Chapter 6

Some Continuous Probability Distributions – part 2

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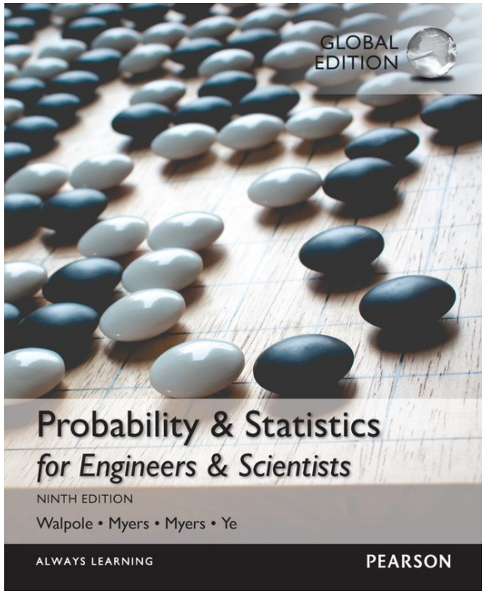

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6.5 Normal Approximation to the Binomial

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Recall from Ch. 5.5: Approximation of Binomial Distribution by a Poisson Distribution

- In the case of the binomial, if *n* is quite large and *p is* small, the conditions begin to simulate the continuous space or time implications of the Poisson process

Let X be a binomial random variable with probability distribution $b(x;n,p)$. When $n \to \infty$, $p \to 0$, and $np \overset{n\to\infty}{\longrightarrow} \mu$ remains constant,

$$b(x;n,p) \overset{n\to\infty}{\longrightarrow} p(x;\mu).$$

- Both Binomial and Poisson distributions are discrete

Question:

- Assume that about 12% of Americans are black.
- Suppose we draw a simple sampling test from 1,500 Americans, to assess whether the representation of blacks in the sample is accurate.
- So, we expect **X**, the number of blacks in the sample, to be around 180 (12% of 1,500).
- What is the probability that the sample contains 170 or fewer blacks?
  - Solution ?:

$$P(X \leq 170) = \sum_{j=0}^{170} P(X = j)$$

$$= \sum_{j=0}^{170} \binom{1500}{j} (0.12)^j (0.88)^{1500-j}$$

That's pretty ugly. Is there an easier way?

It turns out that as *n* gets larger, the Binomial distribution looks increasingly like the Normal distribution.

Normal Approximation to the Binomial

- The normal distribution is often a good approximation to a discrete distribution when the latter takes on a symmetric bell shape.

- From a theoretical point of view, some distributions converge to the normal as their parameters approach certain limits.

Normal Approximation to the Binomial

- Why is this important ?

- The normal distribution is a convenient approximating distribution because the cumulative distribution function is so easily tabled (Table A.3).

  - E.g., The binomial distribution is nicely approximated by the normal in practical problems when one works with the cumulative distribution function.

Review: E[X], Var[X] of binomial dist.

- Let X be a binomial random variable with parameters *n* and *p*.

The mean and variance of the binomial distribution $b(x; n, p)$ are

$$\mu = np \text{ and } \sigma^2 = npq.$$

$$\mu = np$$

$$\sigma = \sqrt{np(1-p)}$$

- Theorem **6.3**

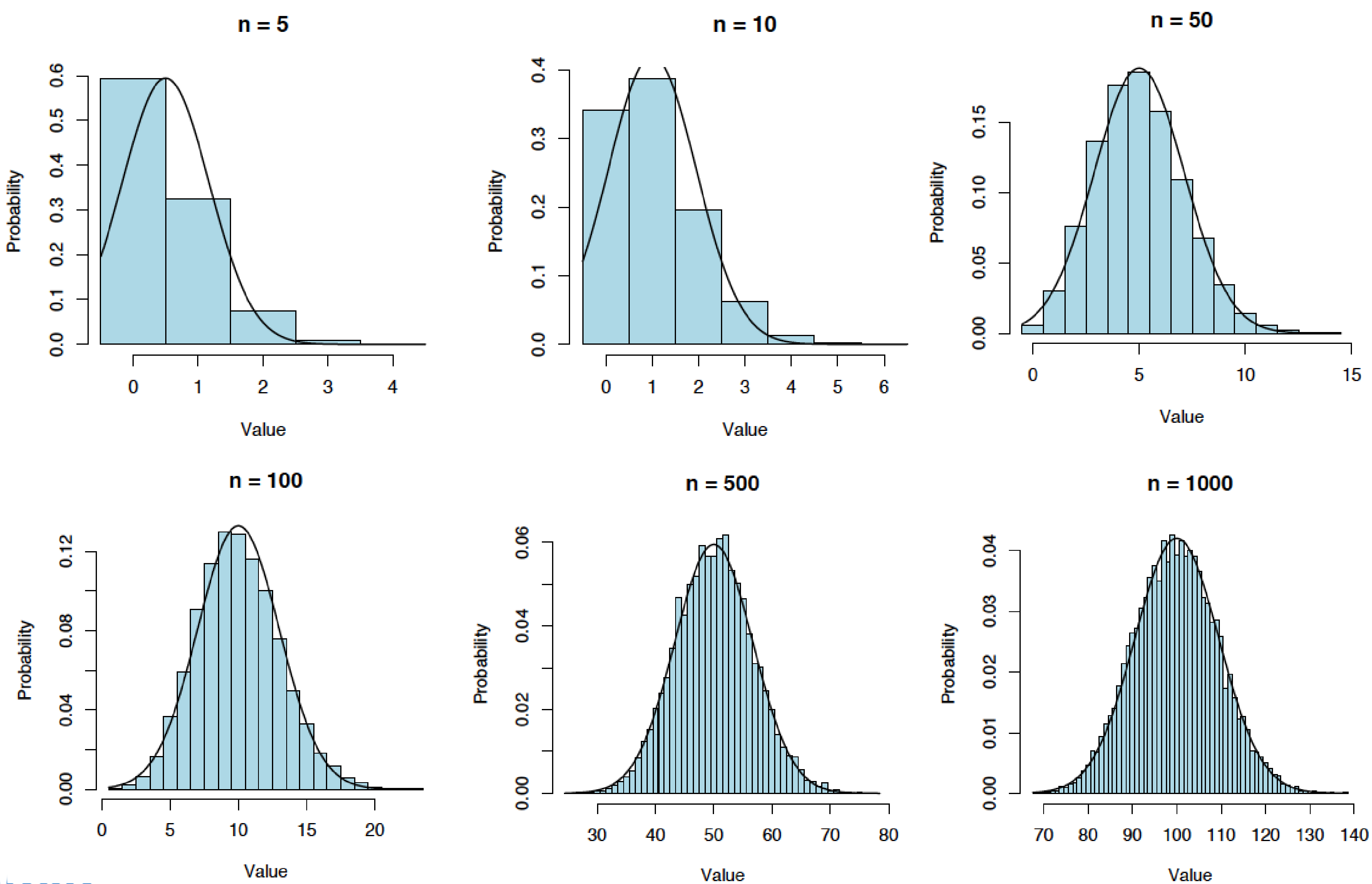
If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then the limiting form of the distribution of

