Chapter 6

Some Continuous Probability Distributions – part 1

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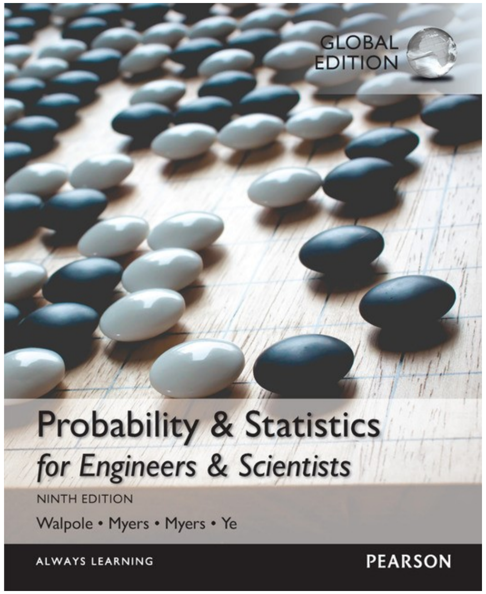

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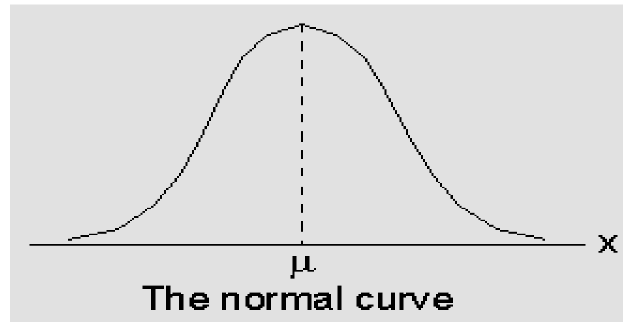
Continuous Probability Distributions

- Many continuous probability distributions, including:
  - Uniform
  - Normal
  - Gamma
  - Exponential
  - Chi-Squared
  - Lognormal
  - Weibull



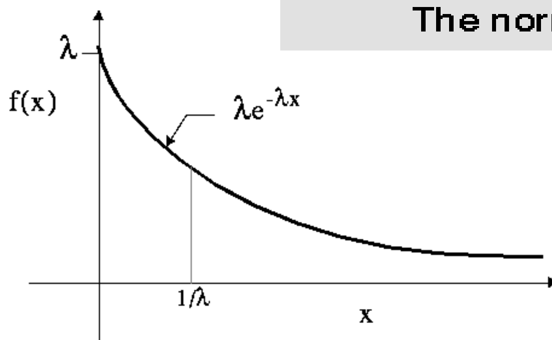
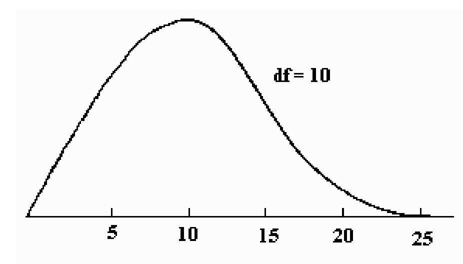
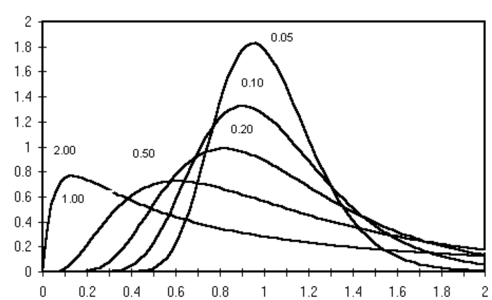
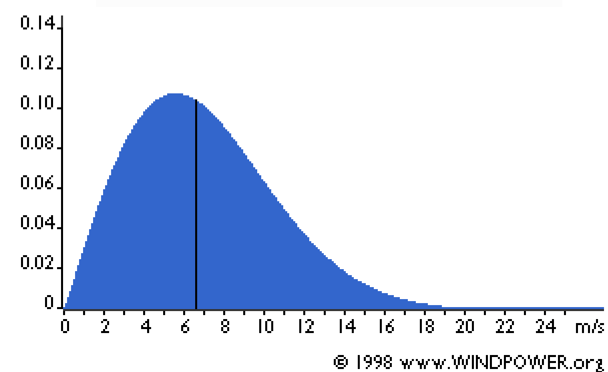


Figure 6. Exponential *pdf*







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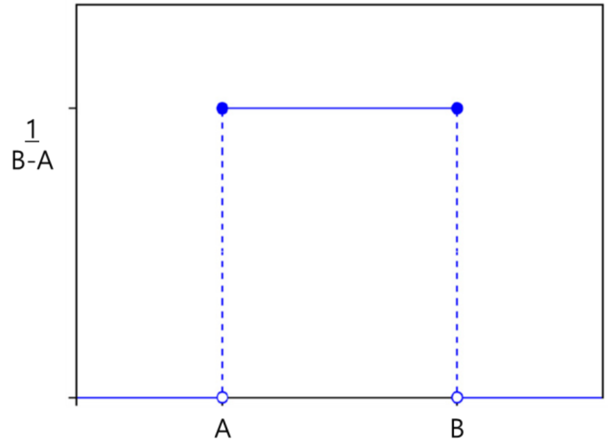
6.1 Continuous Uniform Distribution

