Jacobian矩阵

- 对于两个空间的线性变换 $F(x)$，想用最简单的形式表示变换。

设 $T(x) = Ax + b$，近似 $F(x)$.

因为 P 的聚焦点，$T(P) = F(P) = AP + b$　　$b = F(P) - AP$

则　$T(x) = A(x-P) + F(P)$.

$$\lim_{x \to P} \frac{F(x) - F(P) - A(x-P)}{\|x-P\|} = 0$$

考虑 R^n 的标准基底 $\{e_1, \dots, e_n\}$

$$\lim_{h \to 0} \frac{F(P+he_j) - F(P) - A(he_j)}{h} = 0 \qquad j = 1, 2, \dots, n.$$

因为 $A(he_j) = h(Ae_j)$

$$\lim_{h \to 0} \frac{F(P+he_j) - F(P)}{h} = Ae_j.$$

上式左边即为导数

$$\frac{\partial F}{\partial x_j}(P) = \begin{pmatrix} \frac{\partial f_1}{\partial x_j}(P) \\ \vdots \\ \frac{\partial f_m}{\partial x_j}(P) \end{pmatrix}$$

等式右边即为 A 的第 j 行.

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(P), & \dots, & \frac{\partial f_1}{\partial x_n}(P) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(P), & \dots, & \frac{\partial f_n}{\partial x_n}(P) \end{bmatrix}_{m \times n}$$

称为向量函数 F 在 P 的 Jacobian矩阵，记为 $J(P)$，

因此，F 在 P 的最佳仿射近似是

$$T(x) = F(P) + J(P)(x-P).$$

设 $h = x - P$，则 $F(x) = F(P+h)$ 的泰勒展开为

$$F(P+h) = F(P) + J(P)h + o(\|h\|^2).$$

因为 $F(P) = T(P)$

$$T(x) - T(P) = J(P)(x-P)$$

上式指出，仿射变换的改变量 $T(x) - T(P)$ 是自变量 $x-P$ 的线性函数，$J(P)$ 为线性变换矩阵.

为一个空间到另一个空间的线性变换矩阵，

Jacobian 矩阵

- $F_1(x,y)=(2x,-xy^2,x+3y^2)$ 则

$$J_1(x,y)=\begin{bmatrix}2&0\\-y^2&-2xy\\1&6y\end{bmatrix}$$

若 $F_2(x,y)=(2x,-x+2y,x+3y)$ 则

$$J_2(x,y)=\begin{bmatrix}2&0\\-1&2\\1&3\end{bmatrix}$$

即

$$F_2(x,y)=J_2(x,y)\begin{pmatrix}x\\y\end{pmatrix}$$

- $$J(r,\theta)=\begin{bmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial\theta}\\\frac{\partial y}{\partial r}&\frac{\partial y}{\partial\theta}\end{bmatrix}=\begin{bmatrix}\frac{\partial(r\cos\theta)}{\partial r}&\frac{\partial(r\cos\theta)}{\partial\theta}\\\frac{\partial(r\sin\theta)}{\partial r}&\frac{\partial(r\sin\theta)}{\partial\theta}\end{bmatrix}$$

$$=\begin{bmatrix}\cos\theta&-r\sin\theta\\\sin\theta&r\cos\theta\end{bmatrix}$$

若 $u(t)=\begin{bmatrix}r(t)\\\theta(t)\end{bmatrix}$ 为极坐标平面上的一条曲线，$x(t)=\begin{pmatrix}x(t)\\y(t)\end{pmatrix}$ 是卡氏坐标平面上的映射曲线，

$$\frac{dx}{dt}=\begin{pmatrix}\frac{dx}{dt}\\\frac{dy}{dt}\end{pmatrix}=\begin{bmatrix}\frac{dx}{dr}\frac{dr}{dt}\quad\frac{dx}{d\theta}\frac{d\theta}{dt}\\\frac{dy}{dr}\frac{dr}{dt}\quad\frac{\partial y}{\partial\theta}\frac{d\theta}{dt}\end{bmatrix}=\begin{bmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial\theta}\\\frac{\partial y}{\partial r}&\frac{\partial y}{\partial\theta}\end{bmatrix}\begin{bmatrix}\frac{dr}{dt}\\\frac{d\theta}{dt}\end{bmatrix}$$

$$=J(r,\theta)\frac{du}{dt}$$

从几何上说，Jacobian矩阵将极坐标平面的切向量映至卡氏座标平面的切向量。

Jacobian 行列式

- 若 F 将 $\mathbf{u}=\begin{bmatrix}u\\v\end{bmatrix}$ 映至 $\mathbf{x}=\begin{pmatrix}x\\y\end{pmatrix}$，则F的 Jacobia行列式为.

$$\det(J(u,v))=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

令R表示 $\begin{bmatrix}du\\0\end{bmatrix}$，$\begin{bmatrix}0\\dv\end{bmatrix}$ 所张成的长方形，则F(R)近似下面向量所张成的长方形.