$$Z = \frac{X - np}{\sqrt{npq}},$$

as $n \to \infty$, is the standard normal distribution $n(z; 0, 1)$.

* a very accurate approximation to the binomial distribution when ***n* is large and *p* is not extremely close to 0 or 1** but also provides a fairly good approximation even when ***n* is small and *p* is reasonably close to 1/2**.

Behavior of the Approximation as a Function of *n*, for *p* = 0.1



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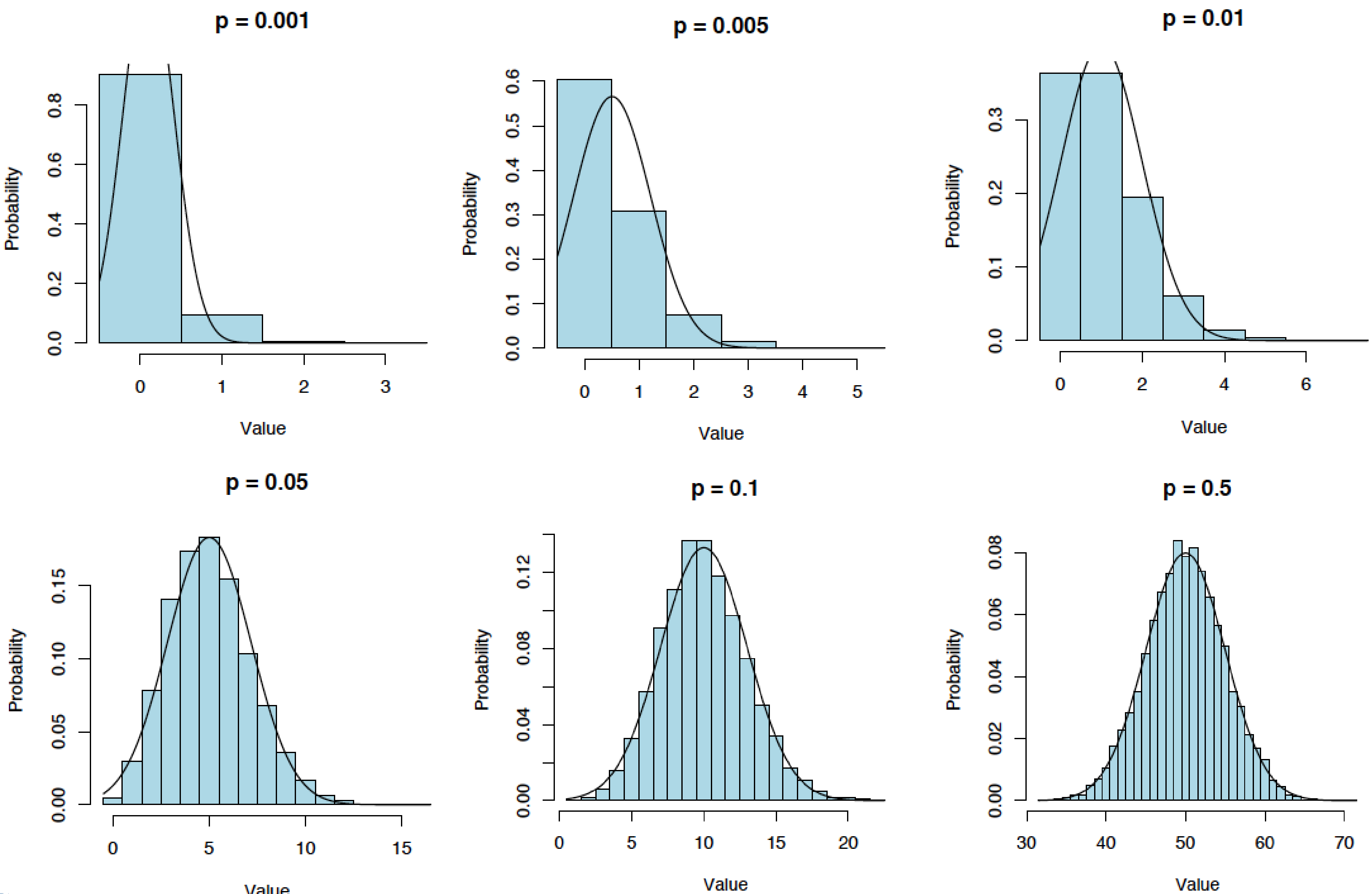
Behavior of the Approximation as a Function of *p*, for *n* = 100