Continuous Uniform Distribution*

- Simplest:
  characterized by the interval endpoints, *A* and *B.*
- The density function of the continuous uniform random variable X on the interval [A, B] is

$$f(x; A, B) = \begin{cases} \frac{1}{B-A}, & A \le x \le B, \\ 0, & \text{elsewhere.} \end{cases}$$



Probability density function for a continuous uniform random variable.

*균일 분포

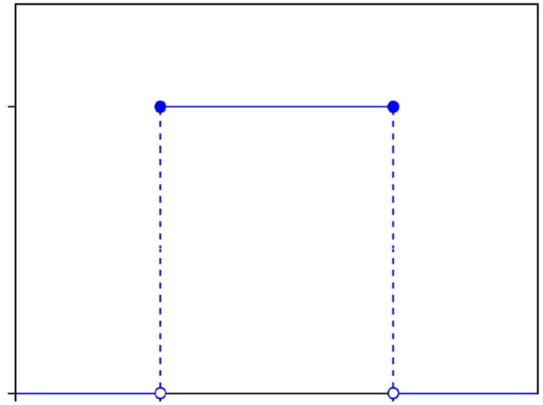
- Example 6.1
  - Suppose that a large conference room at a certain company can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length X of a conference has a uniform distribution on the interval [0, 4].
  - **(a) What is the probability density function?**
  - **(b) What is the probability that any given conference lasts at least 3 hours?**

- Solution
  - (a) The appropriate density function for the uniformly distributed random variable X in this situation is

$$f(x) = \begin{cases} \frac{1}{4}, & 0 \le x \le 4, \\ 0, & \text{elsewhere.} \end{cases}$$

$$\boxed{f(x; A, B) = \begin{cases} \frac{1}{B-A}, & A \le x \le B, \\ 0, & \text{elsewhere.} \end{cases}}$$

  - (b) $P[X \ge 3] = \int_3^4 \frac{1}{4}\, dx = \frac{1}{4}.$



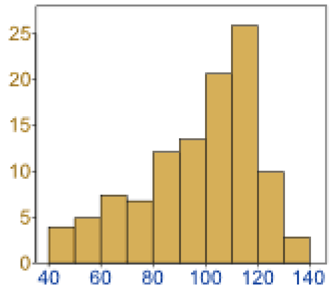
Continuous Uniform Distribution

- Theorem 6.1:
  - The mean and variance of the uniform distribution are

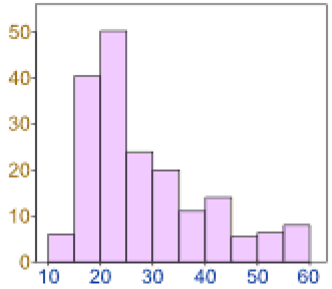
$$\mu = \frac{A+B}{2} \text{ and } \sigma^2 = \frac{(B-A)^2}{12}.$$

Proof: Exercise 6.1

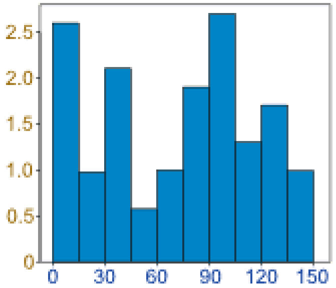
- Data can be "distributed" (spread out) in different ways.



It can be spread out more on the left



Or more on the right



Or it can be all jumbled up

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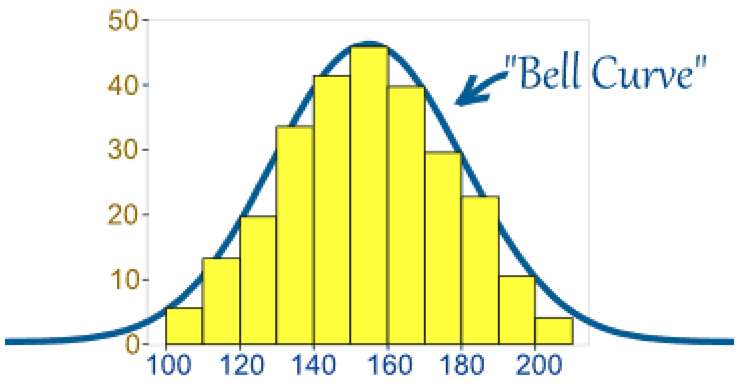
But, in Normal cases

- There are many cases where the data tends to be around a central value (=mean) with no bias left or right



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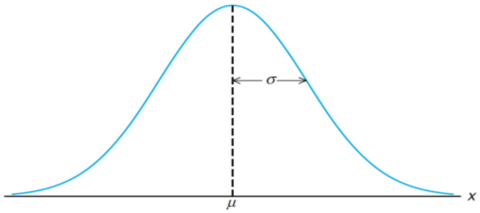
6.2 Normal Distribution

Normal Distribution*

- A symmetric "*bell-shaped curve*"
- Also called the ***Gaussian* distribution**
- The most widely used, important distribution in statistical analysis
  - describes or approximates most phenomena in nature, industry, or research
  - forms the basis for most of the parametric tests we'll perform later in this course.

