

Robust Principal Component Analysis

Maximilian Balandat

Walid Krichene

Chi Pang Lam

Ka Kit Lam

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1 Introduction

Given an observed matrix $M \in \mathbb{R}^{n_1 \times n_2}$ that is formed as a superposition of a low-rank matrix L_0 and a sparse matrix S_0 ,

$$M = L_0 + S_0$$

Robust Principal Component Analysis [5] is the problem of recovering the low-rank and sparse components. Under suitable assumptions on the rank and incoherence of L_0 , and the distribution of the support of S_0 , the components can be recovered exactly with high probability, by solving the Principal Component Pursuit (PCP) problem given by

$$\begin{aligned} & \text{minimize} && \|L\|_* + \lambda \|S\|_1 \\ & \text{subject to} && L + S = M \end{aligned} \tag{1}$$

Principal component pursuit minimizes a linear combination of the nuclear norm of a matrix L and the ℓ_1 norm of $M - L$. Minimizing the ℓ_1 norm is known to favour sparsity, while minimizing the nuclear norm $\|L\|_* = \sum_{\sigma \in \sigma(L)} \sigma$ is known to favour low-rank matrices (intuitively, favours sparsity of the vector of singular values).

One cannot expect to recover the components exactly in the most general case. Assume for example that L_0 is such that $(L_0)_{ij} = \delta_i^1 \delta_j^1$, and $S_0 = -L_0$. Both matrices are sparse and low-rank, and clearly one cannot expect to recover the components in this case, since the observed matrix is $M = 0$. Therefore assumptions are made on the incoherence of L_0 and the support of S_0 .

1.1 Incoherence of the low rank component L_0

The Incoherence conditions describe how much the singular vectors of a given matrix are aligned with the vectors of the canonical basis.

Let the SVD of L_0 be given by

$$L_0 = U \Sigma V^* = \sum_{i=1}^r \sigma_i u_i v_i^* \tag{2}$$

Then the incoherence conditions are given by

$$\max_i \|U^* e_i\|_2^2 \leq \frac{\mu r}{n_1}, \quad \max_i \|V^* e_i\|_2^2 \leq \frac{\mu r}{n_2} \quad (3)$$

and

$$\|UV^*\|_\infty \leq \sqrt{\frac{\mu r}{n_1 n_2}} \quad (4)$$

These conditions require the singular vectors to be “spread” enough with respect to the canonical basis. Intuitively, if the singular vectors of the low-rank matrix L_0 are aligned with a few canonical basis vectors, then L_0 will be sparse and hard to distinguish from the sparse corruption matrix S_0 .

1.2 Support of the sparse component S_0

2 Main Result

Theorem 1. *Suppose $L_0 \in \mathbb{R}^{n \times n}$ satisfies incoherence conditions (3) and (4) and that the support of S_0 is uniformly distributed among all sets of cardinality m . Then $\exists c$ such that with high probability over the choice of support of S_0 (at least $1 - cn^{-10}$), Principal Component Pursuit with $m = 1/\sqrt{n}$ is exact, i.e. $\hat{L} = L_0$ and $\text{hat}S = S_0$ provided that*

$$\text{rank}(L_0) \leq \frac{\rho_r}{\mu} \frac{n}{(\log n)^2} \quad \text{and} \quad m \leq \rho_s n^2 \quad (5)$$

Above, ρ_r and ρ_s are positive numerical constants. Note in particular that no assumptions are made on the magnitudes of the nonzero entries of S_0 .

3 Proof

3.1 Preliminaries

- The subgradient of the ℓ_1 norm at S_0 supported on Ω is of the form $\text{sgn}(S_0) + F$ where $P_\Omega F = 0$ and $\|F\|_\infty \leq 1$.
- The subgradient of the nuclear norm at $L_0 = U\Sigma V^*$ is of the form $UV^* + W$ where

$$\begin{aligned} U^*W &= 0 \\ WV &= 0 \\ \|W\| &\leq 1 \end{aligned} \quad (6)$$

or equivalently

$$\begin{aligned} P_T W &= 0 \\ \|W\| &\leq 1 \end{aligned} \quad (7)$$

where T is the linear space of matrices defined by

$$T = \{UX^* + YV^*, X, Y \in \mathbb{R}^{n \times r}\}$$

Indeed, we have

$$\begin{aligned} P_T W = 0 &\Leftrightarrow W \in T^\perp \\ &\Leftrightarrow \forall M \in T, \text{Tr}(W^* M) = 0 \\ &\Leftrightarrow \forall X, Y \in \mathbb{R}^{n \times r}, \text{Tr}(W^* (UX^* + YV^*)) = 0 \\ &\Leftrightarrow \forall X, Y \in \mathbb{R}^{n \times r}, \text{Tr}((U^* W)^* X^*) + \text{Tr}(W V^* Y) = 0 \\ &\Leftrightarrow U^* W = W V^* = 0 \end{aligned}$$

Note that the projection on the orthogonal of T is given by

$$P_{T^\perp} M = (I - UU^*)M(I - VV^*)$$

Proof. we have □

Note that UU^* is the orthogonal projection on the subspace spanned by the columns of U , and similarly for VV^* . Therefore we have in particular

$$\|P_{T^\perp} M\| \leq \|M\|$$

and for any dyad $e_i e_j^*$, we have

$$\begin{aligned} \|P_{T^\perp} e_i e_j^*\|_F^2 &= \text{Tr}((I - UU^*)e_i e_j^*(I - VV^*)(I - VV^*)^* e_j e_i^*(I - UU^*)^*) \\ &= \text{Tr}(e_i^*(I - UU^*)^*(I - UU^*)e_i e_j^*(I - VV^*)(I - VV^*)^* e_j) \\ &= \text{Tr}(e_i^*(I - UU^*)^*(I - UU^*)e_i) \text{Tr}(e_j^*(I - VV^*)(I - VV^*)^* e_j) \\ &= \|(I - UU^*)e_i\|_2^2 \|(I - VV^*)e_j\|_2^2 \end{aligned}$$

and since UU^* is an orthogonal projection, we have

$$\begin{aligned} \|(I - UU^*)e_i\|_2^2 &= \|e_i\|_2^2 - \|UU^* e_i\|_2^2 \\ &\geq 1 - \mu r/n \end{aligned}$$

where the last inequality results from the incoherence condition ??, $\|U^* e_i\|_2^2 \leq \frac{\mu r}{n}$. Therefore

$$\|P_{T^\perp} e_i e_j^*\|_F^2 \geq (1 - \mu r/n)^2$$

equivalently, using the fact that $\|P_{T^\perp} e_i e_j^*\|_F^2 + \|P_T e_i e_j^*\|_F^2 = \|e_i e_j^*\|_F^2 = 1$, we have

$$\begin{aligned} \|P_T e_i e_j^*\|_F^2 &\leq 1 - (1 - \frac{\mu r}{n})^2 \\ &= \frac{2\mu r}{n} - \left(\frac{\mu r}{n}\right)^2 \\ &\leq \frac{2\mu r}{n} \end{aligned}$$

3.2 Elimination Theorem

The following elimination theorem states the intuitive fact that if PCP exactly recovers the components of $M = L + S$, then it also exactly recovers the components of $M = L + S'$ where S' is a trimmed version of S ($\text{supp}(S') \subset \text{supp}(S)$ and S and S' coincide on $\text{supp}(S')$)

Theorem 2. *Suppose the solution to the PCP problem (1) with input data $M_0 = L_0 + S_0$ is unique and exact, and consider $M'_0 = L_0 + S'_0$ where S'_0 is a trimmed version of S_0 . Then the solution to (1) with input M'_0 is exact as well.*

