Chapter 2

Probability – part 2

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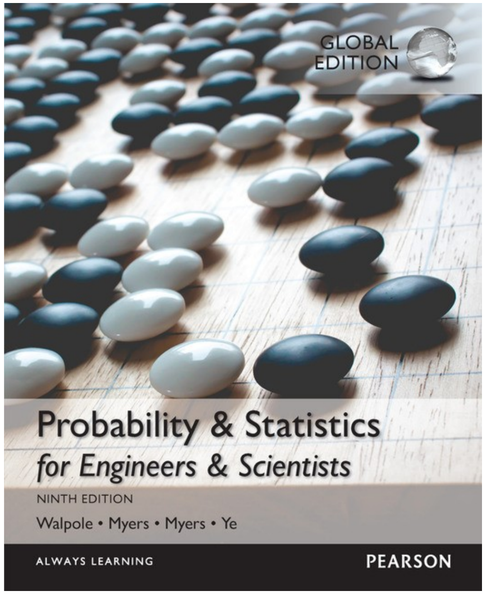

Outline

- Sample Space
- Events
- Counting Sample Points
- **Probability of an Event**
- **Additive Rules**
- **Conditional Probability**
- Bayes' Rule

2.4 Probability of an Event

Def. 2.9. Properties of Probability

- The likelihood of the occurrence of a sample point: **weight** or **probability**
  - Ranges from 0 to 1

Definition 2.9:

The **probability** of an event A is the sum of the weights of all sample points in A. Therefore,

$$0 \le P(A) \le 1, \quad P(\phi) = 0, \quad \text{and} \quad P(S) = 1.$$

Furthermore, if $A_1, A_2, A_3, \ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots .$$

Example 2.24

- A coin is tossed twice. What is the probability that at least one head occurs?

Rule 2.3 (Theorem)

- If an experiment can result in any one of ***N*** **different equally likely outcomes**, and if exactly ***n*** of these outcomes correspond to event *A*, then the probability of event *A* is

$$P(A) = \frac{n}{N}.$$

- That is

$$P(A) = \frac{\text{Number of outcomes favorable to } A}{\text{Total number of outcomes for the experiment}}.$$

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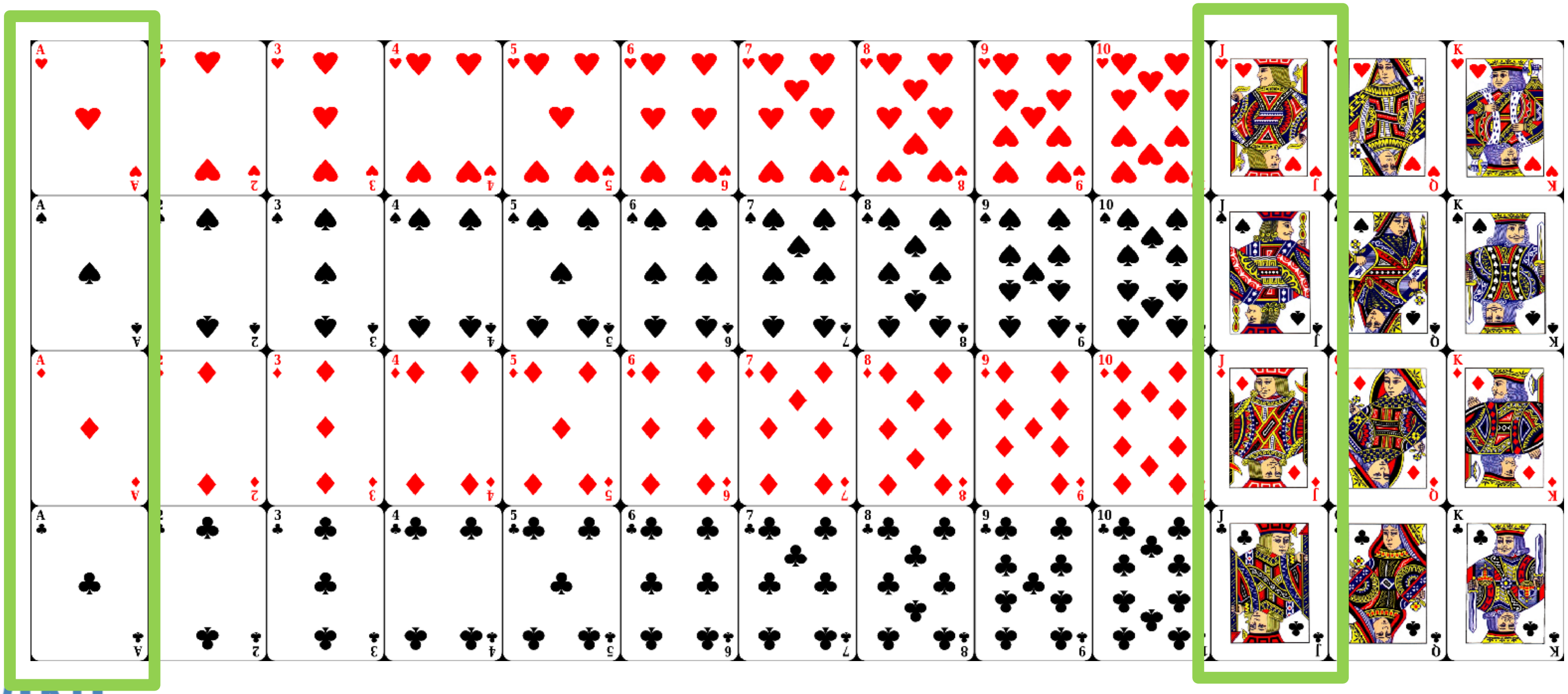
Example 2.28

