# II. Kalman Filter

# II. Kalman Filter

## C. Mathematics of Kalman Filter

- **History:**
  Saver of Apollo 11

- **Assumption:**
  The error of our prediction & observation is under Gaussian distribution

- **Target:**
  Our prediction & observation is not reliable.
  How to make our final result be more accurate?

# II. Kalman Filter

**Current**

## C. Mathematics of Kalman Filter

- **Explanation:**

State: $\hat{\boldsymbol{x}}_k = (\boldsymbol{p}, \boldsymbol{v})$

$\boldsymbol{p}$: position

$\boldsymbol{v}$: velocity

Tips: state can be any form: temperature,
the amount of water
......

**Anything you want to predict**

# II. Kalman Filter

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## C. Mathematics of Kalman Filter

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**Elements in a state may have correlations**

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Anything you want to predict
Elements in a state may have correlations
**We use covariance matrix to describe**

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## C. Mathematics of Kalman Filter

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Tips: state can be any form: temperature,
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……

Anything you want to predict
Elements in a state may have correlations
We use covariance matrix to describe

$$\boldsymbol{S}_k = \begin{bmatrix} \Sigma_{pp} & \Sigma_{pv} \\ \Sigma_{vp} & \Sigma_{vv} \end{bmatrix}$$

# II. Kalman Filter

**Current**

## C. Mathematics of Kalman Filter

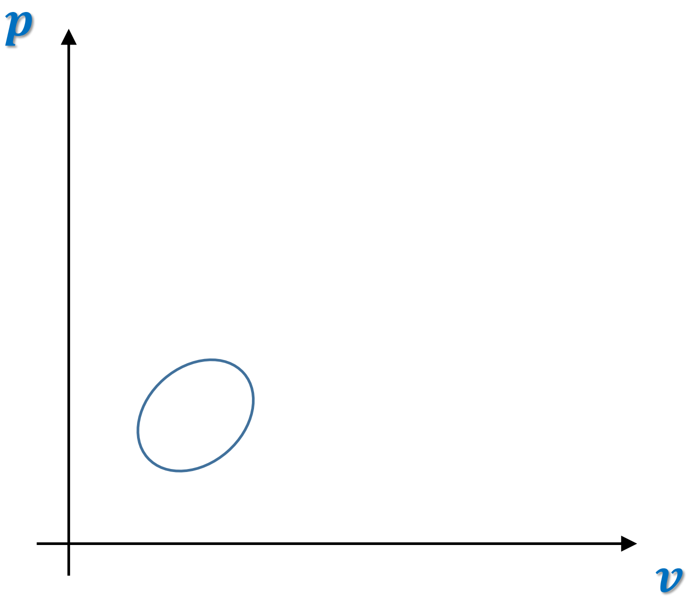
- **Explanation:**

状态

State: $\hat{\boldsymbol{x}}_k = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}$

属性间的关系.

$\boldsymbol{S}_k = \begin{bmatrix} \Sigma_{pp} & \Sigma_{pv} \\ \Sigma_{vp} & \Sigma_{vv} \end{bmatrix}$



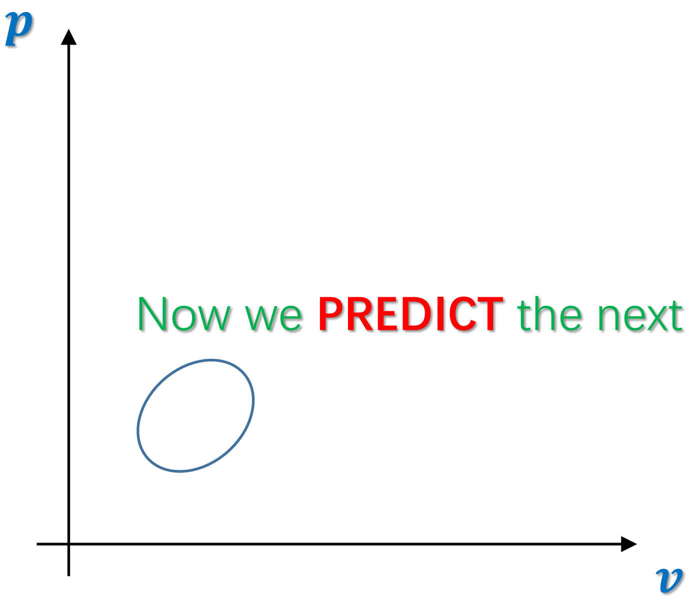
# II. Kalman Filter

**Current**

## C. Mathematics of Kalman Filter

- **Explanation:**

✔ State: $\hat{\boldsymbol{x}}_k = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}$ $\quad \boldsymbol{S}_k = \begin{bmatrix} \Sigma_{pp} & \Sigma_{pv} \\ \Sigma_{vp} & \Sigma_{vv} \end{bmatrix}$



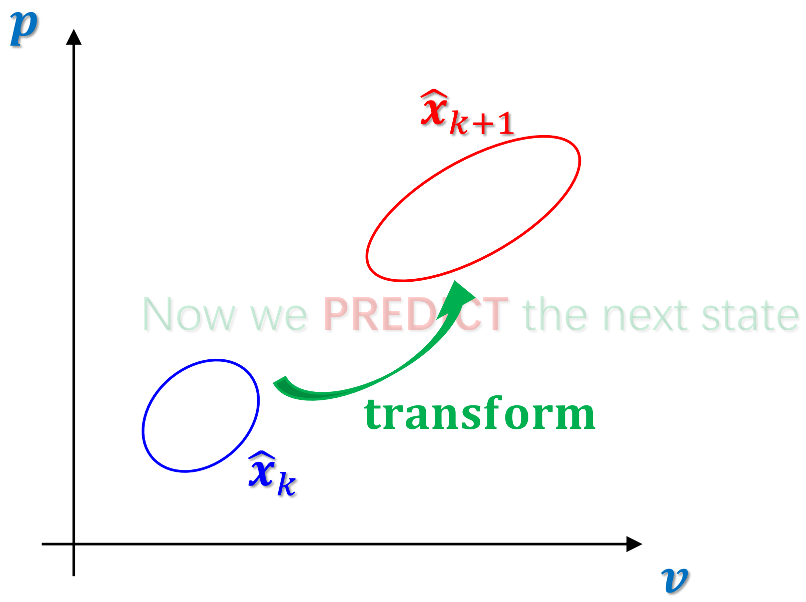
# II. Kalman Filter

**Predict**

## C. Mathematics of Kalman Filter

- **Explanation:**

State: $\hat{\boldsymbol{x}}_k = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}$ $\quad \boldsymbol{S}_k = \begin{bmatrix} \Sigma_{pp} & \Sigma_{pv} \\ \Sigma_{vp} & \Sigma_{vv} \end{bmatrix}$



