Discussion 7

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Properties of Point Estimators and Methods of Estimation

- Properties
  1. Unbiasedness - MSE, Relative Efficiency
  2. Consistency
  3. Sufficiency
- Methods
  1. Methods of Moment (Mom) Estimator
  2. Maximum Likelihood (ML) Estimator
  3. MVUE : Rao-Blackwell Thm

Relative efficiency

- **Def.** Given two unbiased estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ of a parameter θ, with variances $\mathbb{V}(\hat{\theta}_1)$ and $\mathbb{V}(\hat{\theta}_2)$, respectively, then the *efficiency* of $\hat{\theta}_1$ relative to $\hat{\theta}_2$, denoted $\mathrm{eff}(\hat{\theta}_1, \hat{\theta}_2)$, is defined to be the ratio
$$\mathrm{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\mathbb{V}(\hat{\theta}_2)}{\mathbb{V}(\hat{\theta}_1)}.$$

- $\mathrm{eff}(\hat{\theta}_1, \hat{\theta}_2) > 1 \Rightarrow \mathbb{V}(\hat{\theta}_2) > \mathbb{V}(\hat{\theta}_1) \Rightarrow \hat{\theta}_1$ is better!
- $\mathrm{eff}(\hat{\theta}_1, \hat{\theta}_2) < 1 \Rightarrow \mathbb{V}(\hat{\theta}_2) < \mathbb{V}(\hat{\theta}_1) \Rightarrow \hat{\theta}_2$ is better!
- $\mathrm{eff}(\hat{\theta}_1, \hat{\theta}_2) = 1 \Rightarrow \mathbb{V}(\hat{\theta}_1) = \mathbb{V}(\hat{\theta}_2) \Rightarrow$ no preference between the two estimator.

Consistency

- Large Sample Inference
- $\hat{\theta}_n$ **converges to** θ **(**$\hat{\theta}_n \xrightarrow{P} \theta$**)** if, for any positive number ε,

$$\lim_{n\to\infty} \mathbb{P}(|\hat{\theta}_n - \theta| \leq \varepsilon) = 1,$$

  or equivalently,

$$\lim_{n\to\infty} \mathbb{P}(|\hat{\theta}_n - \theta| > \varepsilon) = 0.$$

- It means the probability of the event

$$\{|\hat{\theta}_n - \theta| \geq \varepsilon\} = \{\ \hat{\theta}_n \text{ stays away from } \theta\ \}$$

  gets small as n gets large.

Theorems

1. The Weak Law of Large Numbers (WLLN):
   Let $X_1, X_2, \ldots, X_n$ denote a random sample from a distribution with mean μ and variance $\sigma^2 < \infty$. Then, $\bar{X} \xrightarrow{P} \mu$

2. An unbiased estimator $\hat{\theta}_n$ for θ is a consistent estimator of θ if

$$\lim_{n\to\infty} \mathbb{V}(\hat{\theta}_n) = 0.$$

3. Suppose $X_n \xrightarrow{P} x$ and $Y_n \xrightarrow{P} y$. Then,
   a. $X_n \pm Y_n \xrightarrow{P} x \pm y$
   b. $X_n Y_n \xrightarrow{P} xy$
   c. $X_n / Y_n \xrightarrow{P} x/y$, provided that $y \neq 0$.
   d. If $g(\cdot)$ is a real-valued function, then $g(X_n) \xrightarrow{P} g(x)$.

Sufficiency

- A statistic $U = U(X_1, X_2, \ldots, X_n)$ is a sufficient statistic for a parameter θ if **it contains all of the information about θ** that is available in the sample.

- **Def.** The statistic U is said to be *sufficient* for θ if

$$f_{\mathbf{X}|U}(\mathbf{x}|u) = \frac{f(x_1, x_2, \ldots, x_n, U; \theta)}{f(u; \theta)}$$

  is free of θ.

- **Factorization Thm** The statistic U is said to be *sufficient* for θ if and only if

$$L(x_1, x_2, \ldots, x_n|\theta) = g(u; \theta) \times h(x_1, x_2, \ldots, x_n),$$

  where $g(u, \theta)$ is a function only of u and θ and $h(x_1, x_2, \ldots, x_n)$ is free of θ.

Example

Let $X_1, X_2, \ldots, X_n$ denote a random sample from a uniform distribution on the interval $(0, \theta)$. Consider two unbiased estimators for θ:

$$\hat{\theta}_1 = 2\bar{X} \quad \text{and} \quad \hat{\theta}_2 = \left(\frac{n+1}{n}\right) X_{(n)}.$$

$\hat{\theta}_1$ and $\hat{\theta}_2$ are consistent estimators?

Practice

Let $X_1, X_2, \ldots, X_n$ be i.i.d. $\mathcal{N}(\mu, \sigma^2)$. Prove that

1. $\bar{X}$ is a consistent estimator for μ.
2. X_1 is not a consistent estimator for μ.
3. S_n^2 is a consistent estimator for σ^2.
4. $(X_2 - X_1)^2/2$ is an unbiased estimator, but not a consistent estimator for σ^2