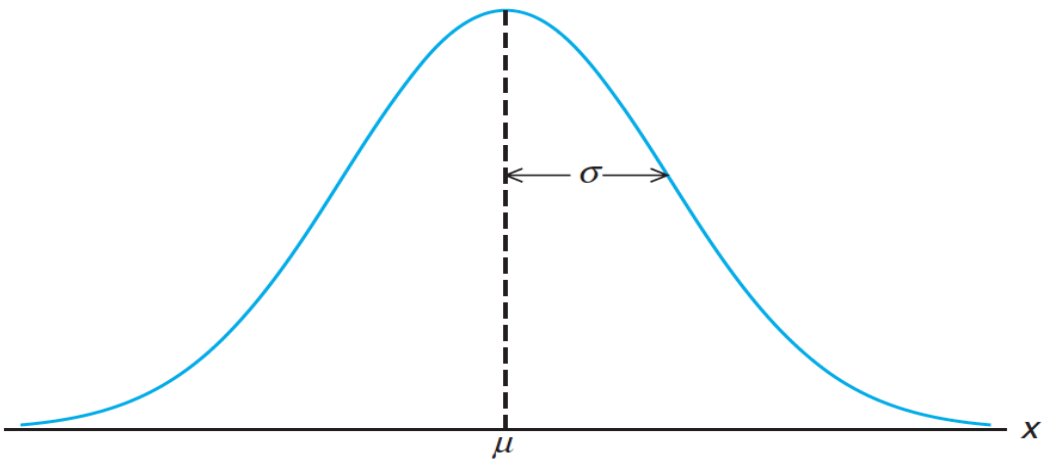


*정규분포

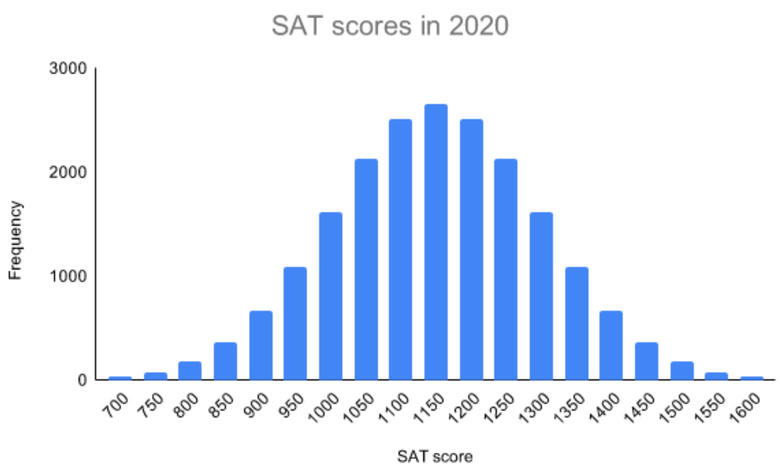
Example

- Physical measurements in areas such as meteorological experiments, rainfall studies, and measurements of manufactured parts are often more than adequately explained with a normal distribution.

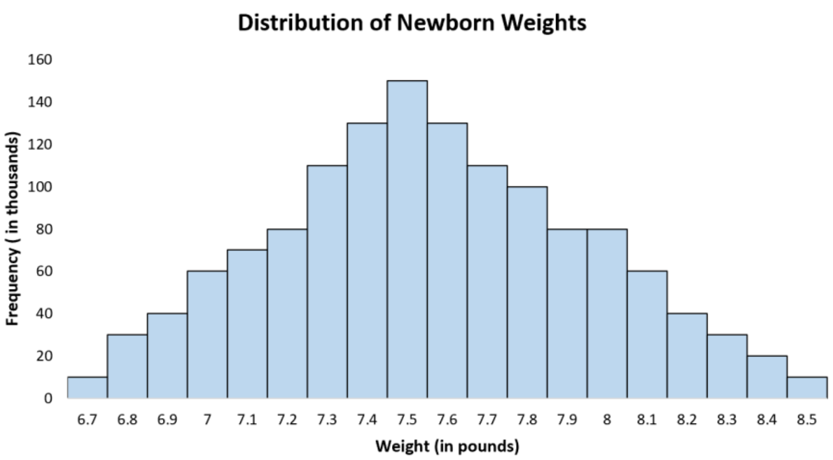
- Errors in scientific measurements are extremely well approximated by a normal distribution.



