Algorithms

Ok-Ran Jeong

School of Computing, Gachon University

5. Graph Algorithms I

Contents

- Graph representation
- Minimum spanning trees
- Prim's algorithm
- Kruskal's algorithm

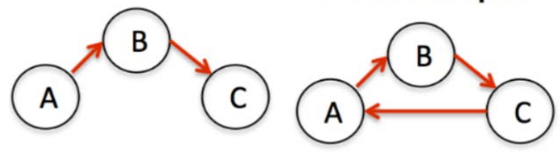
- Problem 9: Saving ink

Graphs (review)

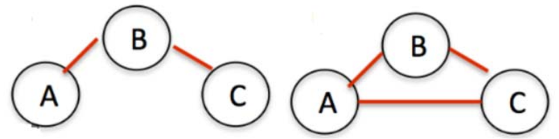
- Definition. A *directed graph (digraph)*

  $G = (V, E)$ is an ordered pair consisting of
  - a set V of *vertices* (singular: *vertex*),
  - a set $E \subseteq V \times V$ of *edges.*

Directed Graphs



Undirected Graphs

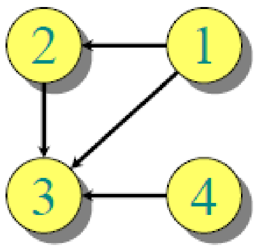


- In an *undirected graph* $G = (V, E)$, the edge set E consists of *unordered* pairs of vertices.
- In either case, we have $|E| = O(V^2)$. Moreover, if G is connected, then $|E| \geq |V| - 1$, which implies that $\lg |E| = \Theta(\lg V)$.

Adjacency-matrix representation

The ***adjacency matrix*** **of a graph *G = (V, E), where*** *V = {1, 2, …, n},* is the matrix *A[1 . . n, 1 . . n]* given by

$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \notin E. \end{cases}$$



A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

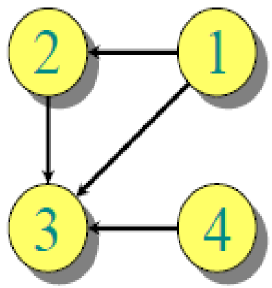
$\Theta(V^2)$ storage

$\Rightarrow$ ***dense***

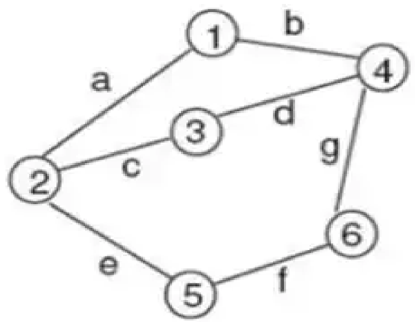
representation.

Adjacency-list representation

An ***adjacency list*** **of a vertex** ***v* ∈ *V*** is the list ***Adj[v]*** of vertices adjacent to *v*.



$Adj[1] = \{2, 3\}$
$Adj[2] = \{3\}$
$Adj[3] = \{\}$
$Adj[4] = \{3\}$



Right Hand Side:
|E| = 7
2 * |E| = 14

Left Hand side:

d(1) = 2 d(4) = 3 d(2) = 3
d(5) = 2 d(3) = 2 d(6) = 2

d(1) + d(2) + d(3) + d(4) + d(5) + d(6) = 2+3+2+3+2+2=14

For undirected graphs, **|Adj[v]|** ***= degree(v).***

For digraphs, **|Adj[v]|** ***= out-degree(v).***

Handshaking Lemma: $\Sigma_{v \in V} = 2\,|E|$ for undirected graphs ⇒ adjacency lists use $\Theta(V + E)$ storage — a **sparse** representation (for either type of graph).

Minimum spanning trees

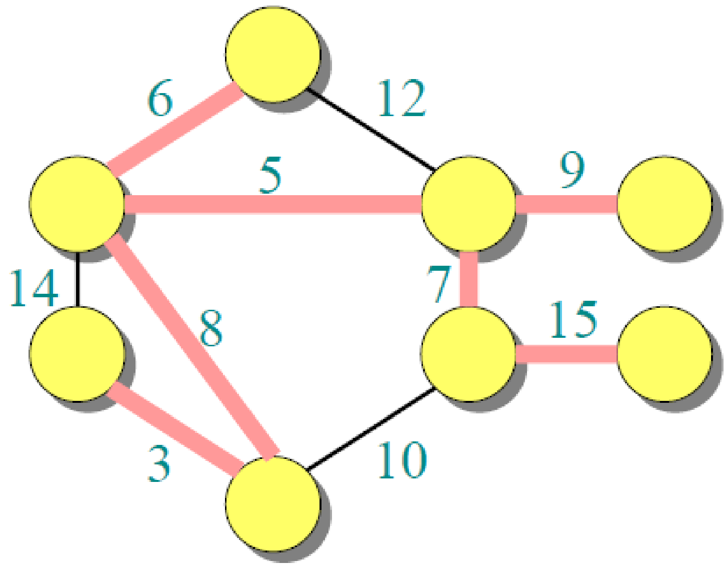
Input: A connected, undirected graph ***G = (V, E)*** with weight function $w : E \to \mathbb{R}$

- For simplicity, assume that all edge weights are distinct.

Output: A *spanning tree T* — a tree that connects all vertices of minimum weight:

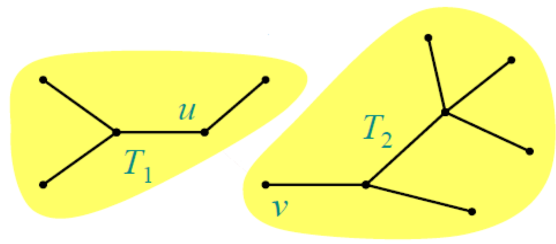
$$w(T) = \sum_{(u,v)\in T} w(u,v) .$$

Example of MST



Optimal substructure

MST T:



(Other edges of G are not shown.)

Remove any edge $(u, v) \in T$. *Then, T is partitioned into two subtrees T_1 and T_2.*

Theorem. The subtree T_1 is an MST of $G_1 = (V_1, E_1)$, the subgraph of G ***induced*** by the vertices of T_1:

V_1 = vertices of T_1,

$E_1 = \{ (x, y) \in E : x, y \in V1 \}$.

Similarly for T_2.

Proof of optimal substructure

Proof. Cut and paste:

$$w(T) = w(u, v) + w(T_1) + w(T_2).$$

If T_1' were a lower-weight spanning tree than T_1 for G_1, then $T' = \{(u, v)\} \cup T_1' \cup T_2$ would be a lower-weight spanning tree than T for G.

Do we also have overlapping subproblems?

- *Yes*

Proof of optimal substructure

Proof. Cut and paste:

$$w(T) = w(u, v) + w(T_1) + w(T_2).$$

If T_1' were a lower-weight spanning tree than T_1 for G_1, then $T' = \{(u, v)\} \cup T_1' \cup T_2$ would be a lower-weight spanning tree than T for G.

Do we also have overlapping subproblems?

Yes

Great, then dynamic programming may work!

