4.7: Cauchy Euler Equation:

Reduction to Constant Coefficients:-

Any Cauchy Euler Equation can always be rewritten as a Linear differential Equation with Constant Coefficients by means of Substitution $x = e^t$. The new D.E is in terms of variable t, and can be solved using Previous Methods.

Ex:- $x^2y'' - xy' + y = \ln x$.

Solu:- with the Substitution $x = e^t$ or $t = \ln x$, it follows that

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\cdot\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x}\left(\frac{d^2y}{dt^2}\cdot\frac{dt}{dx}\right) + \frac{dy}{dt}\left(-1/x^2\right)$$

$$= \frac{1}{x}\left(\frac{d^2y}{dt^2}\cdot\frac{1}{x}\right) + \frac{dy}{dt}\left(-1/x^2\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Substitute value into D.E, we get

$$x^2y'' - xy' + y = \ln x$$

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t \rightarrow$$ Non-Homogeneous D.E in terms of t with Constant Coefficients.

we need to find $y = y_c + y_p$.

Find Complementary Solution :-

Auxiliary Equ is : $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$ Repeated Roots.

$y_c = c_1 e^t + c_2 t e^t$.

Find Particular Solution :-

By Undetermined Coefficient (U.C) Method, we Consider

$y_p = A + Bt$.

$y_p' = B$

$y_p'' = 0$.

D.E $\Rightarrow$ $-2B + A + Bt = t$

Comparing, we get $B = 1$, $A \mp 2B = 0$

$A = 2$

$\therefore$ $y_p = 2 + t$

Therefore; General Solution is

$$y = c_1 e^t + c_2 t e^t + 2 + t$$

Resubstitute values; we get Final Solution

$$\boxed{y = c_1 x + c_2 x \ln x + 2 + \ln x}$$

Practice Problems:-

Use Substitution $x = e^t$ to transform Cauchy Euler Equ to D.E with Constant Coefficients. Then Solve the Equation.

1) $x^2y'' + 10xy' + 8y = x^2$

2) $x^2y'' - 4xy' + 6y = \ln x^2$

3) $4x^2y'' + y = 0; \quad y(-1) = 2; \quad y'(-1) = 4.$

Related Problems:-

Ex 4.7

Q: 31 – 38.