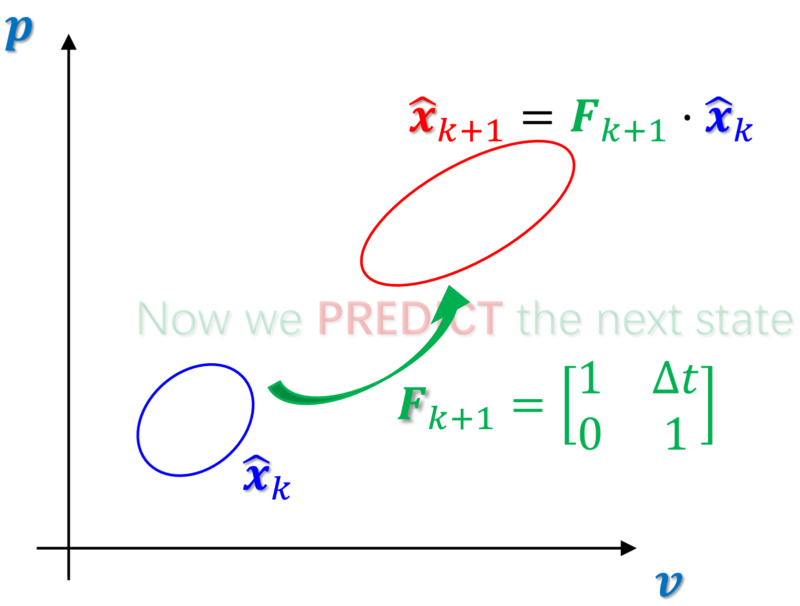
# II. Kalman Filter

**Predict**

## C. Mathematics of Kalman Filter

- **Explanation:**

Predict: 1. state: $\begin{cases} \boldsymbol{p}_{k+1} = 1 \cdot \boldsymbol{p}_k + \Delta t \cdot \boldsymbol{v}_k \\ \boldsymbol{v}_{k+1} = 0 + 1 \cdot \boldsymbol{v}_k \end{cases}$



# II. Kalman Filter

**Predict**

## C. Mathematics of Kalman Filter

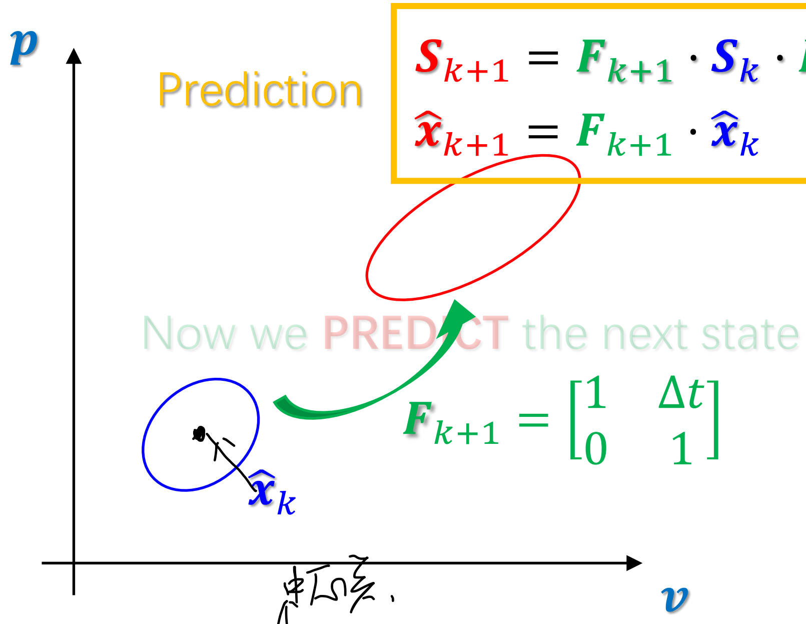
- **Explanation:**

Predict: 2. Correlation: $Cov(\boldsymbol{x}_k) = \boldsymbol{\Sigma}$

$$Cov(\boldsymbol{F}_{k+1}\boldsymbol{x}_k) = \boldsymbol{F}_{k+1}\boldsymbol{x}_k\boldsymbol{F}_{k+1}^T$$

Prediction

$$\boxed{\begin{aligned} \boldsymbol{S}_{k+1} &= \boldsymbol{F}_{k+1} \cdot \boldsymbol{S}_k \cdot \boldsymbol{F}_{k+1}^T \\ \widehat{\boldsymbol{x}}_{k+1} &= \boldsymbol{F}_{k+1} \cdot \widehat{\boldsymbol{x}}_k \end{aligned}}$$




$$\boldsymbol{F}_{k+1} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

中心点.

$(\mu, \Sigma)$

关系.

# II. Kalman Filter

**Predict: Other Influences**

## C. Mathematics of Kalman Filter

- **Explanation:**

Add some **KNOWN** external factors:

Original prediction: $\begin{cases} \boldsymbol{p}_{k+1} = 1 \cdot \boldsymbol{p}_k + \Delta t \cdot \boldsymbol{v}_k \\ \boldsymbol{v}_{k+1} = 0 + 1 \cdot \boldsymbol{v}_k \end{cases}$

**Sometimes, there are some other known external influences like a force to make accelerating**

Add acceleration: $\begin{cases} \boldsymbol{p}_{k+1} = 1 \cdot \boldsymbol{p}_k + \Delta t \cdot \boldsymbol{v}_k + \frac{1}{2}\boldsymbol{a}\Delta t^2 \\ \boldsymbol{v}_{k+1} = 0 + 1 \cdot \boldsymbol{v}_k + \boldsymbol{a}\Delta t \end{cases}$

control matrix / vector

$$\hat{\boldsymbol{x}}_{k+1} = \boldsymbol{F}_{k+1} \cdot \hat{\boldsymbol{x}}_k + \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} \cdot \boldsymbol{a} \quad \Longleftrightarrow \quad \hat{\boldsymbol{x}}_{k+1} = \boldsymbol{F}_{k+1} \cdot \hat{\boldsymbol{x}}_k + \boxed{\boldsymbol{B}_k \cdot \boldsymbol{u}_k}$$

# II. Kalman Filter

## Predict: Other Influences

## C. Mathematics of Kalman Filter

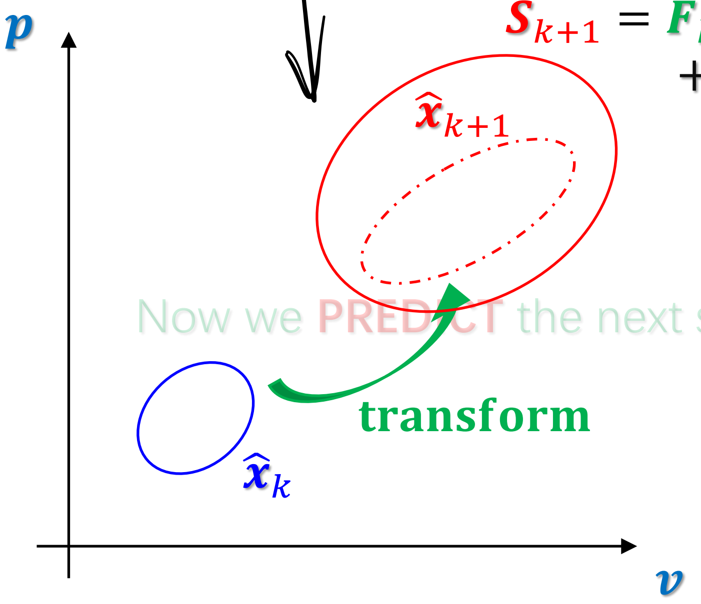
- **Explanation:**

Add some **UNKOWN** external factors:

**Sometimes, there are some other unknown external influences like noises following Gaussian Distribution**

$$S_{k+1} = F_{k+1} \cdot S_k \cdot F_{k+1}^T + Q_{k+1}$$

不确定因素



# II. Kalman Filter

## Predict: Summary

## C. Mathematics of Kalman Filter

- **Explanation:**

Summary of Prediction: 预测分为两部分:

$$\hat{\boldsymbol{x}}_{k+1} = \boldsymbol{F}_{k+1} \cdot \hat{\boldsymbol{x}}_k + \boldsymbol{B}_k \cdot \boldsymbol{u}_k$$ → 当前状态.(确定性)

**Eq. I**

$$\boldsymbol{S}_{k+1} = \boldsymbol{F}_{k+1} \cdot \boldsymbol{S}_k \cdot \boldsymbol{F}_{k+1}^{T} + \boldsymbol{Q}_{k+1}$$ → 属性关系.(不确定性)

New prediction: made from previous estimation
with a correction from **known external control**

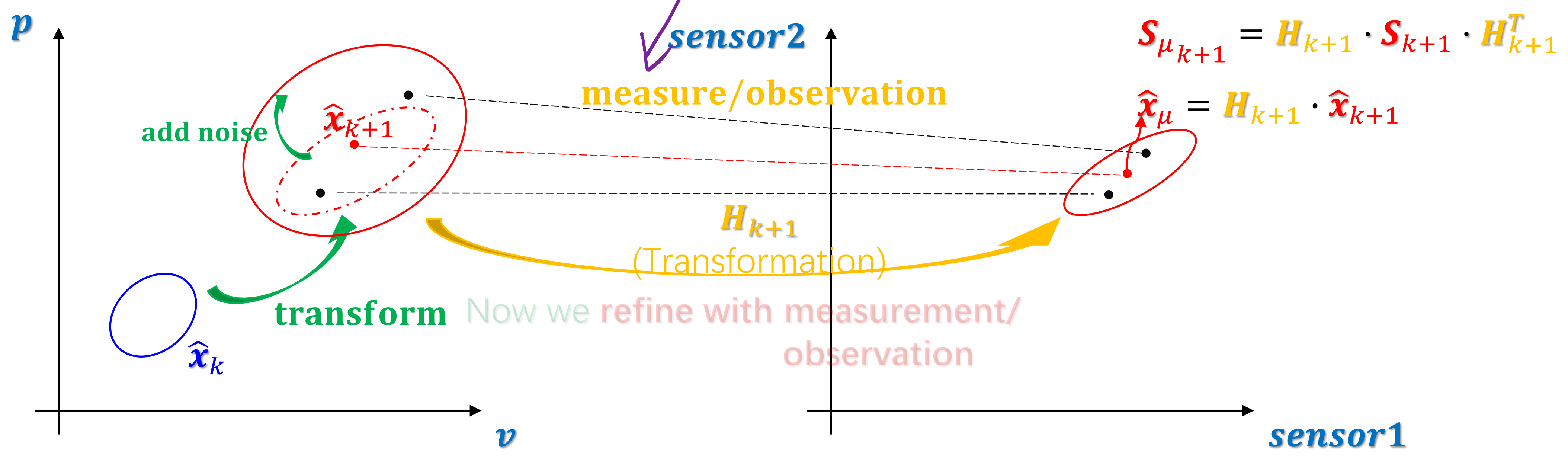
New uncertainty: made from previous uncertainty
with a correction from unknown external influence

# II. Kalman Filter

**Observation: Refine Predict**

## C. Mathematics of Kalman Filter

- **Explanation:** Sensors we have can measure different elements in a state, which can refine our predictions by transforming via a matrix $H$ which maps our estimation to the refined results



# II. Kalman Filter

**Observation: Refine Predict**

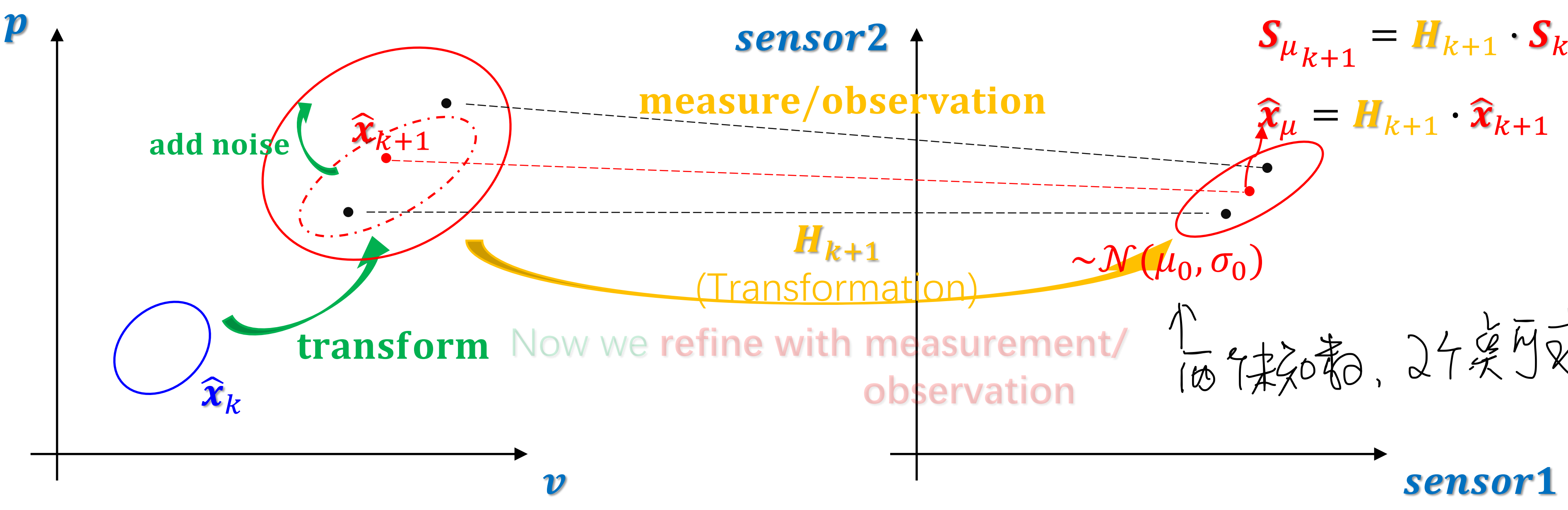
## C. Mathematics of Kalman Filter

- **Explanation:**

Sensors we have can measure different elements in a state,
which can refine our predictions by transforming via a matrix $\boldsymbol{H}$
which maps our estimation to the refined results

$$\boldsymbol{S}_{\mu_{k+1}} = \boldsymbol{H}_{k+1} \cdot \boldsymbol{S}_{k+1} \cdot \boldsymbol{H}_{k+1}^{T}$$

$$\hat{\boldsymbol{x}}_{\mu} = \boldsymbol{H}_{k+1} \cdot \hat{\boldsymbol{x}}_{k+1}$$