- In a poker hand consisting of 5 cards,
  find the probability of holding 2 aces and 3 jacks?

  - c.f. Poker
    - 52 cards = 4 shapes * 13 cards (A,2,3,4, …., 10, J, Q, K)



- Solution:

The number of ways of being dealt 2 aces from 4 cards is

$$\binom{4}{2} = \frac{4!}{2!\,2!} = 6,$$

and the number of ways of being dealt 3 jacks from 4 cards is

$$\binom{4}{3} = \frac{4!}{3!\,1!} = 4.$$

By the multiplication rule (Rule 2.1), there are $n = (6)(4) = 24$ hands with 2 aces and 3 jacks. The total number of 5-card poker hands, all of which are equally likely, is

$$N = \binom{52}{5} = \frac{52!}{5!\,47!} = 2{,}598{,}960.$$

Therefore, the probability of getting 2 aces and 3 jacks in a 5-card poker hand is

$$P(C) = \frac{24}{2,598,960} = 0.9 \times 10^{-5}.$$

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- If the outcomes of an experiment are not equally likely to occur, the probabilities must be assigned on the basis of prior knowledge or experimental evidence.
- For example, consider the case where a die is not balanced.
- Try to solve : **Example 2.25.**

Example 2.25: A die is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.

***Solution*:** The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. We assign a probability of w to each odd number and a probability of $2w$ to each even number. Since the sum of the probabilities must be 1, we have $9w = 1$ or $w = 1/9$. Hence, probabilities of $1/9$ and $2/9$ are assigned to each odd and even number, respectively. Therefore,

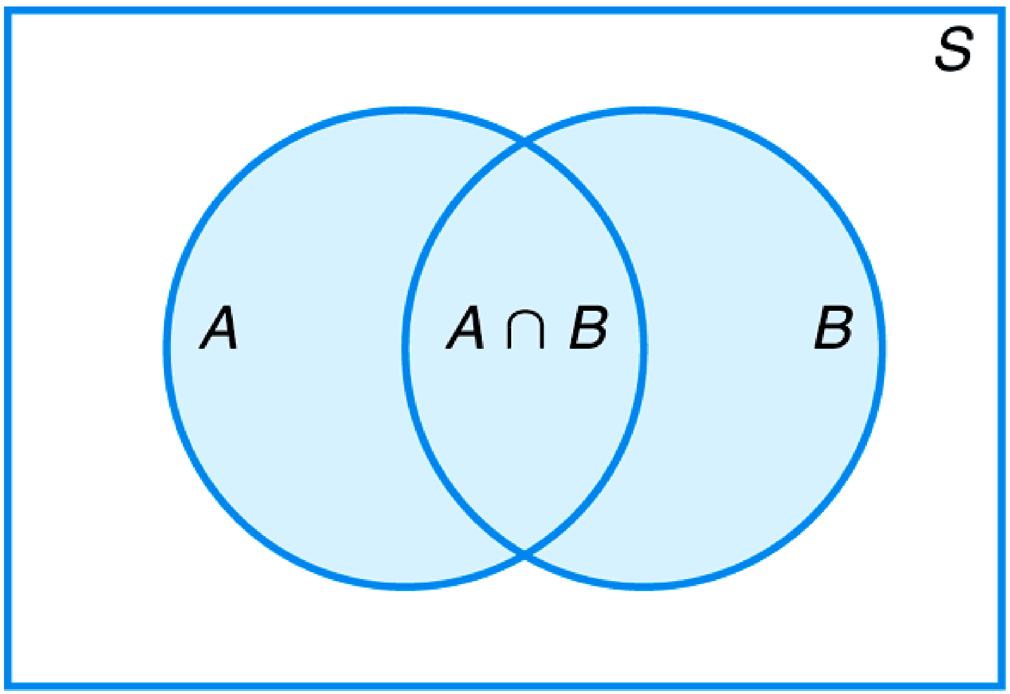
$$E = \{1, 2, 3\} \text{ and } P(E) = \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}.$$