Example

- Binomial b (x; 15, 0.4) approximation to Normal

$$\mu = np = (15)(0.4) = 6 \text{ and } \sigma^2 = npq = (15)(0.4)(0.6) = 3.6.$$

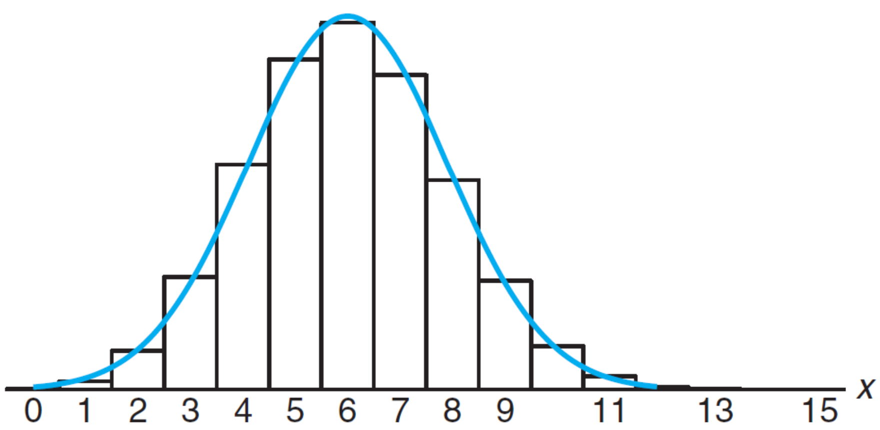


Figure 6.22: Normal approximation of $b(x; 15, 0.4)$.

Example

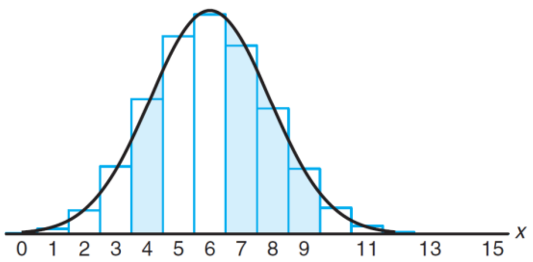


Figure 6.23: Normal approximation of $b(x; 15, 0.4)$ and $\sum_{x=7}^{9} b(x; 15, 0.4)$.

The exact probability that the binomial random variable X assumes a given value x is equal to the area of the bar whose base is centered at x. For example, the exact probability that X assumes the value 4 is equal to the area of the rectangle with base centered at $x = 4$. Using Table A.1, we find this area to be

$$P(X = 4) = b(4; 15, 0.4) = \boxed{0.1268,}$$

which is approximately equal to the area of the shaded region under the normal curve between the two ordinates $x_1 = 3.5$ and $x_2 = 4.5$ in Figure 6.23. Converting to z values, we have

$$z_1 = \frac{3.5 - 6}{1.897} = -1.32 \qquad \text{and} \qquad z_2 = \frac{4.5 - 6}{1.897} = -0.79.$$

If X is a binomial random variable and Z a standard normal variable, then

$$\begin{aligned} P(X = 4) &= b(4; 15, 0.4) \approx P(-1.32 < Z < -0.79) \\ &= P(Z < -0.79) - P(Z < -1.32) = 0.2148 - 0.0934 = \boxed{0.1214.} \end{aligned}$$

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Example

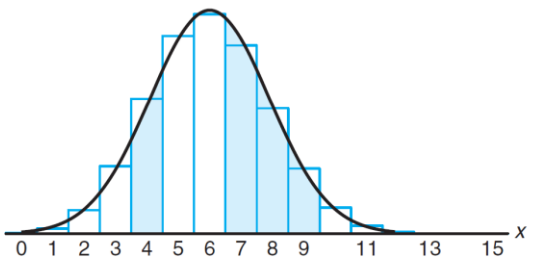

Figure 6.23: Normal approximation of $b(x; 15, 0.4)$ and $\sum_{x=7}^{9} b(x; 15, 0.4)$.

Probability that X assumes a value from 7 to 9 inclusive?

$$P(7 \leq X \leq 9) = \sum_{x=0}^{9} b(x; 15, 0.4) - \sum_{x=0}^{6} b(x; 15, 0.4)$$
$$= 0.9662 - 0.6098 = \boxed{0.3564},$$

which is equal to the sum of the areas of the rectangles with bases centered at $x = 7$, 8, and 9. For the normal approximation, we find the area of the shaded region under the curve between the ordinates $x_1 = 6.5$ and $x_2 = 9.5$ in Figure 6.23. The corresponding z values are

$$z_1 = \frac{6.5 - 6}{1.897} = 0.26 \quad \text{and} \quad z_2 = \frac{9.5 - 6}{1.897} = 1.85.$$

Now,

$$P(7 \leq X \leq 9) \approx P(0.26 < Z < 1.85) = P(Z < 1.85) - P(Z < 0.26)$$
$$= 0.9678 - 0.6026 = \boxed{0.3652.}$$

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Why +0.5 ? : **Continuity Correction**

- The addition of 0.5 in the previous slide is an example of the *continuity correction* which is intended to refine the approximation by accounting for the fact that the Binomial distribution is discrete while the Normal distribution is continuous.

$$P(7 \le X \le 9) = \sum_{x=0}^{9} b(x; 15, 0.4) - \sum_{x=0}^{6} b(x; 15, 0.4)$$
$$= 0.9662 - 0.6098 = 0.3564,$$

$$x_1 = 6.5 \text{ and } x_2 = 9.5$$

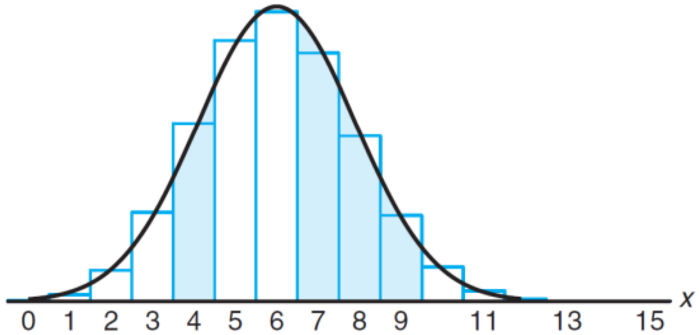

Figure 6.23: Normal approximation of $b(x; 15, 0.4)$ and $\sum_{x=7}^{9} b(x; 15, 0.4)$.

