7.4: Operational Properties II:-

7.4.1: Derivatives of Transforms:-

The Laplace Transform of the product of function $f(t)$ with t Results into differentiating the Laplace Transform of $f(t)$.

If $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt.$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\,dt\right]$$

$$= -\int_0^\infty t e^{-st} f(t)\,dt = -\mathcal{L}\{t f(t)\} \Rightarrow \boxed{\mathcal{L}\{t f(t)\} = -\frac{d}{ds} F(s)}$$

Also

$$\frac{d^2}{ds^2} F(s) = \frac{d}{ds} \cdot \frac{d}{ds} F(s) = -\frac{d}{ds} \mathcal{L}\{t f(t)\}$$

$$= -\frac{d}{ds} \int_0^\infty e^{-st} t f(t)\,dt.$$

$$= -\int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} t f(t)\right] dt$$

$$= \int_0^\infty e^{-st} t^2 f(t)\,dt = \mathcal{L}\{t^2 f(t)\} \Rightarrow \boxed{\mathcal{L}\{t^2 f(t)\} = \frac{d^2}{ds^2} F(s)}$$

In a Similar way we can write.

Theorem:- If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

Example:- Evaluate $\mathcal{L}\{t \sin kt\}$.

$$\mathcal{L}\{t \sin kt\} = -\frac{d}{ds} \mathcal{L}\{\sin kt\} = -\frac{d}{ds}\left[\frac{k}{s^2+k^2}\right] = \frac{2ks}{(s^2+k^2)^2}$$

Example:- Evaluate $\mathcal{L}\{t e^{3t}\}$

1st translation Theorem

(i) $\mathcal{L}\{t e^{3t}\} = \mathcal{L}\{t\}\big|_{s\to s-3} = \frac{1}{s^2}\Big|_{s\to s-3} = \frac{1}{(s-3)^2}$

Derivative of Transform

(ii) $\mathcal{L}\{t e^{3t}\} = -\frac{d}{ds} \mathcal{L}\{e^{3t}\} = -\frac{d}{ds}\left[\frac{1}{s-3}\right] = \frac{1}{(s-3)^2}$.

Example:-

$$\mathcal{L}\{t e^{2t} \sin 6t\} = -\frac{d}{ds} \mathcal{L}\{e^{2t} \sin 6t\}$$

$$= -\frac{d}{ds}\left[\mathcal{L}\{\sin 6t\}\big|_{s\to s-2}\right]$$

$$= -\frac{d}{ds}\left[\frac{6}{s^2+36}\Big|_{s\to s-2}\right]$$

$$= -\frac{d}{ds}\left[\frac{6}{(s-2)^2+36}\right]$$

$$= \frac{6 \cdot 2(s-2)}{[(s-2)^2+36]^2}$$

$$= \frac{12(s-2)}{[(s-2)^2+36]^2}.$$

7.4.2 Transforms of Integral :-

Convolution :- If functions f and g are piecewise Continuous on the interval $[0,\infty)$, then

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

is called the Convolution of f and g.

Convolution Theorem :-

If $f(t)$ and $g(t)$ are piecewise Continuous on $[0,\infty)$ and of exponential order, then

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\} = F(s)\,G(s).$$

Example :- Evaluate $\mathcal{L}\left\{\int_0^t e^{\tau} \sin(t-\tau)\, d\tau\right\}$.

$$\mathcal{L}\left\{\int_0^t e^{\tau} \sin(t-\tau)\, d\tau\right\} = \mathcal{L}\{e^t * \sin t\}$$

$$= \mathcal{L}\{e^t\} \cdot \mathcal{L}\{\sin t\}$$

$$= \frac{1}{s-1} \cdot \frac{1}{s^2+1}$$

$$= \frac{1}{(s-1)(s^2+1)}.$$

NOTE :-

If $g = 1$

Then

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f*1\}$$

$$= F(s) \cdot \frac{1}{s}$$

$$= \frac{F(s)}{s}$$

Example :-

$$\mathcal{L}\left\{t\int_0^t \sin\tau\, d\tau\right\} = -\frac{d}{ds}\,\mathcal{L}\left\{\int_0^t \sin\tau\, d\tau\right\}$$

$$= -\frac{d}{ds}\left(\frac{1}{s} \cdot \frac{1}{s^2+1}\right)$$

$$= -\frac{d}{ds}\left(\frac{1}{s(s^2+1)}\right)$$

$$= \frac{1}{s^2(s^2+1)^2} \cdot 3s^2+1$$

$$= \frac{3s^2+1}{s^2(s^2+1)^2}.$$

Example :-

$$\mathcal{L}\left\{\int_0^t e^{-\tau} \cos\tau\, d\tau\right\} = \frac{1}{s}\,\mathcal{L}\{e^{-t} \cos t\}$$

$$= \frac{1}{s}\,\mathcal{L}\{\cos t\}\Big|_{s \to s+1}$$

$$= \frac{1}{s}\,\frac{s}{s^2+1}\Big|_{s \to s+1}$$

$$= \frac{1}{s}\,\frac{s+1}{(s+1)^2+1}$$

$$= \frac{s+1}{s(s^2+2s+2)}.$$

Inverse Convolution Theorem:-

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f*g.$$

and $$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau.$$

Example:- $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+k^2}\cdot\frac{1}{s^2+k^2}\right\} = \frac{1}{k^2}(\sin kt)*(\sin kt)$$

