LECTURE : 25 :

Reducible to Exact Differential Equation

⇒ The first order differential Equations of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be Exact if it satisfies the Condition

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

⇒ If the first order differential Equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is not Exact, we make it exact by Multiplying with a suitable Integrating factor μ.

Choices for finding I.f μ :-

Choice I:- If $M(x,y)\,dx + N(x,y)\,dy = 0$ is not Exact

and

$\dfrac{M_y - N_x}{N}$ is a function of x only, Then

$$\text{I.f} : \mu = e^{\int \frac{M_y - N_x}{N} dx}$$

Choice II :- If $M(x,y)dx + N(x,y)dy = 0$ is not Exact and $\frac{N_x - M_y}{M}$ is a function of y only. Then

I.F :- $\mu = e^{\int \frac{N_x - M_y}{M}}$

Example : 1 : $(6xy)dx + (4y + 9x^2)dy = 0 \quad -(1)$

Soln $M(x,y) = 6xy$, $N(x,y) = 4y + 9x^2$

$M_y = \frac{\partial M}{\partial y} = 6x$ $\quad N_x = \frac{\partial N}{\partial x} = 18x$

$$\boxed{M_y \neq N_x}$$

Given D.E is not Exact. To make it Exact find Suitable Choice of μ.

Choice I $\quad \frac{M_y - N_x}{N} = \frac{6x - 18x}{4y + 9x^2} = -\frac{12x}{4y + 9x^2}$ depends on x & y both

Choice II $\quad \frac{N_x - M_y}{M} = \frac{18x - 6x}{6xy} = \frac{12x}{6xy} = \frac{2}{y}$ depends on y only.

Therefore

$$\mu = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int 2/y \, dy} = e^{2 \ln y} = e^{\ln y^2} = y^2 .$$

Multiply $M = y^2$ with given D.E to make it Exact, we get

$$\boxed{(6xy^3)\,dx + (4y^3 + 9x^2y^2)\,dy = 0}$$ is an Exact Equ. — (2)

we can Check it as

Let $M_1 = 6xy^3$, $N_1 = 4y^3 + 9x^2y^2$

$(M_1)_y = \frac{\partial M_1}{\partial y} = 18xy^2$ $(N_1)_x = \frac{\partial N_1}{\partial x} = 18xy^2$.

$$\boxed{(M_1)_y = (N_1)_x}$$

Hence D.E (2) is Exact.

To find the Solution, Consider

$$\frac{\partial F}{\partial x} = M_1 = 6xy^3$$

$$\frac{\partial F}{\partial y} = N_1 = 4y^3 + 9x^2y^2.$$

Lets Start with

$$\frac{\partial F}{\partial x} = 6xy^3$$

Integrate w.r.t x.

$$F(x,y) = 3x^2y^3 + g(y).$$

Diff F w.r.t y

$$\frac{\partial F}{\partial y} = 9x^2y^2 + g'(y) = 4y^3 + 9x^2y^2.$$

That is

$$g'(y) = 4y^3$$

Integrate w.r.t y

$$g(y) = y^4 + C_1.$$

Put value of $g(y)$, we get $F(x,y)$

$$F(x,y) = 3x^2y^3 + y^4 + C_1$$

Let $F(x,y) = C_2$

$$3x^2y^3 + y^4 + C_1 = C_2$$

$$3x^2y^3 + y^4 = C_2 - C_1$$

$$\boxed{3x^2y^3 + y^4 = C}$$

Implicit Solution of D.E.

NOTE:-

we Consider $f(x,y) = C$ because

$df = M(x,y)dx + N(x,y)dy$

implies

$M(x,y)dx + N(x,y)dy = 0$

only when

$f = C$

Example:- Solve $(2y^2 + 3x)\,dx + 2xy\,dy = 0$ — (1)

Solu

$M = 2y^2 + 3x$, $N = 2xy$

$M_y = \frac{\partial M}{\partial y} = 4y$ $N_x = \frac{\partial N}{\partial x} = 2y$

$\boxed{M_y \neq N_x}$ D.E is not Exact.

To make D.E an Exact Equ, Find μ.

Choice I: $\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{2y}{2xy} = \frac{1}{x}$ depends on x only

Therefore

$$\mu = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int 1/x\, dx} = e^{\ln x} = x$$

Multiply $\mu = x$ with given D.E

$$\boxed{(2xy^2 + 3x^2)\,dx + 2x^2y\,dy = 0}$$ — (2) is an Exact Equ'n.

That is

$M_1 = 2xy^2 + 3x^2$, $N_1 = 2x^2y$

$(M_1)_y = 4xy$ $(N_1)_x = 4xy$

$$\boxed{(M_1)_y = (N_1)_x}$$

Now to Solve an Exact Equation, we need to find $F(x,y)$

Consider

$$\frac{\partial F}{\partial x} = M_1 = 2xy^2 + 3x^2$$

$$\frac{\partial F}{\partial y} = N_1 = 2x^2y$$

lets Start with

$$\frac{\partial F}{\partial x} = 2xy^2 + 3x^2.$$

Integrate w.r.t x.

$$F(x,y) = x^2y^2 + x^3 + g(y)$$

Diff. w.r.t y

$$\frac{\partial F}{\partial y} = 2x^2y + g'(y) = 2x^2y$$

That is

$$g'(y) = 0$$

Integrate w.r.t y

$$g(y) = c_1$$

Practice Questions
Ex 2.5
Q: 29–30

we get $F(x,y) = x^2y^2 + x^3 + c_1$

Let $F(x,y) = c_2$

$$x^2y^2 + x^3 + c_1 = c_2 \Rightarrow \boxed{x^2y^2 + x^3 = c}$$

Implicit Solu of Exact D.E 2.