Robust Principal Component Analysis

Maximilian Balandat   Walid Krichene   Chi Pang Lam   Ka Kit Lam

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1 Introduction

Given an observed matrix $M \in \mathbb{R}^{n_1 \times n_2}$ that is formed as a superposition of a low-rank matrix L_0 and a sparse matrix S_0,

$$M = L_0 + S_0$$

Robust Principal Component Analysis [5] is the problem of recovering the low-rank and sparse components. Under suitable assumptions on the rank and incoherence of L_0, and the distribution of the support of S_0, the components can be recovered exactly with high probability, by solving the Principal Component Pursuit (PCP) problem given by

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|S\|_1 \\ \text{subject to} & L + S = M \end{array} \tag{1}$$

Principal component pursuit minimizes a linear combination of the nuclear norm of a matrix L and the ℓ_1 norm of $M - L$. Minimizing the ℓ_1 norm is known to favour sparsity, while minimizing the nuclear norm $\|L\|_* = \sum_{\sigma \in \sigma(L)} \sigma$ is known to favour low-rank matrices (intuitively, favours sparsity of the vector of singular values).

The low-rank component S_0 is viewed as a noise matrix, that can represent measurement noise, failure in some sensors that will result in completely corrupting a fraction of the observed entries, or missing data (which translates to having a fraction of the entries equal to zero). In this setting, one would like to be able to recover the original data L_0, without making assumptions on the magnitude $\|S_0\|_\infty$ of the sparse component. PCP achieves recovery with high probability in this setting, under alternate assumptions on the structure of L_0 and sparsity pattern of S_0.

One cannot expect to recover the components exactly in the most general case. Assume for example that L_0 is such that $(L_0)_{ij} = \delta_i^1 \delta_j^1$, and $S_0 = -L_0$. Both matrices are sparse and low-rank, and clearly one cannot expect to recover the components in this case, since the observed matrix is $M = 0$. Therefore assumptions are made on the incoherence of L_0 and the support of S_0.

1.1 Incoherence of the low rank component L_0

The Incoherence conditions describe how much the singular vectors of a given matrix are aligned with the vectors of the canonical basis.

Let the SVD of L_0 be given by

$$L_0 = U\Sigma V^* = \sum_{i=1}^{r} \sigma_i u_i v_i^* \tag{2}$$

where $U \in \mathbb{R}^{n_1 \times r}$ and $V \in \mathbb{R}^{n_2 \times r}$ are the matrices of left and right singular vectors respectivley, $U = [u_1, \ldots, u_r]$, $V = [v_1, \ldots, v_r]$. Then the incoherence conditions are given by

$$\max_i \|U^* e_i\|_2^2 \leq \frac{\mu r}{n_1}, \quad \max_i \|V^* e_i\|_2^2 \leq \frac{\mu r}{n_2} \tag{3}$$

and

$$\|UV^*\|_\infty \leq \sqrt{\frac{\mu r}{n_1 n_2}} \tag{4}$$

Note that the condition $\|U^* e_i\|_2^2 \leq \frac{\mu r}{n_1}$ translates to $\sum_{k=1}^r (u_k)_i^2 \leq \frac{\mu r}{n_1}$. Also note that the orthogonal projection P_U on $\mathrm{Span}(u_1, \ldots, u_r)$ is given by

$$UU^* = [u_1, \ldots, u_r] \begin{bmatrix} u_1^* \\ \vdots \\ u_r^* \end{bmatrix} = \sum_{k=1}^{r} u_k u_k^*$$

and the condition is equivalent to $\|P_U e_i\|_2^2 \leq \frac{\mu r}{n_1}$ since $\|U^* e_i\|_2^2 = e_i^*(UU^*)e_i = e_i^* P_U e_i = (e_i - P_U e_i + P_U e_i)^* P_U e_i = \|P_U e_i\|_2^2$ ($P_U e_i$ and $e_i - P_U e_i$ are orthogonal). Or simply $\|P_U e_i\|_2^2 = e_i^* UU^*UU^* e_i = e_i^* UU^* e_i = \|U^* e_i\|_2^2$ since $U^*U = I_r$.

These conditions require the singular vectors to be “spread” enough with respect to the canonical basis. Intuitively, if the singular vectors of the low-rank matrix L_0 are aligned with a few canonical basis vectors, then L_0 will be sparse and hard to distinguish from the sparse corruption matrix S_0.

1.2 Support of the sparse component S_0

The cardinality of the support of S_0 is denoted m. Guaranteeing exact recovery requires m to be small enough, in a sense that will be defined in the next section. Proving exact recovery will rely on a probabilistic argument on the distribution of sparse matrices S_0 on the set of matrices $\{S \in \mathbb{R}^{n_1 \times n_2} | \mathrm{card}(\mathrm{Supp}(S)) = m\}$ assuming a uniform sampling model. The proof of the main result will use a different sampling model, and prove equivalence with the uniform model.

2 Main Result

Theorem 1. *Suppose $L_0 \in \mathbb{R}^{n \times n}$ satisfies incoherence conditions (3) and (4) and that the support of S_0 is uniformly distributed among all sets of cardinality m. Then $\exists c$ such that with high probability over the choice of support of S_0 (at least $1 - cn^{-10}$), Principal Component Pursuit with $m = 1/\sqrt{n}$ is exact, i.e. $\hat{L} = L_0$ and $hatS = S_0$ provided that*

$$rank(L_0) \leq \frac{\rho_r}{\mu} \frac{n}{(\log n)^2} \quad \text{and} \quad m \leq \rho_s n^2 \tag{5}$$

Above, ρ_r and ρ_s are positive numerical constants. Note in particular that no assumptions are made on the magnitudes of the nonzero entries of S_0.

The first condition in the theorem bounds the rank of L_0, but also how spread the singular vectors have to be, since we need to have $\forall i$ (from the incoherence condition)

$$\|U^* e_i\|_2^2 \leq \frac{\mu \mathrm{rank}(L_0)}{n} \leq \frac{\rho_r}{(\log n)^2}$$

The second condition bounds the size m of the support of S_0.

3 Proof

The main arguments of the proof are the following:

First, change the model of the sparse matrix S_0 from the uniform sampling model, to the Bernoulli sampling model with fixed signs, then to the Bernoulli sampling model with random signs. To show equivalence of the results under the different sampling models, an elimination theorem is used.

Then using the random sign Bernoulli sampling model, it is shown that a dual certificate can be constructed with high probability, proving that (L_0, S_0) is the unique optimizer, by constructing a subgradient that shows that any non-zero perturbation H will result in a strict increase in the objective value $\|L_0 + H\|_* + \lambda \|S_0 - H\|_1$.

3.1 Preliminaries

- The subgradient of of the ℓ_1 norm at S_0 supported on Ω is of the form $\mathrm{sgn}(S_0) + F$ where $P_\Omega F = 0$ and $\|F\|_\infty \leq 1$.
- The subgradient of the nuclear norm (For details see Sec. 3.9) at $L_0 = U \Sigma V^*$ where $U, V \in \mathbb{R}^{n \times r}$ and $\Sigma \in \mathbb{R}^{r \times r}$, is of the form $UV^* + W$, where

$$\begin{aligned} U^* W &= 0 \\ WV &= 0 \\ \|W\| &\leq 1 \end{aligned} \tag{6}$$

  or equivalently

$$\begin{aligned} P_T W &= 0 \\ \|W\| &\leq 1 \end{aligned} \tag{7}$$

  where T is the linear space of matrices defined by

$$T = \{UX^* + YV^*,\ X, Y \in \mathbb{R}^{n \times r}\}$$

Indeed, we have

$$
\begin{aligned}
P_T W = 0 &\Leftrightarrow W \in T^{\perp} \\
&\Leftrightarrow \forall M \in T,\ Tr(W^* M) = 0 \\
&\Leftrightarrow \forall X, Y \in \mathbb{R}^{n\times r},\ Tr(W^*(UX^* + YV^*)) = 0 \\
&\Leftrightarrow \forall X, Y \in \mathbb{R}^{n\times r},\ Tr((U^*W)^* X^*) + Tr((WV)^* Y) = 0 \\
&\Leftrightarrow U^* W = WV^* = 0
\end{aligned}
$$

Note that the projection on the orthogonal of T is given by

$$
P_{T^\perp} M = (I - UU^*) M (I - VV^*) \tag{8}
$$

Proof. Note that UU^* is the orthogonal projection on the subspace spanned by the columns of U, and similarly for VV^*. Let $P_U = UU^*$, $P_{U^\perp} = I - UU^*$, and similarly for V.