2.5 Additive Rules

- **Theorem 2.7: Additive Rule**

If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



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- **Corollary 2.1**
  - **If A and B are mutually exclusive (disjoint)**

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Example

- What is the probability of getting a total of 7 or 11 when a pair of fair dices are tossed?

***Solution*:** Let A be the event that 7 occurs and B the event that 11 comes up. Now, a total of 7 occurs for 6 of the 36 sample points, and a total of 11 occurs for only 2 of the sample points. Since all sample points are equally likely, we have $P(A) = 1/6$ and $P(B) = 1/18$. The events A and B are mutually exclusive, since a total of 7 and 11 cannot both occur on the same toss. Therefore,

$$P(A \cup B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}.$$

This result could also have been obtained by counting the total number of points for the event $A \cup B$, namely 8, and writing

$$P(A \cup B) = \frac{n}{N} = \frac{8}{36} = \frac{2}{9}.$$

- ## Corollary 2.2

If $A_1,\ A_2,\ldots,A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

A collection of events $\{A_1, A_2, \ldots, A_n\}$ of a sample space S is called a **partition** of S if $A_1, A_2, \ldots, A_n$ are mutually exclusive and $A_1 \cup A_2 \cup \cdots \cup A_n = S$. Thus, we have

- ## Corollary 2.3

If $A_1,\ A_2,\ldots,A_n$ is a partition of sample space S, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) = P(S) = 1.$$

Example 2.31

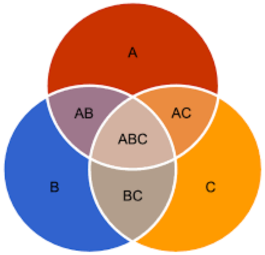
- If the probabilities are, respectively, 0.09, 0.15, 0.21, and 0.23 that a person purchasing a new automobile will choose the color green, white, red, or blue, what is the probability that a given buyer will purchase a new automobile that comes in one of those colors?

$$
\begin{aligned}
P(G \cup W \cup R \cup B) &= P(G) + P(W) + P(R) + P(B) \\
&= 0.09 + 0.15 + 0.21 + 0.23 = 0.68.
\end{aligned}
$$

Corollary 2.2: If $A_1, A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

- **Theorem 2.8**



For three events A, B, and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) \\ - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

- **Theorem 2.9**

If A and A' are complementary events, then

$$P(A) + P(A') = 1.$$

Example 2.32

- If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

***Solution*:** Let E be the event that at least 5 cars are serviced. Now, $P(E) = 1 - P(E')$, where E' is the event that fewer than 5 cars are serviced. Since

$$P(E') = 0.12 + 0.19 = 0.31,$$

it follows from Theorem 2.9 that

$$P(E) = 1 - 0.31 = 0.69.$$

2.6 **Conditional Probability, Independence, and the Product Rule**

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Conditional Probability

- One very important concept in probability theory is conditional probability.

- In some applications, we are interested in the probability structure under certain restrictions.

- The concept of conditional probability has countless uses in both industrial and scientific applications.!!

- # Conditional Probability
  - The probability of an event *B* occurring when it is known that some event *A* has occurred is called a **conditional probability** and is denoted by **P(B|A).**
    The symbol **P(B|A)** is usually read "the probability that *B* occurs given that *A* occurs" or simply "the probability of *B*, given *A*."

- Definition 2.10: the conditional probability B, given A, P(B|A)

The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

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Example

- One of these individuals is to be selected at random
- Probability of choosing a **man**, given that the selected person is **employed**?

Table 2.1: Categorization of the Adults in a Small Town

	Employed	**Unemployed**	**Total**
Male	460	40	500
Female	140	260	400
Total	600	300	900

M: a man is chosen,

E: the one chosen is employed.

