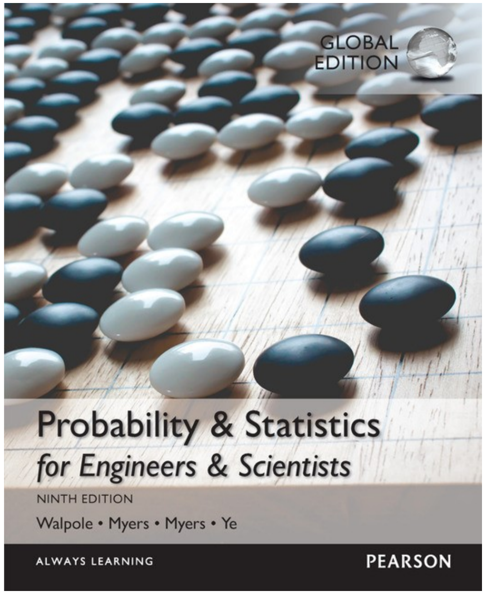
Chapter 9

One- and Two- Sample Estimation (2/2)



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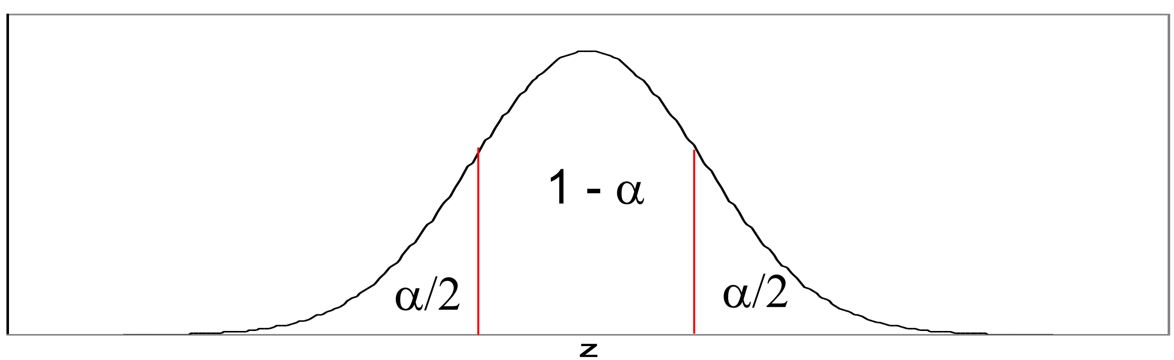
Recall: Confidence Interval of **μ** (if ***σ* is known**)

- Given:
  - σ is known and $\overline{X}$ is the mean of a random sample of size n,
- Then,
  - the $(1 - \alpha)100\%$ *confidence interval* for μ is

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.



Question

- The contents of seven similar containers of sulfuric acid are 9.8, 10.2, 10.4, 9.8, 10.0, 10.2, and 9.6 liters. Find a 95% confidence interval for the mean contents of all such containers, assuming an approximately normal distribution.

???

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

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What if σ^2 is unknown?

If σ^2 is unknown, we must use the *t*-statistic.

Recall

- Recall, if σ^2 is known, by Central Limit Theorem:

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$

is $n(z;\, 0,1)$

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What if we don't know σ^2

- In a situation with σ^2 unknown, then the random variable

$$\boxed{T = \frac{\bar{X} - \mu}{S/\sqrt{n}}.}$$

- has Student's t-distribution with $n-1$ degrees of freedom when the random sample is from a *normal distribution*, and sample size is n.
- T can be used to construct a confidence interval on μ
  - The procedure is the same as that with σ known except that σ is replaced by S

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Recall: t-distribution (Ch 8.6)