$$\sum_{x=0}^{6} b(x; n, p) = P(Y \le \mathbf{6.5}) = P\left(Z \le \frac{\mathbf{6.5} - np}{\sqrt{npq}}\right)$$

$$\sum_{x=0}^{9} b(x; n, p) = P(Y \le \mathbf{9.5}) = P\left(Z \le \frac{\mathbf{9.5} - np}{\sqrt{npq}}\right)$$

Normal Approximation to the Binomial

Let X be a binomial random variable with parameters n and p. For large n, X has approximately a normal distribution with $\mu = np$ and $\sigma^2 = npq = np(1-p)$ and

$$\begin{aligned} P(X \leq x) &= \sum_{k=0}^{x} b(k; n, p) \\ &\approx \text{area under normal curve to the left of } x + 0.5 \\ &= P\left(Z \leq \frac{x + 0.5 - np}{\sqrt{npq}}\right), \end{aligned}$$

and the approximation will be good if np and $n(1-p)$ are greater than or equal to 5.

Recall: Question:

- Assume that about 12% of Americans are black.
- Suppose we draw a simple sampling test from 1,500 Americans, to assess whether the representation of blacks in the sample is accurate.
- So we expect **X**, the number of blacks in the sample, to be around 180 (12% of 1,500).
- What is the probability that the sample contains 170 or fewer blacks?

Recall: the problem (12% black in 1500 people)

- What is the probability that the sample contains 170 or fewer blacks? $P(X \leq 170), \text{ where } X \sim B(1500, 0.12)$

- **How do we calculate this using the Normal approximation?**
  - Issue : Binomial dist. => discrete while Normal dist. => continuous
- If we were to draw a histogram of the **B**(1500**,** 0**.**12) distribution with bars of width one, **P**(**X** >= 170) would be represented by the total area of the bars spanning

(-0.5, 0:5], (0.5, 1.5], … (169.5, 170.5]

- Thus, using the approximating Normal distribution Y ~ N(180, 12.59), we calculate (μ = 180, σ=*12.59)*

$$P(X \leq 170) \approx P(Y \leq 170.5) = 0.2253$$

- For reference, the exact Binomial probability is 0.2265, so the approximation is apparently pretty good.

- Example 6.15
  - The probability that a patient recovers from a rare blood disease is 0.4. If 100 people are known to have contracted this disease, what is the probability that less than 30 survive?

Let the binomial variable X represent the number of patients who survive. Since $n = 100$, we should obtain fairly accurate results using the normal-curve approximation with

$$\mu = np = (100)(0.4) = 40 \text{ and } \sigma = \sqrt{npq} = \sqrt{(100)(0.4)(0.6)} = 4.899.$$

To obtain the desired probability, we have to find the area to the left of $x = 29.5$.

$$\begin{aligned} P(X \le x) &= \sum_{k=0}^{x} b(k; n, p) \\ &\approx \text{area under normal curve to the left of } x + 0.5 \\ &= P\left(Z \le \frac{x + 0.5 - np}{\sqrt{npq}}\right), \end{aligned}$$

The z value corresponding to 29.5 is

$$z = \frac{29.5 - 40}{4.899} = -2.14,$$

and the probability of fewer than 30 of the 100 patients surviving is given by the shaded region in Figure 6.26. Hence,

$$P(X < 30) \approx P(Z < -2.14) = 0.0162.$$

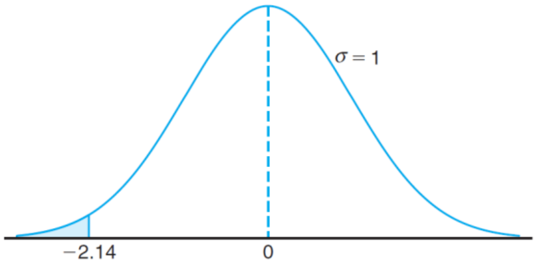

Figure 6.26: Area for Example 6.15.

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- Example 6.16
  - A multiple-choice quiz has 200 questions each with 4 possible answers of which only 1 is the correct answer. What is the probability that sheer guesswork yields from 25 to 30 correct answers for 80 of the 200 problems about which the student has no knowledge?

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The probability of guessing a correct answer for each of the 80 questions is $p = 1/4$. If X represents the number of correct answers resulting from guesswork, then

$$P(25 \leq X \leq 30) = \sum_{x=25}^{30} b(x; 80, 1/4).$$

Using the normal curve approximation with

$$\mu = np = (80)\left(\frac{1}{4}\right) = 20$$

and

$$\sigma = \sqrt{npq} = \sqrt{(80)(1/4)(3/4)} = 3.873,$$

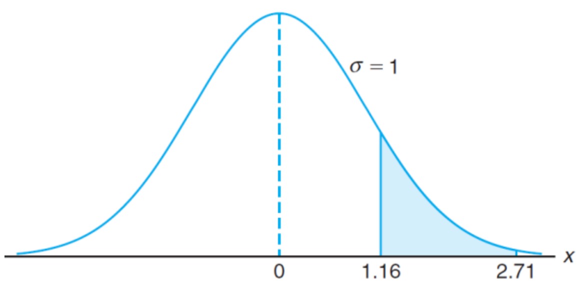

Figure 6.27: Area for Example 6.16.

we need the area between $x_1 = 24.5$ and $x_2 = 30.5$. The corresponding z values are

$$z_1 = \frac{24.5 - 20}{3.873} = 1.16 \text{ and } z_2 = \frac{30.5 - 20}{3.873} = 2.71.$$

The probability of correctly guessing from 25 to 30 questions is given by the shaded region in Figure 6.27. From Table A.3 we find that

$$P(25 \leq X \leq 30) = \sum_{x=25}^{30} b(x; 80, 0.25) \approx P(1.16 < Z < 2.71)$$

$$= P(Z < 2.71) - P(Z < 1.16) = 0.9966 - 0.8770 = 0.1196.$$

6.6 **Gamma** and **Exponential** Distributions

- Although the normal distribution can be used to solve many problems in engineering and science, there are still numerous situations that require different types of density functions.
  - Sometimes we're interested in the **time** until a certain number of events occur.
  - Sometimes we're interested in **time between** two events

- The exponential and gamma distributions
  - play an important role in both queuing theory and reliability problems.