Let $M_1 = (I - UU^*) M (I - VV^*) = P_{U^\perp} M P_{V^\perp}$. We have

$$
P_{T^\perp} M = M_1 \Leftrightarrow (M - M_1 \in T \text{ and } M_1 \perp M - M_1)
$$

and we have $M - M_1 = UU^*M + MVV^* - UU^*MVV^* = U(U^*M) + (MV - UU^*MV)V^* \in T$, and

$$
\begin{aligned}
Tr(M_1^*(M - M_1)) &= Tr((I - VV^*) M^* (I - UU^*)(UU^*M + MVV^* - UU^*MVV^*)) \\
&= Tr(P_{V^\perp} M^* P_{U^\perp}(P_U M + (M - UU^*M) P_V)) \\
&= Tr(P_{V^\perp} M^* P_{U^\perp} P_U M) + Tr((M - UU^*M) P_V P_{V^\perp} M^* P_{U^\perp}) \\
&= 0
\end{aligned}
$$

using the fact that $P_{U^\perp} P_U = P_V P_{V^\perp} = 0$ (projecting consecutively on a subspace and its orthogonal yields 0, or simply expanding, $(I - UU^*)UU^* = UU^* - UU^*UU^* = UU^* - UI_rU^* = 0$). This completes the proof. ☐

Note that since $P_{T^\perp}$ is an orthogonal projection, we have

$$
\|P_{T^\perp} M\| \le \|M\| \tag{9}
$$

and for any dyad $e_i e_j^*$, we have

$$
\begin{aligned}
\|P_{T^\perp} e_i e_j^*\|_F^2 &= Tr\left((I - UU^*) e_i e_j^* (I - VV^*)(I - VV^*)^* e_j e_i^* (I - UU^*)^*\right) \\
&= Tr\left(e_i^* (I - UU^*)^* (I - UU^*) e_i e_j^* (I - VV^*)(I - VV^*)^* e_j\right) \\
&= Tr\left(e_i^* (I - UU^*)^* (I - UU^*) e_i\right) Tr\left(e_j^* (I - VV^*)(I - VV^*)^* e_j\right) \\
&= \|(I - UU^*) e_i\|_2^2 \|(I - VV^*) e_j\|_2^2
\end{aligned}
$$

and since UU^* is an orthogonal projection, we have

$$
\begin{aligned}
\|(I - UU^*) e_i\|_2^2 &= \|e_i\|_2^2 - \|UU^* e_i\|_2^2 \\
&\ge 1 - \mu r / n
\end{aligned}
$$

where the last inequality results form the incoherence condition (4), $\|U^* e_i\|_2^2 \leq \frac{\mu r}{n}$. Therefore

$$\|P_{T^\perp} e_i e_j^*\|_F^2 \geq (1 - \mu r/n)^2 \tag{10}$$

equivalently, using the fact that $\|P_{T^\perp} e_i e_j^*\|_F^2 + \|P_T e_i e_j^*\|_F^2 = \|e_i e_j^*\|_F^2 = 1$, we have

$$\begin{aligned}\|P_T e_i e_j^*\|_F^2 &\leq 1 - (1 - \frac{\mu r}{n})^2 \\ &= \frac{2\mu r}{n} - \left(\frac{\mu r}{n}\right)^2 \\ &\leq \frac{2\mu r}{n}\end{aligned}$$

Thus

$$\|P_T e_i e_j^*\|_F^2 \leq \frac{2\mu r}{n} \tag{11}$$

3.2 Elimination Theorem

The following elimination theorem states the intuitive fact that if PCP exactly recovers the components of $M = L + S$, then it also exactly recovers the components of $M = L + S'$ where S' is a trimmed version of S ($\text{supp}(S') \subset \text{supp}(S)$ and S and S' coincide on $\text{supp}(S')$)

Theorem 2. *Suppose the solution to the PCP problem (1) with input data $M_0 = L_0 + S_0$ is unique and exact, and consider $M_0' = L_0 + S_0'$ where S_0' is a trimmed version of S_0. Then the solution to (1) with input M_0' is exact as well.*

Proof. Let $S_0' = P_{\Omega_0} S_0$ and let $(\hat{L}, \hat{S})$ be the solution to (1) with input $L_0 + S_0'$. Then since (L_0, S_0') is a feasible point for (1), it provides un upper bound on the optimal value

$$\|\hat{L}\|_* + \lambda\|\hat{S}\|_1 \leq \|L_0\|_* + \lambda\|S_0'\|_1$$

then decomposing S_0 into the orthogonal components $S_0 = P_{\Omega_0} S_0 + P_{\Omega_0^\perp} S_0 = S_0' + P_{\Omega_0^\perp} S_0$, we have $\|S_0'\|_1 = \|S_0\|_1 - \|P_{\Omega_0^\perp} S_0\|_1$, thus we have

$$\|\hat{L}\|_* + \lambda\|\hat{S}\|_1 + \lambda\|P_{\Omega_0^\perp} S_0\|_1 \leq \|L_0\|_* + \lambda\|S_0\|_1$$

and using the triangular inequality

$$\|\hat{L}\|_* + \lambda\|\hat{S} + P_{\Omega_0^\perp} S_0\|_1 \leq \|L_0\|_* + \lambda\|S_0\|_1$$

we observe that $(\hat{L}, \hat{S} + P_{\Omega_0^\perp} S_0)$ is feasible for the problem with input $M = L_0 + S_0$, for which the optimal value is precisely $\|L_0\|_* + \lambda\|S_0\|_1$. Therefore by uniqueness of the solution, we have

$$\begin{aligned}\hat{L} &= L_0 \\ P_{\Omega_0^\perp} S_0 &= S_0\end{aligned}$$

the second equality is equivalent to $\hat{S} = S_0 - P_{\Omega_0^\perp} S_0 = P_{\Omega_0} S_0 = S_0'$. This completes the proof. □

3.3 Derandomization

Derandomization is used to show equivalence between the problem where the signs of the entries of S_0 are random, and the problem where the entries of S_0 have fixed signs.

In the setting of Theorem 1, the non-zero entries of the sparse component S_0 are fixed, but the proof will use a stronger assumption: the signs of the non-zero entries are independent Bernoulli variables. The following theorem shows equivalence of the two settings.

Theorem 3. *Suppose L_0 satisfies conditions of Theorem 1, and that the support of S_0 is sampled from a Bernoulli model with parameter $2\rho_s$, and the signs of S_0 are i.i.d. Bernoulli ± 1 with parameter $\frac{1}{2}$, and independent from the support. Then:*

If the PCP solution is exact with high probability, then it is exact with at least the same probability for the model in which values of S_0 are fixed and the support is sampled from a Bernoulli distribution with parameter ρ_s.

Proof. Consider the fixed values model, and let $S_0 = P_\Omega S$ for some matrix S, and the support Ω is sampled from a Bernoulli distribution. Thus the components of S_0 are independent and

$$(S_0)_{ij} = \begin{cases} S_{ij} & \text{w.p. } \rho_s \\ 0 & \text{w.p. } 1 - \rho_s \end{cases}$$

the idea of the proof is to craft a new model, and show that it is equivalent (in terms of probability distribution) to the above model.

Let E be a random sign matrix, with i.i.d. entries

$$E_{ij} = \begin{cases} 1 & \text{w.p. } \rho_s \\ 0 & \text{w.p. } 1 - 2\rho_s \\ -1 & \text{w.p. } \rho_s \end{cases}$$

and $\Delta(E)$ an elimination matrix, function of E, defined as

$$\Delta_{ij} = \begin{cases} 0 & \text{if } E_{ij}S_{ij} < 0 \ (E_{ij} \text{ and } S_{ij} \text{ have different signs}) \\ 1 & \text{otherwise} \end{cases}$$

the entries of Δ are functions of independent variables, and are therefore independent.

Now consider the following variable

$$S_0' = \Delta \circ |S| \circ E$$

where $\circ$ is the component wise product. Then S_0 and S_0' have the same distribution. Indeed, it suffices by independence to check that they have the same marginals:

$$\begin{aligned} P((S_0')_{ij} = S_{ij}) &= P(\Delta_{ij} = 1 \text{ and } E_{ij} = \operatorname{sgn} S_{ij}) \\ &= P(E_{ij}S_{ij} \geq 0 \text{ and } E_{ij} = \operatorname{sgn} S_{ij}) \\ &= P(E_{ij} = \operatorname{sgn} S_{ij}) \\ &= \rho_s \end{aligned}$$

and

$$P(S_0 = S_{ij}) = \rho_s$$

Finally, since, by assumption, PCP recovers $|S| \circ E$ with high probability, then by the elimination theorem, it also recovers $\Delta \circ |S| \circ E$ with at least the same probability. The result follows since S_0' and S_0 have the same distribution. $\square$

We remark that the uniform sampling and the iid Bernoulli sampling model are indeed equivalent and the justification is given in Section (3.8).