- *Yes, but MST exhibits another powerful property which leads to an efficient algorithm.*

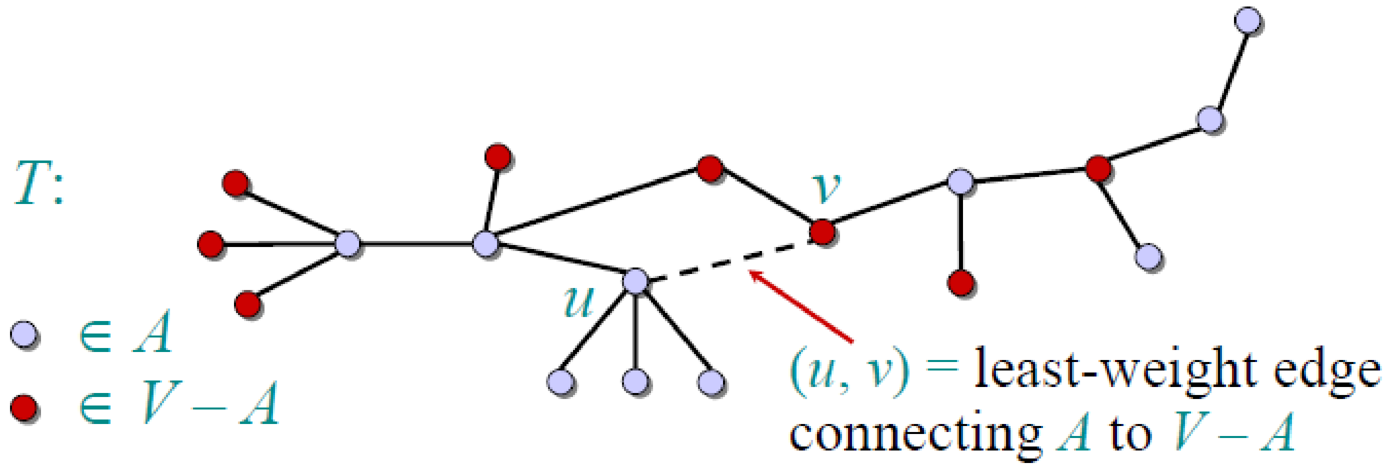
Hallmark for "greedy" algorithms

Greedy-choice property
A locally optimal choice is globally optimal.

Theorem. Let T be the MST of $G = (V, E)$, and let $A \subseteq V$. Suppose that $(u, v) \in E$ is the least-weight edge connecting A to $V - A$. Then, $(u, v) \in T$.

Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.



Proof of theorem

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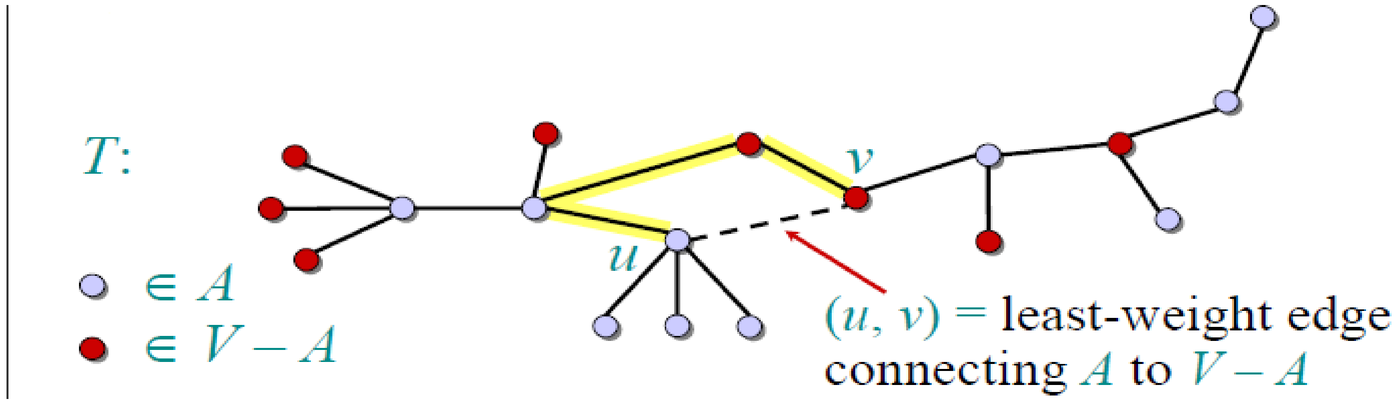

Consider the unique simple path from u to v in T.

Proof of theorem

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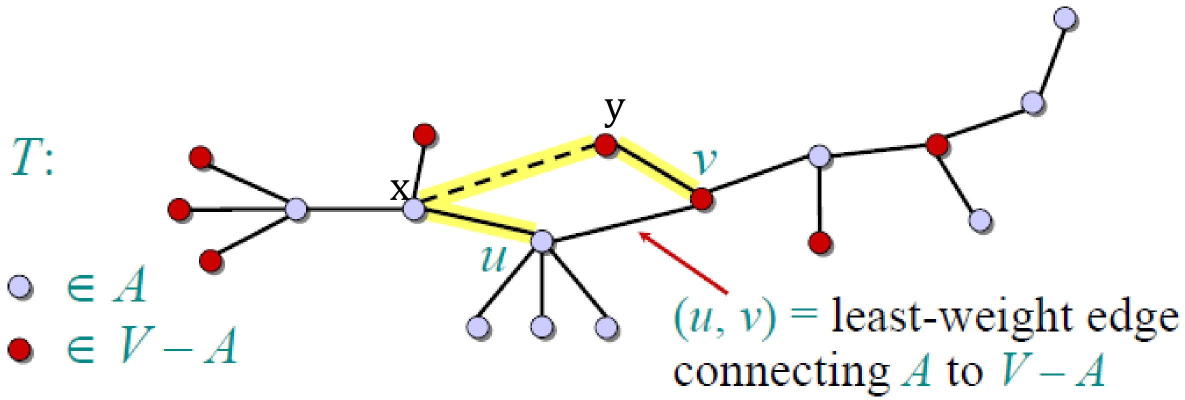

Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V - A$.

Proof of theorem

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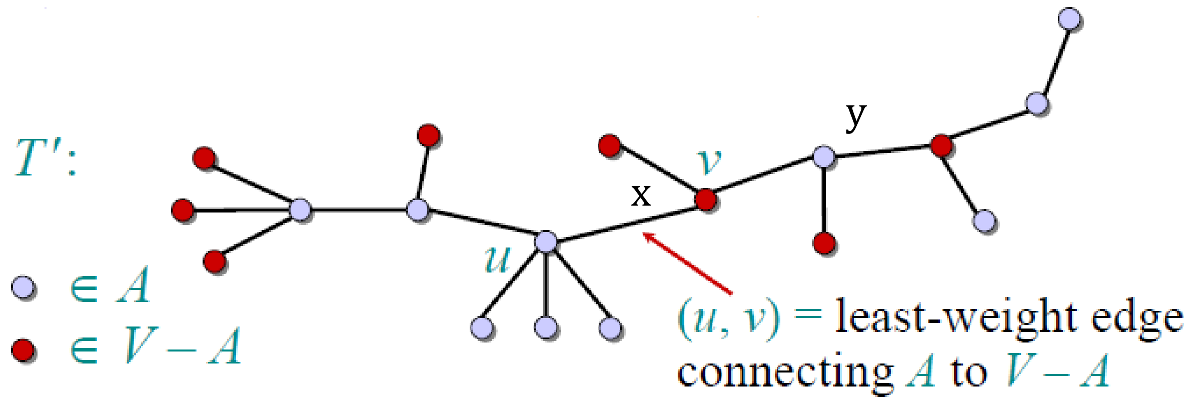


Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V – A$.

A lighter-weight spanning tree than T results.

$$T' = T-\{(x, y)\} \cup \{(u, v)\}$$

$w(u, v) \leq w(x, y)$
$w(T') = w(T)-w(x, y)+w(u, v) \leq w(T)$
$w(T) \leq w(T')$
$\therefore$ *(u, v) is safe for A*

