

The Bigger The Better II

Group 8–29

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1 Introduction

This project aims to find a formula to determine the side length of the largest square that can be inscribed inside a convex n -gon. It is a continuation from a previous project completed in 2021, The Bigger The Better. [3]

1.1 Definitions

placement

A valid location, size and rotation of the square such that all vertices of the square lie on the edges of the polygon.

inscribed

All vertices of the square must lie on the edges of the polygon.

fit All vertices of the square must lie within the polygon.

RQ Research Question

1.2 Research Questions

1. What is the side length of the largest square that can be inscribed in a triangle?
2. What is the side length of the largest square that can be inscribed in a regular n -gon, given $n \neq 4$?
3. What is the side length of the largest square that can be inscribed in a convex n -gon?

1.3 Project Scope

This project will only focus on convex polygons. Also, all four vertices of the square must lie on the perimeter of the polygon.

2 Literature Review

[3] Huo Xi Cheng and Wu Hao (2021) investigated how to find the largest square inscribed in a triangle, splitting it up into acute and obtuse triangles. This would be helpful for RQ1, as it provides insight that only 3 possible placements of an inscribed square exist for an acute triangle. [1] Dilworth and Mane (2011) investigated how to inscribe a regular m -gon in a regular n -gon. It showed that it is always possible to inscribe a square in a regular polygon, as well as provide insights on the possible placements of the square. [2] Emch (1916) investigated how to inscribe a square in a closed curve formed by analytic curves. He managed to prove that for all closed curves that could be split into finite numbers of analytic curves, a square could be inscribed. This is helpful as a polygon is a form of a closed curve, and each side is an analytic curve, showing that it is always possible to inscribe a square in a convex polygon.

3 Research Question 1

What is the side length of the largest square that can be inscribed in a triangle?

RQ1 aims to find out the side length of the largest square that can be inscribed in a triangle, given the side lengths of the triangle, a , b and c .

3.1 Key Insights

Some key insights which greatly aided in solving this problem were found.

1. It can be seen that no more than two vertices of a square can lie on a single side, as a square has at most two co-linear vertices.
2. We notice how a triangle has three sides, and a square has four vertices. In order for all the vertices to lie on the triangle, using the Pigeonhole Principle, there will be at least one side with two vertices lying on it.
3. Combining the above insights, there will be one vertex of the square each lying on two sides of the triangle, with the other two vertices of the square lying on the latter side of the triangle.

3.2 Solution

A figure has been constructed for the purposes of illustrating the following proof.

Let s be the side length of the largest square that can be inscribed in a triangle, with side lengths a , b and c , and circumradius R .

Side c can be formed with the sum of s , $s \cot A$ and $s \cot \angle B$. Hence, we can express s with the side length c , as well as angles A and B .

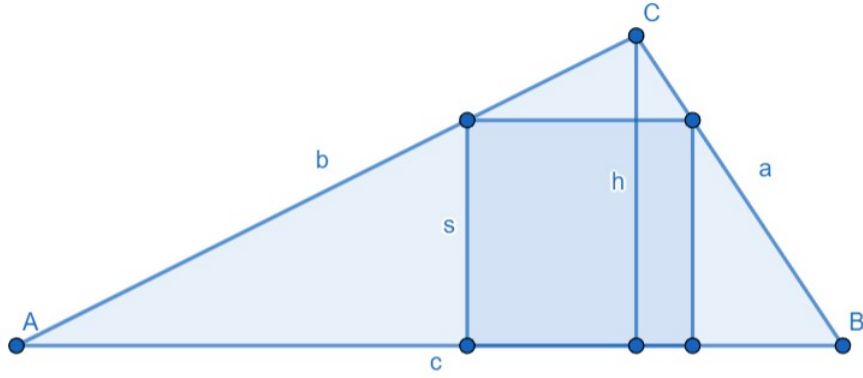


Figure 1: The figure for RQ1.

$$c = s + s \cot \angle A + s \cot \angle B$$

$$s = \frac{c}{1 + \cot \angle A + \cot \angle B}$$

Both sides of the fraction can be multiplied by $\sin A \sin B$. Following which, the Sine Addition Formula can be applied.

