The Bigger The Better II

Group 8-29 *

2022

Contents

1 Introduction

This project aims to find an algorithm to determine the side length of the largest square that can be inscribed inside a convex n-gon. It is a continuation from a previous project completed in 2021, The Bigger The Better. [1]

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1.1 Definitions

placement

    A valid location, size and rotation of the square such that all vertices of the square lie on the edges of the polygon.

inscribed

    All vertices of the square must lie on the edges of the polygon.

RQ   Research Question

1.2 Research Questions

1. What is the side length of the largest square that can be inscribed in a triangle?

2. What is the side length of the largest square that can be inscribed in a regular n-gon, given $n \neq 4$?

3. What is the side length of the largest square that can be inscribed in a convex n-gon?

1.3 Project Scope

This project will only focus on convex polygons.

2 Literature Review

3 Research Question 1

RQ1 aims to find out the side length of the largest square that can be inscribed in a triangle, given the side lengths of the triangle, a, b and c.

Some key insights were found, which are:

1. It can be seen that no more than two vertices of a square can lie on a single side, as a square has at most two co-linear vertices.

2. We notice how a triangle has three sides, and a square has four vertices. In order for all the vertices to lie on the triangle, using the Pigeonhole Principle, there will be at least one side with two vertices lying on it.

3. Combining the above insights, there will be one vertex of the square each lying on two sides of the triangle, with the other two vertices of the square lying on the latter side of the triangle.

3.1 Solution

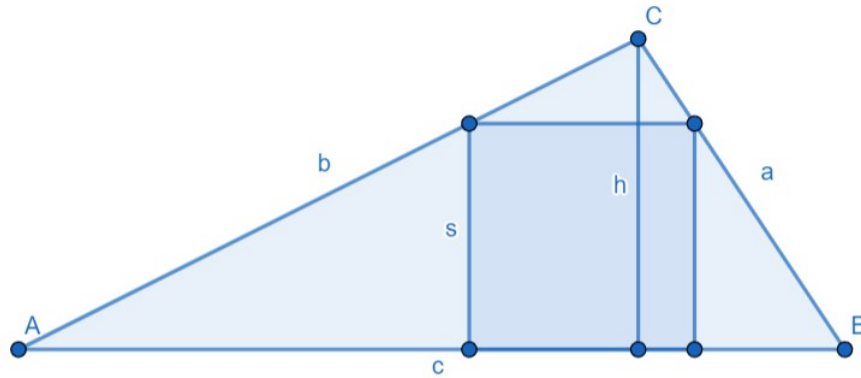A figure has been constructed for the purposes of illustrating the following proof.

Figure 1: The figure for RQ1.

We see that side c can be formed with s as well as $s \cot A$ and $s \cot \angle B$.

$$c = s + s \cot \angle A + s \cot \angle B$$

We would like to express s in terms of side lengths a, b and c, as well as other values directly derived from the side lengths.

$$s = \frac{c}{1 + \cot \angle A + \cot \angle B}$$

$$= \frac{c \sin \angle A}{\sin \angle A + \cos \angle A + \cot \angle B \sin \angle A}$$

$$= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \cos \angle A \sin \angle B + \sin \angle A \cos \angle B}$$

$$= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \sin (\angle A + \angle B)}$$

$$= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \sin (180 - \angle C)}$$

$$= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \sin \angle C}$$

$$= \frac{2Rc \sin \angle A \sin \angle B}{2R \sin \angle A \sin \angle B + 2R \sin \angle C}$$

$$= \frac{ac \sin \angle B}{a \sin \angle B + c}$$

$$= \frac{2Rac \sin \angle B}{2Ra \sin \angle B + 2Rc}$$

$$= \frac{abc}{2Rc + ab}$$

We used formulae and theorems such as the sine addition formula and the law of sines to manipulate the expressions.

Since each of the sides of the triangle, a, b and c can be the longest side, we can take the maximum of the three combinations, hence

$$s_{\max} = \max \left(\frac{abc}{2Rc + ab}, \frac{abc}{2Rb + ac}, \frac{abc}{2Ra + bc} \right) \tag{1}$$

For obtuse triangles, we notice that only one placement exists, when the square lies on the longest side. We have:

$$s = \frac{abc}{2Rc + ab} \tag{2}$$

where c is the longest side.

4 Research Question 2

RQ2 aims to find out the side length of the largest square that can be inscribed in a convex n-gon, given n and the side length, k.

We have noticed that this problem can be split into 3 separate cases:

1. when n is divisible by 4,

2. when n is divisible by 2 but not 4,

3. when n is not divisible by 2.

