Algorithms

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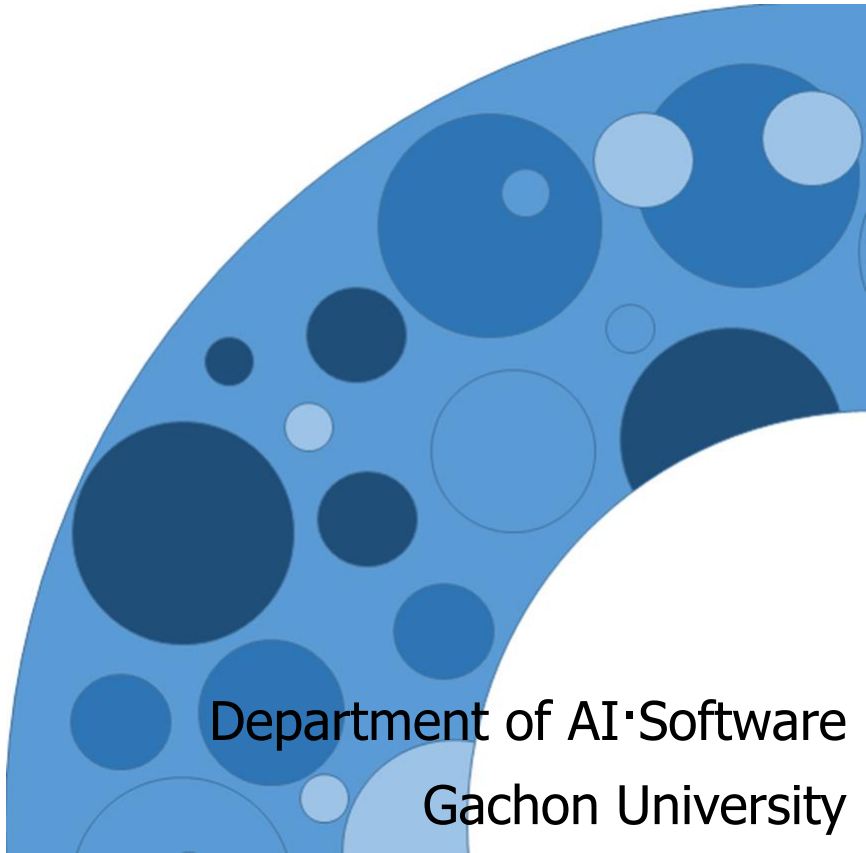

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3. Dynamic Programming I

Contents

- What's dynamic programming
  - Strategy
  - Four-step method
- Elements of dynamic programming
- The Fibonacci Sequence
- Longest common subsequence

- Problem 4: Smart elephant

Dynamic programming

- Dynamic programming is a design technique for solving optimization problems having a recursive structure.
- Dynamic programming is related partially to Divide and Conquer and to Greedy Algorithms.
- Problem at hand is subdivided to partial problems (subproblems) whose results are combined to solve the overall problem

Strategy

- Big Idea: avoid recomputation
- General strategy is to solve all subproblems once and only once, to store all the subsolutions in a table, and to reuse optimal subproblem solutions
- Typically, the total number of distinct subproblems is a polynomial in the input size

Four-step method

- Characterize the structure of an optimal solution.
- Recursively define the value of an optimal solution.
- Compute the value of an optimal solution, typically in a bottom-up fashion.
- Construct an optimal solution from computed information.

Elements of dynamic programming

- Optimal substructure: an optimal solution to the problem contains within it optimal solutions to subproblems.
- Overlapping substructure: a recursive algorithm revisits the same problem over and over again