Proof. Let $S'_0 = P_{\Omega_0} S_0$ and let (\hat{L}, \hat{S}) be the solution to (1) with input $L_0 + S'_0$. Then since (L_0, S'_0) is a feasible point for (1), it provides an upper bound on the optimal value

$$\|\hat{L}\|_* + \lambda \|\hat{S}\|_1 \leq \|L_0\|_* + \lambda \|S'_0\|_1$$

then decomposing S_0 into the orthogonal components $S_0 = P_{\Omega_0} S_0 + P_{\Omega_0^\perp} S_0 = S'_0 + P_{\Omega_0^\perp} S_0$, we have $\|S'_0\|_1 = \|S_0\|_1 - \|P_{\Omega_0^\perp} S_0\|_1$, thus we have

$$\|\hat{L}\|_* + \lambda \|\hat{S}\|_1 + \lambda \|P_{\Omega_0^\perp} S_0\|_1 \leq \|L_0\|_* + \lambda \|S_0\|_1$$

and using the triangular inequality

$$\|\hat{L}\|_* + \lambda \|\hat{S} + P_{\Omega_0^\perp} S_0\|_1 \leq \|L_0\|_* + \lambda \|S_0\|_1$$

we observe that $(\hat{L}, \hat{S} + P_{\Omega_0^\perp} S_0)$ is feasible for the problem with input $M = L_0 + S_0$, for which the optimal value is precisely $\|L_0\|_* + \lambda \|S_0\|_1$. Therefore by uniqueness of the solution, we have

$$\begin{aligned} \hat{L} &= L_0 \\ P_{\Omega_0^\perp} S_0 &= S_0 \end{aligned}$$

the second equality is equivalent to $\hat{S} = S_0 - P_{\Omega_0^\perp} S_0 = P_{\Omega_0} S_0 = S'_0$. This completes the proof. \square

3.3 Derandomization

Derandomization is used to show equivalence between the problem where the signs of the entries of S_0 are random, and the problem where the entries of S_0 have fixed signs.

3.4 Dual certificate

The following lemma gives a simple sufficient condition for the pair (L_0, S_0) to be the unique optimal solution to PCP.

Lemma 1. Assume that $\|P_\Omega P_T\| < 1$. Then (L_0, S_0) is the unique solution to PCP if $\exists(W, F)$ such that

$$\begin{aligned} UV^* + W &= \lambda(\text{sign}(S_0) + F) \\ P_T W &= 0 \\ \|W\| &< 1 \\ P_\Omega F &= 0 \\ \|F\|_\infty &< 1 \end{aligned} \tag{8}$$

Proof. We first prove that the condition $\|P_\Omega P_T\| < 1$ is equivalent to $\Omega \cap T = \{0\}$.

To prove that (L_0, S_0) is the unique optimizer, we show that for any feasible perturbation $(L_0 + H, S_0 - H)$ where $H \neq 0$ strictly increases the objective. Let

- $UV^* + W_0$ be an arbitrary subgradient of the nuclear norm at L_0 , where $\|W_0\| \leq 1$ and $P_T W_0 = 0$
- $\text{sgn}(S_0) + F_0$ be an arbitrary subgradient of the ℓ_1 -norm at S_0 , where $\|F_0\|_\infty \leq 1$ and $P_\Omega F_0 = 0$

Then we can lower bound the value of the objective

$$\|L_0 + H\|_* + \lambda\|S_0 - H\|_1 \geq \|L_0\|_* + \lambda\|S_0\|_1 + \langle UV^* + W_0, H \rangle - \lambda\langle \text{sgn}(S_0) + F_0, H \rangle$$

Now we pick a particular pair (W_0, F_0) such that

- $\langle W_0, H \rangle = \|P_{T^\perp} H\|_*$, for example $W_0 = P_{T^\perp} W$ where W is a normed matrix such that $\langle W, P_{T^\perp} H \rangle = \|P_{T^\perp} H\|_*$ (by duality of $\|\cdot\|$ and $\|\cdot\|_*$)
- $\langle F_0, H \rangle = -\|P_{\Omega^\perp} H\|_1$, for example $F_0 = -\text{sgn}(P_{\Omega^\perp} H)$

then we have

$$\|L_0 + H\|_* + \lambda\|S_0 - H\|_1 \geq \|L_0\|_* + \lambda\|S_0\|_1 + \|P_{T^\perp} H\|_* + \|P_{\Omega^\perp} H\|_1 + \langle UV^* - \lambda \text{sgn}(S_0), H \rangle$$

we can bound the inner product using the definition of W and F ,

$$\begin{aligned} |\langle UV^* - \lambda \text{sgn}(S_0), H \rangle| &= |\langle \lambda F - W, H \rangle| && \text{since } UV^* + W = \lambda(\text{sign}(S_0) + F) \\ &\leq |\langle W, H \rangle| + \lambda |\langle F, H \rangle| && \text{by the triangular inequality} \\ &\leq \beta (\|P_{T^\perp} H\|_* + \lambda \|P_{\Omega^\perp} H\|_1) \end{aligned}$$

where $\beta = \max(\|W\|, \|F\|_\infty) < 1$, and the last inequality follows from the fact that

$$\begin{aligned} \|P_{T^\perp} H\|_* &\geq \langle P_{T^\perp} H, W/\|W\| \rangle \geq \langle H, W/\|W\| \rangle \\ \|P_{\Omega^\perp} H\|_1 &\geq \langle P_{\Omega^\perp} H, F/\|F\|_\infty \rangle \geq \langle H, F/\|F\|_\infty \rangle \end{aligned}$$

Thus

$$\begin{aligned} \|L_0 + H\|_* + \lambda\|S_0 - H\|_1 - \|L_0\|_* - \lambda\|S_0\|_1 &\geq (1 - \beta) (\|P_{T^\perp} H\|_* + \lambda\|P_{\Omega^\perp} H\|_1) \\ &> 0 \end{aligned}$$

since, by assumption, $\Omega \cap T = 0$ and $H \neq 0$. Therefore the objective strictly increases with a non-zero perturbation. This completes the proof. \square

The proof of the main theorem will use a slightly different result, given by the following Lemma

Lemma 2. *Assume that $\|P_\Omega P_T\| \leq 1/2$. Then (L_0, S_0) is the unique solution to PCP if $\exists(W, F)$ such that*

$$\begin{aligned} UV^* + W &= \lambda(\text{sign}(S_0) + F + P_\Omega D) \\ P_T W &= 0 \\ \|W\| &\leq 1/2 \\ P_\Omega F &= 0 \\ \|F\|_\infty &\leq 1/2 \\ \|P_\Omega D\|_F &\leq 1/4 \end{aligned} \tag{9}$$

Proof. Using $\beta = \max(\|W\|, \|F\|_\infty) \leq \frac{1}{2}$ in the previous proof, we have

$$\begin{aligned} \|L_0 + H\|_* + \lambda\|S_0 - H\|_1 - \|L_0\|_* - \lambda\|S_0\|_1 &\geq \frac{1}{2} (\|P_{T^\perp} H\|_* + \lambda\|P_{\Omega^\perp} H\|_1) - \lambda\langle P_\Omega D, H \rangle \\ &\geq \frac{1}{2} (\|P_{T^\perp} H\|_* + \lambda\|P_{\Omega^\perp} H\|_1) - \frac{\lambda}{4} \|P_\Omega H\|_F \end{aligned}$$

the last term can be further bounded

$$\begin{aligned} \|P_\Omega H\|_F &\leq \|P_\Omega P_T H\|_F + \|P_\Omega P_{T^\perp} H\|_F \\ &\leq \frac{1}{2} \|H\|_F + \|P_{T^\perp} H\|_F \\ &\leq \frac{1}{2} \|P_\Omega H\|_F + \frac{1}{2} \|P_{\Omega^\perp} H\|_F + \|P_{T^\perp} H\|_F \end{aligned}$$

therefore

$$\|P_\Omega H\|_F \leq \|P_{\Omega^\perp} H\|_F + 2\|P_{T^\perp} H\|_F$$

and we conclude by lower bounding the increase in the objective

$$\begin{aligned} \|L_0 + H\|_* + \lambda\|S_0 - H\|_1 - \|L_0\|_* - \lambda\|S_0\|_1 &\geq \frac{1}{2} \left((1 - \lambda)\|P_{T^\perp} H\|_* + \frac{\lambda}{2} \|P_{\Omega^\perp} H\|_1 \right) \\ &> 0 \end{aligned}$$

this completes the proof. \square

By the previous Lemma, it suffices to produce a dual certificate W such that

$$\begin{aligned} W &\in T^\perp \\ \|W\| &< 1/2 \\ \|P_\Omega(UV^* - \lambda \text{sgn}(S_0) + W)\|_F &\leq \lambda/4 \\ \|P_{\Omega^\perp}(UV^* + W)\|_\infty &< \lambda/2 \end{aligned} \tag{10}$$

since under these conditions, $D = \frac{1}{\lambda}P_{\Omega}(UV^* - \lambda \text{sgn}(S_0) + W)$ and $F = \frac{1}{\lambda}P_{\Omega^{\perp}}(UV^* + W)$ satisfy the sufficient conditions given by Lemma 2. Indeed we have

$$\begin{aligned} UV^* + W - \lambda \text{sgn}(S_0) &= P_{\Omega}(UV^* + W - \lambda \text{sgn}(S_0)) + P_{\Omega^{\perp}}(UV^* + W - \lambda \text{sgn}(S_0)) \\ &= \lambda D + P_{\Omega^{\perp}}(UV^* + W) \text{ since } \text{sgn}(S_0) \in \Omega \\ &= \lambda(D + F) \end{aligned}$$

and the first condition of Lemma 2 is satisfied. The remaining conditions follow from the definition of F and D .

