Chapter 3

Random Variables and Probability Distributions – part 1

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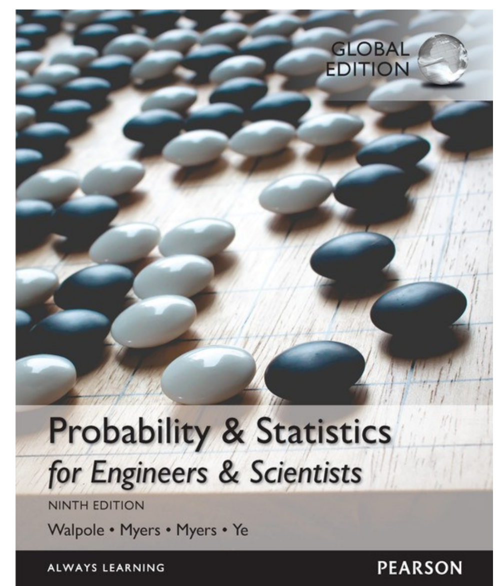

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Outline

- **Concept of a Random Variable**
- **Discrete Probability Distributions**
- Continuous Probability Distributions
- Joint Probability Distribution

3.1 Concept of a Random Variable

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- **Outcome of an experiment**
  - Different experiment may yield a different type of experimental outcome;

- **Description of each possible outcome**
  - is determined by the experiment and/or its purpose
  - **Examples:**
    - Roll a Dice twice :
      - S = {11, 12, 13, … , 16, 21, … 66}
    - Examine three electronic components (Defective or Non-defective)
      - $S = \{NNN, NND, NDN, DNN, NDD, DND, DDN, DDD\}$

- **Example**
  - Examine three electronic components (Defective or Non-defective)

$$S = \{NNN, NND, NDN, DNN, NDD, DND, DDN, DDD\},$$

  - In many cases, it is very useful to assign a numerical value to each possible outcome.
    - Here, the number of defectives that occur
  - Thus, each point in the sample space will be *assigned a numerical value* of 0, 1, 2, or 3.

$$S = \{NNN, NND, NDN, DNN, NDD, DND, DDN, DDD\},$$

$$X = \quad 0 \quad 1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2 \quad 3$$

    - NOTE !!: These values are **random quantities** determined by the outcome of the experiment.

Random Variable (RV)

- A **random variable** is a numerical description of the outcome of an experiment. Each experimental outcome gets assigned a numerical value.

Definition 3.1: A **random variable** is a function that associates a real number with each element in the sample space.

We shall use a capital letter, say X, to denote a random variable and its corresponding small letter, x in this case, for one of its values. In the electronic component testing illustration above, we notice that the random variable X assumes the value 2 for all elements in the subset

$$E = \{DDN, DND, NDD\}$$

of the sample space S. That is, each possible value of X represents an event that is a subset of the sample space for the given experiment.

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- Example:
  - Roll a dice.
  - Suppose X = i if the outcome of the throw is number i. That is, X = 1 represents the "1" showing up. Then X is a random variable.

- Example 3.1:
  - Two balls are drawn in succession without replacement from an urn containing 4 red balls and 3 black balls. The possible outcomes and the values *y* of the random variable *Y* , where *Y* is the number of red balls, are

Sample Space	y
RR	2
RB	1
BR	1
BB	0

- Example:
  - Roll a dice.
  - Suppose X = 1 if the outcome of the throw is an even number, i.e., one of = {2, 4, 6}; and X = 0 otherwise.
  - Then **X is a random variable**.

- Example 3.3:
  - Consider the simple condition in which components are arriving from the production line and they are stipulated to be defective or not defective.
  - We can define the random variable X by

$$X = \begin{cases} 1, & \text{if the component is defective,} \\ 0, & \text{if the component is not defective.} \end{cases}$$

- Example 3.5 :
  - Suppose a sampling plan involves sampling items from a process until a defective is observed. The evaluation of the process will depend on how many consecutive items are observed.
  - In that regard, **let X be a random variable** defined by the number of items observed before a defective is found.
    - With *N* a non-defective and *D* a defective, sample spaces are S = **{***D* **}** given X = 1, S = **{***ND***}** given X = 2, S = **{**NND**}** given X = 3, and so on.