The Fibonacci Sequence

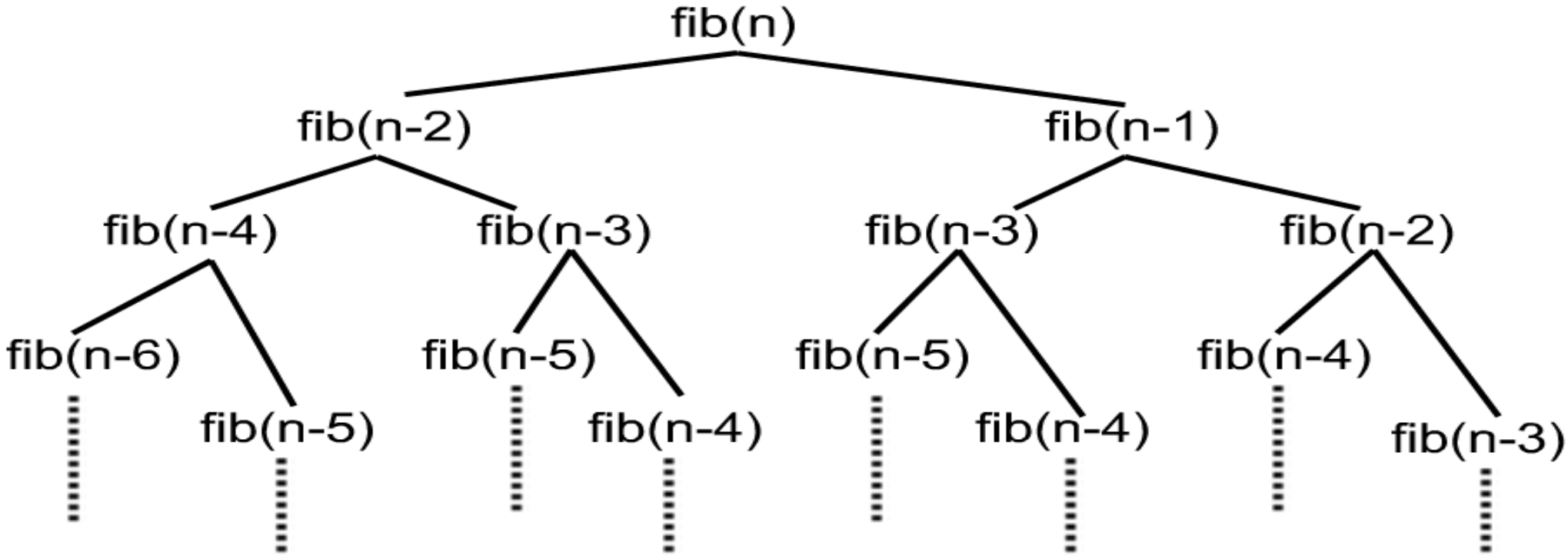
Example : The fibonacci sequence

$$fib(0) = 0$$
$$fib(1) = 1$$
$$fib(2) = 1$$
$$fib(3) = 2$$
$$fib(4) = 3$$
$$fib(5) = 5$$
$$\ldots$$
$$fib(n) = fib(n-1) + fib(n-2)$$

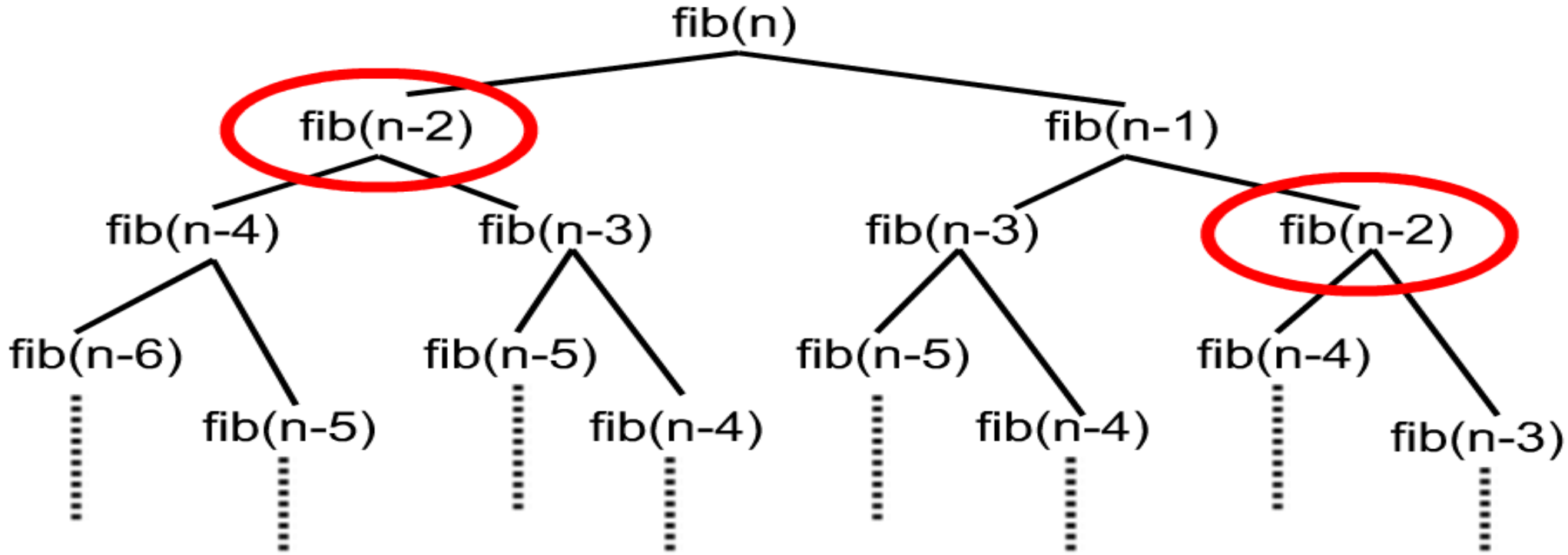
Naïvely computing fib(n)

```
int fib(int n) {
    if (n == 0 || n == 1) {
        return n;
    } else {
        return (fib(n-1) + fib(n-2));
    }
}
```

