Chapter: 12:- Boundary Value Problems in Rectangular Coordinates

≡ Linear Partial Differential Equation:-

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G. \quad \text{— (I)}$$

Where

u : dependent variable

x, y : Independent Variable.

$A, B, \dots, G$: Coefficients, functions of x and y.

Equation (I) is called "Second-order Linear Partial differential Equation".
When $G(x,y) = 0$ Equ (I) is Homogeneous, otherwise, Non Homogeneous.

Definition:- Classification of Equation:-

The Linear second-order differential Equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G.$$

Where A, B, C, D, E, F and G are real Constants, is said to be

- hyperbolic if $B^2 - 4AC > 0$.
- Parabolic if $B^2 - 4AC = 0$
- Elliptic if $B^2 - 4AC < 0$.

Example:- Classify following PDE's.

a) $3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$.

$$3\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0.$$

Here $A = 3$, $B = 0$, $C = 0$.

$B^2 - 4AC = 0 - 4(3)(0) = 0$

∵ The Given Equ is Parabolic

b) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

Here $A = 1$, $B = 0$, $C = -1$

$B^2 - 4AC = 0 - 4(1)(-1) = 4 > 0$

∴ The Equ is Hyperbolic

(c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Here $A = 1$, $B = 0$, $C = 1$

$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$

∵ The Equ is Elliptic

Exercise Problem:- Determine the region in the xy-plane for which the Equation

$$(xy+1)\frac{\partial^2 u}{\partial x^2} + (x+2y)\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + xy^2 u = 0$$

is hyperbolic, Parabolic or Elliptic.

Solu we have $A = xy+1$

$B = x+2y$

$C = 1$

$$B^2 - 4AC = (x+2y)^2 - 4(xy+1)(1)$$
$$= x^2 + 4y^2 + \cancel{4xy} - \cancel{4xy} - 4$$
$$= x^2 + 4y^2 - 4$$

Equation is Parabolic, if

$$B^2 - 4AC = 0$$
$$x^2 + 4y^2 - 4 = 0$$
$$x^2 + 4y^2 = 4$$
$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

$\because \frac{x^2}{(2)^2} + \frac{y^2}{(1)^2} = 1$

is Equation of Ellipse

with $c^2 = a^2 - b^2 = 4 - 1 = 3$

Vertices: $(\pm a, 0) = (\pm 2, 0)$

Foci: $(\pm c, 0) = (\pm\sqrt{3}, 0)$

$= (\pm 1.7, 0)$

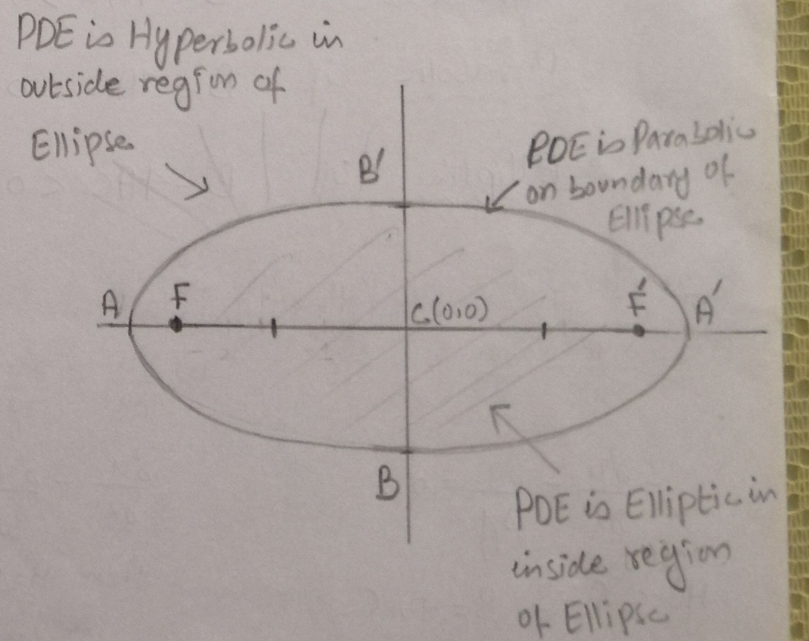
Covertices: $(0, \pm b) = (0, \pm 1)$

Equation is Hyperbolic, if

$$B^2 - 4AC > 0$$
$$\frac{x^2}{4} - \frac{y^2}{1} > 1$$

Equation is Elliptic, if

$$B^2 - 4AC < 0$$
$$\frac{x^2}{4} - \frac{y^2}{4} < 1$$



Seperation of Variables:-

we use Seperation of variables to find the Solution of Partial differential Equation. we consider a Solution of the form

$$u(x,y) = X(x)\, Y(y).$$

with this assumption we can reduce our PDE into two ODE's.

