# Analysis-Suitable Parameterization for Isogeometric Analysis: Isotropic/Anisotropic Methods and Their Applications

Ye Ji (纪野)

School of Mathematical Sciences, Dalian University of Technology, Dalian, China
Delft Institute of Applied Mathematics, Delft University of Technology, the Netherlands

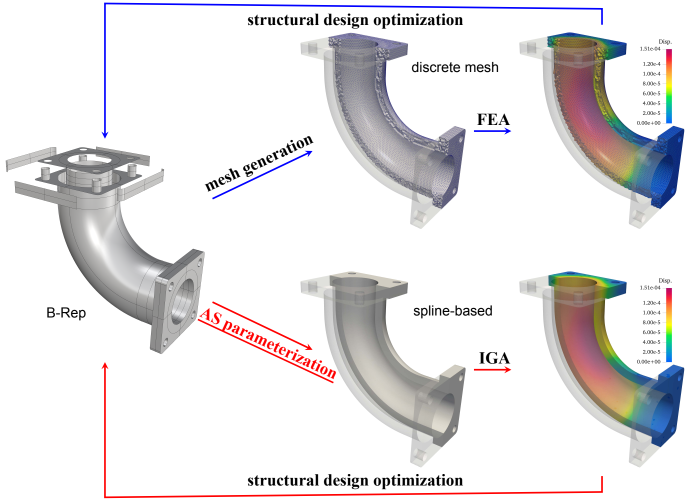
May 20, 2023
2023 年度中国图学学会 “奋发图强” 博士生 workshop

# Contributors


- Ye Ji et al., Constructing high-quality ..., Journal of Computational and Applied Mathematics, 396 (2021), 113615.
- Ye Ji et al., Penalty function-based volumetric ..., Computer Aided Geometric Design, 94 (2022), 102075.
- Ye Ji et al., On an improved PDE-based elliptic ..., Computer Aided Geometric Design, 102 (2023), 102190.
- Ye Ji et al., Curvature-based $r$-adaptive ..., Computer-Aided Design, 150 (2022), 103305.

# Catalogue

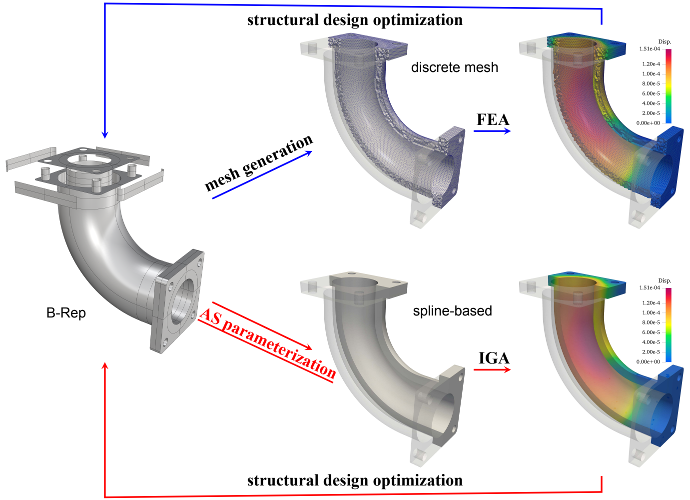
# IsoGeometric Analysis (IGA)




Design-analysis-optimization product development workflow

- Proposed by T.J.R. Hughes et al., 2005.
- **KEY IDEA:** to approximate the physical fields with the same basis functions as that used to generate the CAD model. [1]

# IsoGeometric Analysis (IGA)

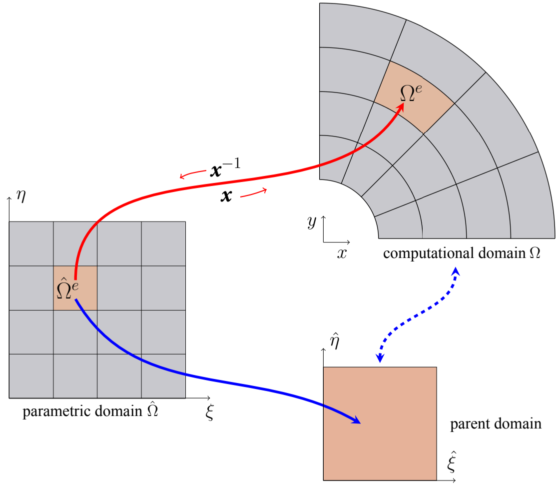



Design-analysis-optimization product development workflow

- Proposed by T.J.R. Hughes et al., 2005.
- **KEY IDEA:** to approximate the physical fields with the same basis functions as that used to generate the CAD model. [1]
- Advantages:
  - Integration of design and analysis;
  - Exact and efficient geometry;
  - No data type transition and mesh generation;
  - Simplified mesh refinement;
  - High order continuous field;
  - Superior approximation properties.
- Extensive applications: shell analysis, fluid-structure interaction, as well as shape and topology optimization etc.

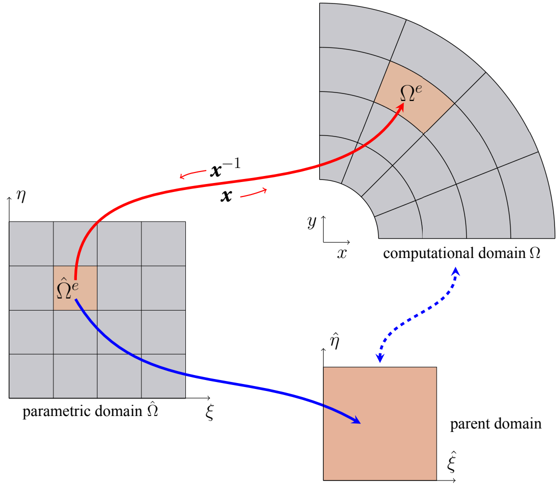
[1] https://games-cn.org/games302/.

