Jiyoon Park jiyoonp

Fall 2022 16-642 MEC Problem Set 1

1 $\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 7 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$

a) So, inorder to check the system's stability, we have to find the eigen values of A. In this case,

$\gg \text{eig}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 7 \end{bmatrix}\right)$ which returns
7.6690
$-0.3345 + 0.1361i$
$-0.3345 - 0.1361i$
in matlab.

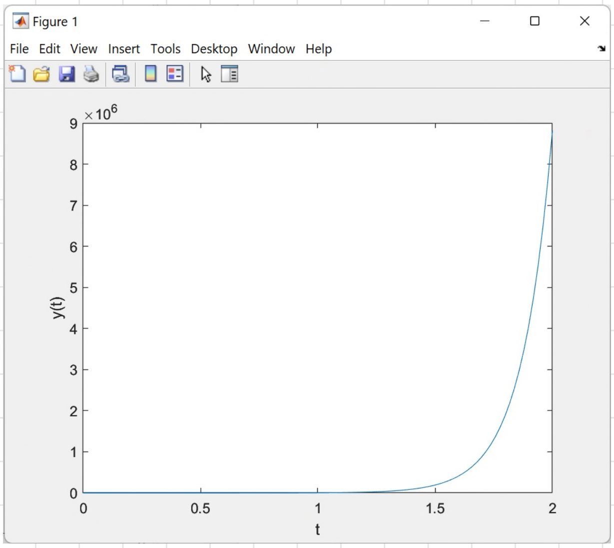
Since there is an eigenvalue greater than 0, the system is unstable.

b) To see if the system is controllable, we have to check if $[B | AB | \cdots A^{n-1}B]$'s rank is n.

$\left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \middle| \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \middle| \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 7 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right] \xrightarrow{\text{(use matlab)}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 7 \end{bmatrix} \xrightarrow[\gg \text{rank}]{\text{(use matlab)}} 3$. Since rank is 3 (=n),

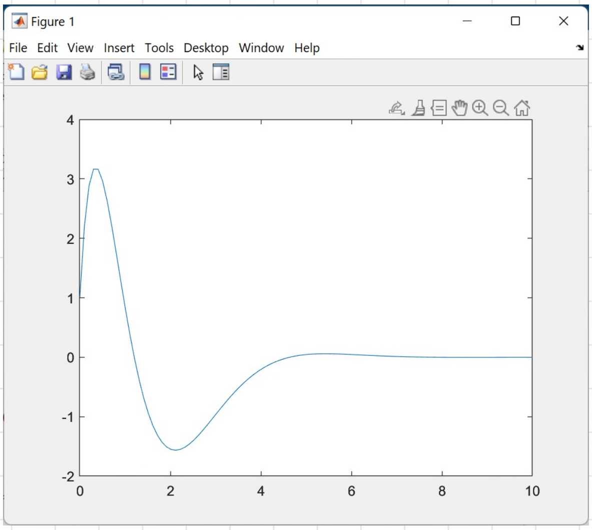
the system is controllable.

c)



d) $K = [11 \quad 60 \quad 88]$

e)



2)

a) $\gamma\ddot{x}_c - \beta\ddot{\theta}\cos\theta + \beta\dot{\theta}^2\sin\theta + \mu\dot{x}_c = F$

$\alpha\ddot{\theta} - \beta\ddot{x}_c\cos\theta - D\sin\theta = 0$

I will put the equation above in a simple form using $M = \begin{bmatrix} \gamma & -\beta\cos\theta \\ -\beta\cos\theta & \alpha \end{bmatrix}$

$$\begin{bmatrix} \gamma & -\beta\cos\theta \\ -\beta\cos\theta & \alpha \end{bmatrix}\begin{bmatrix} \ddot{x}_c \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} \beta\dot{\theta}^2\sin\theta + \mu\dot{x}_c \\ -D\sin\theta \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

since in the question, it says M is invertible,

$$\begin{bmatrix} \ddot{x}_c \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\cos\theta \\ -\beta\cos\theta & \alpha \end{bmatrix}^{-1}\left(\begin{bmatrix} F \\ 0 \end{bmatrix} - \begin{bmatrix} \beta\dot{\theta}^2\sin\theta + \mu\dot{x}_c \\ -D\sin\theta \end{bmatrix}\right)$$

$$= \frac{1}{\gamma\alpha - \beta^2(\cos\theta)^2}\begin{bmatrix} \alpha & \beta\cos\theta \\ \beta\cos\theta & \gamma \end{bmatrix}\begin{bmatrix} F - \beta\dot{\theta}^2\sin\theta - \mu\dot{x}_c \\ D\sin\theta \end{bmatrix}$$