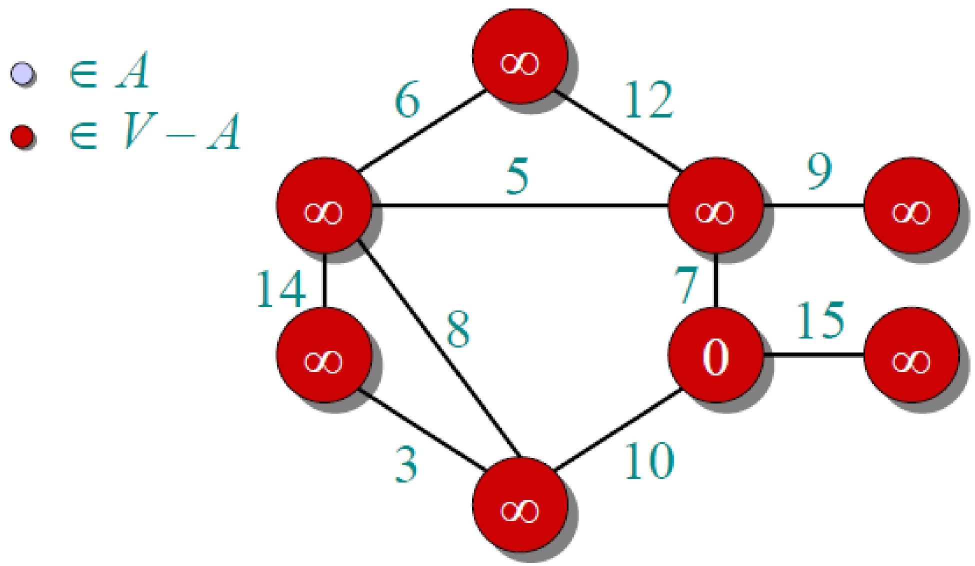
Prim's algorithm

IDEA: Maintain *V – A as a priority queue Q. Key* each vertex in *Q* with the weight of the least-weight edge connecting it to a vertex in *A(MST)*.

```
Q ← V
key[v] ← ∞ for all v ∈ V
key[s] ← 0 for some arbitrary s ∈ V
while Q ≠ ∅
  do u ← EXTRACT-MIN(Q)
   for each v ∈ Adj[u]
          do if v ∈ Q and w(u, v) < key[v]
                then key[v] ← w(u, v)      ⊳ DECREASE-KEY
                     π[v] ← u
```

At the end, $\{(v, \pi[v])\}$ forms the MST.

Example of Prim's algorithm



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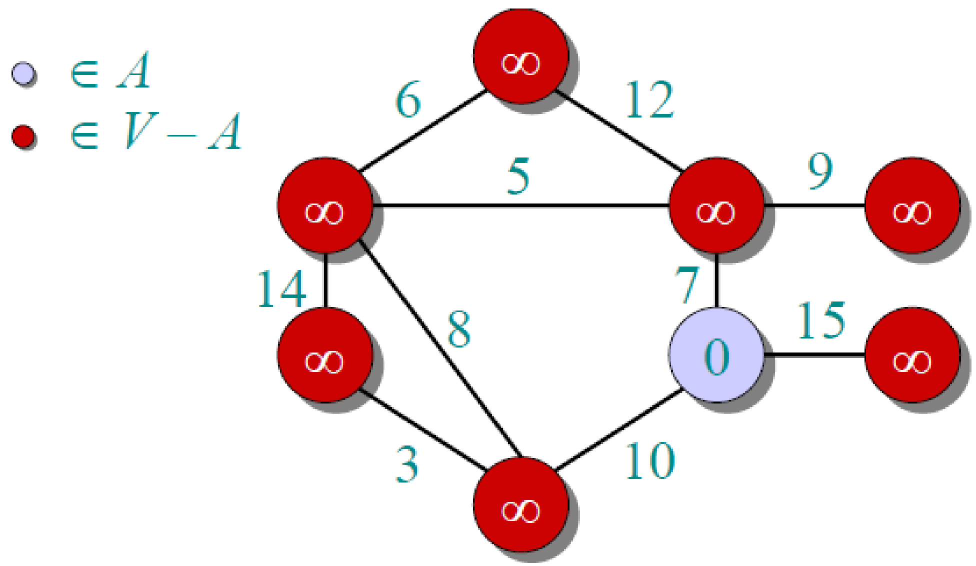

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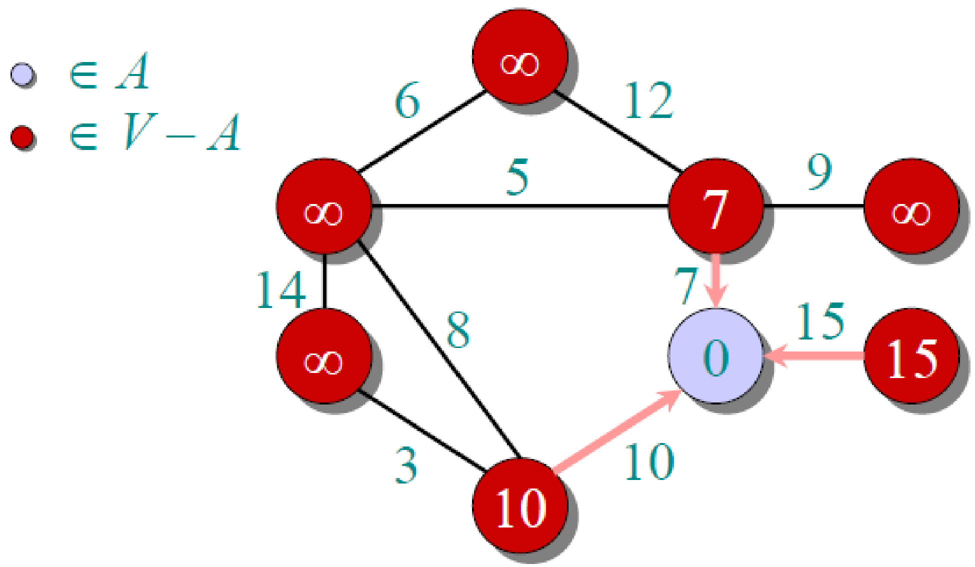

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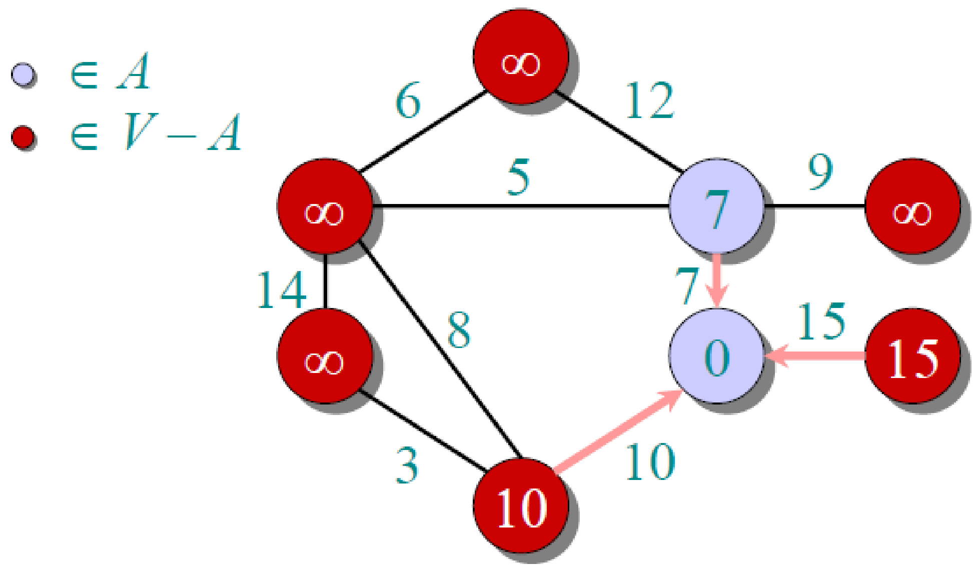

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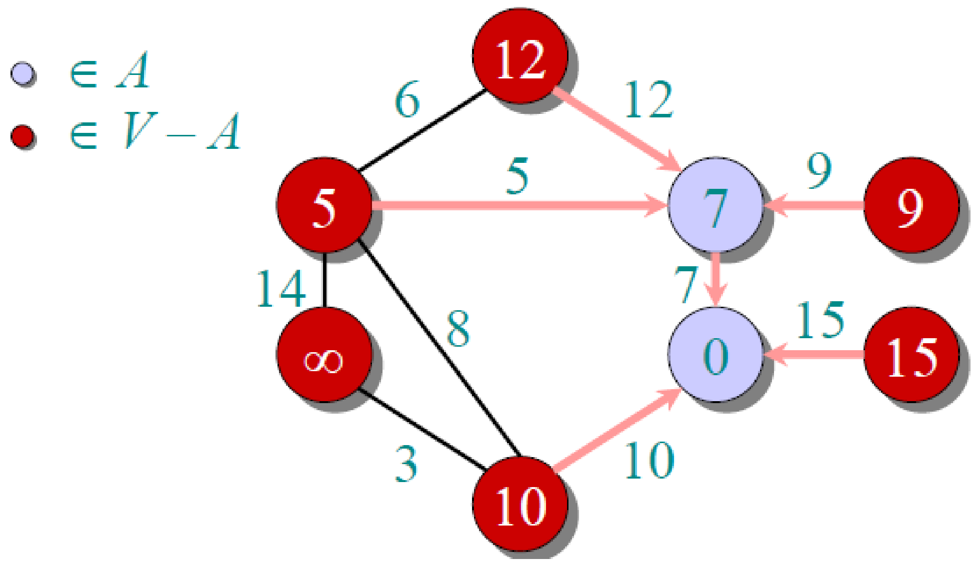