两个未知数，2个式可求得

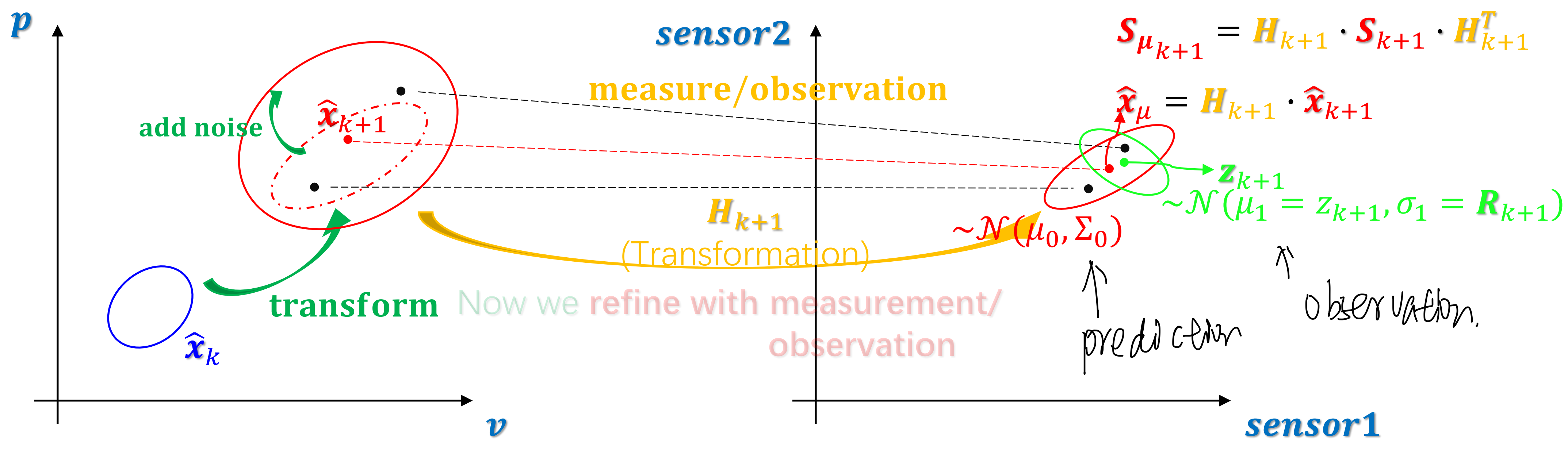
# II. Kalman Filter

**Observation: By Itself**

## C. Mathematics of Kalman Filter

- **Explanation:**

Sensors give out concrete result, which can be regarded as a collapse of uncertainty with a mean $\boldsymbol{z}_{k+1}$ and covariance $\boldsymbol{R}_{k+1}$ which following Gaussian Distribution



从两个不靠谱的结果中找到靠谱的结果.

# II. Kalman Filter

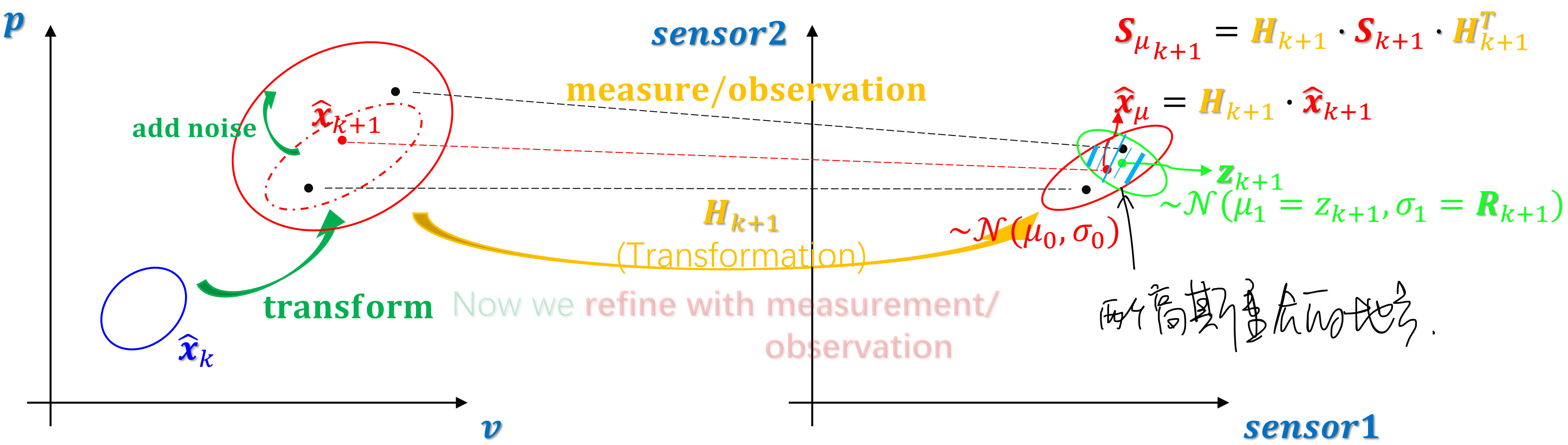
**Combination: Predict & Observation**

## C. Mathematics of Kalman Filter

- **Explanation:**

The blue area is the most likely area that our best results may fall into

**How to find out?**



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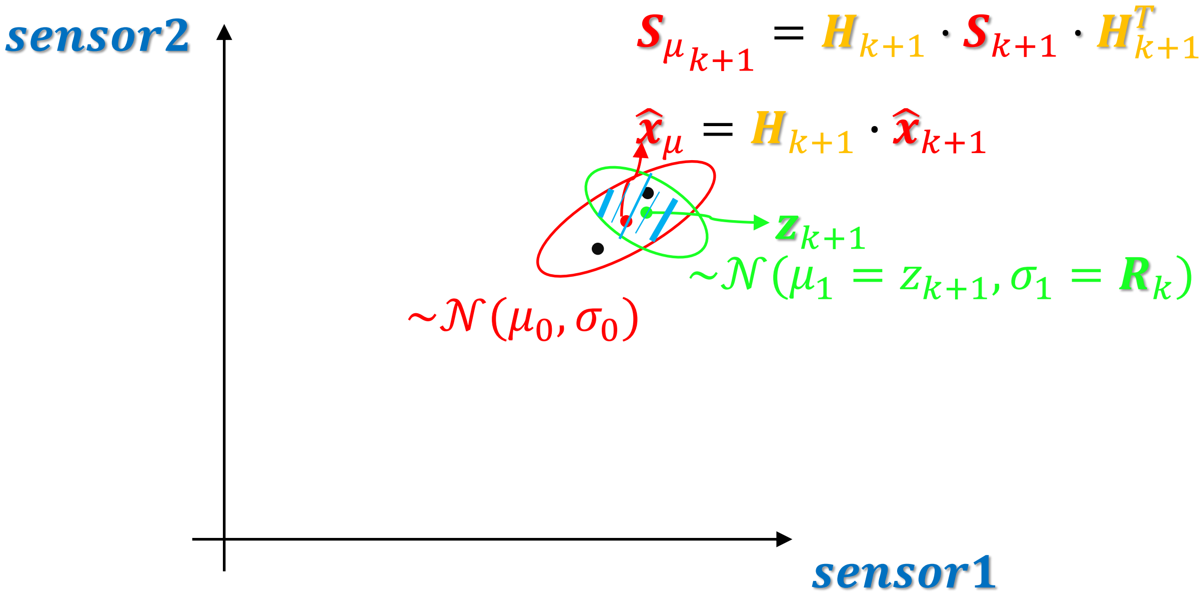
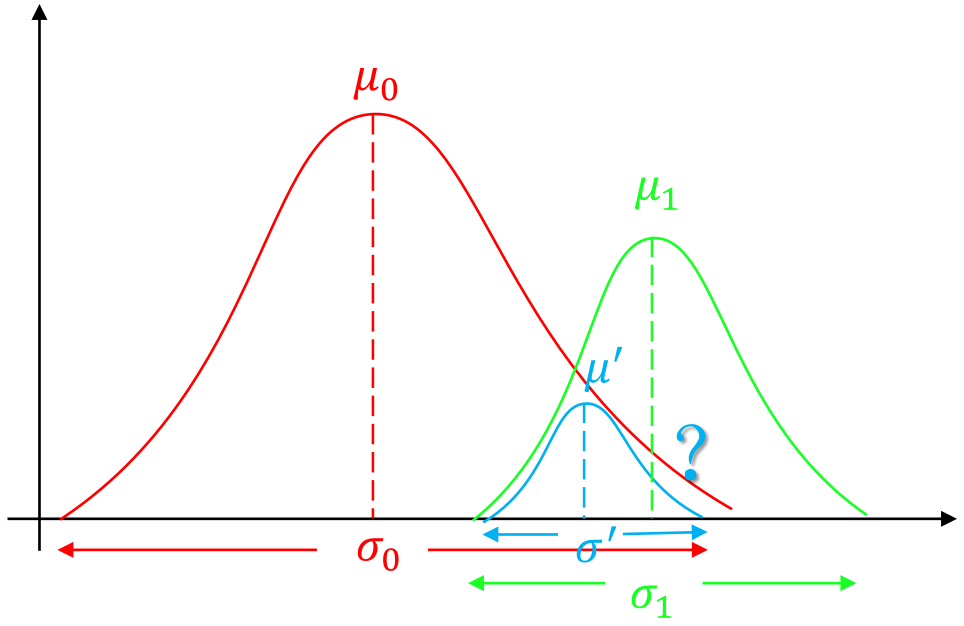



