■ **Power Series** :- A Power Series Centered at a is an infinite Series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots ,$$

■ **Power Series Solution**:-

Definition: Suppose the Linear Second order differential Equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad -(1)$$

is Put into Standard form

$$y'' + P(x)y' + Q(x)y = 0 \; ; \quad P(x) = \frac{a_1(x)}{a_2(x)} \; ; \quad Q(x) = \frac{a_0(x)}{a_2(x)}$$

by dividing by Leading Coefficient $a_2(x)$.

Bessel Equation:-

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

Legendre Equation:-

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0.$$

Standard form of Bessel Equation & Legendre Equation.

Ordinary Point :- A Point $x = x_0$ is called ordinary Point of Eqn (1) if $a_2(x) \neq 0$ at $x = x_0$.

Ex:- In Legendre Equation

At $x=0$; $a_2(x) = 1-x^2$

$a_2(0) = 1-0^2 = 1$

$a_2(0) \neq 0$

$x=0$ is ordinary Point of D.E.

Singular Point:- Regular Singular Point / Irregular Singular Point } Series Solution only exists at regular Singular Point.

A Point $x = x_0$ is called Singular Point of Equ (1) if $a_2(x) = 0$ at $x = x_0$. ~~then~~

Ex:- In Bessels Equation.

At $x=0$; $a_2(x) = x^2$

$a_2(0) = 0$

$x=0$ is a Singular Point of D.E.

Regular Singular Point :- A Point $x = x_0$ is said to be a regular Singular Point of Equ (1) if $a_2(x) = 0$ at $x = x_0$, and

$$\lim_{x \to x_0} x P(x) = \text{finite}$$

$$\lim_{x \to x_0} x^2 Q(x) = \text{finite}$$

NOTE:- A Point $x = x_0$ is irregular Singular Point if it is not regular. i.e if $\lim_{x \to x_0} x P(x)$ and $\lim_{x \to x_0} x^2 Q(x)$ are not finite.

Ex :-

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1-x^2)\,y = 0.$$

Find $x=0$ is ordinary Point or regular Singular Point.

Solu $a_2(x) = 2x^2$.

at $x=0$; $a_2(0) = 0$

i.e $x=0$ is not ordinary Point.

$x=0$ is Singular Point.

$$\lim_{x\to 0} x\,P(x) = \lim_{x\to 0} x\left(\frac{-x}{2x^2}\right) = -\frac{1}{2} \quad (\text{finite})$$

$$\lim_{x\to 0} x^2 Q(x) = \lim_{x\to 0} x^2\left(\frac{1-x^2}{2x^2}\right) = \frac{1}{2} \quad (\text{finite})$$

$\therefore$ $x=0$ is regular Singular Point.

Ex :-

$$x^2(x+1)\frac{d^2y}{dx^2} + (x^2-1)\frac{dy}{dx} + 2y = 0$$

Find $x=0$ is regular singular or irregular Singular Point.

Solu :- $a_2(x) = x^2(x+1)$

at $x=0$; $a_2(0) = 0(0+1) = 0$

i.e, $x=0$ is a Singular Point.

$$\lim_{x\to 0} x\frac{(x^2-1)}{x^2(x+1)} = \lim_{x\to 0}\frac{(x-1)(x+1)}{x(x+1)} = \lim_{x\to 0}\frac{x-1}{x} = -\frac{1}{0} = \infty \quad (\text{Infinite})$$

$$\lim_{x\to 0} x^2\frac{2}{x^2(x+1)} = \lim_{x\to 0}\frac{2}{x+1} = 2 \quad (\text{finite})$$

$\therefore$ $x=0$ is irregular Singular Point.

Theorem:- Existence of Power Series:-

If $x = x_0$ is an ordinary Point of the differential Eqn , we can always find two Linearly independent Solutions of in the form of a Power Series Centered at x_0. that is

$$y = \sum_{n=0}^{\infty} C_n (x - x_0)^n .$$

If $x = x_0$ is a Singular Point of the D.E , a Series Solution Converges at least on Some interval $|x - x_0| < R$; where R is the distance from x_0 to the Closest Singular Point.

Power Series Solutions:-

Example:- Solve $y'' + xy = 0$.

At $x = 0$; $a_2(x) = 1 \neq 0$.