$$J(u,v)\begin{bmatrix}du\\0\end{bmatrix}=\begin{bmatrix}\frac{\partial x}{\partial u}du\\ \frac{\partial y}{\partial u}du\end{bmatrix}$$

$$J(u,v)\begin{bmatrix}0\\dv\end{bmatrix}=\begin{bmatrix}\frac{\partial x}{\partial v}dv\\ \frac{\partial y}{\partial v}dv\end{bmatrix}$$

令dA代表平行四边形 F(R) 的面积，因为二阶行列式的行向量所张成平行四边形面积等于行列式的绝对值.

$$dA=\left|\det\begin{bmatrix}\frac{\partial x}{\partial u}du & \frac{\partial x}{\partial v}dv\\ \frac{\partial y}{\partial u}du & \frac{\partial y}{\partial v}dv\end{bmatrix}\right|=\left|\det\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}\right|du\,dv$$

$$=|\det J(u,v)|\,du\,dv.$$

所以微小区域R，经过 F 映射至F(R)，其面积伸缩 $|\det J(u,v)|$.

Jacobian 计算法

- 外积： $dx_i dx_j = -dx_j dx_i$.

wedge production: $dx_i \wedge dx_j = -dx_j \wedge dx_i$.

- 若：$dy = \begin{pmatrix} dy_1 \\ \vdots \\ dy_m \end{pmatrix}_{m\times 1}$ $\quad dx = \begin{pmatrix} dx_1 \\ \vdots \\ dx_m \end{pmatrix} = B\,dy$, B为非奇异 $m\times m$.

则：$\bigwedge\limits_{i=1}^{m} dx_i = \det(B)\cdot \bigwedge\limits_{i=1}^{m} dy_i$

证：$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} = \begin{pmatrix} B_{11}dy_1 + B_{12}dy_2 \\ B_{21}dy_1 + B_{22}dy_2 \end{pmatrix}$

$$\begin{aligned}
dx_1 \wedge dx_2 &= (B_{11}dy_1 + B_{12}dy_2) \wedge (B_{21}dy_1 + B_{22}dy_2) \\
&= B_{11}B_{21}\cancel{dy_1\wedge dy_1} + B_{12}B_{21}\,dy_2\wedge dy_1 + B_{11}B_{22}\,dy_1\wedge dy_2 + B_{12}B_{22}\cancel{dy_2\wedge dy_2} \\
&= -B_{12}B_{21}\,dy_1\wedge dy_2 + B_{11}B_{22}\,dy_1\wedge dy_2 \\
&= (B_{11}B_{22} - B_{12}B_{21})\,dy_1\wedge dy_2 \\
&= \det(B)\,dy_1 \wedge dy_2
\end{aligned}$$

- 迪卡尔坐标 $x_1 \quad \cdots \quad x_m$

  极坐标 $r, \theta_1 \cdots \quad \theta_{m-1}$

$$x_1 = r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{m-2} \cdot \sin\theta_{m-1}$$

$$x_2 = r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{m-2} \cos\theta_{m-1}$$

$$\vdots$$

$$x_{m-1} = r \sin\theta_1 \cos\theta_2$$

$$x_m = r\cos\theta_1$$

$$J(x \to r_1, \theta_1, \cdots, \theta_{m-1})$$

$$= r^{m-1} \sin^{m-2}\theta_1 \cdot \sin^{m-3}\theta_2 \cdots \sin\theta_{m-2}$$

$x_1^2 = r^2 \sin^2\theta_1 \cdots \sin^2\theta_{m-1}$	$2x_1 dx_1 = 2r^2 \sin^2\theta_1 \cdots d\theta_{m-1} + \text{terms}$
$x_1^2 + x_2^2 = r^2 \sin^2\theta_1 \cdots \sin^2\theta_{m-2}$	$2x_1dx_1 + 2x_2dx_2 = 2r^2 \cdots \sin\theta_{m-2}\cos\theta_{m-2}\, d\theta_{m-2} + \text{terms}$
$\vdots$	$\vdots$
$x_1^2 + x_2^2 + \cdots + x_m^2 = r^2.$	$2x_1dx_1 + 2x_2dx_2 + \cdots + 2x_mdx_m = 2r\,dr$

$$\cancel{2^m} \cdot x_1 \cdots x_m \bigwedge_{i=1}^{m} dx_i = \cancel{2^m} r^{2m-1} \sin^{r-3}\theta_1 \sin^{r-5}\theta_2 \cdots \bigwedge_{i=1}^{m} d\theta_i.$$

- X, Y 是 $n \times m$

  $X = BYC$

$$dX = (\det B)^m (\det C)^n (dY)$$

$$J(X \to Y) = (\det B)^m (\det C)^n.$$

- 讨 $X = BYB^T$, 则 X与Y为对称矩阵, $m \times m$.

  then. $dX = (\det B)^{m+1} dY$

$$J(X \to Y) = (\det B)^{m+1}.$$

- If $X = Y^{-1}$ 则 $dX = (\det Y)^{-2m} dY.$

  若Y为对称 $dX = (\det Y)^{-(m+1)} dY.$

- Thm: A是 $m\times m$ 对称矩阵，$A=T^T T$，T为上三角矩阵

$J(A\to T)$.