Let's see on 1-dim first



# II. Kalman Filter

**Combination: Predict & Observation**

## C. Mathematics of Kalman Filter

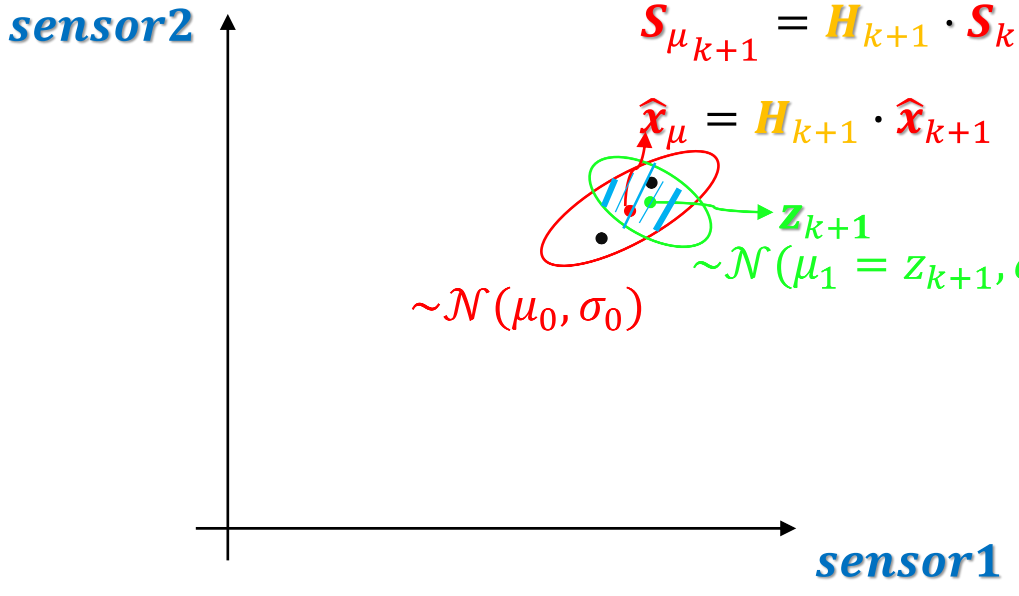
- **Explanation:**

The blue area is the most likely area that our best results may fall into

**How to find out?**

$$S_{\mu_{k+1}} = H_{k+1} \cdot S_{k+1} \cdot H_{k+1}^T$$

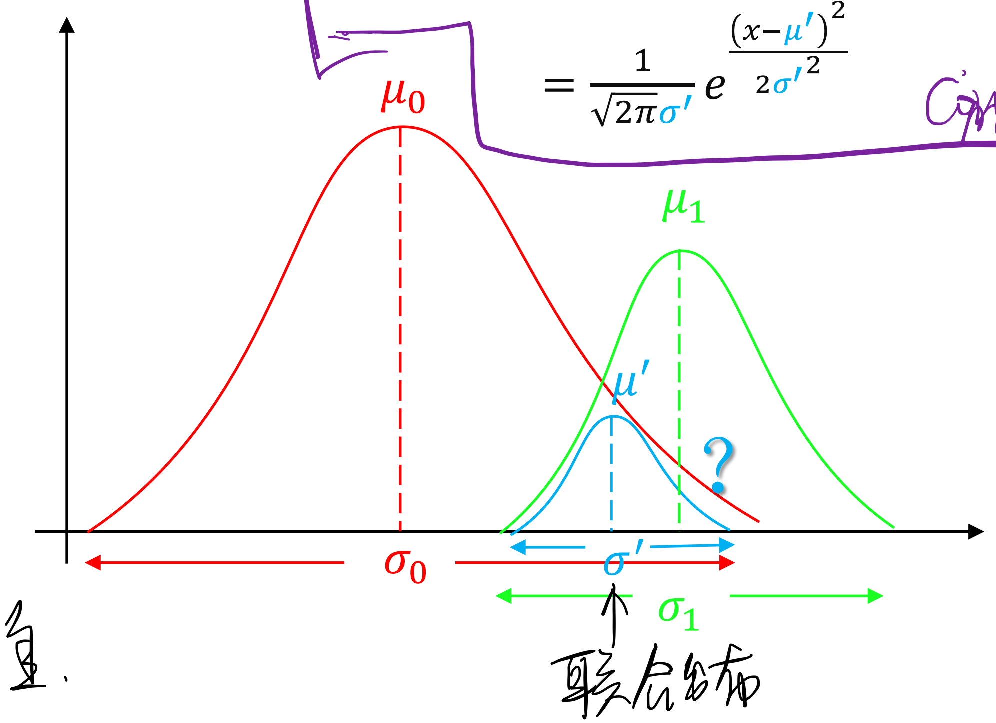
$$\hat{x}_\mu = H_{k+1} \cdot \hat{x}_{k+1}$$




$$\mathcal{N}(\mu', \sigma') = \mathcal{N}(\mu_0, \sigma_0) \cdot \mathcal{N}(\mu_1, \sigma_1)$$

$$= \frac{1}{\sqrt{2\pi}\sigma'} e^{\frac{(x-\mu')^2}{2\sigma'^2}}$$

Combine




一维.