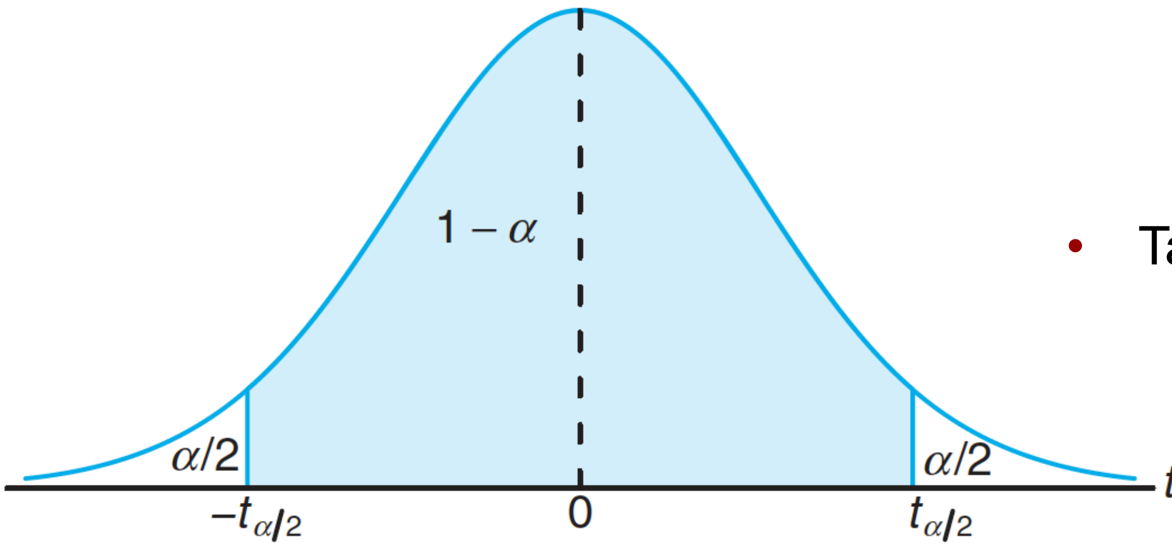

Figure 9.5: $P(-t_{\alpha/2} < T < t_{\alpha/2}) = 1-\alpha$.

- Table A.4

- Symmetry

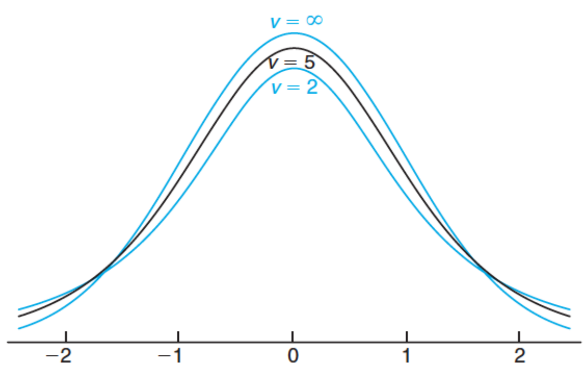

Figure 8.8: The t-distribution curves for $v = 2, 5$, and ∞.

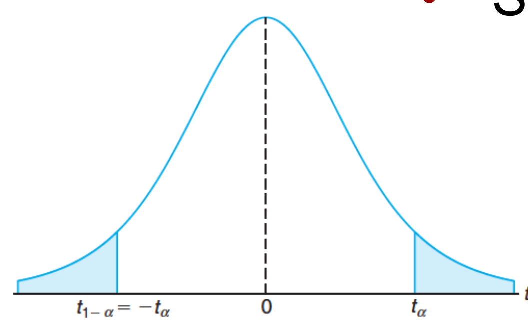

Figure 8.9: Symmetry property (about 0) of the t-distribution.

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What if we don't know σ^2

- *T* can be used to construct a confidence interval on $\boldsymbol{\mu}$.

$$\boxed{T = \frac{\bar{X} - \mu}{S/\sqrt{n}}.}$$

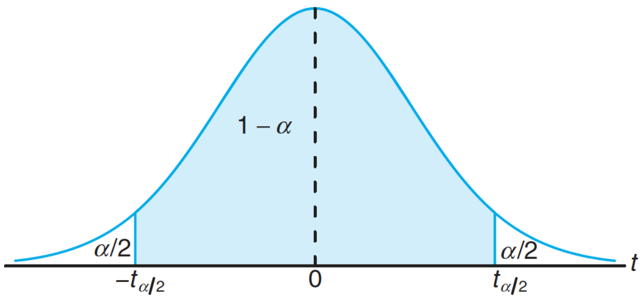

Figure 9.5: $P(-t_{\alpha/2} < T < t_{\alpha/2}) = 1 - \alpha$.

$$\Rightarrow \quad P(-t_{\alpha/2} < T < t_{\alpha/2}) = 1 - \alpha,$$

where $t_{\alpha/2}$ is the t-value with $n-1$ degrees of freedom, above which we find an area of $\alpha/2$. Because of symmetry, an equal area of $\alpha/2$ will fall to the left of $-t_{\alpha/2}$.

What if we don’t know σ^2

- Substituting for T, we write

$$P\left(-t_{\alpha/2} < \frac{\bar{X}-\mu}{S/\sqrt{n}} < t_{\alpha/2}\right) = 1-\alpha.$$

Multiplying each term in the inequality by $S/\sqrt{n}$, and then subtracting $\bar{X}$ from each term and multiplying by -1, we obtain

$$P\left(\bar{X} - t_{\alpha/2}\frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2}\frac{S}{\sqrt{n}}\right) = 1-\alpha.$$

For a particular random sample of size n, the mean $\bar{x}$ and standard deviation s are computed and the following $100(1-\alpha)\%$ confidence interval for μ is obtained.