3.4.1 Bounding $\|P_{\Omega}P_T\|$

Under suitable conditions on the size of the support Ω_0 of the sparse component, a bound can be derived on $\|P_{\Omega}P_T\|$ [4].

Theorem 3. *Suppose Ω_0 is sampled from the Bernoulli model with parameter ρ_0 . Then with high probability,*

$$\|P_T - \rho_0^{-1}P_TP_{\Omega_0}P_T\| \leq \epsilon$$

provided that $\rho_0 \geq C_0\epsilon^{-2\frac{\mu r \log n}{n}}$ where μ is the incoherence parameter and C_0 is a numerical constant.

As a consequence, $\|P_{\Omega}P_T\|$ can be bounded, and if $|\Omega|$ is not too large, then the desired bound $\|P_{\Omega}P_T\| \leq 1/2$ holds.

3.5 Proof of the equivalence of the Bernoulli sampling and uniform sampling model

Theorem 4. *Let E be the event that the recovery of (L_0, S_0) is exact through the RPCA. Then, $\forall \epsilon > 0$,*

- With $\rho = \frac{m}{n^2} + \epsilon$, E holds with high probability when the sparse matrix $S_{i,j} \sim \text{Bern}(\rho)$ iid $\implies E$ holds with high probability when the sparse matrix $S \sim \text{Uniform}(m)$.
- With $\rho = \frac{m}{n^2} - \epsilon$, E holds with high probability when the sparse matrix $S \sim \text{Uniform}(m)$ $\implies E$ holds with high probability when the sparse matrix $S_{i,j} \sim \text{Bern}(\rho)$ iid

Proof. Let us use the notation of subscript to denote the underlying sampling process, e.g., $P_{B(\rho)}(E)$ and $P_{U(m)}(E)$ be the probability of success recovery using Bernoulli sampling and uniform sampling respectively. We then upper and lower bound the difference of $P_{B(\rho)}(E) - P_{U(m)}(E)$ and show that the difference goes to zero as the dimension of the matrix $n \rightarrow \infty$.

$$\begin{aligned}
& P_{B(\rho)}(E) \\
&= \sum_{i=0}^{n^2} P_{B(\rho)}(|\Omega| = i) P_{B(\rho)}(E \mid |\Omega| = i) \\
&= \sum_{i=0}^{n^2} P_{B(\rho)}(|\Omega| = i) P_{U(i)}(E) \\
&\leq \sum_{i=0}^{m-1} P_{B(\rho)}(|\Omega| = i) + \sum_{i=m}^{n^2} P_{U(i)}(E) P_{B(\rho)}(|\Omega| = i) \\
&\leq \sum_{i=0}^{m-1} P_{B(\rho)}(|\Omega| = i) + \sum_{i=m}^{n^2} P_{U(i)}(E) P_{B(\rho)}(|\Omega| = i) \\
&\leq \sum_{i=0}^{m-1} P_{B(\rho)}(|\Omega| = i) + \sum_{i=m}^{n^2} P_{U(m)}(E) P_{B(\rho)}(|\Omega| = i) \\
&\leq P_{B(\rho)}(|\Omega| < m) + P_{U(m)}(E)
\end{aligned}$$

This gives, $P_{B(\rho)}(E) - P_{U(m)}(E) \leq P_{B(\rho)}(|\Omega| < m)$. With $\rho = \frac{m}{n^2} + \epsilon$, by law of large number, when $n \rightarrow \infty$ we get, $P_{B(\rho)}(E) \leq P_{U(m)}(E)$.

On the other hand,

$$\begin{aligned}
& P_{B(\rho)}(E) \\
&\geq \sum_{i=0}^m P_{B(\rho)}(|\Omega| = i) P_{B(\rho)}(E \mid |\Omega| = i) \\
&\geq P_{U(m)}(E) \sum_{i=0}^m P_{B(\rho)}(|\Omega| = i) \\
&= P_{U(m)}(E) (1 - P_{B(\rho)}(|\Omega| > m)) \\
&\geq P_{U(m)}(E) - P_{B(\rho)}(|\Omega| > m)
\end{aligned}$$

This gives, $P_{B(\rho)}(E) - P_{U(m)} \geq -P_{B(\rho)}(|\Omega| > m)$. With $\rho = \frac{m}{n^2} - \epsilon$, by law of large number, when $n \rightarrow \infty$ we get, $P_{B(\rho)}(E) \geq P_{U(m)}(E)$. \square

3.6 Proof of the Lemma about golfing scheme and dual certificate

Golfing scheme:

Recall that the golfing scheme involves creating a W^L according to the following method.

1. Fix $j_0 \geq 1$, define $\Omega_j \sim \text{Bern}(q)$ iid with $1 \leq j \leq j_0$ and $\rho = (1-q)^{j_0}$. Define the complement of support of Ω by $\Omega = \cup_{1 \leq j \leq j_0} \Omega_j^C$.
2. Define a sequence of matrix which finally ends at W^L
 - (a) $Y_0 = 0$

$$(b) \ Y_j = Y_{j-1} + \frac{1}{q} P_{\Omega_j} P_T (UV^* - Y_{j-1}) \text{ for } 1 \leq j \leq j_0$$

$$(c) \ W^L = P_{T^\perp}(Y_{j_0})$$

Fact 1. If we fix $Z \in T$, $\Omega_0 \sim \text{Bern}(\rho_0)$, and $\rho_0 \geq C_0 \epsilon^{-2} \frac{\mu r \log n}{n}$, then with high probability, we will have,

$$\|Z - \rho_0^{-1} P_T P_{\Omega_0}(Z)\|_\infty \leq \epsilon \|Z\|_\infty$$

Fact 2. If we fix Z , $\Omega_0 \sim \text{Bern}(\rho_0)$, and $\rho_0 \geq C_0 \frac{\mu \log n}{n}$, then with high probability, we will have,

$$\|(I - \rho_0^{-1} P_{\Omega_0})Z\| \leq C'_0 \sqrt{\frac{n \log n}{\rho_0}} \|Z\|_\infty$$

Fact 3. If $\Omega_0 \sim \text{Bern}(\rho_0)$, $\rho_0 \geq C_0 \epsilon^{-2} \frac{\mu r \log n}{n}$, then with high probability, we will have,

$$\|P_T - \rho_0^{-1} P_T P_{\Omega_0} P_T\| \leq \epsilon$$

Fact 4. If $\Omega_0 \sim \text{Bern}(\rho)$ and $1 - \rho \geq C_0 \epsilon^{-2} \frac{\mu r \log n}{n}$, then with high probability $\|P_{\Omega_0} P_T\|^2 \leq \rho + \epsilon$

Theorem 5. Let $S \sim \text{Bern}(\rho)$ iid for each entry with Ω as its support set. Set $j_0 = 2 \log n$. With the assumptions in the main theorem of RPCA, W^L satisfies the following with high probability.