$-2\sin\alpha\sin\beta = \cos(\alpha+\beta)-\cos(\alpha-\beta)$

$2\sin\alpha\sin\beta = \cos(\alpha-\beta)-\cos(\alpha+\beta)$

$$= \frac{1}{k^2}\int_0^t \sin k\tau\,\sin k(t-\tau)\,d\tau.$$

$$= +\frac{1}{2k^2}\int_0^t [\cos k(2\tau-t) - \cos kt]\,d\tau.$$

$$= +\frac{1}{2k^2}\int_0^t \cos k(2\tau-t)\,d\tau - \frac{1}{2k^2}\int_0^t \cos kt\,d\tau.$$

$$= +\frac{1}{2k^2}\left[\frac{\sin k(2\tau-t)}{2k}\right]_0^t - \frac{1}{2k^2}\,\tau\cos kt\Big|_0^t$$

$$= \frac{1}{4k^3}\left[\sin kt + \sin kt\right] - \frac{1}{2k^2}\left[t\cos kt - 0\right]$$

$$= \frac{2\sin kt - 2kt\cos kt}{4k^3}.$$

$$= \frac{\sin kt - kt\cos kt}{2k^3}$$

Example:-

i) $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/(s^2+1)}{s}\right\} = \int_0^t \sin\tau\,d\tau = -\cos\tau\Big|_0^t = -\cos t + 1 = 1-\cos t.$

ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/(s(s^2+1))}{s}\right\} = \int_0^t (1-\cos\tau)\,d\tau = t-\sin t$

or $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = t*\sin t = \int_0^t \tau\sin(t-\tau)\,d\tau.$

iii) $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/(s^2(s^2+1))}{s}\right\} = \int_0^t (\tau-\sin\tau)\,d\tau = \frac{1}{2}t^2+\cos t-1$

or $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \frac{1}{2!}t^2*\sin t = \frac{1}{2}\int_0^t \tau^2\sin(t-\tau)\,d\tau.$

Example:- Solve the Integral Equation for $f(t)$.

$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}\,d\tau$$

$$\mathcal{L}\{f(t)\} = 3\mathcal{L}\{t^2\} - \mathcal{L}\{e^{-t}\} - \mathcal{L}\{f(t) * e^t\}$$

$$F(s) = 3\cdot\frac{2!}{s^3} - \frac{1}{s+1} - \mathcal{L}\{f(t)\}\mathcal{L}\{e^t\}$$

$$= \frac{6}{s^3} - \frac{1}{s+1} - F(s)\cdot\frac{1}{s-1}$$

$$\left[1 + \frac{1}{s-1}\right]F(s) = \frac{6}{s^3} - \frac{1}{s+1}$$

$$\frac{s}{s-1}F(s) = \frac{6}{s^3} - \frac{1}{s+1}$$

$$F(s) = \frac{s-1}{s}\left[\frac{6}{s^3} - \frac{1}{s+1}\right]$$

$$= \frac{6(s-1)}{s^4} - \frac{s-1}{s(s+1)}$$

$$\left\{\begin{aligned} &\frac{s-1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \\ &A = \left.\frac{s-1}{s+1}\right|_{s=0} = -1 \\ &B = \left.\frac{s-1}{s}\right|_{s=-1} = 2 \end{aligned}\right.$$

$$= \frac{6s}{s^4} - \frac{6}{s^4} - \left[-\frac{1}{s} + \frac{2}{s+1}\right]$$

$$= \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

$$\mathcal{L}^{-1}\{F(s)\} = 3\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\left\{\frac{1}{s+1}\right\}$$

$$\boxed{f(t) = 3t^2 - t^3 + 1 - 2e^{-t}}.$$

Practice Problems:-

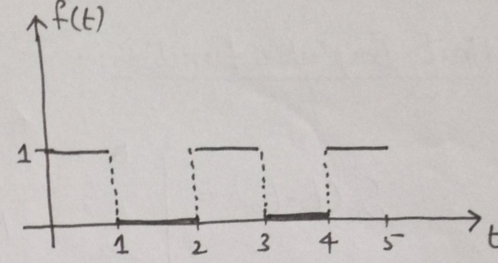
Ex 7.4:

Q: 1-14, 19-34, 37-46, 49-54

7.4.3 Transform of a Periodic Function:-

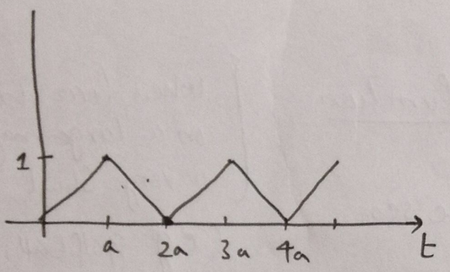
Periodic function:-

$\Rightarrow \quad f(t) = \begin{cases} 1 & : 0 \le t < 2 \\ 0 & : 2 \le t < 3 \end{cases}$



is a Periodic function having Period $T = 3 > 0$; then $f(t+T) = f(t)$

$\Rightarrow$ Also



Here Period $T = 2a$.

Transform of a Periodic function:-

If $f(t)$ is piecewise Continuous on $[0,\infty)$, of exponential order, and Periodic with Period T, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt .$$

Example:- Find the Laplace Transform of the Periodic function.

$$E(t) = \begin{cases} 1 & , 0 \le t < 1 \\ 0 & , 1 \le t < 2 \end{cases}$$

The Function $E(t)$ is called Square wave function and has period $T=2$ on the interval $0 \le t < 2$ and outside the interval $f(t+2) = f(t)$.

Now

$$\mathcal{L}\{E(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} E(t)\, dt = \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} dt + \int_1^2 e^{-st} \cdot 0 \, dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{e^{-st}}{s} \right]_0^1 = \frac{1}{1-e^{-2s}} \cdot \frac{1-e^{-s}}{s} = \frac{\cancel{1-e^{-s}}}{s(\cancel{1-e^{-s}})(1+e^{-s})} = \frac{1}{s(1+e^{-s})}$$