What if we don't know σ^2

Confidence Interval on μ, σ^2 Unknown

If $\bar{x}$ and s are the mean and standard deviation of a random sample from a normal population with unknown variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}},$$

where $t_{\alpha/2}$ is the t-value with $v = n - 1$ degrees of freedom, leaving an area of $\alpha/2$ to the right.

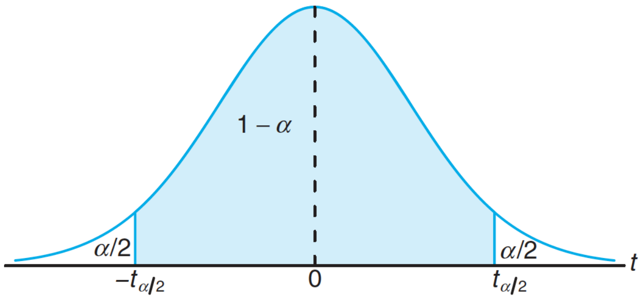

Figure 9.5: $P(-t_{\alpha/2} < T < t_{\alpha/2}) = 1 - \alpha$.

Example 9.5

- The contents of seven similar containers of sulfuric acid are 9.8, 10.2, 10.4, 9.8, 10.0, 10.2, and 9.6 liters. Find a 95% confidence interval for the mean contents of all such containers, assuming an approximately normal distribution.

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Example 9.5

The sample mean and standard deviation for the given data are

$$\bar{x} = 10.0 \qquad \text{and} \qquad s = 0.283.$$

Using Table A.4, we find $t_{0.025} = 2.447$ for $v = 6$ degrees of freedom. Hence, the 95% confidence interval for μ is

$$10.0 - (2.447)\left(\frac{0.283}{\sqrt{7}}\right) < \mu < 10.0 + (2.447)\left(\frac{0.283}{\sqrt{7}}\right),$$

which reduces to $9.74 < \mu < 10.26$.

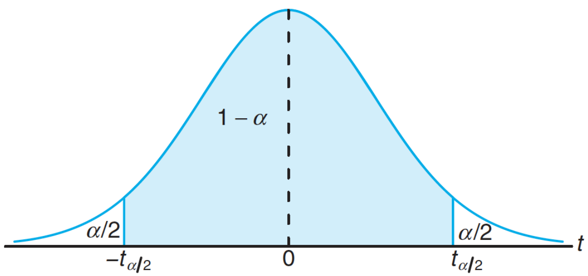

Figure 9.5: $P(-t_{\alpha/2} < T < t_{\alpha/2}) = 1 - \alpha$.

Concept of a Large-Sample Confidence Interval

- Often statisticians recommend that even when normality cannot be assumed, σ is unknown, and $n \geq 30$, s can replace σ and the confidence interval

$$\bar{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}$$

may be used. This is often referred to as a large-sample confidence interval.

- The justification lies only in the presumption that with a sample as large as 30 and the population distribution not too skewed, s will be very close to the true σ and thus the Central Limit Theorem prevails.

Example 9.6

- Scholastic Aptitude Test (SAT) mathematics scores of a random sample of 500 high school seniors in the state of Texas are collected, and the sample mean and standard deviation are found to be 501 and 112, respectively.
- Find a 99% confidence interval on the mean SAT mathematics score for seniors in the state of Texas.

Example 9.6

Since the sample size is large, it is reasonable to use the normal approximation. Using Table A.3, we find $z_{0.005} = 2.575$. Hence, a 99% confidence interval for μ is

$$501 \pm (2.575)\left(\frac{112}{\sqrt{500}}\right) = 501 \pm 12.9,$$

which yields $488.1 < \mu < 513.9$.

9.6 Prediction Intervals

Question

- The following measurements were recorded for the drying time, in hours, of a certain brand of latex paint:
  - 3.4, 2.5, 4.8, 2.9, 3.6,
  - 2.8, 3.3, 5.6, 3.7, 2.8,
  - 4.4, 4.0, 5.2, 3.0, 4.8
- Assuming that measurements represent a random sample from a normal population;
- Find a range of the drying time for the next trial of the paint.
- =Find a **95% prediction interval** for the drying time for the **next trial** of the paint.

Prediction

- Previous focus: (Point or Interval) Estimate of $\boldsymbol{\mu}$
- Sometimes, other than the population mean, the experimenter may also be interested in predicting the **possible value of a future observation (X_0)**.
- Point estimator for future new observation: $\bar{X}$

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Prediction

- What is the mean and variance of the error of the future observation?