- Example 3.7:
  - Let X be the random variable defined by the waiting time, in hours, between successive speeders spotted by a radar unit.
  - The random variable X takes on all values x for which x ≥ 0.

Discrete and Continuous sample space

- **Discrete Sample Space**

If a sample space contains a finite number of possibilities or an unending sequence with as many elements as there are whole numbers, it is called a **discrete sample space**.

- **Continuous Sample Space**

If a sample space contains an infinite number of possibilities equal to the number of points on a line segment, it is called a **continuous sample space**.

Discrete and Continuous r. v.

- **Discrete Random Variable**
  - A random variable is called a **discrete random variable** if its set of possible outcomes is countable.
- **Continuous Random Variable**
  - When a random variable can take on values on a continuous scale, it is called a **continuous random variable**.
- Are the previous examples discrete or continuous RVs?

3.2 **Discrete** Probability Distributions

Probability mass function (pmf)*

- Example
  - In the case of tossing a coin twice, the random variable X, represents the number of heads.
  - The possible value x of X and their probabilities are

x	0	1	2
$f(x) = P(X = x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Note that the values of m exhaust all possible cases and hence the probabilities add to 1.

*확률질량함수

Frequently, it is convenient to represent all the probabilities of a random variable X by a formula. Such a formula would necessarily be a function of the numerical values x that we shall denote by $f(x)$, $g(x)$, $r(x)$, and so forth. Therefore, we write $f(x) = P(X = x)$; that is, $f(3) = P(X = 3)$. The set of ordered pairs $(x, f(x))$ is called the **probability function**, **probability mass function**, or **probability distribution** of the discrete random variable X.

- **Definition 3.4:**

The set of ordered pairs $(x, f(x))$ is a **probability function**, **probability mass function**, or **probability distribution** of the discrete random variable X if, for each possible outcome x,

1. $f(x) \geq 0$,
2. $\sum\limits_x f(x) = 1$,
3. $P(X = x) = f(x)$.

- Example 3.8
  - A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

- **Solution:**
  - Let *X* be a random variable whose values *x* are the possible numbers of defective computers purchased by the school, where x can be 0, 1, or 2.

$$f(0) = P(X=0) = \frac{\binom{3}{0}\binom{17}{2}}{\binom{20}{2}} = \frac{68}{95} \qquad f(1) = P(X=1) = \frac{\binom{3}{1}\binom{17}{1}}{\binom{20}{2}} = \frac{51}{190} \qquad f(2) = P(X=2) = \frac{\binom{3}{2}\binom{17}{0}}{\binom{20}{2}} = \frac{3}{190}$$

Thus, the probability distribution of X is

x	0	1	2
$f(x)$	$\frac{68}{95}$	$\frac{51}{190}$	$\frac{3}{190}$

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Next…

- **There are many problems that want to**
  - compute the probability that the observed value of a random variable X will be **less than** or **equal to** some real number x.

Cumulative Distribution Function

Cumulative Distribution Function (c.d.f.)*

- Cumulative Distribution Function (Discrete)
  - The **cumulative distribution function (c.d.f.)** ***F(x)*** of a discrete random variable X with probability distribution ***f(x)*** is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \quad \text{for} \quad -\infty < x < \infty.$$