Example:- Find Solution of

$$\frac{\partial^2 u}{\partial x^2} = 4 \frac{\partial u}{\partial y}.$$

Solution:- Let $u(x,y) = X(x) Y(y)$

Substitute into Given PDE.

$$\frac{\partial^2}{\partial x^2}(XY) = 4 \frac{\partial}{\partial y}(XY)$$

$$X'' Y = 4 X Y'$$

Now Seperate Variables

$$\frac{X''}{4X} = \frac{Y'}{Y}$$

Consider a Seperation Constant $(-\lambda)$

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$$

we obtain two ODE's

$$X'' + 4\lambda X = 0 \quad ; \quad Y' + \lambda Y = 0$$

We consider three cases for λ: zero, negative or Positive.

Case : 1 :- If $\lambda = 0$, then

$$X'' = 0 \;; \quad Y' = 0.$$

From $X'' = 0$; From $Y' = 0$

$m^2 = 0$; $m = 0$

$m = 0, 0$

$X = C_1 + C_2 X$; $Y = C_3$

General Solution is

$$u = XY = (C_1 + C_2 X) C_3 = A_1 + B_1 X$$

Case : 2 :- If $\lambda = -\alpha^2$, then

$$X'' - 4\alpha^2 X = 0 \;; \quad Y' - \alpha^2 Y = 0$$

From $X'' - 4\alpha^2 X = 0$; From $Y' - \alpha^2 Y = 0$

$m^2 - 4\alpha^2 = 0$; $m - \alpha^2 = 0$

$m = \pm 2\alpha$; $m = \alpha^2$

$$X = C_4 e^{2\alpha x} + C_5 e^{-2\alpha x}$$

$$= C_4 \cosh 2\alpha x + C_5 \sinh 2\alpha x \qquad Y = C_6 e^{\alpha^2 y}.$$

General Solution is

$$u = XY = C_6 e^{\alpha^2 y} (C_4 \cosh 2\alpha x + C_5 \sinh 2\alpha x)$$

$$= A_2 e^{\alpha^2 y} \cosh 2\alpha x + B_2 e^{\alpha^2 y} \sinh 2\alpha x.$$

Case III :- If $\lambda = \alpha^2$, then

$$X'' + 4\alpha^2 X = 0 \quad ; \quad Y' + \alpha^2 Y = 0$$

From $X'' + 4\alpha^2 X = 0$; $Y' + \alpha^2 Y = 0$

$m^2 + 4\alpha^2 = 0$ $\qquad$ $m + \alpha^2 = 0$

$m^2 = -4\alpha^2$ $\qquad$ $m = -\alpha^2$.

$m = \pm 2\alpha i$

$X = C_7 \cos 2\alpha x + C_8 \sin 2\alpha x$ $\qquad$ $Y = C_9 e^{-\alpha^2 y}$.

General Solution is

$$u = XY = (C_7 \cos 2\alpha x + C_8 \sin 2\alpha x) C_9 e^{-\alpha^2 y}$$

$$= A_3 e^{-\alpha^2 y} \cos 2\alpha x + B_3 e^{-\alpha^2 y} \sin 2\alpha x.$$

Classical PDE's and Boundary Value Problems:-

Classical Equations:-

- $k \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $k > 0$ → one-dimensional Heat Equation
- $a \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ → one-dimensional wave Equation
- $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ → two-dimensional Laplace Equation

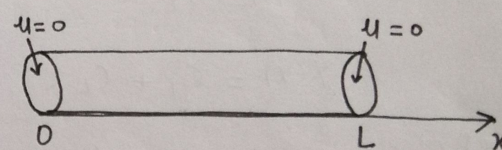
Heat Equation

Consider a thin rod of length L with an initial temperature $f(x)$ and whose ends are held at temperature zero. Then the temperature $u(x,t)$ in the rod is determined from the Boundary-Value Problem:

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < L, \; t > 0 \quad \text{— (1)}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \quad \text{— (2)}$$

$$u(x,0) = f(x), \quad 0 < x < L \quad \text{— (3)}$$



Temperature in a rod of Length L.

Solution of the BVP:-

Consider the Solution

$$u(x,t) = X(x)\,T(t).$$

Put in Equ (1) to Seperate Variables

$$k \frac{\partial^2}{\partial x^2}(XT) = \frac{\partial}{\partial t}(XT)$$

$$kT \frac{d^2 X}{dx^2} = X \frac{dT}{dt}$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{kT}\frac{dT}{dt} = -\lambda \;(\text{Constant}) \quad \text{— (4)}$$

Seperation

It Leads to two ordinary differential Equation

$$\frac{d^2X}{dx^2} + \lambda X = 0 \qquad -(5)$$

$$\frac{dT}{dt} + (\lambda k)T = 0 \qquad -(6)$$

Before Solving (5), we note the B.Cs (2). applied to $u(x,t) = X(x)T(t)$

$u(0,t) = X(0)T(t) = 0$ and $u(L,t) = X(L)T(t) = 0$

$\Rightarrow X(0) = 0 \quad \because T(t) \neq 0$ $\qquad \Rightarrow X(L) = 0 \quad \because T(t) \neq 0$

Therefore we get Homogeneous DE (5) with Homogeneous BC's.

$$\frac{d^2X}{dx^2} + \lambda X = 0 \; ; \; X(0) = 0, \; X(L) = 0. \qquad -(7)$$

Now we consider three Possible cases for λ.