A natural point estimator of a new observation is $\bar{X}$. It is known the variance of $\bar{X}$ is $\frac{\sigma^2}{n}$. To predict a new observation, not only do we need to account for the variation due to the estimating the mean, but also should we account for the **variation of a future observation**. From the assumption, we assume that the variance of the random error in a new observation is σ^2.

- Pointer estimator: $\bar{X} \sim N(\mu, \sigma^2/n)$: sample mean
- Future observation: $X_0 \sim N(\mu, \sigma^2)$: new observation

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- Pointer estimator: $\bar{X} \sim N(\mu, \sigma^2/n)$: sample mean
- Future observation: $X_0 \sim N(\mu, \sigma^2)$: new observation

Theorem 7.11

- The sum of independent normal random variables is also normal random variable with

mean $\sum_{i=1}^{n} \mu_i$ and variance $\sum_{i=1}^{n} \sigma_i^2$

$$X_0 - \bar{X} \sim N\left(0, \sigma^2 + \frac{\sigma^2}{n}\right),$$

$$Z = \frac{X_0 - \bar{X}}{\sqrt{\sigma^2 + \frac{\sigma^2}{n}}} = \frac{X_0 - \bar{X}}{\sigma\sqrt{1 + \frac{1}{n}}}$$

$$P\left(-z_{\alpha/2} < Z < z_{\alpha/2}\right) = 1 - \alpha.$$

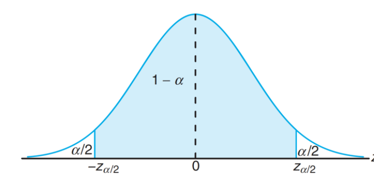

Figure 9.2: $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$.

Hence

$$P\left(-z_{\alpha/2} < \frac{X_0 - \bar{X}}{\sigma\sqrt{1 + \frac{1}{n}}} < z_{\alpha/2}\right) = 1 - \alpha.$$

$$P\left(\bar{X} - z_{\alpha/2} \cdot \sigma \cdot \sqrt{1 + \frac{1}{n}} < X_0 < \bar{X} + z_{\alpha/2} \cdot \sigma \cdot \sqrt{1 + \frac{1}{n}}\right) = 1 - \alpha.$$

Prediction Interval of a Future Observation (**σ is known**)

For a normal distribution of measurements with unknown means μ and **known** variance σ^2 a $100(1-\alpha)\%$ prediction interval of a future observation x_0 is

$$\bar{x} - z_{\alpha/2} \cdot \sigma \cdot \sqrt{1 + \frac{1}{n}} < x_0 < \bar{x} + z_{\alpha/2} \cdot \sigma \cdot \sqrt{1 + \frac{1}{n}}$$

where $z_{\alpha/2}$ is the z-value, leaving an area of $\alpha/2$ to the right.

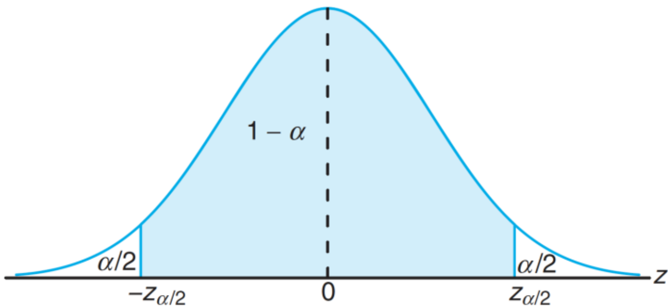

Figure 9.2: $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$.

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$$\bar{x} - z_{\alpha/2}\sigma\sqrt{1+1/n} < x_0 < \bar{x} + z_{\alpha/2}\sigma\sqrt{1+1/n}.$$

- Example 9.7
  - Due to the decrease in interest rates, the first Citizen Bank received a lot of mortgage applications. A recent sample of 50 mortgage loans resulted in an average of \$257,300. Assume a population standard deviation of \$25,000. If the next customer called in for a mortgage loan application, find a 95% prediction interval on this customer's loan amount.

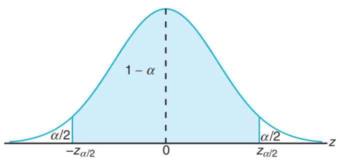

Figure 9.2: $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$.

