16-642 Manipulation, Estimation, and Control Fall 2022
Midterm Exam

Jiyoon Park (andrew ID: jiyoonp)

#1

$$\dddot{y} + \gamma\ddot{y} - (\dot{y} - y^2)^2 = u$$

$$\dddot{y} + \gamma\ddot{y} - (\dot{y}^2 - 2\dot{y}y^2 + y^4) = u$$

$$\dddot{y} + \gamma\ddot{y} - \dot{y}^2 + 2\dot{y}y^2 - y^4 = u$$

$$\dddot{y} = u - \gamma\ddot{y} + \dot{y}^2 - 2\dot{y}y^2 + y^4$$

let $x = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $\dot{x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dddot{y} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \ddot{y} \\ u - \gamma\ddot{y} + \dot{y}^2 - 2\dot{y}y^2 + y^4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ u - \gamma x_3 + x_2^2 - 2x_2x_1^2 + x_1^4 \end{bmatrix}$

$\therefore \dot{x} = \begin{bmatrix} x_2 \\ x_3 \\ u - \gamma x_3 + x_2^2 - 2x_2x_1^2 + x_1^4 \end{bmatrix}$ in state space form

#2

Let $e(t) = x(t) - \hat{x}(t)$. Then for $\hat{x}(t)$ to converge to $x(t)$, we need $e(t)$ to go to 0.
It is given that

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + k_o(C\hat{x}(t) - y(t))$$ (I will write $x(t)$ as x from now on)

$$
\begin{aligned}
e &= x - \hat{x} \\
\dot{e} &= \dot{x} - \dot{\hat{x}} \\
&= Ax + Bu - (A\hat{x} + Bu + k_o(C\hat{x} - y)) \\
&= Ax + Bu - A\hat{x} - Bu - k_oC\hat{x} + k_oy \\
&= A(x - \hat{x}) - k_oC\hat{x} + k_oy \\
&= Ae - k_o(C\hat{x} - Cx) \\
&= Ae - k_oC(-e) \\
&= (A + k_oC)e
\end{aligned}
$$

$\dot{e} = (A + k_oC)e$. ← This is our error dynamics. The eigenvalues for $(A + k_oC)$ should be placed on the left side of all of the closed loop controller eigenvalues so that $\hat{x}$ keeps up with x. (It does not guarantee it will work but it usually does.) The dynamics of x should be slower than the dynamics of $\hat{x}$

$\therefore$ The condition for $\hat{x}(t)$ converging to $x(t)$ would be when eigenvalues for $(A + k_oC)$ are placed further to the left of all eigenvalues for $(A - Bk_c)$
↑ closed loop system

#3 The desired closed loop system has a step respone 1% settling time t_s of 2 sec and percent overshoot M_p 5%.

(1) 1% settling time $t_s = 2$

$$e^{Re(p_1)\times 2} = \frac{1}{100}\sqrt{1-\xi^2}$$

Since we want the settling time to be less than 2, we write

$$2 > \frac{1}{Re(p_1)} \ln\left(\frac{1}{100}\sqrt{1-\xi^2}\right)$$

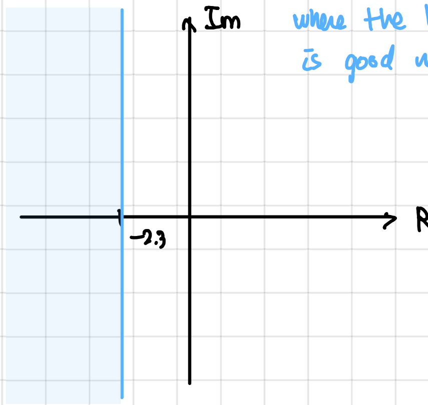
Since $\sqrt{1-\xi^2} < 1$, $Re(p_1) < 0$ we write

$$2 > \frac{1}{Re(p_1)} \ln\left(\frac{1}{100}\right)$$

$$Re(p_1) < \frac{1}{2} \times \ln\left(\frac{1}{100}\right) = \frac{1}{2} \times -4.6 = -2.3$$

$$Re(p_1) < -2.3$$

→ in the complex plane it will look like



where the blue area is good regions.

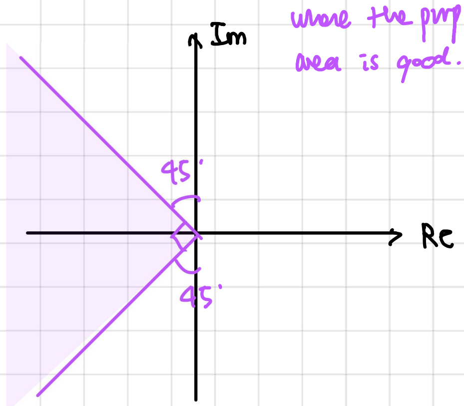
(2) For $M_p < 5\%$

$$\theta = \operatorname{atan}\left(\frac{-Re(p_1)}{Im(p_1)}\right)$$

in english, the percent overshoot is the angle between the positive dominant pole and the imaginary axis. For $M_p < 5\%$, we have to have $\theta > 45°$ (given in notes p(4))