$$\begin{aligned} s &= \frac{c \sin \angle A}{\sin \angle A + \cos \angle A + \cot \angle B \sin \angle A} \\ &= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \cos \angle A \sin \angle B + \sin \angle A \cos \angle B} \\ &= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \sin (\angle A + \angle B)} \\ &= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \sin (180 - \angle C)} \\ &= \frac{c \sin \angle A \sin \angle B}{\sin \angle A \sin \angle B + \sin \angle C} \end{aligned}$$

Both sides of the fraction can be multiplied by $2Rc$ and the Law of Sines can be used

to simplify.

$$\begin{aligned}
s &= \frac{2Rc \sin \angle A \sin \angle B}{2R \sin \angle A \sin \angle B + 2R \sin \angle C} \\
&= \frac{ac \sin \angle B}{a \sin \angle B + c} \\
&= \frac{2Rac \sin \angle B}{2Ra \sin \angle B + 2Rc} \\
&= \frac{abc}{2Rc + ab}
\end{aligned}$$

Since each of the sides of the triangle, a , b and c can be the longest side, the maximum of the three placements can be taken as the solution, hence

$$s_{\max} = \max \left(\frac{abc}{2Rc + ab}, \frac{abc}{2Rb + ac}, \frac{abc}{2Ra + bc} \right) \quad (1)$$

For obtuse triangles, only one placement exists, i.e. when the square lies on the longest side.

$$s = \frac{abc}{2Rc + ab} \quad (2)$$

where c is the longest side.

4 Research Question 2

What is the side length of the largest square that can be inscribed in a regular n -gon, given $n \neq 4$?

RQ2 aims to find out the side length of the largest square that can be inscribed in a convex n -gon, given n and the side length of the n -gon, k .

This problem can be further split into four cases.

1. when $n \equiv 0 \pmod{4}$,
2. when $n \equiv 2 \pmod{4}$,
3. when n is odd.

4.1 Case 1

This case deals with the scenario where the number of sides in the n -gon is divisible by 4.

Firstly, for the side length of the square to be maximised, more than one vertex of the square must coincide with the perimeter of the n -gon. This can be easily seen, because the square can be pushed outwards in the other direction if less than two vertices of the square touch the perimeter of the n -gon.

Let the square be $ABCD$, and the polygon be $V_1V_2 \cdots V_n$. Also, let $n = 4m$, where m is an integer.

Due to such a polygon being symmetrical both horizontally and vertically, the assumption that the centre of the polygon coincides with the centre of the square can be made. From the earlier observation, it can be assumed that one vertex of the square, A , lies on the perimeter of the polygon. Also, due to the polygon being symmetrical both horizontally and vertically, the opposite vertex, C , will also lie on the perimeter of the polygon.

