# Sparse linear-quadratic regulator

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## 1 Problem

We have a system defined by

$$\dot{x} = Ax + Bu + w \tag{1}$$

where $w$ is random with covariance $E[ww^T] = W$. We wish to find a feedback controller

$$u = Kx \tag{2}$$

minimizing the quadratic objective

$$J = \int_0^\infty \left[ x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right] d\tau. \tag{3}$$

This is equivalent to

$$J = \operatorname{tr} W P(K) \tag{4}$$

where $K \in \mathcal{D}_s$, the set of stabilizing feedback gains

$$\mathcal{D}_s = \{K | Re\{\lambda(A + BK)\} < 0\} \tag{5}$$

and $P(K)$ is the unique solution to the Lyapunov equation

$$(A + BK)^T P + P(A + BK) + (Q + K^T RK) = 0. \tag{6}$$

## 2 Differentials

We begin with the differential for the matrix-valued function $P(K)$. Let

$$\psi(P, K) = (A + BK)^T P + P(A + BK) + (Q + K^T RK) \tag{7}$$

and we have the partials

$$
\begin{aligned}
\psi_P(P,K) &= (A+BK)^T dP + dP(A+BK) \\
\psi_K(P,K) &= (BdK)^T P + PBdK + dK^T RK + K^T RdK
\end{aligned}
\tag{8}
$$

which for a given $K$ and $dK$ results in the Lyapunov equation

$$
(A+BK)^T\tilde{P} + \tilde{P}(A+BK) + dK^T(B^TP+RK) + (PB+K^TR)dK = 0 \tag{9}
$$

where $P = P(K)$ and $\tilde{P} = P'(K)dK$ is the unique solution. This gives us an expression for the differential of the objective

$$
J'(K)dK = \operatorname{tr} P'(K)dKW. \tag{10}
$$

However, in order to write the derivative in gradient form we need an expression

$$
J'(K)dK = \operatorname{tr} \nabla J(K)^T dK. \tag{11}
$$

Let $L$ be the solution to the Lyapunov equation

$$
(A+BK)L + L(A+BK)^T + W = 0 \tag{12}
$$

and by multiplying (9) by $L$, (12) by $\tilde{P}$ and taking the trace we get

$$
\operatorname{tr} P'(K)dKW = 2\operatorname{tr} L(PB+K^TR)dK \tag{13}
$$

which results in our equation for the derivative

$$
\nabla J(K) = 2(B^TP+RK)L \tag{14}
$$

Continuing, we can compute the second differential from (13) as

$$
\begin{aligned}
J''(K)(dK,dK) &= 2\operatorname{tr}[L'(K)dK(P(K)B+K^TR)dK + \\
&\qquad L(K)(P'(K)dKB + dK^TR)dK] \\
&= 2\operatorname{tr} H(K,dK)dK
\end{aligned}
\tag{15}
$$

where

$$
H(K,dK) = \tilde{L}(PB+K^TR) + L(\tilde{P}B + dK^TR) \tag{16}
$$

and $\tilde{L} = L'(K)dK$ and $\tilde{P} = P'(K)dK$ are the unique solutions to (9) and

$$
(A+BK)\tilde{L} + \tilde{L}(A+BK)^T + BdKL + LdK^TB^T = 0 \tag{17}
$$

respectively.

# 3 Second-order approximation

In this section we derive an explicit form of the second-order approximation which we will need for the coordinate descent updates and to write down the Hessian. Let $D$ be an arbitrary direction, the second order approximation at $K$ is given by

$$g(K, D) = 2\operatorname{tr} L(PB + K^T R)D + \operatorname{tr} \tilde{L}(PB + K^T R)D + \operatorname{tr} L(\tilde{P}B + D^T R)D. \tag{18}$$

where $\tilde{L}$ and $\tilde{P}$ are the solutions to Lyapunov equations that depend on $D$. In general notation they have the form

$$AX + XA^T + Q(D) = 0 \tag{19}$$

which we can solve for $X$ by taking the eigendecomposition $A = U\Lambda U^{-1}$. This gives

$$\begin{aligned} U\Lambda U^{-1}X + XU^{-T}\Lambda U^T + Q(D) = 0 \\ \Lambda U^{-1}XU^{-T}X + U^{-1}XU^{-T}\Lambda = -U^{-1}Q(D)U^{-T} \end{aligned} \tag{20}$$

which we can solve by noting that $\Lambda$ is diagonal with elements $\lambda_1, \ldots, \lambda_n$ and thus

$$X = U(-U^{-1}Q(D)U^{-T} \circ \Theta)U^T \tag{21}$$

where $\Theta_{ij} = 1/(\lambda_i + \lambda_j)$. Applying the above to the Lyapunov equations (9) and (17) we get the explicit forms

$$\begin{aligned} \tilde{P} &= -U^{-T}\left(U^T(D^T(B^T P + RK) + (PB + K^T R)D)U \circ \Theta\right)U^{-1}, \\ \tilde{L} &= -U\left(U^{-1}(BDL + LD^T B^T)U^{-T} \circ \Theta\right)U^T \end{aligned} \tag{22}$$

where $U\Lambda U^{-1}$ is the eigendecomposition of $A + BK$.

Since $\Theta$ is complex symmetric we can take the Autonne-Takagi factorization and write $\Theta = U\Sigma U^T = XX^T$. Furthermore, we typically have that $\Theta$ is low rank and thus we can take $XX^T \approx \sum_{i=1}^{k} x_i x_i^*$ with $k \ll n$.

