The Bigger The Better II

Group 8-29 *

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Contents

1 Introduction

This project aims to find an algorithm to determine the side length of the largest square that can be inscribed inside a convex n-gon. It is a continuation from a previous project completed in 2021, The Bigger The Better. [1]

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1.1 Definitions

placement
: A valid location, size and rotation of the square such that all vertices of the square lie on the edges of the polygon.

inscribed
: All vertices of the square must lie on the edges of the polygon.

fit All vertices of the square must lie within the polygon.

RQ Research Question

1.2 Research Questions

1. What is the side length of the largest square that can be inscribed in a triangle?
2. What is the side length of the largest square that can be inscribed in a regular n-gon, given $n \neq 4$?
3. What is the side length of the largest square that can be inscribed in a convex n-gon?

1.3 Project Scope

This project will only focus on convex polygons.

2 Literature Review

3 Research Question 1

RQ1 aims to find out the side length of the largest square that can be inscribed in a triangle, given the side lengths of the triangle, a, b and c.

3.1 Key Insights

Some key insights which greatly aided in solving this problem were found.

1. It can be seen that no more than two vertices of a square can lie on a single side, as a square has at most two co-linear vertices.

2. We notice how a triangle has three sides, and a square has four vertices. In order for all the vertices to lie on the triangle, using the Pigeonhole Principle, there will be at least one side with two vertices lying on it.

3. Combining the above insights, there will be one vertex of the square each lying on two sides of the triangle, with the other two vertices of the square lying on the latter side of the triangle.

3.2 Solution

A figure has been constructed for the purposes of illustrating the following proof.

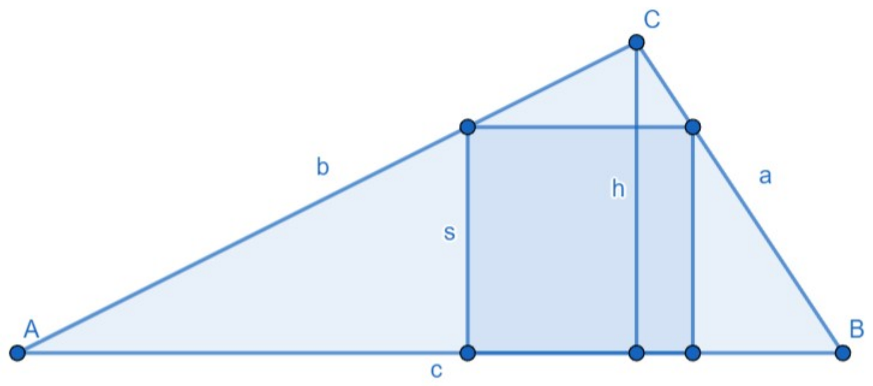

Figure 1: The figure for RQ1.

Let s be the side length of the largest square that can be inscribed in a triangle, with side lengths a, b and c, and circumradius R.

Side c can be formed with the sum of s, $s\cot A$ and $s\cot\angle B$. Hence, we can express s with the side length c, as well as angles A and B.

$$c = s + s\cot\angle A + s\cot\angle B$$
$$s = \frac{c}{1+\cot\angle A+\cot\angle B}$$

Both sides of the fraction can be multiplied by $\sin A\sin B$. Following which, the Sine Addition Formula can be applied.

$$
\begin{aligned}
s &= \frac{c \sin \angle A}{\sin \angle A + \cos \angle A + \cot \angle B \sin \angle A} \\
&= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \cos \angle A \sin \angle B + \sin \angle A \cos \angle B} \\
&= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \sin (\angle A + \angle B)} \\
&= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \sin (180 - \angle C)} \\
&= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \sin \angle C}
\end{aligned}
$$

Both sides of the fraction can be multiplied by $2Rc$ and the Law of Sines can be used to simplify.

$$
\begin{aligned}
s &= \frac{2Rc \sin \angle A \sin \angle B}{2R \sin \angle A \sin \angle B + 2R \sin \angle C} \\
&= \frac{ac \sin \angle B}{a \sin \angle B + c} \\
&= \frac{2Rac \sin \angle B}{2Ra \sin \angle B + 2Rc} \\
&= \frac{abc}{2Rc + ab}
\end{aligned}
$$

Since each of the sides of the triangle, a, b and c can be the longest side, the maximum of the three placements can be taken as the solution, hence

$$
s_{\max} = \max \left(\frac{abc}{2Rc + ab}, \frac{abc}{2Rb + ac}, \frac{abc}{2Ra + bc} \right) \tag{1}
$$

For obtuse triangles, only one placement exists, i.e. when the square lies on the longest side.

$$
s = \frac{abc}{2Rc + ab} \tag{2}
$$

where c is the longest side.