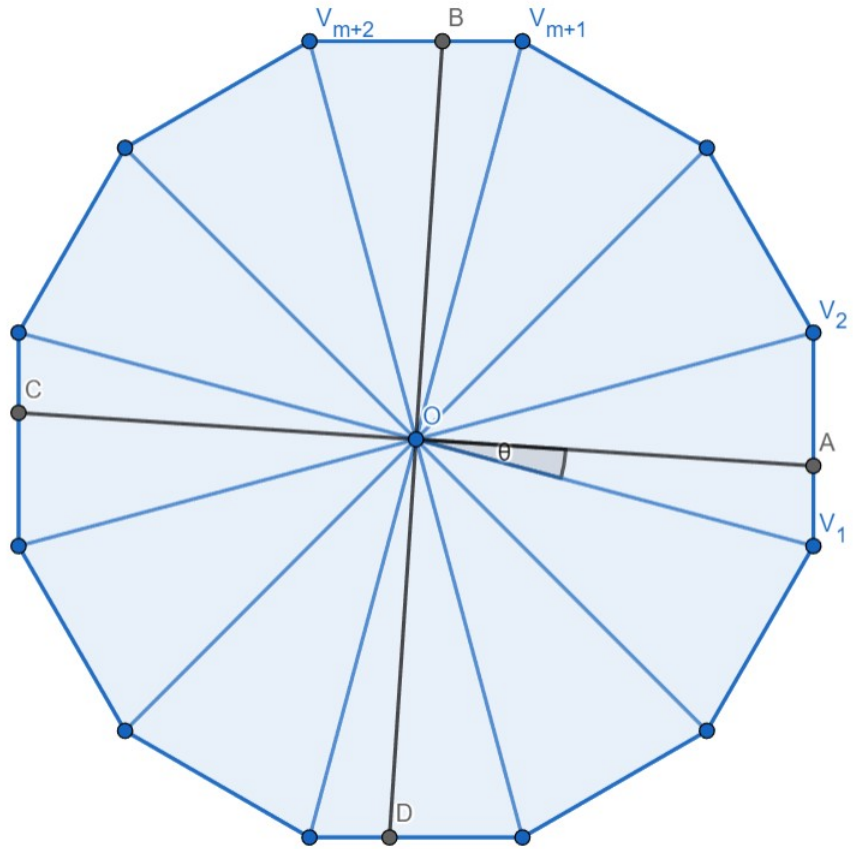


Figure 2: The figure for finding possible placements for Case 1 of RQ2.

Due to the symmetry, as long as vertex B lies on the perimeter of the polygon, the square can be inscribed. If vertex B lies within the polygon, the square can be fit. We shall find the values of θ such that the square can be inscribed or fit, where $\theta = \angle V_1OA$.

Firstly, it can be seen that $V_1OV_2 = V_2OV_3 = \dots = V_nOV_1 = \frac{360^\circ}{n}$. Hence, $0 \leq \theta \leq \frac{360^\circ}{n}$. Furthermore, we can limit this range to $0 \leq \theta \leq \frac{180^\circ}{n}$, as when $\theta > \frac{180^\circ}{n}$, the diagram can be flipped to reduce θ . We can then try to find $\angle V_{m+1}OB$. To do this, we find the slice of the polygon which contains segment OB . We can find the number of triangles that has to be passed through to form $\angle V_1OB$. Let this value be x . This gives the expression:

$$x = \frac{\theta + 90^\circ}{\frac{360^\circ}{n}}$$

Simplifying it,

$$\begin{aligned} x &= \frac{\theta + 90^\circ}{\frac{360^\circ}{n}} \\ &= \frac{\theta n + 90^\circ n}{360^\circ} \\ &= \frac{\theta n}{360^\circ} + \frac{n}{4} \end{aligned}$$

Plugging in the range for θ ,

$$\frac{n}{4} \leq \frac{\theta n}{360^\circ} + \frac{n}{4} \leq \frac{n}{4} + \frac{1}{2}$$

Substituting n for $4m$,

$$m \leq x \leq m + \frac{1}{2}$$

It can be seen that point B is in triangle $V_{m+1}OV_{m+2}$, and is closer to V_{m+1} than V_{m+2} . We can now check for equality between OA and OB , as this would render $ABCD$ as a square. To do so, the equality $\angle V_{m+1}OB = \angle V_1OA$ must be true.

$$\begin{aligned}\angle V_{m+1}OB &= \theta + 90^\circ - m \left(\frac{360^\circ}{n} \right) \\ &= \theta\end{aligned}$$

Hence, a square can always be inscribed.

To find the maximum side length of the square, OA can be maximised. Let $OV_1 = r$. Using the Law of Sines,

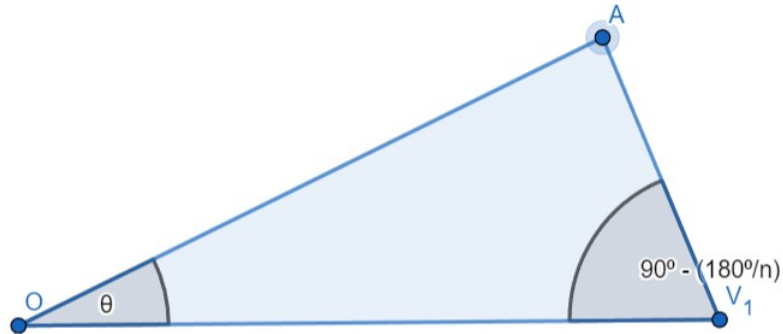


Figure 3: The triangle formed by point A.

$$\begin{aligned}
\angle AV_1O &= 90^\circ - \frac{180^\circ}{n} \\
\frac{OA}{\sin\left(90^\circ - \frac{180^\circ}{n}\right)} &= \frac{r}{\sin\left[180^\circ - \theta - \left(90^\circ - \frac{180^\circ}{n}\right)\right]} \\
\frac{OA}{\cos\frac{180^\circ}{n}} &= \frac{r}{\sin\left(\theta + 90^\circ - \frac{180^\circ}{n}\right)} \\
OA &= \frac{r \cos\frac{180^\circ}{n}}{\cos\left(\frac{180^\circ}{n} - \theta\right)}
\end{aligned}$$

Notice how the numerator is fixed. To maximise OA , $\cos\left(\frac{180^\circ}{n} - \theta\right)$ needs to be minimised. $\frac{180^\circ}{n} - \theta$ needs to be maximised, hence θ must be minimised. This can be done when $\theta = 0$.