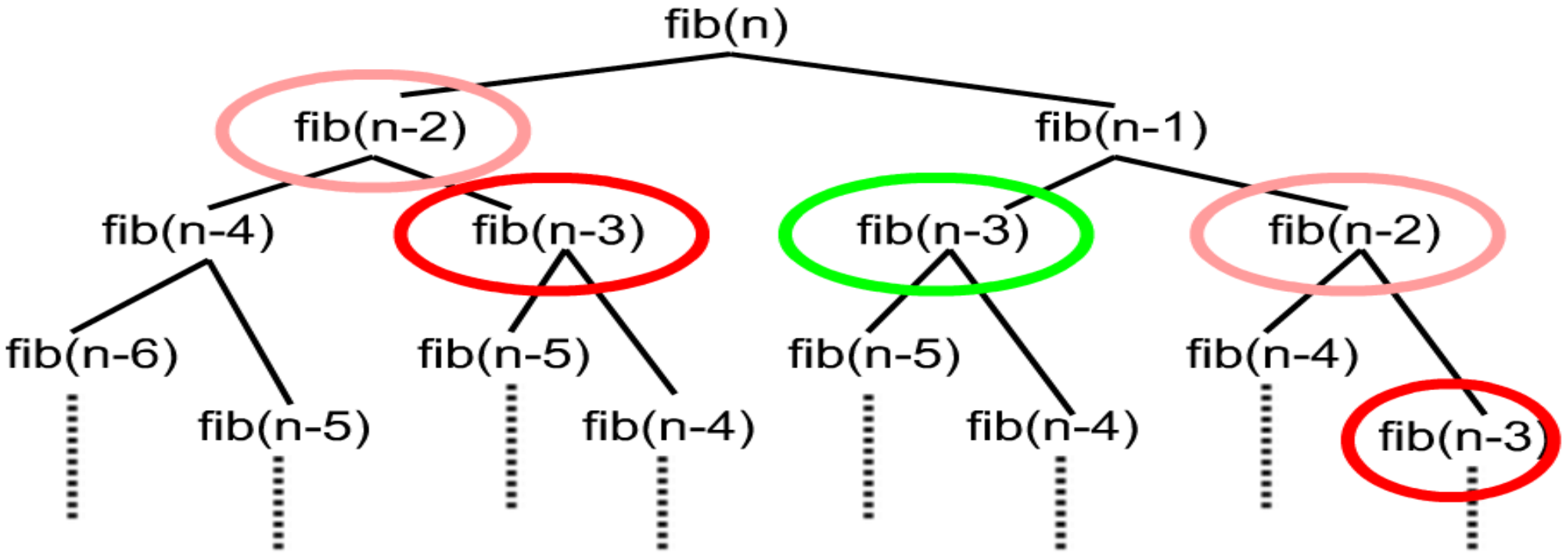
Naïvely computing fib(n)



