LECTURE: 32:-

Section: 4.3: Homogeneous Linear Equations with Constant Coefficients:

Consider a Second order differential Equation with Constant Coefficients.

$$ay'' + by' + cy = 0 \quad - \quad (I)$$

Main Idea:- Let look Equ(I) in 1^{st} order.

$$by' + cy = 0 \quad - \quad (II)$$

$$y' = -\frac{c}{b}y$$

$$y' = my$$

$$\frac{dy}{dx} = my \quad \rightarrow \text{we have Seperable Equ.}$$

$$\frac{dy}{y} = mdx$$

By integrating

$$\ln y = mx + c$$

$$\boxed{y = ce^{mx}}$$ Means Equ (II) has a Solution $y = e^{mx}$.

By this observation; we can say that Equ(I) will also have a Solution like

$$y = e^{mx}.$$

Now $y' = me^{mx}$, $y'' = m^2e^{mx}$.

Put in Equ (I).

$$ay'' + by' + cy = 0$$

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$(am^2 + bm + c)\, e^{mx} = 0$$

$$e^{mx} \neq 0 \; ; \quad am^2 + bm + c = 0 \quad - \text{(III)}$$

Equ (III) is called Auxiliary Equation of D.E (I).

From A. Equ, we get

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \boxed{m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}}$$ Two roots of Equ (III).

Example:- Solve $y'' - 5y' + 6y = 0$.

Auxiliary Equ $m^2 - 5m + 6 = 0$

$$m = \frac{5 \pm \sqrt{25 - 24}}{2} = \frac{5 \pm 1}{2}$$

$$m_1 = \frac{6}{2} = 3, \qquad m_2 = \frac{4}{2} = 2$$

$$y_1 = e^{m_1 x} = e^{3x}, \qquad y_2 = e^{m_2 x} = e^{2x}.$$

$\therefore$

$$\boxed{y = c_1 y_1 + c_2 y_2 = c_1 e^{3x} + c_2 e^{2x}}$$ General Solution.

- Now we discuss three different forms of General Solutions of Equ (I) corresponding to three cases:

Case I :- m_1 and m_2 ; Real and Distinct ($b^2-4ac>0$) :-

In this case; m_1, m_2 are Real

$$m_1 \neq m_2$$

we get two Solutions

$$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$$

$$y = y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Ex :- Solve $2y'' - 5y' - 3y = 0$.

Auxiliary Equ

$$2m^2 - 5m - 3 = 0$$

$$2m^2 - 6m + m - 3 = 0$$

$$2m(m-3) + 1(m-3) = 0$$

$$(2m+1)(m-3) = 0$$

$$2m+1 = 0 \;; \quad m - 3 = 0$$

$$m_1 = -\tfrac{1}{2} \;; \quad m_2 = 3$$

we note here that m_1, m_2 are real and distinct.

$$\therefore \quad y_1 = e^{m_1 x} = e^{-\frac{1}{2}x} \;; \quad y_2 = e^{m_2 x} = e^{3x}.$$

General Solution

$$\boxed{y = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}}$$

Case: II: m_1 and m_2; Real and Equal $(b^2-4ac=0)$:-

we have Second order D.E of the form

$$ay''+by'+cy=0$$

$$\Rightarrow y''+\frac{b}{a}y'+\frac{c}{a}y=0$$

It has one root of the form $y_1=e^{mx}$.

we can find 2nd root by Formula

$$y_2 = y_1\int \frac{e^{-\int P\,dx}}{y_1^2}\,dx$$

$$= e^{mx}\int \frac{e^{-\int b/a\,dx}}{e^{2mx}}\,dx$$

$$= e^{mx}\int \frac{e^{-b/a\,x}}{e^{2mx}}\,dx$$

$\because$ $b^2-4ac=0$

$m=-\frac{b}{2a} \Rightarrow 2m=-b/a$.

$$= e^{mx}\int \frac{e^{2mx}}{e^{2mx}}\,dx$$

$$= e^{mx}\int dx$$

$$= xe^{mx}.$$

Hence we have repeated roots of the form

$$y_1=e^{mx}, \quad y_2=xe^{mx}.$$

General Solution

$$\boxed{y=c_1e^{mx}+c_2xe^{mx}}.$$

Ex:- Solve $y'' - 10y' + 25y = 0$.

Auxiliary Equation.

$$m^2 - 10m + 25 = 0$$

$$(m-5)^2 = 0$$

$$m_1 = 5, \quad m_2 = 5$$

we have

$$y_1 = e^{m_1 x} = e^{5x} \quad ; \quad y_2 = x e^{m_2 x} = x e^{5x}.$$

$\therefore$

$$\boxed{\begin{aligned} y = y_c &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{5x} + c_2 x e^{5x} \end{aligned}}$$

General Solution.