# 4 Coordinate descent updates

Let $E = PB + K^T R$ and we can rewrite (18) as

$$g(K, D) = 2\operatorname{tr} LED + \operatorname{tr} \tilde{L}ED + \operatorname{tr} L\tilde{P}BD + \operatorname{tr} LD^T RD. \tag{23}$$

Plugging in the the explicit solutions to the Lyapunov equations from (22), we have

$$\begin{aligned} \operatorname{tr} \tilde{L}ED &= -\operatorname{tr} U\left(U^{-1}(BDL + LD^T B^T)U^{-T} \circ \Theta\right)U^T ED \\ &\approx -\operatorname{tr} U\left(\sum_{i=1}^{k} \operatorname{diag}(x_i)U^{-1}(BDL + LD^T B^T)U^{-T}) \operatorname{diag}(x_i)\right)U^T ED \\ &= \sum_{i=1}^{k} -\operatorname{tr} X_i BDLX_i^T ED - \operatorname{tr} X_i (BDL)^T X_i^T ED \end{aligned} \tag{24}$$

where $X_i = U \operatorname{diag}(x_i) U^{-1}$. We also have

$$\begin{aligned}
\operatorname{tr} L\tilde{P}BD &= -\operatorname{tr} U^{-T}\left(U^T(D^TE^T + ED)U \circ \Theta\right) U^{-1}BDL \\
&\approx -\operatorname{tr} U^{-T}\left(\sum_{i=1}^{k} \operatorname{diag}(x_i) U^T(D^TE^T + ED)U \operatorname{diag}(x_i)\right) U^{-1}BDL \\
&= \sum_{i=1}^{k} -\operatorname{tr} X_i^T(ED)^T X_i BDL - \operatorname{tr} X_i^T EDX_iBDL
\end{aligned} \tag{25}$$

Noting that these are equivalent, we can rewrite the second-order approximation as

$$g(K, D) = 2\operatorname{tr} LED + \operatorname{tr} LD^TRD - 2\left(\sum_{i=1}^{k} \operatorname{tr} X_i^T(ED)^T X_i BDL + \operatorname{tr} X_i^T EDX_iBDL\right). \tag{26}$$

To derive the coordinatewise updates let $D = D + \mu e_i e_j^T$ and we take the minimum over $\mu$. First, we consider the terms inside the sum and drop the subscript on $X_i$ for brevity and for the first term we have

$$\begin{aligned}
&\arg\min_{\mu} \operatorname{tr} X^T(D + \mu e_i e_j^T)^T E^T XB(D + \mu e_i e_j^T)L \\
&= \arg\min_{\mu} \mu \operatorname{tr} X^T e_j e_i^T E^T XBDL + \mu \operatorname{tr} X^T D^T E^T XBe_i e_j^T L + \mu^2 X^T e_j e_i^T E^T XBe_i e_j^T L \\
&= \arg\min_{\mu} \mu (E^TXBDLX^T)_{ij} + \mu(B^TX^TEDXL)_{ij} + \mu^2(E^TXB)_{ii}(LX^T)_{jj}.
\end{aligned} \tag{27}$$

Similarly, for the second term we have

$$\begin{aligned}
&\arg\min_{\mu} \operatorname{tr} X^T E(D + \mu e_i e_j^T) XB(D + \mu e_i e_j^T)L \\
&= \arg\min_{\mu} \mu \operatorname{tr} X^T Ee_i e_j^T XBDL + \mu \operatorname{tr} X^T EDXBe_i e_j^T L + \mu^2 X^T Ee_i e_j^T XBe_i e_j^T L \\
&= \arg\min_{\mu} \mu (XBDLX^TE)_{ji} + \mu(LX^TEDXB)_{ji} + \mu^2(LX^TE)_{ji}(XB)_{ji}.
\end{aligned} \tag{28}$$

Finally, for the first terms outside of the sum we have

$$\begin{aligned}
&\arg\min_{\mu} 2\operatorname{tr} LE(D + \mu e_i e_j^T) + \operatorname{tr} L(D + \mu e_i e_j^T)^T R(D + \mu e_i e_j^T) \\
&= \arg\min_{\mu} 2\mu \operatorname{tr} LEe_i e_j^T + 2\mu \operatorname{tr} Le_j e_i^T RD + \mu^2 \operatorname{tr} Le_j e_i^T Re_i e_j^T \\
&= \arg\min_{\mu} 2\mu (LE)_{ji} + 2\mu(RDL)_{ij} + \mu^2 L_{jj} R_{ii}
\end{aligned} \tag{29}$$

Combining these expressions and adding $\ell_1$ regularization, we have

$$\arg\min_{\mu} g(K, D + \mu e_i e_j^T) + \lambda\|K + D + \mu e_i e_j^T\|_1 = \arg\min_{\mu} \frac{1}{2}a^2\mu + b\mu + \lambda\|c + \mu\|_1 \tag{30}$$

where

$$
\begin{aligned}
a &= 2R_{ii}L_{jj} - 4\left(\sum_{k=1}^{r}(E^T X_k B)_{ii}(LX_k^T)_{jj} + (LX^T E)_{ji}(XB)_{ji}\right) \\
b &= 2(E^T L)_{ij} + 2(RDL)_{ij} \\
&\quad - 2\left(\sum_{k=1}^{r}(E^T X_k BDLX_k^T)_{ij} + (B^T X_k^T EDX_k L)_{ij} + (X_k BDLX_k^T E)_{ji} + (LX_k^T EDX_k B)_{ji}\right) \\
c &= K_{ij} + D_{ij}
\end{aligned} \tag{31}
$$

which has the closed form solution

$$\mu = -c + S_{\lambda/a}\left(c - \frac{b}{a}\right) \tag{32}$$

where $S_\lambda$ is the soft-thresholding operator.