The side length of the square can be hence found, s , from r , and by expressing r from the side length of the entire polygon, k , s can be found from k and n .

$$\begin{aligned}
OA &= \frac{r \cos\frac{180^\circ}{n}}{\cos\left(\frac{180^\circ}{n}\right)} \\
&= r \\
2r^2 \left(1 - \cos\frac{360^\circ}{n}\right) &= k^2 \\
r &= k \sqrt{\frac{1}{2 - 2\cos\frac{360^\circ}{n}}} \\
s &= OA\sqrt{2} \\
&= r\sqrt{2} \\
&= k \sqrt{\frac{2}{2 - 2\cos\frac{360^\circ}{n}}} \\
&= k \sqrt{\frac{1}{1 - \cos\frac{360^\circ}{n}}}
\end{aligned}$$

Hence, the side length for a square in such a n -gon would be

$$k\sqrt{\frac{1}{1 - \cos \frac{360^\circ}{n}}} \tag{3}$$

where k is the side length of the n -gon.

4.2 Case 2

This case deals with the scenario where the number of sides in the n -gon is divisible by 2 but not 4.

Similar to Case 1 ($4 \mid n$), more than one vertex of the square must coincide with the perimeter of the n -gon.

Let the square be $ABCD$, and the polygon be $V_1V_2 \cdots V_n$. Also, let $n = 4m + 2$, where m is an integer.

Due to such a polygon being symmetrical both horizontally and vertically, the assumption that the centre of the polygon coincides with the centre of the square can be made. From the earlier observation, it can be assumed that one vertex of the square, A , lies on the perimeter of the polygon. Also, due to the polygon being symmetrical both horizontally and vertically, the opposite vertex, C , will also lie on the perimeter of the polygon.

Similarly, due to the symmetry, as long as vertex B lies on the perimeter of the polygon, the square can be inscribed. If vertex B lies within the polygon, the square can be fit. We shall find the values of θ such that the square can be inscribed or fit, where $\theta = \angle V_1OA$.

We still have the range $0 \leq \theta \leq \frac{180^\circ}{n}$. We can then try to find $\angle V_{m+1}OB$. To do this, we can once again find the slice of the polygon which contains segment OB . We can find the number of triangles that has to be passed through to form $\angle V_1OB$. Let this value be x . This gives the expression:

$$x = \frac{\theta + 90^\circ}{\frac{360^\circ}{n}}$$

Simplifying it,

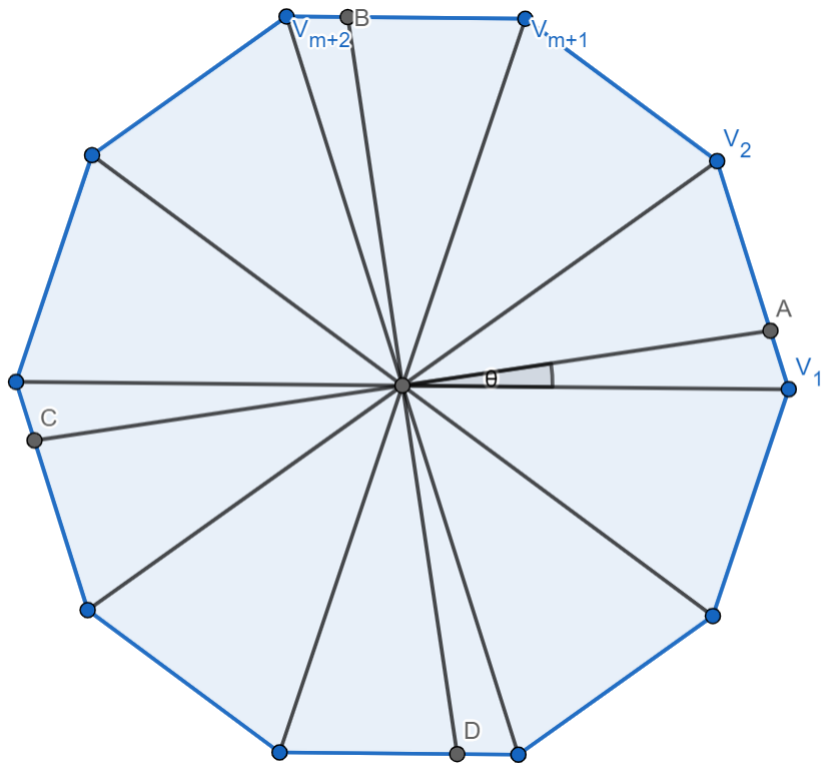


Figure 4: The figure for finding possible placements for Case 2 of RQ2.

$$\begin{aligned}
x &= \frac{\theta + 90^\circ}{\frac{360^\circ}{n}} \\
&= \frac{\theta n + 90^\circ n}{360^\circ} \\
&= \frac{\theta n}{360^\circ} + \frac{n}{4}
\end{aligned}$$

Plugging in the range for θ ,

$$\frac{n}{4} \leq \frac{\theta n}{360^\circ} + \frac{n}{4} \leq \frac{n}{4} + \frac{1}{2}$$

Substituting n for $4m + 2$,

$$m + \frac{1}{2} \leq x \leq m + 1$$

It can be seen that point B is in $\triangle V_{m+1}OV_{m+2}$, and is closer to V_{m+2} than V_{m+1} . We can now check for equality between OA and OB , as this would render $ABCD$ as a square. To do so, the equality $\angle V_{m+2}OB = \angle V_1OA$ must be true.

$$\begin{aligned}
\angle V_{m+2}OB &= (m + 1) \left(\frac{360^\circ}{n} \right) - 90^\circ - \theta \\
2\theta &= \frac{360^\circ (m + 1)}{4m + 2} - 90^\circ \\
2\theta &= \frac{180^\circ}{4m + 2} \\
\theta &= \frac{90^\circ}{n}
\end{aligned}$$

Hence, a square can be inscribed when $\theta = \frac{90^\circ}{n}$.

To find the side length of the square, we can use the Law of Sines, similar to Case 1.

We have:

$$\begin{aligned}
 OA &= \frac{r \cos \frac{180^\circ}{n}}{\cos \left(\frac{180^\circ}{n} - \frac{90^\circ}{n} \right)} \\
 &= \frac{r \cos \frac{180^\circ}{n}}{\cos \frac{90^\circ}{n}} \\
 2r^2 \left(1 - \cos \frac{360^\circ}{n} \right) &= k^2 \\
 r &= k \sqrt{\frac{1}{2 - 2 \cos \frac{360^\circ}{n}}} \\
 s &= OA \sqrt{2} \\
 &= \frac{r \sqrt{2} \cos \frac{180^\circ}{n}}{\cos \frac{90^\circ}{n}} \\
 &= \frac{k \sqrt{\frac{1}{2 - 2 \cos \frac{360^\circ}{n}}} \sqrt{2} \cos \frac{180^\circ}{n}}{\cos \frac{90^\circ}{n}} \\
 &= \frac{k \cos \frac{180^\circ}{n}}{\cos \frac{90^\circ}{n}} \sqrt{\frac{1}{1 - \cos \frac{360^\circ}{n}}}
 \end{aligned}$$

Hence, the expression for the side length of the largest square in such a polygon would be

$$\frac{k \cos \frac{180^\circ}{n}}{\cos \frac{90^\circ}{n}} \sqrt{\frac{1}{1 - \cos \frac{360^\circ}{n}}} \tag{4}$$

where k is the side length of the n -gon.

4.3 Case 3

Unlike the previous two cases, an n -gon with an odd n does not have 2 perpendicular lines of symmetry, hence we cannot claim that the center of the square and the center of the polygon coincide.

Currently, we have no way to assume that the square's vertices lie on the perimeter of the n -gon. This case is still largely a work in progress, and the results obtained so far are inaccurate. However, we have obtained some insights which could aid us in solving this problem.

- The center of the square cannot coincide with the center of the n -gon.
- One side of the square must be parallel to one side of the n -gon.

