LECTURE : 29:

Chap : 04 : Higher - Order Differential Equations

Section: 4.1 : Linear Equations:

1 Initial - Value Problem:-

For Linear differential Equ an n^{th} order Initial value Problem is

$$\left.\begin{array}{l} a_n(x)\dfrac{d^n y}{dx^n} + a_{n-1}(x)\dfrac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\dfrac{dy}{dx} + a_0(x)y = g(x) \\[2ex] \text{Subject to : } y(x_0) = y_0,\ y'(x_0) = y_1,\ \cdots,\ y^{(n-1)}(x_0) = y_{n-1} \end{array}\right\} \rightarrow (1)$$

Theorem: Existance of Unique Solution:-

Let $a_n(x)$, $a_{n-1}(x)$, $\cdots$, $a_1(x)$, $a_0(x)$ and $g(x)$ be Continuous on an Interval I and Let $a_n(x) \neq 0$ for every x in this Interval. If $x = x_0$ is any Point in this interval then a Solution $y(x)$ of I.V.P (1) exists on the interval and is unique.

Ex:- The I.V.P $3y''' + 5y'' - y' + 7y = 0$, $y(1) = 0$, $y'(1) = 0$, $y''(1) = 0$.

Verify that $y = 0$ is the solution of I.V.P

Solu : → Differential Eqn is Linear

→ The Coefficients as well as $g(x)$ are Constant, Therefore they are Continuous.

→ $a_3(x) = 3 \neq 0$ on any Interval Containing $x = 0$.

→ From Theorem, $y = 0$ is unique Solution of Given I.V.P.

Ex:- Verify that $y = 3e^{2x} + e^{-2x} - 3x$ is a Unique Solution of IV.P

$$y'' - 4y = 12x \quad , \quad y(0) = 4, \quad y'(0) = 1.$$

Solu:- → Differential Eqn is Linear

→ Coefficients and $g(x)$ are Continuous.

→ $a_2(x) = 1 \neq 0$ on any Interval Containing $x = 0$

→ From Theorem, Given IVP has a uniqu Solution.

Ex:- Verify that $y = cx^2 + x + 3$ is a unique Solution of I.V.P

$$x^2y'' - 2xy' + 2y = 6, \quad y(0) = 3, \quad y'(0) = 1.$$

Solu :- → Differential Eqn is Linear

→ Coefficients and $g(x)$ are Continuous

→ $a_2(x) = x^2 = 0$ at $x = 0$

→ From Theorem, there is no unique Solution of given IVP.

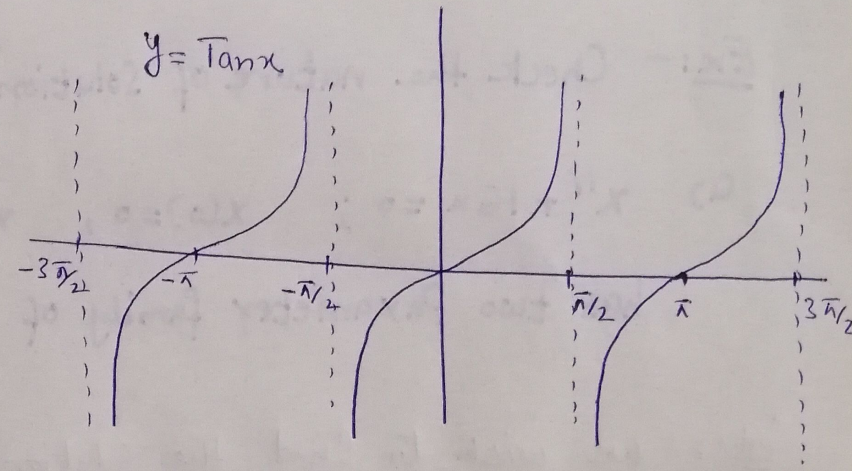
Exercise Ques: Find an Interval Centered at $x=0$ for which given IVP has a unique Solution.

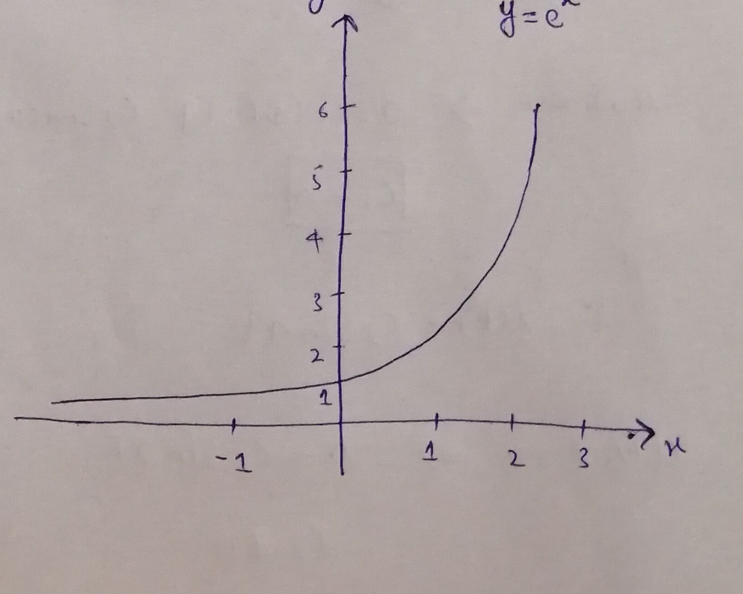
10) $y'' + (\tan x)y = e^x, \quad y(0) = 1, \quad y'(0) = 0.$