Using the reduced sample space E, we find that

$$P(M|E) = \frac{460}{600} = \frac{23}{30}.$$

Definition 2.10: The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

- We can arrive at the same conclusion, by using the conditional Probability,

$$P(E) = \frac{600}{900} = \frac{2}{3} \quad \text{and} \quad P(E \cap M) = \frac{460}{900} = \frac{23}{45}.$$

Hence,

$$P(M|E) = \frac{23/45}{2/3} = \frac{23}{30},$$

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Example

- Roll a dice. What is the chance that you'd get a 6, given that you've gotten an even number?

$$A = \{2, 4, 6\}, \quad P(A) = \frac{1}{2};$$

$$B = \{6\}, \quad P(B) = \frac{1}{6};$$

$$A \cap B = \{6\}, \quad P(A \cap B) = \frac{1}{6};$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{3}$$

Definition 2.10: The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

- **Example 2.34**
  - The probability that a regularly scheduled flight departs on time is $\mathbf{P}(\mathbf{D}) = 0.83$;
  - the probability that it arrives on time is $\mathbf{P}(\mathbf{A}) = 0.82$; and
  - the probability that it departs and arrives on time is $\mathbf{P}(\mathbf{D} \cap \mathbf{A}) = 0.78$.
  - Find the probability that a plane
    - **(a) arrives on time, given that it departed on time, and (b) departed on time, given that it has arrived on time.**

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***Solution*:** Using Definition 2.10, we have the following.

(a) The probability that a plane arrives on time, given that it departed on time, is

$$P(A|D) = \frac{P(D \cap A)}{P(D)} = \frac{0.78}{0.83} = 0.94.$$

(b) The probability that a plane departed on time, given that it has arrived on time, is

$$P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.95.$$

In Example 2.34, it is important to know the probability that the flight arrives on time. One is given the information that the flight did not depart on time. Armed with this additional information, one can calculate the more pertinent probability $P(A|D')$, that is, the probability that it arrives on time, given that it did not depart on time. In many situations, the conclusions drawn from observing the more important conditional probability change the picture entirely. In this example, the computation of $P(A|D')$ is

$$P(A|D') = \frac{P(A \cap D')}{P(D')} = \frac{0.82 - 0.78}{0.17} = 0.24.$$

As a result, the probability of an on-time arrival is diminished severely in the presence of the additional information.

Independent Events

- Consider an experiment in which 2 cards are drawn in succession from an ordinary deck, with replacement. The events are defined as
  - **A** : the first card is an ace,
  - **B** : the second card is a spade.
- Since the first card is replaced, our sample space for both the first and the second draw consists of 52 cards, containing 4 aces and 13 spades. Hence,

$$P(B|A) = \frac{13}{52} = \frac{1}{4} \quad \text{and} \quad P(B) = \frac{13}{52} = \frac{1}{4}.$$

That is, $$P(B|A) = P(B)$$

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- Conditional probability

The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

- **Independent event**

Two events A and B are **independent** if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

  - In other words, the occurrence of B had no impact on the occurrence of A.
  - **VERY important concept!**
    - It plays a vital role in material in virtually all chapters in this book and in all areas of applied statistics.

*Note: Consider Example 2.34. Are A and D dependent or independent?

Multiplicative Rule (or Product Rule)

Definition 2.10: The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

- **Theorem (2.10): Multiplicative Rule**

If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0.$$

  - We can also write:

$$P(A \cap B) = P(B \cap A) = P(B)P(A|B)$$

  - **Why important ?**
    - It enables us to calculate the probability that two events will both occur.

- **Example 2.36:**

  - Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

  - **Solution:**
    - A := event that the first fuse is defective
    - B := event that the second fuse is defective
    - **Our interest → A ∩ B :** the event that A occurs and B occurs after A has occurred.
    - **P(A) = 1 / 4, P(B | A) =** 4 / 19
    - **P(A ∩ B) = P(A) P(B | A) = (1/4)(4/19) = 1 / 19**

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- Example
  - Two cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $\mathbf{A_1 \cap A_2}$ occurs, where $\mathbf{A_1}$ is the event that the first card is a red ace, $\mathbf{A_2}$ is the event that the second card is a 10 or a jack.

$$P(A_1) = \frac{2}{52}, \quad P(A_2|A_1) = \frac{8}{51},$$

and hence, by Multiplicative Rule,

$$P(A_1 \cap A_2) = P(A_1)P(A_2|A_1) = \frac{2}{52} \times \frac{8}{51} = \frac{4}{663}.$$

