Theorem:- Second Translation Theorem:-

If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s).$$

If $f(t) = 1$ then $\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = 1/s = F(s)$.

Then $\mathcal{L}\{\mathcal{U}(t-a)\} = \dfrac{e^{-as}}{s}$.

Example:- Find the Laplace transform of $f(t) = 2 - 3\mathcal{U}(t-2) + \mathcal{U}(t-3)$.

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{1\} - 3\mathcal{L}\{\mathcal{U}(t-2)\} + \mathcal{L}\{\mathcal{U}(t-3)\}$$

$$= \frac{2}{s} - \frac{3e^{-2s}}{s} + \frac{e^{-3s}}{s}.$$

Inverse form of Second Translation Theorem:-

If $f(t) = \mathcal{L}^{-1}\{F(s)\}$ and $a > 0$, then

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a).$$

Example:- Evaluate

(a) $\mathcal{L}^{-1}\left\{\dfrac{1}{s-4}e^{-2s}\right\}$

$a = 2$, $F(s) = 1/(s-4)$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = \underbrace{e^{4t}}_{f(t)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = e^{4(t-2)}\mathcal{U}(t-2).$$

(b) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+9}e^{-\frac{\pi}{2}s}\right\}$

$a = +\pi/2$, $F(s) = \dfrac{s}{s^2+9}$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos 3t.$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\frac{\pi}{2}s}\right\} = \cos 3\left(t-\frac{\pi}{2}\right)\mathcal{U}\left(t-\frac{\pi}{2}\right)$$

NOTE:- To find the Laplace Transform of $t^2 U(t-2)$, we cannot directly apply 2nd Translation Theorem. It is required to first convert $f(t)=t^2$ into $f(t-2)$. we can convert t^2 as

$$t^2 = (t-2)^2 + 4(t-2) + 4$$

Since these simplifications are time consuming, we have a simplest version of 2nd Translation Theorem.

Alternative Form of Second Translation Theorem:-

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = +e^{-as}\mathcal{L}\{g(t+a)\}.$$

Example:- Evaluate $\mathcal{L}\{\cos t\,\mathcal{U}(t-\pi)\}$.

$$g(t) = \cos t, \quad a = \pi$$

$$g(t+\pi) = \cos(t+\pi) = -\cos t$$

$$\mathcal{L}\{g(t+\pi)\} = -\mathcal{L}\{\cos t\} = -\frac{s}{s^2+1}$$

S $\pi-\theta$	A θ
$\pi+\theta$ T	$2\pi-\theta$ C

$-(\theta-\pi)$

$$\mathcal{L}\{\cos t\,\mathcal{U}(t-\pi)\} = -e^{-\pi s}\cdot\frac{s}{s^2+1}$$

Example:- An Initial Value Problem.

Solve $y'+y=f(t)$; $y(0)=5$ where $f(t)=\begin{cases}0 & 0\le t<\pi\\ 3\cos t & t\ge\pi\end{cases}$.

$$= 3\cos t\,\mathcal{U}(t-\pi).$$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 3\mathcal{L}\{\cos t\,\mathcal{U}(t-\pi)\}$$

$$sY(s) - y(0) + Y(s) = -\frac{3s}{s^2+1}e^{-\pi s}$$

$$(s+1)Y(s) - 5 = -\frac{3s}{s^2+1}e^{-\pi s}$$

$$Y(s) = \frac{5}{s+1} - \frac{3s}{(s+1)(s^2+1)}e^{-\pi s} = \frac{5}{s+1} - \left[\frac{-3/2}{s+1} + \frac{3/2\,(s+1)}{s^2+1}\right]e^{-\pi s}$$

$$Y(s) = \frac{5}{s+1} - \frac{3}{2}\left[-\frac{1}{s+1} e^{-\pi s} + \frac{s}{s^2+1} e^{-\pi s} + \frac{1}{s^2+1} e^{-\pi s} \right]$$

- $\mathcal{L}^{-1}\{1/s+1\} = e^{-t}$
- $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s+1}\right\} = e^{-(t-\pi)} \mathcal{U}(t-\pi)$
- $\mathcal{L}^{-1}\left\{\frac{s}{s^2+1} e^{-\pi s}\right\} = \cos(t-\pi)\mathcal{U}(t-\pi)$
- $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} e^{-\pi s}\right\} = \sin(t-\pi)\mathcal{U}(t-\pi)$.

$$\mathcal{L}^{-1}(Y(s)) = \underbrace{5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s+1}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1} e^{-\pi s}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} e^{-\pi s}\right\}$$

$$y(t) = 5e^{-t} + \frac{3}{2} e^{-(t-\pi)} \mathcal{U}(t-\pi) - \frac{3}{2} \cos(t-\pi)\mathcal{U}(t-\pi) - \frac{3}{2}\sin(t-\pi)\mathcal{U}(t-\pi)$$

$$= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \cos t + \sin t\right] \mathcal{U}(t-\pi).$$

$$= \begin{cases} 5e^{-t} & ; \quad 0 \le t < \pi \\ 5e^{-t} + \frac{3}{2} e^{-(t-\pi)} + \frac{3}{2}\cos t + \frac{3}{2}\sin t & ; \quad t \ge \pi \end{cases}$$

Exercise:- Match the given graphs with one of the function in (a)–(f). The graph of $f(t)$ is