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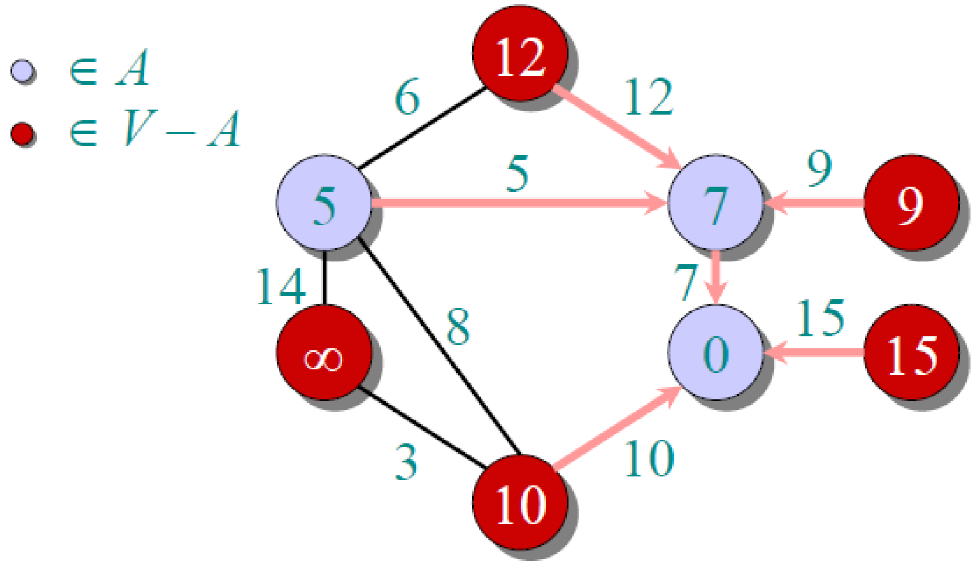

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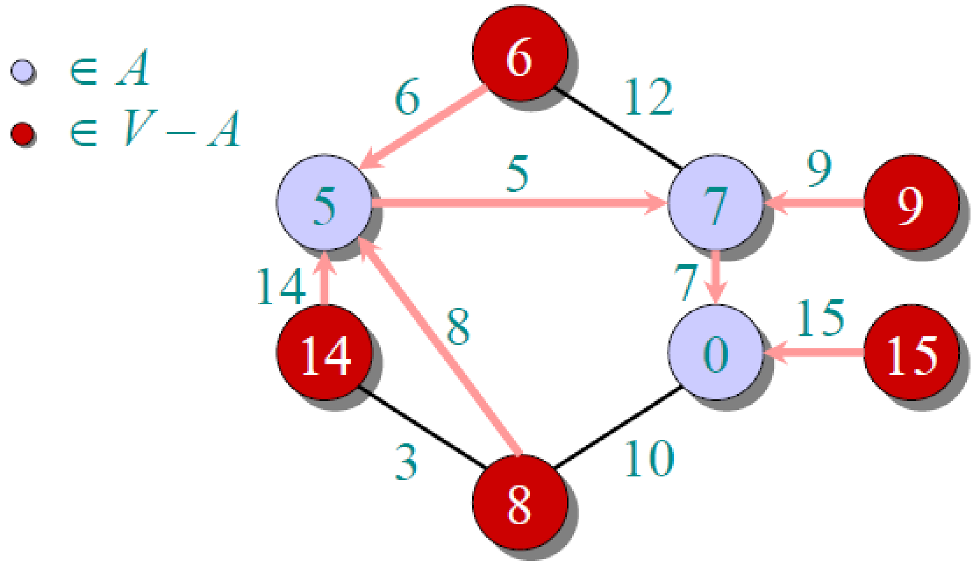

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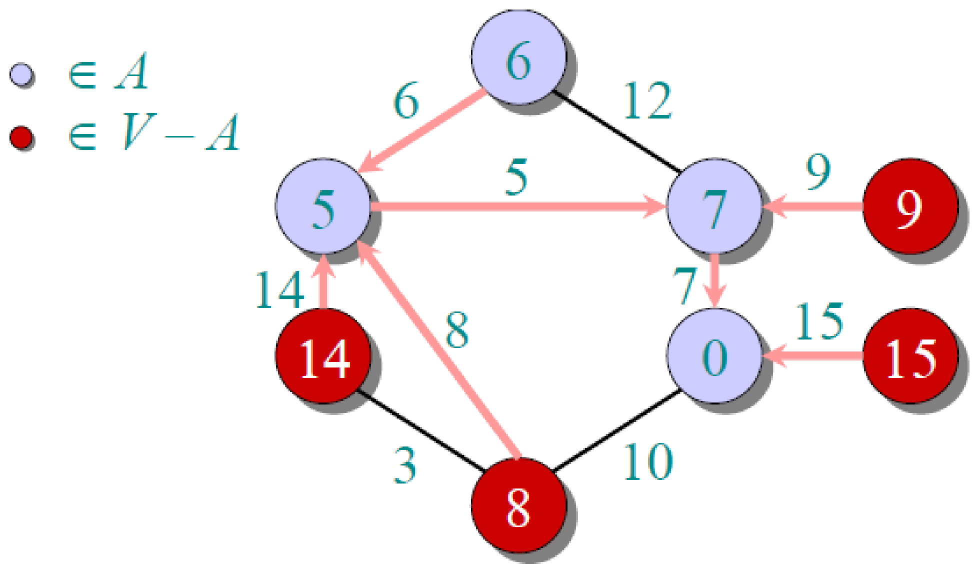

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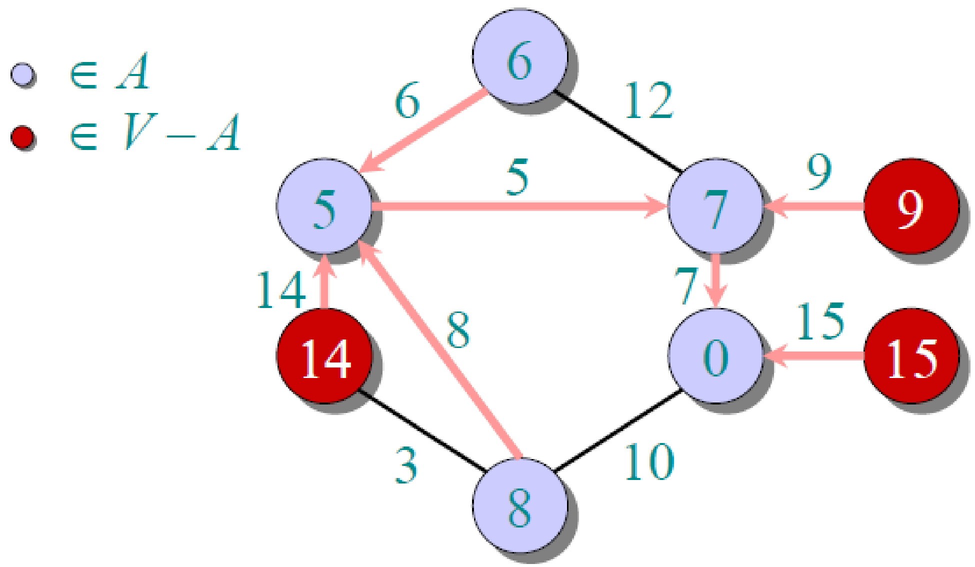