Solu $a_1 = 1 \neq 0$

$a_0 = \tan x$ and $x_0 = 0$.

IVP has a unique Solution for $-\pi/2 < x < \pi/2$.
Because $x_0 = 0$ lies in the interval $(-\pi/2, \pi/2)$ and Coefficients a_1, a_0 and $g(x)$ are Continuous in that Interval.





② Boundary value Problem :-

A Problem Such as

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$y(a) = y_0 \quad , \quad y(b) = y_1$$

is called the boundary value Problem. The values $y(a) = y_0$ and $y(b) = y_1$ are called the Boundary Conditions.

NOTE:- A BVP can have many, one or No Solutions.

Ex:- Check the nature of Solution of Given BVP.

a) $x'' + 16x = 0$; $\quad x(0) = 0, \quad x(\pi/2) = 0$

has two Parameter family of Solution $x = C_1 \cos 4t + C_2 \sin 4t$.

Solu: we wish to find the Solution that Satisfies the Boundary Conditions.

$x(0) = 0 \Rightarrow 0 = C_1 \cos(0) + C_2 \sin(0)$

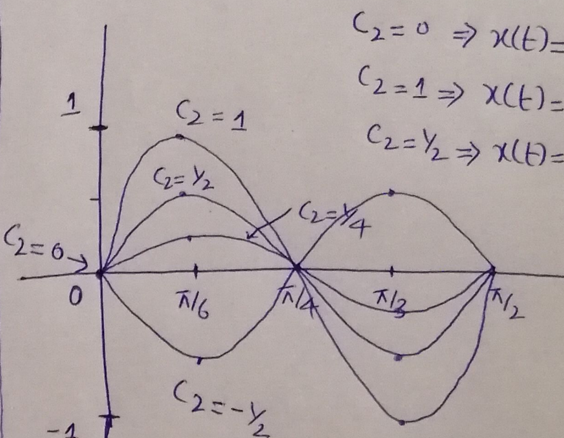
$\boxed{C_1 = 0}$

$\therefore x(t) = C_2 \sin 4t$

$C_2 = 0 \Rightarrow x(t) = 0$

$C_2 = 1 \Rightarrow x(t) = \sin 4t$

$C_2 = \frac{1}{2} \Rightarrow x(t) = \frac{1}{2} \sin 4t$



$x(\pi/2) = 0 \Rightarrow 0 = C_2 \sin 2\pi$

$\because \sin 2\pi = 0$

$C_2 \neq 0$

$C_2 = -\frac{1}{4}, \; x(t) = -\frac{1}{4}\sin 4t$

$C_2 = -\frac{1}{2}, \; x(t) = -\frac{1}{2}\sin 4t$

$\because$ $x(t) = C_2 \sin 4t$ Satisfies for all Choices of C_2

Therefore BVP $x'' + 16x = 0$; $x(0) = 0$, $x(\pi/2) = 0$ has infinitely many Solutions.

(b) $x'' + 16x = 0$; $x(0) = 0$; $x(\pi/8) = 0$.

Apply B.C's on Solution $x(t) = C_1 \cos 4t + C_2 \sin 4t$.

- $x(0) = 0 \Rightarrow$

$$C_1 = 0 \quad \Rightarrow \quad \boxed{x(t) = C_2 \sin 4t}$$

- $x(\pi/8) = 0 \Rightarrow$

$$0 = C_2 \sin 4(\pi/8)$$

$$0 = C_2 \sin \pi/2$$

$$\Rightarrow C_2 = 0 \qquad \because \sin \pi/2 = 1$$

$\because$ $x(t) = 0$ is the only solution of this BVP.

(c) $x'' + 16x = 0$; $x(0) = 0$; $x(\pi/2) = 1$.

Again Apply B.C's on Solution $x(t) = C_1 \cos 4t + C_2 \sin 4t$.

- $x(0) = 0 \Rightarrow C_1 = 0 \quad \Rightarrow \quad x(t) = C_2 \sin 4t$.

- $x(\pi/2) = 1 \Rightarrow \quad 1 = C_2 \sin 4(\pi/2)$

$$1 = C_2 \sin 2\pi$$

$\because \sin 2\pi = 0$

$$1 = C_2 \cdot 0$$

$\boxed{1 = 0}$ which is not True.

