Problem Of The Day 2022

1. **(27 Jun)** If y varies inversely as x and can be represented by the equation $y = (m-1)x^{m^2-2}$, find the value of constant m.

Solution:

$$y = (m-1)x^{m^2-2} = \frac{k}{x}$$

$$k = (m-1)x^{m^2-1}$$

$$= (m-1)x^{(m+1)(m-1)} \quad (x \neq 0)$$

By definition, $y \neq 0$ as well, hence

$$(m-1)x^{(m+1)(m-1)} \neq 0$$

$$\therefore m \neq 1$$

2. **(28 Jun)** Which of the following is a possible plot of $y = x + m$ and $y = \frac{m}{x}$ on the same axes? (The graphs are not drawn to scale.)

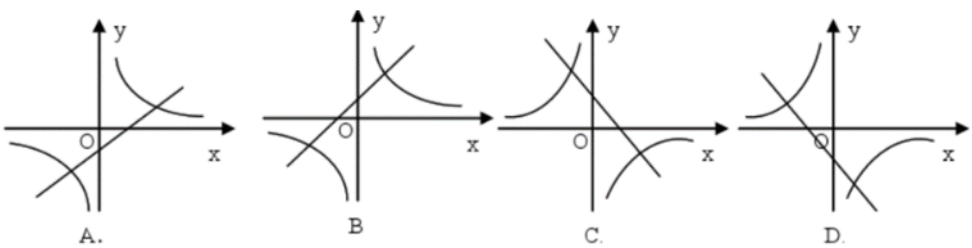

Figure 1: $y = x + m$ and $y = \frac{m}{x}$.

Solution: B.

- The straight line should be increasing, since the coefficient of x is positive. **C** and **D** are eliminated.
- If $m > 0$, the y-intercept of the straight line could not be negative. **A** is eliminated, since the hyperbola in the same graph shows that $m > 0$.

3. (**29 Jun**) Given that points $A\left(-2, y_1\right)$, $B\left(-1, y_2\right)$, $C\left(1, y_3\right)$ are all on the graph of $y = -\frac{1}{x}$, arrange y_1, y_2 and y_3 in ascending order.

Solution: $y_3 < y_1 < y_2$.

Subst. $x = -2$ into $y = -\frac{1}{x}$:

$$\begin{aligned} y_1 &= -\frac{1}{-2} \\ &= \frac{1}{2} \end{aligned}$$

Subst. $x = -1$ into $y = -\frac{1}{x}$:

$$\begin{aligned} y_2 &= -\frac{1}{-1} \\ &= 1 \end{aligned}$$

Subst $x = 1$ into $y = -\frac{1}{x}$:

$$\begin{aligned} y_3 &= -\frac{1}{1} \\ &= -1 \end{aligned}$$

4. (**30 Jun**) Given that points $\left(x_1, y_1\right)$, $\left(x_2, y_2\right)$ and $\left(x_3, y_3\right)$ are all on the graph of $y = \frac{3}{x}$, also $x_1 < x_2 < 0 < x_3$, arrange y_1, y_2 and y_3 in ascending order.

Solution: $y_2 < y_1 < y_3$.

- $y_3 > 0$ since $x_3 > 0$. Hence, y_3 is the greatest.
- $0 > x_2 > x_1$, hence $y_2 < y_1 < 0$.

5. **(1 Jul)** Given that y varies inversely as x such that $y = (a-2)x^{a^2-5}$, also when $x > 0$, as x increases, y increases. Find the equation of the hyperbola.

Solution:

$$\frac{k}{x} = (a-2)x^{a^2-5}$$

$$k = (a-2)x^{a^2-4}$$

$$= (a-2)x^{(a+2)(a-2)}$$

$$\because y = \frac{(a-2)x^{(a+2)(a-2)}}{x}$$

$$\therefore (a-2)x^{(a+2)(a-2)} < 0$$

$$\because a-2 < 0 \text{ and } (a+2)(a-2) \geq 0$$

$$\therefore a+2 = 0$$

$$\therefore a = -2$$

$$\therefore y = -\frac{(-2-2) \times x^{(-2+2)\times(-2-2)}}{x}$$

$$= -\frac{4}{x}$$

6. **(5 Jul)** If a straight line $y = (2m-1)x$ and a hyperbola $y = \frac{3-m}{x}$ has an intersection point each in Quadrant 1 and 3, what is the range of values of the constant m?