4 Research Question 2

RQ2 aims to find out the side length of the largest square that can be inscribed in a convex n-gon, given n and the side length of the n-gon, k.

This problem can be further split into four cases.

1. when $n \equiv 0 \pmod 4$,
2. when $n \equiv 2 \pmod 4$,
3. when $n \equiv 1 \pmod 4$,
4. when $n \equiv 3 \pmod 4$.

4.1 Case 1

This case deals with the scenario where the number of sides in the n-gon is divisible by 4.

Firstly, for the side length of the square to be maximised, more than one vertex of the square must coincide with the perimeter of the n-gon. This can be easily seen, because the square can be pushed outwards in the other direction if less than two vertices of the square touch the perimeter of the n-gon.

Let the square be $ABCD$, and the polygon be $V_1V_2...V_n$. Also, let $n = 4m$, where m is an integer.

Due to such a polygon being symmetrical both horizontally and vertically, the assumption that the centre of the polygon coincides with the centre of the square can be made. From the earlier observation, it can be assumed that one vertex of the square, A, lies on the perimeter of the polygon. Also, due to the polygon being symmetrical both horizontally and vertically, the opposite vertex, C, will also lie on the perimeter of the polygon.

Due to the symmetry, as long as vertex B lies on the perimeter of the polygon, the square can be inscribed. If vertex B lies within the polygon, the square can be fit. We shall find the values of θ such that the square can be inscribed or fit.

Firstly, it can be seen that $V_1OV_2 = V_2OV_3 = ... = V_nOV_1 = \frac{360°}{n}$. Hence, $0 \leq \theta \leq \frac{360°}{n}$. Furthermore, we can limit this range to $0 \leq \theta \leq \frac{180°}{n}$, as when $\theta \geq \frac{180°}{n}$, the diagram can be flipped to reduce θ. Now, we can try to find $\angle V_{m+1}OB$. To do this, we find the slice of the polygon which contains segment OB. We can find the number of triangles