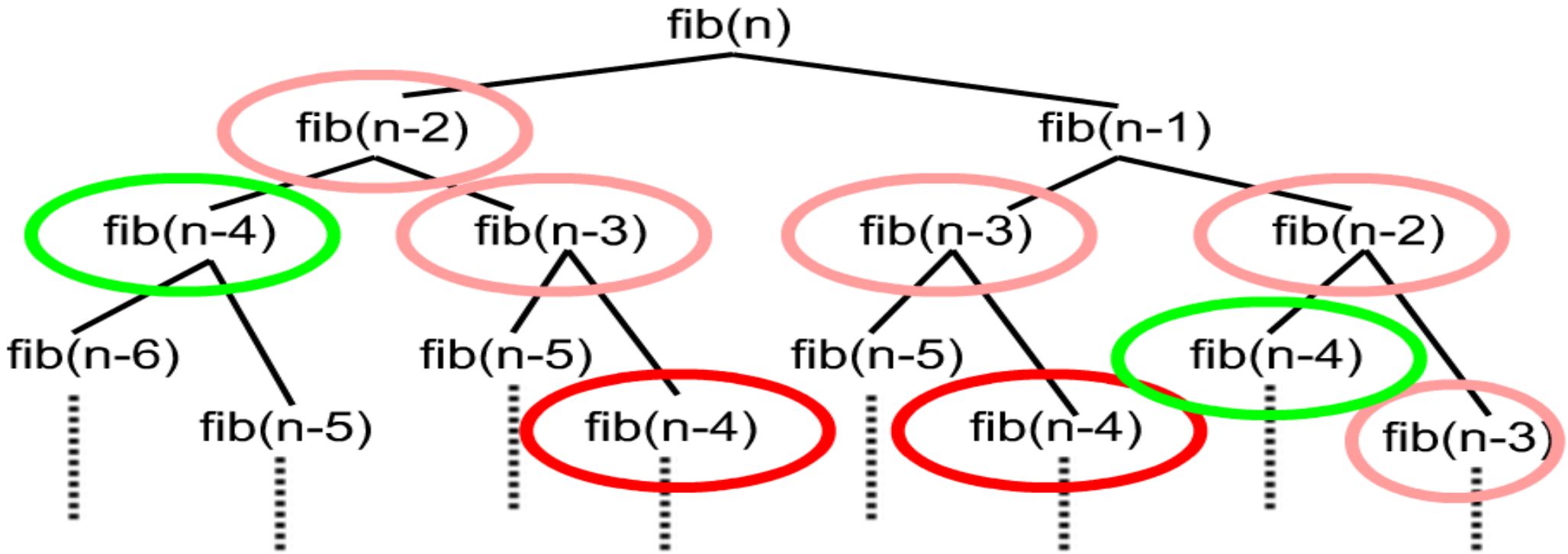
Naïvely computing fib(n)



Naïvely computing fib(n)



Naïvely computing fib(n)



How to fix things...

- The solution to fib(n) depends on:
  - The solution to fib(n - 1).
  - The solution to fib(n - 2).
- Use dynamic programming to organize everything:
  - Each subproblem is characterized by the value of its input.
  - To compute fib(n), we need fib(n - 2) and fib(n - 1).
  - So compute fib(0) first, and work upwards.
  - Use an array to keep track of everything.

Computing fib(n) : Take 2

```
int fib(int n) {
    int[] seq = new int[n+1];
    seq[0] = 0;
    seq[1] = 1;
    for (int i = 2; i <= n; i++) {
        // looks like fib(i) = fib(i - 1) + fib(i - 2)
        seq[i] = seq[i - 1] + seq[i - 2];
    }
    return seq[n];
}
```

Computing fib(n) : Take 2

```
int fib(int n) {

    // This array will keep track of the solutions to
    // all n+1 problems.
    int[] seq = new int[n + 1];

    ...
```

Computing fib(n) : Take 2

```
int fib(int n) {
    ...
    // Assume that (n >= 1).  Otherwise, seq is to small.
    // The smallest subproblems have trivial solutions.
    seq[0] = 0;
    seq[1] = 1;
    ...
```

Computing fib(n) : Take 2

```
int fib(int n) {
    ...
    // Solve other subproblems in order of increasing size.
    // Use the fact that we've already solved even smaller
    // subproblems already.
    for (int i = 2; i <= n; i++) {
        seq[i] = seq[i - 1] + seq[i - 2];
    }
    ...
```

Computing fib(n) : Take 2

```
int fib(int n) {

    ...

    // Finally, return solution to the actual problem.
    return seq[n];
}
```

