Constructing analysis-suitable parameterization and curvature-based r-adaptive parameterization for IsoGeometric Analysis

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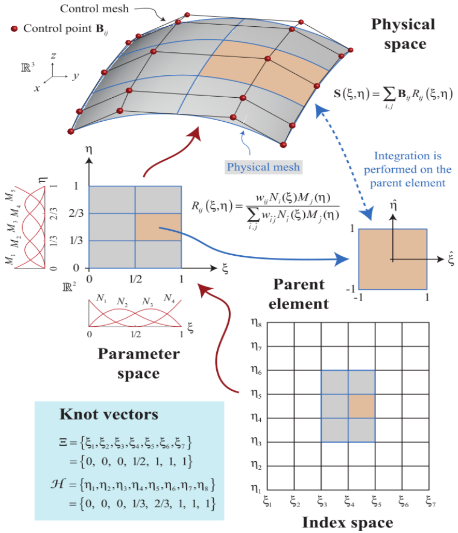
Jan. 3, 2023
2022 岳麓论坛 · 数学分论坛

Contributors

- Ye Ji et al., Constructing high-quality ..., Journal of Computational and Applied Mathematics, 396 (2021), 113615.
- Ye Ji et al., Penalty function-based volumetric ..., Computer Aided Geometric Design, 94 (2022), 102075.
- Ye Ji et al., On an improved PDE-based elliptic parameterization method for isogeometric ..., submitted.
- Ye Ji et al., Curvature-Based r-Adaptive ..., Computer-Aided Design, 150 (2022), 103305.

Catalogue

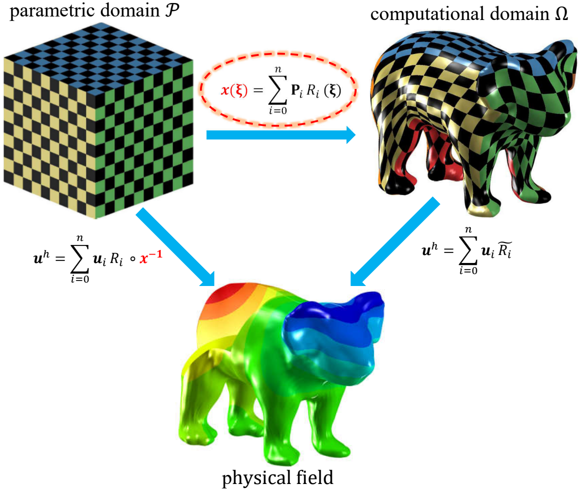
IsoGeometric Analysis (IGA)



Source: Figure from [Cottrell et al. 2009]

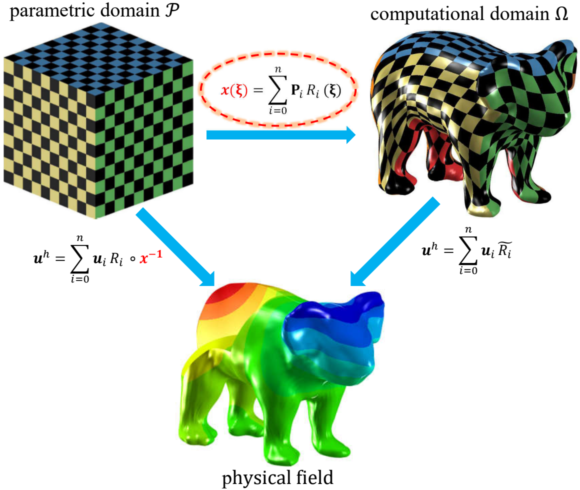
- Proposed by T.J.R. Hughes et al., 2005.
- **KEY IDEA:** to approximate the physical fields with the same basis functions as that used to generate the CAD model.
- Advantages:
  - Integration of design and analysis;
  - Exact and efficient geometry;
  - No data type transition and mesh generation;
  - Simplified mesh refinement;
  - High order continuous field;
  - Superior approximation properties.
- Very broad applications: such as shell analysis, fluid-structure interaction, and shape and topology optimization.

Research motivation



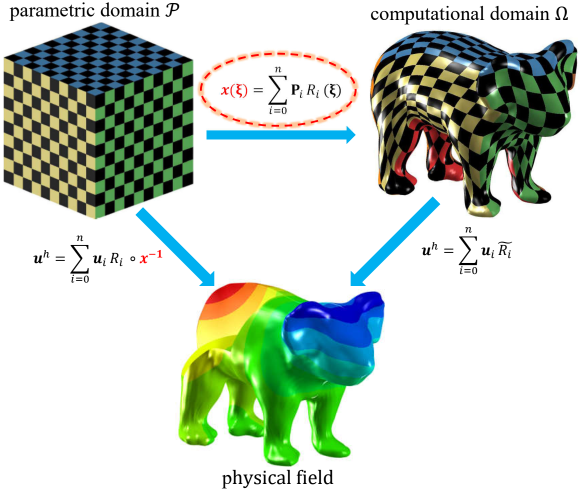
- Most modern CAD systems only focus on boundary representations (B-Reps) in geometry modeling.

Research motivation



- Most modern CAD systems only focus on boundary representations (B-Reps) in geometry modeling.
- **Problem statement**:
  - From a given B-Rep, constructing an analysis-suitable parameterization $\boldsymbol{x}$.

Research motivation



- Most modern CAD systems only focus on boundary representations (B-Reps) in geometry modeling.
- **Problem statement**:
  - From a given B-Rep, constructing an analysis-suitable parameterization $\boldsymbol{x}$.
  - Analysis-suitable parameterizations should
    - be **bijective**;
    - ensure as **low angle and area/volume distortion** as possible.

