4.6 **Variation of Parameters:-**

LECTURE: 34

In Previous Sections we find out the General Solution of a homogeneous and non homogenous differential Equation with the Condition that D.E must have a Constant Coefficients and the input function $g(x)$ must be of the type listed in Table.

In this Section, we examine a method for finding a Particular Solution y_p of a nonhomogenous linear D.E, with no such restriction. i.e

- D.E may have a variable Coefficients
- $g(x)$ may of any type.

Linear Second Order Differential Equation:-

Consider

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x). \quad \text{--- (1)}$$

we have a Standard form

$$y'' + P(x)y' + Q(x)y = f(x).$$

Taking idea from Variation of Parameter method used in 1^{st} order D.E, we assume the Solution of the form.

$$y_p = u_1 y_1(x) + u_2(x) y_2(x).$$

$$y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2.$$

$$y_p'' = u_1 y_1'' + u_1' y_1' + u_1' y_1' + u_1'' y_1 + u_2' y_2' + u_2 y_2'' + u_2'' y_2 + u_2' y_2'.$$

Equ(1) $\Rightarrow$

$$y_p'' + P(x)y_p' + Q(x)y_p = u_1\{y_1'' + Py_1' + Qy_1\} + u_2\{y_2'' + Py_2' + Qy_2\} + y_1 u_1'' + u_1' y_1' + y_2 u_2'' + u_2' y_2' + P\{y_1 u_1' + y_2 u_2'\} + y_1' u_1' + y_2' u_2'.$$

$$= \frac{d}{dx}[y_1 u_1'] + \frac{d}{dx}[y_2 u_2'] + P[y_1 u_1' + y_2 u_2'] + y_1' u_1' + y_2' u_2'$$

$$= \frac{d}{dx}[y_1 u_1' + y_2 u_2'] + P[y_1 u_1' + y_2 u_2'] + y_1' u_1' + y_2' u_2' = f(x)$$

Comparing we get System of Equations.

$$y_1 u_1' + y_2 u_2' = 0$$
$$y_1' u_1' + y_2' u_2' = f(x).$$

By Cramer's Rule, the Solution of Equations expressed as

$$u_1' = \frac{W_1}{W} = \frac{-y_2 f(x)}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W}.$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}.$$

u_1 and u_2 are found by intregrating the results.
final Particular Solution is $y_p = u_1 y_1 + u_2 y_2$.

Example:- $y'' - 4y' + 4y = (x+1)e^{2x}$.

Solu from A.E $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0$

$$y_c = c_1 e^{2x} + c_2 x e^{2x}.$$

where $y_1 = e^{2x}$, $y_2 = xe^{2x}$.

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}$$

$$u_1' = -\frac{y_2 f(x)}{W} = -\frac{xe^{2x}(x+1)e^{2x}}{e^{4x}} = -x(x+1) = -x^2 - x.$$

$$u_1 = \int(-x^2 - x)\,dx = -\frac{x^3}{3} - \frac{x^2}{2}.$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{e^{2x}(x+1)(e^{2x})}{e^{4x}} = x+1$$

$$u_2 = \int(x+1)\,dx = \frac{x^2}{2} + x.$$

$$y = y_c + y_p = c_1 e^{2x} + c_2 x e^{2x} + \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}.$$

Example: 2 $4y'' + 36y = \operatorname{Cosec} 3x$.

Standard form: $y'' + 9y = \frac{1}{4}\operatorname{Cosec} 3x$.

Roots of A.E: $m^2 + 9 = 0$, $m_1 = 3i$, $m_2 = -3i$.

$$y_c = c_1 \cos 3x + c_2 \sin 3x,$$

using $y_1 = \cos 3x$, $y_2 = \sin 3x$.

$$W(\cos 3x, \sin 3x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x = 3$$

$$u_1' = \frac{-y_2 f(x)}{W} = -\frac{\sin 3x \operatorname{Cosec} 3x}{3\times 4} = -\frac{1}{12} \Rightarrow u_1 = -\frac{1}{12}\int dx = -\frac{1}{12}x.$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{\cos 3x \operatorname{Cosec} 3x}{3\times 4} = \frac{1}{12}\frac{\cos 3x}{\sin 3x} \Rightarrow u_2 = \frac{1}{36}\ln|\sin 3x|.$$

$$y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}\ln|\sin 3x|\sin 3x.$$

Example: 3: $y'' - y = \frac{1}{x}$.

$$\left.\begin{array}{l} A.E: m^2 - 1 = 0, \quad m_1 = +1, \quad m_2 = -1 \\ (m+1)(m-1) = 0 \end{array}\right\} y_c = c_1 e^{+x} + c_2 e^{-x}.$$

$$W(e^{+x}, e^{-x}) = \begin{vmatrix} e^{+x} & e^{-x} \\ +e^{+x} & -e^{-x} \end{vmatrix} = -2$$

$$u_1' = \frac{-y_2 f(x)}{W} = -\frac{e^{-x}\cdot 1/x}{-2} \Rightarrow u_1 = -\frac{1}{2}\int \frac{e^{-x}}{x}\,dx.$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{e^{x}\, 1/x}{-2} \Rightarrow u_2 = -\frac{1}{2}\int \frac{e^{x}}{x}\,dx.$$

$$y_p = \left(-\frac{1}{2}\int \frac{e^{-x}}{x}\right)e^{x} - \left(\frac{1}{2}\int \frac{e^{x}}{x}\,dx\right)e^{-x}$$

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{-x} - \frac{1}{2}e^{x}\int \frac{e^{-x}}{x}\,dx - \frac{1}{2}e^{-x}\int \frac{e^{x}}{x}\,dx.$$

Practice Problems :- Solve

1) $y'' - 2y' + y = e^{t} \arctan t$

2) $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$; $y(0) = 1$, $y'(0) = 0$.

3) $x^2y'' + xy' + y = \sec(\ln x)$

where $y_1 = \cos(\ln x)$
$y_2 = \sin(\ln x)$

are known Solutions of associated Homogeneous Equ.

Related Questions :-

Ex 4.6

Q: 1-24.