Excercise 4.2

(2) $y'' + 2y' + y = 0; \quad y_1 = xe^{-x}$.

we use reduction of order method

Let $y = u(x)\, y_1(x)$

$= u(x)\, xe^{-x}$.

$y' = u(xe^{-x})' + xe^{-x}u' = u(e^{-x} - xe^{-x}) + xe^{-x}u'$

$y'' = u(e^{-x} - xe^{-x})' + u'(e^{-x} - xe^{-x}) + (xe^{-x})'u' + (xe^{-x})u''$

$= u[-e^{-x} - (e^{-x} - xe^{-x})] + u'(e^{-x} - xe^{-x}) + (e^{-x} - xe^{-x})u' + xe^{-x}u''$

Given D.E $\Rightarrow$

$y'' + 2y' + y = u[\cancel{-e^{-x}} \cancel{-e^{-x}} + \cancel{xe^{-x}}] + 2u'(e^{-x} - x\cancel{e^{-x}}) + xe^{-x}u'' + 2u(\cancel{e^{-x}} - \cancel{xe^{-x}})$

$+ 2xe^{-x}u' + \cancel{uxe^{-x}}$

$= \cancel{u[xe^{-x} - 2e^{-x}] + 2u'(e^{-x} - xe^{-x}) + xe^{-x}u''}$.

$= \cancel{xe^{-x}[u'' - 2u' + u] + 2u'e^{-x} - 2ue^{-x}}$.

$= xe^{-x}u'' + 2e^{-x}u'$

$= (xu'' + 2u')e^{-x} = 0$

$\Rightarrow \quad xu'' + 2u' = 0 \qquad \because e^{-x} \neq 0$.

$u'' + \frac{2}{x}u' = 0$.

$w' + \frac{2}{x}w = 0$.

I.F: $e^{2\int 1/x\, dx} = e^{2\ln x} = x^2$.

$\frac{d}{dx}[x^2 w] = 0$.

$x^2 w = c_1$

$w = c_1/x^2$

$u' = c_1/x^2$

$u = -c_1/x + c_2$

Let $c_1 = -1, c_2 = 0 \qquad u = +1/x$

$\frac{y_2}{y_1} = +\frac{1}{x}$

$y_2 = +\frac{xe^{-x}}{x} = e^{-x}$.

General Solution.

$$\boxed{y = c_1 xe^{-x} + c_2 e^{-x}}$$

Exercise:-

~~(12) $4x^2y''+y=0$~~ ; ~~$y_1 = x^{1/2}\ln x$~~ .

~~Standard form~~

(5) $y''-y=0;\quad y_1=\cosh x.$

Let $y = uy_1 = u\cosh x.$

$y' = u\sinh x + u'\cosh x.$

$y'' = u\cosh x + u'\sinh x + u'\sinh x + u''\cosh x$

$= u\cosh x + 2u'\sinh x + u''\cosh x.$

Given D.E $\Rightarrow$

$y''-y = \cancel{u\cosh x} + 2u'\sinh x + u''\cosh x - \cancel{u\cosh x}.$

$= u''\cosh x + 2u'\sinh x$

$= 0.$

$\Rightarrow u''\cosh x + 2u'\sinh x = 0.$

$u'' + 2u'\dfrac{\sinh x}{\cosh x} = 0.$

$u'' + 2\tanh x\, u' = 0.$

$w' + 2\tanh x\, w = 0.$

I.F: $e^{2\int \tanh x\, dx} = e^{2\ln|\cosh x|} = \cosh^2 x.$

$\dfrac{d}{dx}[\cosh^2 x\, w] = 0.$

$\cosh^2 x\, w = c_1$

$w = c_1\operatorname{sech}^2 x$

$u' = c_1\operatorname{sech}^2 x.$

$u = c_1\int \operatorname{sech}^2 x\, dx$

$u = c_1\tanh x + c_2$

Let $c_1=1, c_2=0 \Rightarrow u = \tanh x.$

If $u = \tanh x.$

$y_2 = y_1\tanh x.$

$y_2 = \tanh x\cosh x.$

General Solution.

$$\boxed{y = c_1\cosh x + c_2\tanh x\cosh x}$$

(8)

$x^2y'' - 7xy' + 16y = 0, \quad y_1 = x^4.$

Standard form

$$y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0; \qquad P(x) = -7/x.$$

$\left\{ \bullet \int \frac{f'(x)}{f(x)}dx = \ln|f(x)|. \right.$

$\left\{ \circ \int f(x)^n . f'(x)\,dx \quad \frac{f(x)^{n+1}}{n+1} \right.$

Using formula

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$= x^4 \int \frac{e^{-\int -7/x\, dx}}{x^8} dx$$

$$= x^4 \int \frac{e^{7\int 1/x\, dx}}{x^8} dx$$

$$= x^4 \int \frac{e^{7\ln x}}{x^8} dx$$

$$= x^4 \int \frac{x^7}{x^8} dx$$

$$= x^4 \int \frac{1}{x} dx = x^4 \ln x.$$

General Solution

$$y = c_1 x^4 + c_2 x^4 \ln x.$$

(12) $4x^2y'' + y = 0; \quad y_1 = x^{1/2}\ln x.$

Standard form

$$y'' + \frac{1}{4x^2}y = 0; \qquad P(x) = 0$$

Using formula.

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$= x^{1/2}\ln x \int \frac{e^{-\int 0 dx}}{x(\ln x)^2}$$

$$y_2 = x^{1/2}\ln x \int \frac{1}{x(\ln x)^2} dx$$

$$= x^{1/2}\ln x \int (\ln x)^{-2} \cdot \frac{1}{x} dx$$

$$= x^{1/2}\ln x \, \frac{(\ln x)^{-1}}{-1}$$

$$= -x^{1/2}.$$

$\therefore y_2 = x^{1/2}$

$$y = c_1 x^{1/2}\ln x + c_2 x^{1/2}.$$

Use reduction of order to find the Second Solution y_2 of homogeneous Equ^n.
Also find the Particular Solution of given Non-homogeneous Equ.

(17) $y'' - 4y = 2, \quad y = e^{-2x}$

using formula

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^2} dx$$

$$= e^{-2x} \int \frac{e^{+\int x dx}}{e^{-4x}} dx$$

$$= e^{-2x} \int \frac{e^{4x}}{e^{-4x}} dx$$

$$= e^{-2x} \int e^{-8x} dx$$

$$= -\frac{1}{8} e^{-2x} e^{-8x}$$

$$= -\frac{1}{8} e^{-10x}.$$

General Solution / Complementary Solution.

$$y_c = c_1 e^{-2x} - \frac{c_2}{8} e^{-10x}.$$