The Bigger The Better II

Group 8-29 *

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Contents

1 Introduction **1**
1.1 Definitions 2
1.2 Research Questions 2
1.3 Project Scope 2

2 Literature Review **2**

3 Research Question 1 **2**
3.1 Key Insights 2
3.2 Solution 3

4 Research Question 2 **4**
4.1 Case 1 5

1 Introduction

This project aims to find an algorithm to determine the side length of the largest square that can be inscribed inside a convex n-gon. It is a continuation from a previous project completed in 2021, The Bigger The Better. [1]

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1.1 Definitions

placement

A valid location, size and rotation of the square such that all vertices of the square lie on the edges of the polygon.

inscribed

All vertices of the square must lie on the edges of the polygon.

RQ Research Question

1.2 Research Questions

1. What is the side length of the largest square that can be inscribed in a triangle?
2. What is the side length of the largest square that can be inscribed in a regular n-gon, given $n \neq 4$?
3. What is the side length of the largest square that can be inscribed in a convex n-gon?

1.3 Project Scope

This project will only focus on convex polygons.

2 Literature Review

3 Research Question 1

RQ1 aims to find out the side length of the largest square that can be inscribed in a triangle, given the side lengths of the triangle, a, b and c.

3.1 Key Insights

Some key insights which greatly aided in solving this problem were found.

1. It can be seen that no more than two vertices of a square can lie on a single side, as a square has at most two co-linear vertices.

2. We notice how a triangle has three sides, and a square has four vertices. In order for all the vertices to lie on the triangle, using the Pigeonhole Principle, there will be at least one side with two vertices lying on it.

3. Combining the above insights, there will be one vertex of the square each lying on two sides of the triangle, with the other two vertices of the square lying on the latter side of the triangle.

3.2 Solution

A figure has been constructed for the purposes of illustrating the following proof. Let

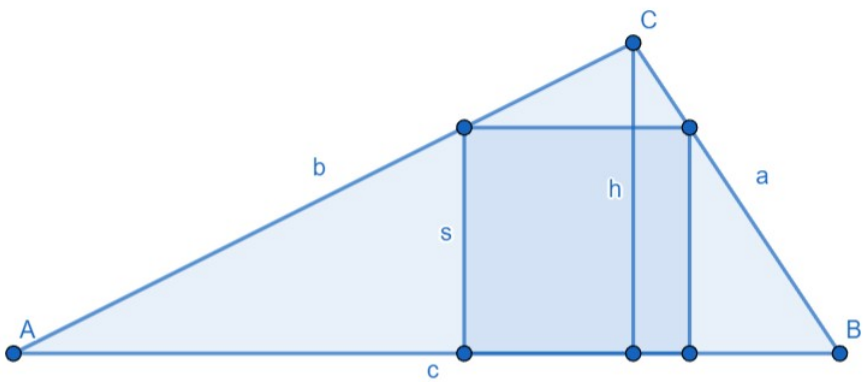

Figure 1: The figure for RQ1.

s be the side length of the largest square that can be inscribed in a triangle, with side lengths a, b and c, and circumradius R.

Side c can be formed with the sum of s, $s \cot A$ and $s \cot \angle B$. Hence, we can express s with the side length c, as well as angles A and B.

$$c = s + s \cot \angle A + s \cot \angle B$$
$$s = \frac{c}{1 + \cot \angle A + \cot \angle B}$$

We multiply both sides of the fraction by $\sin A \sin B$ and use the sine addition formula.

$$\begin{aligned}
s &= \frac{c \sin \angle A}{\sin \angle A + \cos \angle A + \cot \angle B \sin \angle A} \\
&= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \cos \angle A \sin \angle B + \sin \angle A \cos \angle B} \\
&= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \sin (\angle A + \angle B)} \\
&= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \sin (180 - \angle C)} \\
&= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \sin \angle C}
\end{aligned}$$

Now, we can multiply both sides of the fraction to $2Rc$ and use the law of sines to simplify.

$$\begin{aligned}
s &= \frac{2Rc \sin \angle A \sin \angle B}{2R \sin \angle A \sin \angle B + 2R \sin \angle C} \\
&= \frac{ac \sin \angle B}{a \sin \angle B + c} \\
&= \frac{2Rac \sin \angle B}{2Ra \sin \angle B + 2Rc} \\
&= \frac{abc}{2Rc + ab}
\end{aligned}$$

Since each of the sides of the triangle, a, b and c can be the longest side, the maximum of the three placements can be taken as the solution, hence

$$s_{\max} = \max \left(\frac{abc}{2Rc + ab}, \frac{abc}{2Rb + ac}, \frac{abc}{2Ra + bc} \right) \tag{1}$$

For obtuse triangles, only one placement exists, i.e. when the square lies on the longest side.

$$s = \frac{abc}{2Rc + ab} \tag{2}$$

where c is the longest side.

4 Research Question 2

RQ2 aims to find out the side length of the largest square that can be inscribed in a convex n-gon, given n and the side length of the n-gon, k.

This problem can be further split into four cases.

1. when $n \equiv 0(mod 4)$,
2. when $n \equiv 2(mod 4)$,
3. when $n \equiv 1(mod 4)$,
4. when $n \equiv 3(mod 4)$,

4.1 Case 1

This case deals with the scenario where the number of sides in the n-gon is divisible by 4.

Firstly, we can safely state that for the side length of the square to be maximised, more than 1 vertice of the square must coincide with the perimeter of the polygon. This can be easily seen, as if less than 2 vertices of the square touch the perimeter of the polygon, the square can be pushed outwards in the other direction.

Also, due to such a polygon being symmetrical both horizontally and vertically, we can assume that the centre of the polygon coincides with the centre of the square. From the observation earlier, we assume that one vertice of the square, named A, lies on the perimeter of the polygon.

References

[1] H. Wu and X. C. Huo. "The Bigger The Better". In: (2021). URL: `http://projectsday.hci.edu.sg/2021/05-Report/cat-08/8-02/index.pdf`.