Recall: **Poisson Distribution (Ch 5.5)**

- The probability distribution of the Poisson random variable X, representing **the number of outcomes** occurring in a given time interval or specified region denoted by t, is

$$P(X = x) = p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!} \quad x = 0, 1, 2, \ldots,$$

- where λ is the average number of outcomes per unit time, distance, area, or volume, and e = 2.71828 … .

$$E(X) = \mathrm{Var}(X) = \lambda t.$$

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- **Gamma Distribution**
  - Describing the time (or space) occurring until a specified number of Poisson events occur.

- **Exponential Distribution**
  - Describing the time until the first occurrence of a Poisson event (or the time between Poisson events).
  - A special case of Gamma distribution

Gamma Function

- The gamma function is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \qquad \text{for } \alpha > 0.$$

an extension of the factorial function, with its argument shifted down by 1, to real and complex numbers.

Properties

$$\begin{cases} \Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \\ \Gamma(n) = (n-1)! \quad \text{for positive integer } n \\ \Gamma(1) = 1 \qquad \Gamma(0.5) = \sqrt{\pi} \end{cases}$$

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Gamma Distribution

- The continuous random variable X has a gamma distribution, with parameters α and β, if its density function is given by

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$

α = number of occurrences

β = mean time between occurrences

$\Gamma(n) = (n-1)!$ for positive integer n

$\mu = \alpha\beta, \quad \sigma^2 = \alpha\beta^2$

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Gamma Distribution

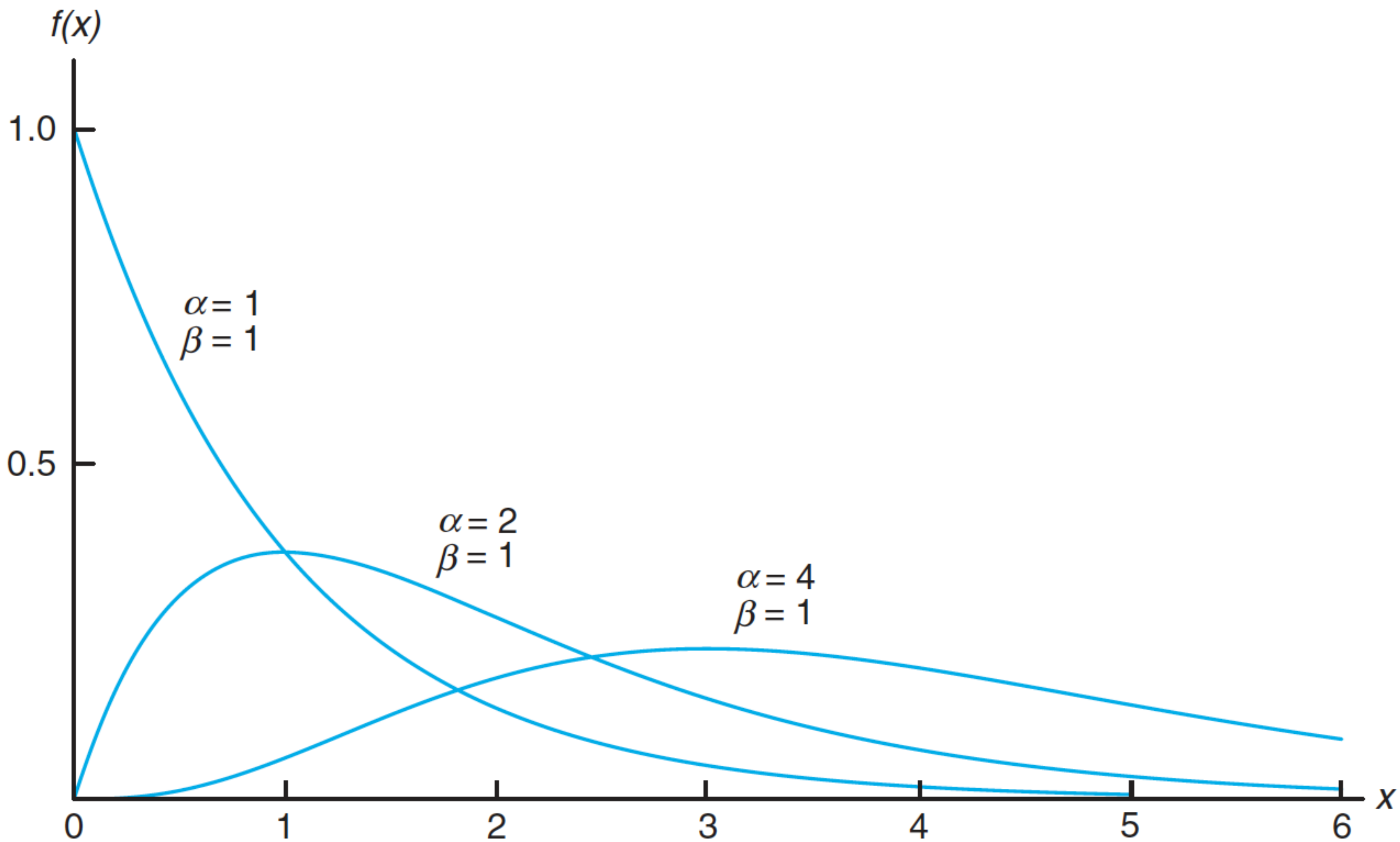

Figure 6.28: Gamma distributions.