(a) $f(t) - f(t)\,\mathcal{U}(t-a)$

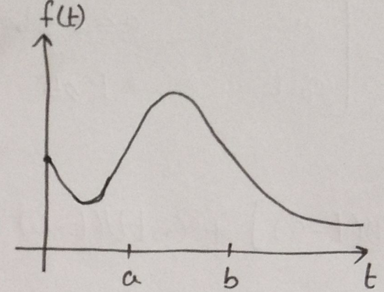
(b) $f(t-b)\,\mathcal{U}(t-b)$

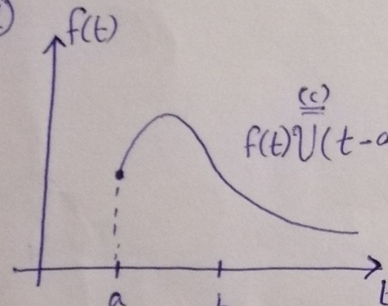
(c) $f(t)\,\mathcal{U}(t-a)$

(d) $f(t) - f(t)\,\mathcal{U}(t-b)$

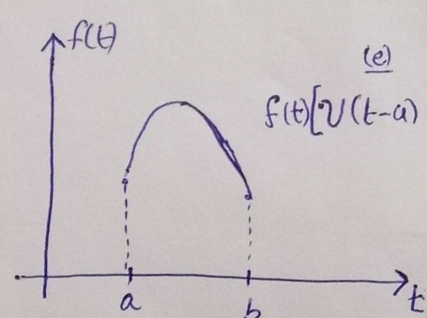
(e) $f(t)\,\mathcal{U}(t-a) - f(t)\,\mathcal{U}(t-b)$

(f) $f(t-a)\,\mathcal{U}(t-a) - f(t-a)\,\mathcal{U}(t-b)$.

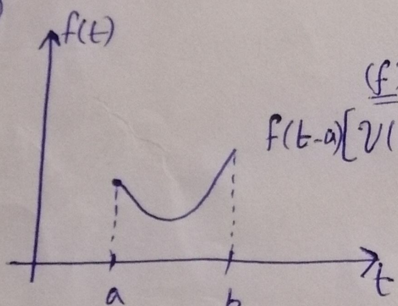


① 

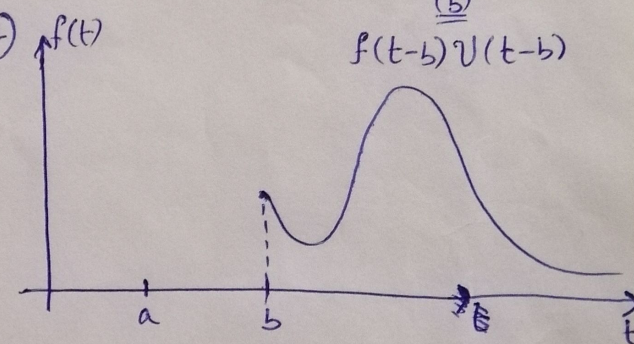
(c) $f(t)\,\mathcal{U}(t-a)$

② 

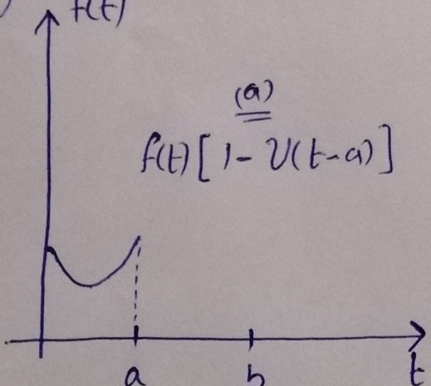
(e) $f(t)[\mathcal{U}(t-a) - \mathcal{U}(t-b)]$

③ 

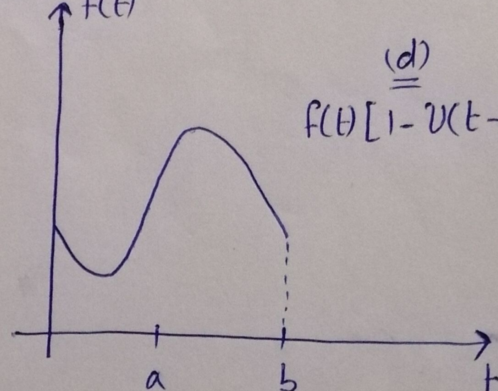
(f) $f(t-a)[\mathcal{U}(t-a) - \mathcal{U}(t-b)]$

④ 

(b) $f(t-b)\,\mathcal{U}(t-b)$

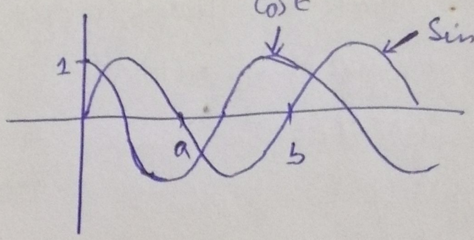
⑤ 

(a) $f(t)[1 - \mathcal{U}(t-a)]$

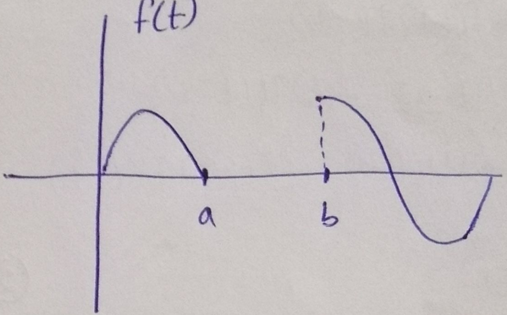
⑥ 

(d) $f(t)[1 - \mathcal{U}(t-b)]$

$$\Rightarrow \quad f(t) = \begin{cases} \sin t & 0 \le t < a \\ 0 & a \le t \le b \\ \cos t & t \ge b \end{cases}$$



$\sin t\,[1 - U(t-a)] + \cos t\, U(t-b)$



Practice Problems:-

Ex 7.3

Q: 33, 37 – 70