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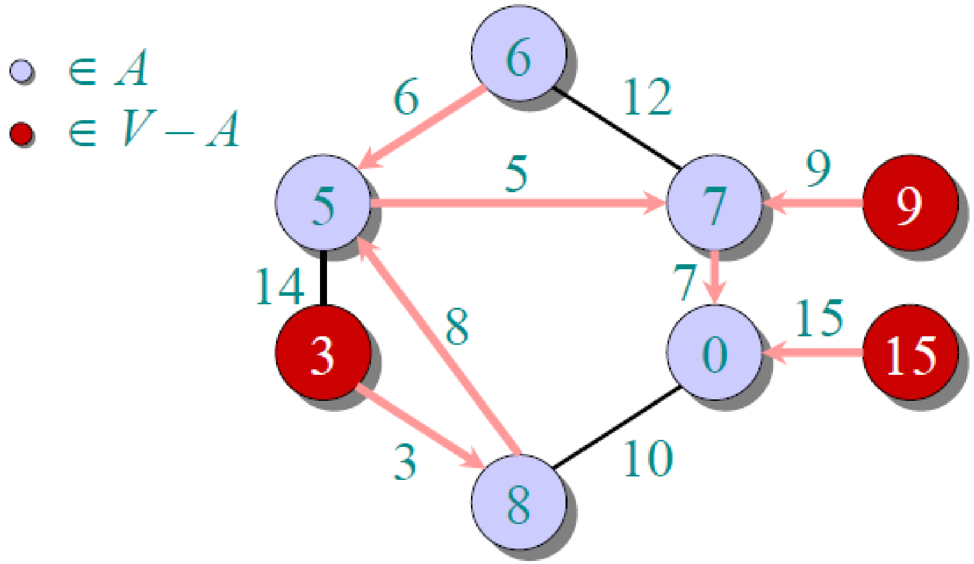

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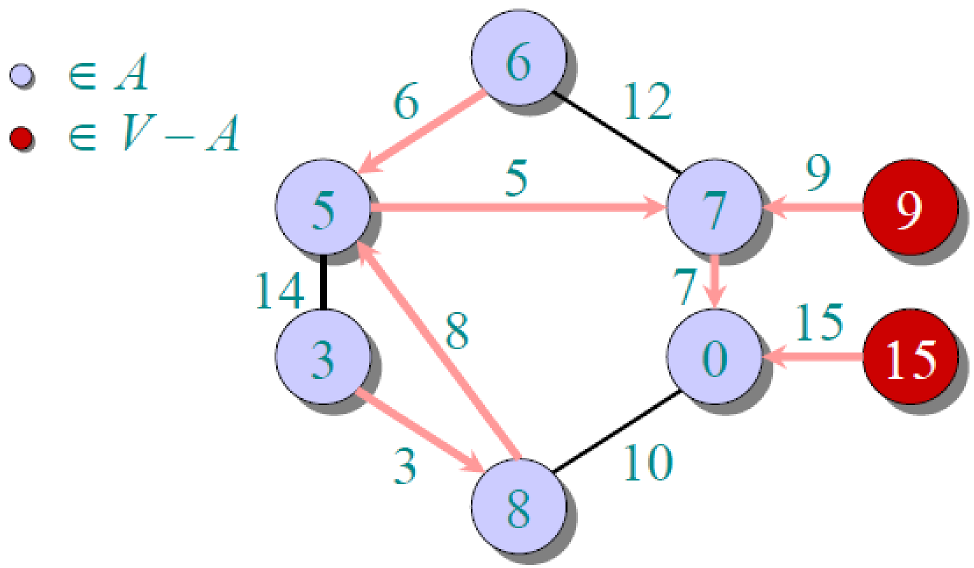
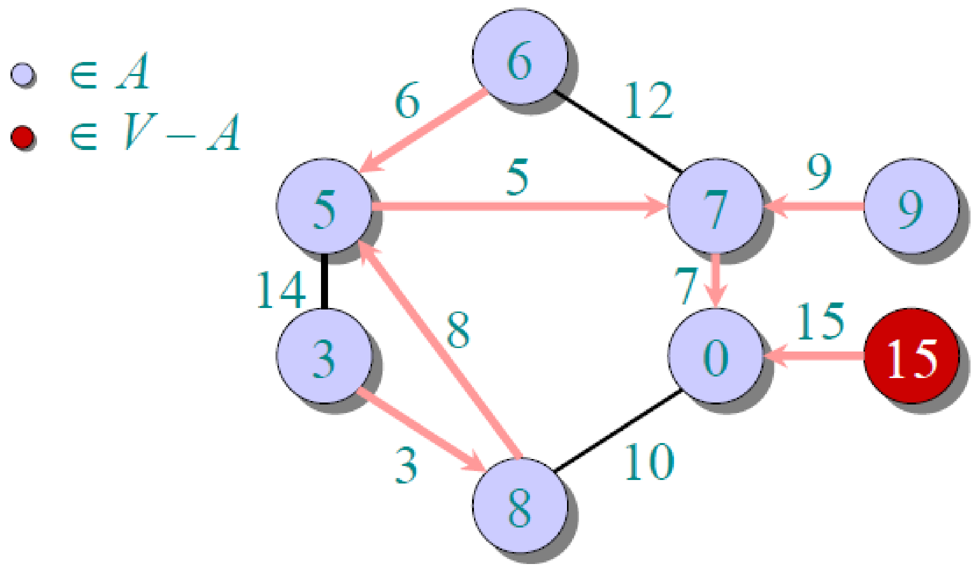


Example of Prim's algorithm



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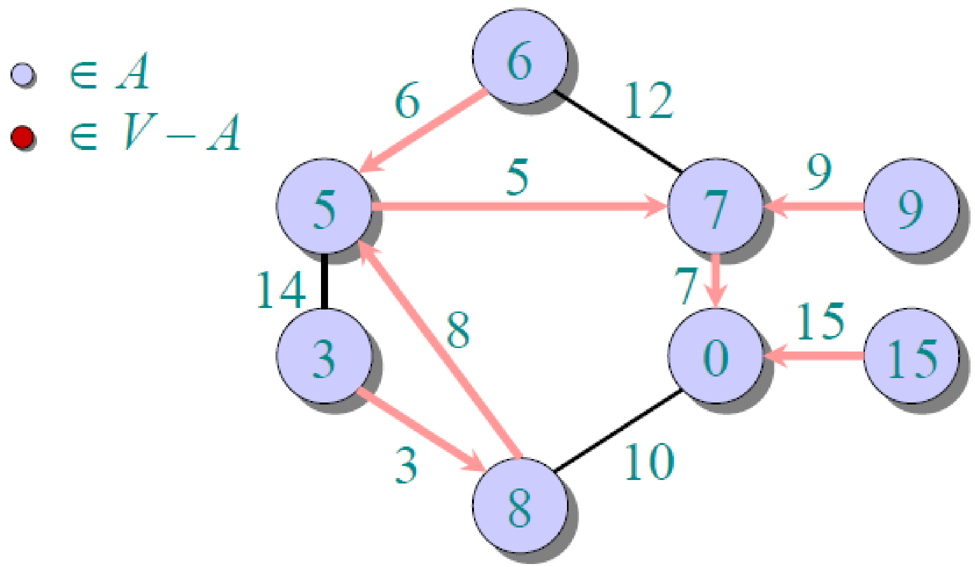


Analysis of Prim

$\Theta(V)$ total

$Q \leftarrow V$

$key[v] \leftarrow \infty$ for all $v \in V$

$key[s] \leftarrow 0$ for some arbitrary $s \in V$

$|V|$ times

while $Q \neq \emptyset$

  do $u \leftarrow$ EXTRACT-MIN(Q) ***Using Binary Heap***

$degree(u)$ times

    for each $v \in Adj[u]$

      do if $v \in Q$ and $w(u, v) < key[v]$

        then $key[v] \leftarrow w(u, v)$

          $\pi[v] \leftarrow u$

Handshaking Lemma $\Rightarrow$ $O(E)$ implicit DECREASE-KEY's.