Example 6.18

Example 6.18: Suppose that telephone calls arriving at a particular switchboard follow a Poisson process with an average of 5 calls coming per minute. What is the probability that up to a minute will elapse by the time 2 calls have come in to the switchboard?

***Solution*:** The Poisson process applies, with time until 2 Poisson events following a gamma distribution with $\beta = 1/5$ and $\alpha = 2$. Denote by X the time in minutes that transpires before 2 calls come. The required probability is given by

$$P(X \leq 1) = \int_0^1 \frac{1}{\beta^2} x e^{-x/\beta}\, dx = 25 \int_0^1 x e^{-5x}\, dx = 1 - e^{-5}(1+5) = 0.96.$$

$$\boxed{f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}}$$

$$\boxed{\Gamma(n) = (n-1)!}$$

Exponential Distribution

- The continuous random variable X has an exponential distribution, with parameter β, if its density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$\boxed{\lambda = 1/\beta}$$

$$f(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere}, \end{cases}$$

$\beta > 0.$ β = mean time between occurrences

- *Special case* of the gamma distribution with α = 1.

$$E(X) = \beta \quad Var(X) = \beta^2$$

✓ Describes the time until the first (α = 1) Poisson event

✓ Describes the time between two Poisson events

CDF and mean/variance

- CDF of exponential distribution

$$F(x) = P(X \le x) = \int_0^x \frac{1}{\beta} e^{-y/\beta} dy = 1 - e^{-\frac{x}{\beta}}, \qquad x \ge 0$$

Theorem 6.4: The mean and variance of the gamma distribution are

$$\mu = \alpha\beta \text{ and } \sigma^2 = \alpha\beta^2.$$

The proof of this theorem is found in Appendix A.26.

Corollary 6.1: The mean and variance of the exponential distribution are

$$\mu = \beta \text{ and } \sigma^2 = \beta^2.$$

Exponential RV: Mean

- Mean $E[X] = \beta$
  - Proof

$$E[X] = \int_0^\infty \frac{1}{\beta} x e^{-x/\beta} dx = \int_0^\infty [(x/\beta - 1) + 1] e^{-x/\beta} dx$$
$$= \int_0^\infty (x/\beta - 1) e^{-x/\beta} dx + \int_0^\infty e^{-x/\beta} dx$$
$$= \left[-xe^{-x/\beta}\right]_0^\infty + \int_0^\infty e^{-x/\beta} dx = (0 - 0) - \beta\left[e^{-x/\beta}\right]_0^\infty$$
$$= 0 + \beta = \beta$$

- Variance

$$Var[X] = \beta^2$$

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Example

- Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10k miles.
- If a person desires to take a 5k mile trip, what is the probability that she will be able to complete her trip without having to replace her car battery?
  - Let β =10

$$
\begin{aligned}
P\{\text{remaining lifetime} > 5\} &= 1 - F(5) \\
&= e^{-5/\beta} \\
&= e^{-1/2} \approx .604
\end{aligned}
$$

Example 6.17

Suppose that a system contains a certain type of component whose time, in years, to failure is given by T. The random variable T is modeled nicely by the exponential distribution with mean time to failure $\beta = 5$. If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

The probability that a given component is still functioning after 8 years is given by

$$P(T > 8) = \frac{1}{5}\int_8^\infty e^{-t/5}\,dt = e^{-8/5} \approx 0.2.$$

Let X represent the number of components functioning after 8 years. Then using the binomial distribution, we have

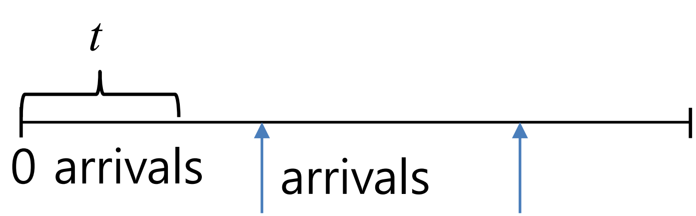
$$P(X \geq 2) = \sum_{x=2}^{5} b(x; 5, 0.2) = 1 - \sum_{x=0}^{1} b(x; 5, 0.2) = 1 - 0.7373 = 0.2627.$$

Relationship between Poisson and Exponential RV

- Poisson distribution
  - probability of specific numbers of "events" during a particular *period of time or span of space*.
  - λ : the mean number of events *per unit "time"*
  - λt: the mean number of events in time t
- Using the Poisson distribution, the probability of ***<u>no</u>*** events occurring in the span up to time t is given by

$$\boxed{p(x;\lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, x = 0, 1, 2, \ldots}$$

$$p(0;\lambda t) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}.$$



Relationship between Poisson and Exponential RV

- Let X be the time to the first Poisson event
- The probability that the length of time until the first event will exceed x ($P(X > x)$) is the same as the probability that no Poisson events will occur in x ($p(0; x)$).

$$P(X > x) = e^{-\lambda x}$$

- The CDF of X is: $P(0 \le X \le x) = 1 - e^{-\lambda x}$

- By differentiating the above CDF, the PDF of X is:

$$f(x) = \lambda e^{-\lambda x}$$

- CDF and PDF are same as ________ distribution with $\lambda = 1/\beta$

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Exponential RV: Example

- The number of points in a baseball game follows a Poisson distribution with rate λ =10 (pts)
  - Compute X = time duration (min) until first point occurs
  - Compute $E[X]$ and Var$[X]$
  - Find $F(x)$

$\beta = 1/\lambda = 1/10$

$$f(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere}, \end{cases}$$

- Answer
  - $f(x) = 10e^{-10x}, x > 0$
  - Since $X \sim Exp(1/10)$, $\mu = \frac{1}{10}$, $\sigma^2 = \frac{1}{100}$ $\quad \boxed{\mu = \beta \text{ and } \sigma^2 = \beta^2}$
  - $F(x) = 1 - e^{-10x}$ $\quad \boxed{P(X \le x) = 1 - e^{-\frac{x}{\beta}}}$