3.4 Dual certificate

The following lemma gives a simple sufficient condition for the pair (L_0, S_0) to be the unique optimal solution to PCP.

Lemma 1. *Assume that $\|P_\Omega P_T\| < 1$. Then (L_0, S_0) is the unique solution to PCP if $\exists (W, F)$ such that*

$$\begin{aligned} &UV^* + W = \lambda(sign(S_0) + F) \\ &P_T W = 0 \\ &\|W\| < 1 \\ &P_\Omega F = 0 \\ &\|F\|_\infty < 1 \end{aligned} \tag{12}$$

Proof. We first prove that the condition $\|P_\Omega P_T\| < 1$ is equivalent to $\Omega \cap T = \{0\}$.

First, if $\Omega \cap T \neq \{0\}$, then let $M_0 \in \Omega \cap T$, $M_0 \neq 0$. We have $\|P_\Omega P_T M_0\| = \|M_0\|$, thus $\|P_\Omega P_T\| = \max_{M \neq 0} \frac{\|P_\Omega P_T M\|}{\|M\|} \geq 1$.

Conversely, if $\|P_\Omega P_T\| \geq 1$, then $\exists M_0 \neq 0$ such that $\|M_0\| \leq \|P_\Omega P_T M_0\|$. But since P_Ω and P_T are orthogonal projections, we have $\|M_0\| \leq \|P_\Omega P_T\| \leq \|P_T M_0\| \leq \|M_0\|$, where inequalities must hold with equality. In particular, we have $\|P_T M_0\| = \|M_0\|$, which implies $P_T M_0 = M_0$ (to prove this, decompose $\|M_0\|$ into the orthogonal components $\|M_0\|^2 = \|M_0 - P_T M_0\|^2 + \|P_T M_0\|^2$, thus $\|P_T M_0\| = \|M_0\| \Rightarrow \|M_0 - P_T M_0\| = 0 \Rightarrow M_0 = P_T M_0$), then similarly, $\|P_\Omega M_0\| = \|M_0\|$, which implies $P_\Omega M_0 = M_0$. Therefore $M_0 \in \Omega \cap T$. This proves the equivalence $\|P_\Omega P_T\| < 1 \Leftrightarrow \Omega \cap T = \{0\}$.

To prove that (L_0, S_0) is the unique optimizer, we show that for any feasible perturbation $(L_0 + H, S_0 - H)$ where $H \neq 0$ strictly increases the objective. Let

- $UV^* + W_0$ be an arbitrary subgradient of the nuclear norm at L_0, where $\|W_0\| \leq 1$ and $P_T W_0 = 0$
- $\text{sgn}(S_0) + F_0$ be an arbitrary subgradient of the ℓ_1-norm at S_0, where $\|F_0\|_\infty \leq 1$ and $P_\Omega F_0 = 0$

Then we can lower bound the value of the objective

$$\|L_0 + H\|_* + \lambda \|S_0 - H\|_1 \geq \|L_0\|_* + \lambda \|S_0\|_1 + \langle UV^* + W_0, H \rangle - \lambda \langle \text{sgn}(S_0) + F_0, H \rangle$$

Now we pick a particular pair (W_0, F_0) such that

- $\langle W_0, H\rangle = \|P_{T^\perp}H\|_*$, for example $W_0 = P_{T^\perp}W$ where W is a normed matrix such that $\langle W, P_{T^\perp}H\rangle = \|P_{T^\perp}H\|_*$ (by duality of $\|.\|$ and $\|.\|_*$)
- $\langle F_0, H\rangle = -\|P_{\Omega^\perp}H\|_1$, for example $F_0 = -\text{sgn}(P_{\Omega^\perp}H)$

then we have

$$\|L_0 + H\|_* + \lambda\|S_0 - H\|_1 \geq \|L_0\|_* + \lambda\|S_0\|_1 + \|P_{T^\perp}H\|_* + \|P_{\Omega^\perp}H\|_1 + \langle UV^* - \lambda\text{sgn}(S_0), H\rangle$$

we can bound the inner product using the definition of W and F,

$$\begin{aligned}
|\langle UV^* - \lambda\text{sgn}(S_0), H\rangle| &= |\langle \lambda F - W, H\rangle| && \text{since } UV^* + W = \lambda(\text{sign}(S_0) + F)\\
&\leq |\langle W, H\rangle| + \lambda|\langle F, H\rangle| && \text{by the triangular inequality}\\
&\leq \beta(\|P_{T^\perp}H\|_* + \lambda\|P_{\Omega^\perp}H\|_1)
\end{aligned}$$

where $\beta = \max(\|W\|, \|F\|_\infty) < 1$, and the last inequality follows from the fact that

$$\begin{aligned}
\|P_{T^\perp}H\|_* &\geq \langle P_{T^\perp}H, W/\|W\|\rangle \geq \langle H, W/\|W\|\rangle\\
\|P_{\Omega^\perp}H\|_1 &\geq \langle P_{\Omega^\perp}H, F/\|F\|_\infty\rangle \geq \langle H, F/\|F\|_\infty\rangle
\end{aligned}$$

Thus

$$\begin{aligned}
\|L_0 + H\|_* + \lambda\|S_0 - H\|_1 - \|L_0\|_* - \lambda\|S_0\|_1 &\geq (1-\beta)\left(\|P_{T^\perp}H\|_* + \lambda\|P_{\Omega^\perp}H\|_1\right)\\
&> 0
\end{aligned}$$

since $\|P_{T^\perp}H\|_* = \|P_{\Omega^\perp}H\|_1 = 0$ only if $P_{T^\perp}H = P_{\Omega^\perp}H = 0$, i.e. $H \in \Omega \cap T$, and, by assumption, $\Omega \cap T = 0$ and $H \neq 0$. Therefore the objective strictly increases with a non-zero perturbation. This completes the proof.

□

The proof of the main theorem will use a slightly different result, given by the following Lemma

Lemma 2. *Assume that $\|P_\Omega P_T\| \leq 1/2$. Then (L_0, S_0) is the unique solution to PCP if $\exists(W, F)$ such that*

$$\begin{aligned}
&UV^* + W = \lambda(sign(S_0) + F + P_\Omega D)\\
&P_T W = 0\\
&\|W\| \leq 1/2\\
&P_\Omega F = 0\\
&\|F\|_\infty \leq 1/2\\
&\|P_\Omega D\|_F \leq 1/4
\end{aligned} \tag{13}$$

Proof. Using $\beta = \max(\|W\|, \|F\|_\infty) \leq \frac{1}{2}$ in the previous proof, we have for a non-zero perturbation H

$$\begin{aligned}
\|L_0 + H\|_* + \lambda\|S_0 - H\|_1 - \|L_0\|_* - \lambda\|S_0\|_1 &\geq \frac{1}{2}\left(\|P_{T^\perp}H\|_* + \lambda\|P_{\Omega^\perp}H\|_1\right) - \lambda\langle P_\Omega D, H\rangle\\
&\geq \frac{1}{2}\left(\|P_{T^\perp}H\|_* + \lambda\|P_{\Omega^\perp}H\|_1\right) - \frac{\lambda}{4}\|P_\Omega H\|_F
\end{aligned}$$

the last term can be further bounded

$$\begin{aligned}\|P_\Omega H\|_F &\leq \|P_\Omega P_T H\|_F + \|P_\Omega P_{T^\perp} H\|_F \\ &\leq \frac{1}{2}\|H\|_F + \|P_{T^\perp} H\|_F && \text{using } \|P_\Omega P_T\| \leq \frac{1}{2} \text{ and } \|P_\Omega\| \leq 1 \\ &\leq \frac{1}{2}\|P_\Omega H\|_F + \frac{1}{2}\|P_{\Omega^\perp} H\|_F + \|P_{T^\perp} H\|_F\end{aligned}$$

therefore

$$\|P_\Omega H\|_F \leq \|P_{\Omega^\perp} H\|_F + 2\|P_{T^\perp} H\|_F$$

and we conclude by lower bounding the increase in the objective

$$\begin{aligned}\|L_0 + H\|_* + \lambda\|S_0 - H\|_1 - \|L_0\|_* - \lambda\|S_0\|_1 &\geq \frac{1}{2}\left((1-\lambda)\|P_{T^\perp} H\|_* + \frac{\lambda}{2}\|P_{\Omega^\perp} H\|_1\right) \\ &> 0\end{aligned}$$

since $\|P_{T^\perp} H\|_* = \|P_{\Omega^\perp} H\|_1 = 0$ only if $P_{\Omega^\perp} H = P_{T^\perp} H = 0$, i.e. $H \in \Omega \cap T$, and, by assumption, $\Omega \cap T = \{0\}$ ($\|P_\Omega P_T\| \leq \frac{1}{2} < 1$). This completes the proof.