The first of these insights has a simple proof: there are no two parallel lines of symmetry for such a regular polygon, hence if the center of the square were to coincide with the center of the polygon, at least two vertices of the square would not lie on the perimeter of the polygon.

The second of these insights will be explained in further detail in RQ3, below. For now, we shall take it that this statement is true. We can hence calculate the side length using a similar concept to the previous two cases. Furthermore, we also have to account for when θ lies on the left of the triangle instead of the right.

$$\begin{aligned}\theta &= \min \left[45^\circ - \frac{360^\circ}{n} \left\lfloor \frac{45^\circ}{\frac{360^\circ}{n}} \right\rfloor, \frac{360^\circ}{n} - \left(45^\circ - \frac{360^\circ}{n} \left\lfloor \frac{45^\circ}{\frac{360^\circ}{n}} \right\rfloor \right) \right] \\ &= \min \left[45^\circ - \frac{360^\circ}{n} \left\lfloor \frac{n}{8} \right\rfloor, \frac{360^\circ}{n} - \left(45^\circ - \frac{360^\circ}{n} \left\lfloor \frac{n}{8} \right\rfloor \right) \right]\end{aligned}$$

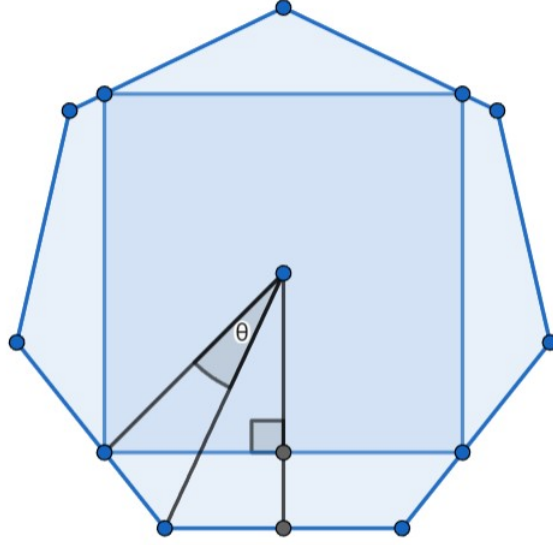


Figure 5: The construction for case 3 of RQ2.

$$OA = \frac{r \cos \frac{180^\circ}{n}}{\cos \left\langle \frac{180^\circ}{n} - \left(\min \left[45^\circ - \frac{360^\circ}{n} \left\lfloor \frac{n}{8} \right\rfloor, \frac{360^\circ}{n} - \left(45^\circ - \frac{360^\circ}{n} \left\lfloor \frac{n}{8} \right\rfloor \right) \right] \right) \right\rangle}$$

$$k^2 = 2r^2 \left(1 - \cos \frac{360^\circ}{n} \right)$$

$$r = k \sqrt{\frac{1}{2 - 2 \cos \frac{360^\circ}{n}}}$$

$$\begin{aligned}
s &= OA\sqrt{2} \\
&= \frac{r\sqrt{2} \cos \frac{180^\circ}{n}}{\cos \left\langle \frac{180^\circ}{n} - \left(\min \left[45^\circ - \frac{360^\circ}{n} \left\lfloor \frac{n}{8} \right\rfloor, \frac{360^\circ}{n} - \left(45^\circ - \frac{360^\circ}{n} \left\lfloor \frac{n}{8} \right\rfloor \right) \right] \right) \right\rangle} \\
&= \frac{k \sqrt{\frac{1}{2-2\cos \frac{360^\circ}{n}}} \sqrt{2} \cos \frac{180^\circ}{n}}{\cos \left\langle \frac{180^\circ}{n} - \left(\min \left[45^\circ - \frac{360^\circ}{n} \left\lfloor \frac{n}{8} \right\rfloor, \frac{360^\circ}{n} - \left(45^\circ - \frac{360^\circ}{n} \left\lfloor \frac{n}{8} \right\rfloor \right) \right] \right) \right\rangle} \\
&= \frac{k \cos \frac{180^\circ}{n}}{\cos \left\langle \frac{180^\circ}{n} - \left(\min \left[45^\circ - \frac{360^\circ}{n} \left\lfloor \frac{n}{8} \right\rfloor, \frac{360^\circ}{n} - \left(45^\circ - \frac{360^\circ}{n} \left\lfloor \frac{n}{8} \right\rfloor \right) \right] \right) \right\rangle} \sqrt{\frac{1}{1 - \cos \frac{360^\circ}{n}}}
\end{aligned}$$

5 Research Question 3

What is the side length of the largest square that can be inscribed in a convex n -gon?