Catalogue

Problem statement

- A spline-based parameterization $\boldsymbol{x}$ from a parametric domain $\mathcal{P} = [0,1]^d$ $(d = 2,3)$ to computational domain Ω is of the following form

$$\boldsymbol{x}(\boldsymbol{\xi}) = \mathbf{R}^{\mathrm{T}}\mathbf{P} = \underbrace{\sum_{i\in\mathcal{I}_I} \mathbf{P}_i R_i(\boldsymbol{\xi})}_{\text{unknown}} + \underbrace{\sum_{j\in\mathcal{I}_B} \mathbf{P}_j R_j(\boldsymbol{\xi})}_{\text{known}}, \tag{1}$$

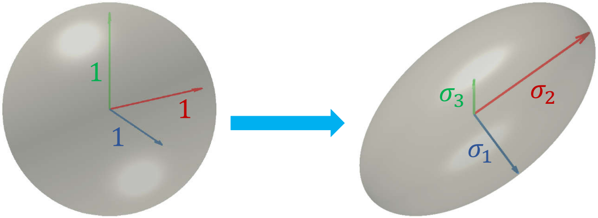
  where $\mathbf{P}_i$ are unknown inner control points and $\mathbf{P}_j$ are given boundary control points.
- **GOAL:** To construct the **unknown inner control points $\mathbf{P}_i$** such that $\boldsymbol{x}$ is **bijective** and has the **lowest possible angle and area/volume distortion**.

Objective function: angle distortion

- Most-Isometric ParameterizationS (MIPS) energy [Hormann and Greiner 2000, Fu+2015]:

$$E_{\text{angle}}(\boldsymbol{x}) = \begin{cases} \dfrac{\sigma_1}{\sigma_2} + \dfrac{\sigma_2}{\sigma_1}, & 2D, \\ \dfrac{1}{8}\left(\dfrac{\sigma_1}{\sigma_2} + \dfrac{\sigma_2}{\sigma_1}\right)\left(\dfrac{\sigma_2}{\sigma_3} + \dfrac{\sigma_3}{\sigma_2}\right)\left(\dfrac{\sigma_1}{\sigma_3} + \dfrac{\sigma_3}{\sigma_1}\right), & 3D. \end{cases} \tag{2}$$

  where σ_i are the singular values of the Jacobian matrix $\mathcal{J}$ of the parameterization $\boldsymbol{x}$.
- When $\sigma_1 = \sigma_2 = \ldots = \sigma_d$, $\boldsymbol{x}$ is conformal and E_{angle} reaches its minimum value.

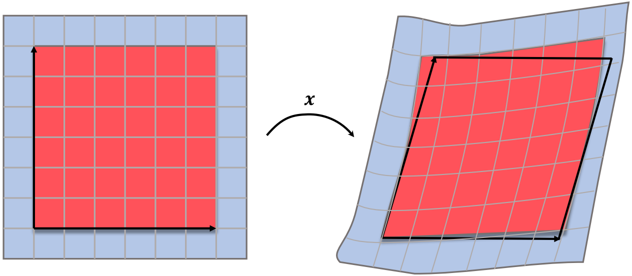


Objective function: area/volume distortion

- Area/volume distortion energy:

$$E_{\mathrm{vol}}(\boldsymbol{x}) = \frac{|\mathcal{J}|}{vol(\Omega)} + \frac{vol(\Omega)}{|\mathcal{J}|}, \tag{3}$$

where $vol(\Omega)$ denotes the area/volume of the computational domain Ω;



Objective function: variational formulation

- **Basic idea**: to solve the following constrained optimization problem:

$$
\begin{aligned}
\underset{\mathbf{P}_i,\ i\in\mathcal{I}_I}{\arg\min} \quad & E(\boldsymbol{x}) = \int_{\mathcal{P}} \left(\lambda_1 E_{\text{angle}}(\boldsymbol{x}) + \lambda_2 E_{\text{vol}}(\boldsymbol{x})\right) \, \mathrm{d}\mathcal{P}, \\
s.t. \quad & \boldsymbol{x} \text{ is bijective.}
\end{aligned} \tag{4}
$$

Objective function: variational formulation

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- Suppose that the given B-Rep is bijective. $\boldsymbol{x}$ is bijective $\Leftrightarrow |\mathcal{J}(\boldsymbol{x}(\boldsymbol{\xi}))| \neq 0, \ \forall \boldsymbol{\xi} \in \mathcal{P}$.

Objective function: variational formulation

- **Basic idea**: to solve the following constrained optimization problem:

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- Suppose that the given B-Rep is bijective. $\boldsymbol{x}$ is bijective $\Leftrightarrow |\mathcal{J}(\boldsymbol{x}(\boldsymbol{\xi}))| \neq 0, \ \forall \boldsymbol{\xi} \in \mathcal{P}$.
- Due to the high-order continuity of $\boldsymbol{x}$, we need $|\mathcal{J}| > 0 \ (< 0), \ \forall \boldsymbol{\xi} \in \mathcal{P}$.

Equivalence problem: unconstrained optimization

- Recall the planar MIPS energy,

$$E^{2D}_{angle}(\boldsymbol{x}) = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1\sigma_2} = \frac{trace(\mathcal{J}^{\mathrm{T}}\mathcal{J})}{|\mathcal{J}|}.$$

Since the Jacobian determinant appears in its denominator, it proceeds to infinity if the Jacobian determinant $|\mathcal{J}|$ approaches zero.

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- Remove the constraints and solve the following **unconstrained optimization problem**:

$$\underset{\mathbf{P}_i,\ i\in\mathcal{I}_I}{\arg\min} \quad E(\boldsymbol{x}) = \int_{\mathcal{P}} \left(\lambda_1 E_{\text{angle}}(\boldsymbol{x}) \ + \lambda_2 E_{\text{vol}}(\boldsymbol{x})\right) \ \mathrm{d}\mathcal{P}. \tag{5}$$

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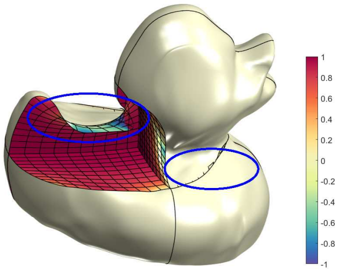
- **Prerequisite: need an already bijective initialization.**