1. $\|W^L\| < \frac{1}{4}$
2. $\|P_{\Omega}(UV^* + W^L)\|_F < \frac{\lambda}{4}$
3. $\|P_{\Omega^\perp}(UV^* + W^L)\| < \frac{\lambda}{4}$

Proof. We define another sequence of matrix $Z_j = UV^* - P_T(Y_j)$. There are some properties about Z_j which allows us to establish the proof. We survey them here and provides the proof of them.

i) Note that $Z_j = (P_T - \frac{1}{q} P_T P_{\Omega_j} P_T) Z_{j-1}$. The reason is as follows.

$$\begin{aligned} Z_j &= UV^* - P_T(Y_{j-1} + \frac{1}{q} P_{\Omega_j} P_T(UV^* - Y_{j-1})) \\ &= UV^* - P_T(Y_{j-1}) - \frac{1}{q} P_T P_{\Omega_j} P_T(UV^* - Y_{j-1}) \\ &= Z_{j-1} - q^{-1} (P_T P_{\Omega_j} (UV^* - P_T(Y_{j-1}))) \text{ because } P_T(UV^*) = UV^* \\ &= P_T(Z_{j-1}) - q^{-1} (P_T P_{\Omega_j} P_T Z_{j-1}) \text{ because } Z_{j-1} \in T \\ &= (P_T - q^{-1} P_T P_{\Omega_j} P_T) Z_{j-1} \end{aligned}$$

ii) If $q \geq C_0 \epsilon^{-2} \frac{\mu r \log n}{n}$, then $\|Z_j\|_\infty \leq \epsilon^j \|UV^*\|_\infty$ with high probability. The reason is as follows. By Fact (1), we have,

$$\begin{aligned} \|Z_{j-1} - q^{-1} P_T P_{\Omega_j} Z_{j-1}\|_\infty &\leq \epsilon \|Z_{j-1}\|_\infty \\ \|Z_j\|_\infty &\leq \epsilon \|Z_{j-1}\|_\infty \end{aligned}$$

Inductively, we get desired.

iii) If $q \geq C_0 \epsilon^{-2} \frac{\mu r \log n}{n}$, then $\|Z_j\|_F \leq \epsilon^j \sqrt{r}$. The reason is as follows. By Fact (3), we have,

$$\begin{aligned} \|(P_T - q^{-1} P_T P_{\Omega_0} P_T) \left(\frac{Z_{j-1}}{\|Z_{j-1}\|_F} \right)\|_F &\leq \epsilon \\ \|(P_T - q^{-1} P_T P_{\Omega_0} P_T) Z_{j-1}\|_F &\leq \epsilon \|Z_{j-1}\|_F \\ \|Z_{j-1}\| &\leq \epsilon \|Z_{j-1}\|_F \end{aligned}$$

Inductively, we get desired.

After establishing these properties, we are ready to prove that golfing scheme yields W^L that satisfies the desired properties.

1) Proof of (1):

$$\begin{aligned} \|W^L\| &= \|P_{T^\perp}(Y_{j_0})\| \\ &\leq \sum_{j=1}^{j_0} \frac{1}{q} \|P_{T^\perp} P_{\Omega_j} Z_{j-1}\| \text{ because } Y_j = Y_{j-1} + q^{-1} P_{\Omega_j}(Z_{j-1}) \\ &= \sum_{j=1}^{j_0} \|P_{T^\perp} \left(\frac{1}{q} P_{\Omega_j} Z_{j-1} - Z_{j-1} \right)\| \text{ because } Z_j \in T \\ &\leq \sum_{j=1}^{j_0} \left\| \left(\frac{1}{q} P_{\Omega_j} Z_{j-1} - Z_{j-1} \right) \right\| \text{ because } \|P_{T^\perp}(M)\| \leq \|M\| \\ &\leq C'_0 \sqrt{\frac{n \log n}{q}} \sum_{j=1}^{j_0} \|Z_{j-1}\|_\infty \text{ because Fact(2)} \\ &\leq C'_0 \sqrt{\frac{n \log n}{q}} \sum_{j=1}^{j_0} \epsilon^j \|UV^*\|_\infty \\ &\leq C'_0 \sqrt{\frac{n \log n}{q}} \frac{1}{1-\epsilon} \|UV^*\|_\infty \\ &\leq C'_0 \sqrt{\frac{n \log n}{q}} \frac{1}{1-\epsilon} \frac{\sqrt{\mu r}}{n} \\ &\leq C'' \epsilon < \frac{1}{4} \text{ for some constant } C'' \end{aligned}$$

2) Proof of (2) : First, we expand,

$$\|P_\Omega(UV^* + W^L)\|_F = \|P_\Omega(UV^* + P_{T^\perp} Y_{j_0})\|_F$$

Then, because $P_\Omega(Y_{j_0}) = P_\Omega(\sum_j P_{\Omega_j} Z_{j-1}) = 0$ and $P_\Omega(P_T(Y_{j_0}) + P_{T^\perp}(Y_{j_0})) = 0$, we have,

$$\|P_\Omega(UV^* + W^L)\|_F = \|P_\Omega(UV^* - P_T Y_{j_0})\|_F$$

Continuing,

$$\begin{aligned}
\|P_\Omega(UV^* + W^L)\|_F &= \|P_\Omega(Z_{j_0})\|_F \\
&\leq \|Z_{j_0}\|_F \\
&\leq \epsilon^{j_0} \sqrt{r} \\
&\leq \sqrt{r} \frac{1}{n^2} \leq \frac{\lambda}{4}
\end{aligned}$$

3) Proof of (3) :

$$\begin{aligned}
\|P_{\Omega^\perp}(UV^* + W^L)\|_\infty &= \|P_{\Omega^\perp}(Z_{j_0} + Y_{j_0})\|_\infty \\
&\leq \|Z_{j_0}\|_\infty + \|Y_{j_0}\|_\infty \\
&\leq \|Z_{j_0}\|_F + \|Y_{j_0}\|_\infty \\
&\leq \frac{\lambda}{8} + \|Y_{j_0}\|_\infty
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\|Y_{j_0}\|_\infty &\leq q^{-1} \sum_j \|P_{\Omega_j} Z_{j-1}\|_\infty \\
&\leq q^{-1} \sum_j \|Z_{j-1}\|_\infty \\
&\leq q^{-1} \sum_j \epsilon^j \frac{\sqrt{\mu r}}{n} \\
&\leq \frac{\lambda}{8} \text{ if } \epsilon \text{ is sufficiently small}
\end{aligned}$$

□

3.7 Proof of the Lemma about least square construction and dual certificate

Construction of W^S :

$$W_S = \lambda P_{T^\perp} ((P_\Omega - P_\Omega P_T P_\Omega)^{-1} \text{sign}(S_0))$$

Theorem 6. *Let $S \sim \text{Bern}(\rho)$ iid for each entry with Ω as its support set. With the assumptions in the main theorem of RPCA, W^S satisfies the following with high probability.*

1. $\|W^S\| < \frac{1}{4}$
2. $\|P_{\Omega^\perp}(W^S)\| < \frac{\lambda}{4}$

Proof. We consider the sign of S_0 to be distributed as follows

$$\text{sign}(S_0)_{i,j} = \begin{cases} 1 & \text{wp } \frac{\rho}{2} \\ 0 & \text{wp } 1 - \rho \\ -1 & \text{wp } \frac{\rho}{2} \end{cases}$$

1) Proof of (1) :

I) We note the we can separate W^S into two parts and then bound them separately.

$$W^S = \lambda P_{T^\perp}(\text{sign}(S_0)) + \lambda P_{T^\perp}\left(\sum_{k \geq 1} (P_\Omega P_T P_\Omega)^k (\text{sign}(S_0))\right)$$

II) Then, we have

$$\begin{aligned} \lambda P_{T^\perp}(\text{sign}(S_0)) &\leq \lambda \|\text{sign}(S_0)\| \\ &= \frac{1}{\sqrt{n}} \|\text{sign}(S_0)\| \\ &\leq 4\sqrt{\rho} \text{ with high probability} \end{aligned}$$

where the last inequality uses the fact that for the entry-wise distribution of $\text{sign}(S_0)$, we can have $\|\text{sign}(S_0)\| \leq 4\sqrt{n\rho}$ holds with high probability.