□

3.4.1 Bounding $\|P_\Omega P_T\|$

Under suitable conditions on the size of the support Ω_0 of the sparse component, a bound can be derived on $\|P_\Omega P_T\|$ [4].

Theorem 4. *Suppose Ω_0 is sampled from the Bernoulli model with parameter ρ_0. Then with high probability,*

$$\|P_T - \rho_0^{-1} P_T P_{\Omega_0} P_T\| \leq \epsilon$$

provided that $\rho_0 \geq C_0 \epsilon^{-2} \frac{\mu r \log n}{n}$ where μ is the incoherence parameter and C_0 is a numerical constant.

As a consequence, $\|P\Omega P_T\|$ can be bounded, and if $|\Omega|$ is not too large, then the desired bound $\|P_\Omega P_T\| \leq 1/2$ holds.

3.5 Probabilistic Guarantee via Dual Certification

We now present the proof that, under the assumptions of Robust PCA, then , with high probability, we can find a dual certificate (W, F, D) such that it can satisfy Lemma (?) . This can then finish the proof of the probabilistic guarantee of recovery of (L_0, S_0).

In order to accomplish that, we construct W by $W = W^L + W^S$ where the constructed W^L and W^S will satisfy the following properties. First , $P_\Omega(W^S) = \lambda sign(S_0)$. Then it also need to satisfies the followings.

Lemma 3. *Let $S_0 \sim Bern(\rho)$ iid for each entry with Ω as its support set. Set $j_0 = 2\log n$. With the assumptions in the main theorem of RPCA, W^L satisfies the following with high probability.*

1. $||W^L|| < \frac{1}{4}$
2. $||P_\Omega(UV^* + W^L)||_F < \frac{\lambda}{4}$
3. $||P_{\Omega^\perp}(UV^* + W^L)||_\infty < \frac{\lambda}{4}$

Lemma 4. *Let $S_0 \sim Bern(\rho)$ iid for each entry with Ω as its support set. With the assumptions in the main theorem of RPCA, W^S satisfies the following with high probability.*

1. $||W^S|| < \frac{1}{4}$
2. $||P_{\Omega^\perp}(W^S)||_\infty < \frac{\lambda}{4}$

After having these two lemmas, we are ready to show that they help to justify the probabilistic guarantee. We note that

$$\begin{aligned} UV^* + W &= W^L + W^S + UV^* \\ &= \lambda[P_\Omega(\frac{UV^* + W^L}{\lambda}) + sign(S_0) + P_{\Omega^\perp}(\frac{W^L + W^S + UV^*}{\lambda})] \end{aligned}$$

Thus we take

$$\begin{aligned} D &= \frac{UV^* + W^L}{\lambda} \\ W &= W^L + W^S \\ F &= P_\Omega(\frac{W^L + W^S + UV^*}{\lambda}) \end{aligned}$$

From Lemma (3) and Lemma (4), we can check that (W, F, D) satisfies the conditions of Lemma (2), thus establishing the probabilistic guarantee.

The following section will discuss in depth how to construct $W^L$and W^S that can will satisfy Lemma (3) and Lemma (4).

3.6 Proof of the Lemma about golfing scheme and dual certificate

Golfing scheme:

The golfing scheme involves creating a W^L according to the following method.

1. Fix $j_0 \geq 1$, define $\Omega_j \sim Bern(q)$ iid with $1 \leq j \leq j_0$ and $\rho = (1-q)^{j_0}$. Define the complement of support of Ω by $\Omega = \cup_{1\leq j\leq j_0}\Omega_j^C$.
2. Define a sequence of matrix which finally ends at W^L
   (a) $Y_0 = 0$
   (b) $Y_j = Y_{j-1} + \frac{1}{q}P_{\Omega_j}P_T(UV^* - Y_{j-1})$ for $1 \leq j \leq j_0$
   (c) $W^L = P_{T^\perp}(Y_{j_0})$

We first list a number of facts that will be used in the proof of Lemma (3).

Fact 1. *If we fix $Z \in T$, $\Omega_0 \sim Bern(\rho_0)$, and $\rho_0 \geq C_0\epsilon^{-2}\frac{\mu r \log n}{n}$, then with high probability, we will have,*

$$||Z - \rho_0^{-1} P_T P_{\Omega_0}(Z)||_\infty \leq \epsilon ||Z||_\infty$$

Fact 2. *If we fix Z, $\Omega_0 \sim Bern(\rho_0)$, and $\rho_0 \geq C_0 \frac{\mu \log n}{n}$, then with high probability, we will have,*

$$||(I - \rho_0^{-1} P_{\Omega_0})Z|| \quad \leq \quad C_0' \sqrt{\frac{n \log n}{\rho_0}} ||Z||_\infty$$

Fact 3. *If $\Omega_0 \sim Bern(\rho_0)$,$\rho_0 \geq C_0 \epsilon^{-2} \frac{\mu r \log n}{n}$, then with high probability, we will have,*

$$||P_T - \rho_0^{-1} P_T P_{\Omega_0} P_T|| \leq \epsilon$$

Fact 4. *If $\Omega_0 \sim Bern(\rho)$ and $1 - \rho \geq C_0 \epsilon^{-2} \frac{\mu r \log n}{n}$, then with high probability $||P_\Omega P_T||^2 \leq \rho + \epsilon$*

Now we present the proof of Lemma (3)

Proof. We define another sequence of matrix $Z_j = UV^* - P_T(Y_j)$. There are some properties about Z_j which allows us to establish the proof. We survey them here and provides the proof of them.

i) Note that

$$Z_j = (P_T - \frac{1}{q} P_T P_{\Omega_j} P_T) Z_{j-1}. \tag{14}$$

The reason is as follows.

$$\begin{aligned}
& Z_j \\
= \quad & UV^* - P_T(Y_{j-1} + \frac{1}{q} P_{\Omega_j} P_T(UV^* - Y_{j-1})) \text{because of construction of } Y_j \\
= \quad & UV^* - P_T(Y_{j-1}) - \frac{1}{q} P_T P_{\Omega_j} P_T(UV^* - Y_{j-1}) \text{because of linearity of } P_T \\
= \quad & Z_{j-1} - q^{-1}(P_T P_{\Omega_j}(UV^* - P_T(Y_{j-1}))) \text{ because } P_T(UV^*) = UV^* \\
= \quad & P_T(Z_{j-1}) - q^{-1}(P_T P_{\Omega_j} P_T Z_{j-1}) \text{ because } Z_{j-1} \in T \\
= \quad & (P_T - q^{-1} P_T P_{\Omega_j} P_T) Z_{j-1} \text{because of linearity}
\end{aligned}$$

ii) If $q \geq C_0 \epsilon^{-2} \frac{\mu r \log n}{n}$, then with high probability,

$$||Z_j||_\infty \leq \epsilon^j ||UV^*||_\infty \tag{15}$$

The reason is as follows. By Fact (1), we have,

$$\begin{aligned}
||Z_{j-1} - q^{-1} P_T P_{\Omega_j} Z_{j-1}||_\infty \quad & \leq \quad \epsilon ||Z_{j-1}||_\infty \\
||Z_j||_\infty \quad & \leq \quad \epsilon ||Z_{j-1}||_\infty \text{because of (14)}
\end{aligned}$$

Inductively, we get desired.

iii) If $q \geq C_0 \epsilon^{-2} \frac{\mu r \log n}{n}$, then

$$||Z_j||_F \leq \epsilon^j \sqrt{r} \tag{16}$$

The reason is as follows. By Fact (3), we have,

$$\begin{aligned}
||(P_T - q^{-1}P_T P_{\Omega_0} P_T)(\frac{Z_{j-1}}{||Z_{j-1}||_F})||_F &\leq \epsilon \\
||(P_T - q^{-1}P_T P_{\Omega_0} P_T)Z_{j-1}||_F &\leq \epsilon ||Z_{j-1}||_F \text{by rearranging terms} \\
||Z_j||_F &\leq \epsilon ||Z_{j-1}||_F \text{by (14)}
\end{aligned}$$

Inductively, we get desired.

After establishing these properties, we are ready to prove that golfing scheme yields W^L that satisfies the desired properties.