$x = 0$ is an ordinary Point.

There is no finite Singular Point.

Distance from x_0 to Singular Point is 0

∴ Two Series Solutions Converges for $|x| < \infty$.

$$\left\{ \begin{array}{l} \left| \lim_{n\to\infty} \frac{x^{n+1}}{x^n} \right| < 1 \\ \lim_{n\to\infty} |x| < 1 \\ \lim_{n\to\infty} |x| \to 0 \ \forall x. \\ R = \infty . \\ \text{i.e } \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{array} \right.$$

Let $y = \sum_{n=0}^{\infty} C_n x^n$.

$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2}$$

$$y'' + xy = 0$$

$$\sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} + x \sum_{n=0}^{\infty} C_n x^n = 0.$$

$$\sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1} = 0.$$

$$2C_2 + \underbrace{\sum_{n=3}^{\infty} C_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1}} = 0.$$

Both Series Starts with x.

Let $k = n-2$ in 1st Series

$k = n+1$ in 2nd Series.

$$2C_2 + \sum_{k=1}^{\infty} (k+2)(k+1) C_{k+2} x^k + \sum_{k=1}^{\infty} C_{k-1} x^k = 0.$$

$$2C_2 + \sum_{k=1}^{\infty} \left[(k+1)(k+2) C_{k+2} + C_{k-1} \right] x^k = 0.$$

Comparing Coefficients

$2C_2 = 0 \Rightarrow C_2 = 0.$

$(k+1)(k+2) C_{k+2} + C_{k-1} = 0; \quad k = 1, 2, 3, \ldots$

$$C_{k+2} = -\frac{C_{k-1}}{(k+1)(k+2)} \; ; \quad k = 1, 2, 3, \ldots$$

$k=1;\quad c_3 = -\dfrac{c_0}{2\cdot 3}$

$k=2;\quad c_4 = -\dfrac{c_1}{3\cdot 4}$

$k=3;\quad c_5 = -\dfrac{c_2}{4\cdot 5} = 0$

$k=4;\quad c_6 = -\dfrac{c_3}{5\cdot 6} = \dfrac{1}{2\cdot 3\cdot 5\cdot 6}\, c_0$

$k=5;\quad c_7 = -\dfrac{c_4}{6\cdot 7} = \dfrac{1}{3\cdot 4\cdot 6\cdot 7}\, c_1$

$k=6;\quad c_8 = -\dfrac{c_5}{7\cdot 8} = 0$

$k=7;\quad c_9 = -\dfrac{c_6}{8\cdot 9} = \dfrac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}\, c_0$

$k=8;\quad c_{10} = -\dfrac{c_7}{9\cdot 10} = \dfrac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}\, c_1$

$k=9,\quad c_{11} = -\dfrac{c_8}{10\cdot 11} = 0$

Now Substituting Coefficients.

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8$$
$$+ c_9 x^9 + c_{10} x^{10} + c_{11} x^{11} + \dots$$

$$y = c_0 + c_1 x + 0 - \frac{c_0}{2\cdot 3}x^3 - \frac{c_1}{3\cdot 4}x^4 + 0 + \frac{c_0}{2\cdot 3\cdot 5\cdot 6}x^6 + \frac{c_1}{3\cdot 4\cdot 6\cdot 7}x^7$$

$$+ 0 - \frac{c_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^9 - \frac{c_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}x^{10} + 0 + \cdots$$

After grouping the terms containing c_0 and terms containing c_1, we obtain $y = c_0 y_1(x) + c_1 y_2(x)$. a General Solution.

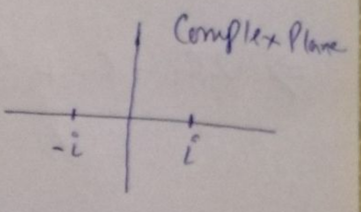
where

$$y_1(x) = 1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^9 + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2\cdot 3 \cdots\cdots (3k-1)(3k)} x^{3k}$$

$$y_2(x) = x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}x^{10} + \cdots$$

$$= x + \sum_{k=1}^{\infty} \frac{(-1)^k}{3\cdot 4 \cdots\cdots (3k)(3k+1)} x^{3k+1}.$$



Example:- Solve $(x^2+1)y'' + xy' - y = 0$.

