5.1 : Examples:-

Free Damped Motion:

Example:- (overdamped Motion)

The Mass is initially released from a Position 1 unit below the equilibrium Position with a downward velocity of 1 ft/s.

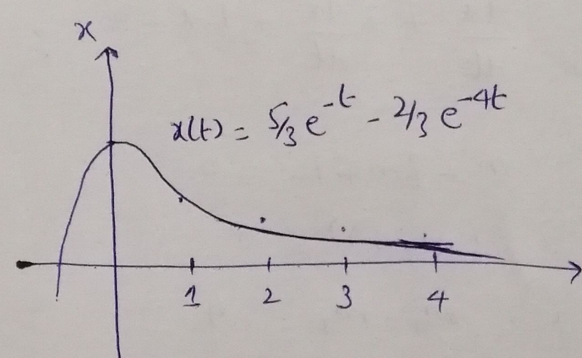
we have a D.E

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0 ; \quad x(0) = 1, \; x'(0) = 1 .$$

with solution $x(t) = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t}$.

To Check the behaviour of a moving mass.

t	x(t)
1	0.601
1.5	0.370
2	0.225
2.5	0.137
3	0.083



Over Damped System.

Equ Shows smooth and non oscillatory motion.

Example:- (Critically Damped Motion)

A Mass weighing 8 Pounds stretches a spring 2 feet. Assuming that a damping force numerically equal to 2 times the intantaneous velocity acts on the system; determine the equation of motion if the mass is initially released from the equilibrium Position with an upward velocity of 3 ft/s.

$W = 8\,lb, \quad S = 2\,ft, \quad \beta = 2$

$W \propto S$

$W = kS$

$k = W/S = 8/2 = 4\ lb/ft$

$W = mg$

$m = W/g = 8/32 = 1/4$ slug

Equ of Motion

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

$$\frac{1}{4}\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = 0$$

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0 \ ; \quad x(0) = 0; \quad x'(0) = -3$$

A.E:- $m^2 + 8m + 16 = 0$

$(m+4)^2 = 0$

$m_1 = m_2 = -4$

Hence System is Critically Damped, and

$$x(t) = c_1 e^{-4t} + c_2 t e^{-4t}.$$

Apply IC's.

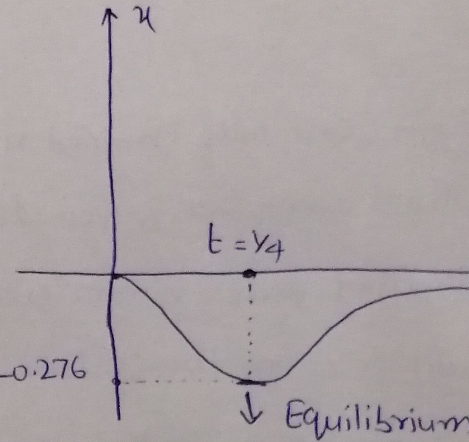
$x(0) = 0 \Rightarrow c_1 = 0$

$x'(t) = -4c_1 e^{-4t} + c_2(e^{-4t} - 4te^{-4t})$

$x'(0) = -3 \Rightarrow -3 = -4c_1 + c_2$

$c_2 = -3.$

Equ of Motion is $\boxed{x(t) = -3te^{-4t}}$



Critically Damped

Example:- (Underdamped Motion)

A mass weighing 16 Pounds is attached to a 5-foot-long spring. At equilibrium the spring measures 8.2 feet. If the mass is initially released from rest at a point 2 feet above the equilibrium position. find $x(t)$.

Solu: Elongation of spring after mass is attached is $8.2 - 5 = 3.2\,ft = s$.

$$W = ks$$
$$k = W/s = 16/3.2 = 5\ lb/ft$$

$$W = mg$$
$$m = W/g = 16/32 = \tfrac{1}{2}\ \text{slug}.$$

Equ of Motion

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

$$\frac{1}{2}\frac{d^2x}{dt^2} + \frac{dx}{dt} + 5x = 0$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0, \qquad x(0) = -2,\ x'(0) = 0$$

A.E:- $m^2 + 2m + 10 = 0$

$m_1 = -1+3i,\ m_2 = -1-3i$. System is underdamped.

Equ of Motion:-

$$\boxed{x(t) = e^{-t}(C_1\cos 3t + C_2\sin 3t)}$$

Apply IC's: $x(0) = -2 \Rightarrow -2 = C_1$

$x'(t) = -e^{-t}(C_1\cos 3t + C_2\sin 3t) + e^{-t}(-3C_1\sin 3t + 3C_2\cos 3t)$

$x'(0) = 0 \Rightarrow 0 = -C_1 + 3C_2 \Rightarrow C_2 = -2/3.$

$$\boxed{x(t) = e^{-t}\left(-2\cos 3t - \frac{2}{3}\sin 3t\right)}$$ Eqn of Motion.

Forced Undamped Motion:-

Solve IVP:

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin\gamma t \; ; \quad x(0)=0 \; ; \; x'(0)=0 .$$

where F_0 is a constant and $\gamma \neq \omega$.

Soln:- Homogeneous Eqn: $\frac{d^2x}{dt^2} + \omega^2 x = 0$

A.E:- $m^2 + \omega^2 = 0$

$m = \pm i\omega$

$$\boxed{x_c(t) = C_1 \cos\omega t + C_2 \sin\omega t}$$

Let $x_p = A\cos\gamma t + B\sin\gamma t$

$x_p' = -A\gamma \sin\gamma t + B\gamma\cos\gamma t$

$x_p'' = -A\gamma^2\cos\gamma t - B\gamma^2 \sin\gamma t .$

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin\gamma t$$

$$-A\gamma^2\cos\gamma t - B\gamma^2\sin\gamma t + \omega^2(A\cos\gamma t + B\sin\gamma t) = F_0\sin\gamma t$$

$$A(\omega^2-\gamma^2)\cos\gamma t + B(\omega^2-\gamma^2)\sin\gamma t = F_0 \sin\gamma t .$$

Equating Coefficients $A=0$, $B = \frac{F_0}{\omega^2-\gamma^2}$.

$$\therefore \quad \boxed{x_p = \frac{F_0}{\omega^2-\gamma^2} \sin \gamma t}$$

Apply I.Cs to General Solution.

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t + \frac{F_0}{\omega^2-\gamma^2} \sin \gamma t$$

$$x(0) = 0 \Rightarrow \boxed{C_1 = 0}$$

$$x'(t) = -C_1 \omega \sin \omega t + C_2 \omega \cos \omega t + \frac{\gamma F_0}{\omega^2-\gamma^2} \cos \gamma t.$$

$$x'(0) = 0 \Rightarrow C_2 \omega + \frac{\gamma F_0}{\omega^2-\gamma^2} = 0 \Rightarrow \boxed{C_2 = \frac{-\gamma F_0}{\omega(\omega^2-\gamma^2)}}$$

Final Solution.

$$x(t) = \frac{-\gamma F_0}{\omega(\omega^2-\gamma^2)} \sin \omega t + \frac{F_0}{\omega^2-\gamma^2} \sin \gamma t.$$

$$\boxed{x(t) = \frac{F_0}{\omega(\omega^2-\gamma^2)} (-\gamma \sin \omega t + \omega \sin \gamma t) ; \quad \gamma \neq \omega}$$

Equ of Motion.
General Solution in forced undamped Motion.

Practice Problems:-

Ex: 5.1

Q 1-11.