Curvature-based r-Adaptive Planar NURBS Parameterization Method for Isogeometric Analysis Using Bi-Level Approach

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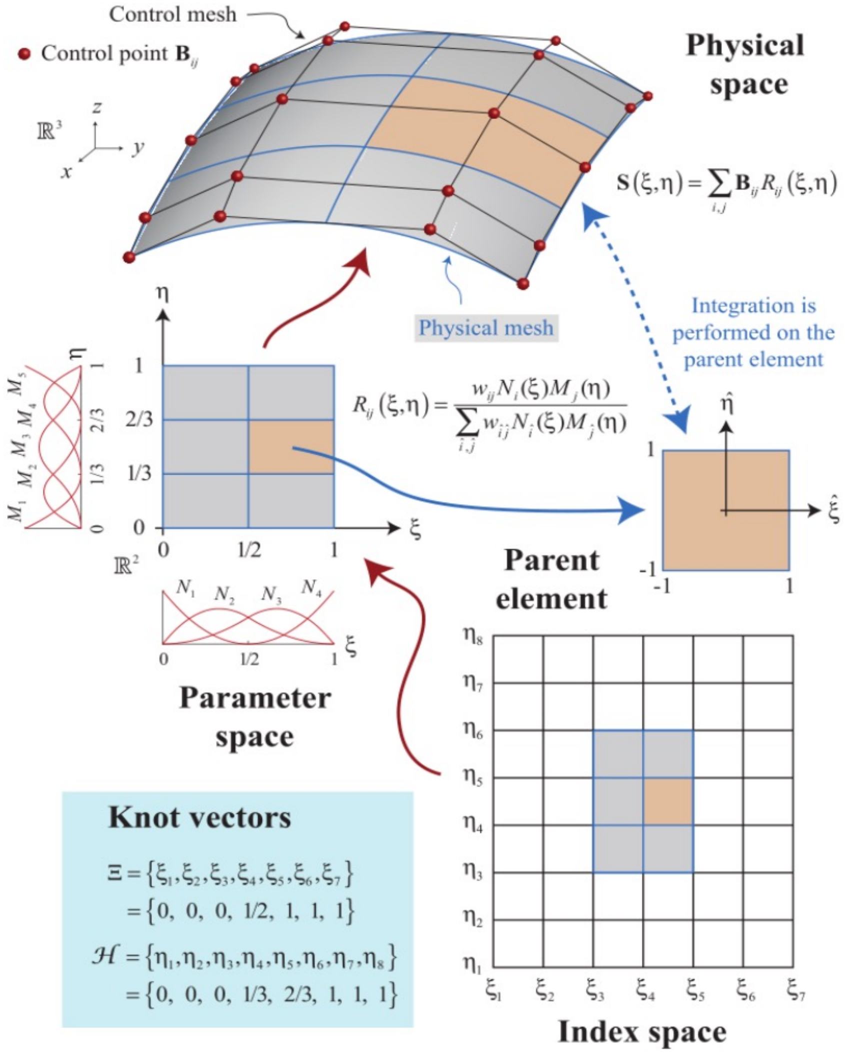
Catalogue

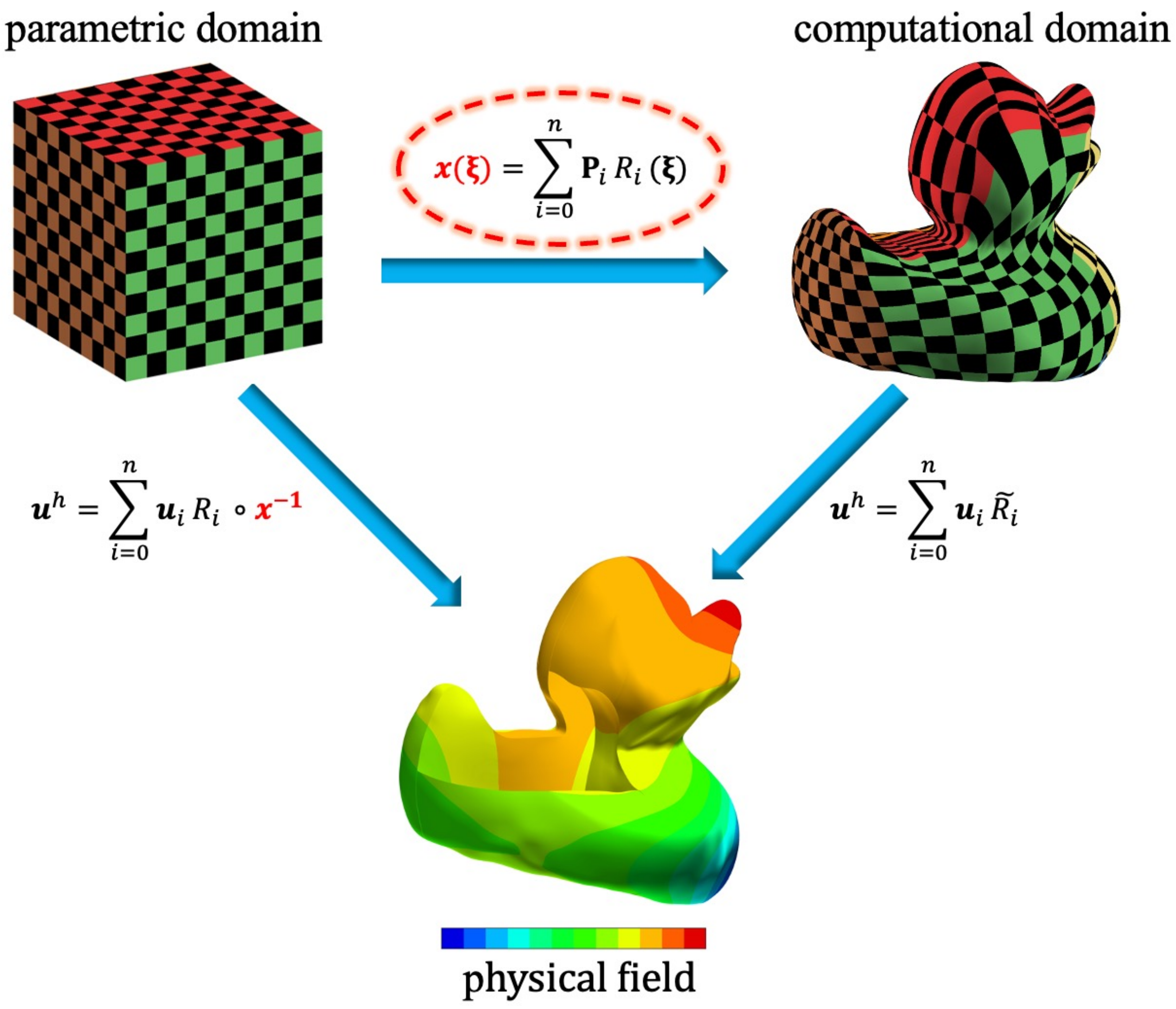
IsoGeometric Analysis (IGA)



Source: Figure from [Cottrell et al. 2009]

- Proposed by T.J.R. Hughes et al., 2005.
- KEY IDEA: approximate physical field with the same basis functions as that used to generate CAD models.
- Advantages:
  - Integration of design and analysis;
  - Exact and efficient geometry;
  - No data type transition and mesh generation;
  - Simplified mesh refinement;
  - High order continuous field;
  - Superior approximation properties.
- Very broad applications: such as shell analysis, fluid-structure interaction, and structural shape and topology optimization.

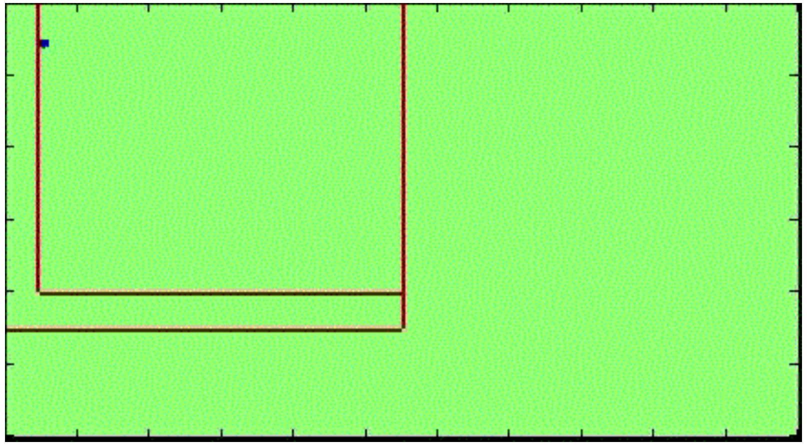
Isotropic parameterization problem for IGA



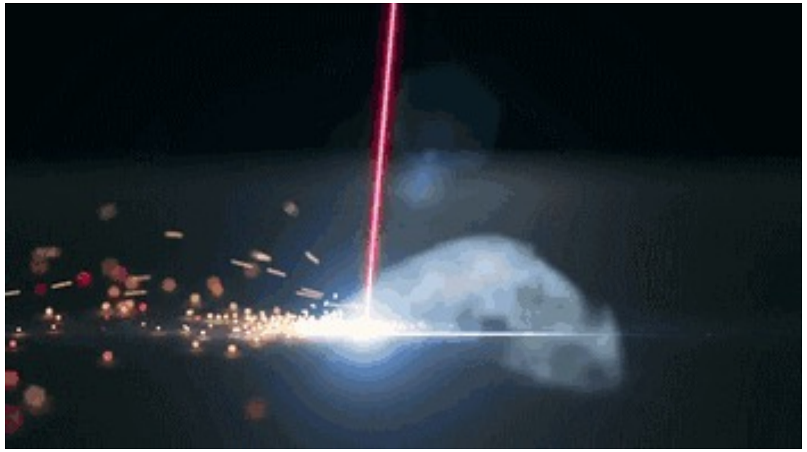
Parameterization problem [Ji et al. 2022]

- Bijective parameterization construction is crucial
  - Similar to mesh generation in FEM;
  - Parameterization quality significantly affects the accuracy and efficiency of the analysis [Cohen et al. 2010, Xu et al. 2013, Pilgerstorfer and Jüttler 2013].
- Isotropic parameterization:
  - Good orthogonality and uniformity;
  - Independent of the governing PDE and thus more efficient to generate.

