Olimpiada Colombiana de
Astronomía, Astrofísica
y Astronáutica

OCAAA

Prueba Clasificatoria

XVI OCAAA

Autores:

Isaac Newton
Henrietta Leavitt
Edwin Hubble
Albert Einstein
Georges Lemaître
Edwin Hubble

UNIVERSIDAD ANTONIO NARIÑO

Oficina de Olimpiadas Colombianas (OC) - Universidad Antonio Nariño

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Olimpiada Colombiana de Astronomía, Astrofísica y Astronáutica
OCAAA

PT-SEL-202500

UNIVERSIDAD ANTONIO NARIÑO

Datos del Participante

APELLIDOS Y NOMBRES ______________________________

GRADO ______________ **COLEGIO** ______________________

EMAIL ______________________ **TELÉFONO** ______________

CIUDAD ______________ **DEPT.** ______________ **EDAD:** ________

1	2	3	4	5

Total

Cuadro para calificación (no llenar)

Instrucciones

1. No abra los enunciados hasta que se le indique.
2. Llene los datos del participante en los espacios asignados.
3. Se dispone de $\boxed{90}$ minutos para el desarrollo de la prueba.
4. Es válido el uso de calculadoras (No se permite el uso de calculadoras graficadoras o calculadoras programables).
5. Es permitido el uso de instrumentos de escritura y hojas en blanco, las cuales se le proporcionarán.
6. Escriba sus respuestas y operaciones de manera clara y ordenada en las hojas que se le proporcionarán, a menos que el enunciado indique lo contrario.
7. Está prohibido el uso de apuntes o ayudas. En caso de dudas en cuanto a los enunciados de la prueba, el profesor de la OCAAA, podrá resolverlas.
8. Recuerde que está permitido el uso de la tabla de constantes (TC-ES-2024-01).
9. El jurado se reserva el derecho de no considerar soluciónes confusas o ilegibles. Así como el derecho de rechazar soluciones que generen dudas de honestidad en su aplicación.

Código de Honor

Con mi firma certifico que este es mi trabajo personal y no recibí ayuda o colaboración ajena en el desarrollo de la prueba.

Firma del participante: ______________________

1 Co-orbital satellites (50 Points)

This question applies a method of determining the masses of two approximately co-orbital satellites developed by Dermott and Murray in 1981.
Suppose that two small satellites of masses m_1 and m_2 are approximately co-orbital (moving on very similar orbits) around a large central body of mass M, with $m_1,\ m_2 \ll M$. At any instant, the orbits of the satellites may be approximated as circular Keplerian orbits with radii r_1 and r_2 respectively, although r_1 and r_2 will vary slightly over time due to the mutual gravitational interaction between the satellites.
Figure 4 depicts the shapes of the orbits in the rotating reference frame with zero angular momentum, centred on the central body. We denote by θ the angle $\measuredangle m_1 M m_2$, while R, x_1, and x_2 denote the mean orbital radius and radial deviations of the satellites.
Throughout this problem, write all answers in an inertial reference frame.
Hint: $(1+x)^\alpha \approx 1 + \alpha x$ for $\alpha x \ll 1$
First we will determine the value of $\frac{m_1}{m_2}$.

(a) Write down the angular momentum L_i of the satellite with mass m_i when its circular orbit has radius r_i. **(3pt)**

(b) The satiletes total angular momentum $L_1 + L_2$ is conserved. Let $x_1,\ x_2 \ll R$ be the distances as shown in Figure 4. Find a simple relation between the ratios $\frac{m_1}{m_2}$ and $\frac{x_1}{x_2}$. **(8pt)**

Solución:

Esta es la solución de a

Now, we will try and determine the value of $m_1 + m_2$. For next parts, we will use the actual barycenter of the system, which may not be exactly at the center of the planet.

(c) The individual angular momenta of the satellites m_1 and m_2 will vary over time due to their gravitational interactions. Show that the rate of change of the angular momentum of the second satellite is given by

$$\frac{\Delta L_2}{\Delta t} \approx -\frac{G m_1 m_2}{R} h(\theta) \quad \text{where} \quad h(\theta) = \left[\frac{\cos\left(\frac{\theta}{2}\right)}{4\sin^2\left(\frac{\theta}{2}\right)} - \sin\theta\right]$$ **(18pt)**

(d) Show that $s = r_2 - r_1$ satisfies

$$\frac{\Delta s}{\Delta t} \approx -2\sqrt{\frac{G}{MR}}(m_1+m_2)h(\theta)$$ **(8pt)**

(e) For the angle $\theta = \measuredangle m_1 M m_2$ as indicated on Figure 4, find an expression for $\dfrac{\Delta\theta}{\Delta t}$ **(5pt)**

(f) Using the results above, find the relation between Δs and $\Delta\theta$. **(2pt)**

(g) After integrating the expression above, we will obtain the result,

$$\overline{x}^2 \approx \frac{4R^2}{3}\frac{m_1+m_2}{M}\left(\frac{1}{\sin\left(\frac{\theta_{\min}}{2}\right)} - 2\cos\theta_{\min} - 3\right)$$

where $\overline{x} = \dfrac{x_1+x_2}{2}$.