Initialization

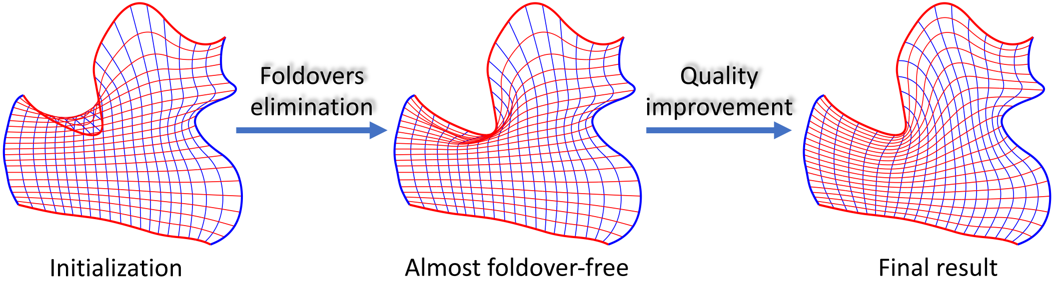


Initial parameterization.

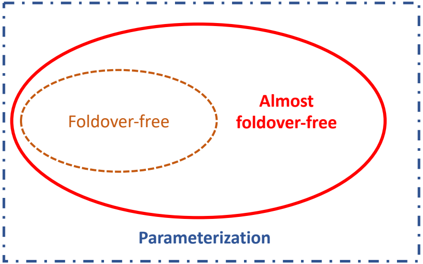
- Many algebraic methods can be adopted to initialize:
  - Discrete Coon's patch [Farin and Hansford 1999];
  - Spring patch [Gravesen et al. 2012];
  - Smoothness energy minimization [Wang et al. 2003, Pan et al. 2020];
  - ...
- No guarantee of bijectivity.
- However, an already bijective parameterization is needed in our optimization problem (5).

Barrier function-based parameterization construction

- Three-step strategy.



Foldovers elimination: almost foldover-free parameterization



- We solve the following optimization problem:

$$\underset{\mathbf{P}_i,\ i\in\mathcal{I}_I}{\arg\min} \quad E(\boldsymbol{x}) = \int_{\mathcal{P}} \max\left(0, \delta - |\mathcal{J}|\right) \ \mathrm{d}\mathcal{P},$$

where δ is a threshold ($\delta = 5\% vol(\Omega)$ as default).

Quality improvement: robustness consideration

- Recall that E_{angle} proceeds to infinity if the Jacobian determinant $\mathcal{J}$ approaches zero.
- DANGER!: discontinuous function value change in numerical optimization.

Quality improvement: robustness consideration

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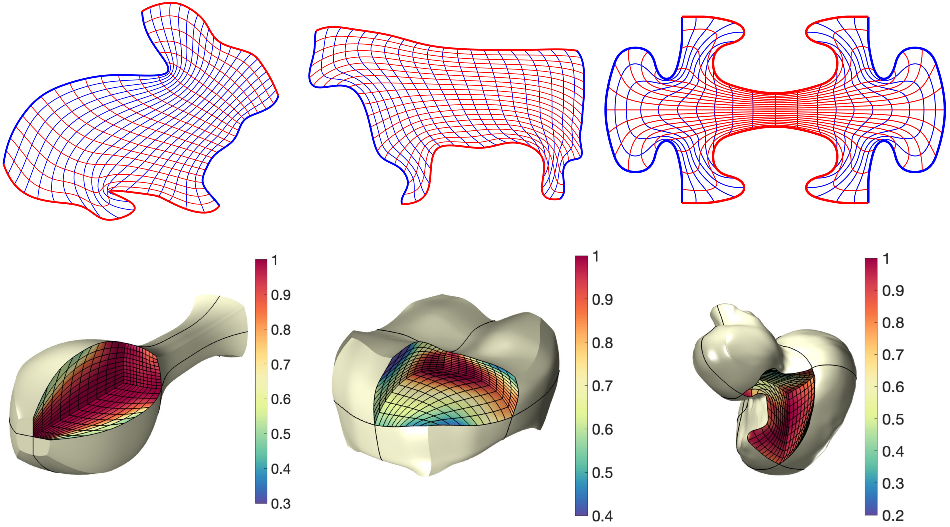
- Line search ensures sufficient reduction, e.g., strong Wolfe condition.

Quality improvement: robustness consideration

- Recall that E_{angle} proceeds to infinity if the Jacobian determinant $\mathcal{J}$ approaches zero.
- DANGER!: discontinuous function value change in numerical optimization.
- Line search ensures sufficient reduction, e.g., strong Wolfe condition.
- With this feature, we simply revise the objective function (barrier function):

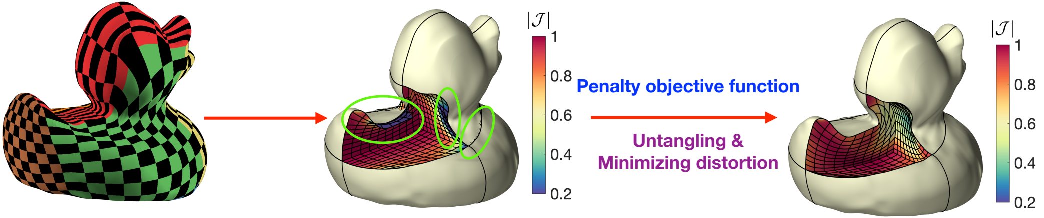
$$E^c = \begin{cases} \int_{\mathcal{P}} \left(\lambda_1 E_{\text{angle}}(\boldsymbol{x}) + \lambda_2 E_{\text{vol}}(\boldsymbol{x})\right) \, \mathrm{d}\mathcal{P}, & \text{if } \min |\mathcal{J}| > 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Gallery: barrier function-based method



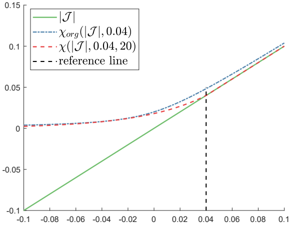
Penalty function-based parameterization construction