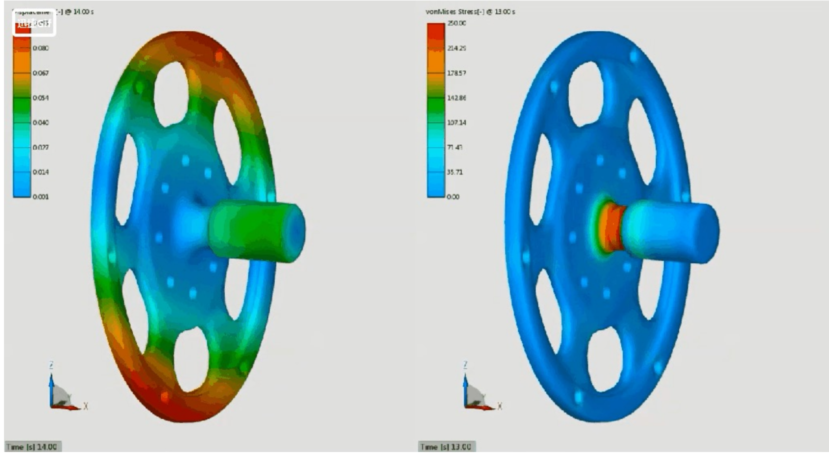
Anisotropic phenomena in physics



Wave propagation. source [wiki/ Wave_propagation]



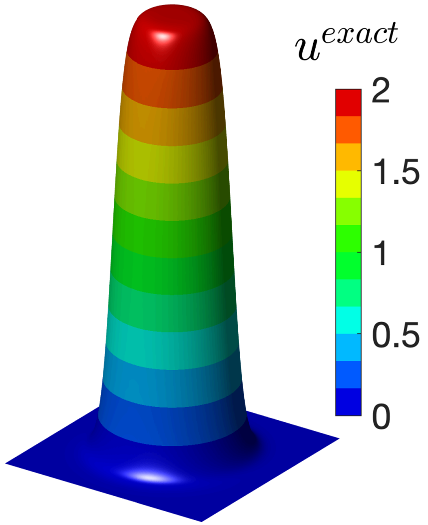
Laser printing. source



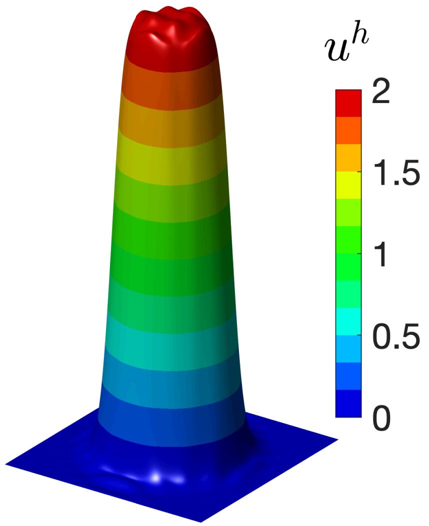
Stress concentration. source

- Localized and anisotropic features extensively exist in various physical phenomena;
- For such problems, isotropic parameterizations are computationally uneconomical;
- Anisotropic parameterizations (r-adaptivity): increase per-degree-of-freedom accuracy while keeping the total degrees-of-freedom (DOFs) constant.

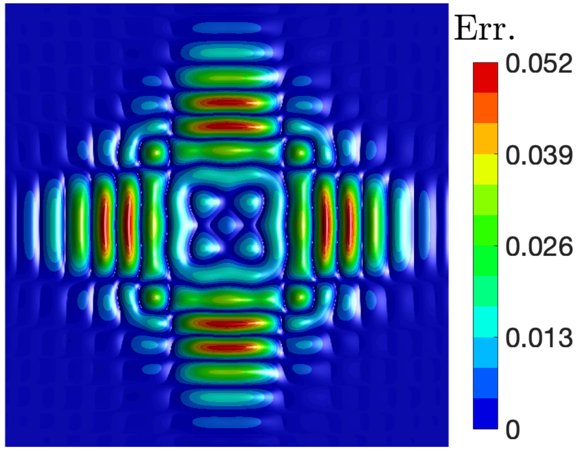
Our basic idea



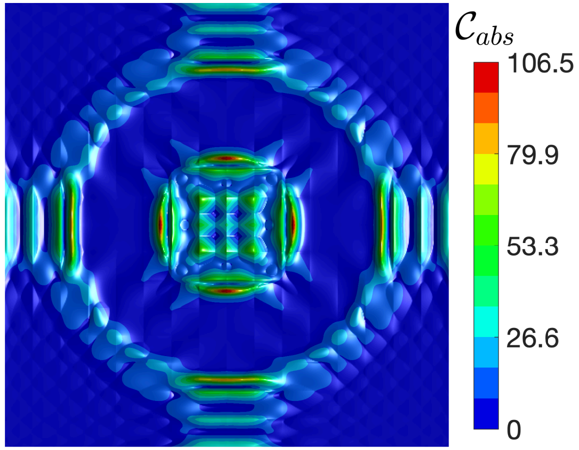
Exact solution



IGA solution



Absolute error

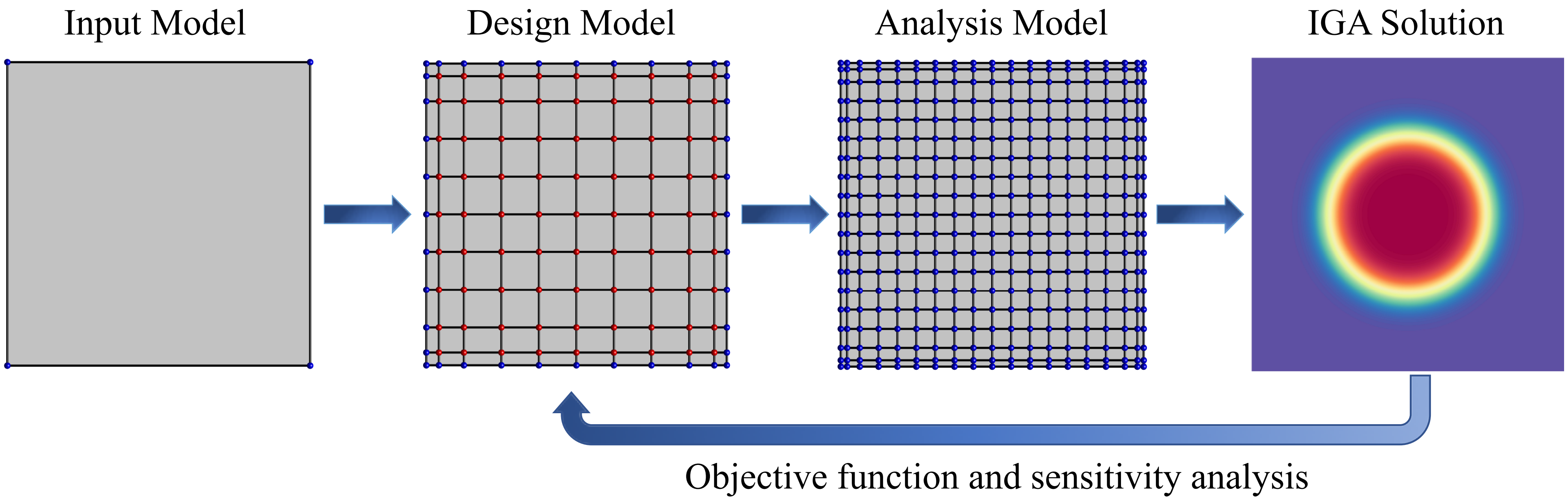


Absolute principal curvature

- Absolute principal curvature: to characterize the variations of the isogeometric solution;
- A tight relationship between geometric quantity and isogeometric solution is established;
- Absolute error and absolute principal curvature show similar performance (left figure);
- Absolute principal curvature is a good error estimator.

Our basic idea - cont'd

- **Anisotropic parameterizations are often solution-dependent:**
  - Need **good numerical solution accuracy** to drive parameterization;
  - Adjust as few control points as possible **for high efficiency**;
  - **Bi-level strategy:** a coarse level (design model) to update the parameterization for efficiency's sake and a fine level (analysis model) to perform the isogeometric simulation for accuracy's sake.