III) Now, for the other part, $\lambda P_{T^\perp}(\sum_{k \geq 1} (P_\Omega P_T P_\Omega)^k (\text{sign}(S_0)))$, we bound it by first expressing it in the form of $\langle X, \text{sign}(S_0) \rangle$ and then claim that with high probability, this term is bounded above as desired. Let $R = \sum_{k \geq 1} (P_\Omega P_T P_\Omega)^k$, then we have,

$$\begin{aligned} \|P_{T^\perp}(R(\text{sign}(S_0)))\| &\leq \|R(\text{sign}(S_0))\| \\ &\leq 4 \sup_{x, y \in N} \langle y, R(\text{sign}(S_0))x \rangle \end{aligned}$$

where the last inequality uses the fact that there exists a $\frac{1}{2}$ -net of the Euclidean ball and it has at most 6^n elements. Continuing, we have

$$\begin{aligned} \|P_{T^\perp}(R(\text{sign}(S_0)))\| &\leq 4 \sup_{x, y \in N} \langle y, R(\text{sign}(S_0))x \rangle \\ &= 4 \sup_{x, y \in N} \langle yx^*, R(\text{sign}(S_0)) \rangle \\ &= 4 \sup_{x, y \in N} \langle R(yx^*), \text{sign}(S_0) \rangle \end{aligned}$$

and that we denote $X(x, y) = \langle R(yx^*), \text{sign}(S_0) \rangle$ afterwards.

Note that, by Hoeffding's inequality, we have,

$$Pr(|X(x, y)| > t \mid \Omega) \leq 2 \exp\left(-\frac{t^2}{2\|R(yx^*)\|_F^2}\right)$$

This gives,

$$\begin{aligned} Pr(\|P_{T^\perp}(R(\text{sign}(S_0)))\| > 4t \mid \Omega) &\leq Pr(\|R(\text{sign}(S_0))\| > 4t \mid \Omega) \\ &\leq Pr(\sup_{x, y} |X(x, y)| > t \mid \Omega) \\ &\leq 2N^2 \exp\left(-\frac{t^2}{2\|R\|_F^2}\right) \text{ because } \|yx^*\|_F \leq 1 \end{aligned}$$

Now, we proceed to bound the probability without the condition on Ω .

First, note that the event of $\|P_\Omega P_T\| \leq \sigma = \rho + \epsilon$, implies that $\|R\| \leq (\frac{\sigma^2}{1-\sigma^2})^2$. Thus, unconditionally, we have

$$\begin{aligned} Pr(\|R(sign(S_0))\| > 4t) &\leq 2|N|^2 \exp\left(\frac{-t^2}{2(\frac{\sigma^2}{1-\sigma^2})^2}\right) + Pr(\|P_\Omega P_T\| > \sigma) \\ &\leq 2 \cdot 6^{2n} \exp\left(\frac{-t^2}{2(\frac{\sigma^2}{1-\sigma^2})^2}\right) + Pr(\|P_\Omega P_T\| > \sigma) \end{aligned}$$

Thus, where we finally put $t = \frac{1}{16}$

$$Pr(\lambda \|R(sign(S_0))\| > 4t) \leq 2 \cdot 6^{2n} \exp\left(\frac{-\frac{t^2}{\lambda^2}}{2(\frac{\sigma^2}{1-\sigma^2})^2}\right) + Pr(\|P_\Omega P_T\| > \sigma)$$

With $\lambda = \sqrt{\frac{1}{n}}$, we have this probability $\rightarrow 0$ as $n \rightarrow \infty$. Thus with high probability $\|W^S\| \leq \frac{1}{4}$

2) Proof of (2) :

The idea is that we first express $P_{\Omega^\perp}(W^S)$ in the form of $\langle X, sign(S_0) \rangle$ and we can derive upper bound on it if highly probable event of $\{\|P_\Omega P_T\| \leq \sigma\}$ for some small $\sigma = \rho + \epsilon$ holds.

I) First,

$$\begin{aligned} P_{\Omega^\perp}(W^S) &= P_{\Omega^\perp}(\lambda(I - P_T)\left(\sum_{k \geq 0} (P_\Omega P_T P_\Omega)^k\right) sign(S_0)) \\ &= -\lambda P_{\Omega^\perp} P_T (P_\Omega - P_\Omega P_T P_\Omega)^{-1} sign(S_0) \end{aligned}$$

For $(i, j) \in \Omega^C$, we have

$$\begin{aligned} e_i^* W^S e_j &= \langle e_i e_j^*, W^S \rangle \\ &= \langle e_i e_j^*, -\lambda P_{\Omega^\perp} P_T (P_\Omega - P_\Omega P_T P_\Omega)^{-1} sign(S_0) \rangle \\ &= -\lambda \langle e_i e_j^*, P_T (P_\Omega - P_\Omega P_T P_\Omega)^{-1} sign(S_0) \rangle \\ &= -\lambda \langle e_i e_j^*, P_T P_\Omega (P_\Omega - P_\Omega P_T P_\Omega)^{-1} sign(S_0) \rangle \\ &= -\lambda \langle e_i e_j^*, P_T \sum_{k \geq 0} (P_\Omega P_T P_\Omega)^k sign(S_0) \rangle \end{aligned}$$

Noting that P_Ω, P_T are self-adjoint, thus, we have

$$e_i^* W^S e_j = \lambda \langle -(P_\Omega - P_\Omega P_T P_\Omega)^{-1} P_\Omega P_T (e_i e_j^*), sign(S_0) \rangle$$

where we now denote $X(i, j) = -(P_\Omega - P_\Omega P_T P_\Omega)^{-1} P_\Omega P_T (e_i e_j^*)$

II) We now consider, where we put $t = \frac{1}{4}$,

$$\begin{aligned}
 Pr(\|P_{\Omega^\perp}(W^S)\|_\infty > t\lambda \mid \Omega) &\leq \sum_{(i,j) \in \Omega^C} Pr(|e_i^* W^S e_j| > t\lambda \mid \Omega) \\
 &\leq n^2 Pr(|e_i^* W^S e_j| > t\lambda \mid \Omega) \text{ for some } (i,j) \\
 &= n^2 Pr(| < X_{i,j}, \text{sign}(S_0) > | > t \mid \Omega) \\
 &\leq 2n^2 \exp\left(-\frac{2t^2}{4\|X(i,j)\|_F}\right) \text{ because of Hoeffding's inequality}
 \end{aligned}$$

III) We then proceed to bound the $\|X(i,j)\|$. On the event of $\{\|P_\Omega P_T\| \leq \sigma\}$, we have ,

$$\begin{aligned}
 \|P_\Omega P_T(e_i e_j^*)\|_F &\leq \|P_\Omega P_T\| \cdot \|P_T(e_i e_j^*)\|_F \\
 &\leq \sigma \sqrt{\frac{2\mu r}{n}}
 \end{aligned}$$

Moreover, we have

$$\begin{aligned}
 \|(P_\Omega - P_\Omega P_T P_\Omega)^{-1}\| &\leq \sum_{k \geq 0} \|(P_\Omega P_T P_\Omega)^k\| \\
 &\leq \frac{1}{1 - \sigma}
 \end{aligned}$$

Finally, we have

$$\|X(i,j)\|_F \leq 2\sigma^2 \frac{\frac{\mu r}{n}}{(1 - \sigma)^2}$$

Combining, we have

$$\begin{aligned}
 Pr(\|P_{\Omega^\perp} W^S\| > t\lambda) &\leq 2n^2 \exp\left(\frac{-t^2 n(1 - \sigma)^2}{4\sigma^2(\mu r)}\right) + Pr(\|P_\Omega P_T\| \geq \sigma) \\
 &\leq \epsilon \text{ if } \mu r < \rho'_r \frac{n}{\log n}
 \end{aligned}$$

□

3.8 Proof of the form of sub-differential of nuclear norm

Definition 1. For matrix norms $\|\cdot\|$ which satisfy $\|UAV\| = \|A\| \forall U, V$ being orthonormal, then they are called orthogonally invariant norm.

Definition 2. For orthogonally invariant norm $\|\cdot\|$ which is defined by its singular values $\|A\| = \phi(\vec{\sigma})$ where $\vec{\sigma}$ are the singular values of A , we call the function ϕ as a symmetric gauge function if it is a norm and it satisfies $\phi(\vec{\sigma}) = \phi(\epsilon_1 \sigma_{i_1}, \dots, \epsilon_n \sigma_{i_n})$ for any permutation of (i_1, \dots, i_n) of $(1, \dots, n)$ and $\epsilon_i = \pm 1$.