# Research motivation



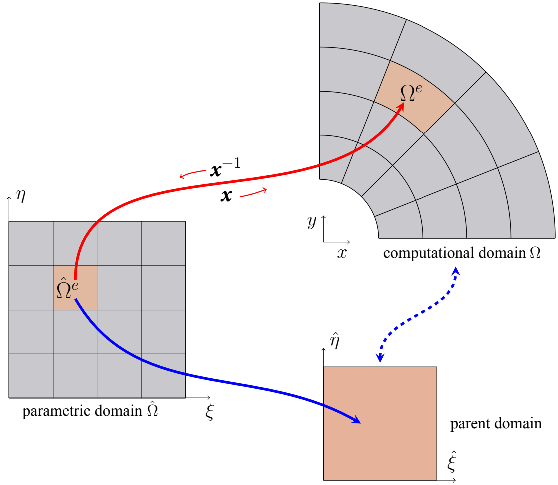

- From a given B-Rep, constructing an analysis-suitable parameterization $\boldsymbol{x}$.

# Research motivation

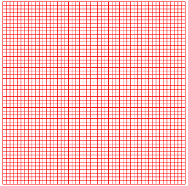


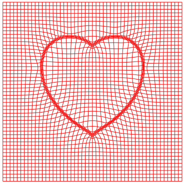

- From a given B-Rep, constructing an analysis-suitable parameterization $\boldsymbol{x}$.
- Analysis-suitable parameterizations should
  - be **bijective**;

# Research motivation



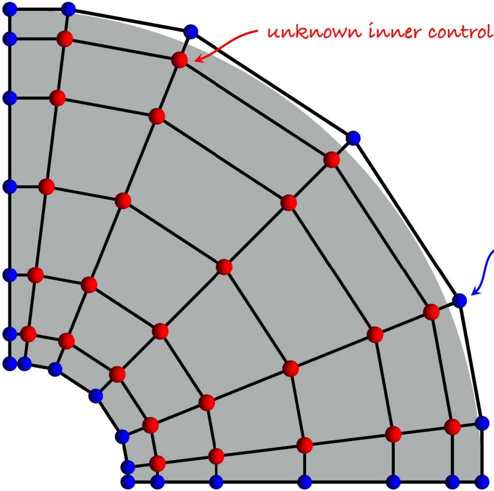

- From a given B-Rep, constructing an analysis-suitable parameterization $\boldsymbol{x}$.
- Analysis-suitable parameterizations should
  - be **bijective**;
  - ensure as **low angle and area/volume distortion** as possible $\rightarrow$ isotroptic.
  - ensure as **low numerical error** as possible $\rightarrow$ anisotroptic.





# Catalogue

# Problem statement: isotroptic parameterization



*unknown inner control points* $\mathbf{P}_i$

$$\boldsymbol{x}(\boldsymbol{\xi}) = \underbrace{\sum_{i\in\mathcal{I}_I} \mathbf{P}_i R_i(\boldsymbol{\xi})}_{\textbf{unknown}} + \underbrace{\sum_{j\in\mathcal{I}_B} \mathbf{P}_j R_j(\boldsymbol{\xi})}_{\textbf{known}}.$$

*known boundary control points* $\mathbf{P}_j$

**GOAL:** To construct the **unknown inner control points $\mathbf{P}_i$** (or basis functions $R_i(\boldsymbol{\xi})$) such that $\boldsymbol{x}$ ensures **bijective** and exhibits the **minimal possible angle and area/volume distortion**.

# Objective functions

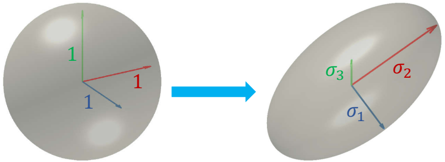
## angle distortion

- MIPS energy [Hormann and Greiner 2000, Fu+2015]:

$$\mathscr{E}^{\text{angle}} = \begin{cases} \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}, \\ \frac{1}{8}\left(\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}\right)\left(\frac{\sigma_2}{\sigma_3} + \frac{\sigma_3}{\sigma_2}\right)\left(\frac{\sigma_1}{\sigma_3} + \frac{\sigma_3}{\sigma_1}\right), \end{cases}$$

where $\sigma_i$ are the singular values of $\mathcal{J}$.

- Ideally, $\sigma_1 = \sigma_2 = \cdots = \sigma_d$.

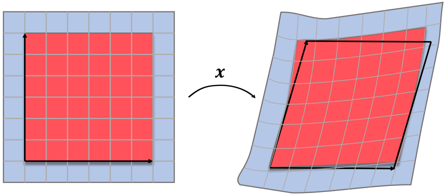



## area/volume distortion

- Area/volume distortion energy:

$$\mathscr{E}^{\text{unif.}}(\boldsymbol{x}) = \frac{|\mathcal{J}|}{vol(\Omega)} + \frac{vol(\Omega)}{|\mathcal{J}|}, \tag{1}$$

where $vol(\Omega)$ denotes the area/volume of the computational domain $\Omega$;



# A natural method: constrained optimization problem

- Solve the following constrained optimization problem:

$$
\begin{aligned}
\underset{\mathbf{P}_i,\ i\in\mathcal{I}_I}{\arg\min} \quad & \mathscr{E}(\boldsymbol{x}) = \int_{\hat{\Omega}} \left(\lambda_{\text{angle}}\mathscr{E}^{\text{angle}}(\boldsymbol{x}) \; + \lambda_{\text{unif.}}\mathscr{E}^{\text{unif.}}(\boldsymbol{x})\right) \; \mathrm{d}\hat{\Omega}, \\
\text{s.t.} \quad & \left|\boldsymbol{\mathcal{J}}\right|_i > 0,
\end{aligned}
\tag{2}
$$

where $\left|\boldsymbol{\mathcal{J}}\right|_i$ are the control coefficients of the Jacobian determinant $\left|\boldsymbol{\mathcal{J}}\right| = \sum_i \left|\boldsymbol{\mathcal{J}}\right|_i R_i(\boldsymbol{\xi})$.