Catalogue

Related work

- Analysis-suitable isotropic parameterization (tremendous amount):
  - [Cohen+2010], [Xu+2011 2013], [Pilgerstorfer and Jüttler 2013], [Pan and Chen 2018], [Ji et al. 2021 2022], etc;
- Adaptive refinement:
  - h-adaptivity: Hierarchical B-splines [Forsey and Bartels 1988, Vuong+2011], THB-splines [Giannelli+2012 2016, Carraturo+2019], T-splines [Bazilevs+2010, Zhang+2012 2013], Truncated T-splines [Wei+2017];
  - p, k-adaptivity: hybrid-degree weighted T-splines [Liu+2016];

- ***r*-Adaptivity (reparameterization):**
  - Posterior error estimate-based method [Xu+2011];
  - Analysis-oriented method [Gravesen+2012];
  - L_2-error based method [Xu+2013];
  - Winslow's mapping method [Xu+2019];
  - IGA-Collocation-PDE method [Ali and Ma 2021];
  - Related finite element moving mesh method: [Huang and Russell 1998, 2010].

Catalogue

Model problem: Poisson's equation

- Poisson's equation with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\nabla^2 u(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \mathrm{u}(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \partial\Omega, \end{cases}$$

where $f(\boldsymbol{x}) \in L^2(\Omega)$ is a given source term, and $\mathrm{u}(\boldsymbol{x})$ is unknown.

- Isogeometric framework (geometry and physical field are defined by the same spline bases):
  - Parameterization (geometry):

$$\boldsymbol{x} = \mathbf{N}^{\mathbf{T}}\mathbf{P} = \sum_{\mathrm{i}=1}^{\boldsymbol{n_e}} N_i \mathbf{P}_{\boldsymbol{i}},$$

  - Physical field:

$$\boldsymbol{u} = \mathbf{N}^{\mathbf{T}}\mathbf{u} = \sum_{\mathrm{i}=1}^{\boldsymbol{n_e}} N_i \boldsymbol{u}_{\boldsymbol{i}}.$$

Model problem: notations

- Some auxiliary matrices:

$$\boldsymbol{G}=\begin{bmatrix}\frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial x} & \cdots & \frac{\partial N_1}{\partial x}\\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial y} & \cdots & \frac{\partial N_1}{\partial y}\end{bmatrix}=\begin{bmatrix}\boldsymbol{x}_\xi & \boldsymbol{y}_\xi\\ \boldsymbol{x}_\eta & \boldsymbol{y}_\eta\end{bmatrix}^{-1}\begin{bmatrix}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_1}{\partial \xi}\\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_1}{\partial \eta}\end{bmatrix}=\boldsymbol{J}^{-1}\widehat{\boldsymbol{G}}.$$

- After isogeometric discretization, we obtain the following **discrete equilibrium equation**

$$\mathop{\mathbf{A}}_{i=1}^{n_{el}}\boldsymbol{K_e u}=\mathop{\mathbf{A}}_{i=1}^{n_{el}}\boldsymbol{f_e},$$

where $\mathop{\mathbf{A}}_{i=1}^{n_{el}}$ is the element assembly operator, $\boldsymbol{K_e}$ is the element stiffness matrix

$$\boldsymbol{K_e}=\int_{\widehat{\boldsymbol{\Omega}}_{\boldsymbol{e}}}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}|\boldsymbol{J}|\,\boldsymbol{d}\widehat{\boldsymbol{\Omega}}_{\boldsymbol{e}},$$

and $\boldsymbol{f_e}$ is the element force vector

$$\boldsymbol{f_e}=\int_{\widehat{\boldsymbol{\Omega}}_{\boldsymbol{e}}}\boldsymbol{N}f|\boldsymbol{J}|\,\boldsymbol{d}\widehat{\boldsymbol{\Omega}}_{\boldsymbol{e}}.$$

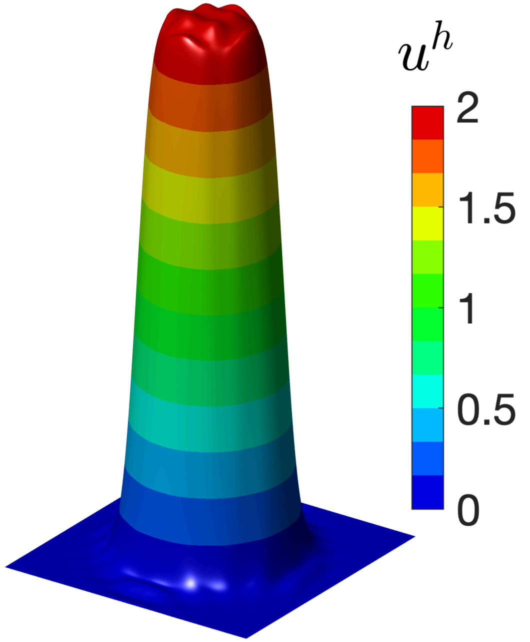
Isogeometric solution surface

- Isogeometric solution is considered as **a parametric surface** in $\mathbb{R}^3$:
  - More precisely, we define an **isogeometric solution surface**

$$\mathcal{S}(\xi,\eta) = \big(x(\xi,\eta), y(\xi,\eta), u(\xi,\eta)\big) = \boldsymbol{N}^{\mathrm{T}}\widetilde{\boldsymbol{P}},$$

where $\widetilde{\boldsymbol{P}} = [\boldsymbol{P}, \boldsymbol{u}]$ are the control points of $\mathcal{S}(\xi,\eta)$.

- **Curvature information** is used to characterize the solution variations.



Isogeometric solution surface in $\mathbb{R}^3$

Absolute principal curvature

- The first and the second fundamental forms of $\mathcal{S}(\xi,\eta)$:

$$I=\begin{bmatrix}E & F\\ F & G\end{bmatrix}=\begin{bmatrix}\langle\mathcal{S}_\xi,\mathcal{S}_\xi\rangle & \langle\mathcal{S}_\xi,\mathcal{S}_\eta\rangle\\ \langle\mathcal{S}_\xi,\mathcal{S}_\eta\rangle & \langle\mathcal{S}_\eta,\mathcal{S}_\eta\rangle\end{bmatrix},\ \text{and}\ II=\begin{bmatrix}L & M\\ M & N\end{bmatrix}=\begin{bmatrix}\langle\mathcal{S}_{\xi\xi},\boldsymbol{n}\rangle & \langle\mathcal{S}_{\xi\eta},\boldsymbol{n}\rangle\\ \langle\mathcal{S}_{\xi\eta},\boldsymbol{n}\rangle & \langle\mathcal{S}_{\eta\eta},\boldsymbol{n}\rangle\end{bmatrix}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product, and $\boldsymbol{n}$ is the unit normal vector of $\mathcal{S}(\xi,\eta)$.

- **Absolute principal curvature:**

$$\mathcal{C}_{abs}=|\kappa_{max}|+|\kappa_{min}|=\left|\mathcal{H}+\sqrt{\mathcal{H}^2-\mathcal{K}}\right|+\left|\mathcal{H}-\sqrt{\mathcal{H}^2-\mathcal{K}}\right|$$

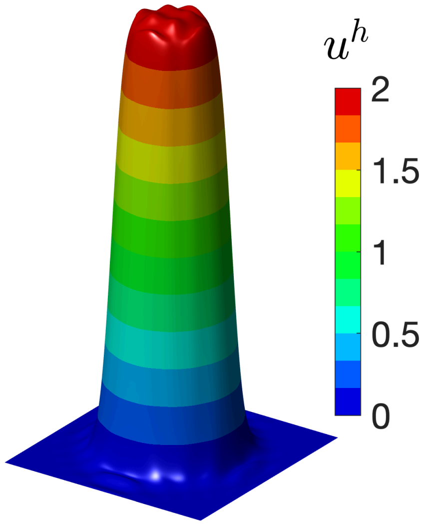
where

$$\mathcal{K}=\frac{LN-M^2}{EG-F^2}$$

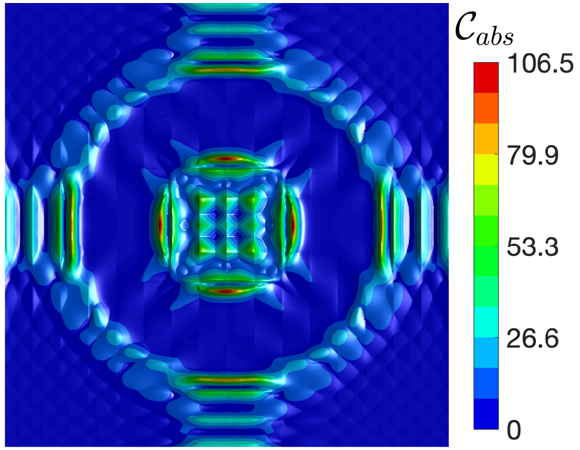
is Gaussian curvature, and

$$\mathcal{H}=\frac{LG-2MF+NE}{2(EG-F^2)}$$

is the mean curvature.



Isogeometric solution surface



Absolute principal curvature

Objective function

- **Absolute principal curvature measure:**

$$\hat{\Phi}(\boldsymbol{x}) = \Phi\big(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})\big) = \begin{cases} \int_{\hat{\Omega}} \mathcal{C}_{abs}\, d\hat{\Omega}, & \text{if } |J| > 0, \text{for } \forall(\xi, \eta) \in \hat{\Omega}, \\ +\infty, & \text{otherwise.} \end{cases}$$

- Main ideas:
  - Fundamental prerequisite for an analysis-suitable parameterization is bijectivity;
  - Most modern line search conditions need a sufficient reduction of the objective function;
  - Simply set the objective function value $\hat{\Phi}(\boldsymbol{x})$ to $+\infty$ if the bijective constraints violate;
  - If the bijectivity condition violates, the current step size will be rejected and recompute a new step size.