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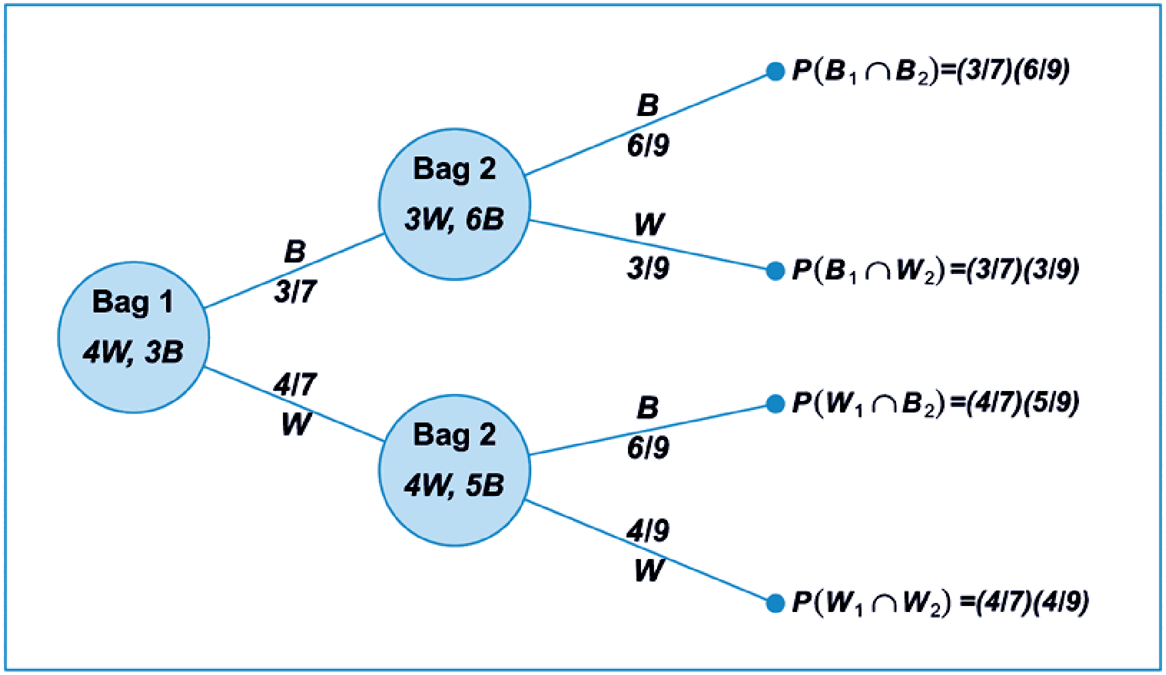
- ## Example 2.37:
  - One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?



$$\begin{aligned} P[(B_1 \cap B_2) \text{ or } (W_1 \cap B_2)] &= P(B_1 \cap B_2) + P(W_1 \cap B_2) \\ &= P(B_1)P(B_2|B_1) + P(W_1)P(B_2|W_1) \\ &= \left(\frac{3}{7}\right)\left(\frac{6}{9}\right) + \left(\frac{4}{7}\right)\left(\frac{5}{9}\right) = \frac{38}{63}. \end{aligned}$$

- **Example 2.36:**

  - Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

If, in Example 2.36, the first fuse is replaced and the fuses thoroughly rearranged before the second is removed, then the probability of a defective fuse on the second selection is still $1/4$; that is, $P(B|A) = P(B)$ and the events A and B are independent. When this is true, we can substitute $P(B)$ for $P(B|A)$ in Theorem 2.10 to obtain the following special multiplicative rule.

- **Theorem 2.11**

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

- Example 2.38
  - A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that **both the ambulance and the fire engine** will be available, assuming they operate independently.