RQ3 aims to find the largest square that can be inscribed in a convex polygon, given the side lengths of the polygon. We were unable to find this due to the vagueness of the question. However, we are still able to prove certain things.

5.1 Proof that a square can always be inscribed

We have proved that a square can always be inscribed in a convex polygon.

We first construct an arbitrary line A_1A_2 .

Then, at every vertex, we draw a line parallel to A_1A_2 .

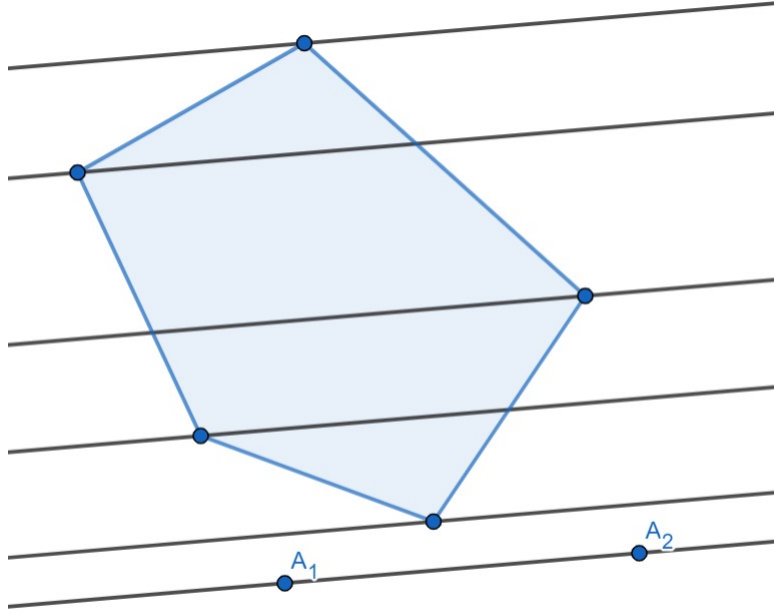


Figure 6: The construction for RQ3.

We can get the midpoint of the newly formed line segments in the polygon, and connect them together.

We shall call the line segments that connect the midpoints together the **first group** of line segments, as shown in Figure 7.

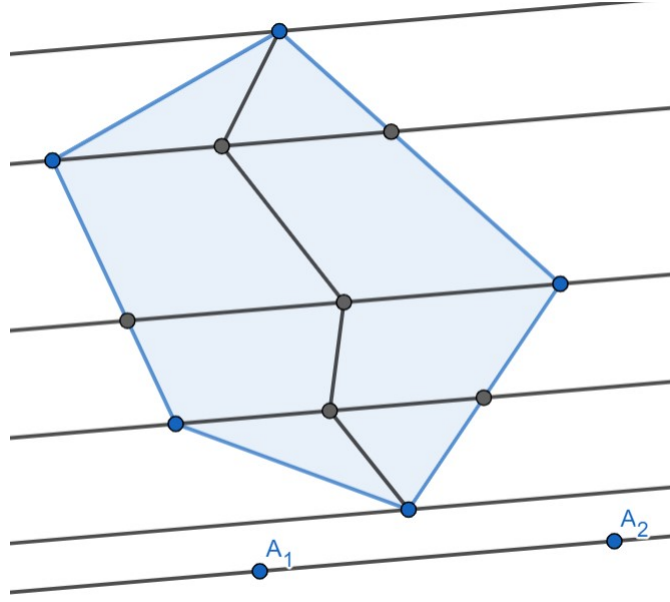


Figure 7: The construction of the first group of line segments.

Now, we draw a line perpendicular to A_1A_2 at every vertex, get the midpoint of the newly formed line segments, and connect them together.

We shall call the line segments that connect the midpoints together the **second group** of line segments, as shown in Figure 8.