Objective function

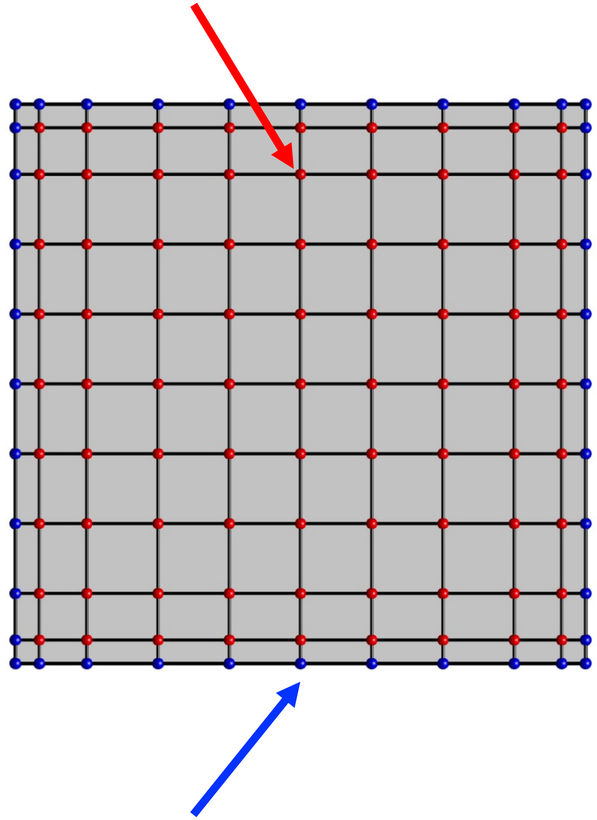
- Finally, our problem boils down to the following optimization:

$$\underset{\boldsymbol{P}_j \in \mathcal{I}_I}{\operatorname{argmin.}} \quad \widehat{\Phi}(\boldsymbol{x}),$$
$$s.t. \;\; \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}),$$

where $\mathcal{I}_I$ is the indices set of the inner control points.

- **Exact geometry:** reposition the inner control points (in red) while leaving the boundary control points (in blue) unchanged.

moveable inner control points



fixed boundary control points

Sensitivity analysis - Sensitivity propagation from design model to analysis model

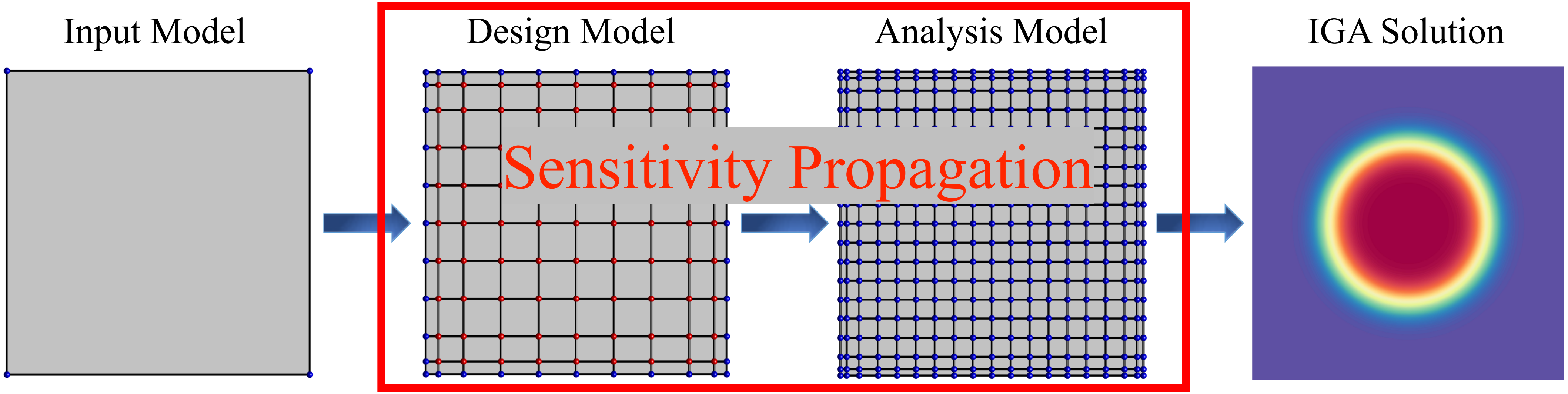
- Spline refinement:

$$\boldsymbol{Q} = \widetilde{\boldsymbol{R}}\boldsymbol{P},$$

where $\boldsymbol{Q}$ are the control points of the analysis model, $\boldsymbol{P}$ are the control points of the design model, and $\widetilde{\boldsymbol{R}} = diag(\omega_Q)^{-1}Rdiag(\omega_Q)$ is a linear transformation operator.

- **Sensitivity propagation** from the design model to the analysis model:

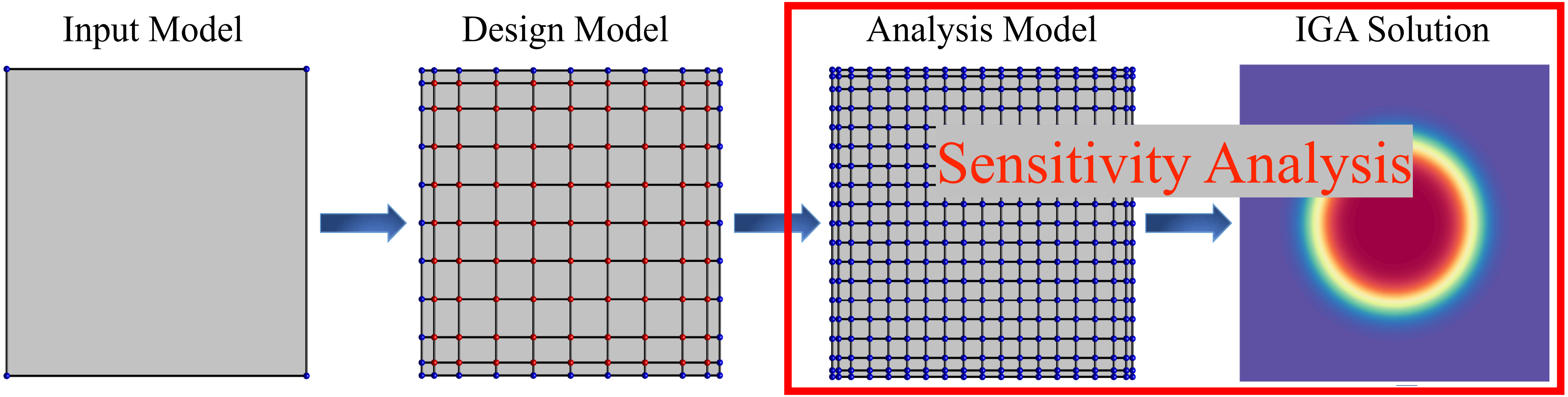
$$\frac{\partial \boldsymbol{Q}}{\partial \alpha_s} = \widetilde{\boldsymbol{R}}\frac{\partial \boldsymbol{P}}{\partial \alpha_s}.$$



Sensitivity analysis in the analysis model

- Sensitivity analysis **performs mainly in the analysis model:**

$$\frac{\partial \widehat{\Phi}(\boldsymbol{x})}{\partial \alpha_s} = \underbrace{\frac{\partial \Phi(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))}{\partial \alpha_s}}_{\text{geometry}} + \underbrace{\frac{\partial \Phi(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))}{\partial \boldsymbol{u}}}_{\text{geometry}} \underbrace{\frac{\partial \boldsymbol{u}(\boldsymbol{x})}{\partial \alpha_s}}_{\text{analysis}}.$$



Sensitivity analysis – geometry part

- Sensitivity for the Gaussian curvature and the mean curvature:

$$\mathcal{K}' = \frac{(LN - M^2)' - \mathcal{K}(EG - F^2)'}{EG - F^2},$$

$$\mathcal{H}' = \frac{1}{2}\frac{(LG - 2MF + NE)' - 2\mathcal{H}(EG - F^2)'}{EG - F^2}.$$

- The **derivative of the absolute principal curvature** is of the following form

$$\mathcal{C}'_{abs} = \begin{cases} 2\mathcal{H}' & \text{if } \kappa_{min} \geq 0, \\ \dfrac{2\mathcal{H}\mathcal{H}' - \mathcal{K}'}{\sqrt{\mathcal{H}^2 - \mathcal{K}}} & \text{if } \kappa_{max} \geq 0 \text{ and } \kappa_{min} < 0, \\ -2\mathcal{H}' & \text{otherwise.} \end{cases}$$