Solution:

$$\because y = \frac{3-m}{x}$$

$$\therefore m < 3$$

$$\because y = x(2m-1)$$

$$\therefore 2m - 1 > 0$$

$$\therefore 0.5 < m < 3$$

7. **(6 Jul)** Points A and B are on the hyperbola $y = \frac{k}{x}$. Right $\triangle AOC$ and $\triangle BOD$ are drawn by perpendicular lines drawn from the two points and connecting the points with the origin. Compare the sizes of the areas of the two triangles.

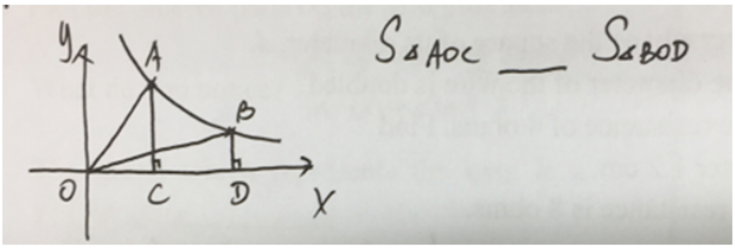

Figure 2: $\triangle AOC$ and $\triangle BOD$.

Solution: $S_{\triangle AOC} < S_{\triangle BOD}$. As Point B's y-coordinate approaches 0, its x-coordinate approaches infinity, resulting in a triangle which approaches an infinite area. Point A is further from $(\infty, 0)$ than Point B, so $S_{\triangle AOC} < S_{\triangle BOD}$.

8. **(7 Jul)** Points A and B are on the hyperbola $y = \frac{3}{x}$. Rectangles are drawn from the two points as shown. The areas bounded by the two rectangles are labelled S_1 and S_2. If the shaded area is 1 sq. unit, find the sum $S_1 + S_2$.

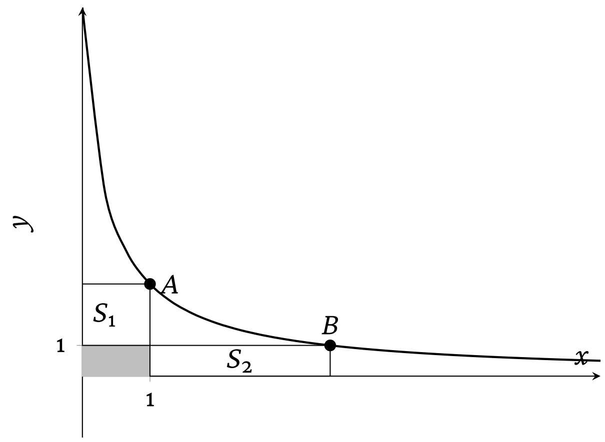

Figure 3: The shaded areas, S_1 and S_2.

Solution:

$$S_1 = (1-0) \times \left(\frac{3}{1} - 1\right)$$
$$= 2 \text{ sq. units}$$
$$S_2 = \left(\frac{3}{3} - 0\right) \times (3-1)$$
$$= 2 \text{ sq. units}$$
$$S_1 + S_2 = 2 + 2$$
$$= 4 \text{ sq. units}$$

9. **(8 Jul)** If a hyperbola $y = -\dfrac{3m}{x}$ and a straight line $y = kx - 1$ both pass through the point $P(m, -3m)$,

(a) find the coordinates of P and the equations of the hyperbola and the straight line.

Solution:

$$y = -\frac{3m}{x} \tag{1}$$
$$y = kx - 1 \tag{2}$$

Subst. $x = m, y = -3m$ into (1):

$$-3m = -\frac{3m}{m}$$

$$\therefore m = 1$$

$$\therefore P(1, -3) \qquad (3)$$

We can substitute the values obtained in (3) into (2):

$$-3 = k - 1$$

$$\therefore k = -2$$

The equations of the hyperbola and the straight line, are, thus:

$$y = -\frac{3}{x}$$

$$y = -2x - 1$$

(b) If the points $M(a, y_1)$ and $N(a+1, y_2)$ are both on the straight line, explain clearly why $y_1 > y_2$.