The point prediction of the next customer's loan amount is $\bar{x} = \$257,300$. The z-value here is $z_{0.025} = 1.96$. Hence, a 95% prediction interval for the future loan amount is

Prediction Interval
(if σ is **unknown**)

- For a normal distribution of unknown mean μ, and standard deviation σ, a 100(1-α)% *prediction interval* of a future observation, x_0 is

$$\boxed{\overline{X} - z_{\alpha/2}\,\sigma\sqrt{1+\frac{1}{n}} < x_0 < \overline{X} + z_{\alpha/2}\,\sigma\sqrt{1+\frac{1}{n}}}$$

if σ is known, and

$$\boxed{\overline{X} - t_{\alpha/2,n-1}\,s\sqrt{1+\frac{1}{n}} < x_0 < \overline{X} + t_{\alpha/2,n-1}\,s\sqrt{1+\frac{1}{n}}}$$

if σ is unknown

- Example 9.8

$$\bar{x} - t_{\alpha/2}s\sqrt{1+1/n} < x_0 < \bar{x} + t_{\alpha/2}s\sqrt{1+1/n}$$

  - A meat inspector has randomly selected 30 packs of 95% lean beef. The sample resulted in a mean of 96.2% with a sample standard deviation of 0.8%.
  - Find a 99% prediction interval for the leanness of a new pack. Assume normality.

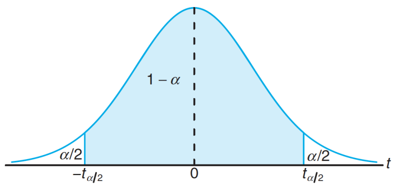

Figure 9.5: $P(-t_{\alpha/2} < T < t_{\alpha/2}) = 1 - \alpha$.

For $v = 29$ degrees of freedom, $t_{0.005} = 2.756$. Hence, a 99% prediction interval for a new observation x_0 is

Outlier Detection

- **Use of Prediction Limits for Outlier Detection**
  - an observation is an **outlier** if it falls outside the prediction interval computed without including the questionable observation in the sample

- Example 9.8 : if a new pack of beef is measured and its leanness is outside the interval (93.96, 98.44), that observation can be viewed as an outlier

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9.8. Two Samples: Estimating the Difference Between Two Means

Example

- A study was conducted in which two types of engines, *A* and *B*, were compared. Gas mileage, in miles per gallon, was measured.
- Fifty experiments were conducted using engine type *A* and 75 experiments were done with engine type *B*. The gasoline used and other conditions were held constant. The average gas mileage was 36 miles per gallon for engine *A* and 42 miles per gallon for engine *B*.
- Find a 96% confidence interval on $\mu_B - \mu_A$, where μ_A and μ_B are population mean gas mileages for engines *A* and *B*, respectively. Assume that the population standard deviations are 6 and 8 for engines *A* and *B*, respectively.

Estimating Difference between two means

- Two normal populations $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$
- Point estimator of the difference between μ_1 and μ_2 is given by the statistic $\overline{X_1} - \overline{X_2}$
- According to Theorem 8.3,
- Then, probability $1 - \alpha$ that the standard normal variable,

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

will fall between $-z_{\alpha/2}$ and $z_{\alpha/2}$.

Theorem 8.3: If independent samples of size n_1 and n_2 are drawn at random from two populations, discrete or continuous, with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively, then the sampling distribution of the differences of means, $\bar{X}_1 - \bar{X}_2$, is approximately normally distributed with mean and variance given by

$$\mu_{\bar{X}_1 - \bar{X}_2} = \mu_1 - \mu_2 \text{ and } \sigma^2_{\bar{X}_1 - \bar{X}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}.$$

Hence,

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2/n_1) + (\sigma_2^2/n_2)}}$$

is approximately a standard normal variable.

$$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha.$$

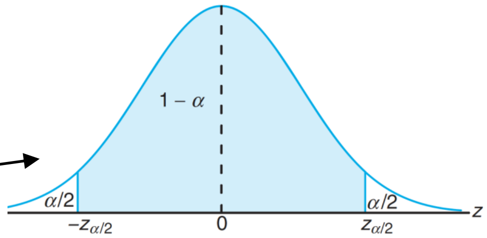

Figure 9.2: $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$.

Substituting for Z, we state equivalently that

$$P\left(-z_{\alpha/2} < \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} < z_{\alpha/2}\right) = 1 - \alpha,$$

which leads to the following $100(1 - \alpha)\%$ confidence interval for $\mu_1 - \mu_2$.

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Confidence Interval for $\mu_1 - \mu_2$

If $\bar{x}_1$ and $\bar{x}_2$ are means of independent random samples of sizes n_1 and n_2 from populations with known variances σ_1^2 and σ_2^2, respectively, a $100(l-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is given by

$$(\bar{x}_1 - \bar{x}_2) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < (\bar{x}_1 - \bar{x}_2) + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}},$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

The degree of confidence is exact when samples are selected from normal populations. For nonnormal populations, the Central Limit Theorem allows for a good approximation for reasonable size samples.