*누적분포함수

C. D. F.

- Example
  - In the case of tossing a coin twice, the random variable X, represents the number of heads.
  - The possible value x of X and their probabilities are

x	0	1	2
$f(x) = P(X = x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

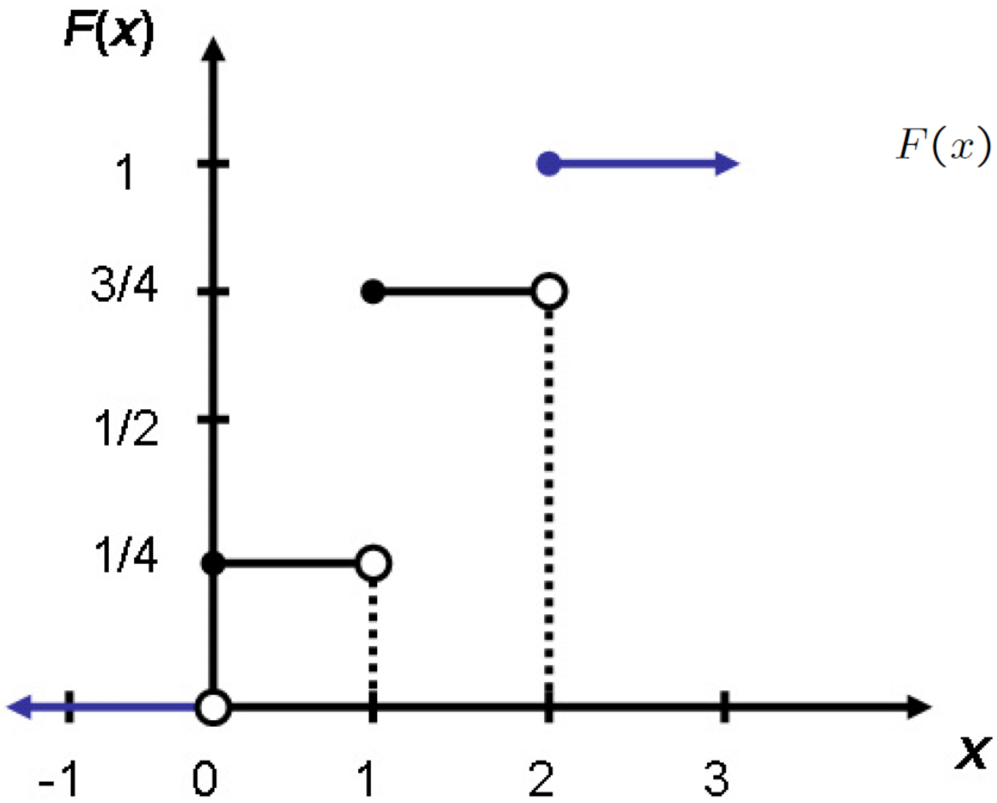
- **Find the cumulative distribution function of the random variable X :**

$$F(0) = P(X \le 0) = f(0) = \frac{1}{4},$$

$$F(1) = P(X \le 1) = f(0) + f(1) = \frac{1}{4} + \frac{2}{4} = \frac{3}{4},$$

$$F(2) = P(X \le 2) = f(0) + f(1) + f(2) = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1.$$

$$F(x) = \begin{cases} 0, & \text{for } x < 0; \\ \frac{1}{4}, & \text{for } 0 \le x < 1; \\ \frac{3}{4}, & \text{for } 1 \le x < 2; \\ 1, & \text{for } x \ge 2. \end{cases}$$



$$F(x) = \begin{cases} 0, & \text{for } x < 0; \\ \frac{1}{4}, & \text{for } 0 \le x < 1; \\ \frac{3}{4}, & \text{for } 1 \le x < 2; \\ 1, & \text{for } x \ge 2. \end{cases}$$

- **Example 3.10:**
  - Find the cumulative distribution function of the random variable X whose function is given follow:

$$f(x) = \frac{1}{16}\binom{4}{x}, \quad \text{for } x = 0, 1, 2, 3, 4.$$

***Solution*:** Direct calculations of the probability distribution of Example 3.9 give $f(0) = 1/16$, $f(1) = 1/4$, $f(2) = 3/8$, $f(3) = 1/4$, and $f(4) = 1/16$. Therefore,

$$\begin{aligned}
F(0) &= f(0) = \frac{1}{16},\\
F(1) &= f(0) + f(1) = \frac{5}{16},\\
F(2) &= f(0) + f(1) + f(2) = \frac{11}{16},\\
F(3) &= f(0) + f(1) + f(2) + f(3) = \frac{15}{16},\\
F(4) &= f(0) + f(1) + f(2) + f(3) + f(4) = 1.
\end{aligned}$$