# A natural method: constrained optimization problem

- Solve the following constrained optimization problem:

$$\begin{aligned} \underset{\mathbf{P}_i,\ i\in\mathcal{I}_I}{\arg\min} \quad & \mathscr{E}(\boldsymbol{x}) = \int_{\hat{\Omega}} \left(\lambda_{\text{angle}}\mathscr{E}^{\text{angle}}(\boldsymbol{x}) \ + \lambda_{\text{unif.}}\mathscr{E}^{\text{unif.}}(\boldsymbol{x})\right) \ \mathrm{d}\hat{\Omega}, \\ \text{s.t.} \quad & |\boldsymbol{\mathcal{J}}|_i > 0, \end{aligned} \tag{2}$$

  where $|\boldsymbol{\mathcal{J}}|_i$ are the control coefficients of the Jacobian determinant $|\boldsymbol{\mathcal{J}}| = \sum_i |\boldsymbol{\mathcal{J}}|_i R_i(\boldsymbol{\xi})$.

- However, the number of constraints can be significantly large!!! ◁ [Pan+2020, Ji+2021]. (To a bi-cubic planar NURBS parameterization with $20 \times 20$ control points, the number of inequality constraints is over $34$**k**.)

# Equivalence problem: unconstrained optimization

- Recall the planar MIPS energy,

$$\mathscr{E}_{2D}^{\text{angle}}(\boldsymbol{x}) = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 \sigma_2} = \frac{trace(\mathcal{J}^{\mathrm{T}}\mathcal{J})}{|\mathcal{J}|}.$$

Since the Jacobian determinant appears in its denominator, it proceeds to infinity if the Jacobian determinant $|\mathcal{J}|$ approaches zero.

# Equivalence problem: unconstrained optimization

- Recall the planar MIPS energy,

$$\mathscr{E}_{2D}^{\text{angle}}(\boldsymbol{x}) = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 \sigma_2} = \frac{trace(\mathcal{J}^{\mathrm{T}}\mathcal{J})}{|\mathcal{J}|}.$$

  Since the Jacobian determinant appears in its denominator, it proceeds to infinity if the Jacobian determinant $|\mathcal{J}|$ approaches zero.

- Remove the constraints and solve the following **unconstrained optimization problem**:

$$\arg\min_{\mathbf{P}_i,\ i\in\mathcal{I}_I} \quad \mathscr{E}(\boldsymbol{x}) = \int_{\hat{\Omega}} \left(\lambda_{\text{angle}}\mathscr{E}^{\text{angle}}(\boldsymbol{x}) + \lambda_{\text{vol}}\mathscr{E}^{\text{vol}}(\boldsymbol{x})\right)\, \mathrm{d}\hat{\Omega}. \tag{3}$$

# Equivalence problem: unconstrained optimization

- Recall the planar MIPS energy,

$$\mathscr{E}_{2D}^{\text{angle}}(\boldsymbol{x}) = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 \sigma_2} = \frac{trace(\boldsymbol{\mathcal{J}}^{\mathrm{T}}\boldsymbol{\mathcal{J}})}{|\boldsymbol{\mathcal{J}}|}.$$

  Since the Jacobian determinant appears in its denominator, it proceeds to infinity if the Jacobian determinant $|\boldsymbol{\mathcal{J}}|$ approaches zero.

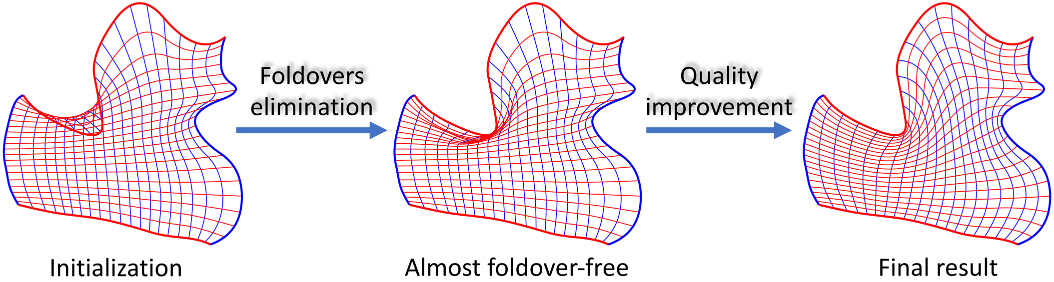
- Remove the constraints and solve the following **unconstrained optimization problem**:

$$\underset{\mathbf{P}_i,\ i \in \mathcal{I}_I}{\arg\min} \quad \mathscr{E}(\boldsymbol{x}) = \int_{\hat{\Omega}} \left(\lambda_{\text{angle}} \mathscr{E}^{\text{angle}}(\boldsymbol{x}) + \lambda_{\text{vol}} \mathscr{E}^{\text{vol}}(\boldsymbol{x})\right) \, \mathrm{d}\hat{\Omega}. \qquad (3)$$

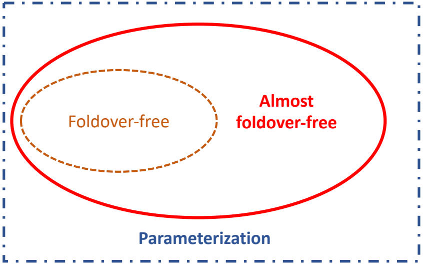
- **Prerequisite: need an already bijective initialization.**

# Barrier function-based parameterization construction

- Three-step strategy.



# Foldovers elimination: almost foldover-free parameterization

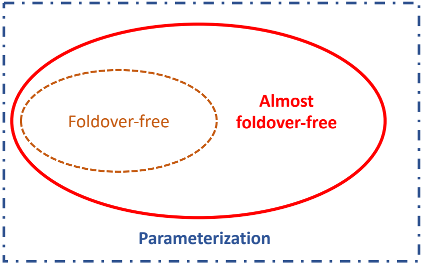



- Solve the following optimization problem:

$$\underset{\mathbf{P}_i,\ i\in\mathcal{I}_I}{\arg\min}\quad E(\boldsymbol{x}) = \int_{\hat{\Omega}} \max\left(0, \delta - |\mathcal{J}|\right)\ \mathrm{d}\hat{\Omega},$$

where $\delta$ is a threshold ($\delta = 5\% vol(\Omega)$ as default).