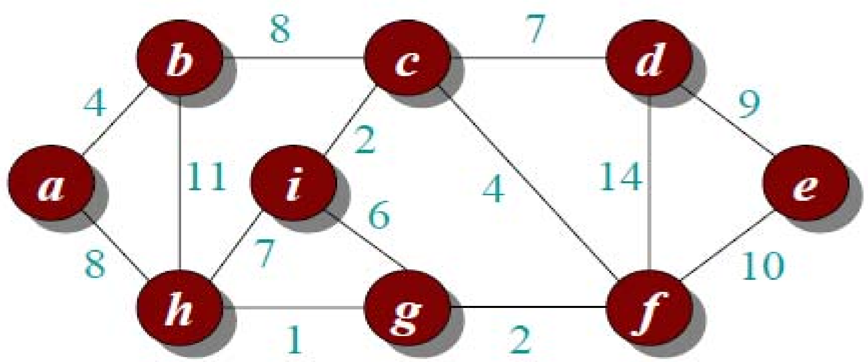
Time = $O(V) \cdot T_{\text{EXTRACT-MIN}} + O(E) \cdot T_{\text{DECREASE-KEY}} = O(V\log V) + O(E\log V) = O(E\log V)$

Kruskal’s algorithm

Idea: Start with a forest of single-node trees. Grow MST by repeatedly adding a light edge from all edges connecting two different trees in the forest. Such a light edge is safe by corollary.

Implementation: Sort the edges in non-decreasing order of weights. Using a *disjoint-set (union-find) data structure*, *u* and *v* are vertices of the same tree if and only if FIND-SET(u) = FINDSET (v).

Kruskal's algorithm



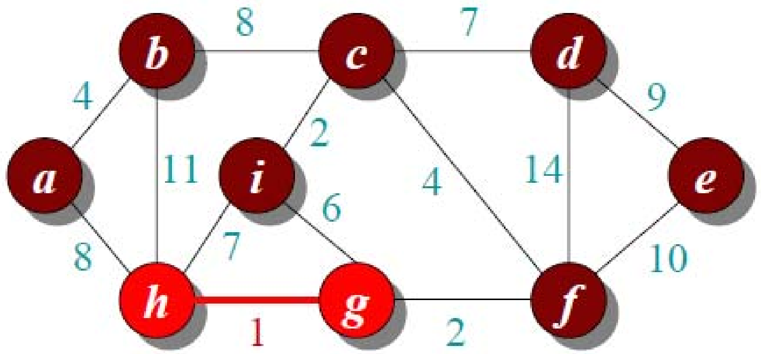

```
MST-KRUSKAL(G, w)
A ← ∅
for each vertex v ∈ V[G] do
  MAKE-SET(v)
sort E in nondecreasing order of w values
for each edge (u, v) ∈ E, in sorted order, do
  if FIND-SET(u) ≠ FIND-SET(v) then
    A ← A ∪ {(u, v)}
    UNION(u, v)
return A
```

Disjoint-set collection:

$\{\{a\},\{b\},\{c\},\{d\},\{e\},\{f\},\{g\},\{h\},\{i\}\}$

Kruskal's algorithm

- Edge (g, h) is safe.



MST-KRUSKAL(G, w)
  $A \leftarrow \emptyset$
  for each vertex $v \in V[G]$ **do**
    MAKE-SET(v)
  sort E in nondecreasing order of w values
  for each edge $(u, v) \in E$, in sorted order, **do**
    if FIND-SET$(u) \neq$ FIND-SET(v) **then**
      $A \leftarrow A \cup \{(u, v)\}$
      UNION(u, v)
  return A

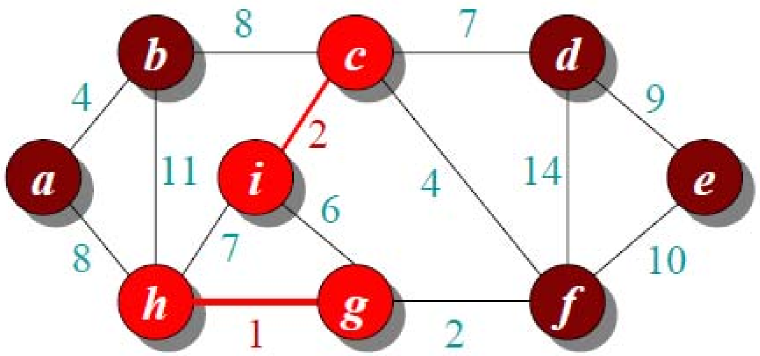
{1, 2, 2, 4, 4, 6, 7, 7, 8, 8, 9, 10, 11, 14}

Disjoint-set collection:

$\{\{a\},\{b\},\{c\},\{d\},\{e\},\{f\},\{g, h\},\{i\}\}$

Kruskal's algorithm

- Edge (c, i) is safe.



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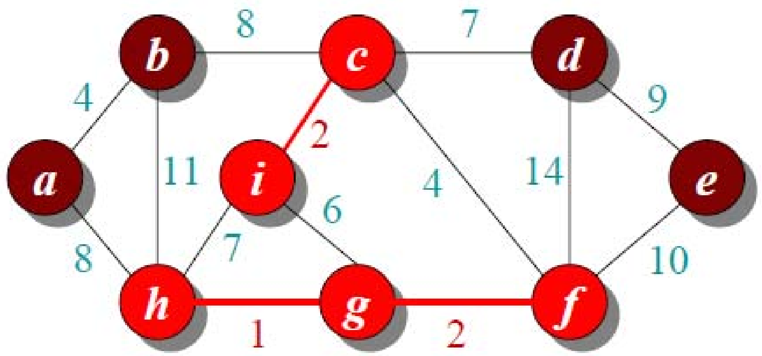
  for each edge $(u, v) \in E$, in sorted order, **do**
    if FIND-SET$(u) \neq$ FIND-SET(v) **then**
      $A \leftarrow A \cup \{(u, v)\}$
      UNION(u, v)
  return A

Disjoint-set collection:

$\{\{a\},\{b\},\{c, i\},\{d\},\{e\},\{f\},\{g, h\}\}$

Kruskal's algorithm

- Edge (f, g) is safe.



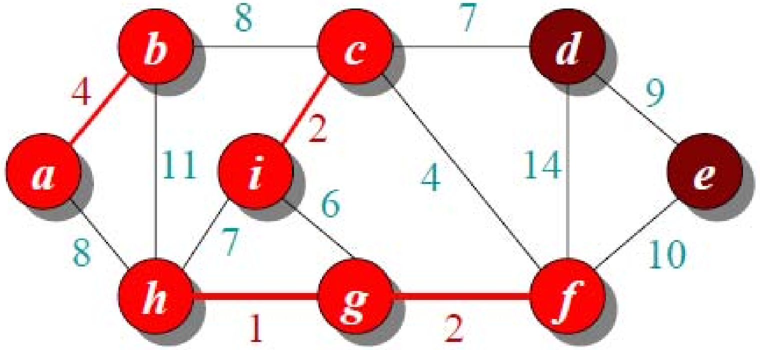
MST-KRUSKAL(G, w)
  $A \leftarrow \varnothing$
  for each vertex $v \in V[G]$ **do**
    MAKE-SET(v)
  sort E in nondecreasing order of w values
  for each edge $(u, v) \in E$, in sorted order, **do**
    if FIND-SET$(u) \neq$ FIND-SET(v) **then**
      $A \leftarrow A \cup \{(u, v)\}$
      UNION(u, v)
  return A

Disjoint-set collection:

$\{\{a\},\{b\},\{c, i\},\{d\},\{e\},\{f, g, h\}\}$

Kruskal's algorithm

- Edge (a, b) is safe.