Hence,

$$F(x) = \begin{cases} 0, & \text{for } x < 0,\\ \frac{1}{16}, & \text{for } 0 \le x < 1,\\ \frac{5}{16}, & \text{for } 1 \le x < 2,\\ \frac{11}{16}, & \text{for } 2 \le x < 3,\\ \frac{15}{16}, & \text{for } 3 \le x < 4,\\ 1 & \text{for } x \ge 4. \end{cases}$$

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- **Example 3.10:**
  - Find the cumulative distribution function of the random variable X whose function is given follow:

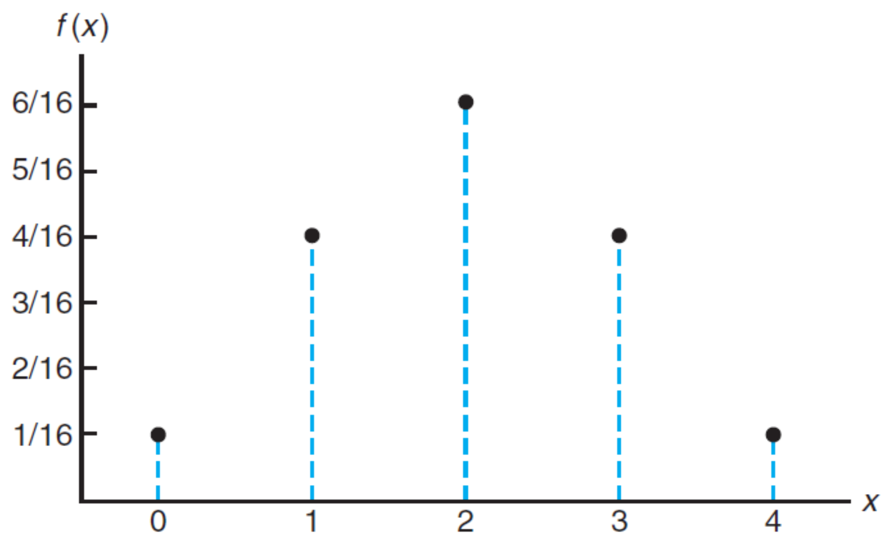
$$f(x) = \frac{1}{16}\binom{4}{x}, \quad \text{for } x = 0, 1, 2, 3, 4.$$

  - Using $F(x)$, verify that $f(2) = 3/8$.

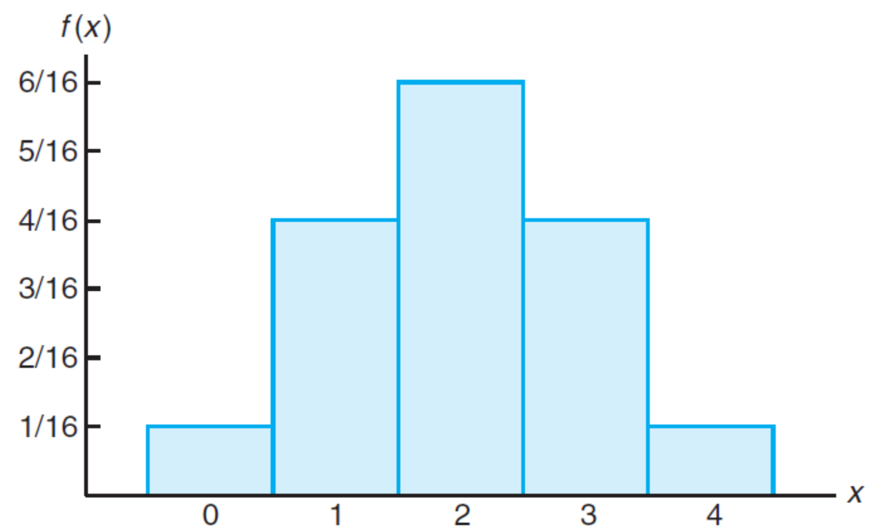
$$F(1) = f(0) + f(1) = \frac{5}{16},$$
$$F(2) = f(0) + f(1) + f(2) = \frac{11}{16},$$