- **Foldovers elimination does not improve sufficient to the parameterization quality.**
- **Untangling and minimizing distortion perform simultaneously!!!**

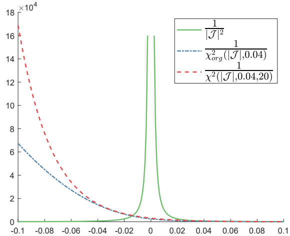


B-Rep **Initialization** **Optimized parameterization**

Basic idea: Penalty function



$\chi(|\mathcal{J}|, \varepsilon, \beta)$



$\frac{1}{\chi^2(|\mathcal{J}|,\varepsilon,\beta)}$

- **Penalty function:**

$$\chi(|\mathcal{J}|, \varepsilon, \beta) = \begin{cases} \varepsilon \cdot e^{\beta(|\mathcal{J}|-\varepsilon)} & \text{if } |\mathcal{J}| \le \varepsilon \\ |\mathcal{J}| & \text{if } |\mathcal{J}| > \varepsilon \end{cases}, \tag{6}$$

where ε is a small positive number and β is a penalty factor;

- $\chi(|\mathcal{J}|, \varepsilon, \beta)$ equals a small positive number if $|\mathcal{J}| < \varepsilon$, and strictly equals the Jacobian determinant $|\mathcal{J}|$ if $|\mathcal{J}| \ge \varepsilon$;

- $\frac{1}{\chi^2(|\mathcal{J}|,\varepsilon,\beta)}$ have **very large values to penalize the negative Jacobians and small values to accept positive Jacobians**.

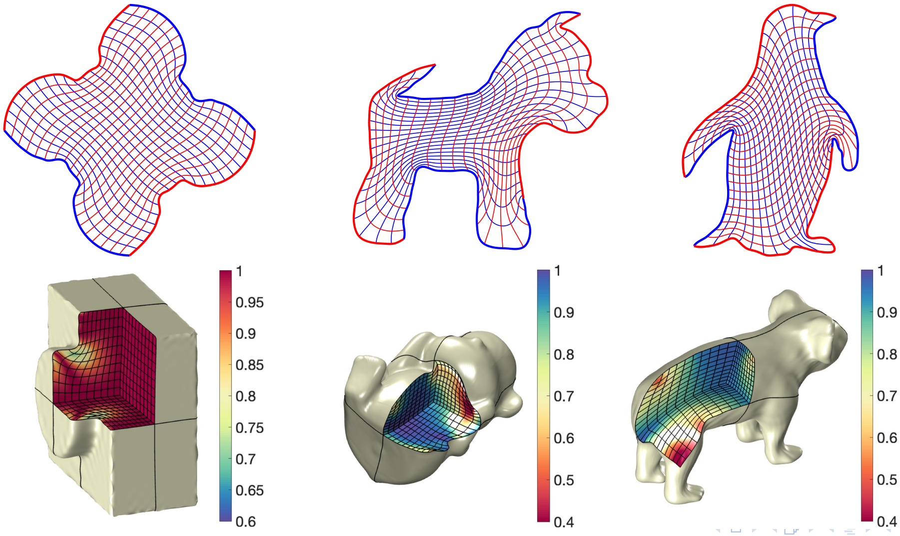
Jacobian regularization and revised objective function

- With this basic idea, we solve the following optimization problem:

$$\begin{aligned}\underset{\mathbf{P}_i,\ i\in\mathcal{I}_I}{\arg\min} E^c &= \int_{\mathcal{P}} \left(\lambda_1 E^c_{\text{mips}} + \lambda_2 E^c_{\text{vol}}\right) \,\mathrm{d}\mathcal{P} \\ &= \int_{\mathcal{P}} \left(\frac{\lambda_1}{8} \kappa_F^2(\mathcal{J}) \cdot \frac{|\mathcal{J}|^2}{\chi^2(|\mathcal{J}|,\varepsilon,\beta)} + \lambda_2 \left(\frac{vol(\Omega)}{\chi(|\mathcal{J}|,\varepsilon,\beta)} + \frac{\chi(|\mathcal{J}|,\varepsilon,\beta)}{vol(\Omega)} \right) \right) \mathrm{d}\mathcal{P},\end{aligned} \tag{7}$$

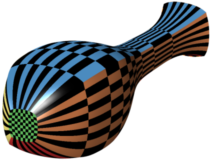
  where $\mathbf{P}_i,\ i \in \mathcal{I}_I$ are the unknown inner control points.
- Now, **only one optimization problem is solved.**

