c_1 e^{4x}\cos 2x + c_2 e^{4x}\sin 2x$.

$\Rightarrow y_p = (Ax^2 + Bx + C) + (Dx + E)e^x$.

$\Rightarrow y = y_c + y_p = c_1 e^{4x}\cos 2x + c_2 e^{4x}\sin 2x + 5x^2 + 4x + \frac{11}{10} + \left(-2x - \frac{12}{13}\right)e^x$.

(13) $y'' + 4y = 3\sin 2x$.

$\Rightarrow$ A.E: $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$y_c = c_1\cos 2x + c_2 \sin 2x$.

$\Rightarrow y_p = (A\cos 2x + B\sin 2x)x = Ax\cos 2x + Bx\sin 2x$.

$\Rightarrow y = c_1\cos 2x + c_2\sin 2x - \frac{3}{4}x\cos 2x$.

(30) $y'' + 4y' + 4y = (3+x)e^{-2x}$; $y(0) = 2$, $y'(0) = 5$.

$\Rightarrow$ A.E: $m^2 + 4m + 4 = 0$.

$(m+2)^2 = 0$.

$m + 2 = 0$.

$m = -2, -2$.

$y_c = c_1 e^{-2x} + c_2 x e^{-2x}$.

$\Rightarrow y_p = (Ax + B)e^{-2x} = (Axe^{-2x} + Be^{-2x})x$.

$= (Ax^2e^{-2x} + Bxe^{-2x})x$.

$= Ax^3e^{-2x} + Bx^2e^{-2x}$

$= (Ax^3 + Bx^2)e^{-2x}$

$y = 2e^{-2x} + 9xe^{-2x} + \left(\frac{1}{6}x^3 + \frac{3}{2}x^2\right)e^{-2x}$

(32) $y'' - y = \cosh x \quad , \quad y(0) = 2, \; y'(0) = 12.$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\sinh x = \dfrac{e^x - e^{-x}}{2}$

$\Rightarrow$ A.E : $m^2 - 1 = 0$

$m^2 = 1$

$m = 1, 1.$

$y_c = c_1 e^x + c_2 \cancel{x} e^x.$

$\Rightarrow y_p = Ae^{-x} + Bx^2 e^x.$

Related Questions:-

Ex 4.4

Q: 1 – 40.