Discussion 8

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March 13, 2020

Sufficiency

- A statistic $U = U(X_1, X_2, \ldots, X_n)$ is a sufficient statistic for a parameter θ if **it contains "all of the information" about θ** that is available in the sample.

- **Def.** The statistic U is said to be *sufficient* for θ if

$$f_{\mathbf{X}|U}(\mathbf{x}|u) = \frac{f(x_1, x_2, \ldots, x_n, U; \theta)}{f(u; \theta)}$$

  is free of θ.

- **Factorization Thm** The statistic U is said to be *sufficient* for θ if and only if

$$L(x_1, x_2, \ldots, x_n|\theta) = g(u; \theta) \times h(x_1, x_2, \ldots, x_n),$$

  where $g(u, \theta)$ is a function only of u and θ and $h(x_1, x_2, \ldots, x_n)$ is free of θ.

MVUE: minimum-variance unbiased estimator

- **The Rao-Blackwell Theorem**
  Let $\hat{\theta}$ be an unbiased estimator for θ such that $\mathbb{V}[\hat{\theta}] < \infty$. If T is a sufficient statistic for θ, define $\hat{\theta}^* = \mathbb{E}[\hat{\theta}|T]$. Then,
  1. $\hat{\theta}^*$ is an unbiased estimator for θ.
  2. The variance of $\hat{\theta}^*$ is no greater than the variance of $\hat{\theta}$, i.e., $\mathbb{V}[\hat{\theta}^*] \leq \mathbb{V}[\hat{\theta}]$.

- R.B. Thm implies that an unbiased estimator for θ with a small variance is function of the sufficient statistic.

$$\hat{\theta}^* = g(T)$$

1. $E(\hat{\theta}^*) = \theta$
2. $\hat{\theta}^*$ has the smallest variance among all unbiased estimators.

Practice

Let $X_1, X_2, \ldots, X_n$ be i.i.d. $\mathcal{N}(\mu, 1)$.

1. Find a sufficient statistic for μ.
2. Find the MVUE for μ.

Practice

Let $X_1, X_2, \dots, X_n$ be i.i.d. $Gam(\alpha, \beta)$.

1. If $Gam(\alpha, 4)$ and α is unknown, find a sufficient statistic for α.
2. If $Gam(2, \beta)$ and β is unknown, find a sufficient statistic and the MVUE for β.

Practice

Let $X_1, X_2, \ldots, X_n$ be i.i.d. $Geo(p)$. $f(x; p) = (1-p)^{x-1}p, \; x = 1, 2, 3...$

1. Find a sufficient statistic for p.
2. Find the MVUE for p.