$$= \frac{1}{\gamma\alpha - \beta^2(\cos\theta)^2}\begin{bmatrix} F\alpha - \alpha\beta\dot{\theta}^2\sin\theta - \alpha\mu\dot{x}_c + \beta D\cos\theta\sin\theta \\ F\beta\cos\theta - \beta^2\dot{\theta}^2\cos\theta\sin\theta - \beta\mu\dot{x}_c\cos\theta + D\gamma\sin\theta \end{bmatrix}$$

let $x = \begin{bmatrix} x_c \\ \theta \\ \dot{x}_c \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

so, $$\dot{x} = \begin{bmatrix} \dot{x}_c \\ \dot{\theta} \\ \ddot{x}_c \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x}_c \\ \dot{\theta} \\ \dfrac{F\alpha - \alpha\beta\dot{\theta}^2\sin\theta - \alpha\mu\dot{x}_c + \beta D\cos\theta\sin\theta}{\gamma\alpha - \beta^2(\cos\theta)^2} \\ \dfrac{F\beta\cos\theta - \beta^2\dot{\theta}^2\cos\theta\sin\theta - \beta\mu\dot{x}_c\cos\theta + D\gamma\sin\theta}{\gamma\alpha - \beta^2(\cos\theta)^2} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \dfrac{u\alpha - \alpha\beta x_4^2\sin x_2 - \alpha\mu x_3 + \beta D\cos x_2\sin x_2}{\gamma\alpha - \beta^2(\cos x_2)^2} \\ \dfrac{u\beta\cos x_2 - \beta^2 x_4^2\cos x_2\sin x_2 - \beta\mu x_3\cos x_2 + D\gamma\sin x_2}{\gamma\alpha - \beta^2(\cos x_2)^2} \end{bmatrix}$$

b) To get the equilibrium points, we have to solve for $\dot{x} = 0$ when $F(=u) = 0$

$$\dot{x} = \begin{bmatrix} \dot{x}_c \\ \dot{\theta} \\ \dfrac{F\alpha - \alpha\beta\dot{\theta}^2\sin\theta - \alpha\mu\dot{x}_c + \beta D\cos\theta\sin\theta}{\gamma\alpha - \beta^2(\cos\theta)^2} \\ \dfrac{F\beta\cos\theta - \beta^2\dot{\theta}^2\cos\theta\sin\theta - \beta\mu\dot{x}_c\cos\theta + D\gamma\sin\theta}{\gamma\alpha - \beta^2(\cos\theta)^2} \end{bmatrix} = 0$$

naturally, $\dot{x}_c = \dot{\theta} = 0$ and

$$\frac{F\alpha - \alpha\beta\dot{\theta}^2\sin\theta - \alpha\mu\dot{x}_c + \beta D\cos\theta\sin\theta}{\gamma\alpha - \beta^2(\cos\theta)^2} \Rightarrow \frac{\beta D\cos\theta\sin\theta}{\gamma\alpha - \beta^2(\cos\theta)^2} = 0$$

$$\frac{F\beta\cos\theta - \beta^2\dot{\theta}^2\cos\theta\sin\theta - \beta\mu\dot{x}_c\cos\theta + D\gamma\sin\theta}{\gamma\alpha - \beta^2(\cos\theta)^2} \Rightarrow \frac{D\gamma\sin\theta}{\gamma\alpha - \beta^2(\cos\theta)^2} = 0$$

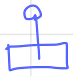
$$\frac{\beta D\cos\theta\sin\theta}{\gamma\alpha - \beta^2(\cos\theta)^2} = 0 \Rightarrow \cos\theta\sin\theta = 0 \qquad \theta = \frac{\pi}{2}n \quad (n = 0, 1, \cdots)$$

$$\frac{D\gamma\sin\theta}{\gamma\alpha - \beta^2(\cos\theta)^2} = 0 \Rightarrow \sin\theta = 0 \qquad \theta = \pi n \quad (n = 0, 1, \cdots)$$

Thus, $\theta = \pi n$ $(n = 0, 1, \cdots)$ and no conditions for x_c is the equilibrium points.

$\therefore$ equilibrium points $\begin{bmatrix} x_c \\ \pi n \\ 0 \\ 0 \end{bmatrix}$ for $x_c \in \mathbb{R}$, $n = 0, 1, \cdots$

In english, this means that the system has equilibrium points when the pendulum is straight down (πn for n is odd) or straight up (πn for n is even). At this point, the change rate will be zero and the system will remain the same (unchanged) in those two positions.