Sensitivity analysis – analysis part

- **Differentiation of the discrete equilibrium equation** and some algebraic operation give

$$\frac{\partial \boldsymbol{u}(\boldsymbol{x})}{\partial \alpha_s} = \boldsymbol{K}(\boldsymbol{x})^{-1}\left(\frac{\partial \boldsymbol{F}(\boldsymbol{x})}{\partial \alpha_s} - \frac{\partial \boldsymbol{K}(\boldsymbol{x})}{\partial \alpha_s}\boldsymbol{u}(\boldsymbol{x})\right),$$

where

$$\boldsymbol{K}'(\boldsymbol{x}) = \overset{n_{el}}{\underset{i=1}{\mathbf{A}}}\boldsymbol{K}'_e,$$
$$\boldsymbol{F}'(\boldsymbol{x}) = \overset{n_{el}}{\underset{i=1}{\mathbf{A}}}\mathbf{f}'_e,$$

$$\boldsymbol{K}'_e = \int_{\widehat{\boldsymbol{\Omega}}_e} \left(\boldsymbol{G}'^{\mathrm{T}}\boldsymbol{G}|\boldsymbol{J}| + \boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}'|\boldsymbol{J}| + \boldsymbol{G}^{T}\,\boldsymbol{G}|\boldsymbol{J}|'\right)d\widehat{\boldsymbol{\Omega}}_e\,,$$

and

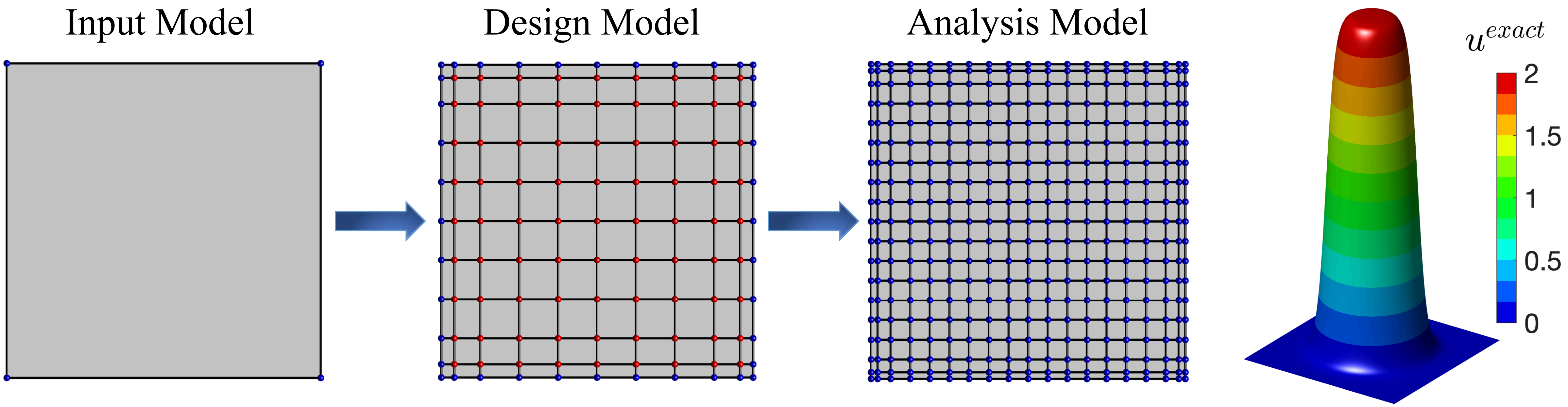
$$\mathbf{f}'_e = \int_{\widehat{\boldsymbol{\Omega}}_e} \boldsymbol{N}\left(\boldsymbol{\nabla}_{\boldsymbol{x}} f \cdot \boldsymbol{\nabla}_{\alpha_s}\boldsymbol{x}|\boldsymbol{J}| + f\,|\boldsymbol{J}|'\right)d\widehat{\boldsymbol{\Omega}}_e\,.$$

Catalogue

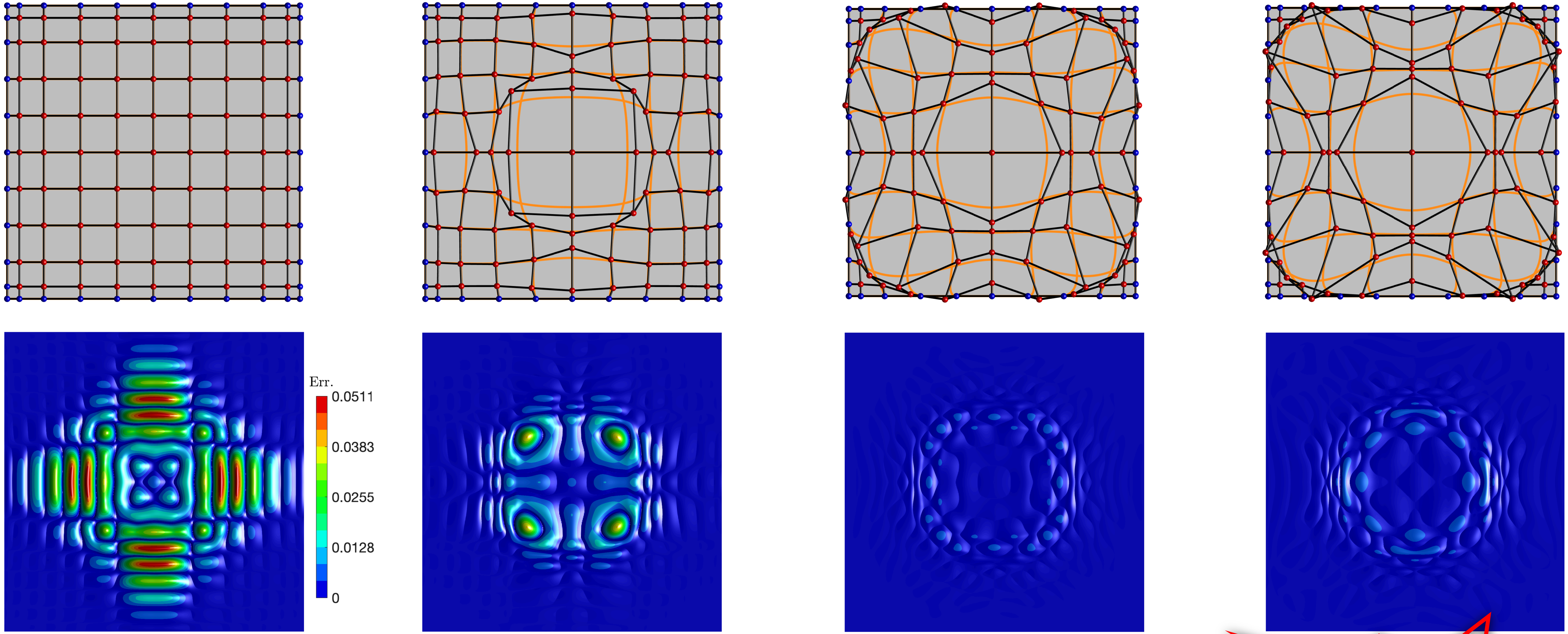
Toy problem: the square case

- Consider Poisson's equation over $\Omega = [0,1]^2$ with the following exact solution

$$u(\boldsymbol{x}) = \tanh\left(\frac{0.25 - \sqrt{(x-0.5)^2 + (y-0.5)^2}}{0.05}\right) + 1.$$