1) Proof of condition (1):

$$\begin{aligned}
||W^L|| &= ||P_{T^\perp}(Y_{j_0})|| \text{by definition} \\
&\leq \sum_{j=1}^{j_0} \frac{1}{q} ||P_{T^\perp} P_{\Omega_j} Z_{j-1}|| \text{ because } Y_j = Y_{j-1} + q^{-1} P_{\Omega_j}(Z_{j-1}) \\
&= \sum_{j=1}^{j_0} ||P_{T^\perp}(\frac{1}{q} P_{\Omega_j} Z_{j-1} - Z_{j-1})|| \text{ because } Z_j \in T \\
&\leq \sum_{j=1}^{j_0} ||(\frac{1}{q} P_{\Omega_j} Z_{j-1} - Z_{j-1})|| \text{ because } ||P_{T^\perp}(M)|| \leq ||M|| \\
&\leq C_0' \sqrt{\frac{n \log n}{q}} \sum_{j=1}^{j_0} ||Z_{j-1}||_\infty \text{ because Fact(2)} \\
&\leq C_0' \sqrt{\frac{n \log n}{q}} \sum_{j=1}^{j_0} \epsilon^j ||UV^*||_\infty \text{by (15)} \\
&\leq C_0' \sqrt{\frac{n \log n}{q}} \frac{1}{1-\epsilon} ||UV^*||_\infty \text{by bound on geometric series} \\
&\leq C_0' \sqrt{\frac{n \log n}{q}} \frac{1}{1-\epsilon} \frac{\sqrt{\mu r}}{n} \text{by RPCA assumptions} \\
&\leq C'' \epsilon < \frac{1}{4} \text{for some constant } C''
\end{aligned}$$

2) Proof of condition (2) : First, we expand,

$$||P_\Omega(UV^* + W^L)||_F = ||P_\Omega(UV^* + P_{T^\perp} Y_{j_0})||_F$$

Then, because $P_\Omega(Y_{j_0}) = P_\Omega(\sum_j P_{\Omega_j} Z_{j-1}) = 0$ and $P_\Omega(P_T(Y_{j_0}) + P_{T^\perp}(Y_{j_0})) = 0$, we have,

$$||P_\Omega(UV^* + W^L)||_F = ||P_\Omega(UV^* - P_T Y_{j_0})||_F$$

Continuing,

$$\begin{aligned}
||P_\Omega(UV^*+W^L)||_F &= ||P_\Omega(Z_{j_0})||_F \text{by definition} \\
&\leq ||Z_{j_0}||_F \text{because of summing over larger set} \\
&\leq \epsilon^{j_0}\sqrt{r} \text{by (16)} \\
&\leq \sqrt{r}\frac{1}{n^2} \leq \frac{\lambda}{4} \text{by the choice of} \lambda \text{and} \epsilon
\end{aligned}$$

3) Proof of condition (3) :

$$\begin{aligned}
||P_{\Omega^\perp}(UV^*+W^L)||_\infty &= ||P_{\Omega^\perp}(Z_{j_0}+Y_{j_0})||_\infty \text{by definition} \\
&\leq ||Z_{j_0}||_\infty + ||Y_{j_0}||_\infty \text{by triangle inequality and summing over larger set} \\
&\leq ||Z_{j_0}||_F + ||Y_{j_0}||_\infty \text{by the properties of Frobenius and infinite norms} \\
&\leq \frac{\lambda}{8} + ||Y_{j_0}||_\infty \text{similar argument as in Proof of condition (2)}
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
||Y_{j_0}||_\infty &\leq q^{-1}\sum_j ||P_{\Omega_j}Z_{j-1}||_\infty \text{ by triangle inequality} \\
&\leq q^{-1}\sum_j ||Z_{j-1}||_\infty \text{ by summing over a larger set} \\
&\leq q^{-1}\sum_j \epsilon^j\frac{\sqrt{\mu r}}{n} \text{ by (15)} \\
&\leq \frac{\lambda}{8} \text{ if } \epsilon \text{ is sufficiently small}
\end{aligned}$$

□

3.7 Proof of the Lemma about least square construction and dual certificate

Construction of W^S:

$$W^S = \lambda P_{T^\perp}((P_\Omega - P_\Omega P_T P_\Omega)^{-1} sign(S_0))$$

Now we present the proof of Lemma(4).

Proof. We consider the sign of S_0 to be distrbuted as follows

$$sign(S_0)_{i,j} = \begin{cases} 1 & \text{wp } \frac{\rho}{2} \\ 0 & \text{wp } 1-\rho \\ -1 & \text{wp } \frac{\rho}{2} \end{cases}$$

1) Proof of condition (1) :

I) We note the we can separate W^Sinto two parts and then bound them separately.

$$W^S \quad = \quad \lambda P_{T^\perp}(sign(S_0)) + \lambda P_{T^\perp}(\sum_{k\geq 1}(P_\Omega P_T P_\Omega)^k(sign(S_0)))$$

II) Then, we have

$$\begin{aligned}\lambda||P_{T^\perp}(sign(S_0))|| &\leq \lambda||sign(S_0)|| \text{ by (9)} \\ &= \frac{1}{\sqrt{n}}||sign(S_0)|| \text{ by the choice of}\lambda \\ &\leq 4\sqrt{\rho}\text{with high probability}\end{aligned}$$

where the last inequality uses the fact that for the entry-wise distribution of $sign(S_0)$, we can have $||sign(S_0)|| \leq 4\sqrt{n\rho}$ holds with high probability.

III) Now, for the other part, $\lambda P_{T^\perp}(\sum_{k\geq 1}(P_\Omega P_T P_\Omega)^k(sign(S_0)))$, we bound it by first expressing it in the form of $< X, sign(S_0) >$ and then claim that with high probability, this term is bounded above as desired. Let $R = \sum_{k\geq 1}(P_\Omega P_T P_\Omega)^k$, then we have,

$$\begin{aligned}||P_{T^\perp}(R(sign(S_0)))|| &\leq ||R(sign(S_0))|| \\ &\leq 4 \sup_{x,y\in N} < y, R(sign(S_0)x) >\end{aligned}$$

where the last inequality uses the fact that there exists a $\frac{1}{2} - net$ of the Eucledean ball and it has at most 6^n elements. Continuing, we have

$$\begin{aligned}||P_{T^\perp}(R(sign(S_0)))|| &\leq 4 \sup_{x,y\in N} < y, R(sign(S_0)x) > &(17)\\ &= 4 \sup_{x,y\in N} < yx^*, R(sign(S_0)) > &(18)\\ &= 4 \sup_{x,y\in N} < R(yx^*), sign(S_0) > &(19)\end{aligned}$$

and that we denote $X(x,y) =< R(yx^*), sign(S_0) >$ afterwards.

Note that, by Hoeffding's inequality, we have,

$$Pr(|X(x,y)| > t \mid \Omega) \quad \leq \quad 2exp(-\frac{t^2}{2||R(xy^*)||_F^2})$$

This gives,

$$\begin{aligned}Pr(||P_{T^\perp}(R(sign(S_0)))|| > 4t \mid \Omega) &\leq Pr(||R(sign(S_0))|| > 4t \mid \Omega) \\ &\leq Pr(\sup_{x,y}|X(x,y)| > t \mid \Omega) \text{ by (19)} \\ &\leq 2N^2exp(-\frac{t^2}{2||R||_F^2})\text{because } ||yx^*||_F \leq 1\end{aligned}$$

Now, we proceed to bound the probability without the condition on Ω.

First, note that the event of $||P_\Omega P_T|| \leq \sigma = \rho + \epsilon$, implies that $||R|| \leq (\frac{\sigma^2}{1-\sigma^2})^2$. Thus, unconditionally, we have

$$\begin{aligned} Pr(||R(sign(S_0))|| > 4t) &\leq 2|N|^2 exp(\frac{-t^2}{2(\frac{\sigma^2}{1-\sigma^2})^2}) + Pr(||P_\Omega P_T|| > \sigma) \\ &\leq 2 \cdot 6^{2n} exp(\frac{-t^2}{2(\frac{\sigma^2}{1-\sigma^2})^2}) + Pr(||P_\Omega P_T|| > \sigma) \end{aligned}$$

Thus, where we finally put $t = \frac{1}{16}$

$$Pr(\lambda||R(sign(S_0))|| > 4t) \leq 2 \cdot 6^{2n} exp(\frac{-\frac{t^2}{\lambda^2}}{2(\frac{\sigma^2}{1-\sigma^2})^2}) + Pr(||P_\Omega P_T|| > \sigma)$$

With $\lambda = \sqrt{\frac{1}{n}}$,we have this probability$\rightarrow 0$ as $n \rightarrow \infty$. Thus with high probability $||W^S|| \leq \frac{1}{4}$

2) Proof of condition (2) :

The idea is that we first express $P_{\Omega^\perp}(W^S)$ in the form of $< X, sign(S_0) >$and we can derive upper bound on it if highly probably event of $\{||P_\Omega P_T|| \leq \sigma\}$ for some small $\sigma = \rho + \epsilon$ holds .

