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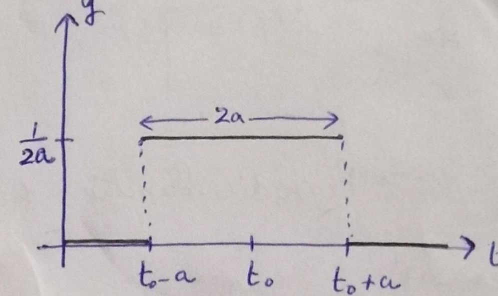
7.5 The Dirac Delta Function :-

Unit Impulse Function :-

$$\delta_a(t-t_0) = \begin{cases} 0 & 0 \le t < t_0 - a \\ \frac{1}{2a} & t_0 - a \le t < t_0 + a \\ 0 & t \ge t_0 + a . \end{cases}$$



$a > 0, \; t_0 > 0$.

$\delta_a(t-t_0)$ is called a Unit Impulse function.

when force is applied on a large magnitude for a very short Period of time (e.g golf ball, tennis ball, base ball, Heart beat shown in machine)

Unit step function $\delta_a(t-t_0)$ also possesses the integration Property

$$\int_0^\infty \delta_a(t-t_0)\,dt = 1 .$$

Dirac Delta Function :- It is Convenient to work with another type of unit Impulse function, that approximates $\delta_a(t-t_0)$ as Limit $a \to 0$.

$$\delta(t-t_0) = \lim_{a \to 0} \delta_a(t-t_0) = \begin{cases} \infty & ; \; t = t_0 \\ 0 & ; \; t \ne t_0 . \end{cases}$$

The Unit-Impulse function $\delta(t-t_0)$ is called the Dirac Delta function.

Dirac Delta function also satisfies

$$\int_0^\infty \delta(t-t_0)\,dt = 1 .$$

Transform of the Dirac Delta function :-

For $t_0 > 0$, $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$.

Proof:- we can write Unit Impulse function $\delta_a(t-t_0)$ in terms of Unit Step function as

$$\delta_a(t-t_0) = \frac{1}{2a}\left[\mathcal{U}(t-(t_0-a)) - \mathcal{U}(t-(t_0+a))\right]$$

$$\mathcal{L}\{\delta_a(t-t_0)\} = \frac{1}{2a}\left[\mathcal{L}\{\mathcal{U}(t-(t_0-a))\} - \mathcal{L}\{\mathcal{U}(t-(t_0+a))\}\right]$$

$$= \frac{1}{2a}\left[\frac{e^{-(t_0-a)s}}{s} - \frac{e^{-(t_0+a)s}}{s}\right]$$

$$= e^{-st_0}\left[\frac{e^{sa}-e^{-sa}}{2as}\right]$$

$$\mathcal{L}\{\delta(t-t_0)\} = \lim_{a\to 0}\mathcal{L}\{\delta_a(t-t_0)\} = e^{-st_0}\lim_{a\to 0}\left[\frac{e^{sa}-e^{-sa}}{2as}\right]$$

$$\overset{L.H}{=} e^{-st_0}\lim_{a\to 0}\frac{se^{as}+se^{-as}}{2s}$$

$$= e^{-st_0}\cdot\frac{2s}{2s}$$

$$= e^{-st_0}.$$

__Example__:- Initial Value Problem.

$$y''+y = 4\delta(t-2\pi)\ ;\ y(0)=1,\ y'(0)=0.$$

$$\mathcal{L}\{y''\}+\mathcal{L}\{y\} = 4\mathcal{L}\{\delta(t-2\pi)\}$$

$$s^2Y(s)-sy(0)-y'(0)+Y(s) = 4e^{-2\pi s}$$

$$(1+s^2)Y(s)-s = 4e^{-2\pi s}$$

$$Y(s) = \frac{s}{s^2+1}+\frac{4e^{-2\pi s}}{s^2+1}.$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+4\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\} = \cos t + 4\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\}$$

Using inverse form of Second Translation Theorem.

$$\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\} = \sin(t-2\pi)\,\mathcal{U}(t-2\pi).$$

$$\left\{\begin{array}{l} a = 2\pi, \; F(s) = \frac{1}{s^2+1} \\ f(t) = \sin t. \end{array}\right.$$

$\therefore$

$$y(t) = \cos t + 4\sin(t-2\pi)\,\mathcal{U}(t-2\pi).$$

$$= \cos t + 4\sin t\,\mathcal{U}(t-2\pi)$$

$\because \sin(t-2\pi) = \sin t.$

$$y(t) = \begin{cases} \cos t & 0 \le t < 2\pi. \\ \cos t + 4\sin t & t \geq 2\pi \end{cases}$$

Practice Problems:-

Ex 7.5

Q: 1-12.

7.6 System of Linear differential Equations:-

Example:-

$$\left.\begin{aligned} x_1'' + 10x_1 - 4x_2 &= 0 \quad -(1) \\ -4x_1 + x_2'' + 4x_2 &= 0 \quad -(2) \\ \text{Subject to } x_1(0)=0,\ x_1'(0)=1,\ &x_2(0)=0,\ x_2'(0)=-1 \end{aligned}\right\} \to (A)$$

Solu:- Taking Laplace Transform of both Equations seperately.