$$f(2) = F(2) - F(1) = \frac{11}{16} - \frac{5}{16} = \frac{3}{8}.$$

Plotting Ex. 3.10



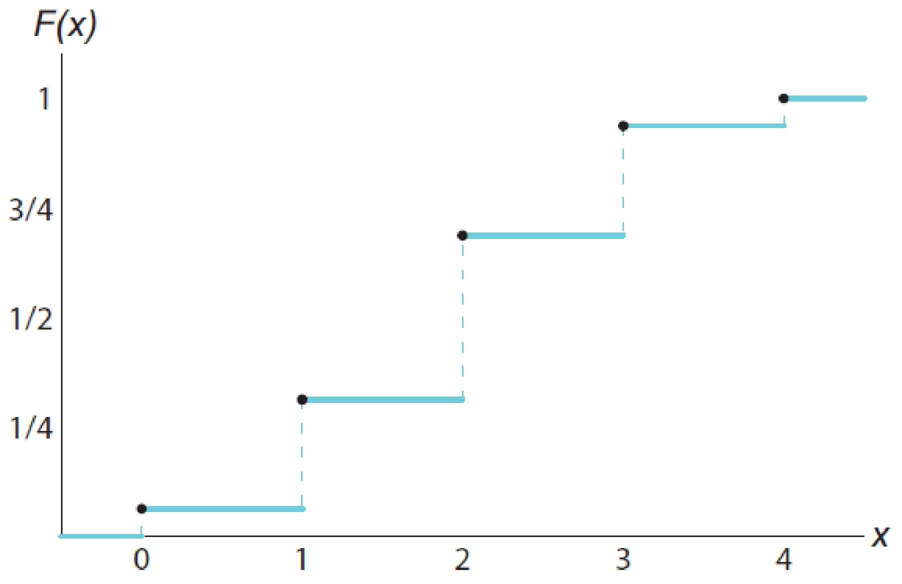
Probability mass function plot.
(Bar chart)



Probability histogram.

Direct calculations of the probability distribution of Example 3.9 give $f(0) = 1/16$, $f(1) = 1/4$, $f(2) = 3/8$, $f(3) = 1/4$, and $f(4) = 1/16$. Therefore,

Cont.



Discrete cumulative distribution function (C.D.F)

$$F(0) = f(0) = \frac{1}{16},$$
$$F(1) = f(0) + f(1) = \frac{5}{16},$$
$$F(2) = f(0) + f(1) + f(2) = \frac{11}{16},$$
$$F(3) = f(0) + f(1) + f(2) + f(3) = \frac{15}{16},$$
$$F(4) = f(0) + f(1) + f(2) + f(3) + f(4) = 1.$$

TRY OUT: Find probability using CDF

- Example
  - The cumulative distribution function of X is given by

$$F(x) = \begin{cases} 0, & x < 1; \\ \frac{1}{5}, & 1 \leq x < 5; \\ \frac{1}{3}, & 5 \leq x < 7; \\ \frac{3}{5}, & 7 \leq x < 10; \\ 1, & x \geq 10. \end{cases}$$

  - **Find**
    - 1. P(X=6)
    - 2. P(X>7)
    - 3. P(2.5 < X < 9.2)

- Solution

$$f(x) = P(X = x) = \begin{cases} \frac{1}{5}, & \text{if } x = 1; \\ \frac{2}{15}, & \text{if } x = 5; \\ \frac{4}{15}, & \text{if } x = 7; \\ \frac{2}{5}, & \text{if } x = 10. \end{cases}$$

1. **P(X=6)**
2. **P(X>7)**
3. **P(2.5 < X < 9.2)**

1. $P(X = 6) = 0.$

2. $P(X > 7) = 1 - P(X \leq 7) = 1 - F(7) = \frac{2}{5}.$

3. $P(2.5 < X < 9.2) = P(X = 5) + P(X = 7) = \frac{2}{5}.$

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