Solution: Substitute the x- and y-coordinates of both points into the equation of the line.

$$y_1 = -2a - 1$$

$$y_2 = -2(a+1) - 1$$

$$= -2a - 3$$

$$\because -2a - 1 > -2a - 3, \text{ where } a \in \mathbb{R}$$

$$\therefore y_1 > y_2$$

10. (**12 Jul**) The line $y = x$ meets the hyperbola $y = \frac{1}{x}$ at points A and C. Vertical lines from A and C meet the x-axis at points B and D respectively. Find the area of the quadrilateral $ABCD$. (The diagram is not drawn to scale.)

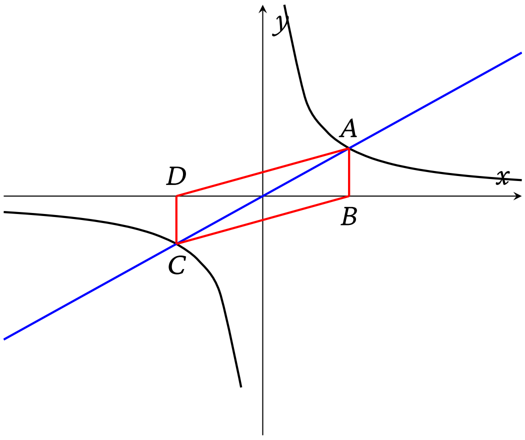

Figure 4: Quadrilateral $ABCD$.

Solution:

$$x = \frac{1}{x}$$

$$\therefore x = \pm 1$$

$$\therefore A(1, 1) \text{ and } C(-1, -1)$$

$$S_{ABCD} = 1 \times [1 - (-1)]$$

$$= 2 \text{ sq. units}$$

11. (**13 Jul**) A ladder AB of length 2.5 m has its foot B 1.5 m away from a wall. The ladder is then moved to a new position ED. The foot of the ladder is moved 0.5 m from the original position B. Find the distance the top of the ladder drops, the length of AE.

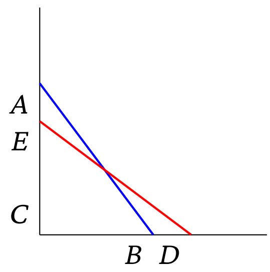

Figure 5: The ladder, before and after.

Solution:

$$\begin{aligned} AC &= \sqrt{2.5^2 - 1.5^2} \\ &= 2\,\text{m} \\ EC &= \sqrt{2.5^2 - (1.5 + 0.5)^2} \\ &= 1.5\,\text{m} \\ \text{height dropped} &= 2 - 1.5 \\ &= 0.5\,\text{m} \end{aligned}$$

12. **(14 Jul)** In $\triangle ABC$, $\angle B = 22.5^\circ$. The perpendicular bisector of AB intersects BC at point D and $BD^2 = 72$. $AE \perp BC$. Find AE.

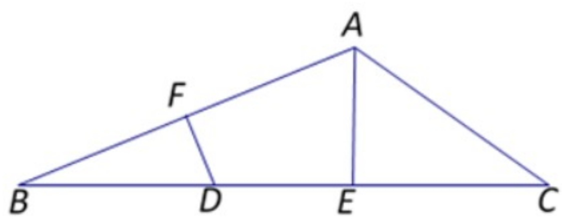

Figure 6: Triangle $\triangle ABC$.

Solution:

$$
\begin{aligned}
\angle FAD &= \angle B \\
&= 22.5^\circ \\
BD &= AD \\
&= \sqrt{72} \\
\angle FDA &= 90 - 22.5 \\
&= 67.5^\circ \\
\angle FDB &= \angle FDA = 67.5^\circ \\
\therefore \angle ADE &= 180 - 67.5 \times 2 \\
&= 45^\circ \\
\sin \angle ADE &= \frac{AE}{\sqrt{72}} \\
AE &= \sqrt{72} \times \sin 45^\circ \\
&= \frac{\sqrt{72}}{\sqrt{2}} \\
&= 6
\end{aligned}
$$

13. (**15 Jul**) In $\triangle ABC$, $\angle A = 90^\circ$. The point P is the midpoint of AC. $PD \perp BC$, $BC = 9$ and $DC = 3$. Find AB.