Epimetheus (m_1) and Janus (m_2) are two approximately co-orbital moons of Saturn. Detailed observations of their orbits have been performed by the Voyager 1 and Cassini spacecraft, which found that $R = 150\,000\,\mathrm{km}$, $x_1 = 76\,\mathrm{km}$ and $x_2 = 21\,\mathrm{km}$. The minimum distance between Janus and Epimetheus is $13\,000\,\mathrm{km}$. The mass of Saturn is known to be $5.7 \times 10^{26}\,\mathrm{kg}$. Estimate the masses of Epimetheus and Janus. **(6pt)**

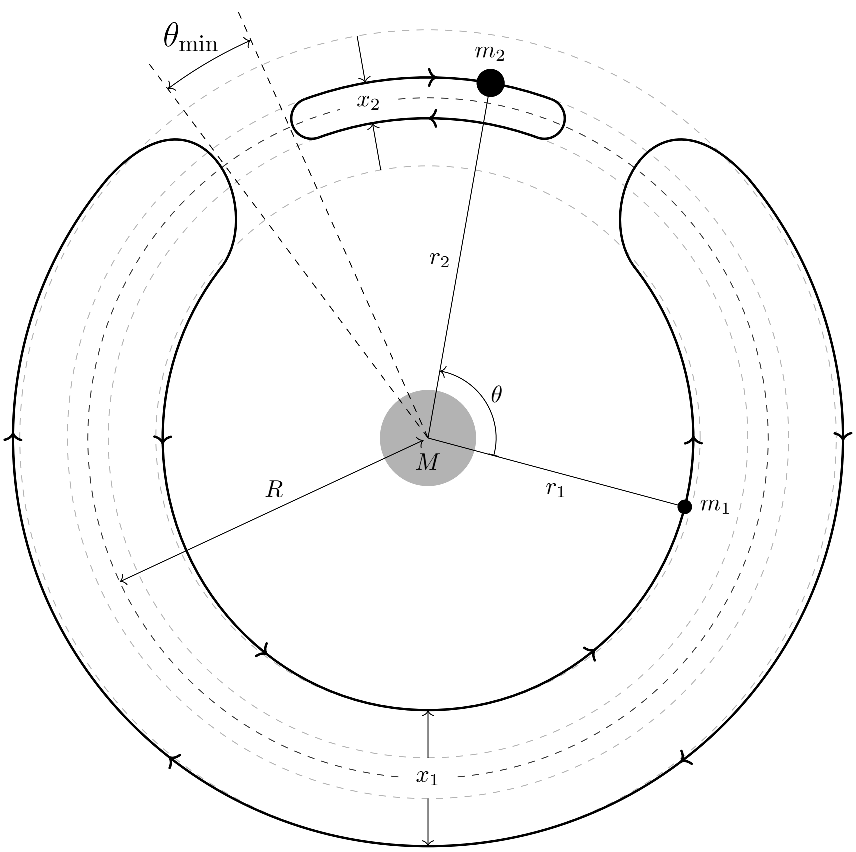


Figure 1: This figure schematically depicts the shapes of the orbits in the rotating reference frame, selected such that in this frame the total angular momentum of the two satellites is zero.

2 Relativistic Beaming (50 Points)

Consider an isotropic light source of frequency f_R in a frame which is fixed to the source (i.e. rest frame). In this rest frame, consider a light ray emitted from the source that makes an angle θ_R with the X-axis. The light source is moving along positive X direction with (relativistic) speed v as measured in the lab frame.