Fact 7. For orthogonally invariant norm $\|\cdot\|$ with symmetric gauge function ϕ , the sub-differential is given by

$$\partial\|A\| = \{U \text{diag}(\vec{d})V \mid A = U\Sigma V^T, \vec{d} \in \partial\phi(\vec{d}), U \in R^m, V \in R^n\}$$

Theorem 8. Let $A = U^{(1)}\Sigma V^{(1)T}$ then $\partial\|A\|_* = \{U^{(1)}V^{(1)T} + W : \|W\| \leq 1, U^{(1)T}W = 0, WV^{(1)} = 0\}$

Proof. We take the symmetric gauge function as $\|\cdot\|_1$ and then apply the Fact (7) and will obtain desired. \square

4 Related Problems and Extensions

4.1 Exact Matrix completion

Robust PCA is an extension of the exact matrix completion problem introduced in [4], where one seeks to recover a low-rank matrix L_0 from a small fraction of its entries. More precisely, assume one is given $\{(L_0)_{ij}, (i, j) \in \Omega\}$ where Ω is a subset of $[n] \times [n]$.

Problem to solve

$$\begin{aligned} & \text{minimize} && \text{rank}(L) \\ & \text{subject to} && P_\Omega L = P_\Omega L_0 \end{aligned} \tag{11}$$

A heuristic is to minimize the nuclear norm of L

4.1.1 Incoherence

Singular vectors have to be sufficiently spread

$$\mu(U) = \frac{n}{r} \max_i \|P_U e_i\|_2^2 = \frac{n}{r} \max_i \left[\sum_{k=1}^r u_{ki}^2 \right] \tag{12}$$

Assumptions:

- $\max\{\mu(U), \mu(V)\} \leq \mu_0$
- $(\sum_k u_k v_k^*)_{ij} \leq \mu_1 \sqrt{\frac{r}{n_1 n_2}}$ (true for $\mu_1 = \mu_0 \sqrt{r}$)
- $m \geq c \max\{\mu_1^*, \sqrt{\mu_0} \mu_1, \mu_0 n^{1/4}\} n r \beta \log n$

Under these assumptions, recovery is exact with high probability (at least $1 - \frac{c}{n\beta}$)

Incoherent matrices:

- sampled from the incoherent basis model
- sampled from the random orthogonal model: if $M = \sum_k \sigma_k u_k v_k^*$, then $\{u_1, \dots, u_r\}$ and $\{v_1, \dots, v_r\}$ are assumed to be selected at random.

4.1.2 Main result

4.1.3 Comparing results to Robust PCA

Robust PCA can be thought of as an extension of the matrix completion problem, where instead of having a known subset of the entries $\{(L_0)_{ij}, (i, j) \in \Omega\}$ and the rest is missing, we have an unknown subset of the entries and the rest is corrupted. In this sense, Robust PCA is a harder problem.

Note that the matrix L_0 can be recovered by Principal Component Pursuit, solving a different problem:

$$\begin{aligned} & \text{minimize} && \|L\|_* + \lambda\|S\|_1 \\ & \text{subject to} && P_\Omega(L + S) = P_\Omega L_0 \end{aligned} \tag{13}$$

4.2 Stable Principal Component Pursuit

4.2.1 Overview

The paper studies the problem of recovering a low-rank matrix (the principal components) from a high-dimensional data matrix despite both small entry-wise noise and gross sparse errors. It proves that the solution to a convex program (a relaxation of classic Robust PCA) gives an estimate of the low-rank matrix that is simultaneously stable to small entry-wise noise and robust to gross sparse errors. The result shows that the proposed convex program recovers the low-rank matrix even though a positive fraction of its entries are arbitrarily corrupted, with an error bound proportional to the noise level.

4.2.2 Main result

The paper consider a matrix $M \in \mathbb{R}^{n_1 \times n_2}$ of the form $M = L_0 + S_0 + Z_0$, where L_0 is (non-sparse) low rank, S_0 is sparse (modeling gross errors) and Z_0 is “small” (modeling a small noisy perturbation). The assumption on Z_0 is simply that $\|Z_0\|_F \leq \delta$ for some small known δ . Hence at least for the theory part of the paper the authors do not assume anything about the distribution of the noise other than it is bounded (however they will gloss over this in their algorithm).

The convex program to be solved is a slight modification of the standard Robust PCA problem and given by

$$\begin{aligned} & \min_{L, S} \|L\|_* + \lambda\|S\|_1 \\ & \text{s.t.} \quad \|M - L - S\|_F \leq \delta \end{aligned} \tag{14}$$

where $\lambda = 1/\sqrt{n_1}$. Under a standard incoherence assumption on L_0 (which essentially means that L_0 should not be sparse) and a uniformity assumption on the sparsity pattern of S_0 (which means that the support of S_0 should not be too concentrated) the main result states that, with high probability in the support of S_0 , for any Z_0 with $\|Z_0\|_F \leq \delta$, the solution (\hat{L}, \hat{S}) to (14) satisfies

$$\|\hat{L} - L_0\|_F^2 + \|\hat{S} - S_0\|_F^2 \leq C n_1 n_2 \delta^2$$

where C is a numerical constant. The above claim essentially states that the recovered low-rank matrix \hat{L} is stable with respect to non-sparse but small noise acting on all entries of the matrix.

In order to experimentally verify the predicted performance to their formulation, the author provide a comparison with an oracle. This oracle is assumed to provide information about the support of S_0 and the row and column spaces of L_0 , which allows the computation of the MMSE estimator which otherwise would be computationally intractable (strictly speaking it of course is not really the MMSE, since it uses additional information from the oracle). Simulation results that show that the RMS error of the solution obtained through (14) in the non-breakdown regime (that is, for the

support of S_0 sufficiently small) is only about twice as large as that of the oracle-based MMSE. This suggests that the proposed algorithm works quite well in practice.

4.2.3 Relations to existing work

The result of the paper can be seen from two different view points. On the one hand, it can be interpreted from the point of view of standard PCA. In this case, the result states that standard PCA, which can in fact be shown to be statistically optimal w.r.t. i.i.d Gaussian perturbations, can also be made robust with respect to sparse gross corruptions. On the other hand, the result can be interpreted from the point of view of Robust PCA. In this case, it essentially states that the classic Robust PCA solution can itself be made robust with respect to some small but non-sparse noise acting on all entries of the matrix.

Conceptually, the work presented in the paper is similar to the development of results for “imperfect” scenarios in compressive sensing where the measurements are noisy and the signal is not exact sparse. In this body of literature, l_1 -norm minimization techniques are adapted to recover a vector $x_0 \in \mathbb{R}^n$ from contaminated observations $y = Ax_0 + z$, where $A \in \mathbb{R}^{m \times n}$ with $m \ll n$ and z is the noise term.

4.2.4 Algorithm

For the case of a noise matrix Z_0 whose entries are i.i.d. $\mathcal{N}(0, \sigma^2)$, the paper suggests to use an Accelerated Proximal Gradient (APG) algorithm (see algorithms section for details) for solving (14). Note that for $\delta = 0$ the problem reduces to the standard Robust PCA problem with an equality constraint on the matrices. For this case the APG algorithm proposed in [6] solves an approximation of the form

$$\min_{L, S} \|L\|_* + \lambda \|S\|_1 + \frac{1}{2\mu} \|M - L - S\|_F^2$$

For the Stable PCP problem where $\delta > 0$ the authors advocate using the same algorithm with fixed but carefully chosen parameter μ (similar to [3]). In particular, they point out¹ that for $Z_0 \in \mathbb{R}^{n \times n}$ with $(Z_0)_{ij} \sim \mathcal{N}(0, \sigma^2)$ i.i.d. it holds that $n^{-1/2} \|Z_0\|_2 \rightarrow \sqrt{2}\sigma$ almost surely as $n \rightarrow \infty$. They then choose the parameter μ such that if $M = Z_0$, i.e. if $L_0 = S_0 = 0$, the minimizer of the above problem is likely to be $\hat{L} = \hat{S} = 0$. The claim is that this is the case for $\mu = \sqrt{2n}\sigma$.