I) First,

$$\begin{aligned} P_{\Omega^\perp}(W^S) &= P_{\Omega^\perp}(\lambda(I - P_T)(\sum_{k\geq 0}(P_\Omega P_T P_\Omega)^k)sign(S_0)) \text{ by } P_{T^\perp = I - P_T} && (20) \\ &= -\lambda P_{\Omega^\perp} P_T (P_\Omega - P_\Omega P_T P_\Omega)^{-1} sign(S_0) \text{ by summing over terms and canceling} && (21) \end{aligned}$$

For $(i,j) \in \Omega^C$, we have

$$\begin{aligned} e_i^* W^S e_j &= < e_i e_j^*, W_S > \text{ by property of trace} \\ &= < e_i e_j^*, -\lambda P_{\Omega^\perp} P_T (P_\Omega - P_\Omega P_T P_\Omega)^{-1} sign(S_0) > \text{ by (21)} \\ &= -\lambda < e_i e_j^*, P_T (P_\Omega - P_\Omega P_T P_\Omega)^{-1} sign(S_0) > \text{ by rearranging terms} \\ &= -\lambda < e_i e_j^*, P_T P_\Omega (P_\Omega - P_\Omega P_T P_\Omega)^{-1} sign(S_0) > \text{ by the property of the inverse} \\ &= -\lambda < e_i e_j^*, P_T \sum_{k\geq 0}(P_\Omega P_T P_\Omega)^k sign(S_0) > \text{ by infinite sum representation} \end{aligned}$$

Noting that P_Ω, P_T are self-adjoint, thus, we have

$$e_i^* W^S e_j = \lambda < -(P_\Omega - P_\Omega P_T P_\Omega)^{-1} P_\Omega P_T (e_i e_j^*), sign(S_0) > \quad (22)$$

where we now denote $X(i,j) = -(P_\Omega - P_\Omega P_T P_\Omega)^{-1} P_\Omega P_T (e_i e_j^*)$

II)We now consider, where we put $t = \frac{1}{4}$,

$$\begin{aligned} Pr(||P_{\Omega^\perp}(W^S)||_\infty > t\lambda \mid \Omega) &\leq \sum_{(i,j)\in\Omega^C} Pr(|e_i^* W^S e_j| > t\lambda|\Omega) \text{ by union bound} \\ &\leq n^2 Pr(|e_i^* W^S e_j| > t\lambda|\Omega) \text{ for some (i,j) by taking the maximum} \\ &= n^2 Pr(| < X(i,j), sign(S_0) > | > t|\Omega) \text{ by (22)} \\ &\leq 2n^2 exp(-\frac{2t^2}{4||X(i,j)||_F}) \text{ because of Hoeffding's inequality} \end{aligned}$$

III) We then proceed to bound the $||X(i,j)||$. On the event of $\{||P_\Omega P_T|| \leq \sigma\}$, we have ,

$$\begin{aligned}
||P_\Omega P_T(e_i e_j^*)||_F &\leq ||P_\Omega P_T|| \cdot ||P_T(e_i e_j^*)||_F \text{ by property of spectral norm} \\
&\leq \sigma\sqrt{\frac{2\mu r}{n}} \text{ by (11) and the bound on } ||P_\Omega P_T||
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
||(P_\Omega - P_\Omega P_T P_\Omega)^{-1}|| &\leq \sum_{k\geq 0} ||(P_\Omega P_T P_\Omega)^k|| \text{ by triangle inequality} \\
&\leq \frac{1}{1-\sigma} \text{ by the bound on } ||P_\Omega P_T||
\end{aligned}$$

Finally, we have

$$||X(i,j)||_F \leq 2\sigma^2 \frac{\frac{\mu r}{n}}{(1-\sigma)^2}$$

Combining, we have

$$\begin{aligned}
Pr(||P_{\Omega^\perp} W^S|| > t\lambda) &\leq 2n^2 exp(\frac{-t^2 n(1-\sigma)^2}{4\sigma^2(\mu r)}) + Pr(||P_\Omega P_T|| \geq \sigma) \\
&\leq \epsilon \text{ if } \mu r < \rho_r^{'} \frac{n}{\log n}
\end{aligned}$$

□

3.8 Proof of the equivalence of the Bernoulli sampling and uniform sampling model

To complete the story about the equivalence of sampling model, we present theorem.
Theorem 5. *Let E be the event that the recovery of (L_0, S_0) is exact through the RPCA. Then, $\forall\epsilon > 0$,*

- With $\rho = \frac{m}{n^2} + \epsilon$, E holds with high probability when the sparse matrix $S_{i,j} \sim Bern(\rho)$ iid $\Longrightarrow$ E holds with high probability when the sparse matrix $S \sim Uniform(m)$.
- With $\rho = \frac{m}{n^2} - \epsilon$, E holds with high probability when the sparse matrix $S \sim Uniform(m)$ $\Longrightarrow$ E holds with high probability when the sparse matrix $S_{i,j} \sim Bern(\rho)$ iid

Proof. Let us use the notation of subscrpt to denote the underlying sampling process, e.g. ,$P_{B(\rho)}(E)$ and $P_{U(m)}(E)$ be the probability of success recovery using Bernoulli sampling and uniform sampling respectively. We then upper and lower bound the difference of $P_{B(\rho)}(E) - P_{U(m)}(E)$ and show that the difference goes to zero as the dimension of the matrix $n \to \infty$.

$$
\begin{aligned}
& P_{B(\rho)}(E) \\
= & \sum_{i=0}^{n^2} P_{B(\rho)}(|\Omega| = i) P_{B(\rho)}(E \mid |\Omega| = i) \\
= & \sum_{i=0}^{n^2} P_{B(\rho)}(|\Omega| = i) P_{U(i)}(E) \\
\leq & \sum_{i=0}^{m-1} P_{B(\rho)}(|\Omega| = i) + \sum_{i=m}^{n^2} P_{U(i)}(E) P_{B(\rho)}(|\Omega| = i) \\
\leq & \sum_{i=0}^{m-1} P_{B(\rho)}(|\Omega| = i) + \sum_{i=m}^{n^2} P_{U(i)}(E) P_{B(\rho)}(|\Omega| = i) \\
\leq & \sum_{i=0}^{m-1} P_{B(\rho)}(|\Omega| = i) + \sum_{i=m}^{n^2} P_{U(m)}(E) P_{B(\rho)}(|\Omega| = i) \\
\leq & P_{B(\rho)}(|\Omega| < m) + P_{U(m)}(E)
\end{aligned}
$$

This gives, $P_{B(\rho)}(E) - P_{U(m)}(E) \leq P_{B(\rho)}(|\Omega| < m)$. With $\rho = \frac{m}{n^2} + \epsilon$, by law of large number, when $n \to \infty$ we get, $P_{B(\rho)}(E) \leq P_{U(m)}(E)$.

On the other hand,

$$
\begin{aligned}
& P_{B(\rho)}(E) \\
\geq & \sum_{i=0}^{m} P_{B(\rho)}(|\Omega| = i) P_{B(\rho)}(E \mid |\Omega| = i) \\
\geq & P_{U(m)}(E) \sum_{i=0}^{m} P_{B(\rho)}(|\Omega| = i) \\
= & P_{U(m)}(E)(1 - P_{B(\rho)}(|\Omega| > m)) \\
\geq & P_{U(m)}(E) - P_{B(\rho)}(|\Omega| > m)
\end{aligned}
$$

This gives, $P_{B(\rho)}(E) - P_{U(m)} \geq -P_{B(\rho)}(|\Omega| > m)$. With $\rho = \frac{m}{n^2} - \epsilon$, by law of large number, when $n \to \infty$ we get, $P_{B(\rho)}(E) \geq P_{U(m)}(E)$. $\square$

3.9 Proof of the form of sub-differential of nuclear norm

To complete the story on the structure of subdifferential of nuclear norm, we present the following justifications.

Definition 1. For matrix norms $||\cdot||$ which satisfy $||UAV|| = ||A||\ \forall U, V$ being orthonormal, then they are called orthogonally invariant norm.

Definition 2. For orthogonally invariant norm $||\cdot||$ which is defined by its singular values $||A|| = \phi(\vec{\sigma})$ where $\vec{\sigma}$ are the singular values of A, we call the function ϕ as a symmetric gauge function if it is a norm and it satisfies $\phi(\vec{\sigma}) = \phi(\epsilon_1\sigma_{i_1}, ..., \epsilon_n\sigma_{i_n})$ for any permulation of $(i_1, ..., i_n)$ of $(1, ..., n)$ and $\epsilon_i = \pm 1$.

