MT-224
Differential Equations (Cal II)

National University of Computer and Emerging Sciences

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Reducible to Exact Differential Equation

Exact Differential Equations

The first order differential equations

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be exact if we can find a function $f(x, y)$ such that

$$df = M(x, y)dx + N(x, y)dy.$$

Solution of Exact differential equation is

$$f(x, y) = constant.$$

Criteria for Exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Non Exact Differential Equations

If the first order differential equations

$$M(x, y)dx + N(x, y)dy = 0$$

is not exact
but

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is exact.

i.e., we want to find $\mu(x, y)$ such that

$$(\mu M)_y = (\mu N)_x.$$

What are the possibilities of μ?

Non Exact Differential Equations

Possibilities of $\mu(x, y)$

$$\mu_x N - \mu_y M = (M_y - N_x)\mu$$

Case-I

$$\mu(x, y) = \mu(x)$$

we get

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu,$$

which is first order linear differential equation in x, if

$$\frac{M_y - N_x}{N} \text{ is only function of x,}$$

and in that case

$$\mu = e^{\int \frac{M_y - N_x}{N} dx}.$$

Non Exact Differential Equations

Case-II

$$\mu(x, y) = \mu(y)$$

we get

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu,$$

which is first order linear differential equation in y, if

$$\frac{N_x - M_y}{M} \text{ is only function of y,}$$

and in that case

$$\mu = e^{\int \frac{N_x - M_y}{M} dy}.$$

Summary

- If $\dfrac{M_y - N_x}{N}$ is function of x only, then integrating factor

$$\mu = e^{\int \frac{M_y - N_x}{N} dx}$$

.

- If $\frac{N_x - M_y}{M}$ is function of y only, then integrating factor

$$\mu = e^{\int \frac{N_x - M_y}{M} dy}$$

.

- Multiplying given equation with integrating factor μ, the resulting equation would be exact.
- Use the procedure discussed in previous lecture to get the solution.

Examples

$$(6xy)dx + (4y + 9x^2)dy = 0 \tag{1}$$

Solution: Notice that here $M(x, y) = 6xy$ and $N(x, y) = 4y + 9x^2$. Since

$$M_y = 6x \neq 18x = N_y,$$

Now, we will check whether it is reducible to exact or not. Notice that

$$\frac{N_x - M_y}{M} = \frac{2}{y} \quad \text{(which is function of } y\text{)}$$

Hence, integrating factor is

$$\mu = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int \frac{2}{y} dy} = y^2$$

Multiplying (1) with $\mu = y^2$, we get

$$6xy^3dx + (4y^3 + 9x^2y^2)dy = 0 \tag{2}$$

with $M_1 = 6xy^3$ and $N_1 = 4y^3 + 9x^2y^2$, also notice

$$(M_1)_y = 18xy^2 = (N_1)_x$$

Hence, (2) is exact.

So, we need to find $F(x, y)$(solution) such that

$$\frac{\partial F}{\partial x} = M_1 = 6xy^3, \tag{3}$$

$$\frac{\partial F}{\partial y} = N_1 = 4y^3 + 9x^2y^2 \tag{4}$$

Integrating M_1 i.e., given by (4), w.r.t x to get

$$F(x, y) = 3x^2y^3 + g(y) \tag{5}$$

Differentiating F w.r.t y to get $\frac{\partial F}{\partial y} = 9x^2y^2 + g'(y)$.
But by looking into (4), $\frac{\partial F}{\partial y} = 4y^3 + 9x^2y^2$.
Hence, we must have

$$9x^2y^2 + g'(y) = 4y^3 + 9x^2y^2$$

That is

$$g'(y) = 4y^3$$

Integrating w.r.t y, we get $g(y) = y^4 + C$ Using value of $g(y)$ in (5), to get the solution

$$F(x, y) = 3x^2y^3 + y^4 + C.$$

Examples

Solve the following differential equation:

$$(2y^2 + 3x)dx + 2xydy = 0.$$

Solution: $M = (2y^2 + 3x),\ N = 2xy.$

$$M_y = 4y \neq 2y = N_x.$$

So, given differential equation is not exact.

$$\frac{N_x - M_y}{M} = \frac{-2y}{(2y^2 + 3x)},\ \frac{M_y - N_x}{N} = \frac{1}{x}.$$

Integrating Factor is

$$\mu = e^{\int \frac{1}{x} dx} = x.$$

Given, differential equation can be written as

$$(2xy^2 + 3x^2)dx + 2x^2ydy = 0,$$

which is exact differential equation

$$(2xy^2 + 3x^2)_y = 4xy = (2x^2y)_x.$$

$$M_1 = (2xy^2 + 3x^2),\ N_1 = 2x^2y.$$

We want to find a function $F(x, y)$ such that

$$dF = (2xy^2 + 3x^2)dx + 2x^2ydy.$$

$$\frac{\partial F}{\partial x} = 2xy^2 + 3x^2, \tag{6}$$

$$\frac{\partial F}{\partial y} = 2x^2y. \tag{7}$$

Integrate (6), w.r.t x

$$F = x^2y^2 + x^3 + g(y).$$

Differentiate w.r.t y, we get

$$\frac{\partial F}{\partial y} = 2x^2y + g'(y).$$

Comparing with (7), we get

$$g'(y) = 0,$$

which implies

$$g(y) = c_1.$$

So, $F(x, y) = x^2y^2 + x^3 + c_1$.
Hence, solution of the differential equation is

$$F(x, y) = c_2$$

, where c_1, c_2 are arbitrary constants.

Practice Questions

Exercise: 2.5

$$Q\#29 - 39$$