Toy problem: the square case – cont'd



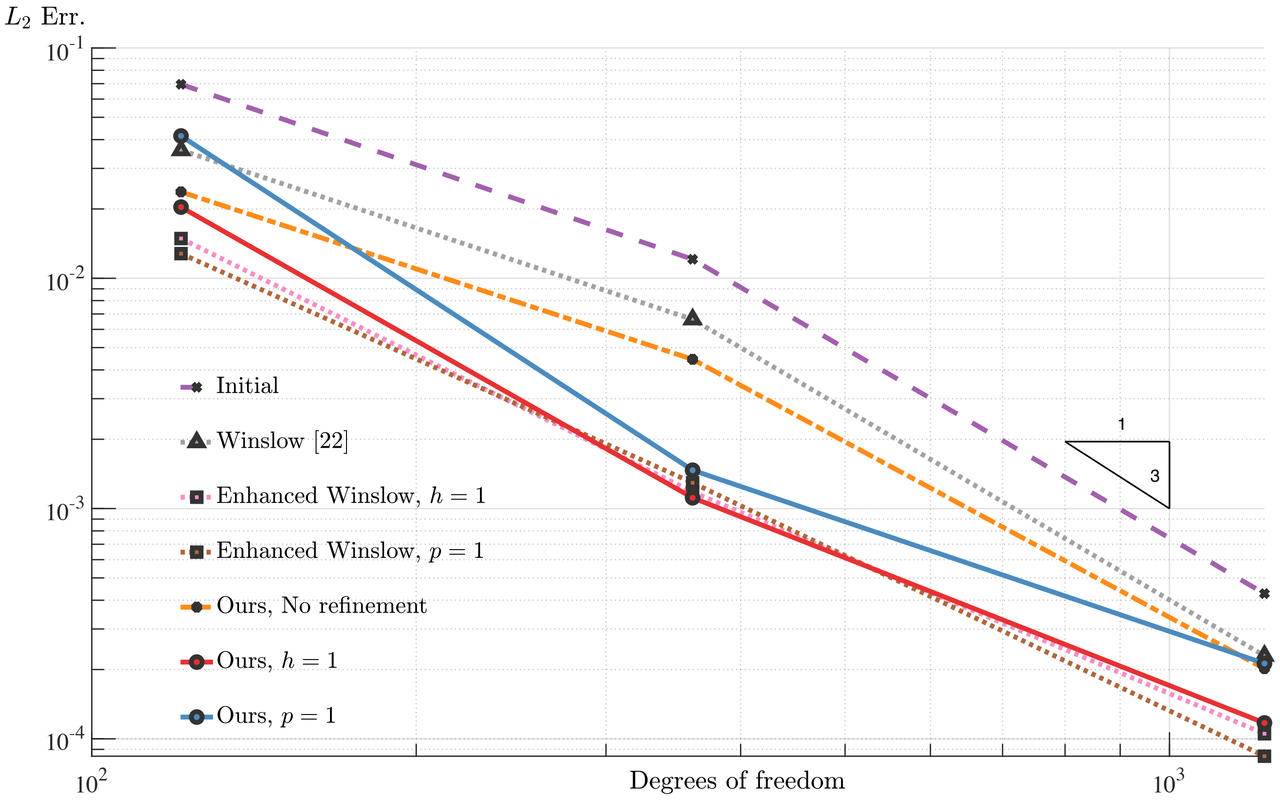
Initial (isotropic)
Err. = 1.2093e-02

No refinment
Err. = 4.4468e-03

$p = 1$
Err. = 1.4674e-03

$h = 1$
Err. = 1.1141e-03

Toy problem: the square case – cont'd



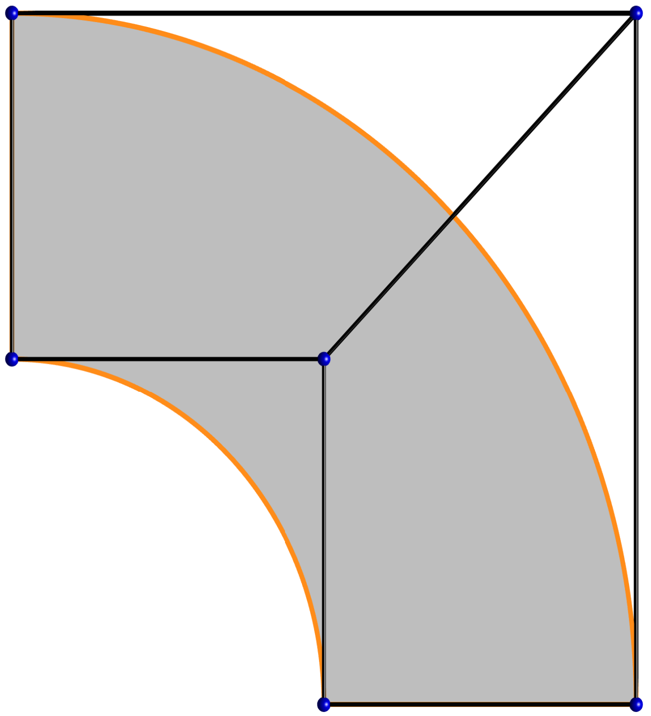
- High numerical accuracy **remains in finer meshes;**
- **Our method shows better performance** than the original Winslow method [Xu et al. 2019].

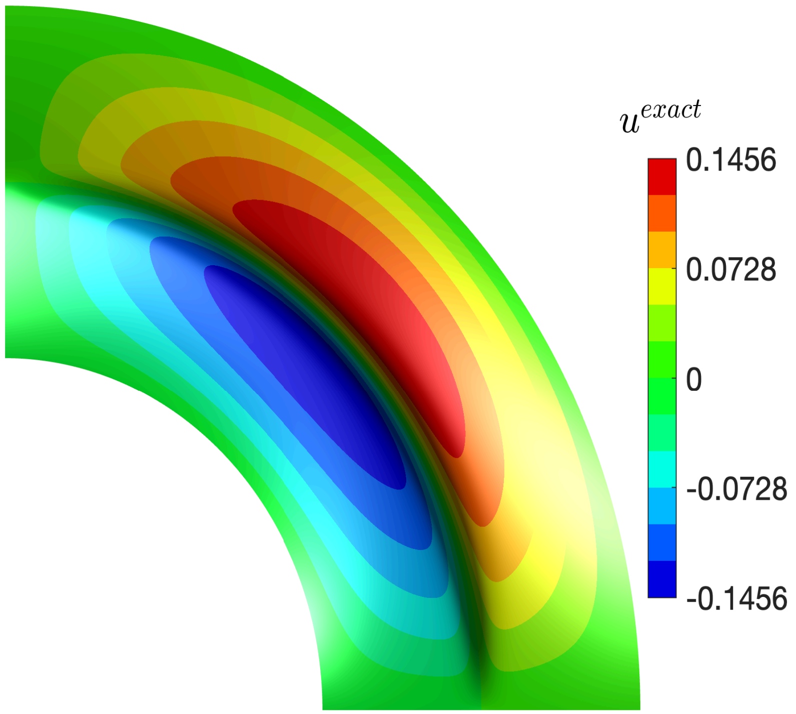
NURBS geometry: quarter annulus

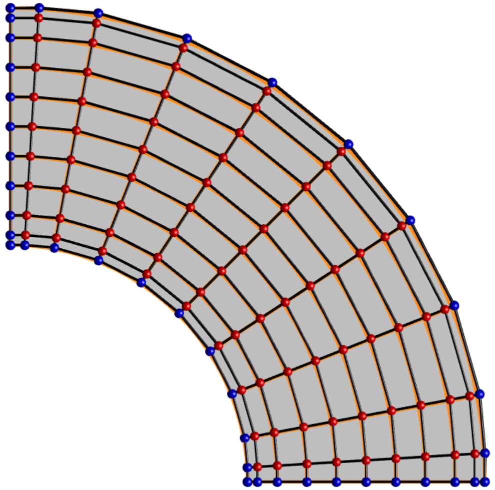
- **NURBS representations are mandatory** to exactly recover computational domains with conic sections;
- Exact solution:

$$u(x) = \theta\left(\frac{\pi}{2} - \theta\right)(r-1)(r-2)\tanh\big(24(r-1.5)\big),$$

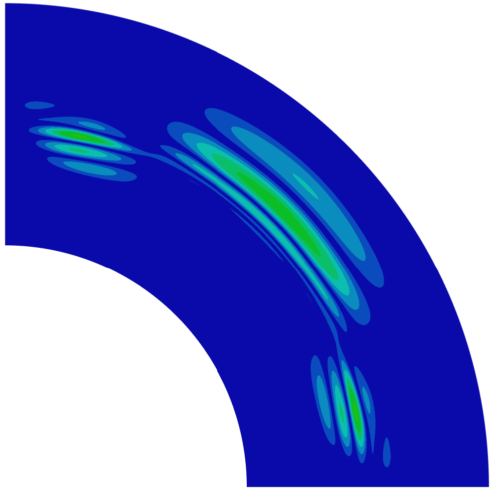
where $r = \sqrt{x^2 + y^2}$ and $\theta = atan(y/x)$.



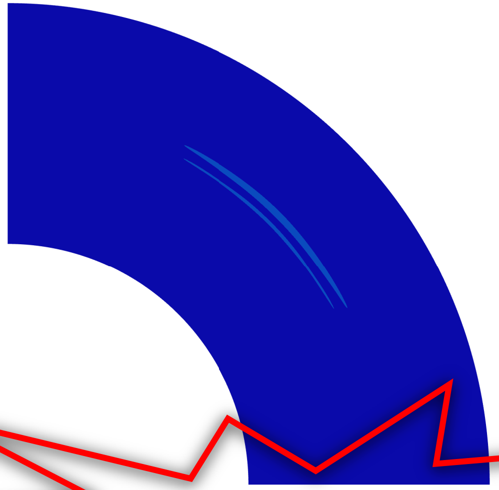




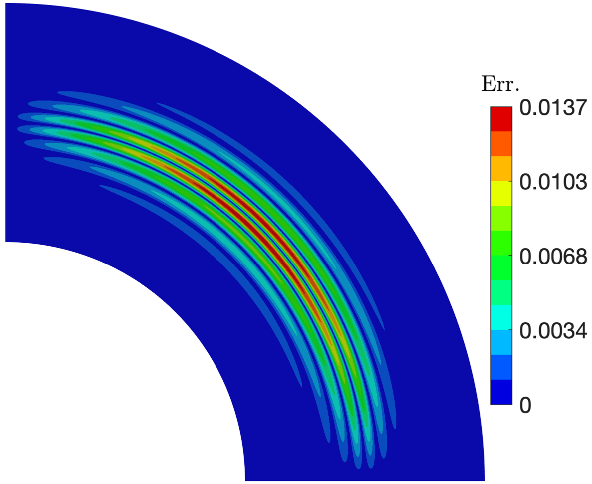
Design model



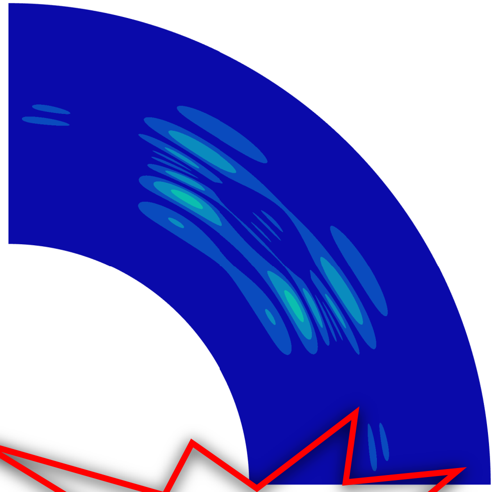
Ours, no Refinement
Err.=2.0980e-03



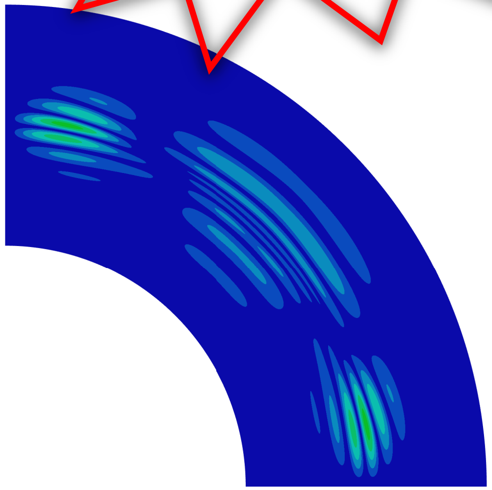
L2 error [Xu et al. 2013]
Err.=5.5329e-04