Gallery: penalty function-based results

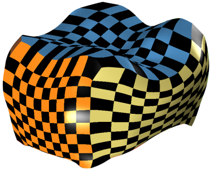


Catalogue

More complicated models

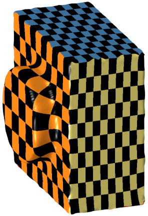


Vase

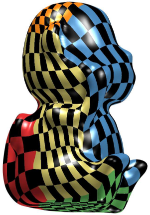


Tooth

Duck



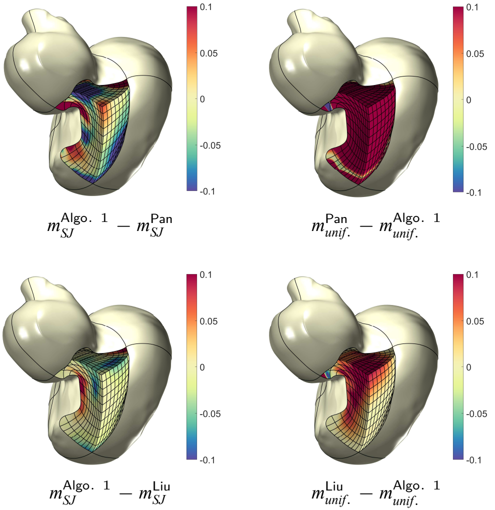
Component



Monkey

Koala

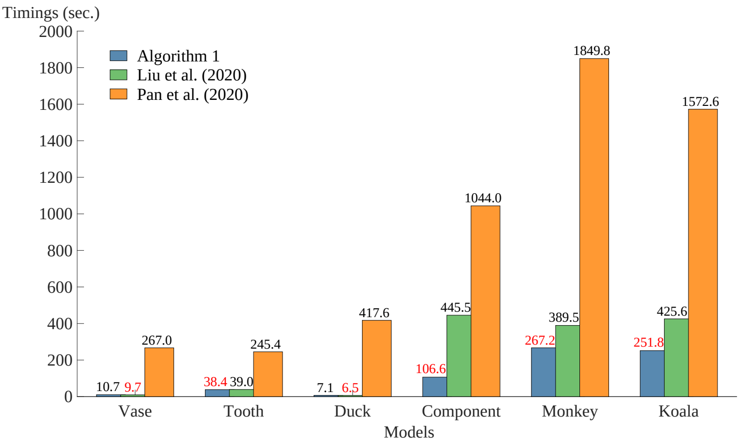
Comparison: Our method vs. current competitive approaches



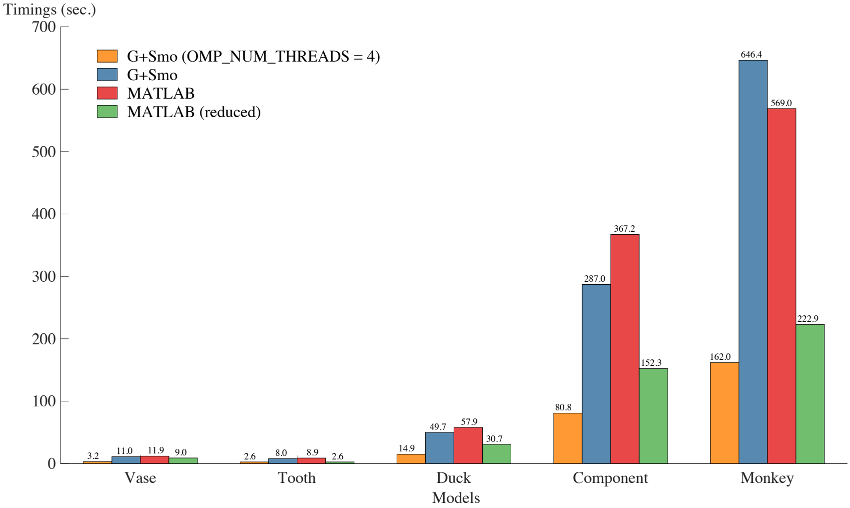
- We compare our method with two current competitors, i.e., Pan et al. 2020 and Liu et al. 2020.
- Positive values (red regions) indicate our method has lower angle distortion and/or lower volume distortion.

Efficiency: our method vs. current competitive approaches

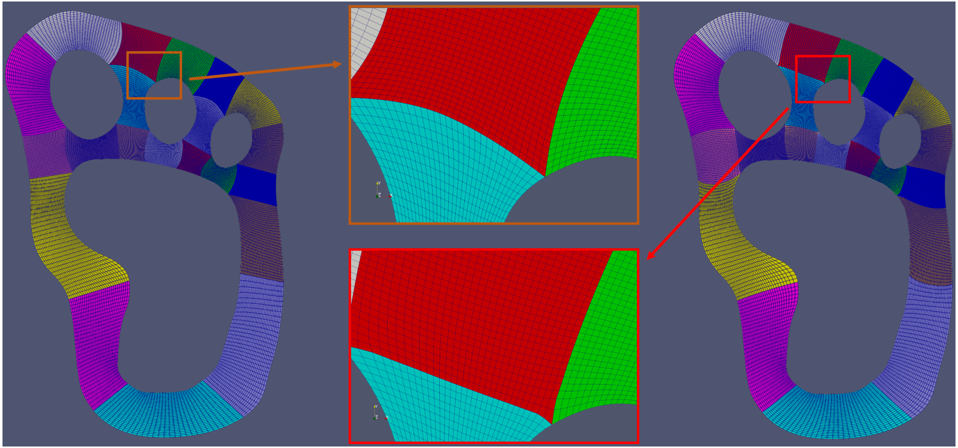
- Our method $\gg$ Pan et al. (2020);
- First three small-scale models, our method $\approx$ Liu et al. (2020);
- Last three large-scale models, our method $>$ Liu et al. (2020).