# II. Kalman Filter

**Combination: Predict & Observation**

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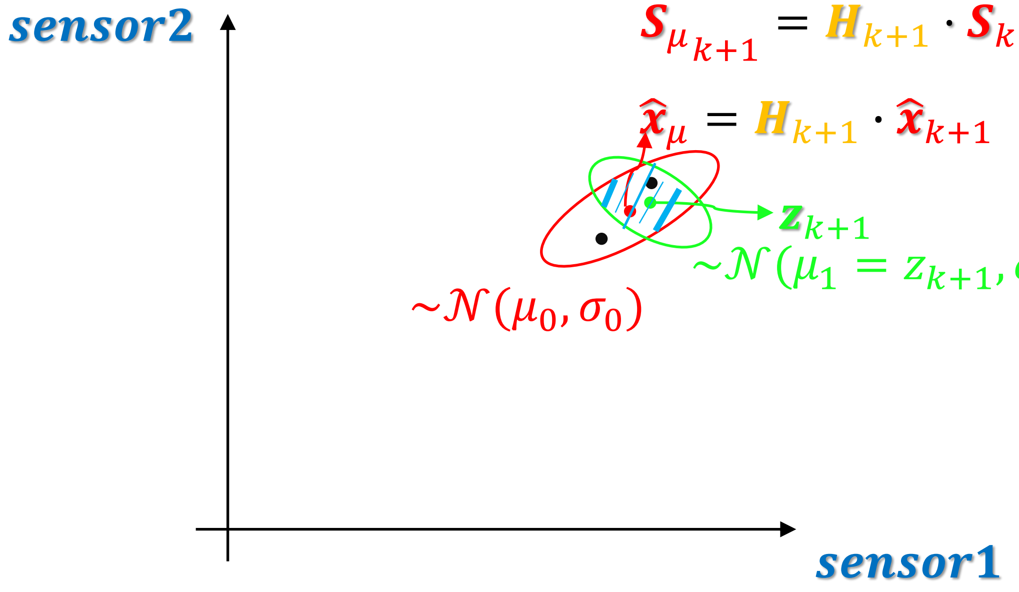



$$S_{\mu_{k+1}} = H_{k+1} \cdot S_{k+1} \cdot H_{k+1}^T$$

$$\hat{x}_\mu = H_{k+1} \cdot \hat{x}_{k+1}$$

$$z_{k+1}$$

$$\sim \mathcal{N}(\mu_1 = z_{k+1}, \sigma_1 = R_k)$$

$$\sim \mathcal{N}(\mu_0, \sigma_0)$$

$$\mathcal{N}(\mu', \sigma') = \mathcal{N}(\mu_0, \sigma_0) \cdot \mathcal{N}(\mu_1, \sigma_1) = \frac{1}{\sqrt{2\pi}\sigma'} e^{\frac{(x-\mu')^2}{2\sigma'^2}}$$

$$\begin{cases} \mu' = \mu_0 + \dfrac{\sigma_0^2(\mu_1 - \mu_0)}{\sigma_0^2 + \sigma_1^2} \\ \sigma'^2 = \sigma_0^2 - \dfrac{\sigma_0^4}{\sigma_0^2 + \sigma_1^2} \end{cases}, \quad \text{then define } K \equiv \frac{\sigma_0^2}{\sigma_0^2 + \sigma_1^2}$$

$$\Downarrow$$

$$\begin{cases} \mu' = \mu_0 + K(\mu_1 - \mu_0) \\ \sigma'^2 = \sigma_0^2 - K\sigma_0^2 \end{cases}$$

联合分布

# II. Kalman Filter

## Combination: Predict & Observation

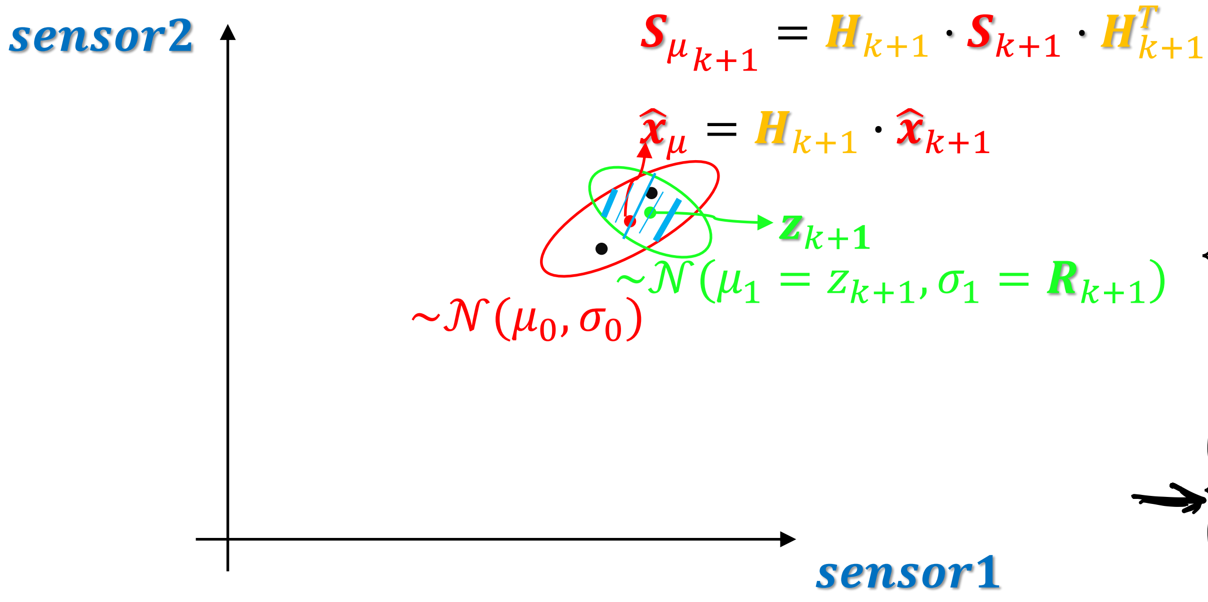
## C. Mathematics of Kalman Filter

- **Explanation:**

The blue area is the most likely area that our best results may fall into

**How to find out?**

Now we're dealing with multi-variables/matrix




$$\mathcal{N}(\mu', \Sigma') = \mathcal{N}(\mu_0, \Sigma_0) \cdot \mathcal{N}(\mu_1, \Sigma_1)$$

$$\begin{cases} K = \Sigma_0(\Sigma_0 + \Sigma_1)^{-1} & \textbf{Eq. II} \\ \mu' = \mu_0 + K(\mu_1 - \mu_0) \\ \Sigma' = \Sigma_0 - K\Sigma_0 & \textbf{Eq. III} \end{cases}$$

$$\begin{cases} (\mu_0, \Sigma_0) = (H_{k+1}\hat{x}_{k+1}, H_{k+1} \cdot S_{k+1} \cdot H_{k+1}^T) \\ (\mu_1, \Sigma_1) = (z_{k+1}, R_{k+1}) \end{cases} \quad \textbf{Eq. IV}$$

# II. Kalman Filter

**Combination: Predict & Observation**

## C. Mathematics of Kalman Filter

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**How to find out?**

Now we're dealing with multi-variables/matrix

$$\mathcal{N}(\mu', \Sigma') = \underbrace{\mathcal{N}(\mu_0, \Sigma_0)}_{} \cdot \underbrace{\mathcal{N}(\mu_1, \Sigma_1)}_{}$$

predict observation

$$\hat{\boldsymbol{x}}_{k+1} = \boldsymbol{F}_{k+1} \cdot \hat{\boldsymbol{x}}_k + \boldsymbol{B}_k \cdot \boldsymbol{u}_k$$

$$\boldsymbol{S}_{k+1} = \boldsymbol{F}_{k+1} \cdot \boldsymbol{S}_k \cdot \boldsymbol{F}_{k+1}^T + \boldsymbol{Q}_{k+1}$$