# Foldovers elimination: almost foldover-free parameterization




- Solve the following optimization problem:

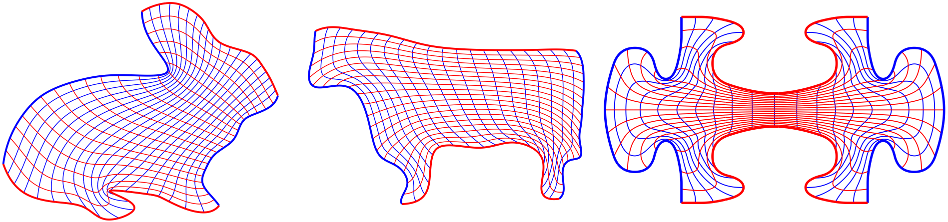
$$\underset{\mathbf{P}_i,\ i\in\mathcal{I}_I}{\arg\min} \quad E(\boldsymbol{x}) = \int_{\hat{\Omega}} \max\left(0, \delta - |\mathcal{J}|\right) \, \mathrm{d}\hat{\Omega},$$

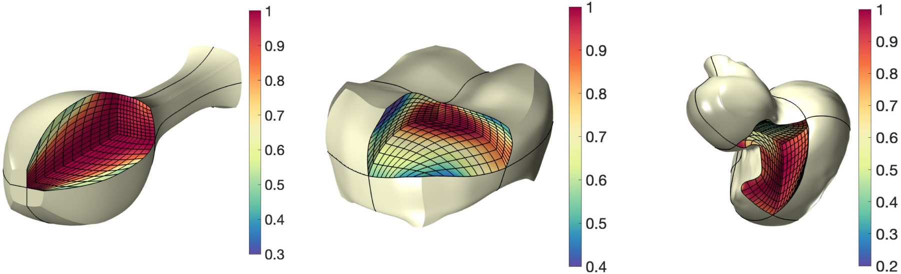
  where $\delta$ is a threshold ($\delta = 5\% vol(\Omega)$ as default).
- Quality improvement using barrier function (robustness consideration):

$$\mathscr{E}^c = \begin{cases} \int_{\hat{\Omega}} \left(\lambda_{\text{angle}} \mathscr{E}^{\text{angle}}(\boldsymbol{x}) + \lambda_{\text{vol}} \mathscr{E}^{\text{vol}}(\boldsymbol{x})\right) \, \mathrm{d}\hat{\Omega}, & \min|\mathcal{J}| > 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

- Analytical gradient: for numerical stability and computational efficiency aspects.

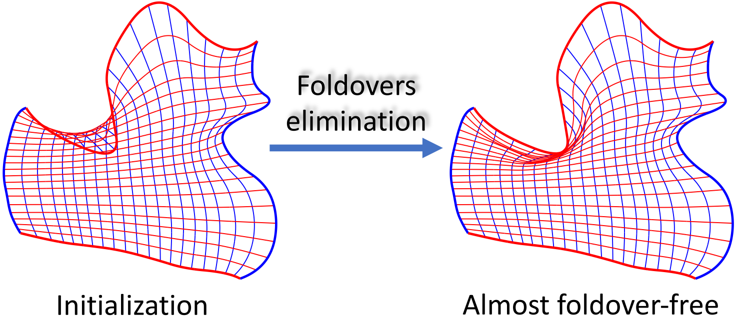
# Gallery: barrier function-based method





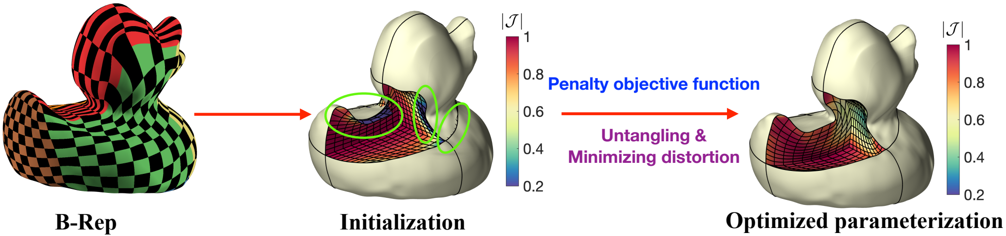
# Penalty function-based parameterization construction

- **Foldover elimination does not significantly improve the quality of parameterization.**
- Is it possible to skip the foldover elimination step?



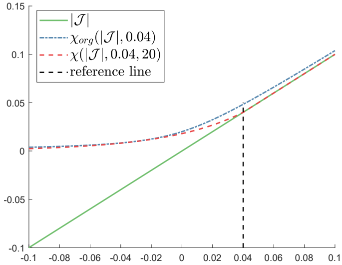
# Penalty function-based parameterization construction

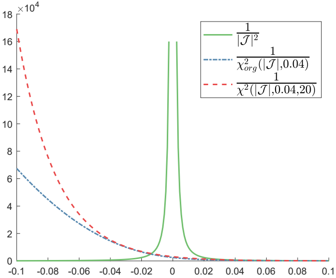
- **Foldover elimination does not significantly improve the quality of parameterization.**
- Is it possible to skip the foldover elimination step?




- **Certainly! Simultaneously untangling and minimizing distortion!!!**

# Basic idea: Penalty function




- **Penalty function:**

$$\chi(|\mathcal{J}|, \varepsilon, \beta) = \begin{cases} \varepsilon \cdot e^{\beta(|\mathcal{J}|-\varepsilon)} & \text{if } |\mathcal{J}| \le \varepsilon \\ |\mathcal{J}| & \text{if } |\mathcal{J}| > \varepsilon \end{cases}, \tag{4}$$

where $\varepsilon$ is a small positive number and $\beta$ is a penalty factor;

- $\chi(|\mathcal{J}|, \varepsilon, \beta)$ equals a small positive number if $|\mathcal{J}| < \varepsilon$, and strictly equals the Jacobian determinant $|\mathcal{J}|$ if $|\mathcal{J}| \ge \varepsilon$;

- $\frac{1}{\chi^2(|\mathcal{J}|,\varepsilon,\beta)}$ have **very large values to penalize the negative Jacobians and small values to accept positive Jacobians**.