(a) Find an expression for the frequency f_L of this ray in the lab frame, and the cosine of the angle that this ray makes with the X-axis in the lab frame. **(11pt)**
Hint: In relativistic mechanics, energy E and momentum p of a particle between rest and lab frame are related in the following way:

$$\begin{aligned}\frac{E_L}{c} &= \gamma\left(\frac{E_R}{c} + p_{x_R}\frac{v}{c}\right)\\ p_{x_L} &= \gamma\left(p_{x_R} + \frac{E_R v}{c^2}\right)\\ p_{y_L} &= p_{y_R}\\ p_{z_L} &= p_{z_R}\end{aligned}$$

where:

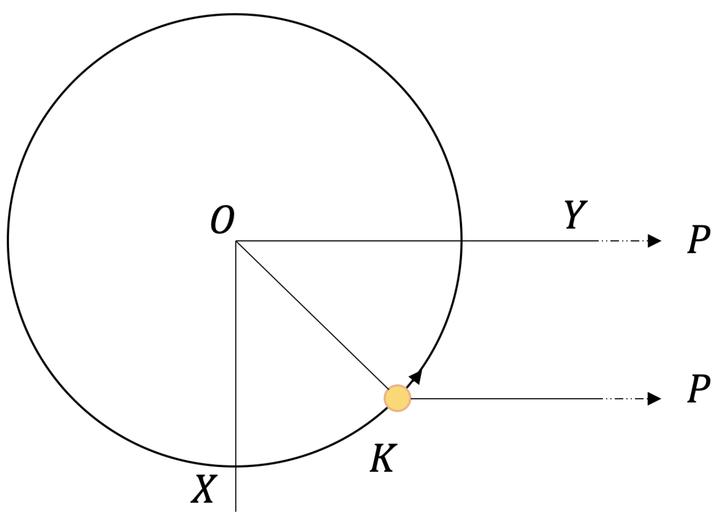
$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

(b) For the following cases:

i) $\theta_R = 0°$
ii) $\theta_R = \cos^{-1}(-v/c)$
iii) $\theta_R = 90°$
iv) $\theta_L = 180°$

draw direction vectors of the beam in XY plane of the rest frame as well as separately in $X'Y'$ plane of the lab frame. **(4 pt)**

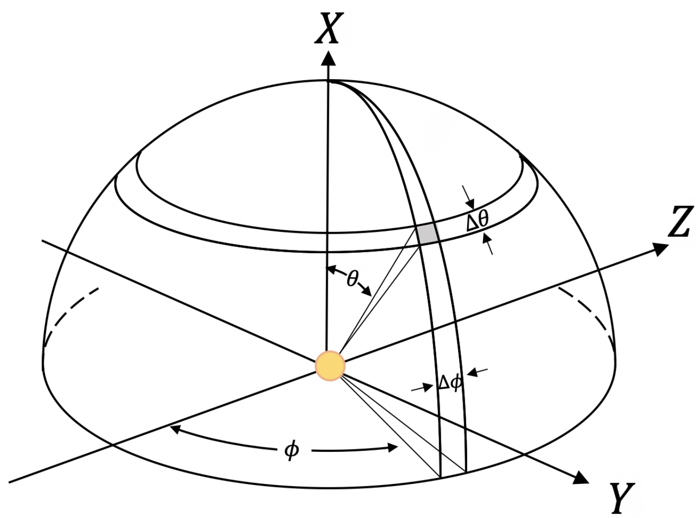
In accretion disks around black holes, the charged particles are orbiting at relativistic speeds and in their rest frames may be considered as isotropic point sources of light. Consider such a particle K in a circular orbit of radius r and angular speed ω around a central object located at O (see figure).



Let us assume that our lab frame is fixed to an observer located at a point P on the OY axis, which is stationary with respect to O. $OP = R \gg r$. Let $t_{L0} = t_{R0} = 0$ correspond to the moment when K is seen crossing the OX axis. As K is moving with relativistic speed, the duration Δt_R measured by an observer in the rest frame of the source K is related to the duration measured in the lab frame Δt_L at P by the expression $\Delta t_L = \gamma \Delta t_R$.

(c) Derive an expression for f_L as a function of t_L ($t_L > R/c$)? **(7pt)**

Let us consider a fraction of the light from the source that is emitted in an infinitesimal solid angle $\Delta\Omega_R = -\Delta(\cos\theta_R) \cdot \Delta\phi$ in the direction making an angle θ_R with respect to the X axis in the rest frame, as it is shown on the figure below.



(d) Show that, as measured in the lab frame

$$\Delta\Omega_L = \frac{\Delta\Omega_R}{\gamma^2 \left(1 + \frac{v}{c}\cos\theta_R\right)^2}$$ **(10pt)**

(e) If the intrinsic luminosity of the light source is L, what is the energy flux F_L observed by the observer at point P at the moment t_L ($t_L > R/c$)? **(15pt)**
Hint: In the rest frame of the source, you may assume N_R number of photons are directed within the solid angle $\Delta\Omega_R$ during the time interval Δt_R.

(f) Charged particles in the relativistic beam shot from the supermassive black hole at the centre of the galaxy M87 have speeds up to $0.95c$. What would be the maximum and minimum amplification factor for the energy flux for a relativistic beam from M87? **(3pt)**