**Eq. I**

$$\begin{cases} K = \Sigma_0(\Sigma_0 + \Sigma_1)^{-1} & \textbf{Eq. II} \\ \mu' = \mu_0 + K(\mu_1 - \mu_0) \\ \Sigma' = \Sigma_0 - K\Sigma_0 & \textbf{Eq. III} \end{cases}$$

$$\begin{cases} (\mu_0, \Sigma_0) = (\boldsymbol{H}_{k+1}\hat{\boldsymbol{x}}_{k+1}, \boldsymbol{H}_{k+1} \cdot \boldsymbol{S}_{k+1} \cdot \boldsymbol{H}_{k+1}^T) \\ (\mu_1, \Sigma_1) = (z_{k+1}, \boldsymbol{R}_{k+1}) \end{cases}$$

**Eq. IV**

# II. Kalman Filter

**Combination: Predict & Observation**

## C. Mathematics of Kalman Filter

- **Explanation:**

The blue area is the most likely area that our best results may fall into

How to find out?

Now we're dealing with multi-variables/matrix

$$\mathcal{N}(\mu', \Sigma') = \mathcal{N}(\mu_0, \Sigma_0) \cdot \mathcal{N}(\mu_1, \Sigma_1)$$

$$K = \Sigma_0(\Sigma_0 + \Sigma_1)^{-1}$$

**Eq. II**

$$\begin{cases} \mu' = \mu_0 + K(\mu_1 - \mu_0) \\ \Sigma' = \Sigma_0 - K\Sigma_0 \end{cases}$$

**Eq. III**

$$\begin{cases} (\mu_0, \Sigma_0) = (\boldsymbol{H}_{k+1}\hat{x}_{k+1}, \boldsymbol{H}_{k+1} \cdot \boldsymbol{S}_{k+1} \cdot \boldsymbol{H}^T_{k+1}) \\ (\mu_1, \Sigma_1) = (z_{k+1}, \boldsymbol{R}_{k+1}) \end{cases}$$

**Eq. IV**

**Eq. IV** ⟶ **Eq. III**

$$\begin{cases} \boldsymbol{H}_{k+1}\hat{\boldsymbol{x}}'_{k+1} = (\boldsymbol{H}_{k+1}\hat{\boldsymbol{x}}_{k+1} + K(z_{k+1} - \boldsymbol{H}_{k+1}\hat{\boldsymbol{x}}_{k+1})) \\ \boldsymbol{H}_{k+1}\boldsymbol{S}'_{k+1}\boldsymbol{H}^T_{k+1} = (\boldsymbol{H}_{k+1}\boldsymbol{S}_{k+1}\boldsymbol{H}^T_{k+1}, K\boldsymbol{H}_{k+1}\boldsymbol{S}_{k+1}\boldsymbol{H}^T_{k+1}) \end{cases}$$

**Eq. V**

**Eq. IV** ⟶ **Eq. II**

$$K = \boldsymbol{H}_{k+1}\boldsymbol{S}_{k+1}\boldsymbol{H}^T_{k+1}(\boldsymbol{H}_{k+1}\boldsymbol{S}_{k+1}\boldsymbol{H}^T_{k+1} + \boldsymbol{R}_{k+1})^{-1}$$

**Eq. VI**

$\boldsymbol{H}^T_{k+1}$ · **Eq. V / Eq. VI**

**Eq. VIII** **Eq. VII**

# II. Kalman Filter

**Kalman Filter:**

## C. Mathematics of Kalman Filter

- **Explanation:**

$$\left\{\begin{aligned} \hat{\boldsymbol{x}}_{k+1} &= \boldsymbol{F}_{k+1} \cdot \hat{\boldsymbol{x}}_k + \boldsymbol{B}_k \cdot \boldsymbol{u}_k \\ \boldsymbol{S}_{k+1} &= \boldsymbol{F}_{k+1} \cdot \boldsymbol{S}_k \cdot \boldsymbol{F}_{k+1}^T + \boldsymbol{Q}_{k+1} \end{aligned}\right.$$

**Eq. I** **Prediction**

$$\rightarrow K' = \boldsymbol{S}_{k+1} \cdot \boldsymbol{H}_{k+1}^T \cdot \left(\boldsymbol{H}_{k+1} \cdot \boldsymbol{S}_{k+1} \cdot \boldsymbol{H}_{k+1}^T + \boldsymbol{R}_{k+1}\right)^{-1}$$

**Eq. VII** **Gain**

$$\left\{\begin{aligned} \hat{\boldsymbol{x}}'_{k+1} &= \hat{\boldsymbol{x}}_{k+1} + K'(z_{k+1} - \boldsymbol{H}_{k+1} \cdot \hat{\boldsymbol{x}}_{k+1}) \\ \hat{\boldsymbol{S}}'_{k+1} &= \boldsymbol{S}_{k+1} - K' \cdot \boldsymbol{H}_{k+1} \cdot \boldsymbol{S}_{k+1} \end{aligned}\right.$$

**Eq. VIII** **Correction**

# II. Kalman Filter

**Kalman Filter:**

## C. Mathematics of Kalman Filter

增益

方向

- **Explanation:**

Diff between observation mean & corrected predicted state

$$\hat{\boldsymbol{x}}'_{k+1} = \hat{\boldsymbol{x}}_{k+1} + K' \boxed{(z_{k+1} - \boldsymbol{H}_{k+1} \cdot \hat{\boldsymbol{x}}_{k+1})}$$

$$\hat{\boldsymbol{S}}'_{k+1} = \boldsymbol{S}_{k+1} + K' \cdot \boldsymbol{H}_{k+1} \cdot \boldsymbol{S}_{k+1}$$

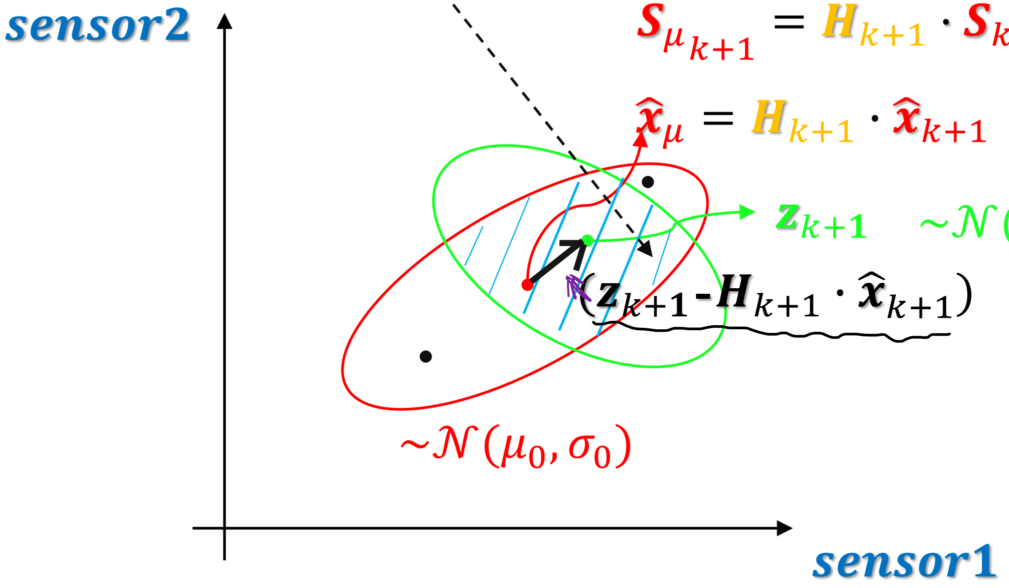
**Eq. VIII**

**Correction**

$$\boldsymbol{S}_{\mu_{k+1}} = \boldsymbol{H}_{k+1} \cdot \boldsymbol{S}_{k+1} \cdot \boldsymbol{H}^{T}_{k+1}$$

$$\hat{\boldsymbol{x}}_{\mu} = \boldsymbol{H}_{k+1} \cdot \hat{\boldsymbol{x}}_{k+1}$$ 预测值.