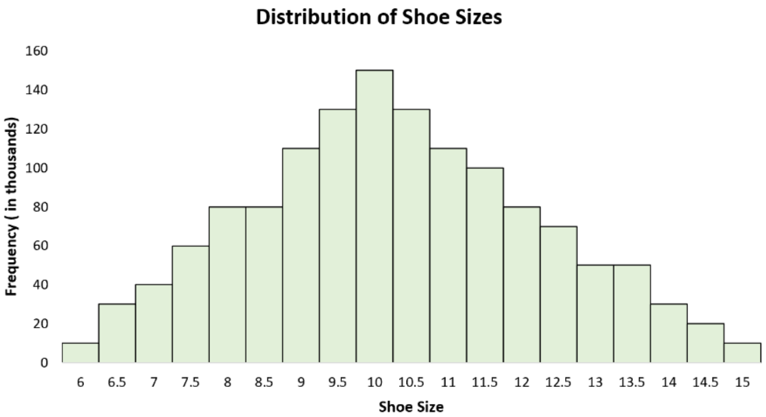
Normal Distribution Examples



https://www.scribbr.com/statistics/normal-distribution/



https://www.statology.org/example-of-normal-distribution/



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Normal Distribution

- Random variables (***X***) following this distribution are called ***normal random variables***.
  - the parameters of the normal distribution are **μ and σ** (sometimes μ and σ^2.)

Normal Distribution The density of the normal random variable X, with mean μ and variance σ^2, is

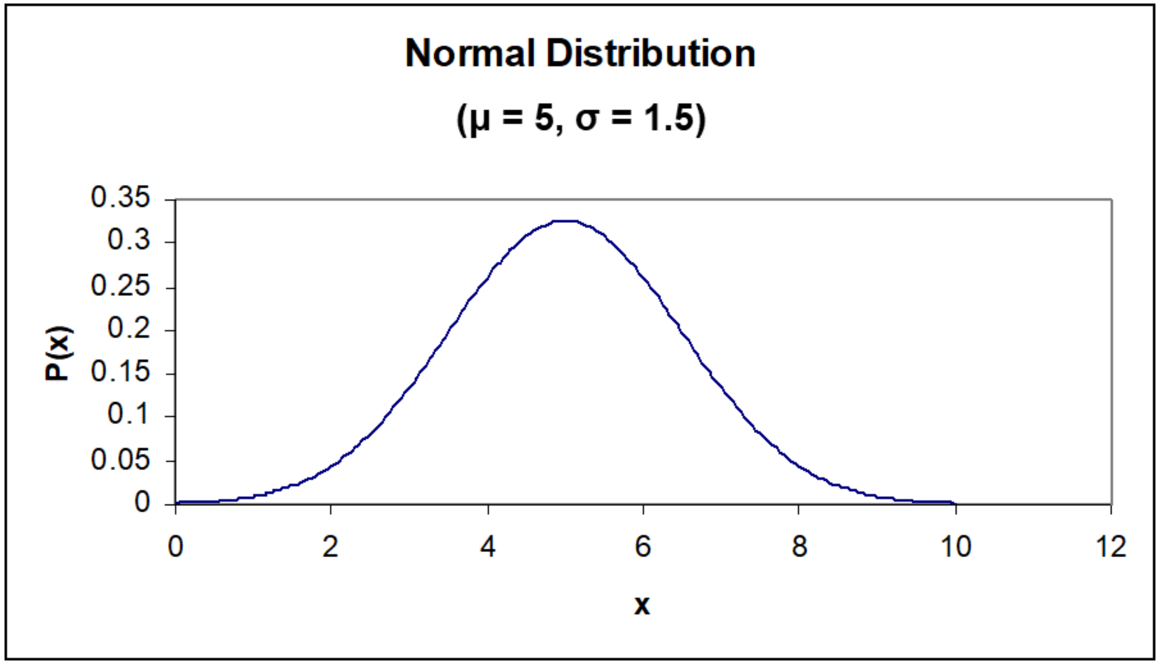
$$n(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \qquad -\infty < x < \infty,$$

where $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$.

So, !!