Example

- A study was conducted in which two types of engines, *A* and *B*, were compared. Gas mileage, in miles per gallon, was measured.
- Fifty experiments were conducted using engine type *A* and 75 experiments were done with engine type *B*. The gasoline used and other conditions were held constant. The average gas mileage was 36 miles per gallon for engine *A* and 42 miles per gallon for engine *B*.
- Find a 96% confidence interval on $\mu_B - \mu_A$, where μ_A and μ_B are population mean gas mileages for engines *A* and *B*, respectively. Assume that the population standard deviations are 6 and 8 for engines *A* and *B*, respectively.

- Solution

The point estimate of $\mu_B - \mu_A$ is $\bar{x}_1 - \bar{x}_2 = 42 - 36 = 6$. Using $\alpha = 0.04$, we find $z_{0.02} = 2.05$. Hence with substitution in the formula above, the 96% confidence interval is

$$6 - 2.05 \cdot \sqrt{\frac{24}{75} + \frac{36}{50}} < \mu_B - \mu_A < 6 + 2.05 \cdot \sqrt{\frac{24}{75} + \frac{36}{50}}$$

or simply $3.43 < \mu_B - \mu_A < 8.57$.

$$\boxed{(\bar{x}_1 - \bar{x}_2) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < (\bar{x}_1 - \bar{x}_2) + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}},}$$

Active Learning

9.14 Maximum Likelihood Estimator (MLE)

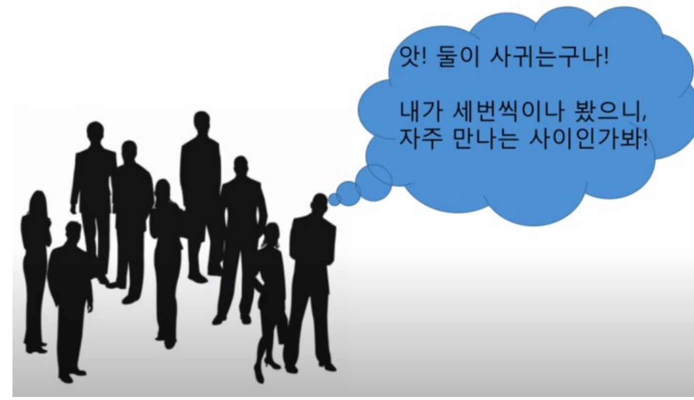
- Watch the YouTube video and learn by yourself
- Q/A with each other via cyber campus or other communities
- We do not discuss this content in regular class but may be part of your exam

Maximum Likelihood Estimator (MLE)

팀장님의 관측

팀장님은 최근에 같은 팀 남녀 사원이 카페, 영화관, 식당에서
함께 있는 것을 보게 되었습니다.

팀장님의 결론



- 팀장님은 세번만 봤으면서 왜 자주 만났을 것이라고 생각했을까

➔ 그들의 데이트 사건은 흔한 일이므로 본인이 목격한 것이라 생각함

- 즉, 목격한 사건

➔ 흔한 사건 ➔ 빈도가 높은 사건 ➔ 발생 확률이 높은 사건

Source: https://www.youtube.com/watch?v=sOtkPm_1GYw

Maximum Likelihood Estimator (MLE)

1) Set a model
2) Express the probability of observed events in terms of the model equations.
3) <u>Find the model parameters that maximizes the likelihood.</u>

This methodology makes sense.
You can't be the only one who observed the event.
It is reasonable to interpret this as a likely event.

[Source: https://www.youtube.com/watch?v=sOtkPm_1GYw](https://www.youtube.com/watch?v=sOtkPm_1GYw)

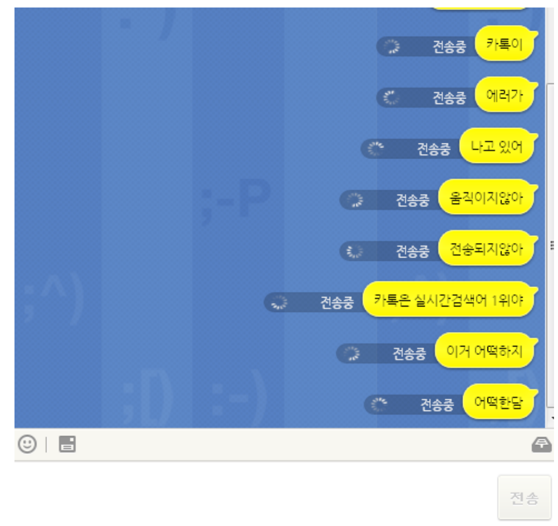
Motivation

- Motivation
  - In real world problems, some parameters are not given as a prior
  - Example
    - Parameter p in geometric RV

✓ Assumption

$$\text{Success} \;:\; P(S) = p, \quad \text{Failure} \;:\; P(F) = 1 - p$$

$$\Pr(X = k) = (1-p)^{k-1}p \quad \text{for } k = 1, 2, 3, \ldots.$$



✓ Problem: we do not know this p in practice!