$\because$ BVP has no Solution.

Practice Problems:-

The given family of functions is the general Solution of the differential Equ on the indicated interval. find the member of the family that is a Solution of the IVP.

Related Questions
Ex 4.1
Q: 1-14

(1) $y = c_1 e^{x} + c_2 e^{-x}$, $(-\infty, \infty)$

$y'' - y = 0$, $y(0) = 0$, $y'(0) = 1$.

(3) $y = c_1 x + c_2 x \ln x$, $(0, \infty)$

$x^2 y'' - x y' + y = 0$, $y(1) = 3$, $y'(1) = -1$.

(5) $y = c_1 + c_2 x^2$; $(-\infty, \infty)$

$x y'' - y' = 0$, $y(0) = 0$, $y'(0) = 1$

(13) $y = c_1 e^{x} \cos x + c_2 e^{x} \sin x$

$y'' - 2y' + 2y = 0$; $y(0) = 1$, $y(\pi) = 0$

(9) Find an interval Centered about $x = 0$ for which IVP has a unique Solution

$(x-2) y'' + 3y = x$, $y(0) = 0$, $y'(0) = 1$.

(Hint: $a_2 = (x-2)$ $x_0 = 0$ unique Solu for $-\infty < x < 2$)

Exercise: 4.1:-

find member of the family that is a solution of I.V.P.

① $\underbrace{y = C_1e^x + C_2e^{-x}}$; $(-\infty,\infty)$ two Parameter family of solution.

$y'' - y = 0$; $y(0) = 0$; $y'(0) = 1$.

$C_1 = ?$, $C_2 = ?$

$y = C_1e^x + C_2e^{-x}$.

$y(0) = 0 \Rightarrow 0 = C_1e^0 + C_2e^{-0}$.

$\boxed{0 = C_1 + C_2}$ — (i)

$y' = C_1e^x - C_2e^{-x}$.

$y'(0) = 1 \Rightarrow 1 = C_1e^0 - C_2e^{-0}$

$\boxed{1 = C_1 - C_2}$ — (ii)

$C_1 + \cancel{C_2} = 0$
$C_1 - \cancel{C_2} = 1$

$2C_1 = 1$

$\boxed{C_1 = \frac{1}{2}}$

$C_1 + C_2 = 0$

$C_2 = -C_1$

$\boxed{C_2 = -\frac{1}{2}}$

$y = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$.

③ $y = C_1x + C_2x\ln x$, $(0,\infty)$

$x^2y'' - xy' + y = 0$, $y(1) = 3$, $y'(1) = -1$.

$y = C_1x + C_2x\ln x$

$y(1) = 3 \Rightarrow 3 = C_1 + C_2\ln 1$

$\boxed{3 = C_1}$

~~$y = 3x + C_2\ln x$~~ $y = 3x + C_2x\ln x$

~~$y' = 3 + \frac{C_2}{x}$~~ $y' = 3 + C_2[\ln x + 1]$

$y'(1) = -1 \Rightarrow -1 = 3 + C_2[\ln 1 + 1]$

$\boxed{-4 = C_2}$

$\boxed{y = 3x - 4x\ln x}$

$a_2 = x^2$

$a_2\big|_{x=1} = 1 \neq 0$.

There is a unique solution of given IVP.

⑤ $y = C_1 + C_2x^2$; $(-\infty,\infty)$

$xy'' - y' = 0$ $\quad y(0) = 0$, $y'(0) = 1$

$a_2 = x$

$a_2\big|_{x=0} = 0 \Rightarrow$ no unique solution.

$y(0) = 0 \Rightarrow \boxed{0 = C_1}$; $y' = 2xC_2$

$y'(0) = 1 \Rightarrow 1 = 0$ not true

It means that there are many solutions / no unique solution

$y = C_2x^2$

(13) $y = c_1 e^x \cos x + c_2 e^x \sin x$.

$y'' - 2y' + 2y = 0$. $\quad y(0) = 1, \; y(\pi) = 0$.

$y(0) = 1 \Rightarrow \quad 1 = c_1 e^0 \cos 0 + c_2 e^0 \sin 0$

$\boxed{1 = c_1} \Rightarrow \quad y = e^x \cos x + c_2 e^x \sin x \; ; \; y' = e^x \cos x - e^x \sin x + c_2 (e^x \sin x + e^x \cos x)$

$y(\pi) = 0 \Rightarrow \quad 0 = e^\pi \cos \pi \cancel{+} \cancel{c_2} e^\pi \sin \pi + c_2 (e^\pi \sin \pi + e^\pi \cos \pi)$

$0 = -e^\pi \cancel{+} c_2 e^\pi$

$c_2 e^\pi = -e^\pi$

$c_2 = \frac{-e^\pi}{e^\pi} \Rightarrow \boxed{c_2 = -1}$

$\boxed{y = e^x \cos x - e^x \sin x.}$