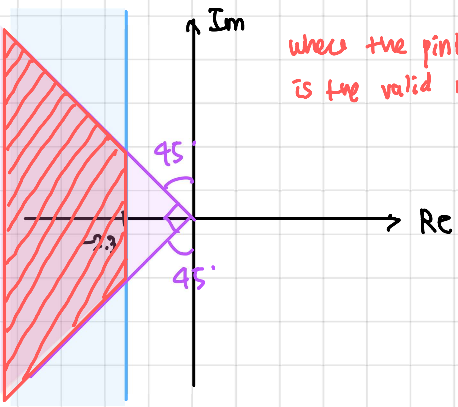
$$\operatorname{atan}\left(\frac{-Re(p_1)}{Im(p_1)}\right) > 45°$$

→ in the complex plane it will be



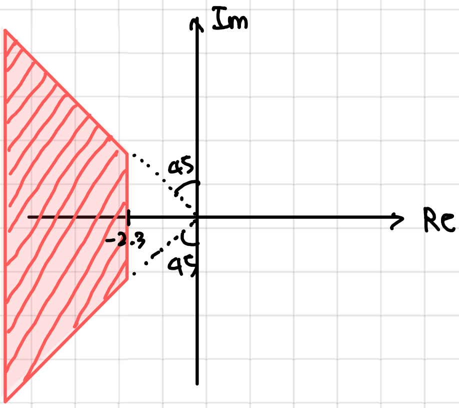
where the purple area is good.

(3) If we combine the two valid areas,



where the pink region is the valid regions

∴ The closed loop poles need to be placed within the pink region to meet all the requirements.



#4

$$\dot{x} = \begin{bmatrix} x_2^2 + x_1 \cos x_2 - u \\ x_2 + (x_1+1)x_1 + x_1 \sin x_2 + u \end{bmatrix} = \begin{bmatrix} x_2^2 + x_1 \cos x_2 \\ x_2 + (x_1+1)x_1 + x_1 \sin x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

a) To check if there is an equilibrium point at $x=0$, we see if $\dot{x}=0$ with $x=0$ $u=0$.

$$\dot{x} = \begin{bmatrix} x_2^2 + x_1 \cos x_2 \\ x_2 + (x_1+1)x_1 + x_1 \sin x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$\dot{x}\Big|_{x=0\ u=0} = \begin{bmatrix} 0 + 0 \cos 0 \\ 0 + (0+1)0 + 0 \sin 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0.$$

So we checked that $\dot{x}=0$ at $x=0$. So the system does have an equilibrium point at $x=0$.

(b) To linearize the system at $x=0$, we will take 3 steps (although for this case $x=0$, the first two steps can be skipped)

① Change to new coordinates

$$z = x - x_e = x - 0 = x$$

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

② rewrite f in new coordinates

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^2 + x_1 \cos x_2 - u \\ x_2 + (x_1+1)x_1 + x_1 \sin x_2 + u \end{bmatrix}$$

$$= \begin{bmatrix} z_2^2 + z_1 \cos z_2 - u \\ z_2 + (z_1+1)z_1 + z_1 \sin z_2 + u \end{bmatrix}$$

③ linear approximation is

$$\dot{z} \approx \frac{\partial f_z}{\partial z}\Big|_{z=0\ u=0} z + \frac{\partial f_z}{\partial u}\Big|_{z=0\ u=0} u$$

$$\frac{\partial f_z}{\partial z}\Big|_{z=0\ u=0} = \begin{bmatrix} \cos z_2 & 2z_2 - z_1 \sin z_2 \\ 2z_1 + 1 + \sin z_2 & 1 + z_1 \cos z_2 \end{bmatrix}\Bigg|_{z=0\ u=0} = \begin{bmatrix} \cos 0 & 0 - 0 \sin 0 \\ 0 + 1 + \sin 0 & 1 + 0 \cdot \cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\frac{\partial f_z}{\partial u}\Big|_{z=0\ u=0} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\Bigg|_{z=0\ u=0} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\therefore$ The system can be approximated to

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} z + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

(c) To check the stability of the nonlinear system at $x=0$, we first check the stability of the linearized system at $x=0$. To do that we look at the eigenvalues of $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ which is $[1, 1]$ (used matlab). Since it has $Re(\lambda_i) > 0$ for some i (for this case all i), we can say that the system is unstable. Since the linearized system is unstable, the nonlinear system is automatically unstable at $x=0$.

$\therefore$ The nonlinear system is not stable at equilibrium point 0.

#5 To check the stability of the unforced system, we look at the eigenvalues of $\begin{bmatrix} 0 & 1 \\ -3 & -6 \end{bmatrix}$. The eigenvalues are (used matlab) $[-0.5505, -5.4495]$. Since $Re(\lambda_i) \leq 0$ for all i, the system is stable at equilibrium point at 0. Further, since $Re(\lambda_i) < 0$ for $\forall i$, we can even say it is asymptotically stable at equilibrium point at 0.