Case III :- m_1 and m_2 ; Conjugate Complex numbers ($b^2 - 4ac < 0$).

$$\therefore \quad m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

If m_1 and m_2 are complex, we can write $m = \alpha \pm i\beta$

that is

$$m_1 = \alpha + i\beta \quad ; \quad m_2 = \alpha - i\beta \quad ; \quad \alpha, \beta > 0 \text{ are real}$$

$$i^2 = -1.$$

Then Solution is of the form

$$\begin{aligned} y_1 &= e^{m_1 x} & ; \quad y_2 &= e^{m_2 x} \\ &= e^{(\alpha + i\beta)x} & &= e^{(\alpha - i\beta)x} \\ &= e^{\alpha x} . e^{i\beta x} & &= e^{\alpha x} . e^{-i\beta x} \\ &= e^{\alpha x} . (\cos\beta x + i \sin\beta x) & &= e^{\alpha x} . (\cos\beta x - i \sin\beta x) \end{aligned}$$

where we have used the Euler Formula $e^{i\theta} = \cos\theta + i\sin\theta$.

Now if we add and Substract y_1 & y_2.

- $y_1 + y_2 = 2e^{\alpha x}\cos\beta x$

  $\frac{1}{2}y_1 + \frac{1}{2}y_2 = e^{\alpha x}\cos\beta x$

- $y_1 - y_2 = 2ie^{\alpha x}\sin\beta x$

  $-\frac{i}{2}y_1 + \frac{i}{2}y_2 = e^{\alpha x}\sin\beta x$

It gives an Idea

Our Solution is of the form

$$y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$$
$$= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

we can check that these two Solutions $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ also forms a fundamental set of Solutions.

<u>Ex</u>:- Solve $y'' + 4y' + 7y = 0$

Auxiliary Equ

$$m^2 + 4m + 7 = 0$$

$$m = \frac{-4 \pm \sqrt{16 - 4(1)(7)}}{2}$$

$$= \frac{-4 \pm \sqrt{16-28}}{2}$$

$$= \frac{-4 \pm \sqrt{-12}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}i}{2} = -2 \pm \sqrt{3}i$$
$$= \alpha \pm i\beta$$

we get

$$m_1 = \alpha + i\beta = -2 + i\sqrt{3}\ ; \quad m_2 = \alpha - i\beta = -2 - i\sqrt{3}$$

General Solution

$$\boxed{\begin{aligned} y &= c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x \\ &= e^{-2x}(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x) \end{aligned}}$$

Practice Problems:-

1) Solve IVP; $4y'' + 4y' + 17y = 0$; $y(0) = -1$, $y'(0) = 2$.

2) Solve $y''' + 3y'' - 4y = 0$

3) Solve $\frac{d^4y}{dx^2} + 2\frac{d^2y}{dx^2} + y = 0$.

4) Solve $2\frac{d^5x}{ds^5} - 7\frac{d^4x}{ds^4} + 12\frac{d^3x}{ds^3} + 8\frac{d^2x}{ds^2} = 0$.

5) The roots of Cubic Auxiliary Equ are $m_1 = 4$, and $m_2 = m_3 = -5$. what is the Corresponding Homogeneous Linear D.E.

6) Two roots of Cubic Auxiliary Equ with real Coefficients are $m_1 = -1/2$, $m_2 = 3+i$. what is D.E.

7) Find the General Solution of $y''' + 6y'' + y' - 34y = 0$. if it is known that $y_1 = e^{-4x}\cos x$ is one Solution.

Related Questions:-

Ex: 4.3

Q: 1 - 42, 49, 50, 51

Ex:- ~~$m^2+m+7=0$~~

$y''+4y'+7y=0$.

Auxiliary Equation.

$m^2+4m+7=0$.

$m=\dfrac{-4\pm\sqrt{16-4(1)(7)}}{2}$

$=\dfrac{-4\pm\sqrt{16-28}}{2}$

$=\dfrac{-4\pm\sqrt{-12}}{2}$

$=\dfrac{-4\pm 2\sqrt{3}i}{2}$

$=-2\pm\sqrt{3}i$

$m_1=-2+\sqrt{3}i,\quad m_2=-2-\sqrt{3}i$

$y=c_1e^{-2x}\cos\sqrt{3}x+c_2e^{-2x}\sin\sqrt{3}x$.

$y=e^{-2x}(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x)$

Ex:- $y'''+3y''-4y=0$.

$m^3+3m^2-4=0$.

$(m-1)(m^2+4m+4)=0$.

$(m-1)(m+2)^2=0$.

$m_1=1,\ m_2=m_3=-2$

$y=c_1e^{x}+c_2e^{-2x}+c_3xe^{-2x}$.

1	1	3	0	-4
		1	4	4
	1	4	4	0

Ex:- $\dfrac{d^4y}{dx^4}+2\dfrac{d^2y}{dx^2}+y=0$.

$m^4+2m^2+1=0$.

$(m^2+1)^2=0$.