# Jacobian regularization and revised objective function

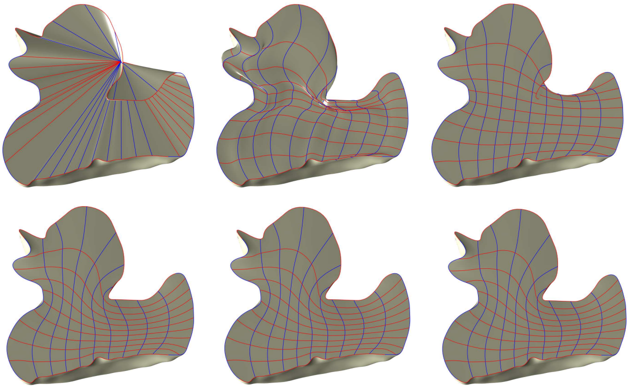
- With this basic idea, **only one optimization problem needs to be solved:**

$$\begin{aligned}\mathop{\arg\min}_{\mathbf{P}_i,\ i\in\mathcal{I}_I} \mathscr{E}^c &= \int_{\hat{\Omega}} \left(\lambda_{\text{mips}}\mathscr{E}^c_{\text{mips}} + \lambda_{\text{vol}}\mathscr{E}^c_{\text{vol}}\right)\, \mathrm{d}\hat{\Omega} \\ &= \int_{\hat{\Omega}} \left(\frac{\lambda_{\text{mips}}}{8}\kappa_F^2(\boldsymbol{\mathcal{J}}) \cdot \frac{|\boldsymbol{\mathcal{J}}|^2}{\chi^2(|\boldsymbol{\mathcal{J}}|,\varepsilon,\beta)} + \lambda_{\text{vol}}\left(\frac{vol(\Omega)}{\chi(|\boldsymbol{\mathcal{J}}|,\varepsilon,\beta)} + \frac{\chi(|\boldsymbol{\mathcal{J}}|,\varepsilon,\beta)}{vol(\Omega)}\right)\right)\, \mathrm{d}\hat{\Omega}, \end{aligned} \tag{5}$$

where $\mathbf{P}_i,\ i \in \mathcal{I}_I$ are the unknown inner control points.

- Analytical gradient, reduced numerical integration, pre-computation etc.

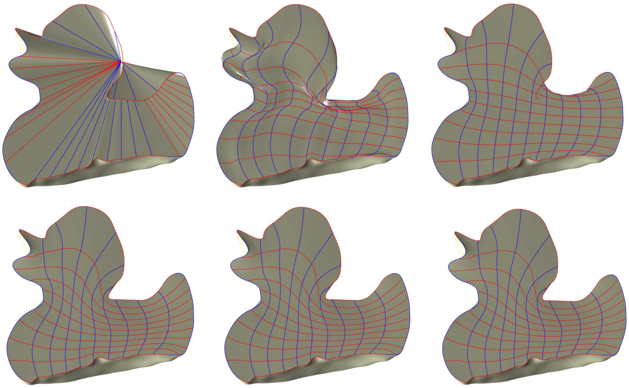
# Parameterization results



Same point Discrete Coons Smoothness energy

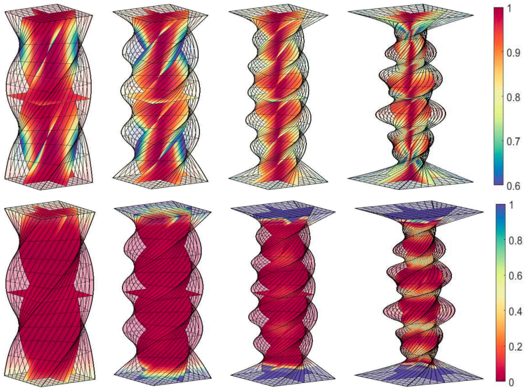
- The results are nearly identical.
- Insensitive to different initializations.

# Parameterization results



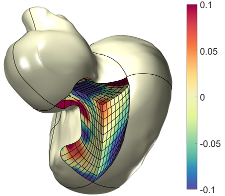
Same point Discrete Coons Smoothness energy

- The results are nearly identical.
- Insensitive to different initializations.

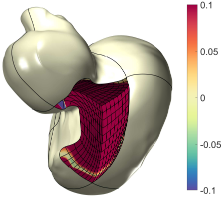



Rot180 Rot360 Rot540 Rot720

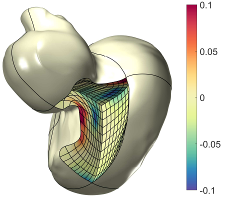
# Comparison: Our method vs. current competitive approaches



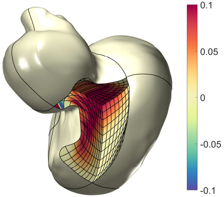

$m_{SJ}^{\text{Algo. 1}} - m_{SJ}^{\text{Pan}}$




$m_{unif.}^{\text{Pan}} - m_{unif.}^{\text{Algo. 1}}$




$m_{SJ}^{\text{Algo. 1}} - m_{SJ}^{\text{Liu}}$




$m_{unif.}^{\text{Liu}} - m_{unif.}^{\text{Algo. 1}}$

- Comparisons to Pan et al. 2020 [1] and Liu et al. 2020 [2].
- Positive values (red regions) indicate our method has lower angle distortion and/or lower volume distortion.