$x_e = \pi n$ (n%2==0)

$x_e = \pi n$ (n%2==1)

c)

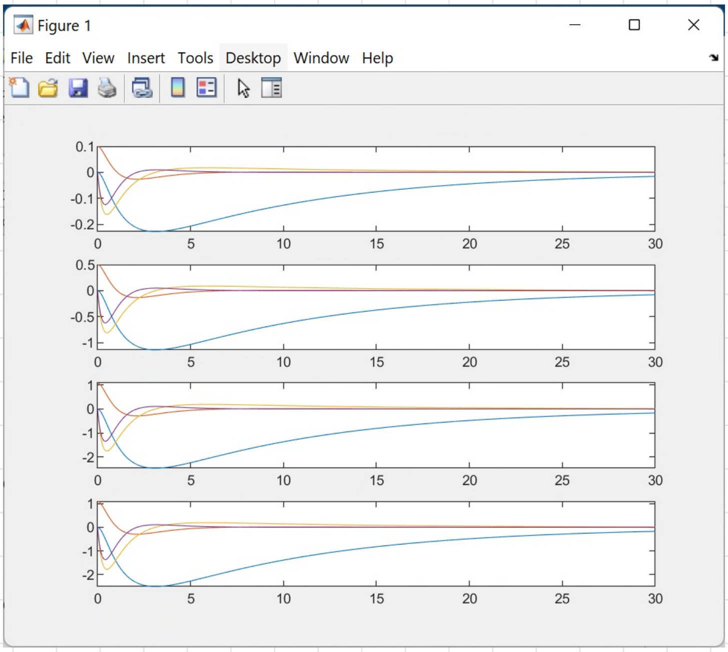
$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u(t)$$

If I use matlab to calculate the eigenvalues of A,

$$\gg \mathrm{eig}(A) = \begin{bmatrix} 0 \\ -3.3301 \\ 1.1284 \\ -0.7984 \end{bmatrix}$$

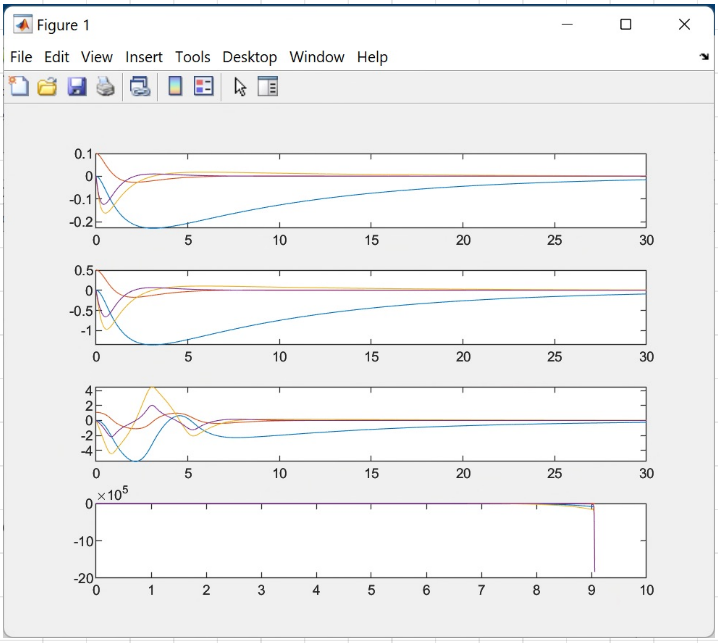
Since for the system to be stable, we need to have $Re(\lambda_i) < 0 \ \forall i$. However we have $\lambda_3 = 1.1284$. Thus the linear system is not stable at the eq. point. Automatically, the original non-linear system is not stable as well at the equilibrium point $x=0$.

d)



Using matlab, $K = [-0.3162 \quad 10.2723 \quad -6.7857 \quad 9.2183]$

e)



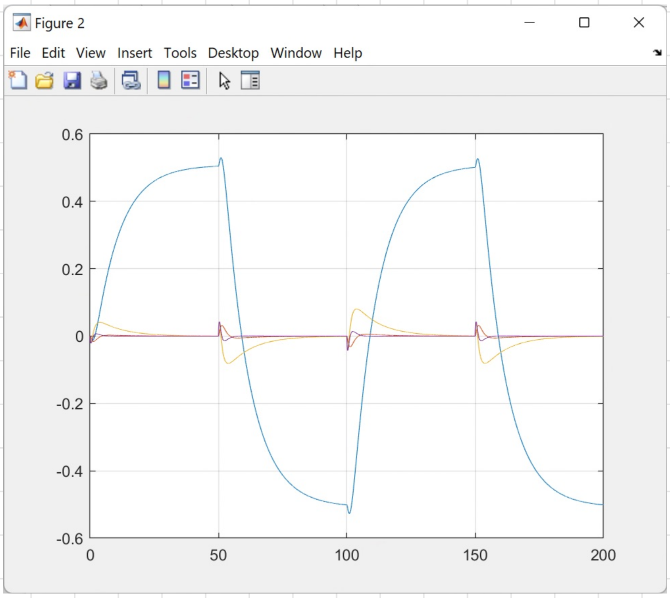
As we are using K that we got from linearizing the system at $x_e = 0$, we can only predict how the system will behave near $x=0$. We can see that as θ becomes larger than 0, the nonlinear system (not stable) acts differently than the linearized system. Matlab even throws this error for the 4th input (not stable)