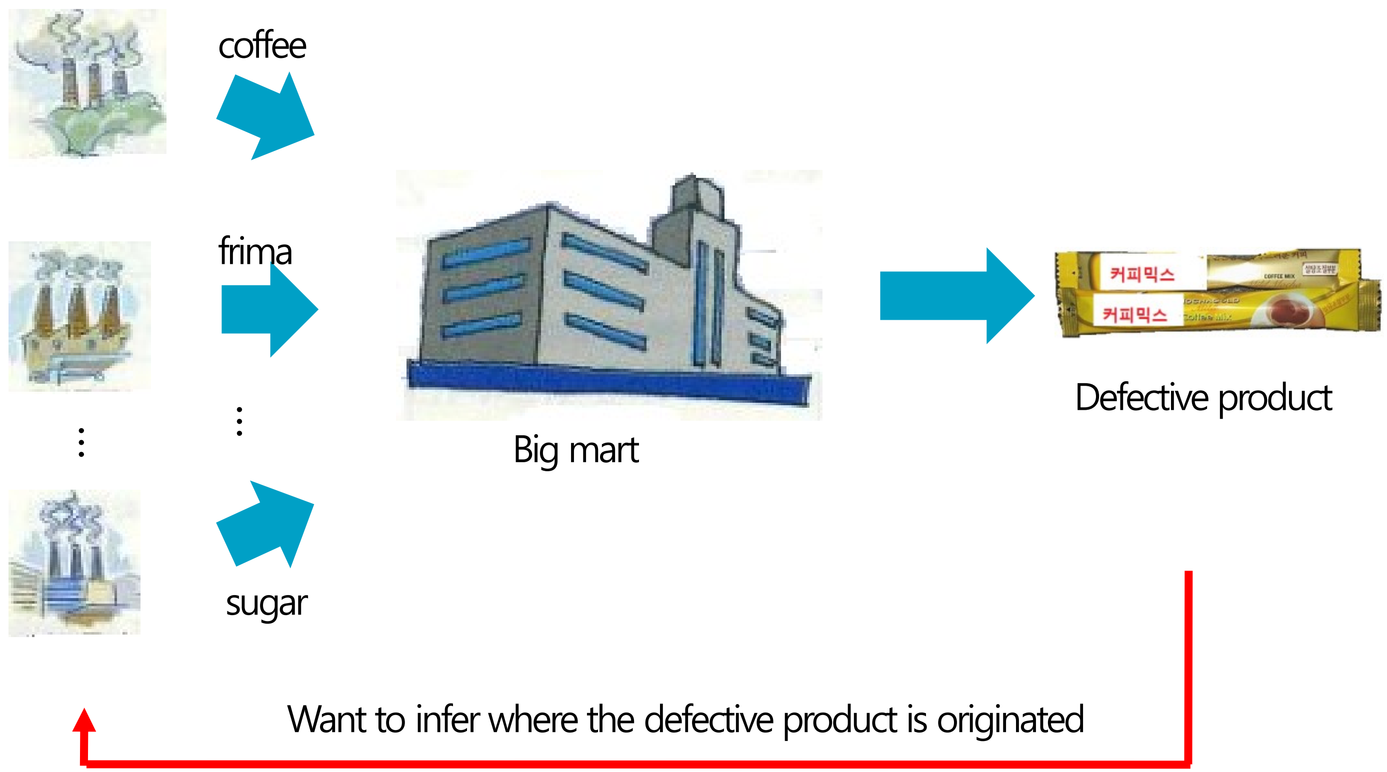
Let A and B represent the respective events that the fire engine and the ambulance are available. Then

$$P(A \cap B) = P(A)P(B) = (0.98)(0.92) = 0.9016.$$

2.6 Bayes' Formula

Bayes' formula (Rule)

- Motivation example



*Source: Probability & Statistics, Jaewon Lee, Hanbit Academy

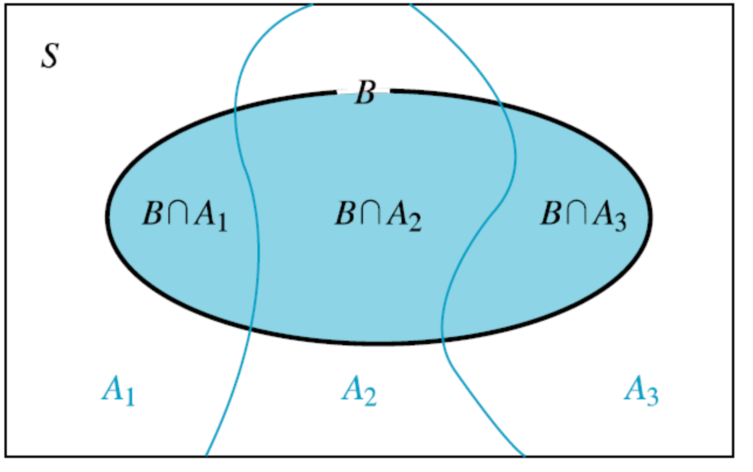
Total probability law

- Partition of sample space S
  - Three partition A_1, A_2, A_3 and $P(A_i) > 0$, i = 1, 2, 3
  - Arbitrary event B (defective)

$$
\begin{aligned}
P(B) &= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) \\
&= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)
\end{aligned}
$$

Product Rule

Theorem 2.10: If in an experiment the events A and B can both occur, then
$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0.$$