It is worth noting that the assumption of a Gaussian noise matrix Z_0 is reasonable but not always satisfied. If it is not, then it is not clear if using the APG algorithm to solve the associated approximate problem is a good idea and different algorithms may be needed. The problem (14) can be expressed as an SDP and can therefore in principle be solved using general purpose interior point solvers. However, the same scalability issues as in the standard Robust PCA problem will limit prohibit to use these methods for high-dimensional data. The paper [1] focuses on efficient first-order algorithms for solving (14).

¹this based on the strong Bai Yin Theorem [2], which implies that for an $n \times n$ real matrix with entries $\xi_{ij} \sim \mathcal{N}(0, 1)$ it holds that $\limsup_{n \rightarrow \infty} \|Z_0\|_2 / \sqrt{n} = 2$ almost surely

4.2.5 Conclusion and Outlook

The paper addresses a problem of potentially very high practical relevance. While it is reasonable to assume that in many applications the low-rank component L_0 will only be corrupted by a comparatively small number of gross errors (caused by rare and isolated events), the assumption of perfect measurements for the rest of the data outside the support of S_0 that is made in classic Robust PCA will generally not hold for example due to sensor noise. This paper asserts that if the non-sparse noise component Z_0 is sparse, then with high probability the recovered components are “close” to the actual ones.

For simplicity, the paper models the non-sparse noise simply as an additive perturbation that is bounded in the Frobenius norm. In cases where one has additional information available about this noise, for example its distribution or some bounds on the absolute value of each entry, it might be possible to derive better bounds on the resulting errors. One possible extension could therefore be to look at exploiting structure in the noise.

One thing the paper claims is that “at a cost not so much higher than the classical PCA, [the] result is expected to have significant impact on many practical problems”. As mentioned above I do agree that the result has a significant impact on many practical problems. However, the claim concerning the computational complexity is very optimistic. The fastest solver for the special case $\delta = 0$ (classic Robust PCA) currently seems to be a alternating directions augmented Lagrangian method. This method requires an SVD at each iteration, and for problems involving large-scale data the number of iterations can be very large. The standard PCP algorithm on the other hand is based on a single SVD, hence it can be computed much faster.

4.3 Robust Alignment by Sparse and Low-rank Decomposition

The convex optimization framework for low-rank matrix recovery has been employed successfully. However, in practice, much more data can be viewed as low-rank only after some transformation is applied. The new formulation of this problem as Robust Alignment by Sparse and Low-rank Decomposition (RASL) [7]:

$$\min_{A, E, \tau} \|A\|_* + \lambda \|E\|_1 \quad \text{s.t. } D \circ \tau = A + E \quad (15)$$

where $A \in \mathbb{R}^{m \times n}$ is low-rank matrix, $E \in \mathbb{R}^{m \times n}$ is sparse matrix, D is our measurements, which is the result of $(A + E)$ subjecting to transformation τ^{-1} . Here we assume that the transformation is invertible. We define $D \circ \tau$ as: $D \circ \tau = [D_1 \circ \tau_1 \mid D_2 \circ \tau_2 \mid \dots \mid D_n \circ \tau_n]$, which is the measurements $D = [D_1 \mid D_2 \mid \dots \mid D_n]$ subjects to set of transformations $\tau = [\tau_1 \mid \tau_2 \mid \dots \mid \tau_n] \in \mathbb{G}^n$, where \mathbb{G} is a group of certain type of invertible transformations, which could be affine transform, rotation transform, etc.

The main difficulty in solving (15) is the nonlinearity of constraint $D \circ \tau = A + E$. When the change in τ is small, we can approximate this constraint by linearizing about the current estimate of τ . Here, we assume that \mathbb{G} is some p -parameter group and identify $\tau = [\tau_1 \mid \tau_2 \mid \dots \mid \tau_n] \in \mathbb{R}^{p \times n}$ with the parameterizations of all of the transformations. For $\Delta\tau = [\Delta\tau_1 \mid \Delta\tau_2 \mid \dots \mid \Delta\tau_n]$, write $D \circ (\tau + \Delta\tau) \approx D \circ \tau + \sum_{i=1}^n J_i \Delta\tau_i \epsilon_i$, where $J_i \doteq \frac{\partial}{\partial \zeta} (D_i \circ \zeta)|_{\zeta=\tau_i}$ is the Jacobian of the i -th measurement with respect to the transformation parameters τ_i . $\{\epsilon_i\}$ denotes the standard basis for

\mathbb{R}^n . This leads to a convex optimization problem in unknowns $A, E, \Delta\tau$:

$$\min_{A, E, \Delta\tau} \|A\|_* + \lambda \|E\|_1 \quad \text{s.t.} \quad D \circ \tau + \sum_{i=1}^n J_i \Delta\tau \epsilon_i \epsilon_i^T = A + E \quad (16)$$

It leads to algorithm 1

Algorithm 1: RASL

Input: $D = [D_1 \mid D_2 \mid \dots \mid D_n]$, initial transformation $\tau_1, \tau_2, \dots, \tau_n$ in a certain parametric group \mathbb{G} , weight $\lambda > 0$.

while *not converged* **do**

Step 1: compute Jacobian matrices w.r.t. transformation:

$$J_i \leftarrow \frac{\partial}{\partial \zeta} (D_i \circ \zeta)|_{\zeta=\tau_i}$$

Step 2 (inner loop): solve the linearized convex optimization:

$$(A^*, E^*, \Delta\tau^*) \leftarrow \arg \min_{A, E, \Delta\tau} \|A\|_* + \lambda \|E\|_1 \quad \text{s.t.} \quad D \circ \tau + \sum_{i=1}^n J_i \Delta\tau \epsilon_i \epsilon_i^T = A + E$$

Step 3: update the transformation: $\tau \leftarrow \tau + \Delta\tau^*$

Output: A^*, E^*, τ^*

4.4 Robust Matrix Decomposition With Sparse Corruptions: D Hsu et. al.

4.4.1 Question being addressed

Under deterministic setting, it studies how much sparsity is allowed for accurate recovery of the sparse-lowrank pairs.

4.4.2 Main ideas

Given M as the observed matrix, it analyze the following two optimization problems. With arguments as (L, S) ,

$$\begin{aligned} \min_{(L, S)} \quad & \|L\|_* + \lambda \|S\|_1 \\ \text{s.t.} \quad & \|L + S - M\|_1 \leq \epsilon_1 \\ & \|L + S - M\|_* \leq \epsilon_* \end{aligned} \quad (17)$$

and

$$\min_{(L, S)} \quad \|L\|_* + \lambda \|S\|_1 + \frac{1}{2\mu} \|L + S - M\|_F \quad (18)$$

It is remarked that the M is a perturbed observation outcome of the original (L_0, S_0) pairs.

4.4.3 Contributions

1. It provides sufficient conditions on sparsity of the original (L_0, S_0) pairs that allow accurate recovery in the sense that $\|L_0 - \hat{L}\|_\infty, \|S_0 - \hat{S}\|_\infty$ is small.
2. If the observed matrix M is perturbed from $L_0 + S_0$ by a small amount (i.e. ϵ), the optimizer (\hat{L}, \hat{S}) will be ϵ -close to the original (L_0, S_0) pairs.