Equ(1) Implies

$$L\{x_1''\} + 10L\{x_1\} - 4L\{x_2\} = 0.$$

$$s^2X_1(s) - s\underbrace{x_1(0)}_{0} - \underbrace{x_1'(0)}_{1} + 10X_1(s) - 4X_2(s) = 0$$

$$\boxed{(s^2+10)X_1(s) - 4X_2(s) = 1} \quad -(3)$$

Equ(2) Implies

$$-4L\{x_1\} + L\{x_2''\} + 4L\{x_2\} = 0.$$

$$-4X_1(s) + s^2X_2(s) - s\underbrace{x_2(0)}_{0} - \underbrace{x_2'(0)}_{-1} + 4X_2(s) = 0$$

$$\boxed{-4X_1(s) + (s^2+4)X_2(s) = -1} \quad -(4).$$

Multiply Equ(3) by (s^2+4) and Equ(4) by 4, and Adding the resulting Equations

$$(s^2+10)(s^2+4)X_1(s) - \cancel{4(s^2+4)X_2(s)} = s^2+4$$

$$-16X_1(s) + \cancel{4(s^2+4)X_2(s)} = -4$$

$$[(s^2+10)(s^2+4) - 16]X_1(s) = s^2$$

$$\boxed{X_1(s) = \frac{s^2}{(s^2+2)(s^2+12)}}$$

$$\left\{\begin{aligned} \because\ &(s^2+4)(s^2+10) - 16 \\ &= s^4 + 14s^2 + 40 - 16 \\ &= s^4 + 14s^2 + 24 \\ &= s^4 + 12s^2 + 2s^2 + 24 \\ &= (s^2+2)(s^2+12) \end{aligned}\right.$$

To find $x_1(t)$:

$$X_1(s) = \frac{s^2}{(s^2+2)(s^2+12)} = \frac{-1/5}{s^2+2} + \frac{6/5}{s^2+12}$$

$$\mathcal{L}^{-1}\{X_1(s)\} = -\frac{1}{5\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} + \frac{6}{5\sqrt{12}}\mathcal{L}^{-1}\left\{\frac{\sqrt{12}}{s^2+12}\right\}$$

$$\boxed{x_1(t) = -\frac{\sqrt{2}}{10}\sin\sqrt{2}\,t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}\,t.}$$

Now Solve for $X_2(s)$:-

Put $X_1(s) = s^2/(s^2+2)(s^2+12)$ in Equ (3), we get.

$$(s^2+10)X_1(s) - 4X_2(s) = 1$$

$$\frac{s^2(s^2+10)}{(s^2+2)(s^2+12)} - 4X_2(s) = 1.$$

$$4X_2(s) = \frac{s^2(s^2+10)}{(s^2+2)(s^2+12)} - 1$$

$$X_2(s) = \frac{s^2(s^2+10) - (s^2+2)(s^2+12)}{4(s^2+2)(s^2+12)}$$

$$= \frac{-\cancel{4}(s^2+6)}{\cancel{4}(s^2+2)(s^2+12)}$$

$$\left\{\begin{aligned} \because\ & s^2(s^2+10) - (s^2+2)(s^2+12) \\ &= \cancel{s^4} + 10s^2 - \cancel{s^4} - 14s^2 - 24 \\ &= -4s^2 - 24 \\ &= -4(s^2+6) \end{aligned}\right.$$

$$\boxed{X_2(s) = -\frac{s^2+6}{(s^2+2)(s^2+12)}}$$

To find $x_2(t)$:

$$X_2(s) = -\frac{s^2+6}{(s^2+2)(s^2+12)} = \frac{-2/5}{s^2+2} - \frac{3/5}{s^2+12}$$

$$\mathcal{L}^{-1}\{X_2(s)\} = -\frac{2}{5\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} - \frac{3}{5\sqrt{12}}\mathcal{L}^{-1}\left\{\frac{\sqrt{12}}{s^2+12}\right\}$$

$$\boxed{x_2(t) = -\frac{\sqrt{2}}{5}\sin\sqrt{2}\,t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}\,t}$$

Finally the solution to the given system (A) is

$$\left.\begin{aligned} x_1(t) &= -\frac{\sqrt{2}}{10}\sin\sqrt{2}\,t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}\,t \\ x_2(t) &= -\frac{\sqrt{2}}{\sqrt{10}}\sin\sqrt{2}\,t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}\,t. \end{aligned}\right\}$$

Example:-

$$\left.\begin{aligned} \frac{dx}{dt} &= 2y + e^t \\ \frac{dy}{dt} &= 8x - t \\ x(0) &= 1,\ y(0) = 1 \end{aligned}\right\} \rightarrow (A)$$

Solu:- we can write System (A) as

$$x' - 2y = e^t \quad \text{— (1)}$$
$$y' - 8x = -t \quad \text{— (2)}.$$

Taking Laplace Transform.

Equ (1) implies

$$L\{x'\} - 2L\{y\} = L\{e^t\}$$

$$sX(s) - x(0) - 2Y(s) = \frac{1}{s-1}$$

$$sX(s) - 2Y(s) = \frac{1}{s-1} + 1.$$

$$\boxed{sX(s) - 2Y(s) = \frac{s}{s-1}} \quad \text{— (3)}.$$

Equ(2) Implies

$$\mathcal{L}\{y'\} - 8\mathcal{L}\{x\} = -\mathcal{L}\{t\}$$

$$sY(s) - y(0) - 8X(s) = -\frac{1}{s^2}$$

$$\boxed{-8X(s) + sY(s) = 1 - \frac{1}{s^2}} \quad -(4)$$

Multiply Equ(3) by s and Equ(4) by 2 and Add.

$$s^2X(s) - \cancel{2sY(s)} = \frac{s^2}{s-1}$$

$$-16X(s) + \cancel{2sY(s)} = \frac{2(s^2-1)}{s^2}$$

$$(s^2-16)X(s) = \frac{s^2}{s-1} + \frac{2(s^2-1)}{s^2}$$

$$\boxed{X(s) = \frac{s^2}{(s-1)(s^2-16)} + \frac{2(s^2-1)}{s^2(s^2-16)}}$$

Practice Problems:-

Ex 7.6

Q: 1-12.