$m^2+1=0,\quad m^2+1=0$

$m^2=-1 \qquad m=\pm i$

$m=\pm\sqrt{-1}$

$=\pm i$

$m_1=i,\ m_2=-i,\ m_3=i,\ m_4=i$

$y=C_1e^{ix}+C_2e^{-ix}+C_3xe^{ix}+C_4xe^{-ix}$.

$$\boxed{y=C_1\cos x+C_2\sin x+C_3x\cos x+C_4x\sin x}$$

$\because C_1e^{ix}+C_2e^{-ix}=C_1\cos x+C_2\sin x$

$x(C_3e^{ix}+C_4e^{-ix})=x(C_3\cos x+C_4\sin x)$.

Ex:- $4y''+4y'+17y=0;\ y(0)=-1$
$y'(0)=2$.

$4m^2+4m+17=0$.

$m=\dfrac{-4\pm\sqrt{16-4(4)(17)}}{2(4)}$

$=\dfrac{-4\pm\sqrt{16-272}}{8}$

$=\dfrac{-4\pm\sqrt{-226}}{8}$

$=\dfrac{-4\pm 16i}{8}$

$=-\frac{1}{2}\pm 2i$

$y=e^{-\frac{1}{2}x}(C_1\cos 2x+C_2\sin 2x)$

$y(0)=-1 \Rightarrow C_1=-1$

$y'=e^{-x/2}(-\cos 2x+C_2\sin 2x)$

$=-\frac{1}{2}e^{-x/2}(-\cos 2x+C_2\sin 2x)$

$+e^{-x/2}(2\sin 2x+2C_2\cos 2x)$

$y'(0)=2 \Rightarrow 2=\frac{1}{2}+2C_2$

$3/2=2C_2$

$3/4=C_2$.

$y=e^{-x/2}(-\cos 2x+\frac{3}{4}\sin 2x)$

Ex:- The roots of Cubic Auxiliary Equ are $m_1 = 4$ and $m_2 = m_3 = -5$ what is the Corresponding Homogeneous Linear D.E .

Since $(m-4)(m+5)^2 = m^3 + 6m^2 - 15m - 100$.

D.E is $y''' + 6y'' - 15y' - 100y = 0$.

Ex:- Two roots of Cubic Auxiliary Equ with real Coefficients are $m_1 = -\frac{1}{2}$, $m_2 = 3+i$. What is D.E .

Third root must be $m_3 = 3-i$.

Auxiliary Equ .

$$(m+\tfrac{1}{2})(m-(3+i))(m-(3-i))$$

$$= (m+\tfrac{1}{2})(m^2 - 6x + 10)$$

$$= m^3 - \frac{11}{2}m^2 + 7m + 5$$

D.E $y''' - \frac{11}{2}y'' + 7y' + 5y = 0$

Ex:- Find the General Solution of $y''' + 6y'' + y' - 34y = 0$.
if it is known that $y_1 = e^{-4x}\cos x$ is one Solution .

From the Solution $y_1 = e^{-4x}\cos x$, we Conclude that $m_1 = -4+i$ and $m_2 = -4-i$ are roots of auxiliary Equation .

Hence another Solution must be $y_2 = e^{-4x}\sin x$.

Now dividing the Polynomial $m^3 + 6m^2 + m - 34 = 0$ by $[m-(-4+i)][m-(-4-i)] = m^2 + 8m + 17$

2	1	6	1	-34
		2	16	34
	1	8	17	0

$\Rightarrow m^3 + 6m^2 + m - 34 = (m-2)(m^2 + 8m + 17)$.

$\Rightarrow \dfrac{(m-2)\cancel{(m^2+8m+17)}}{\cancel{(m^2+8m+17)}} = m-2$.

$m_3 = 2$ is third root .

$$y = C_1 e^{-4x}\cos x + C_2 e^{-4x}\sin x + C_3 e^{2x}$$

Ex:- $2\frac{d^5x}{ds^5} - 7\frac{d^4x}{ds^4} + 12\frac{d^3x}{ds^3} + 8\frac{d^2x}{ds^2} = 0.$

$-2\left(\frac{1}{8}\right) - 7\left(\frac{1}{4}\right)$

$-\frac{1}{4} - \frac{7}{4}$

$-\frac{1-7}{4} = -\frac{8}{4} = -2$

Auxiliary Equation:-

$2m^5 - 7m^4 + 12m^3 + 8m^2 = 0.$

$m^2(2m^3 - 7m^2 + 12m + 8) = 0.$

$m^2\left(m + \frac{1}{2}\right)(2m^2 - 8m + 16) = 0.$

$-\frac{1}{2}$	2	-7	12	8
		-1	4	-8
	2	-8	16	0

$m = 0, 0$

$m = -\frac{1}{2}$

$m = 2 \pm 2i.$

$x = C_1 + C_2 s + C_3 e^{-s/2} + e^{2s}(C_4 \cos 2s + C_5 \sin 2s).$