Fact 5. *For orthogonally invariant norm $||\cdot||$ with symmetric gauge function ϕ, the sub-differential is given by*

$$\partial||A|| \quad = \quad \{U diag(\vec{d})V \mid A = U\Sigma V^T, \vec{d} \in \partial\phi(\vec{d}), U \in R^m, V \in R^n\}$$

Theorem 6. *Let $A = U^{(1)}\Sigma {V^{(1)}}^T$ then $\partial||A||_* = \{U^{(1)}{V^{(1)}}^T + W : ||W|| \leq 1, {U^{(1)}}^T W = 0, WV^{(1)} = 0\}$*

Proof. We take the symmetric gauge function as $||\cdot||_1$and then apply the Fact (5) and will obtain desired. □

4 Related Problems and Extensions

4.1 Exact Matrix completion

Robust PCA is an extension of the exact matrix completion problem introduced in [4], where one seeks to recover a low-rank matrix L_0 from a small fraction of its entries. More precisely, assume one is given $\{(L_0)_{ij}, (i,j) \in \Omega\}$ where Ω is a subset of $[n] \times [n]$. The observed matrix in this case is

$$M = P_\Omega L_0$$

where P_Ω denotes the sampling operator, i.e. the orthogonal projection on the subspace of matrices supported on Ω. One seeks to solve the problem

$$\begin{array}{ll} \text{minimize} & \operatorname{rank}(L) \\ \text{subject to} & P_\Omega L = P_\Omega L_0 \end{array} \tag{23}$$

A heuristic is to minimize the nuclear norm of L, $\|L\|_* = \|\sigma(L)\|_1$ which encourages sparsity of the vector of singular components of L, and can thus be considered an approximation of the rank operator, similarly to the ℓ_1-norm that can be considered an approximation of the ℓ_0 count operator.

$$\begin{array}{ll} \text{minimize} & \|L\|_* \\ \text{subject to} & P_\Omega L = P_\Omega L_0 \end{array} \tag{24}$$

4.1.1 Incoherence

In order to guarantee recovery with high probability, an incoherence condition is introduced, similar to the one in the Robust PCA framework, though slightly different. First consider an orthogonal matrix $U = [u_1, \ldots, u_n]$, and define its coherence of $\mu(U)$ with respect to the canonical basis to be

$$\mu(U) = \frac{n}{r} \max_i \|P_U e_i\|_2^2 = \frac{n}{r} \max_i \left[\sum_{k=1}^{r} u_{ki}^2\right] \tag{25}$$

$\mu(U)$ is a measure of spread of the vectors $u_1, \ldots, u_n$ with respect to the canonical basis. One seeks matrices with low coherence, since intuitively, those matrices will have low probability to be in the null space of the sampling operator P_Ω.

4.1.2 Main result

Theorem 6. *Let the SVD of original matrix L_0 be given by $L_0 = U\Sigma V^T$, and assume that the following conditions hold:*

- $\max\{\mu(U), \mu(V)\} \le \mu_0$
- $\left(\sum_k u_k v_k^T\right)_{ij} \le \mu_1 \sqrt{\frac{r}{n_1 n_2}}$ *(true for $\mu_1 = \mu_0\sqrt{r}$)*

- $m \geq c \max\{\mu_1^2, \sqrt{\mu_0}\mu_1, \mu_0 n^{1/4}\} n r \beta \log n$

Then recovery is exact with high probability (at least $1 - \frac{c}{n\beta}$*)*

The authors also give a list of models that can be used to generate incoherent matrices. Let the SVD of L_0 be given by $L_0 = \sum_k \sigma_k u_k v_k^*$. Then L_0 is incoherent with high probability if it is sampled from:

- The incoherent basis model: U and V satisfy the size property

  $$\|U\|_\infty \leq \sqrt{\mu_B/n} \quad \|V\|_\infty \leq \sqrt{\mu_B/n}$$

  for some numerical constant μ_B. Observe that under these conditions, one can bound the coherence $\max(\mu(U), \mu(V)) \leq \mu_B$, and it can be shown that the second condition of Theorem 6 holds for $\mu_1 = O(\sqrt{\log n})$.

- The random orthogonal model: if , then $\{u_1, \ldots, u_r\}$ and $\{v_1, \ldots, v_r\}$ are assumed to be selected at random.

4.1.3 SDP formulation

Observe that the problem is equivalent to the SDP

$$\begin{array}{ll} \text{minimize}_{L,W_1,W_2} & tr(W_1) + tr(W_2) \\ \text{subject to} & P_\Omega L = P_\Omega L_0 \\ & \begin{bmatrix} W_1 & L \\ L^T & W_2 \end{bmatrix} \succeq 0 \end{array} \tag{26}$$

4.1.4 Comparing results to Robust PCA

Robust PCA can be thought of as an extension of the matrix completion problem, where instead of having a known subset of the entries $\{(L_0)_{ij}, (i,j) \in \Omega\}$ and the rest is missing, we have an unknown subset of the entries and the rest is corrupted. In this sense, Robust PCA is a harder problem.

Note that the matrix L_0 can be recovered by Principal Component Pursuit, solving a different problem:

$$\begin{array}{ll} \text{minimize} & \|L\|_* + \lambda \|S\|_1 \\ \text{subject to} & P_\Omega(L + S) = M \end{array} \tag{27}$$

where now the observed matrix M is assumed to be given by

$$M = P_\Omega(L_0 + S_0) = P_\Omega(L_0) + S_0'$$

Here the original data matrix L_0 is assumed to be corrupted with the noise matrix S_0 in addition to being under-sampled. The exact matrix completion problem however, assumes that the observed data is perfect $S_0 = 0$. Under the assumptions of Theorem 1, recovery is exact with high probability, in particular for $S_0 = 0$ (support of the sparse matrix has cardinality 0).

4.2 Stable Principal Component Pursuit

4.2.1 Overview

The paper studies the problem of recovering a low-rank matrix (the principal components) from a high- dimensional data matrix despite both small entry-wise noise and gross sparse errors. It proves that the solution to a convex program (a relaxation of classic Robust PCA) gives an estimate of the low-rank matrix that is simultaneously stable to small entry- wise noise and robust to gross sparse errors. The result shows that the proposed convex program recovers the low-rank matrix even though a positive fraction of its entries are arbitrarily corrupted, with an error bound proportional to the noise level.

4.2.2 Main result

The paper consider a matrix $M \in \mathbb{R}^{n_1 \times n_2}$ of the from $M = L_0 + S_0 + Z_0$, where L_0 is (non-sparse) low rank, S_0 is sparse (modeling gross errors) and Z_0 is "small" (modeling a small noisy perturbation). The assumption on Z_0 is simply that $||Z_0||_F \leq \delta$ for some small known δ. Hence at least for the theory part of the paper the authors do not assume anything about the distribution of the noise other than it is bounded (however they will gloss over this in their algorithm).

The convex program to be solved is a slight modification of the standard Robust PCA problem and given by

$$\begin{aligned} \min_{L,S} \; & ||L||_* + \lambda ||S||_1 \\ \text{s.t.} \quad & ||M - L - S||_F \leq \delta \end{aligned} \tag{28}$$

where $\lambda = 1/\sqrt{n_1}$. Under a standard incoherence assumption on L_0 (which essentially means that L_0 should not be sparse) and a uniformity assumption on the sparsity pattern of S_0 (which means that the support of S_0 should not be too concentrated) the main result states that, with high probability in the support of S_0, for any Z_0 with $||Z_0||_F \leq \delta$, the solution $(\hat{L}, \hat{S})$ to (28) satisfies

$$||\hat{L} - L_0||_F^2 + ||\hat{S} - S_0||_F^2 \leq C n_1 n_2 \delta^2$$

where C is a numerical constant. The above claim essentially states that the recovered low-rank matrix $\hat{L}$ is stable with respect to non-sparse but small noise acting on all entries of the matrix.

In order to experimentally verify the predicted performance to their formulation, the author provide a comparison with an oracle. This oracle is assumed to provide information about the support of S_0 and the row and column spaces of L_0, which allows the computation of the MMSE estimator which otherwise would be computationally intractable (strictly speaking it of course is not really the MMSE, since it uses additional information from the oracle). Simulation results that show that the RMS error of the solution obtained through (28) in the non-breakdown regime (that is, for the support of S_0 sufficiently small) is only about twice as large as that of the oracle-based MMSE. This suggests that the proposed algorithm works quite well in practice.

4.2.3 Relations to existing work

The result of the paper can be seen from two different view points. On the one hand, it can be interpreted from the point of view of standard PCA. In this case, the result states that standard PCA, which can in fact be shown to be statistically optimal w.r.t. i.i.d Gaussian perturbations, can also be made robust with respect to sparse gross corruptions. On the other hand, the result can be interpreted from the point of view of Robust PCA. In this case, it essentially states that the classic Robust PCA solution can itself be made robust with respect to some small but non-sparse noise acting on all entries of the matrix.

Conceptually, the work presented in the paper is similar to the development of results for "imperfect" scenarios in compressive sensing where the measurements are noisy and the signal is not exact sparse. In this body of literature, l_1-norm minimization techniques are adapted to recover a vector $x_0 \in \mathbb{R}^n$ from contaminated observations $y = Ax_0 + z$, where $A \in \mathbb{R}^{m \times n}$ with $m \ll n$ and z is the noise term.