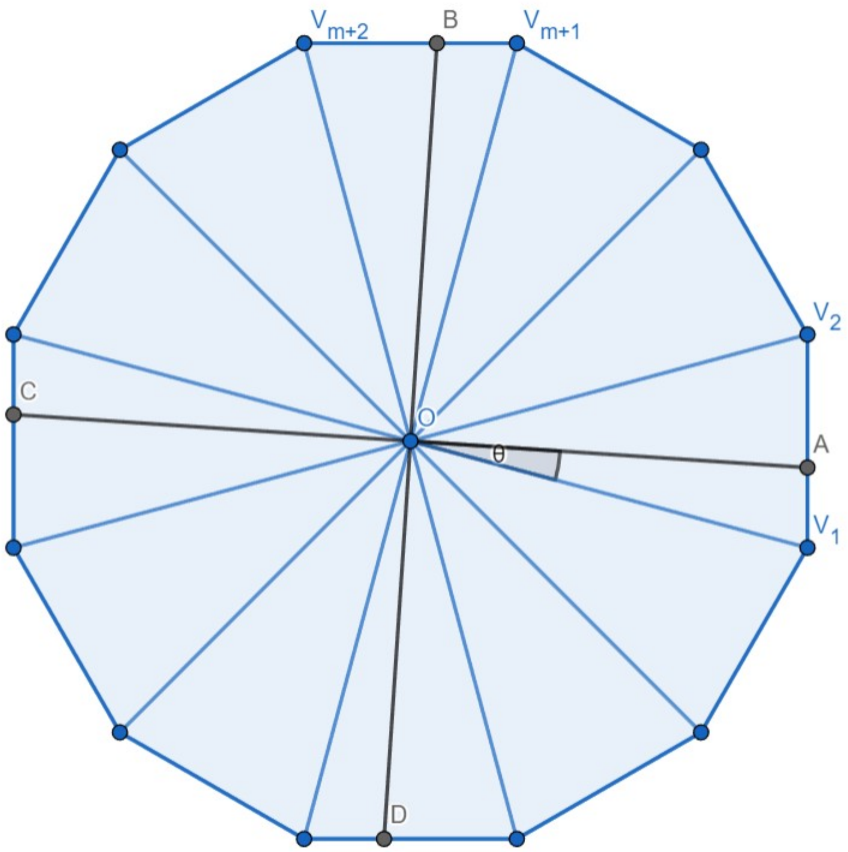

Figure 2: The figure for finding possible placements for case 1 of RQ2.

that has to be passed through to form $\angle V_1OB$. Let this value be x. This gives the expression:

$$x = frac\theta + 90^\circ frac360^\circ n$$

Simplifying it,

$$\begin{aligned} x &= frac\theta + 90^\circ frac360^\circ n \\ &= frac\theta n + 90^\circ n360^\circ \\ &= frac\theta n360^\circ + fracn4 \end{aligned}$$

Plugging in the range for θ,

$$fracn4 \leq frac\theta n360^\circ + fracn4 \leq fracn4 + frac12$$

Substituting n for $4m$,

$$m \leq x \leq m + frac12$$

It can be seen that point B is in triangle $V_{m+1}OV_{m+2}$, and is closer to V_{m+1} than V_{m+2}. We can now check for equality between OA and OB, as this would render $ABCD$ as a square. To do that, the equality $\angle V_{m+1}OB = \angle V_1OA$ must be true.

$$\begin{aligned} \angle V_{m+1}OB &= \theta + 90^\circ - m\left(\frac{360^\circ}{n}\right) \\ &= \theta \end{aligned}$$

Hence, a square can always be inscribed.

To find the maximum side length of the square, we can maximise OA. Let $OV_1 = r$. Using the sine law,

$$
\begin{aligned}
\angle AV_1O &= 90^\circ - \frac{180^\circ}{n} \\
\frac{OA}{\sin\left(90^\circ - \frac{180^\circ}{n}\right)} &= \frac{r}{\sin\left(180^\circ - \theta - \left(90^\circ - \frac{180^\circ}{n}\right)\right)} \\
\frac{OA}{\cos\frac{180^\circ}{n}} &= \frac{r}{\sin\left(\theta + 90^\circ - \frac{180^\circ}{n}\right)} \\
OA &= \frac{r\cos\frac{180^\circ}{n}}{\cos\left(\frac{180^\circ}{n} - \theta\right)}
\end{aligned}
$$

Notice how the numerator is fixed. To maximise OA, we need to minimise $\cos\left(\frac{180^\circ}{n} - \theta\right)$. We need to maximise $\frac{180^\circ}{n} - \theta$, hence we have to minimise θ. This can be done when $\theta = 0$.

Now, we can find the side length of the square, s, from r, and by expressing r from the side length of the entire polygon, k, we can find s from k and n.

$$
\begin{aligned}
OA &= \frac{r\cos\frac{180^\circ}{n}}{\cos\left(\frac{180^\circ}{n}\right)} \\
&= r \\
s &= OA\sqrt{2} \\
&= r\sqrt{2} \\
2r^2\left(1 - \cos\frac{360^\circ}{n}\right) &= k^2 \\
r &= k\sqrt{\frac{1}{2 - 2\cos\frac{360^\circ}{n}}} \\
s &= k\sqrt{\frac{2}{2 - 2\cos\frac{360^\circ}{n}}} \\
&= k\sqrt{\frac{1}{1 - 1\cos\frac{360^\circ}{n}}}
\end{aligned}
$$

References

[1] H. Wu and X. C. Huo. “The Bigger The Better”. In: (2021). URL: `http://projectsday.hci.edu.sg/2021/05-Report/cat-08/8-02/index.pdf`.