Exponential RV: Memoryless property

- Memoryless
  - The probabilities would not be influenced by the history of the process

$$P(X > a+b \mid X > a) = P(X > b)$$

  - Exponential RV ≈ Geometric RV

	Memoryless property
Discrete	Geometric RV
Continuous	Exponential RV

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Exponential RV: Memoryless property

- Exponential random variables are memoryless if,

$$P\{X > s + t | X > t\} = P\{X > s\} \qquad \text{for all } s, t \geq 0$$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

- Proof

$$P(X > s + t | X > t) = \frac{P(X > t, X > s + t)}{P(X > t)} = \frac{P(X > s + t)}{P(X > t)}$$

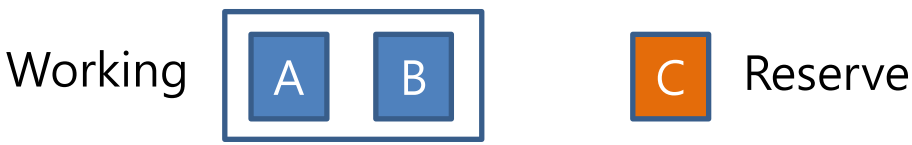
Since $P\{X > x\} = e^{-\lambda x}$ and $e^{-\lambda(s+t)} = e^{-\lambda s} e^{-\lambda t}$,

Thus, $P\{X > s + t\} = P\{X > s\} P\{X > t\}$

Finally, $P(X > s + t | X > t) = \frac{P(X>s+t)}{P(X>t)} = \frac{P(X>s)P(X>t)}{P(X>t)} = P(X > s)$

Exponential RV: Example

- 3 interchangeable machines, of which **2 must be working** to do its job.
- When in use, each machine will function for an **exponentially** distributed time having parameter λ before breaking down.
- initially to use machines A and B and keep machine C in reserve to replace whichever of A or B breaks down first.



- When the crew is forced to stop working because only one of the machines has not yet broken down, what is the probability that the still operable machine is machine C?

- Answer
  - Prob = 1/2

Chi-squared (χ^2) Distribution*

- Definition
  - If Z_1, ..., Z_n are independent, standard normal random variables, then the sum of their squares,

$$X = Z_1^2 + Z_2^2 + \cdots + Z_n^2$$

  - is distributed according to the **chi-squared distribution** with n degrees of freedom.
  - Denoted as

$$X \sim \chi_n^2 \text{ or } X \sim \chi^2(n)$$

  - The chi-squared distribution has one parameter:
    - ***n*** — a positive integer that specifies the number of degrees of freedom (i.e. the number of standard normal deviates being summed)

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*카이제곱분포

The Chi-Squared Distribution

Special case of the gamma distribution is obtained by letting $\alpha = v/2$ and $\beta = 2$, where v is a positive integer.

The continuous random variable X has a **chi-squared distribution**, with v **degrees of freedom**, if its density function is given by

$$f(x; v) = \begin{cases} \frac{1}{2^{v/2}\Gamma(v/2)} x^{v/2-1} e^{-x/2}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

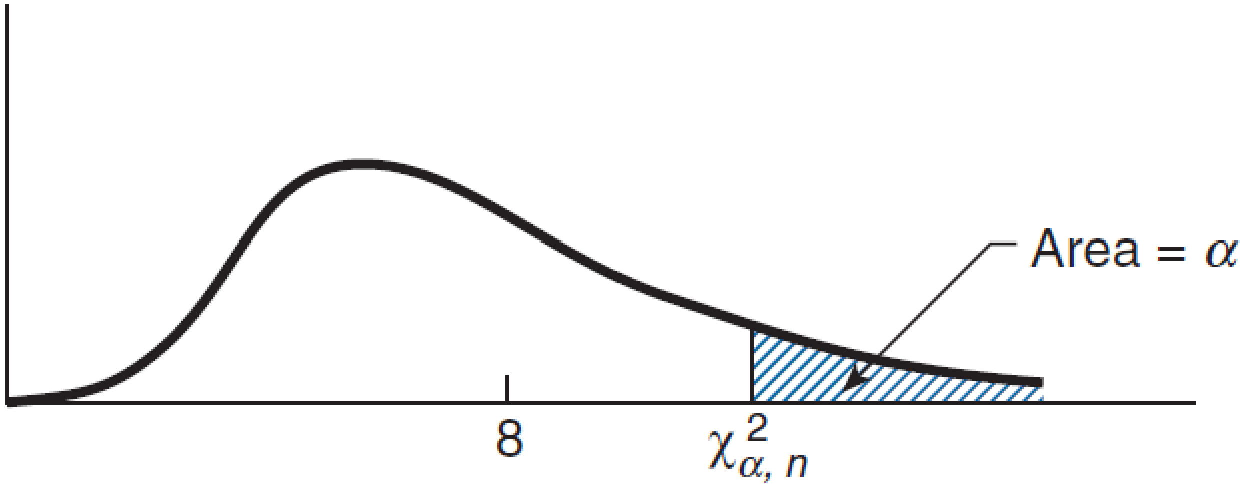
where v is a positive integer.

The Chi-Squared distribution plays a vital role in statistical inference (Ch. 8, 9). It has considerable application in both methodology and theory. It is an important component of statistical hypothesis testing and estimation.