4.2.4 Algorithm

For the case of a noise matrix Z_0 whose entries are i.i.d. $\mathcal{N}(0, \sigma^2)$, the paper suggests to use an Accelerated Proximal Gradient (APG) algorithm (see algorithms section for details) for solving (28). Note that for $\delta = 0$ the problem reduces to the standard Robust PCA problem with an equality constraint on the matrices. For this case the APG algorithm proposed in [6] solves an approximation of the form

$$\min_{L,S} ||L||_* + \lambda ||S||_1 + \frac{1}{2\mu} ||M - L - S||_F$$

For the Stable PCP problem where $\delta > 0$ the authors advocate using the same algorithm with fixed but carefully chosen parameter μ (similar to [3]). In particular, they point out[1] that for $Z_0 \in \mathbf{R}^{n \times n}$ with $(Z_0)_{ij} \sim \mathcal{N}(0, \sigma^2)$ i.i.d. it holds that $n^{-1/2}\|Z_0\|_2 \to \sqrt{2}\sigma$ almost surely as $n \to \infty$. They then choose the parameter μ such that if $M = Z_0$, i.e. if $L_0 = S_0 = 0$, the minimizer of the above problem is likely to be $\hat{L} = \hat{S} = 0$. The claim is that this is the case for $\mu = \sqrt{2n}\sigma$.

It is worth noting that the assumption of a Gaussian noise matrix Z_0 is reasonable but not always satisfied. If it is not, then it is not clear if using the APG algorithm to solve the associated approximate problem is a good idea and different algorithms may be needed. The problem (28) can be expressed as an SDP and can therefore in principle be solved using general purpose interior point solvers. However, the same scalability issues as in the standard Robust PCA problem will limit prohibit to use these methods for high-dimensional data. The paper [1] focuses on efficient first-order algorithms for solving (28).

4.2.5 Conclusion and Outlook

The paper addresses a problem of potentially very high practical relevance. While it is reasonable to assume that in many applications the low-rank component L_0 will only be corrupted by a comparatively small number of gross errors (caused by rare and isolated events), the assumption

[1] this based on the strong Bai Yin Theorem [2], which implies that for an $n \times n$ real matrix with entries $\xi_{ij} \sim \mathcal{N}(0,1)$ the it holds that $\limsup_{n\to\infty} \|Z_0\|_2/\sqrt{n} = 2$ almost surely

of perfect measurements for the rest of the data outside the support of S_0 that is made in classic Robust PCA will generally not hold for example due to sensor noise. This paper asserts that if the non-sparse noise component Z_0 is sparse, then with high probability the recovered components are "close" to the actual ones.

For simplicity, the paper models the non-sparse noise simply as an additive perturbation that is bounded in the Frobenius norm. In cases where one has additional information available about this noise, for example its distribution or some bounds on the absolute value of each entry, it might be possible to derive better bounds on the resulting errors. One possible extension could therefore be to look at exploiting structure in the noise.

One thing the paper claims is that "at a cost not so much higher than the classical PCA, [the] result is expected to have significant impact on many practical problems". As mentioned above I do agree that the result has a significant impact on many practical problems. However, the claim concerning the computational complexity is very optimistic. The fastest solver for the special case $\delta = 0$ (classic Robust PCA) currently seems to be a alternating directions augmented Lagrangian method. This method requires an SVD at each iteration, and for problems involving large-scale data the number of iterations can be very large. The standard PCP algorithm on the other hand is based on a single SVD, hence it can be computed much faster.

4.3 Robust Alignment by Sparse and Low-rank Decomposition

The convex optimization framework for low-rank matrix recovery has been employed successfully. However, in practice, much more data can be viewed as low-rank only after some transformation is applied. The new formulation of this problem as Robust Alignment by Sparse and Low-rank Decomposition (RASL) [7]:

$$\min_{A,E,\tau} ||A||_* + \lambda ||E||_1 \quad \text{s.t. } D \circ \tau = A + E \tag{29}$$

where $A \in \mathbb{R}^{m \times n}$ is low-rank matrix, $A \in \mathbb{R}^{m \times n}$ is sparse matrix, D is our measurements, which is the result of $(A + E)$ subjecting to transformation τ^{-1}. Here we assume that the transformation is invertible. We define $D \circ \tau$ as: $D \circ \tau = [\ D_1 \circ \tau_1 \mid D_2 \circ \tau_2 \mid \ \dots \ \mid D_n \circ \tau_n\ ]$, which is the measurements $D = [\ D_1 \mid D_2 \mid \ \dots \ \mid D_n\ ]$subjects to set of transformations $\tau = [\ \tau_1 \mid \tau_2 \mid \ \dots \ \mid \tau_n\ ] \in \mathbb{G}^n$, where $\mathbb{G}$ is a group of certain type of invertible transformations, which could be affine transform, rotation transform, etc.

The main difficulty in solving (29) is the nonlinearity of constraint $D \circ \tau = A + E$. When the change in τ is small, we can approximate this constraint by linearizing about the current estimate of τ. Here, we assume that $\mathbb{G}$ is some p-parameter group and identify $\tau = [\ \tau_1 \mid \tau_2 \mid \ \dots \ \mid \tau_n\ ] \in \mathbb{R}^{p \times n}$ with the parameterizations of all of the transformations. For $\Delta\tau = [\ \Delta\tau_1 \mid \Delta\tau_2 \mid \ \dots \ \mid \Delta\tau_n\ ]$, write $D \circ (\tau + \Delta\tau) \approx D \circ \tau + \sum_{i=1}^{n} J_i \Delta\tau_i \epsilon_i$, where $J_i \doteq \frac{\partial}{\partial \zeta}(D_i \circ \zeta)|_{\zeta = \tau_i}$ is the Jacobian of the i-th measurement with respect to the transformation parameters τ_i. $\{\epsilon_i\}$ denotes the standard basis for $\mathbb{R}^n$. This leads to a convex optimization problem in unknowns $A, E, \Delta\tau$:

$$\min_{A,E,\Delta\tau} ||A||_* + \lambda ||E||_1 \quad \text{s.t. } D \circ \tau + \sum_{i=1}^{n} J_i \Delta\tau \epsilon_i \epsilon_i^T = A + E \tag{30}$$

It leads to algorithm 1

Algorithm 1: RASL

Input: $D = [\ D_1 \mid D_2 \mid \ \ldots \ \mid D_n]$, initial transformation $\tau_1, \tau_2, \ldots, \tau_n$ in a certain parametric group $\mathbb{G}$, weight $\lambda > 0$.

while *not converged* **do**

Step 1: compute Jacobian matrices w.r.t. transformation:

$$J_i \leftarrow \frac{\partial}{\partial \zeta}(D_i \circ \zeta)|_{\zeta=\tau_i}$$

Step 2 (inner loop): solve the linearized convex optimization:

$$(A^*, E^*, \Delta\tau^*) \leftarrow \underset{A,E,\Delta\tau}{\arg\min} ||A||_* + \lambda||E||_1 \quad \text{s.t. } D \circ \tau + \sum_{i=1}^{n} J_i \Delta\tau \epsilon_i \epsilon_i^T = A + E$$

Step 3: update the transformation: $\tau \leftarrow \tau + \Delta\tau^*$

Output: A^*, E^*, τ^*

4.4 Robust Matrix Decomposition With Sparse Corruptions: D Hsu et. al.

4.4.1 Question being addressed

Under deterministic setting, it studies how much sparsity is allowed for accurate recovery of the sparse-lowrank pairs.

4.4.2 Main ideas

Given M as the observed matrix, it analyze the following two optimization problems. With arguments as (L, S),

$$\begin{aligned} \min_{(L,S)} \quad & ||L||_* + \lambda||S||_1 \\ s.t. \quad & ||L + S - M||_1 \leq \epsilon_1 \\ & ||L + S - M||_* \leq \epsilon_* \end{aligned} \tag{31}$$

and

$$\min_{(L,S)} \quad ||L||_* + \lambda||S||_1 + \tfrac{1}{2\mu}||L + S - M||_F \tag{32}$$

It is remarked that the M is a perturbed observation outcome of the original (L_0, S_0) pairs.

4.4.3 Contributions

1. It provides sufficient conditions on sparsity of the original (L_0, S_0) pairs that allow accurate recovery in the sense that $||L_0 - \hat{L}||_\infty, ||S_0 - \hat{S}||_\infty$ is small.

2. If the observed matrix M is pertured from $L_0 + S_0$ by a small amount (i.e.ϵ), the optimizer $(\hat{L}, \hat{S})$ will be $\epsilon-$close to the orginal (L_0, S_0) pairs.

5 Generalization

6 Algorithms

7 Applications

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