We shall tackle the cases in increasing order of difficulty.

4.1 Case 1

The first case is when n is divisible by 4.

This is the simplest case, as it can be seen that the diagonal of the square must be the same as the diagonal of the polygon.

Proof: The diagonal of the square connects 2 vertices of the square opposite to each other. It is the longest line segment in a square. Since the square has to be inscribed in the polygon, both ends of the diagonal have to lie on the edges of the polygon. It can be seen that the 2 furthest points on the edges of the polygon would be any vertex of the polygon and the opposite vertex. Hence, the diagonal of the square can be at most the length of the diagonal of the polygon. We can show that such a construction exists.
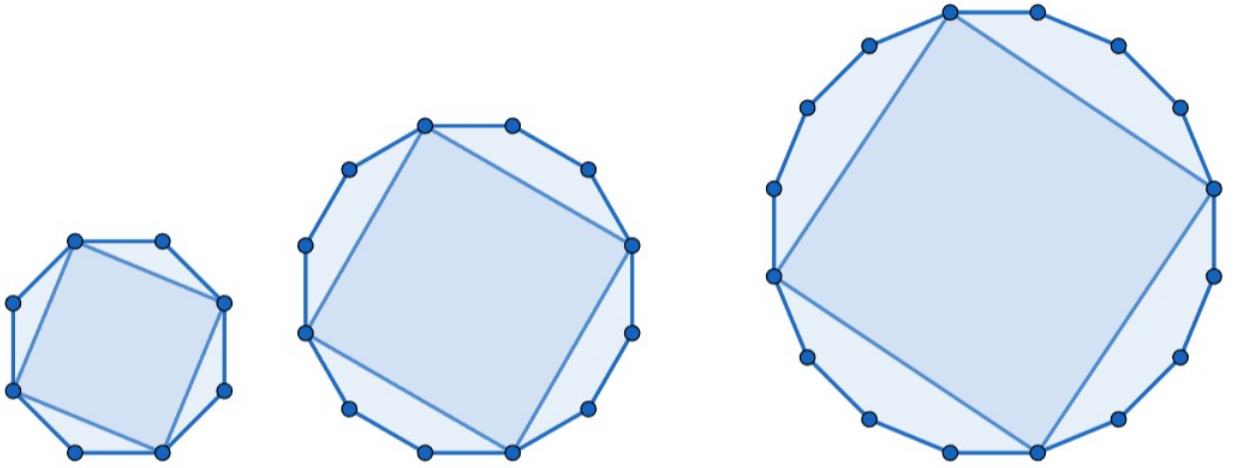

Figure 2: The constructions for case 1 of RQ2.

Hence, we can calculate the side length of the square. With reference to the figure, we shall calculate l using the law of cosines.
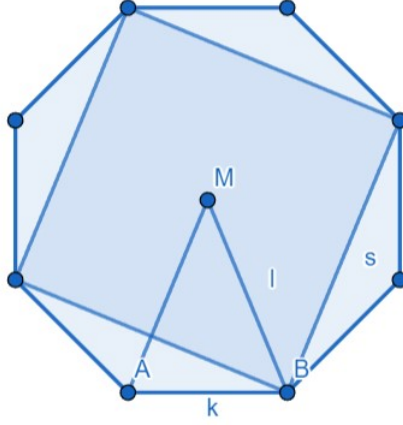
Figure 3: The figure for case 1 of RQ2.

$$\angle AMB = \frac{360°}{2n}$$
$$= \frac{180°}{n}$$
$$l^2 + l^2 - 2\,(l)\,(l)\cos\left(\angle AMB\right) = k^2$$
$$2l^2\left(1 - \cos\left(\angle AMB\right)\right) = k^2$$
$$l^2 = \frac{k^2}{2\left(1 - \cos\left(\frac{180°}{n}\right)\right)}$$
$$l = k\sqrt{\frac{1}{2\left(1 - \cos\left(\frac{180°}{n}\right)\right)}}$$

Now, we can find s using the Pythagoras theorem.

$$2s^2 = (2l)^2$$
$$s^2 = 2l^2$$
$$s = l\sqrt{2}$$
$$= k\sqrt{\frac{2}{2\left(1 - \cos\left(\frac{180°}{n}\right)\right)}}$$
$$= k\sqrt{\frac{1}{\left(1 - \cos\left(\frac{180°}{n}\right)\right)}}$$

References

[1]   H. Wu and X. C. Huo. "The Bigger The Better". In: (2021). URL: `http://projectsday.hci.edu.sg/2021/05-Report/cat-08/8-02/index.pdf`.