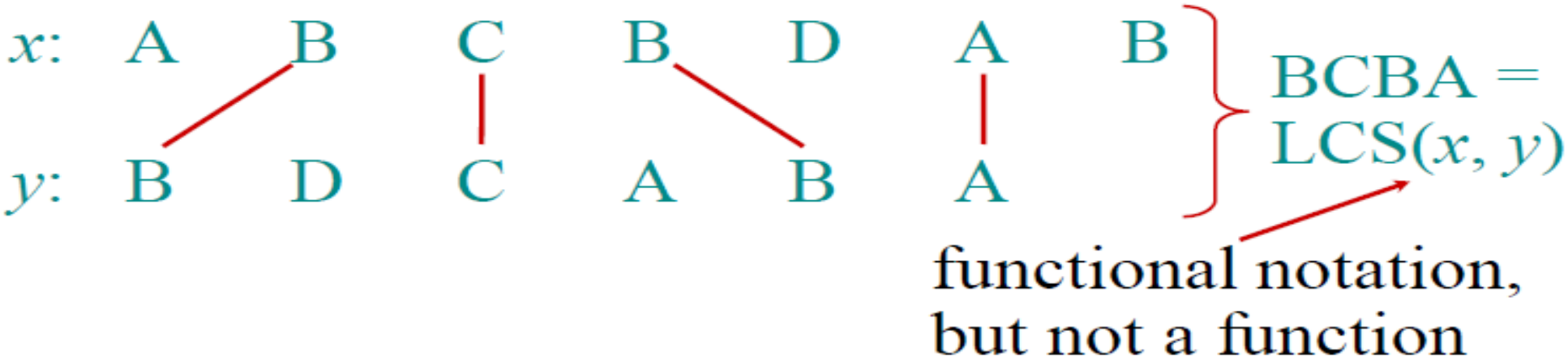
Longest common subsequence

Design technique, like divide-and-conquer.

Example: *Longest Common Subsequence (LCS)*

- Given two sequences $x[1\ .\ .\ m]$ and $y[1\ .\ .\ n]$, find a longest subsequence common to them both.

"a" *not* "the"



Longest common subsequence

Optimal substructure

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

If $z = \mathrm{LCS}(x, y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y.

Longest common subsequence

$$
\begin{aligned}
&\text{LCS}(x, y, i, j)\\
&\quad \textbf{if } x[i] = y[j]\\
&\qquad \textbf{then } c[i, j] \leftarrow \text{LCS}(x, y, i{-}1, j{-}1) + 1\\
&\qquad \textbf{else } c[i, j] \leftarrow \max\{\text{LCS}(x, y, i{-}1, j),\\
&\qquad\qquad\qquad\qquad\qquad \text{LCS}(x, y, i, j{-}1)\}
\end{aligned}
$$

Worst-case: $x[i] \neq y[j]$, in which case the algorithm evaluates two subproblems, each with only one parameter decremented.

Longest common subsequence

Overlapping subproblems

A recursive solution contains a "small" number of distinct subproblems repeated many times.

The number of distinct LCS subproblems for two strings of lengths m and n is only mn.

Longest common subsequence

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

```
LCS(x, y, i, j)
    if c[i, j] = NIL
        then if x[i] = y[j]
                then c[i, j] ← LCS(x, y, i–1, j–1) + 1
                else c[i, j] ← max{ LCS(x, y, i–1, j),
                                    LCS(x, y, i, j–1) }
```

same as before

Longest common subsequence

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

```
LCS(x, y, i, j)
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                else c[i, j] ← max{ LCS(x, y, i–1, j),
                                    LCS(x, y, i, j–1) }
```

same as before

Time = $\Theta(mn)$ = constant work per table entry.
Space = $\Theta(mn)$.

Longest common subsequence

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

Reconstruct LCS by tracing backwards.

Space = $\Theta(mn)$.

Exercise:

$O(\min\{m, n\})$.

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

Implementation

Implementation of Longest common subsequence

$$
\begin{aligned}
&\mathrm{LCS}(x, y, i, j)\\
&\quad \textbf{if } x[i] = y[j]\\
&\qquad \textbf{then } c[i,j] \leftarrow \mathrm{LCS}(x, y, i{-}1, j{-}1) + 1\\
&\qquad \textbf{else } c[i,j] \leftarrow \max\{\mathrm{LCS}(x, y, i{-}1, j),\\
&\qquad\qquad\qquad\qquad \mathrm{LCS}(x, y, i, j{-}1)\}
\end{aligned}
$$