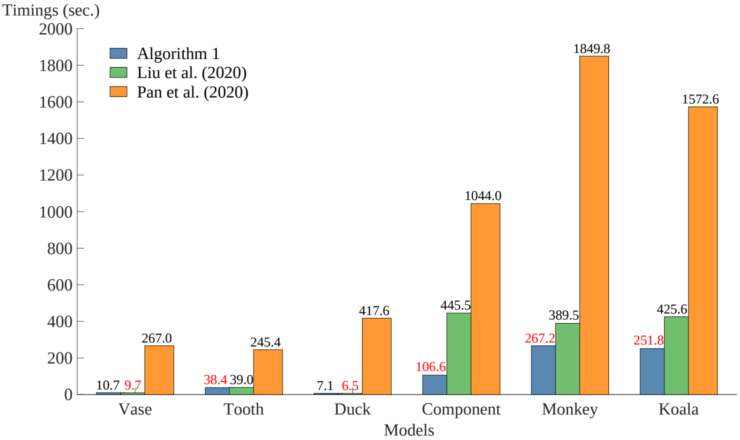
---

[1] Pan, M., Chen, F., & Tong, W. (2020). Volumetric spline parameterization for isogeometric analysis. Computer Methods in Applied Mechanics and Engineering, 359, 112769.

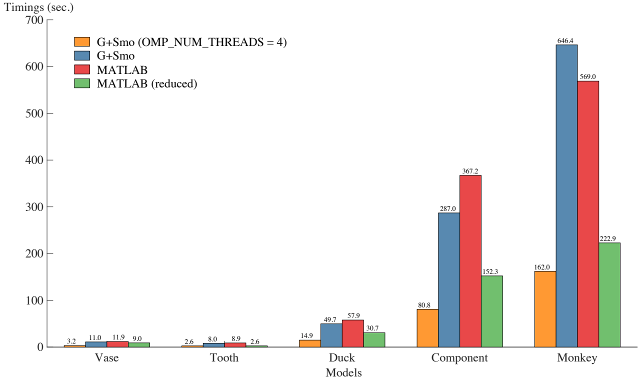
[2] Liu, H., Yang, Y., Liu, Y., & Fu, X. M. (2020). Simultaneous interior and boundary optimization of volumetric domain parameterizations for IGA. Computer Aided Geometric Design, 79, 101853.

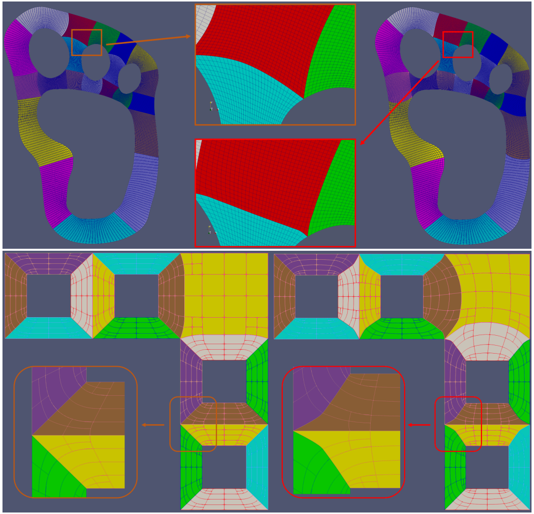
# Efficiency: our method vs. current competitive approaches

- Thanks to avoiding numerous constraints, our method $\gg$ Pan et al. (2020);
- For the last three large-scale models, our method surpasses Liu et al. (2020);
- Unfair: Our method relies on MATLAB, while their methods utilize a C++ implementation.



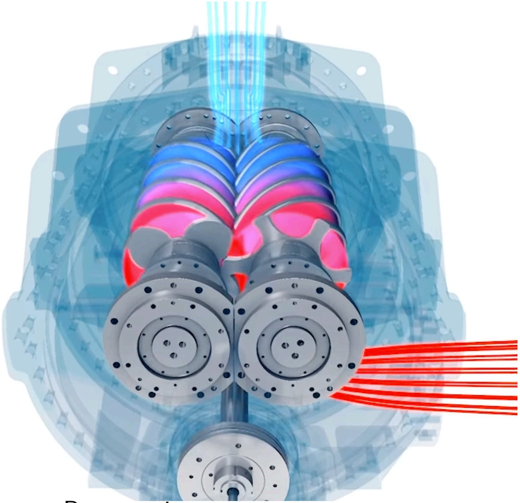
# G+Smo implementation



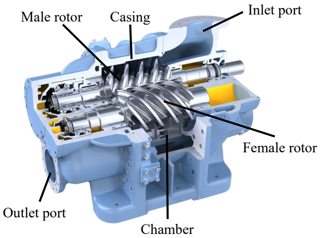



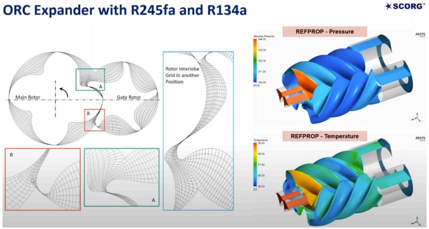
# Catalogue

# Application: rotary twin-screw machines



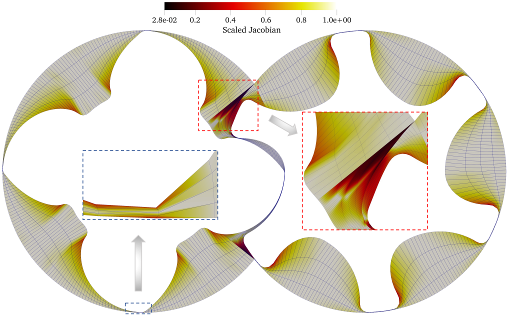
Rotary twin-screw compressor source



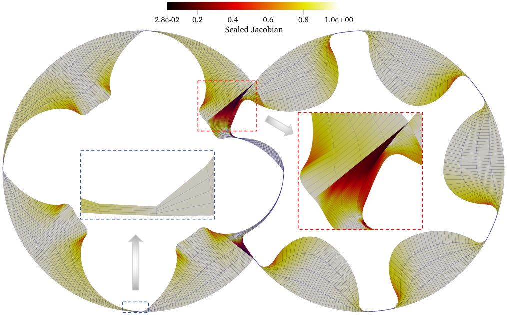



# Parameterizations by using different methods

- This challenging geometry exhibits an **extremely high aspect ratio**.
- The aforementioned penalty function-based method may encounter difficulties, but the elliptic grid generation method proves to be effective.