Maximum Likelihood Estimator (MLE)

Let's apply the previous story to the problem below.

- Motivating example
  - Suppose that following 10 coin tosses were simulated using a Bernoulli distribution

0 0 0 1 1 1 0 1 1 1

  - Recall the Bernoulli $X_i = \begin{cases} 1 & \text{if trial } i \text{ is a success} \\ 0 & \text{otherwise} \end{cases}$

같은 표현

$$P\{X_i = 0\} = 1 - p$$

$$P\{X_i = 1\} = p$$

$$P\{X_i = x\} = p^x(1-p)^{1-x}, \quad x = 0, 1$$

  - We have

✓ Problem: We do not know this p!

Maximum Likelihood Estimator (MLE)

Let's calculate the probability that we will observe the below.

- Motivating example:
  - We know the result (0 0 0 1 1 1 0 1 1 1) comes from Bernoulli RV
  - Let x_i denote the results for 10 simulations (ex, $x_1 = 0, x_{10} = 1$)
  - From the Bernoulli distribution, for $1 \le i \le 10$,

$$P(X_i = x_i) = p^{x_i}(1-p)^{1-x_i}$$

  - Since each X_i is <u>independent</u>, we have

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_{10} = x_{10}) = P(X_1 = x_1)P(X_2 = x_2) \ldots P(X_{10} = x_{10})$$

$$= \prod_{i=1}^{10} p^{x_i}(1-p)^{1-x_i}$$

Maximum Likelihood Estimator (MLE)

- Motivating example
  - Let the **likelihood function** be:

$$L(p) = f(x_1, \dots, x_{10}|p) := \prod_{i=1}^{10} p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=1}^{10} x_i}(1-p)^{10-\sum_{i=1}^{10} x_i}$$

  - For the samples (0 0 0 1 1 1 0 1 1 1), we have

$$L(p) = p^6(1-p)^4 \text{ (likelihood)}$$

✓ Objective: Find p that maximizes the L(p)!

  - Hence, we call such the p by Maximum Likelihood Estimator (MLE)

Bernoulli RV - MLE

- Bernoulli
  - Suppose that n independent trials with probability p are performed. What is the MLE of p?

$$
\begin{aligned}
f(x_1,\ldots,x_n|p) &= P\{X_1 = x_1,\ldots,X_n = x_n|p\} \\
&= p^{x_1}(1-p)^{1-x_1}\cdots p^{x_n}(1-p)^{1-x_n} \\
&= p^{\Sigma_1^n x_i}(1-p)^{n-\Sigma_1^n x_i}, \quad x_i = 0,1, \quad i = 1,\ldots,n
\end{aligned}
$$

$$
\log f(x_1,\ldots,x_n|p) = \sum_1^n x_i \log p + \left(n - \sum_1^n x_i\right)\log(1-p)
$$

  - Log likelihood

Bernoulli RV - MLE

$$\log f(x_1,\ldots,x_n|p) = \sum_1^n x_i \log p + \left(n - \sum_1^n x_i\right)\log(1-p)$$

- Bernoulli
  - Suppose that n independent trials with probability p are performed. What is the MLE of p?

$$\frac{d}{dp}\log f(x_1,\ldots,x_n|p) = \frac{\sum_1^n x_i}{p} - \frac{\left(n - \sum_1^n x_i\right)}{1-p}$$

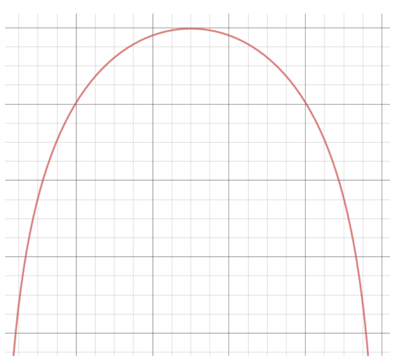
$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

$$\longrightarrow \qquad \hat{p} = \frac{\sum_{i=1}^n x_i}{n}$$

Find maximum value
= differentiation is zero

$$f(x) = \log x + \log(1-x)$$



Recall previous example

- Motivating example
  - Suppose that following 10 coin tosses were simulated using a Bernoulli distribution