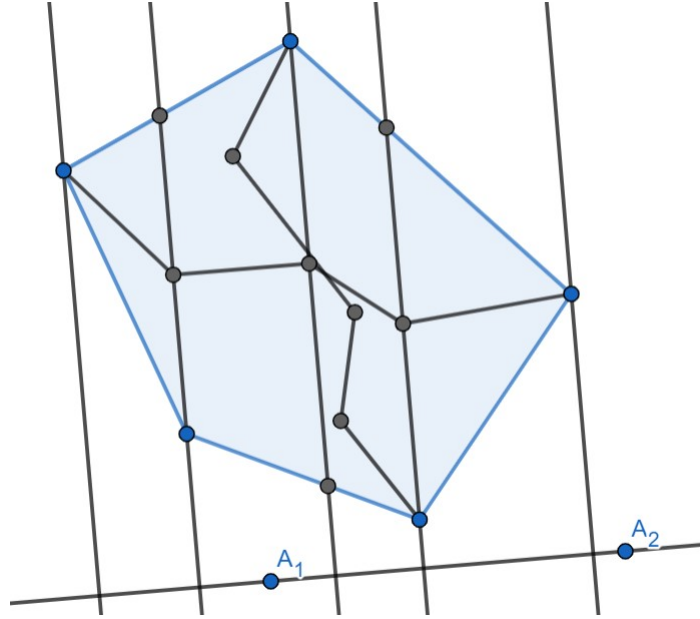


Figure 8: The construction of the second group of line segments.

Notice how the two groups intersect at one point. Let this point be O . We draw a line parallel to A_1A_2 at point O . We then draw another line perpendicular to A_1A_2 at O .

Notice how the four intersections between these two lines and the perimeter of the polygon form a rhombus. Let this rhombus be $ABCD$. This is illustrated in Figure 9.

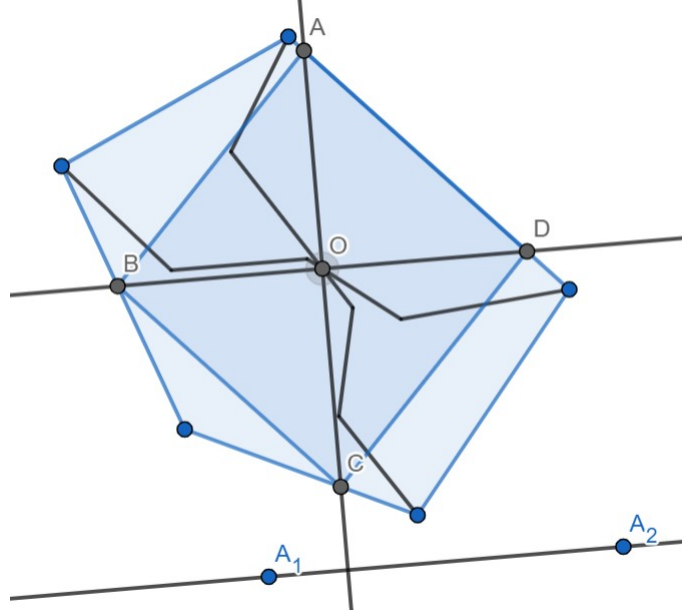


Figure 9: The construction of the rhombus.

Do note that this rhombus need not be a square. However, using the Intermediate Value Theorem, we can show that if A_1A_2 is rotated clockwise enough, the rhombus will be a square.

5.2 Proof for Case 3 of RQ2

In RQ2, we mentioned that one side of the square had to be parallel to one side of the polygon. This can be proven using the concept presented above: We know that a construction where one side of the square is parallel to one side of the polygon exists. In this case, A_1A_2 is 45° away from the base of the polygon. It can be seen that by making the angle between A_1A_2 and the base of the polygon smaller, 2 of the angles of the rhombus will increase, and 2 will decrease and vice versa. Hence, the only occurrence where the rhombus is a square and all four angles of the rhombus are equal is when A_1A_2 is 45° away from the base of the polygon.

6 Conclusion and Further Work

All in all, we were able to solve most of our research questions. However, as RQ3 was very vague, it was more difficult than the others, leading to us being unable to complete it. One potential continuation of the project is that, besides completing RQ3, we can also address concave polygons, or even curves. Such a continuation would be able to address things such as the inscribed square problem. Other possible extension methods would be to extend this problem to 3-dimensional space, for example, by fitting cubes into prisms with different n -gons as bases.

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