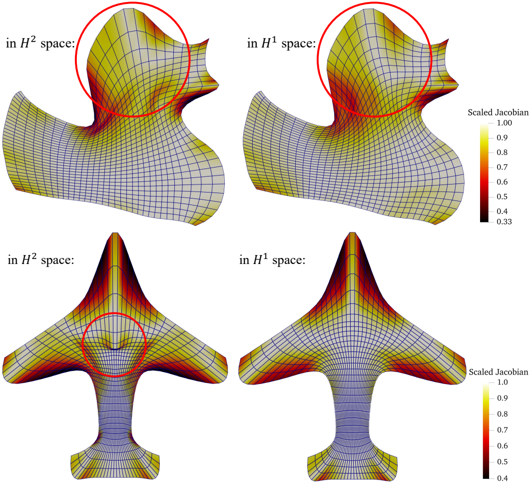



Penalty function-based method




Elliptic Grid Generation (EGG) method

# Problem 1: parameterization quality for general domains




- Essentially, compute a harmonic mapping

$$\begin{cases} \Delta\xi(x,y) = 0 \\ \Delta\eta(x,y) = 0 \end{cases} \quad \text{s.t. } \boldsymbol{x}^{-1}|_{\partial\Omega} = \partial\hat{\Omega}.$$

- Our discretization in $H^1$ space:

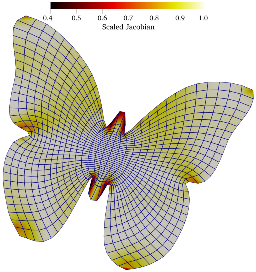
$$\forall R_i \in \Sigma_0 : \begin{cases} \mathbf{F}^x_{H^1} = \mathbf{0}, \\ \mathbf{F}^y_{H^1} = \mathbf{0}, \end{cases} \quad \text{s.t. } \boldsymbol{x}|_{\partial\hat{\Omega}} = \partial\Omega.$$

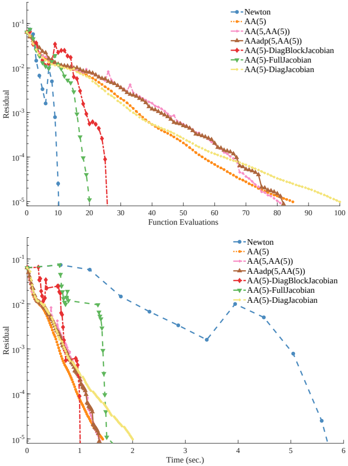
where

$$\mathbf{F}^x_{H^1} = \int_{\hat{\Omega}} \nabla_{\boldsymbol{x}}\mathbf{R} \cdot \nabla_{\boldsymbol{x}}\xi \; d\hat{\Omega},$$

$$\mathbf{F}^y_{H^1} = \int_{\hat{\Omega}} \nabla_{\boldsymbol{x}}\mathbf{R} \cdot \nabla_{\boldsymbol{x}}\eta \; d\hat{\Omega},$$

# Problem 2: computational efficiency - preconditioned AA




- Nonlinear system $\rightarrow$ Fixed point problem

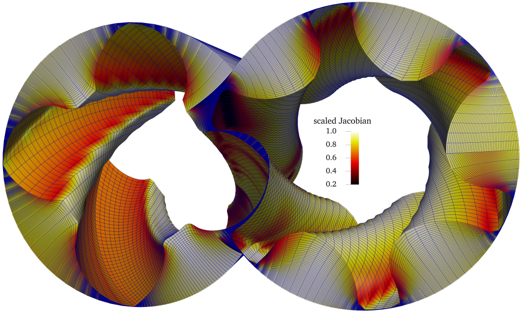
$$\mathbf{F}(\mathbf{x}) = \mathbf{0}, \ \mathbf{x} \in \mathbb{R}^n, \ \mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

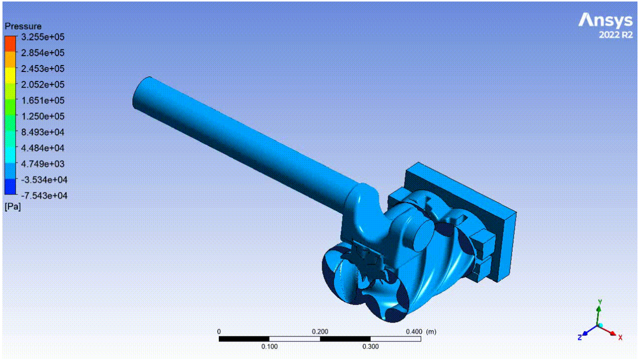
$$\rightarrow \mathbf{x} = \mathbf{x} + \mathbf{F}(\mathbf{x}).$$

- Preconditioned Anderson acceleration:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{M}_k^{-1}\mathbf{F}(\mathbf{x}_k),$$

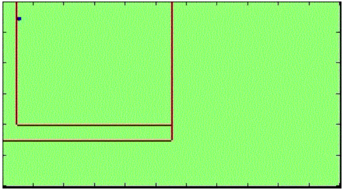
# Back to rotary twin-screw compressor application