Solu:- At $x = 0$; $a_2(x) = x^2 + 1 = 1$ $\therefore$ $x = 0$ is an ordinary Point.

$a_2(0) = 1 \neq 0$.

$x^2 + 1 = 0 \Rightarrow x = \pm i$ $\therefore$ $x = \pm i$ are Singular Points.

Power Series Solution Centered at 0, will Converge at $|x| < 1$.
where 1 is the distance in Complex Plane from 0 to either i or $-i$.

Let $y = \sum_{n=0}^{\infty} c_n x^n$.

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}.$$

$$(x^2+1)y'' + xy' - y = 0.$$

$$\Rightarrow (x^2+1)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0.$$

$$\Rightarrow \sum_{n=2}^{\infty} c_n n(n-1)x^n + \underbrace{\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}} + \underbrace{\sum_{n=1}^{\infty} c_n n x^n} - \underbrace{\sum_{n=0}^{\infty} c_n x^n} = 0$$

$$\cancel{2c_2x^0 + 6c_3x - c_0x^0 - c_1x}$$

$$\Rightarrow \sum_{n=2}^{\infty} c_n n(n-1)x^n + 2c_2x^0 + 6c_3x + \sum_{n=4}^{\infty} c_n n(n-1)x^{n-2} + \cancel{c_1 x}$$

$$+ \sum_{n=2}^{\infty} c_n n x^n - c_0x^0 - \cancel{c_1x} - \sum_{n=2}^{\infty} c_n x^n = 0$$

$$\Rightarrow 2c_2 - c_0 + 6c_3x + \underbrace{\sum_{n=2}^{\infty} c_n n(n-1)x^n}_{k=n} + \underbrace{\sum_{n=4}^{\infty} c_n n(n-1)x^{n-2}}_{k=n-2}$$

$$+ \underbrace{\sum_{n=2}^{\infty} n c_n x^n}_{k=n} - \underbrace{\sum_{n=2}^{\infty} c_n x^n}_{k=n}$$

$$\Rightarrow 2c_2 - c_0 + 6c_3x + \sum_{k=2}^{\infty} \left[k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k\right]x^k = 0$$

$$\Rightarrow 2C_2 - C_0 + 6C_3 x + \sum_{k=2}^{\infty} \left[(k+1)(k-1)C_k + (k+2)(k+1)C_{k+2} \right] x^k = 0$$

Comparing :-

$$2C_2 - C_0 = 0$$

$$6C_3 = 0 .$$

$$(k+1)(k-1)C_k + (k+2)(k+1)C_{k+2} = 0 ,$$

Thus

$$C_2 = \frac{1}{2} C_0$$

$$C_3 = 0 .$$

$$C_{k+2} = \frac{1-k}{k+2} C_k \ ; \quad k = 2, 3, 4, \ldots$$

Putting values of k.

$$C_4 = -\frac{1}{4} C_2 = -\frac{1}{2 \cdot 4} C_0 = -\frac{1}{2^2 2!} C_0 .$$

$$C_5 = -\frac{2}{5} C_3 = 0 .$$

$$C_6 = -\frac{3}{6} C_4 = \frac{3}{2 \cdot 4 \cdot 6} C_0 = \frac{1 \cdot 3}{2^3 3!} C_0$$

$$C_7 = -\frac{4}{7} C_5 = 0 .$$

$$C_8 = -\frac{5}{8} C_6 = -\frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} C_0 = -\frac{1 \cdot 3 \cdot 5}{2^4 4!} C_0$$

$$C_9 = -\frac{6}{7} C_7 = 0 .$$

$$C_{10} = -\frac{7}{10} C_8 = \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} C_0 = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5 5!} C_0 .$$

Therefore $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7$

$+ c_8 x^8 + c_9 x^9 + c_{10} x^{10} + \cdots$

$$= c_0 \left[1 + \frac{1}{2} x^2 - \frac{1}{2^2 2!} x^4 + \frac{1.3}{2^3 3!} x^6 - \frac{1.3.5}{2^4 4!} x^8 + \frac{1.3.5.7}{2^5 5!} x^{10} - \cdots \right]$$

$+ c_1 x$

$= c_0 y_1(x) + c_1 y_2(x)$

Solutions Converges for $|x| < 1$.

Practice Problems:-

Ex 6.1

Q: 17–32.