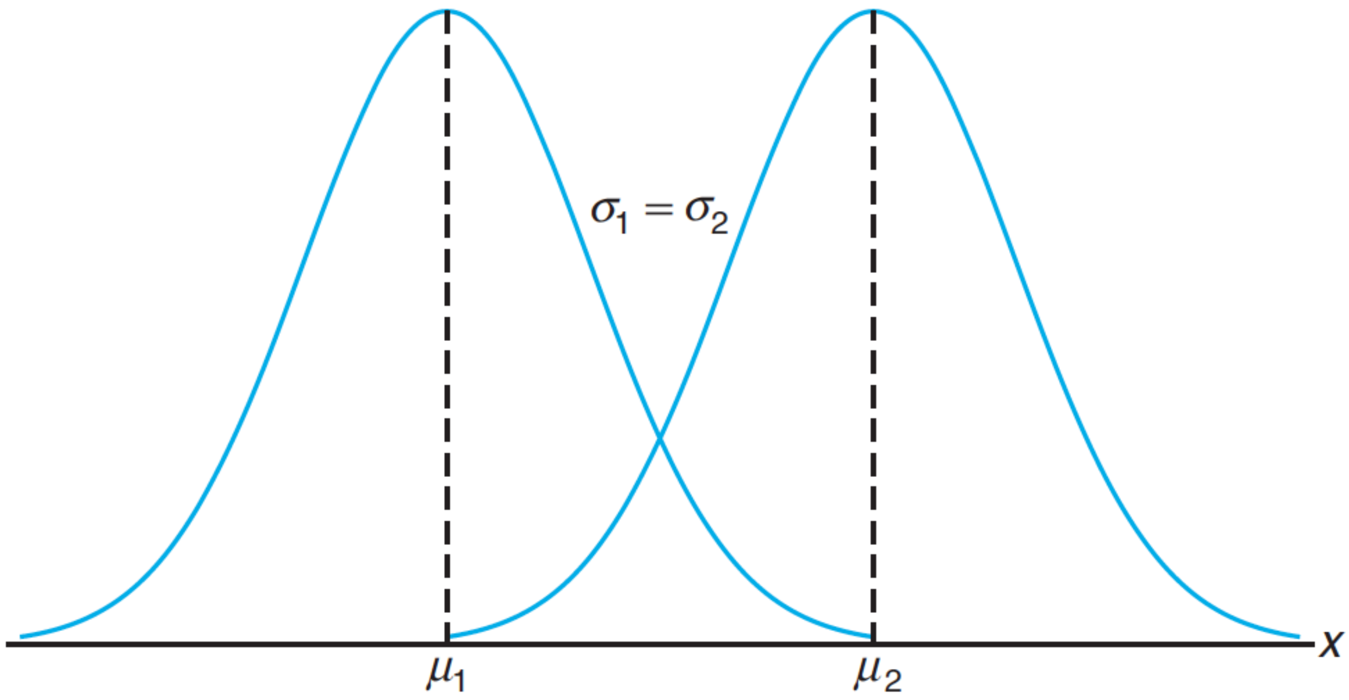
Once μ and σ are specified, the normal curve is completely determined.

- (μ = 5, σ = 1.5)



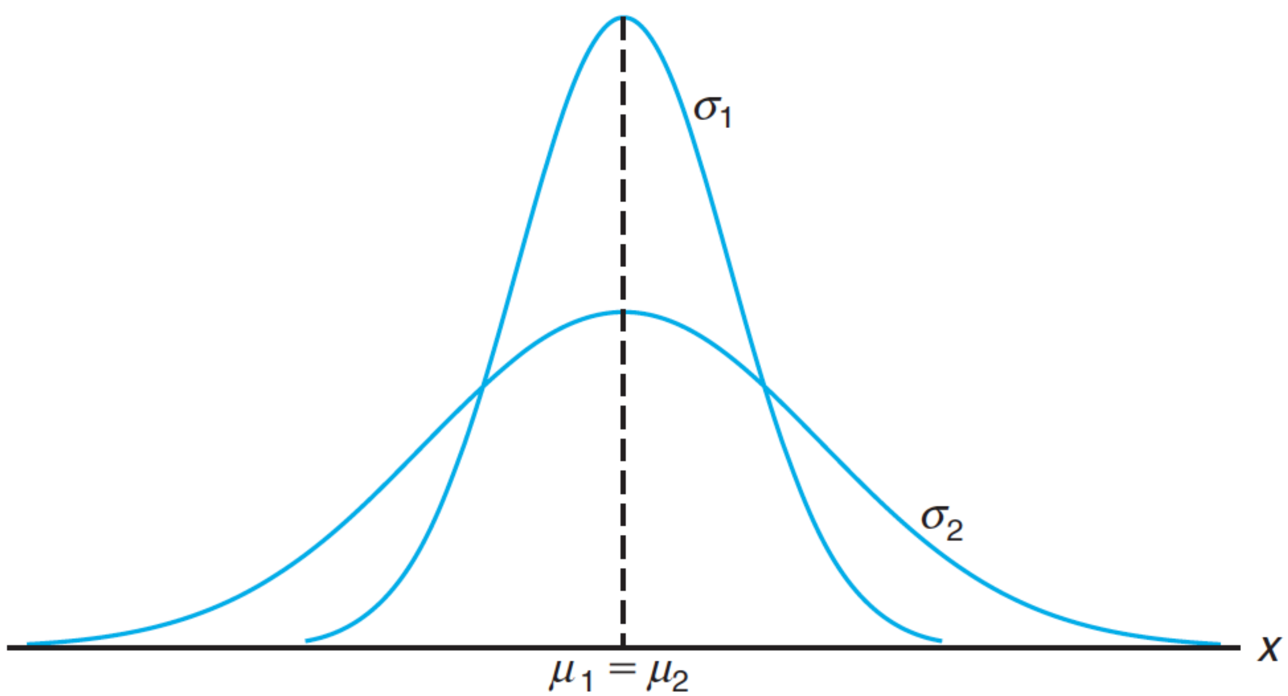
Example

- Normal curves with $\boldsymbol{\mu}_1 < \boldsymbol{\mu}_2$ and $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2$.