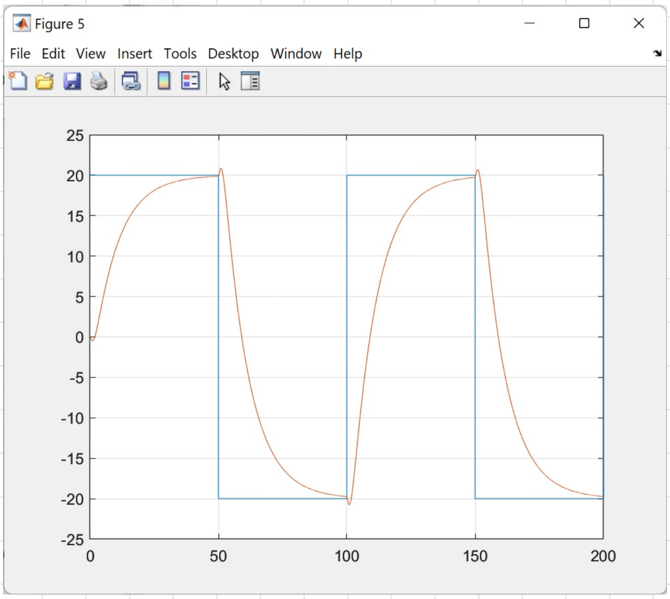
```
>> Q2_e
Warning: Failure at t=9.061899e+00.  Unable to meet
integration tolerances without reducing the step size
below the smallest value allowed (2.842171e-14) at time
t.
> In ode45 (line 352)
In Q2_e (line 30)
```

From this we can see that linearization of the system and what we can say about its stability is only valid for points very close to the equilibrium point used.

f) Since we only want the cart position in y, C will look something like $C = [c \; 0 \; 0 \; 0]$
But since we want to convert meters into inches, $C = \frac{1 \text{meter}}{1 \text{inch}} = 39.3700787$ (according to google)
So, $C = [39.3700787 \;\; 0 \;\; 0 \;\; 0]$

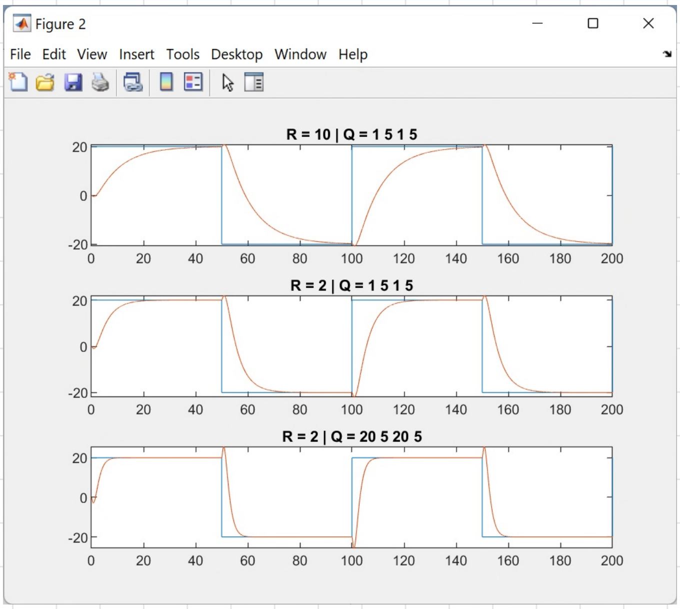
g)





We can see that there are greatly changing vales for each states on 0, 50, 100, 150. This is because the system is using the exponential function to track y_{des}. But overall, the y seems to follow y_{des} fairly well.

h)



So, instead of the initial R=10 and
$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$
I will allow the system to have a bigger input (u). Meaning I will reduce R to 2 to not penalize as much. Then, I get a better tracking system in the sense that it follows y_{des} more accurately on the curves.

To make it even better, I increased the penalty for $x_c, \dot{x}_c$ in Q so that it will track y_{des} faster.

Now we use
$$Q = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$
and R=2.

In the third image, we can see that it follows y_{des} better than the first image.