The Central Limit Theorem

- Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed random variables with $E[Y_i] = \mu$ and $Var[Y_i] = \sigma^2 < \infty$.

$$U_n = \frac{\sum_{i=1}^{n} Y_i - n\mu}{\sigma\sqrt{n}} = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \approx N(0, 1).$$

- *No matter what the shape of the population distribution,* **sample means** *approaches a normal distribution as the sample size gets larger.*

- $\bar{Y} \approx N(\mu, \frac{\sigma^2}{n}$

Example

Consider an i.i.d sample $X_1, \ldots, X_n$ of size n. In each situation, construct a statistic such that the distribution of the statistic is standard normal.

$n = 100$ and $X_i \sim unif(0, 1)$

$n = 100$ and $X_i \sim Exp(1)$

$n = 100$ and $X_i \sim Bern(1/2)$

$n = 144$ and $X_i \sim Gamma(1, 2)$

$n = 144$ and $X_i \sim \chi^2_{(4)}$

Estimation Theory

- Statistical **inference** aims at learning characteristics of the population from a sample; the population characteristics are parameters and sample characteristics are statistics.
- **Estimation** represents ways or a process of learning and determining the population parameter based on the model fitted to the data.
- If the parameter is denoted by θ, then the **estimator** is traditionally written as $\widehat{\theta}$ or $\widehat{\theta}(X)$.
- How can we establish some criteria of goodness to compare estimators? How can we pick up the best estimator?
- **One intuitive rule**: we would like the expected value of the distribution of the estimates to equal the parameter estimated.

Unbiased Estimator

- Let $\hat{\theta}$ be an estimator for a parameter θ. Then $\hat{\theta}$ is an *unbiased estimator* if
$$\mathbb{E}[\hat{\theta}] = \theta.$$
  - If $\mathbb{E}[\hat{\theta}] \neq \theta.$, then $\hat{\theta}$ is an biased estimator.
- Bias
$$\mathbb{B}(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta.$$
  - If $\hat{\theta}$ is unbiased, then $\mathbb{B}(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta = 0$.
- How do we choose the best estimator among unbiased estimators? $\Rightarrow$ *MSE*

Mean Square Error (MSE)

- MSE of a point estimator $\hat{\theta}$ is

$$\mathrm{MSE}_\theta(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \theta)^2].$$

- The MSE can be expressed in terms of the variance and the bias of a estimator as:

$$\mathrm{MSE}_\theta(\hat{\theta}) = \mathbb{V}[\hat{\theta}] + [\mathbb{B}(\hat{\theta})]^2.$$

  That is, for an unbiased estimator $\mathbb{B}(\hat{\theta}) = 0$

$$\mathrm{MSE}_\theta(\hat{\theta}) = \mathbb{V}[\hat{\theta}].$$

Let $X_1, \ldots, X_n$ be i.i.d. $\mathcal{N}(\mu, \sigma^2)$ random variables, where $-\infty < \mu < \infty$ and $0 < \sigma^2 < \infty$ are unknown.

a) $\bar{X}$, X_1 and $\frac{X_1+X_2}{2}$ are unbiased estimators of μ.

b) S^2 and $\frac{(X_1-X_2)^2}{2}$ are unbiased estimators of σ^2

c) S_n^2 is a biased estimator of σ^2. Note however that $n/(n-1)S_n^2$ is an unbiased estimator of σ^2.