4.5 Characterization of the subdifferential of some matrix norms : GA Watson et. al.

4.5.1 Significance

Subdifferential of nuclear norm is a critical element in the formulation of dual certificates. This paper proves why the subdifferential of nuclear norm is of the form.

$$\{U^{(1)}V^{(1)T} + W|U^{(1)T}W = 0, WV^{(1)}, \|W\|_2 \leq 1\}$$

where $A = [U^{(1)}|U^{(2)}][\begin{smallmatrix} \Sigma & 0 \\ 0 & 0 \end{smallmatrix}][\begin{smallmatrix} V^{(1)T} \\ V^{(2)T} \end{smallmatrix}]$

4.5.2 Details

Lemma 3. *If a norm $\|\cdot\|$ is orthogonal invariant and is characterized by a symmetric gauge function, then*

$$\partial\|A\| = \text{conv}\{U\text{diag}(\vec{d})V|A = U[\begin{smallmatrix} \Sigma & 0 \\ 0 & 0 \end{smallmatrix}]V^T, \vec{d} \in \partial\phi(\Sigma)\}$$

Proof. Sketch: For $RHS \subset LHS$, one can check that they satisfy the dual norm characterization of $\partial\|A\|$. For $LHS \subset RHS$, one can use maximum representation of the directional derivative of the $\|A\|$ to show that subdifferential not inside RHS will give a contradiction. \square

Proposition 9.

$$\partial\|A\|_* = \{U^{(1)}V^{(1)T} + W|U^{(1)T}W = 0, WV^{(1)}, \|W\|_2 \leq 1\}$$

Proof. Sketch: Take $\phi(\vec{\sigma}) = \|\sigma\|_1$ and apply Lemma ?? to the convex hull of all SVD of A . \square

5 Generalization of the Robust PCA scheme with known rank

5.1 Introduction

Recall that the nuclear norm introduced in the PCP scheme is resulted because we would like to extract the low rank component from gross random noise. Nuclear norm is used because it is a heuristic for penalizing high rank matrix. Now, we consider the case when we have extra

information/guess from the data set that we know precisely what the rank of the matrix are. Therefore, it is natural to introduce the following the following heuristics.

$$E^* = \min_{\{p_i\}\{q_j\}, \|p_j\|_2, 1 \leq j \leq r} \|M - \sum_{j=1}^r p_j q_j^T\|_1$$

We specialize the discussion to the case of rank-1 approximation (when $r = 1$) and shows that this scheme has a probabilistic guarantee of recovery and demonstrate that an efficient power iteration can be employed to solve this problem. Empirical results are also provided to support this method. We remark that we can generalize this and obtain similar results.

5.2 Problem statement.

We are now interested in solving the following problem.

$$E^* = \min_{p,q} \|M - pq^T\|_1 \quad (19)$$

5.3 Algorithm derivation

Let $M = (a_{i,j}) \in R^{m \times n}$. We now employ the block-coordinate descent method to solve this problem. Note that

$$\begin{aligned} \min_p \|M - pq^T\| &= \sum_{i=1}^m \min_t \left(\sum_{j=1}^n |a_{i,j} - tq_j| \right) \\ &= \sum_{i=1}^m \min_t \left(\sum_{j=1}^n |q_j| \left| t - \frac{a_{i,j}}{q_j} \right| \right) \end{aligned} \quad (20)$$

$$\begin{aligned} \min_q \|M - pq^T\| &= \sum_{j=1}^n \min_t \left(\sum_{i=1}^m |a_{i,j} - tp_i| \right) \\ &= \sum_{j=1}^n \min_t \left(\sum_{i=1}^m |p_i| \left| t - \frac{a_{i,j}}{p_i} \right| \right) \end{aligned} \quad (21)$$

And for solving the subproblem of finding

$$\min_t \sum_{k=1}^{k_0} c_i |t - d_i|$$

where $c_i \geq 0$ is basically finding the weighted median and can be done by the following method with complexity $O(k_0 \log k_0)$ mostly on sorting the sequence. We call it WMH.

Algorithm 2: WMH (k_0, \vec{c}, \vec{d})

1. We first sort \vec{d} s.t. $d_{i_1} \leq d_{i_2} \leq \dots \leq d_{i_{k_0}}$
2. We then find k' s.t.

$$\begin{aligned} \sum_{\theta=1}^{k'-1} c_{i_\theta} &\leq \sum_{\theta=k'}^{k_0} c_{i_\theta} \\ \sum_{i=1}^{k'} c_{i_\theta} &\geq \sum_{i=k'+1}^{k_0} c_{i_\theta} \end{aligned}$$

3. We then set t^* to be $d_{i_{k'}}$
-

This algorithm is optimal in finding t . This is justified by using the property of sub-differential of $\|\cdot\|_1$ and note that $0 \in \partial(\sum_{k=1}^{k_0} c_i |t^* - d_i|)$.

Now we are ready to state the power iteration method to solve the optimization rank-1 optimization problem. We call it Poweriteration.

Algorithm 3: Poweriteration(M)

Repeat

1. $p_i \leftarrow \text{wmh}(n, \text{abs}(q), M(i, :)/q)$ for each i
2. $p \leftarrow \frac{p}{\max(p)}$
3. $q_j \leftarrow \text{wmh}(n, \text{abs}(p), M(:, j)/p)$ for each j

Until stopping criterion is met

5.4 Performance Guarantee

Here are two results about performance guarantee. One is a result for deterministic case and the other is for the random case.

Proposition 10. *Let $M = pq^T + S$ with $\frac{2}{\epsilon} \|S\|_1 < \|pq^T\|_1$. Then the (\hat{p}, \hat{q}) recovered from (19) satisfy*

$$\frac{\|pq^T - \hat{p}\hat{q}^T\|}{\|pq^T\|_1} \leq \epsilon$$

Proof. If $(\hat{p}, \hat{q}) = (p, q)$ then the proposition is trivially true. Now assume $(\hat{p}, \hat{q}) \neq (p, q)$. Then we have ,

$$\begin{aligned} \|S\|_1 &\geq \|pq^T + S - \hat{p}\hat{q}^T\|_1 \\ &\geq \|pq^T - \hat{p}\hat{q}^T\|_1 - \|S\|_1 \end{aligned}$$

Now if the conclusion is false, we will have

$$\|S\|_1 \geq \epsilon \|pq^T\|_1 - \|S\|_1$$

This gives a contradiction of,

$$2\|S\|_1 \geq \epsilon\|pq^T\|_1$$

□

Proposition 11. *Let $M = pq^T + S$ be $R^{n \times n}$ and S has k possibly non-zero elements. If $S_i \sim U[-x_s, x_s]$, $p_i \sim U[-x_p, x_p]$, $q_i \sim U[-x_p, x_p]$ independently distributed, with $\frac{n}{k} \geq \frac{8x_s}{\epsilon x_p x_q}$, then*

$$\lim_{n \rightarrow \infty} Pr\left(\frac{\|pq^T - \hat{p}\hat{q}^T\|}{\|pq^T\|_1} > \epsilon\right) = 0$$

Proof. Let the event that $\frac{\|pq^T - \hat{p}\hat{q}^T\|}{\|pq^T\|_1} > \epsilon$ be an error event. If error occurs, then

$$\begin{aligned} kx_s &\geq \|pq^T - \hat{p}\hat{q}^T + S\|_1 \\ &\geq \|pq^T - \hat{p}\hat{q}^T\|_1 - \|S\|_1 \\ &\geq \epsilon\|pq^T\|_1 - kx_s \end{aligned}$$

We now bounding the probability of error.

$$\begin{aligned} &Pr\left(\frac{\|pq^T - \hat{p}\hat{q}^T\|}{\|pq^T\|_1} > \epsilon\right) \\ &\leq Pr(kx_s \geq \epsilon\|pq^T\|_1 - kx_s) \\ &\leq Pr\left(\frac{2kx_s}{\epsilon} \geq \|pq^T\|_1\right) \\ &\leq \left(1 - \left(\frac{t}{x_q}\right)^n\right) Pr\left(\frac{1}{n} \sum_{i=1}^n |p_i| \leq \frac{2kx_s}{tn\epsilon}\right) + \left(\frac{t}{x_q}\right)^n \end{aligned}$$

Then by Chebyshev inequality, we have, by putting $t = \frac{1}{2x_q}$

$$\begin{aligned} &Pr\left(\frac{\|pq^T - \hat{p}\hat{q}^T\|}{\|pq^T\|_1} > \epsilon\right) \\ &\leq \left(1 - \left(\frac{1}{2}\right)^n\right) \frac{\frac{1}{n} \left(\frac{x_p^2}{12}\right)}{\left(\frac{x_p}{2} - \frac{2kx_s}{n\epsilon x_q}\right)} + \left(\frac{1}{2}\right)^n \end{aligned}$$

And this goes to zero as $n \rightarrow \infty$.

□

5.5 Empirical Evidence

We simulate the power iteration in recovering sparse noise. In this simulation, we use randomly generated the entries of p and q as $N(0,1)$ iid. And then we randomly generate sparse matrix with sparse support uniformly distributed across the $n \times n$ matrix. And each sparse entry has a value with distribution of $N(0,1)$. We then plot the graph of different degree of sparsity and the corresponding effectiveness of the optimization heuristic in extracting the original pq^T . The result is as follows.

Figure 1: Simulation result

6 Algorithms

7 Applications

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