Initial (isotropic)
Err.=4.3519e-03

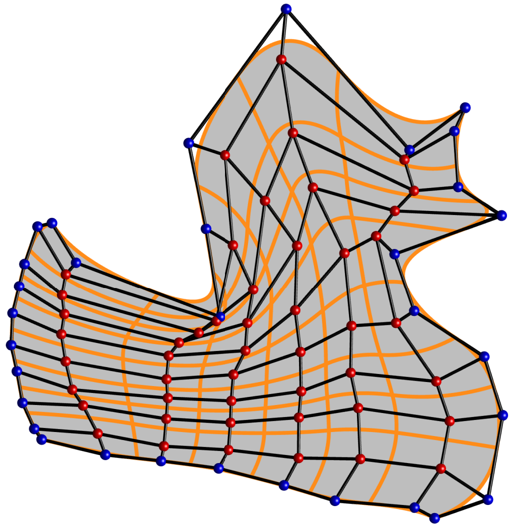


Ours, h=1
Err.=1.2247e-03

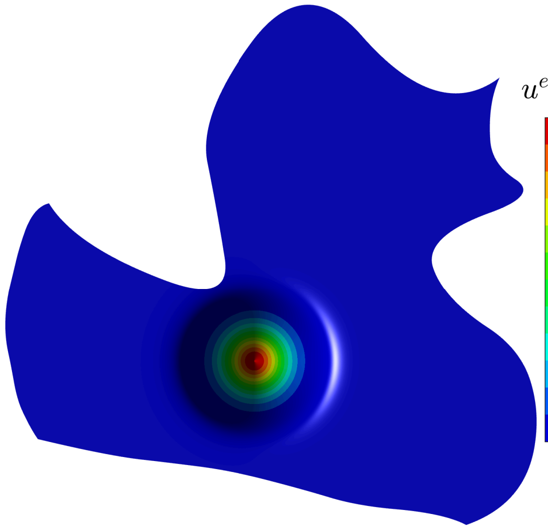


Winslow [Xu et al. 2019]
Err.=1.7134e-03

More complicated geometry



Initial parameterization [Ji et al. 2021]

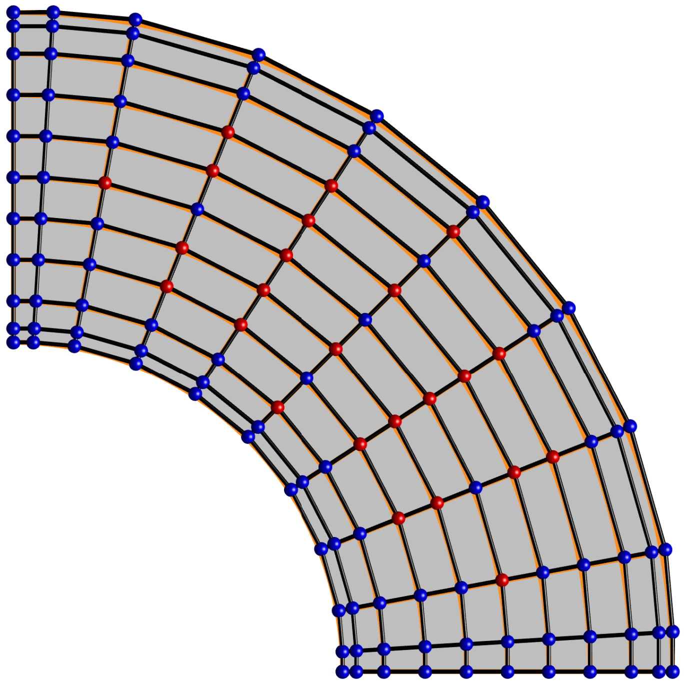


Exact solution

Refinement	Average error	L_2 error	Running time (sec.)
Initialization	1.5032e−02	2.7269e−02	0.21
NO	6.2290e−03	1.1213e−02	6.03
$p = 1$	**2.1423e−03**	**4.5594e−03**	9.28
$h = 1$	2.4356e−03	7.0504e−03	15.60

Acceleration strategy

- Setting all the inner control points as optimization variables is **computationally uneconomical**;
- We **compute the sensitivity once** before iteration and **select the ones with large derivatives**.

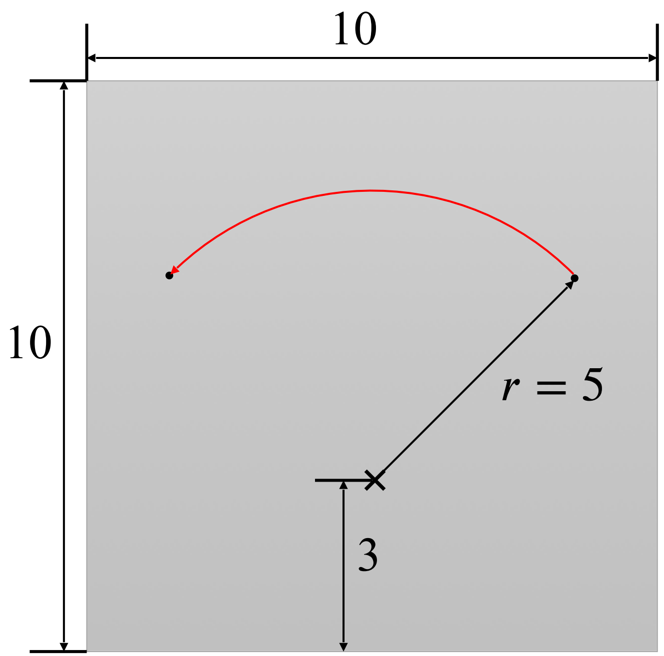


Method	Refine	L_2 error	timing(sec.)
Unaccelerated	NO	2.0980e-3	12.20
	$p=1$	1.3501e-3	25.68
	$h=1$	1.2247e-3	48.72
Accelerated	NO	1.4312e-3	4.70
	$p=1$	1.4252e-3	7.94
	$\boldsymbol{h=1}$	**1.1296e-3**	15.79

Application: Time-dependent dynamic PDE

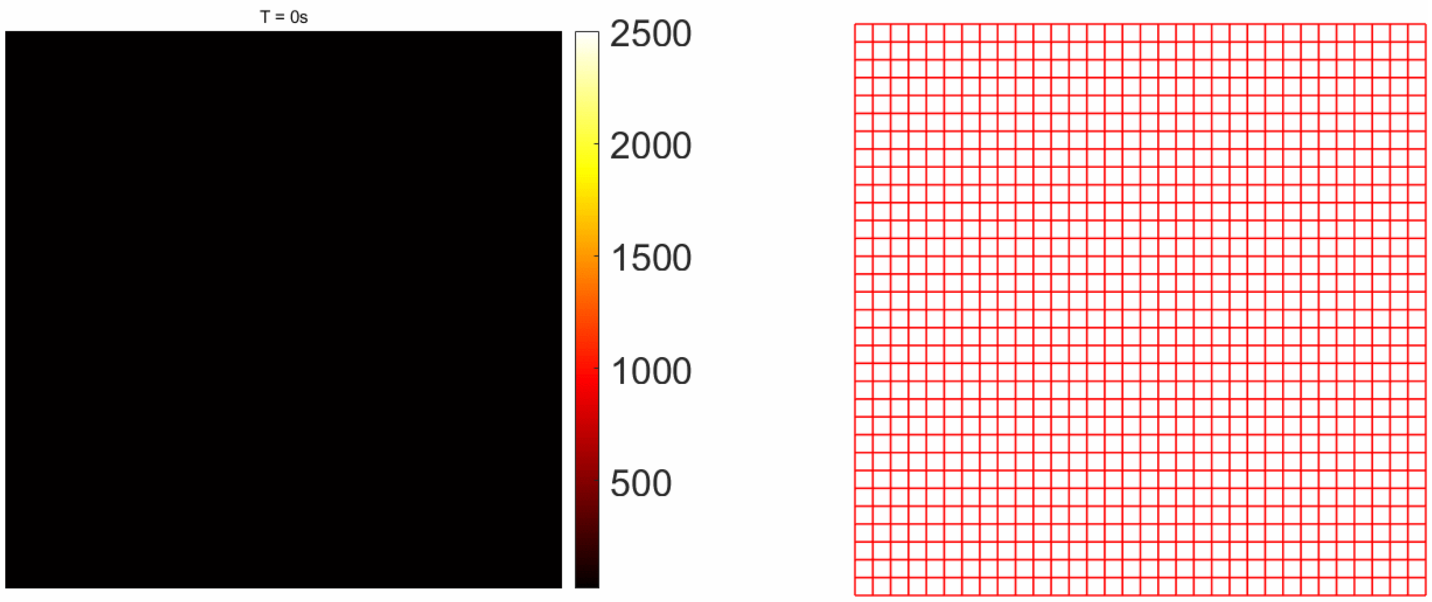
- Consider a two-dimensional **linear heat transfer problem** with a moving Gaussian heat source [12]:

$$\begin{cases} C_p\rho\dfrac{\partial u(\boldsymbol{x},t)}{\partial t}-\nabla\cdot\left(\kappa\nabla u(\boldsymbol{x},t)\right)=f(\boldsymbol{x},t), & \text{in } \Omega\times T,\\ u(\boldsymbol{x},t)=u_0, & \text{in } \Omega,\\ \kappa\nabla u(\boldsymbol{x},t)=0, & \text{on } \partial\Omega\times T. \end{cases}$$