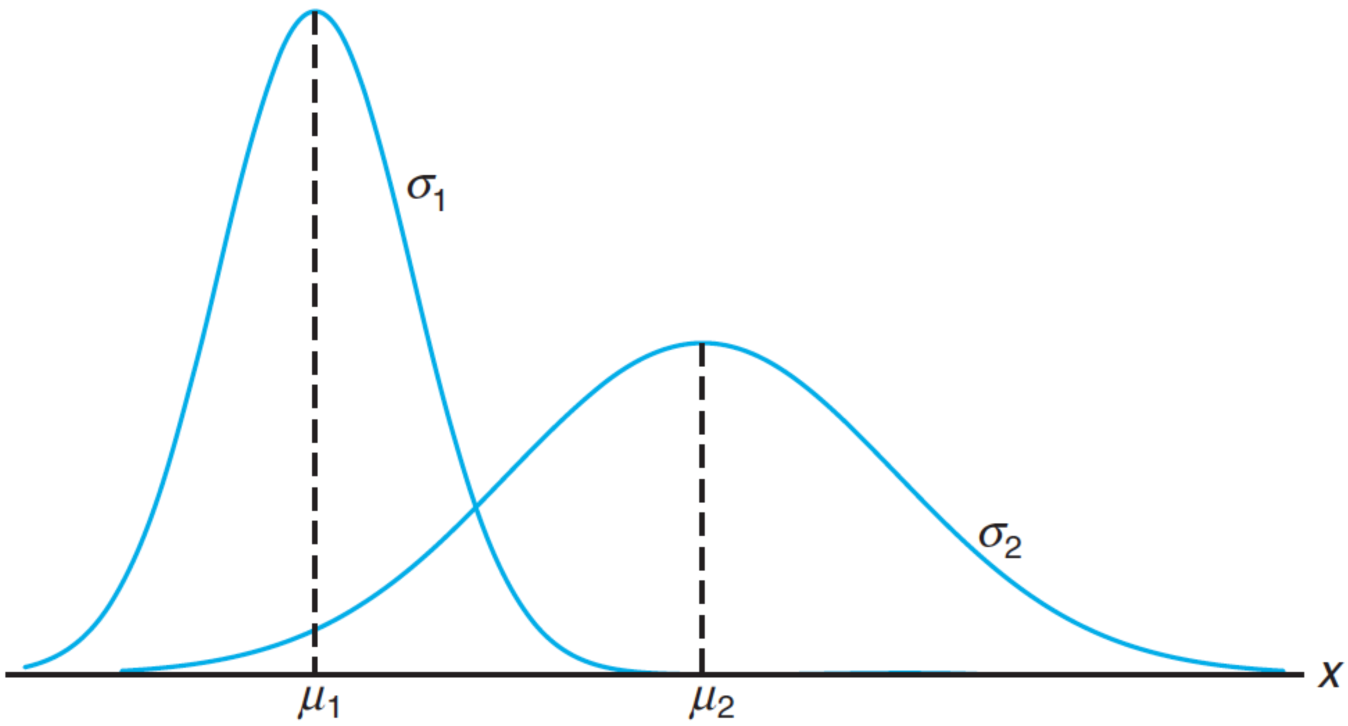
Example

- Normal curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$.



Example

- Normal curves with $\boldsymbol{\mu}_1 < \boldsymbol{\mu}_2$ and $\boldsymbol{\sigma}_1 < \boldsymbol{\sigma}_2$.



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Mean and **Variance***

- The mean and variance of **n(x ; μ, σ**) are **μ** and **σ²**, respectively. Hence, the standard deviation is **σ**.

***Proof*:** To evaluate the mean, we first calculate

$$E(X-\mu)=\int_{-\infty}^{\infty}\frac{x-\mu}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx.$$

Setting $z=(x-\mu)/\sigma$ and $dx=\sigma\ dz$, we obtain

$$E(X-\mu)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}ze^{-\frac{1}{2}z^2}dz=0,$$

since the integrand above is an odd function of z. Using Theorem 4.5 we conclude that

$$\boxed{E(aX+b)=aE(X)+b.}$$

$$E(X)=\mu.$$

*Proof of variance in Appendix

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Question 1

- A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that the battery lives are normally distributed, find the probability that a given battery will last less than 2.3 years.