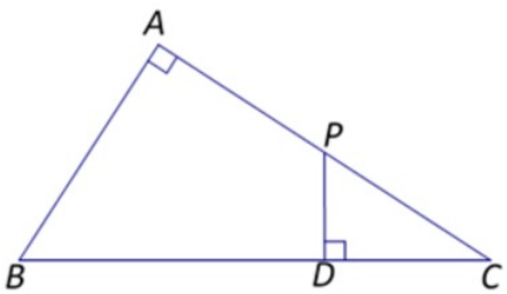

Figure 7: Triangle $\triangle ABC$.

Solution:

$$\sqrt{3^2 + PD^2} = \sqrt{6^2 + PD^2 - AB^2}$$

$$PD^2 + 9 = PD^2 + 36 - AB^2$$

$$27 - AB^2 = 0$$

$$\therefore AB = \sqrt{27}$$

14. (**18 Jul**) In $\triangle ABC$, $AB = 7$, $BC = 6$, $AC = 4$, and AD and AE are the height and the median on the side BC, such that $BE = EC$ and $AD \perp BC$. Find the length of DE.

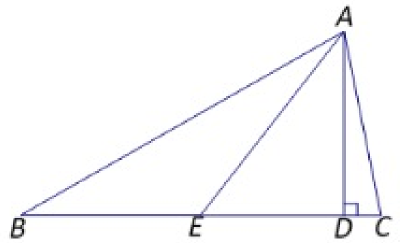

Figure 8: $\triangle ABC$.

Solution:

$$3^2 + DE^2 = 7^2$$

$$DE^2 = 7^2 - 3^2$$

$$= 40$$

$$DE = \sqrt{40}$$

15. (**19 Jul**) The lengths of three sides of a triangle are $m^2 - n^2$, $m^2 + n^2$ and $2mn$, where m and n are positive integers and $m > n$. Determine if this triangle is a right triangle or not. Show your reason clearly.

Solution: Yes. Assuming that the triangle **is** a right triangle, $m^2 + n^2$ would be the hypotenuse.

$$(2mn)^2 + \left(m^2 - n^2\right)^2 = \left(m^2 + n^2\right)^2$$

$$4m^2n^2 + m^4 - 2m^2n^2 + n^4 = m^4 + 2m^2n^2 + n^4 \text{ (true, proven.)}$$

16. (**20 Jul**) The lengths of the three sides of a triangle satisfy $a^2c^2 - b^2c^2 = a^4 - b^4$. Determine the type of triangle it is. Show your reason clearly.

Solution: It is a right triangle.

$$a^2c^2 - b^2c^2 = a^4 - b^4$$

$$c^2(a-b)(a+b) = \left(a^2 + b^2\right)(a-b)(a+b)$$

$$c^2 = a^2 + b^2$$

17. (**21 Jul**) A corner of a rectangle $ABCD$ is folded along the line AE, such that the vertex D lands exactly on the opposite side BC at point F. If $AB = 8$ and $BC = 10$, find the length of EC.

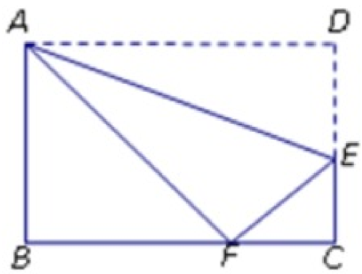

Figure 9: Rectangle $ABCD$.

Solution:

$$BC = AD = AF = 10$$

$$BF = \sqrt{10^2 - 8^2}$$

$$= 6$$

$$\therefore FC = 10 - 6$$

$$= 4$$

$$\angle BAF = \angle EFC \text{ (alt. ext. angles)}$$

$$\angle ECF = \angle ABF \text{ (given)}$$

$$\therefore \triangle ABF \simeq \triangle FCE \text{ (AA)}$$

$$\therefore \frac{EC}{AB} = \frac{CF}{BF}$$

$$\frac{EC}{8} = \frac{4}{6}$$

$$\therefore EC = \frac{16}{3}$$

18. (**22 Jul**) In an isosceles $\triangle ABC$, $AB = AC$. P is a random point on BC. Show that $AB^2 - AP^2 = PB \times PC$.