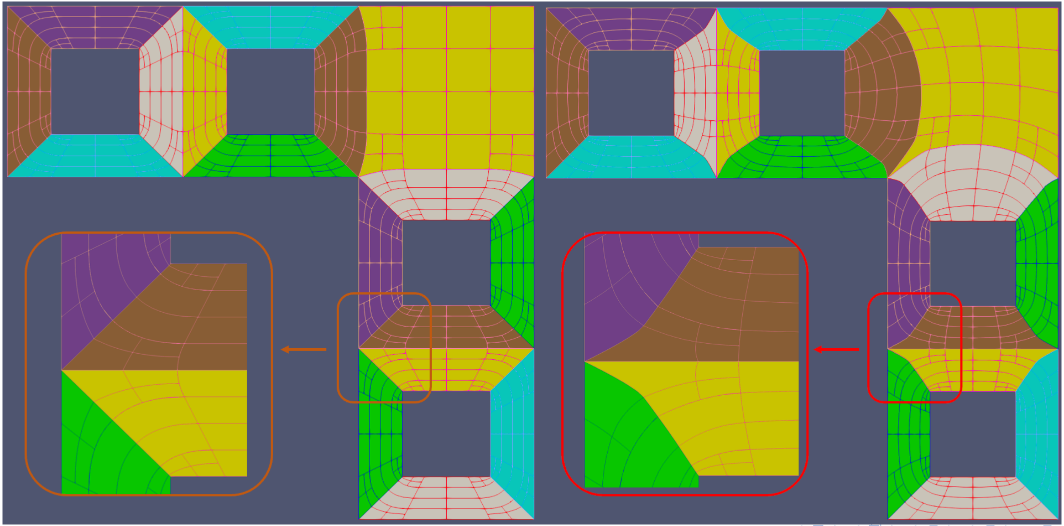
G+Smo implementation with OPENMP



Multi-patch result: `yeti_footprint.xml`

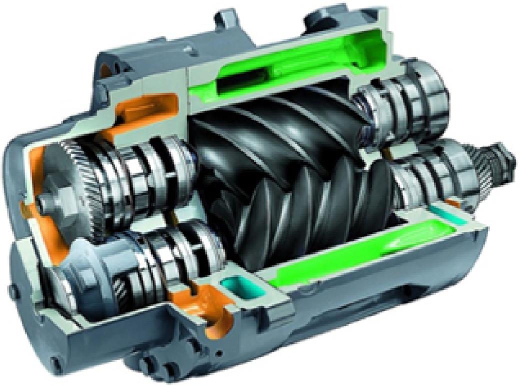


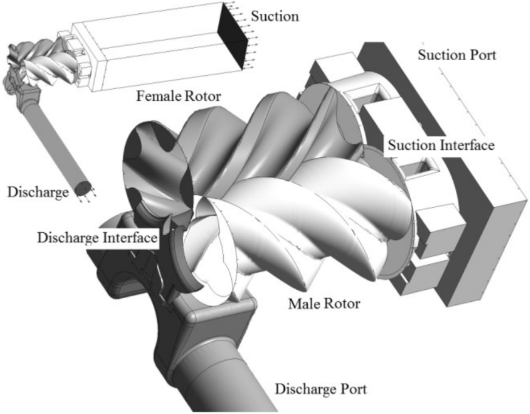
Compatible to multi-patch THB parameterization



Catalogue

Application: twin-screw rotary compressor





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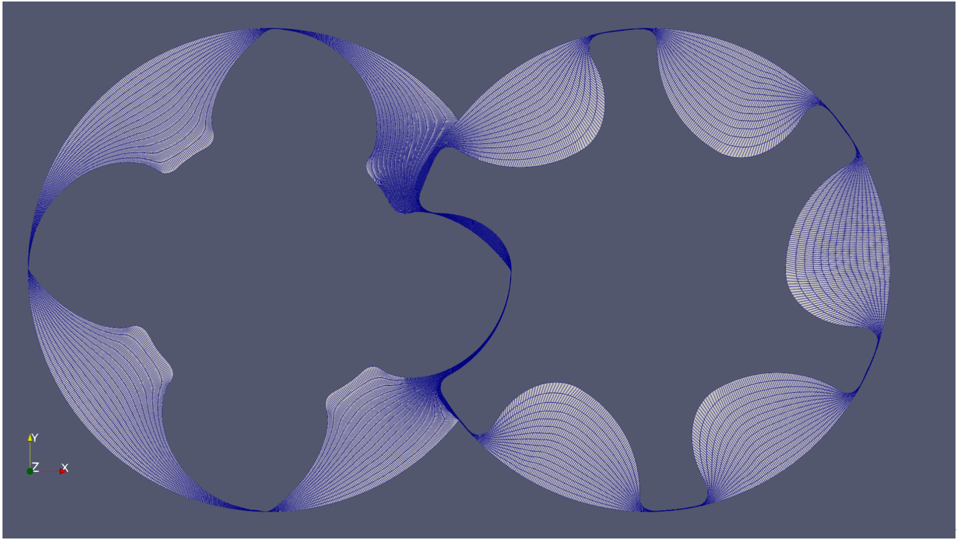
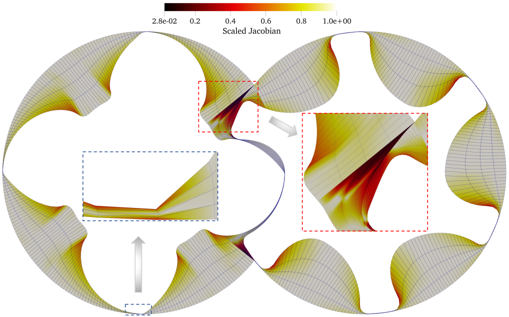
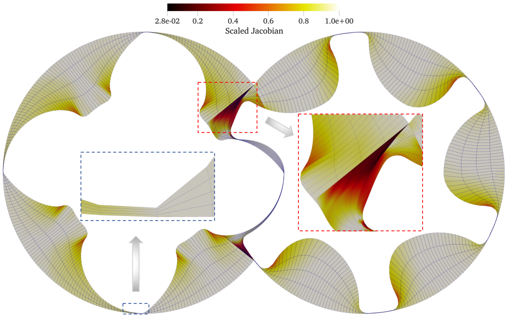