Question 2

- The average grade for a midterm exam is 74, and the standard deviation is 7. If 25% of the class is given A's, and the grades are curved to follow a normal distribution what is the lowest possible A and the highest possible B?

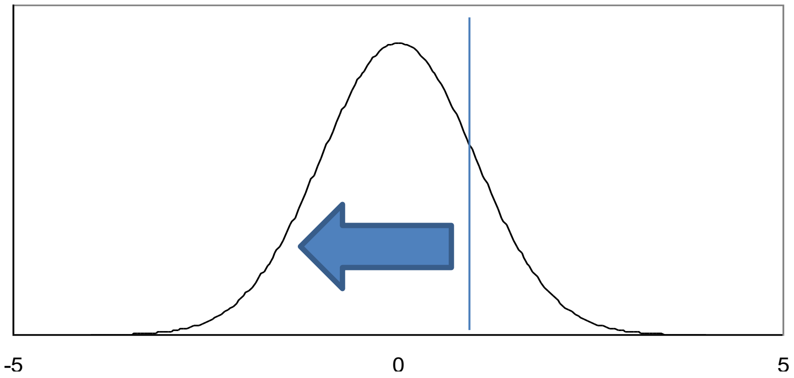
You should know ..

$$P(X \leq x)$$

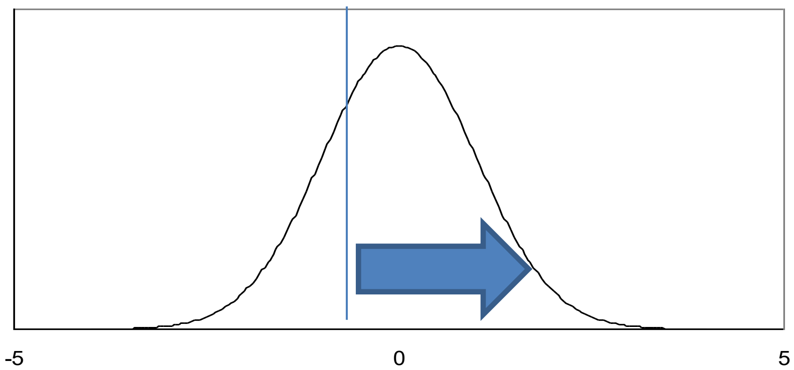
- What does this mean in the curve ?
- How to calculate ?

Examples (μ=0)