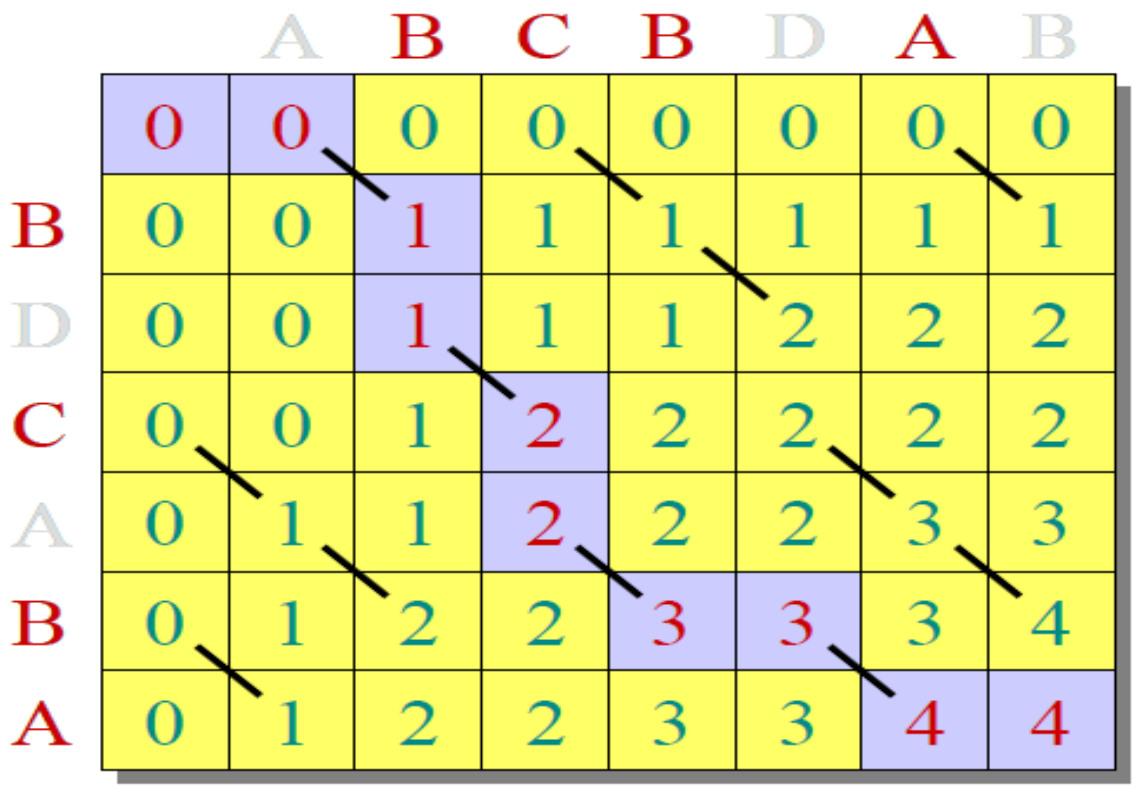
```python
def lcs(X, Y, m, n):
    if m == 0 or n == 0:
       return 0;
    elif X[m-1] == Y[n-1]:
       return 1 + lcs(X, Y, m-1, n-1);
    else:
       return max(lcs(X, Y, m, n-1), lcs(X, Y, m-1, n));


# Input texts
X = "ABCBDAB"
Y = "BDCABA"

# Print results
print("Length of LCS is ", lcs(X, Y, len(X), len(Y)))
```

```
Length of LCS is  4
```

Implementation of Longest common subsequence



		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

```python
def lcs(X, Y):
    # Length
    m = len(X)
    n = len(Y)

    # Array
    L = [[None]*(n + 1) for i in range(m + 1)]

    # Filling array from the bottom
    for i in range(m + 1):
        for j in range(n + 1):
            if i == 0 or j == 0 :
                L[i][j] = 0
            elif X[i-1] == Y[j-1]:
                L[i][j] = L[i-1][j-1]+1
            else:
                L[i][j] = max(L[i-1][j], L[i][j-1])
    return L[m][n]

# Input texts
X = "ABCBDAB"
Y = "BDCABA"

# Print results
print("Length of LCS is ", lcs(X, Y))
```

```
Length of LCS is  4
```

Example code test

- Code test: https://www.acmicpc.net/problem/11004
- Solving the problem using quick sort
- Example result of submission

제출 번호	아이디	문제	결과	메모리	시간	언어	코드 길이	제출한 시간
48920370	aikiho	9251	맞았습니다!!	55032 KB	732 ms	Python 3 / 수정	404 B	1분 전

THANK YOU