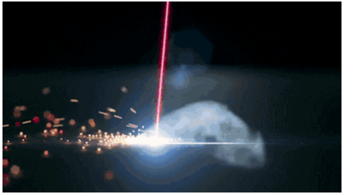



# Catalogue

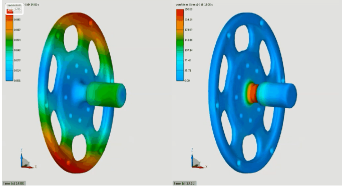
# Anisotropic phenomena in physics



Wave propagation. source



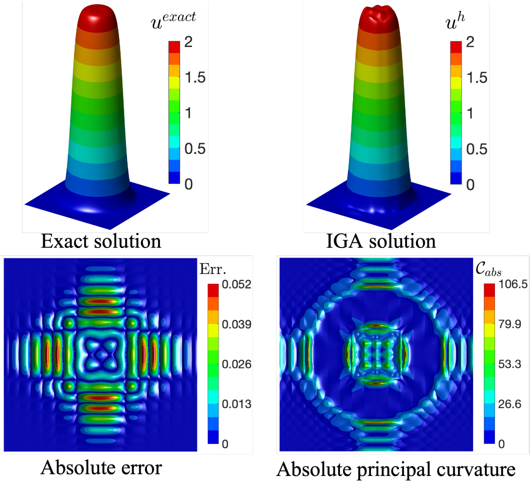
Laser printing. source



Stress concentration. source

- **Localized and anisotropic features extensively exist** in various physical phenomena;
- For such problems, **isotropic parameterizations are computationally uneconomical**;
- **Anisotropic parameterizations ($r$-adaptivity)**: increase per-degree-of-freedom accuracy while keeping the total degrees-of-freedom (DOFs) constant.

# Basic idea

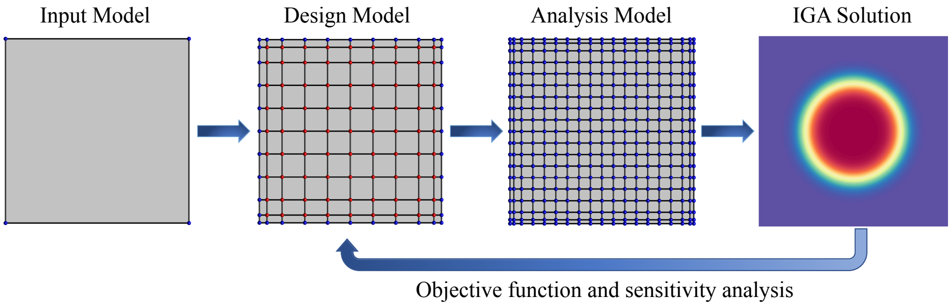



- **Absolute principal curvature**: to characterize the variations of the isogeometric solution;
- A tight relationship between geometric quantity and isogeometric solution is established;
- Absolute error and absolute principal curvature **show similar performance** (left figure);
- Absolute principal curvature is a good error estimator.

# Basic idea - cont'd

**Anisotropic parameterizations are often solution-dependent**:

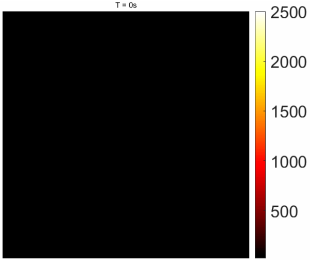
- Need **good numerical solution accuracy** to drive parameterization;
- Adjust as few control points as possible **for high efficiency**;
- **Bi-level strategy**: a coarse level (design model) to update the parameterization for efficiency's sake and a fine level (analysis model) to perform the isogeometric simulation for accuracy's sake.

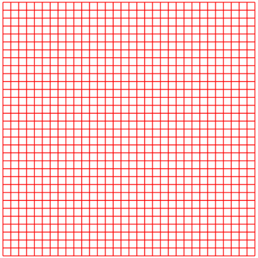


# Application: Time-dependent dynamic PDE

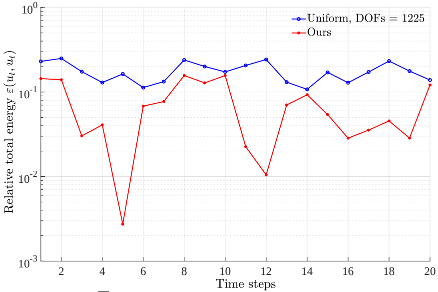
- Consider a 2D **linear heat transfer problem** with a moving Gaussian heat source:

$$\begin{cases} C_p\rho\frac{\partial u(\boldsymbol{x},t)}{\partial t} - \nabla\cdot(\kappa\nabla u(\boldsymbol{u},t)) = f(\boldsymbol{x},t) & \text{in } \Omega\times T \\ u(\boldsymbol{x},t) = u_0 & \text{in } \Omega \\ \kappa\nabla u(\boldsymbol{x},t) = 0 & \text{on } \partial\Omega\times T \end{cases}$$



$u(\boldsymbol{x},t)$ and the parameterizations




Errors vs. time instants $t$

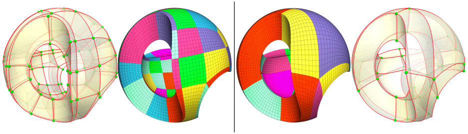
# Catalogue

# Conclusions and future work

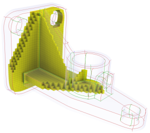
- Conclusions:
  - **Barrier function-based parameterization method** is proposed;
  - **Penalty function-based parameterization method** is proposed;
  - **A preconditioned Anderson Acceleration framework** is proposed for EGG;
  - **A curvature-based $r$-adaptive parameterization approach using a bi-level strategy** is proposed for anisotropic parameterization.

# Conclusions and future work

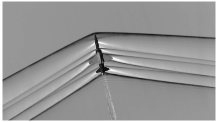
- Conclusions:
  - **Barrier function-based parameterization method** is proposed;
  - **Penalty function-based parameterization method** is proposed;
  - **A preconditioned Anderson Acceleration framework** is proposed for EGG;
  - **A curvature-based $r$-adaptive parameterization approach using a bi-level strategy** is proposed for anisotropic parameterization.
- Future work:
  - **Topology computation for multi-patch parameterization**;
  - **High-efficient adaptive parameterization methods**.



[Gao+2015]



[HexaLab.net]



[Wiki]

# Thanks for your attention!

# Q&A.

jiye@mail.dlut.edu.cn

# Analysis-Suitable Parameterization for Isogeometric Analysis: Isotropic/Anisotropic Methods and Their Applications

Ye Ji (纪野)

School of Mathematical Sciences, Dalian University of Technology, Dalian, China
Delft Institute of Applied Mathematics, Delft University of Technology, the Netherlands

May 20, 2023
2023 年度中国图学学会 “奋发图强” 博士生 workshop