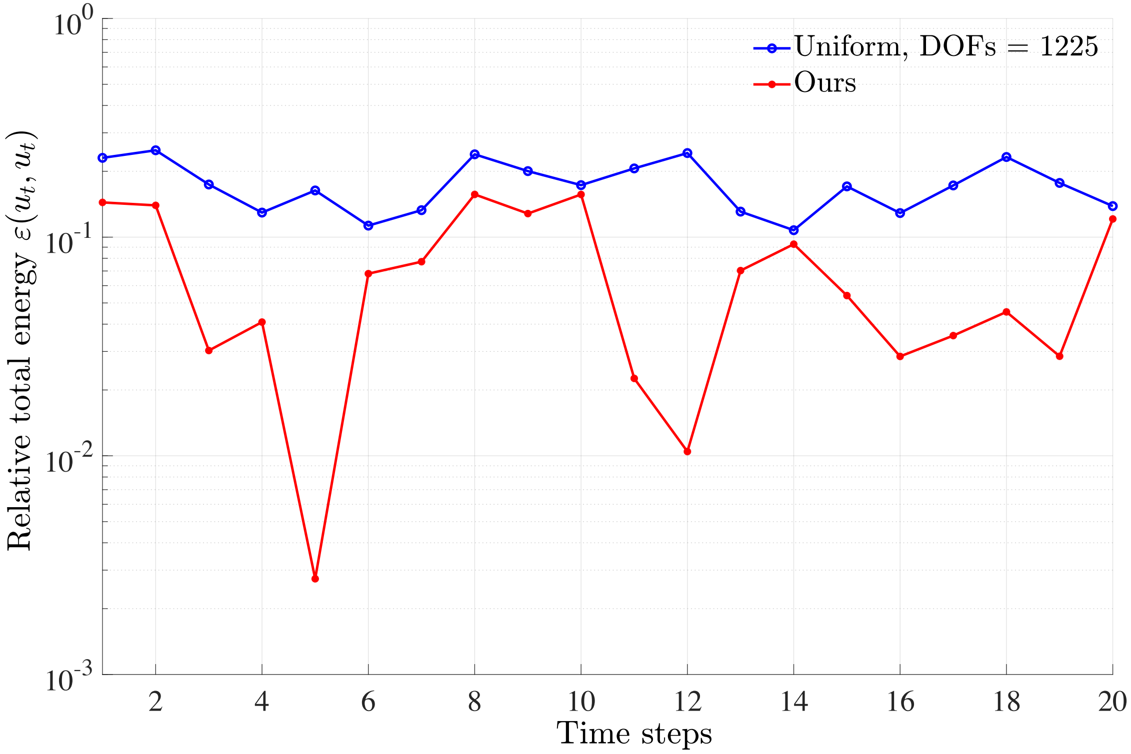


Laser Power P	9×10^5	$[W]$
Laser speed	1.57	$[mm/s]$
Absorptivity η	0.33	
Source radius r_h	100	$[\mu m]$
Conductivity κ	1.0	$[\mathrm{W}/mm/K]$
Heat capacity C_p	1.0	$[\mathrm{J/kg}/K]$
Density ρ	1.0	$[kg/mm^3]$
Initial temperature u_0	20.0	[°C]

Results



Solution and the corresponding parameterization



Error vs. Different instants t

Catalogue

Conclusions and future work

- Conclusions:
  - An **r-adaptivity based on absolute principal curvature** is proposed for isogeometric analysis;
  - **Bi-level approach** using coarse and fine refinements is employed to balance efficiency and accuracy;
  - **Bijectivity is guaranteed** by regularization technique and common line search criteria;
  - Examples show the **effectiveness, efficiency, and superiority** of the proposed method.

- Future work:
  - Extending the proposed method to **shell and solid problems;**
  - To further acceleration, combine our method with spline techniques **with local refinement capability;**
  - We will **release our reference implementation in Geometry + Simulation Modules (G+Smo) library**.

Thanks for your attention!

Q&A

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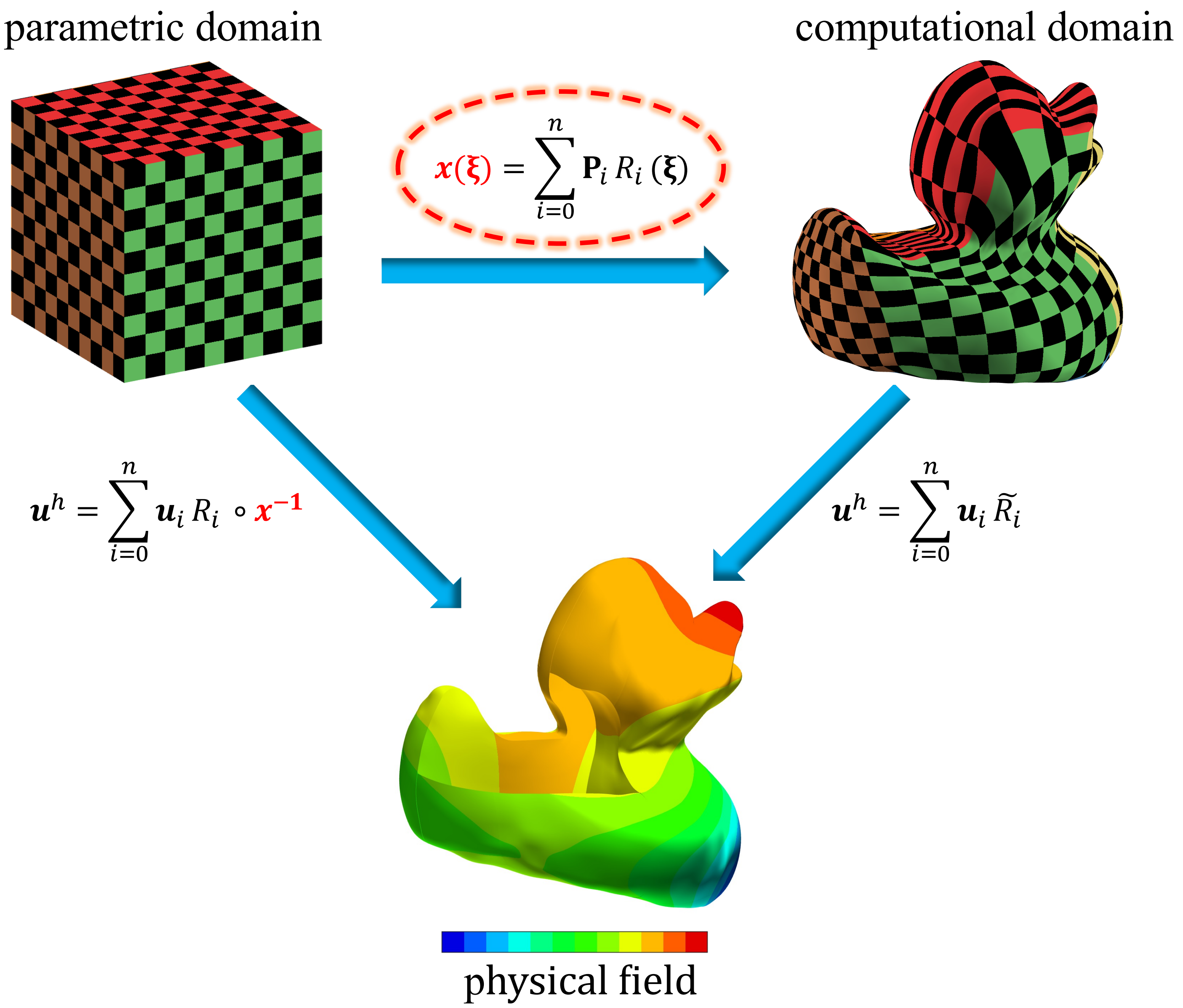
parametric domain
computational domain
$\boldsymbol{x}(\boldsymbol{\xi}) = \sum_{i=0}^{n} \mathbf{P}_i \, R_i\,(\boldsymbol{\xi})$
$\boldsymbol{u}^h = \sum_{i=0}^{n} \boldsymbol{u}_i \, R_i \circ \boldsymbol{x}^{-1}$
$\boldsymbol{u}^h = \sum_{i=0}^{n} \boldsymbol{u}_i \, \widetilde{R}_i$
physical field