- $\underline{\lambda = 0}$: $m^2 + \lambda = 0$

$m^2 = 0$

$m = 0, 0$

$X(x) = C_1 + C_2 x$

BC's :- $X(0) = 0 \Rightarrow C_1 = 0 \quad \Rightarrow X(x) = C_2 x$

$X(L) = 0 \Rightarrow C_2 \cdot L = 0 \quad \Rightarrow \quad X(x) = 0$

$C_2 = 0$

$\therefore u(x,t) = X(x)T(t) = 0$.

- $\underline{\lambda = -\alpha^2 < 0}$: $m^2 - \alpha^2 = 0$

$m^2 = \alpha^2$

$m = \pm \alpha$.

$X(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$.

BC's :- $X(0) = 0 \Rightarrow C_1 = -C_2 \qquad \Rightarrow X(x) = C_1 e^{\alpha x} - C_1 e^{-\alpha x} = C_1 \dfrac{(e^{2\alpha x} - 1)}{e^{\alpha x}}$.

$C_2 = -C_1$

$X(L) = 0 \Rightarrow C_1\left(\dfrac{e^{2\alpha L} - 1}{e^{\alpha L}}\right) = 0 \Rightarrow C_1(e^{2\alpha L} - 1) = 0$

$c_1 = 0 \quad , \quad e^{2\alpha L} - 1 = 0$

$\because 2\alpha L \neq 0$

$\Rightarrow e^{2\alpha L} - 1 \neq 0$

$\therefore X(x) = 0$

and $u(x,t) = X(x)\,T(t) = 0$.

- $\underline{\lambda = \alpha^2 > 0}$:- $m^2 + \lambda = 0$

$m^2 = -\lambda$

$m = \pm\sqrt{\lambda}\, i$

$$X(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$$

BC's :- $X(0) = 0 \Rightarrow c_1 = 0 \qquad \therefore X(x) = c_2 \sin\sqrt{\lambda}\, x$

$X(L) = 0 \Rightarrow c_2 \sin\sqrt{\lambda}\, L = 0$

$c_2 = 0 \quad , \quad \sin\sqrt{\lambda}\, L = 0$.

Suppose $c_2 \neq 0$, Since $c_2 = 0$ leads us to the obvious solution.

$\sin\sqrt{\lambda}\, L = 0$

$\sqrt{\lambda}\, L = \sin^{-1}(0)$

$\sqrt{\lambda}\, L = n\pi \quad : n = 1, 2, \ldots$

$\sqrt{\lambda} = \dfrac{n\pi}{L}$

$\lambda = \dfrac{n^2\pi^2}{L^2}, \quad n = 1, 2, \ldots \rightarrow$ These values of λ are the Eigen values of the Problem.

$\boxed{X(x) = c_2 \sin \frac{n\pi}{L} x}$; $n = 1, 2, 3, \ldots \rightarrow$ values of $X(x)$ are the Eigen Functions.

Now Consider ODE (6)

$$\frac{dT}{dt} + (\lambda k)T = 0.$$

For $\lambda > 0$:- $m + k\lambda = 0$

$$m = -k\lambda$$

$$T(t) = C_3 e^{(-k\lambda)t}.$$

Therefore

$$u(x,t) = X(x)T(t)$$

$$= C_2 C_3 \sin\frac{n\pi}{L}x \, e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t}$$

$$u_n(x,t) = A_n \sin\frac{n\pi}{L}x \cdot \exp\left(-k\frac{n^2\pi^2}{L^2}t\right) \quad ; \quad n = 1, 2, 3, \ldots$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \exp\left(-k\frac{n^2\pi^2}{L^2}t\right)$$

Now Apply Condition (2)

$$u(x,0) = f(x)$$

$$\Rightarrow \quad f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

Multiply with $\sin\frac{m\pi}{L}x$ and integrate from $0 \to L$

$$\sum_{n=1}^{\infty} A_n \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi}{L}x \, dx = \int_0^L f(x) \sin\frac{m\pi}{L}x \, dx$$

For $m = n$

$$A_n\left(\frac{L}{2}\right) = \int_0^L f(x) \sin\frac{n\pi}{L}x \, dx$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x \, dx.$$

$$\because \int_0^L \sin\frac{m\pi}{L}x \sin\frac{n\pi}{L}x \, dx = \begin{cases} 0 & : m \neq n \\ L/2 & : m = n \end{cases}$$

we Conclude that the Solution of the Boundary-value Problem is given by infinite Series

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x) \sin\frac{n\pi}{L}x \, dx\right) \sin\frac{n\pi}{L}x \, \exp\left(-k\frac{n^2\pi^2}{L^2}t\right).$$

NOTE:- If Second Condition Changes from $f(x)$

$$u(0,t)=0 \quad , \quad u(L,t)=0$$

$$u(x,0)=\begin{cases} 1 & 0<x<L/2 \\ 0 & L/2<x<L \end{cases}$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

$$= \frac{2}{L}\int_0^{L/2} \sin \frac{n\pi}{L} x \, dx .$$

Practice Problems:-

Ex 12.1 ; Q: 1-26

Ex 12.3 ; Q: 1-6