Parameterizations by different methods

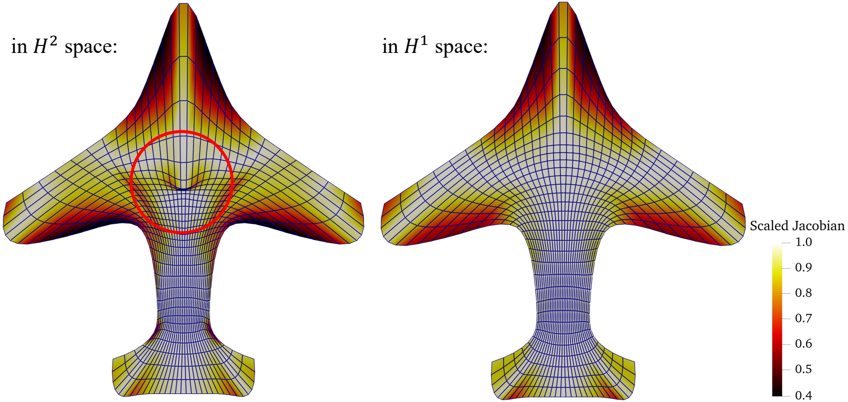


Penalty function-based method



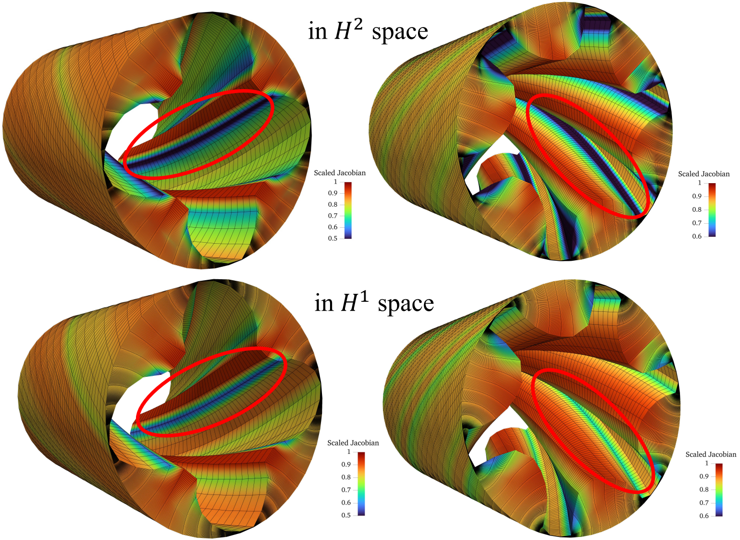
Elliptic Grid Generation (EGG) method

Problem 1: parameterization quality for general domains



- To improve the quality of the parameterization, we develop a novel scaled version of harmonic maps in the Sobolev space H^1.

Screw example

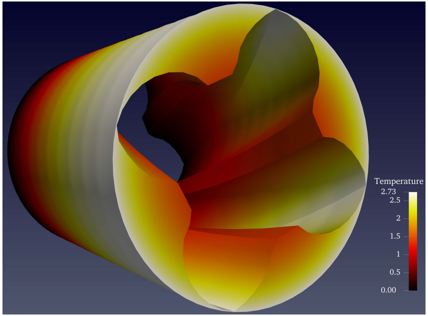


Application to IGA simulation

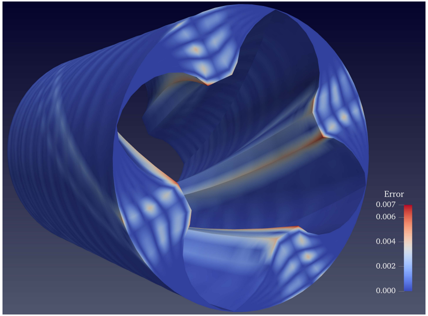
- Poisson problem:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \Gamma_D, \\ \mathbf{n} \cdot \nabla u = h, & \text{on } \Gamma_N, \end{cases}$$

where $f \in L^2(\Omega) : \Omega \to \mathbb{R}$ is a given source term.

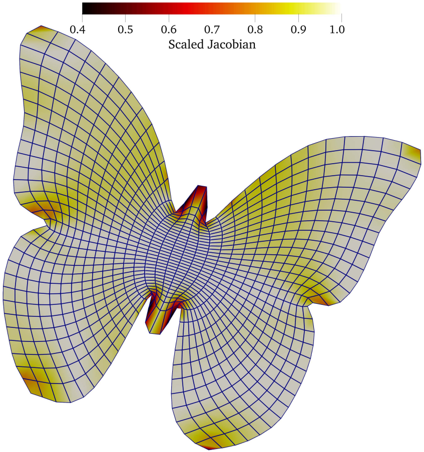


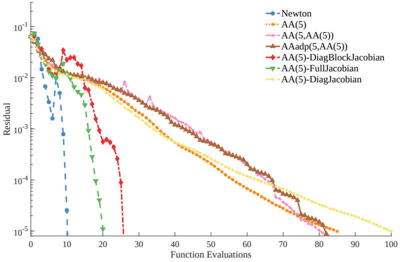
IGA solution

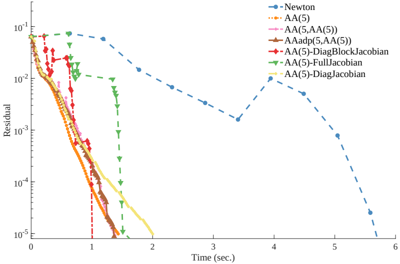


Absolute error

Problem 2: computational efficiency - preconditioned AA

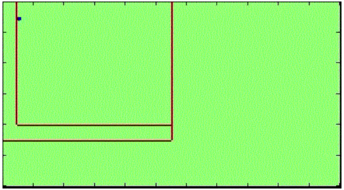




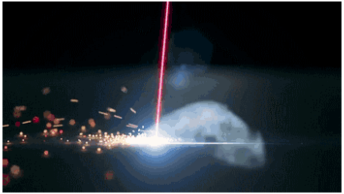


Catalogue

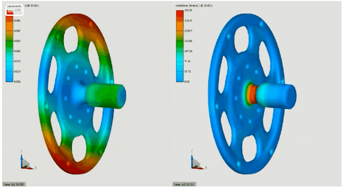
Anisotropic phenomena in physics



Wave propagation. source



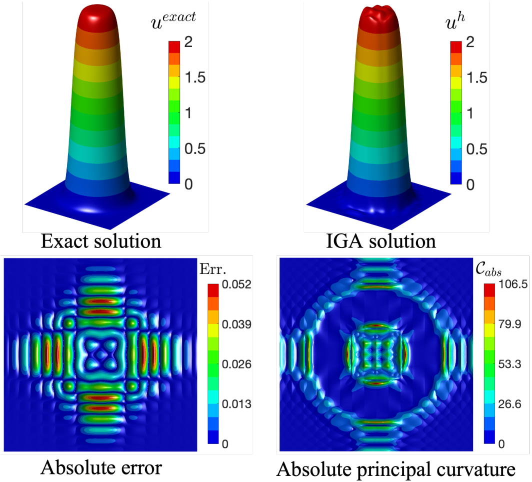
Laser printing. source



Stress concentration. source

- **Localized and anisotropic features extensively exist** in various physical phenomena;
- For such problems, **isotropic parameterizations are computationally uneconomical**;
- **Anisotropic parameterizations (r-adaptivity)**: increase per-degree-of-freedom accuracy while keeping the total degrees-of-freedom (DOFs) constant.