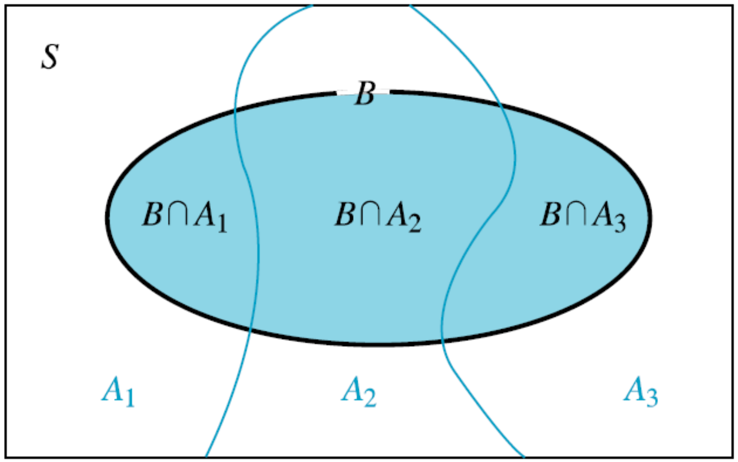


Total probability law

- General case
  - n partition $A_1, A_2 \ldots, A_n$ and $P(A_i) > 0$, $i = 1, 2, \ldots, n$
  - Arbitrary event B

Theorem (2.13):

$$\boxed{P(B) = \sum_{i=1}^{n} P(A_i) P(B \mid A_i)}$$



Example 1

- Defective product

Production lines	A	B	C	D
Production rates	20%	40%	30%	10%
Defection rates	0.04	0.02	0.01	0.05

- What is the probability that a product is defective?

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i)$$

$$\begin{aligned} P(F) &= P(A)P(F \mid A) + P(B)P(F \mid B) + P(C)P(F \mid C) + P(D)P(F \mid D) \\ &= 0.2 \times 0.04 + 0.4 \times 0.02 + 0.3 \times 0.01 + 0.1 \times 0.05 = 0.024 \end{aligned}$$

Bayes' formula

- Setting
  - n partition $A_1, A_2 \dots, A_n$ and $P(A_i) > 0$, $i = 1, 2, \dots, n$
  - $P(B) > 0$ for the event B, what is $P(A_i | B)$?

Product Rule

$$\boxed{P(A \cap B) = P(A)P(B|A)}$$

Theorem (2.14): $$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)}$$

$$\boxed{P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i)}$$

Total probability law

Posterior distribution

Likelihood

Prior distribution

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Evidence

Example 2

Production lines	A	B	C	D
Production rates	20%	40%	30%	10%
Defection rates	0.04	0.02	0.01	0.05

- Defective product
  - One defective (F) comes out.
  - Find the probabilities that it was produced on production lines A,B,C,D
  - $P(A|F)$, $P(B|F)$, $P(C|F)$, and $P(D|F)$
  - From Ex.1, $P(F) = P(A)P(F \mid A) + P(B)P(F \mid B) + P(C)P(F \mid C) + P(D)P(F \mid D)$
    $= 0.2 \times 0.04 + 0.4 \times 0.02 + 0.3 \times 0.01 + 0.1 \times 0.05 = 0.024$

$$\Rightarrow P(A \mid F) = \frac{P(A \cap F)}{P(F)} = \frac{0.2 \times 0.04}{0.024} = \frac{1}{3}$$

$$\Rightarrow P(B \mid F) = \frac{P(B \cap F)}{P(F)} = \frac{0.4 \times 0.02}{0.024} = \frac{1}{3}$$

$$\Rightarrow P(C \mid F) = \frac{P(C \cap F)}{P(F)} = \frac{0.3 \times 0.01}{0.024} = \frac{1}{8}$$

$$\Rightarrow P(D \mid F) = \frac{P(D \cap F)}{P(F)} = \frac{0.1 \times 0.05}{0.024} = \frac{5}{24}$$

$$\boxed{P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)}}$$

Bayes' formula

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Question

- Suppose that Bob can decide to go to school by one of three modes of transportation:
  - car, bus, or subway. (select one of them equally with **1/3.**)
- The probability of being late is :
  - if he decides to go by car, there is a 50% chance he will be late.
  - If he goes by bus, the probability of being late is only 20%.
  - the subway is almost never late, with a probability of only 1%, but is more expensive than the bus.
- Questions:
  - Suppose that **Bob** is late one day, we want to calculate the probability that he drove to work that day by car.