$\boldsymbol{z}_{k+1}$ $\sim\mathcal{N}(\mu_1 = z_{k+1}, \sigma_1 = \boldsymbol{R}_{k+1})$ 观测点



# II. Kalman Filter

**Kalman Filter:**

## C. Mathematics of Kalman Filter

- **Explanation:**

How far the corrected predicted state is from the ideal observation

$$\hat{\boldsymbol{x}}'_{k+1} = \hat{\boldsymbol{x}}_{k+1} + K' \boxed{(z_{k+1} - \boldsymbol{H}_{k+1} \cdot \hat{\boldsymbol{x}}_{k+1})}$$

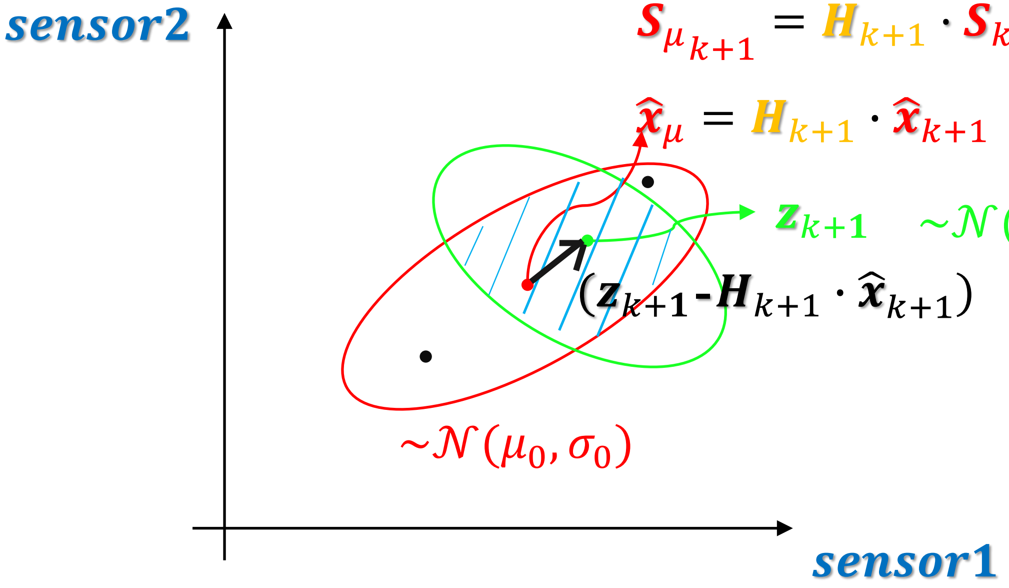
$$\hat{\boldsymbol{S}}'_{k+1} = \boldsymbol{S}_{k+1} + K' \cdot \boldsymbol{H}_{k+1} \cdot \boldsymbol{S}_{k+1}$$

**Eq. VIII**

**Correction**

$$\boldsymbol{S}_{\mu_{k+1}} = \boldsymbol{H}_{k+1} \cdot \boldsymbol{S}_{k+1} \cdot \boldsymbol{H}^{T}_{k+1}$$

$$\hat{\boldsymbol{x}}_{\mu} = \boldsymbol{H}_{k+1} \cdot \hat{\boldsymbol{x}}_{k+1}$$




$\boldsymbol{z}_{k+1} \quad \sim\mathcal{N}(\mu_1 = z_{k+1}, \sigma_1 = \boldsymbol{R}_{k+1})$

# II. Kalman Filter

**Kalman Filter:**

## C. Mathematics of Kalman Filter

- **Explanation:**

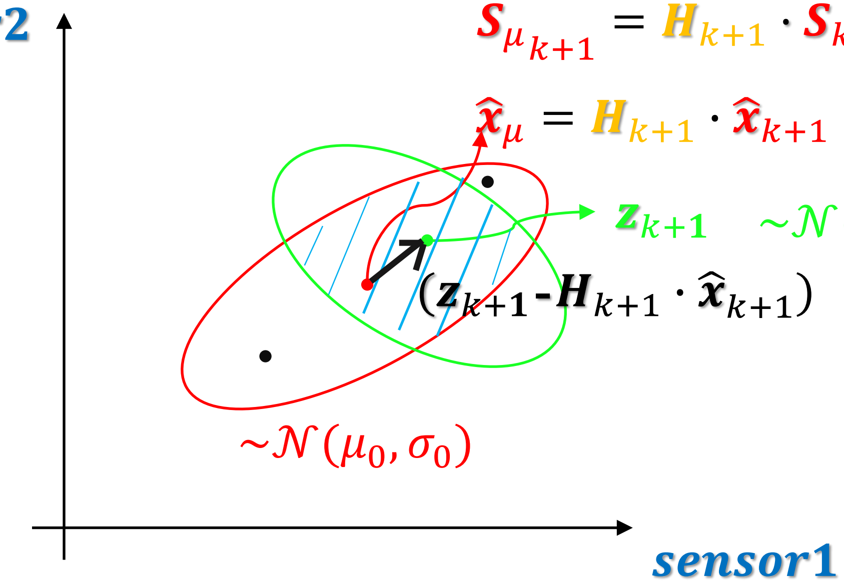
$$\hat{\boldsymbol{x}}'_{k+1} = \hat{\boldsymbol{x}}_{k+1} + K'(z_{k+1} - \boldsymbol{H}_{k+1} \cdot \hat{\boldsymbol{x}}_{k+1})$$

$$\hat{\boldsymbol{S}}'_{k+1} = \boldsymbol{S}_{k+1} + \boxed{K'} \cdot \boldsymbol{H}_{k+1} \cdot \boldsymbol{S}_{k+1}$$

**Eq. VIII**

**Correction**

How much we need to move the corrected predicted state to the ideal observation




$$\boldsymbol{S}_{\mu_{k+1}} = \boldsymbol{H}_{k+1} \cdot \boldsymbol{S}_{k+1} \cdot \boldsymbol{H}^T_{k+1}$$

$$\hat{\boldsymbol{x}}_{\mu} = \boldsymbol{H}_{k+1} \cdot \hat{\boldsymbol{x}}_{k+1}$$

$$\boldsymbol{z}_{k+1} \quad \sim\mathcal{N}(\mu_1 = z_{k+1}, \sigma_1 = \boldsymbol{R}_{k+1})$$