The Chi-Squared Model - Properties

- For, $X \sim \chi^2_n$ and any $\alpha \in (0, 1)$, $\chi^2{}_{\alpha,n}$ is defined as

$$P\{X \geq \chi^2_{\alpha,n}\} = \alpha$$



The chi-square density function with 8 degrees of freedom

Chi-square density function: Example (1)

- Find $\chi^2_{0.05,15}$
  - 24.996
- Determine $P\{\chi^2_{16} \le 32\}$
  - 0.01

$$P\{X \geq \chi^2_{\alpha,n}\} = \alpha$$

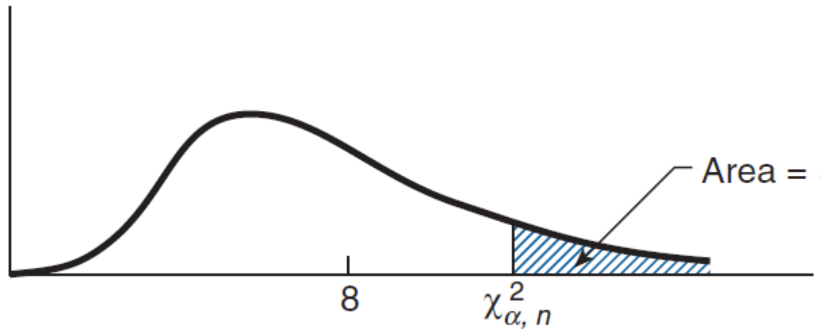


Table A.5 (continued) Critical Values of the Chi-Squared Distribution

	α									
v	0.30	0.25	0.20	0.10	0.05	0.025	0.02	0.01	0.005	0.001
1	1.074	1.323	1.642	2.706	3.841	5.024	5.412	6.635	7.879	10.827
2	2.408	2.773	3.219	4.605	5.991	7.378	7.824	9.210	10.597	13.815
3	3.665	4.108	4.642	6.251	7.815	9.348	9.837	11.345	12.838	16.266
4	4.878	5.385	5.989	7.779	9.488	11.143	11.668	13.277	14.860	18.466
5	6.064	6.626	7.289	9.236	11.070	12.832	13.388	15.086	16.750	20.515
6	7.231	7.841	8.558	10.645	12.592	14.449	15.033	16.812	18.548	22.457
7	8.383	9.037	9.803	12.017	14.067	16.013	16.622	18.475	20.278	24.321
8	9.524	10.219	11.030	13.362	15.507	17.535	18.168	20.090	21.955	26.124
9	10.656	11.389	12.242	14.684	16.919	19.023	19.679	21.666	23.589	27.877
10	11.781	12.549	13.442	15.987	18.307	20.483	21.161	23.209	25.188	29.588
11	12.899	13.701	14.631	17.275	19.675	21.920	22.618	24.725	26.757	31.264
12	14.011	14.845	15.812	18.549	21.026	23.337	24.054	26.217	28.300	32.909
13	15.119	15.984	16.985	19.812	22.362	24.736	25.471	27.688	29.819	34.527
14	16.222	17.117	18.151	21.064	23.685	26.119	26.873	29.141	31.319	36.124
15	17.322	18.245	19.311	22.307	24.996	27.488	28.259	30.578	32.801	37.698
16	18.418	19.369	20.465	23.542	26.296	28.845	29.633	32.000	34.267	39.252
17	19.511	20.489	21.615	24.769	27.587	30.191	30.995	33.409	35.718	40.791
18	20.601	21.605	22.760	25.989	28.869	31.526	32.346	34.805	37.156	42.312
19	21.689	22.718	23.900	27.204	30.144	32.852	33.687	36.191	38.582	43.819
20	22.775	23.828	25.038	28.412	31.410	34.170	35.020	37.566	39.997	45.314
21	23.858	24.935	26.171	29.615	32.671	35.479	36.343	38.932	41.401	46.796
22	24.939	26.039	27.301	30.813	33.924	36.781	37.659	40.289	42.796	48.268
23	26.018	27.141	28.429	32.007	35.172	38.076	38.968	41.638	44.181	49.728
24	27.096	28.241	29.553	33.196	36.415	39.364	40.270	42.980	45.558	51.179
25	28.172	29.339	30.675	34.382	37.652	40.646	41.566	44.314	46.928	52.619
26	29.246	30.435	31.795	35.563	38.885	41.923	42.856	45.642	48.290	54.051
27	30.319	31.528	32.912	36.741	40.113	43.195	44.140	46.963	49.645	55.475
28	31.391	32.620	34.027	37.916	41.337	44.461	45.419	48.278	50.994	56.892
29	32.461	33.711	35.139	39.087	42.557	45.722	46.693	49.588	52.335	58.301
30	33.530	34.800	36.250	40.256	43.773	46.979	47.962	50.892	53.672	59.702
40	44.165	45.616	47.269	51.805	55.758	59.342	60.436	63.691	66.766	73.403
50	54.723	56.334	58.164	63.167	67.505	71.420	72.613	76.154	79.490	86.660
60	65.226	66.981	68.972	74.397	79.082	83.298	84.58	88.379	91.952	99.608

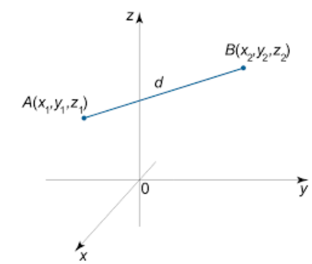
Wireless Intelligent Networking LAB

Chi-square density function: Example (2)

- Suppose that we are attempting to locate a target in three-dimensional space, and that the three coordinate errors (in meters) of the point chosen are independent normal random variables with mean 0 and standard deviation 2. Find the probability that the distance between the point chosen and the target exceeds 3 meters.

Distance Formula in Three Dimensions The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



SOLUTION If D is the distance, then

$$D^2 = X_1^2 + X_2^2 + X_3^2$$

where X_i is the error in the ith coordinate. Since $Z_i = X_i/2, i = 1, 2, 3$, are all standard normal random variables, it follows that

$$\begin{aligned} P\{D^2 > 9\} &= P\{Z_1^2 + Z_2^2 + Z_3^2 > 9/4\} \\ &= P\{\chi_3^2 > 9/4\} \\ &= .5222 \end{aligned}$$

where the final equality was obtained from Program 5.8.1a. ■

End of Ch. 6