0 0 0 1 1 1 0 1 1 1

$$P\{X_i = x\} = p^x(1-p)^{1-x}, \quad x = 0, 1$$

$$\hat{p} = \frac{\sum_{i=1}^{n} x_i}{n}$$

  - $\hat{p}$ = 6/10 = 0.6

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Example

- RAM chip
  - Suppose that each RAM chip is, independently, of acceptable quality with probability p.
  - If out of a sample of 1,000 tested 921 are acceptable. What is the maximum likelihood estimate of p??
  - Answer: $\hat{p} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{921}{1000} = 0.921$

Poisson RV - MLE

- Poisson RV
  - Suppose that $X_1, \ldots, X_n$ are independent Poisson random variables each having mean λ. Determine the MLE of λ.

$$P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}$$

$$\Downarrow$$

$$f(x_1, \ldots, x_n|\lambda) = \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \cdots \frac{e^{-\lambda}\lambda^{x_n}}{x_n!}$$

$$= \frac{e^{-n\lambda}\lambda^{\Sigma_1^n x_i}}{x_1! \ldots x_n!}$$

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Poisson RV - MLE

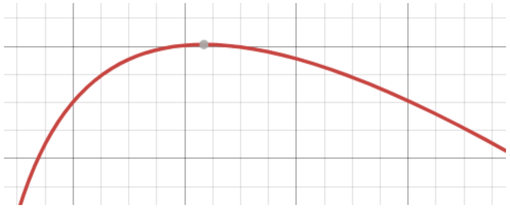
- Poisson RV
  - Suppose that $X_1, \ldots, X_n$ are independent Poisson random variables each having mean λ. Determine the MLE of λ.

$$\log f(x_1, \ldots, x_n|\lambda) = -n\lambda + \sum_1^n x_i \log\lambda - \log c$$

where $c = \prod_{i=1}^n x_i!$ does not depend on λ, and

$$\frac{d}{d\lambda}\log f(x_1, \ldots, x_n|\lambda) = -n + \frac{\sum_1^n x_i}{\lambda} \longrightarrow \hat{\lambda} = \frac{\sum_1^n x_i}{n}$$

$f(x) = -x + \log x$



Example

- Accident
  - The number of traffic accidents in a city in 10 randomly chosen days in 1998 is 4, 0, 6, 5, 2, 1, 2, 0, 4, 3
  - What is the probability that there will be 2 or fewer accidents in a day?

  - Sol
    - First, find MLE of λ by

$$\hat{\lambda} = \frac{\sum_{1}^{n} x_i}{n} = \frac{27}{10} = 2.7$$

- Apply this MLE for the Poisson RV to get the probability:

$$P\{X = i\} = e^{-\hat{\lambda}} \frac{\hat{\lambda}^i}{i!}$$

$$e^{-2.7}(1 + 2.7 + (2.7)^2/2) = .4936$$

End of Ch 9.

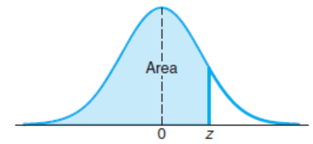


Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

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Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Table A.4 (continued) Critical Values of the t-Distribution

	α						
v	0.02	0.015	0.01	0.0075	0.005	0.0025	0.0005
1	15.894	21.205	31.821	42.433	63.656	127.321	636.578
2	4.849	5.643	6.965	8.073	9.925	14.089	31.600
3	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	2.328	2.491	2.718	2.879	3.106	3.497	4.437
12	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	2.249	2.397	2.602	2.746	2.947	3.286	4.073
16	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	2.197	2.336	2.528	2.661	2.845	3.153	3.850
21	2.189	2.328	2.518	2.649	2.831	3.135	3.819
22	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	2.167	2.301	2.485	2.612	2.787	3.078	3.725
26	2.162	2.296	2.479	2.605	2.779	3.067	3.707
27	2.158	2.291	2.473	2.598	2.771	3.057	3.689
28	2.154	2.286	2.467	2.592	2.763	3.047	3.674
29	2.150	2.282	2.462	2.586	2.756	3.038	3.660
30	2.147	2.278	2.457	2.581	2.750	3.030	3.646
40	2.123	2.250	2.423	2.542	2.704	2.971	3.551
60	2.099	2.223	2.390	2.504	2.660	2.915	3.460
120	2.076	2.196	2.358	2.468	2.617	2.860	3.373
∞	2.054	2.170	2.326	2.432	2.576	2.807	3.290

Table A.4 Critical Values of the t-Distribution

	α						
v	**0.40**	**0.30**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.538	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960