MST-KRUSKAL(G, w)
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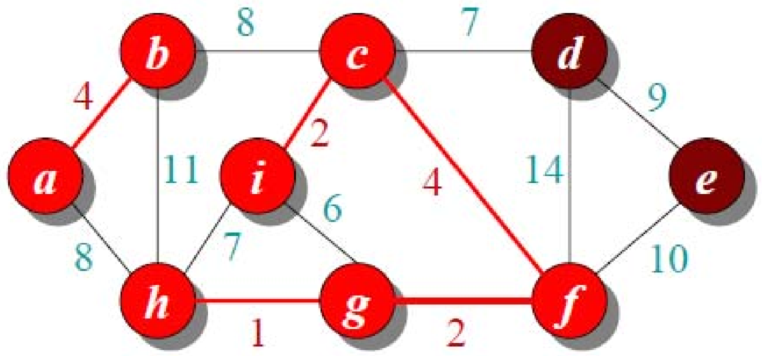
    if FIND-SET$(u) \neq$ FIND-SET(v) **then**
      $A \leftarrow A \cup \{(u, v)\}$
      UNION(u, v)
  return A

Disjoint-set collection:

$\{\{a, b\}, \{c, i\}, \{d\}, \{e\}, \{f, g, h\}\}$

Kruskal's algorithm

- Edge (c, f) is safe.



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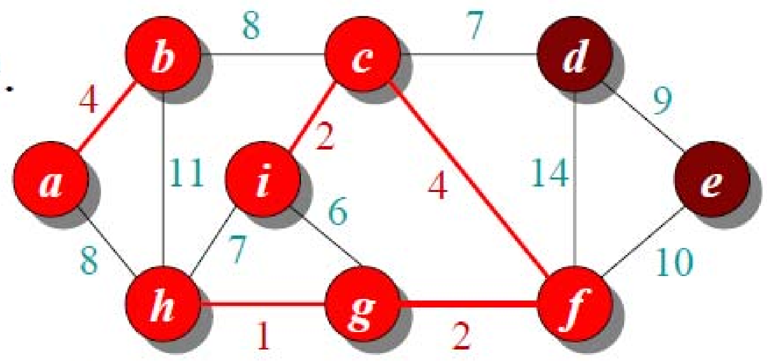
      UNION(u, v)
  return A

Disjoint-set collection:

$\{\{a, b\}, \{c, f, g, h, i\}, \{d\}, \{e\}\}$

Kruskal's algorithm

- Edge (g, i) is unsafe.



MST-KRUSKAL(G, w)
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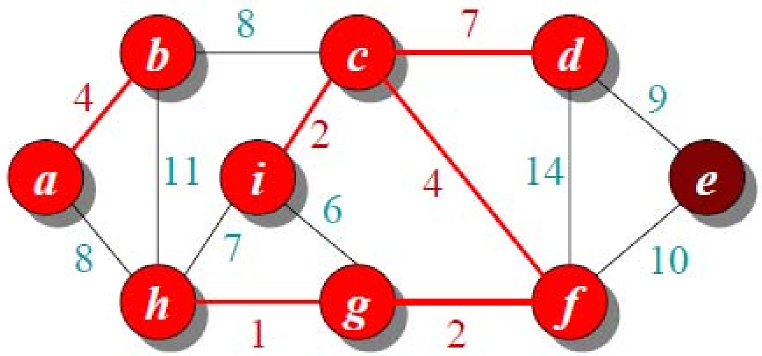
  for each edge $(u, v) \in E$, in sorted order, **do**
    if FIND-SET$(u) \neq$ FIND-SET(v) **then**
      $A \leftarrow A \cup \{(u, v)\}$
      UNION(u, v)
  return A

Disjoint-set collection:

$\{\{a, b\}, \{c, f, g, h, i\}, \{d\}, \{e\}\}$

Kruskal's algorithm

- Edge (c, d) is safe.



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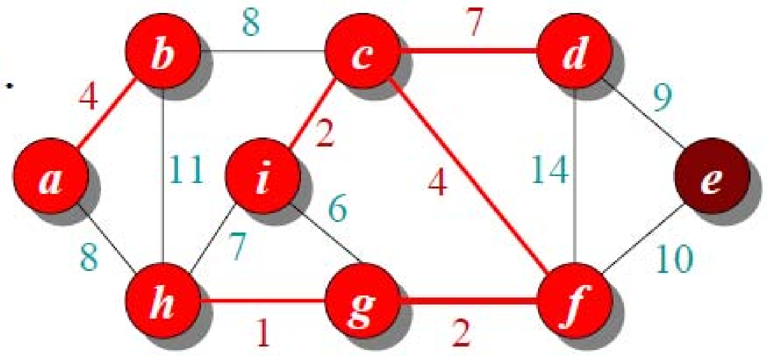
      $A \leftarrow A \cup \{(u, v)\}$
      UNION(u, v)
  return A

Disjoint-set collection:

$\{\{a, b\}, \{c, d, f, g, h, i\}, \{e\}\}$

Kruskal's algorithm

- Edge (h, i) is unsafe.



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  $A \leftarrow \varnothing$
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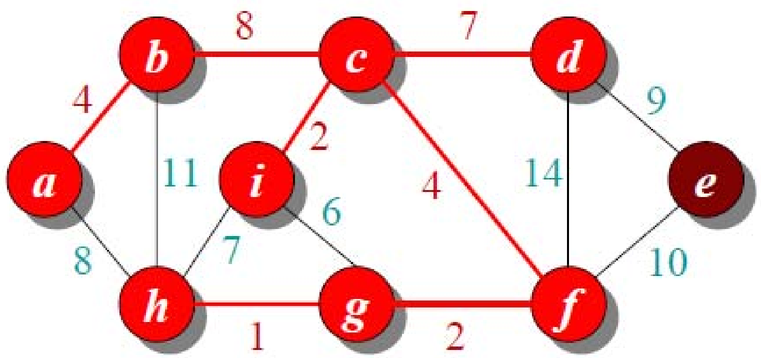
  sort E in nondecreasing order of w values
  for each edge $(u, v) \in E$, in sorted order, **do**
    if FIND-SET$(u) \neq$ FIND-SET(v) **then**
      $A \leftarrow A \cup \{(u, v)\}$
      UNION(u, v)
  return A

Disjoint-set collection:

$\{\{a, b\},\{c, d, f, g, h, i\},\{e\}\}$

Kruskal's algorithm

- Edge (b, c) is safe.



MST-KRUSKAL(G, w)
  $A \leftarrow \emptyset$
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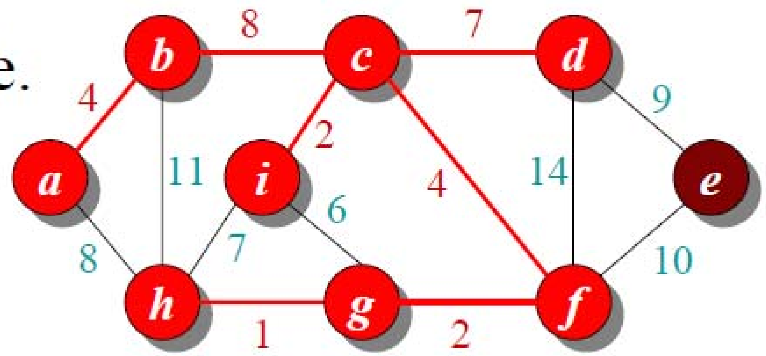
  for each edge $(u, v) \in E$, in sorted order, **do**
    if FIND-SET$(u) \neq$ FIND-SET(v) **then**
      $A \leftarrow A \cup \{(u, v)\}$
      UNION(u, v)
  return A

Disjoint-set collection:

$\{\{a, b, c, d, f, g, h, i\}, \{e\}\}$

Kruskal’s algorithm

- Edge (a, h) is unsafe.