$\therefore$ Yes it is (asymptotically) stable at equilibrium point at zero.

#6 To check if the system is controllable, we need to see if the rank $(Q) = n$ for $Q = [B \;\; AB \;\; A^2B \cdots A^{n-1}B]$. In our case, $n=3$ and $Q = [B \;\; AB \;\; A^2B]$.

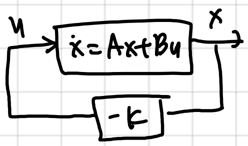
(used matlab)

$Q = \begin{bmatrix} 0 & 3 & 42 \\ 0 & 6 & 96 \\ 1 & 9 & 150 \end{bmatrix}$ and rank $(Q) = 3$. Since rank $(Q) = 3$, the system is controllable.

#7 I first tried to blindly use the place command and realized that the system itself is uncontrollable rank $([B \;\; AB \;\; A^2B \;\; A^3B \;\; A^4B]) = 3 < 5$ because the input cannot effect the top left corner of A. I realized the top left part of the matrix was already stable (proven in question 5) and I only needed to control the bottom right part of A to make the system stable. So I realized that my K should look something like

$K = [0 \;\; 0 \;\; a \;\; b \;\; c]$ where $a, b, c \in \mathbb{R}$. To get K, I used the lqr way of eigenvalue placement with $Q = I$, $R = 1$. My $K = lqr(A, B, Q, R)$ gave $K = [0 \;\; 0 \;\; 22.4512 \;\; 28.8078 \;\; 32.3150]$.

Now when I look at my closed loop system,



$u = -Kx$

$\dot{x} = Ax + B(-Kx) = (A - BK)x$

The system has to have $Re(\lambda_i) < 0$ for $\forall i$ to be asymptotically stable. Now if I check

$eig(A - BK) = \begin{bmatrix} -5.4495 \\ -0.5505 \\ -16.1376 \\ -1.1969 \\ -0.3804 \end{bmatrix}$. Since $Re(\lambda_i) < 0$ for $\forall i$ for the closed loop system, the new system is asymptotically stable.

$\therefore$ The K is $[0 \;\; 0 \;\; 22.4512 \;\; 28.8078 \;\; 32.3150]$

Jiyoon Park MEC Exam 1 Matlab code

(andrew id: jiyoonp)

Question 4

```
A = [1 0;
    1 1];
B = [-1;
    1];

eig(A)
```

```
ans = 2×1
     1
     1
```

Question 5

```
A5 = [0 1;
    -3 -6];

eig(A5)
```

```
ans = 2×1
   -0.5505
   -5.4495
ans = 2×2
    0.6410    0.1168
   -0.3505   -0.0600
```

Question 6

```
A6 = [1 2 3;
    4 5 6;
    7 8 9];
B6 = [0 0 1]';

Q = [B6 A6*B6 A6*A6*B6]
```

```
Q = 3×3
     0     3    42
     0     6    96
     1     9   150
```

```
rank(Q)
```

```
ans = 3
```

```
rank(ctrb(A6, B6))
```

```
ans = 3
```

Question 7

```
A7 = [0 1 0 0 0;
    -3 -6 0 0 0 ;
    0 0 1 2 3;
    0 0 4 5 6;
    0 0 7 8 9];
B7 = [0 0 0 0 1]';
C7 = [ 1 1 1 1 1];

rank(ctrb(A7, B7))
```

```
ans = 3
```

```
eig(A7)
```

```
ans = 5×1
   -0.5505
   -5.4495
   16.1168
   -1.1168
   -0.0000
```

```
% p = [- 10 -100 -120 -500 -600];
% K = place(A7, B7, p)

Q7 = [1 0 0 0 0;
    0 1 0 0 0;
    0 0 1 0 0;
    0 0 0 1 0;
    0 0 0 0 1];
R7 = 1;

K_lqr = lqr(A7, B7, Q7, R7)
```

```
K_lqr = 1×5
   -0.0000   -0.0000   22.4512   28.8078   32.7150
```

```
u = (A7-B7*K_lqr)
```

```
u = 5×5
         0    1.0000         0         0         0
   -3.0000   -6.0000         0         0         0
         0         0    1.0000    2.0000    3.0000
         0         0    4.0000    5.0000    6.0000
    0.0000    0.0000  -15.4512  -20.8078  -23.7150
```

```
A72 = A7-B7*K_lqr;
rank(ctrb(A72, B7))
```

```
ans = 3
```

```
p_lqr = eig(u)
```

```
p_lqr = 5×1
   -5.4495
```

```
   -0.5505
  -16.1376
   -1.1969
   -0.3804
```

```
K = place(A7, B7, p_lqr)
```

```
K = 1×5
         0         0   22.4512   28.8078   32.7150
```

```
eig(A7-B7*K)
```

```
ans = 5×1
   -0.5505
   -5.4495
  -16.1376
   -1.1969
   -0.3804
```