• Solution

- Suppose that Bob can decide to go to school by one of three modes of transportation:
  - car, bus, or subway. (select one of them equally with **1/3.**)
- The probability of being late is :
  - if he decides to go by car, there is a 50% chance he will be late.
  - If he goes by bus, the probability of being late is only 20%.
  - the subway is almost never late, with a probability of only 1%, but is more expensive than the bus.
- Questions:
  - Suppose that Bob is late one day, we want to calculate the probability that he drove to work that day by car.

$$\boxed{P(A_i|B)=\frac{P(A_i\cap B)}{P(B)}=\frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n}P(A_j)P(B|A_j)}}$$

- P(car | late) = ?

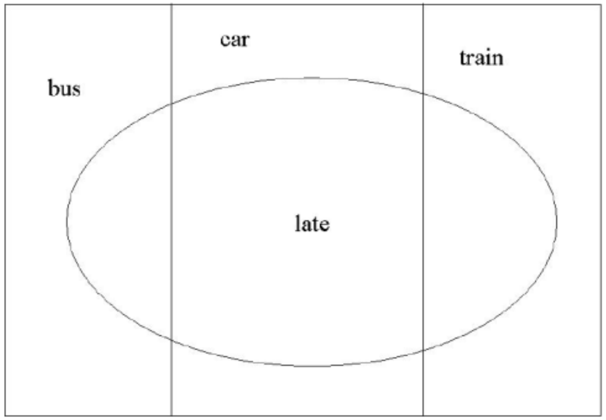
= { P(car) P(late | car) } / P(late)

$$= \frac{0.5\times 1/3}{0.5\times 1/3+0.2\times 1/3+0.01\times 1/3}$$

$$= 0.7042$$

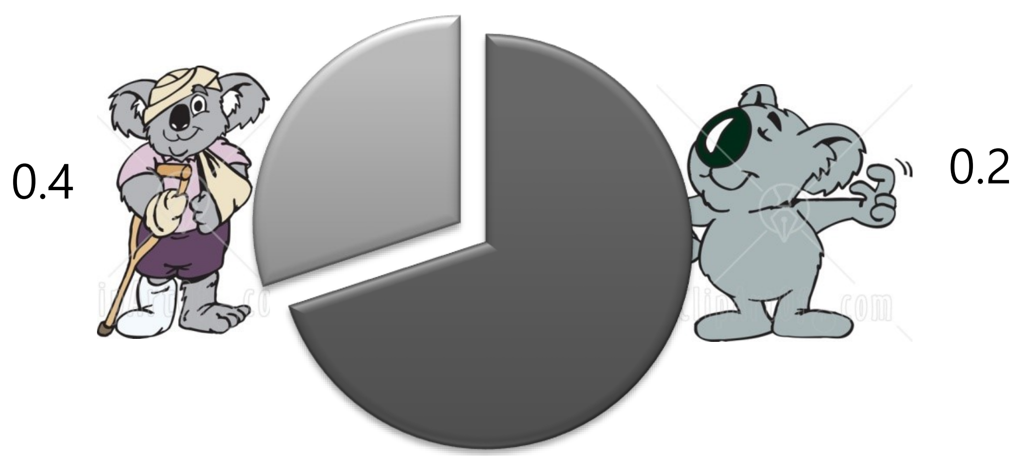
- P(bus) = P(car) = P(subway) = 1/3

- P(late | car) = 0.5
- P(late | bus) = 0.2
- P(late | subway) = 0.01



Example

- An insurance company divides people into two classes those that are accident prone and those that are not.
  - An accident-prone person will have an accident within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a non-accident-prone person.
  - Q1 : If we assume that 30 percent of population is accident prone, what is the probability that a new policy holder will have an accident within a year?
  - Q2: Suppose that a new policy holder has an accident within a year of purchasing his policy. What is the probability that he is accident prone?



- Q1 : If we assume that 30 percent of population is accident prone, what is the probability that a new policy holder will have an accident within a year?

$$P(A_1) = P(A_1|A)P(A) + P(A_1|A^c)P(A^c)$$
$$= (.4)(.3) + (.2)(.7) = .26 \quad \blacksquare$$

$$\boxed{P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i)}$$

- Q2: Suppose that a new policy holder has an accident within a year of purchasing his policy. What is the probability that he is accident prone?

$$
\begin{aligned}
P(A|A_1) &= \frac{P(AA_1)}{P(A_1)} \\
&= \frac{P(A)P(A_1|A)}{P(A_1)} \\
&= \frac{(.3)(.4)}{.26} = \frac{6}{13} = .4615
\end{aligned}
$$

$$\boxed{P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)}}$$

https://www.psycom.net/bipolar-questions-answers