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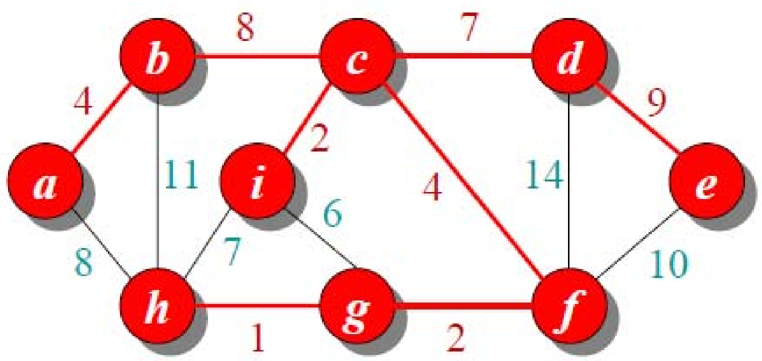
  return A

Disjoint-set collection:

$\{\{a, b, c, d, f, g, h, i\}, \{e\}\}$

Kruskal's algorithm

- Edge (d, e) is safe.



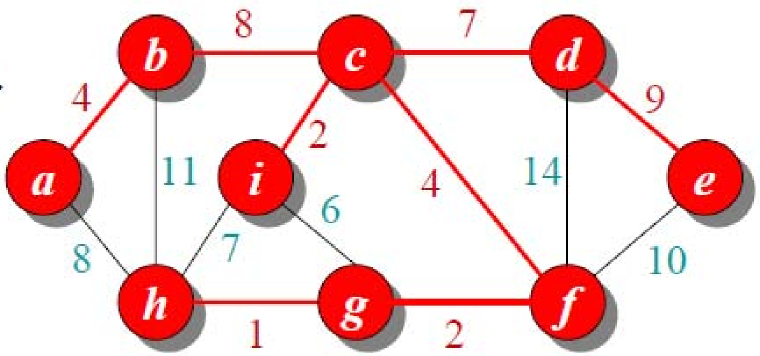
MST-KRUSKAL(G, w)
  $A \leftarrow \varnothing$
  for each vertex $v \in V[G]$ **do**
    MAKE-SET(v)
  sort E in nondecreasing order of w values
  for each edge $(u, v) \in E$, in sorted order, **do**
    if FIND-SET$(u) \neq$ FIND-SET(v) **then**
      $A \leftarrow A \cup \{(u, v)\}$
      UNION(u, v)
  return A

Disjoint-set collection:

$\{\{a, b, c, d, e, f, g, h, i\}\}$

Kruskal's algorithm

- Edge (e, f) is unsafe.



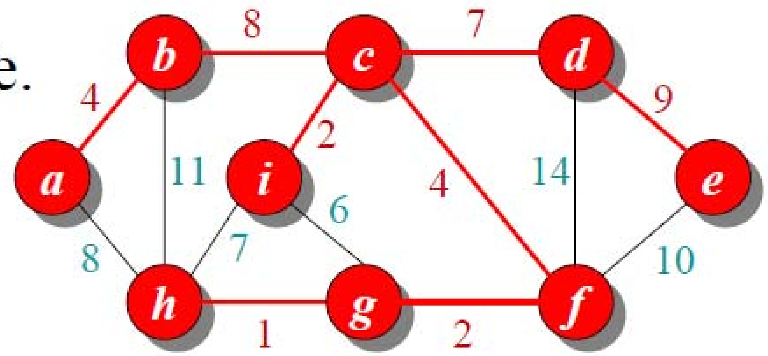
MST-KRUSKAL(G, w)
  $A \leftarrow \emptyset$
  for each vertex $v \in V[G]$ **do**
    MAKE-SET(v)
  sort E in nondecreasing order of w values
  for each edge $(u, v) \in E$, in sorted order, **do**
    if FIND-SET$(u) \neq$ FIND-SET(v) **then**
      $A \leftarrow A \cup \{(u, v)\}$
      UNION(u, v)
  return A

Disjoint-set collection:

$\{\{a, b, c, d, e, f, g, h, i\}\}$

Kruskal's algorithm

- Edge (b, h) is unsafe.



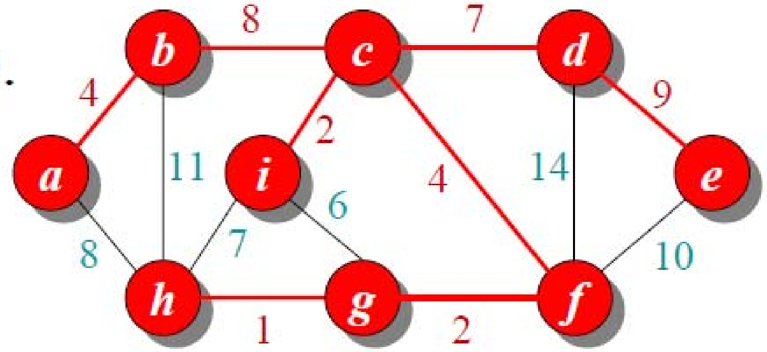
MST-KRUSKAL(G, w)
  $A \leftarrow \emptyset$
  for each vertex $v \in V[G]$ **do**
    MAKE-SET(v)
  sort E in nondecreasing order of w values
  for each edge $(u, v) \in E$, in sorted order, **do**
    if FIND-SET$(u) \neq$ FIND-SET(v) **then**
      $A \leftarrow A \cup \{(u, v)\}$
      UNION(u, v)
  return A

Disjoint-set collection:

$\{\{a, b, c, d, e, f, g, h, i\}\}$

Kruskal's algorithm

- Edge (d, f) is unsafe.



MST-KRUSKAL(G, w)
  $A \leftarrow \emptyset$
  for each vertex $v \in V[G]$ **do**
    MAKE-SET(v)
  sort E in nondecreasing order of w values
  for each edge $(u, v) \in E$, in sorted order, **do**
    if FIND-SET$(u) \neq$ FIND-SET(v) **then**
      $A \leftarrow A \cup \{(u, v)\}$
      UNION(u, v)
  return A

Disjoint-set collection:

$\{\{a, b, c, d, e, f, g, h, i\}\}$

MST-KRUSKAL Pseudocode

- MST-KRUSKAL(G, w)

```
1. A = { }
2. For each vectex v  G.V
3.       MAKE-SET(v)
4.   sort the edges of G.E into nondecreasing order by weight w
5. For each edge (u, v)  G.E, taken in nondecreasing order by weight
6.       if FIND-SET(u) ≠ FIND-SET(v)
7.               A = A U {(u,v)}
8.                UNION(u,v)
9. return A
```

Kruskal's algorithm

Theorem On a disjoint-set forest with union by rank and path compression, a sequence of m MAKE-SET, FIND-SET, and UNION operations, n *of* which are MAKE-SET operations, take $O(m\,\alpha(n))$ time, where α is a very slowly growing function.

Kruskal's algorithm makes $|V|$ MAKE-SET operations, $2|E|$ FIND-SET operations, and $|V| - 1$ UNION operations. Since G is connected, $|E| \geq |V| - 1$, and so the total number of disjoint set operations is $2|E|+2|V|-1 = O(E)$. Running time is thus $O(E\,\alpha(V)) = O(E \lg V) = O(E \lg E)$.

Problem 9: Saving ink

- Susan likes to make a line drawing with ink. There‛re several dots on drawing paper. Your job is to tell Susan how to connect the dots so as to minimize the amount of ink used.
  - Susan connects the dots by drawing straight lines between pairs, possibly lifting the pen between lines.
  - When Susan is done there must be a sequence of connected lines from any dot to any other dot.
- Test using 3 different data sets (test cases).

Problem 9: Saving ink

- Input
  - The input begins with a single positive integer on a line by itself indicating the number of dots ($0<n<30$) on drawing paper. For each dots, a line follows; each following line contains two real numbers indicating the (x, y) coordinates of the dots.
- Output
  - Your program must print a single real number to two decimal places: the minimum total length of ink lines that can connect all the dots.

Problem 9: Saving ink

- Sample input
  - 3
  - 1.0 1.0
  - 2.0 2.0
  - 2.0 4.0

- Sample output
  - ✓3.41

Problem 9: Data sets for Saving ink

- Data sets

3	
1	1
2	2
2	4

3.41

4	
1	4
2	5
3	6
1	1

5.83

5	
1	1
1	2
2	1
2	2
5	5

7.24

THANK YOU