Basic Idea

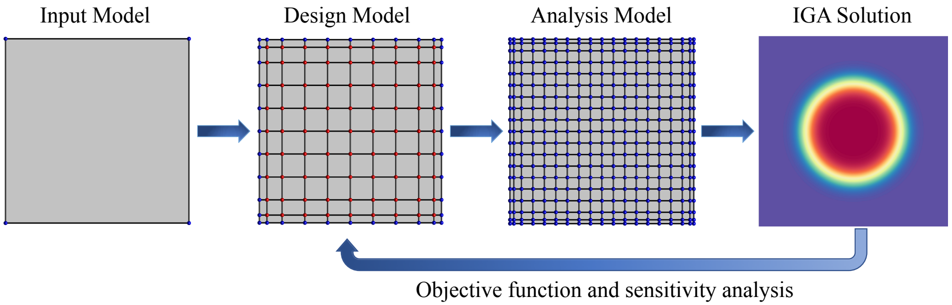


- **Absolute principal curvature**: to characterize the variations of the isogeometric solution;
- A tight relationship between geometric quantity and isogeometric solution is established;
- Absolute error and absolute principal curvature **show similar performance** (left figure);
- Absolute principal curvature is a good error estimator.

Basic idea - cont'd

Anisotropic parameterizations are often solution-dependent:

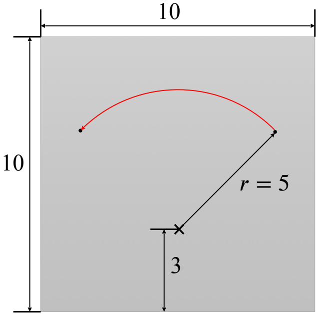
- Need **good numerical solution accuracy** to drive parameterization;
- Adjust as few control points as possible **for high efficiency**;
- **Bi-level strategy**: a coarse level (design model) to update the parameterization for efficiency's sake and a fine level (analysis model) to perform the isogeometric simulation for accuracy's sake.



Application: Time-dependent dynamic PDE

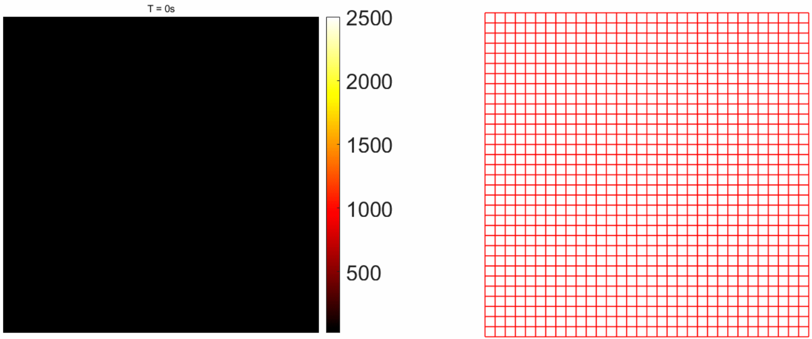
- Consider a 2D **linear heat transfer problem** with a moving Gaussian heat source:

$$\begin{cases} C_p\rho\frac{\partial u(\boldsymbol{x},t)}{\partial t} - \nabla\cdot(\kappa\nabla u(\boldsymbol{u},t)) & \text{in } \Omega\times T \\ u(\boldsymbol{x},t) = u_0 & \text{in } \Omega \\ \kappa\nabla u(\boldsymbol{x},t) = 0 & \text{on } \partial\Omega\times T \end{cases}$$



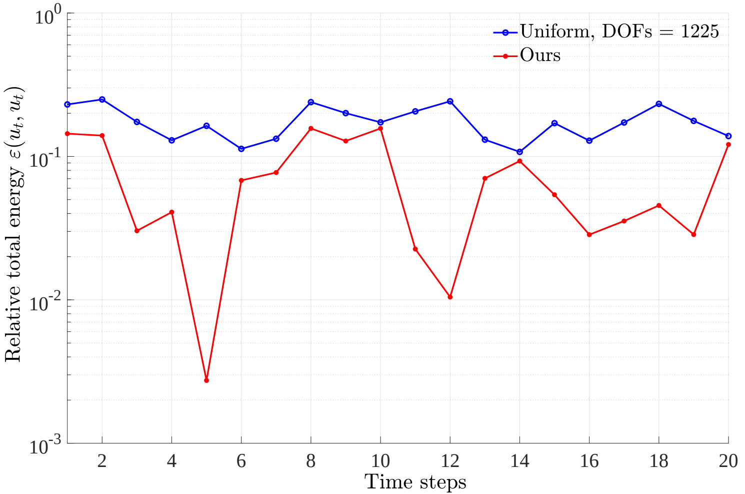
Laser Power P	9×10^5	$[W]$
Laser speed	1.57	$[mm/s]$
Absorptivity η	0.33	
Source radius r_h	100	$[\mu m]$
Conductivity κ	1.0	$[\mathrm{W}/mm/K]$
Heat capacity C_p	1.0	$[\mathrm{J/kg}/K]$
Density ρ	1.0	$[kg/mm^3]$
Initial temperature u_0	20.0	[°C]

Application: Time-dependent dynamic PDE



$u(\boldsymbol{x}, t)$ and their corresponding parameterizations on different time instants t

Application: Time-dependent dynamic PDE



Errors vs. Different time instants t

Catalogue

Conclusions and future work

- Conclusions:
  - **Barrier function-based NURBS parameterization method** is proposed for both the planar and volumetric cases;
  - **Penalty function-based NURBS parameterization method** is proposed for both the planar and volumetric cases;
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  - **All the proposed parameterization approaches work for the multi-patch cases**;
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  - In addition, we will **release all of the models and our reference implementation** in Geometry + Simulation Modules (**G+Smo**) library.

Thanks for your attention!

Q&A.

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Constructing analysis-suitable parameterization and curvature-based r-adaptive parameterization for IsoGeometric Analysis

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Jan. 3, 2023
2022 岳麓论坛 · 数学分论坛