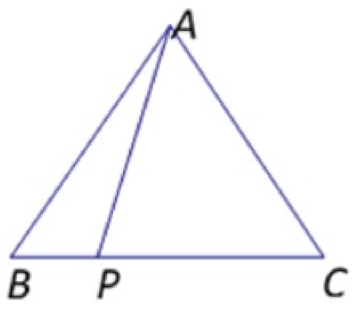

Figure 10: Isosceles $\triangle ABC$.

Solution: We can construct a line $AO \perp BC$, thereby bisecting BC.

$$\begin{aligned} AB^2 - AP^2 &= AO^2 + OB^2 - \left(AO^2 + PO^2\right) \\ &= OB^2 - PO^2 \\ &= (OB + PO)(OB - PO) \\ &= PC \times PB \\ &= PB \times PC \end{aligned}$$

19. (**25 Jul**) In $\triangle ABC$, $\angle A = 30^\circ$ and $\angle B = 45^\circ$. Points D and E are on sides AB and AC respectively, such that $AE = ED = EC$. Find the ratio $\dfrac{AD^2}{BC^2}$.

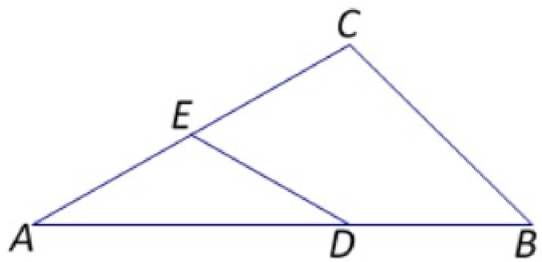

Figure 11: $\triangle ABC$.

20. (**26 Jul**) In $\triangle ABC$, a, b and c are the three sides. $\angle C = 90^\circ$, $\angle A = 60^\circ$, $a + b = 3 + \sqrt{3}$. Find a, b and c.

Solution: By using trigonometric ratios,

$$AB : AC : BC = 2 : 1 : \sqrt{3}$$

Multiply the ratio by $\sqrt{3}$:

$$AB : AC : BC = 2\sqrt{3} : \sqrt{3} : 3$$

$$\because a + b = 3 + \sqrt{3}$$

$$\therefore BC + AC = a + b$$

$$\therefore c = AB = 2\sqrt{3}$$

However, we cannot determine a and b because of the commutative law of addition.

$$\begin{cases} a & = 3 \text{ if } b = \sqrt{3} \text{ else } \sqrt{3} \\ b & = 3 \text{ if } a = \sqrt{3} \text{ else } \sqrt{3} \\ c & = 2\sqrt{3} \end{cases}$$

21. (**27 Jul**) In $\triangle ABC$, $\angle ACB = 90^\circ$, $AB \perp CD$, $AC = \sqrt{5}$, $BC = 2$. Find $\sin \angle ACD$.

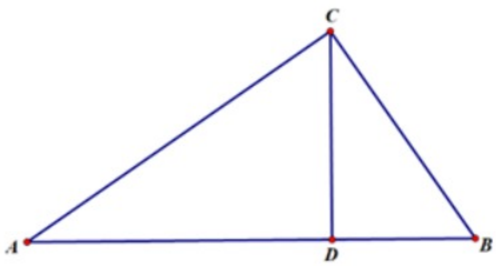

Figure 12: $\triangle ABC$.

Solution:

$$\angle CAB = \arctan \frac{2\sqrt{5}}{5}$$
$$\angle ACD = \left(90 - \arctan \frac{2\sqrt{5}}{5}\right)^{\circ}$$
$$\therefore \sin \angle ACD = \sin\left(90 - \arctan \frac{2\sqrt{5}}{5}\right)$$
$$= \frac{2}{3}$$

22. (**28 Jul**) For acute angle α, fill in the blanks with >, =, or <.

Solution:

$$\begin{cases} \because \alpha = 45^{\circ} \quad \therefore \sin\alpha = \cos\alpha \\ \because \alpha < 45^{\circ} \quad \therefore \sin\alpha < \cos\alpha \\ \because \alpha > 45^{\circ} \quad \therefore \sin\alpha > \cos\alpha \end{cases}$$