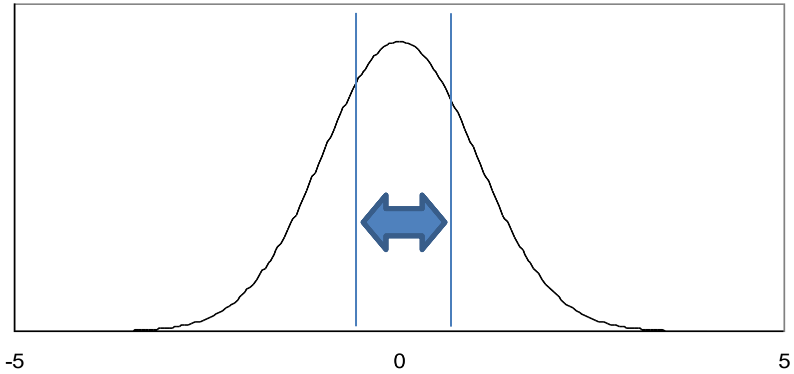
- $P(X \leq 1) =$



- $P(X \geq -1) =$



- $P(-0.45 \leq X \leq 0.36) =$



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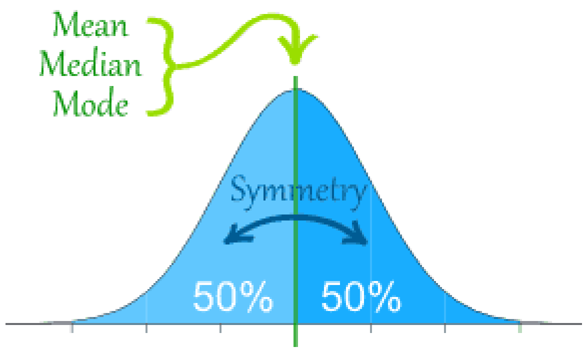
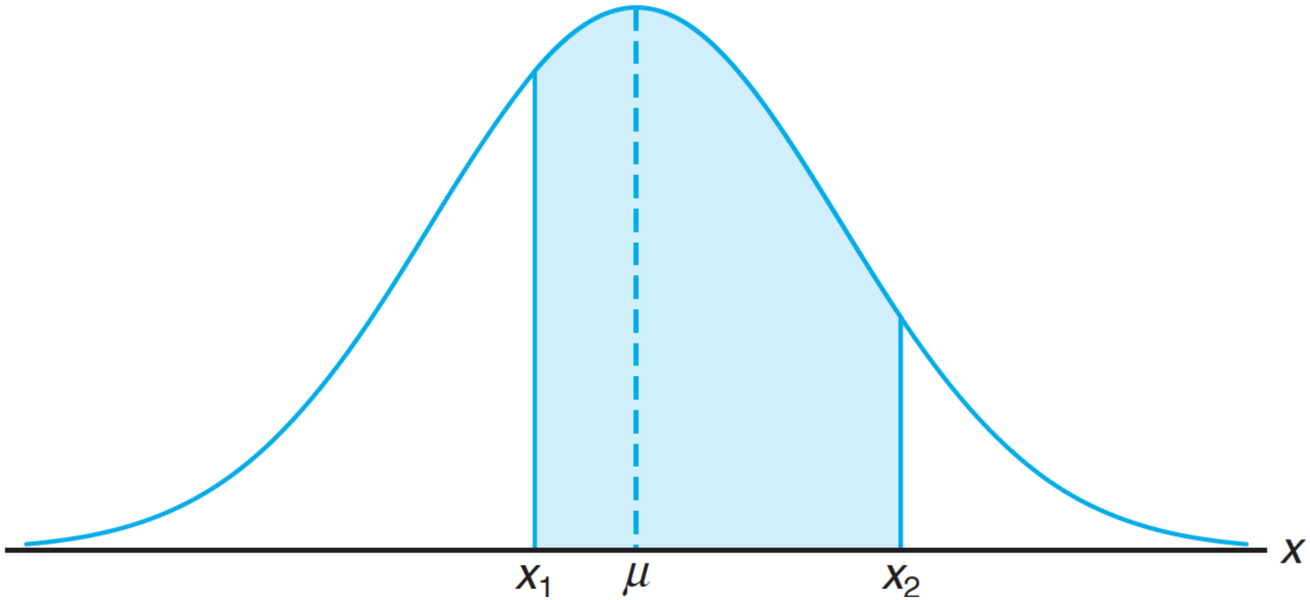
Remember Key Feature : Normal dist. is **Symmetric**!



- The **Normal Distribution** has:
- mean = median = mode
- symmetry about the center
- 50% of values less than the mean and 50% greater than the mean

6.3. Areas under the Normal Curve

$\mathbf{P}(x_1 < X < x_2)$



- **$P(x_1 < X < x_2)$ = area of the shaded region**

Areas Under the Normal Curve

- The curve of any continuous probability distribution or density function is constructed so that the area under the curve bounded by the two ordinates $x = x_1$ and $x = x_2$ equals the probability that the random variable X assumes a value between x_1 and x_2.
- Thus, for the normal curve

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

- is represented by the area of the shaded region.

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How to compute ?

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

- Use the reference Table (A.3)

Tables for all μ and σ ?

- The difficulty encountered in solving integrals of normal density functions necessitates the tabulation of normal curve areas for quick reference. (e.g., like Tables A.1, A.2, …)
- However, it would be a hopeless task to attempt to set up separate tables for every conceivable values of μ and σ.

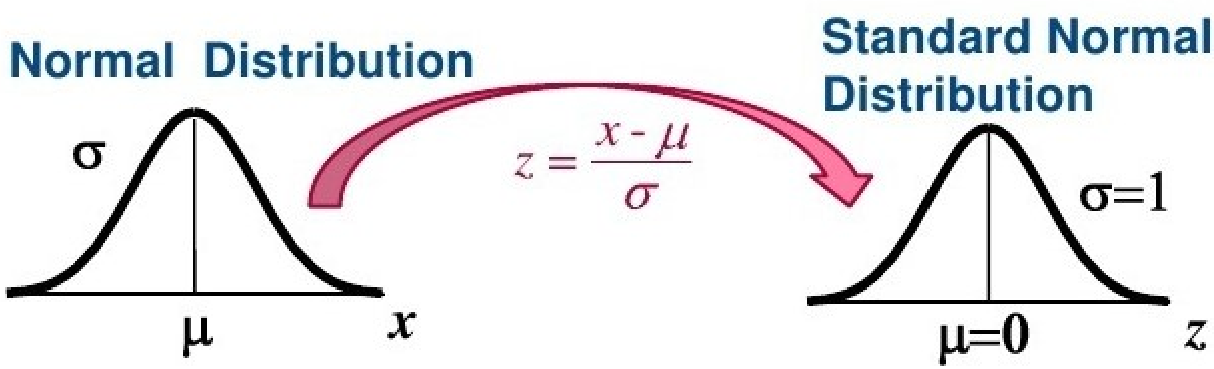
Fortunately, we are able to transform all the observations of any normal random variable X to a new set of observation of
a normal random variable Z with mean 0 and variance 1.

Standard Normal Distribution

- To ease calculations, we define a normal random variable

$$Z = \frac{X - \mu}{\sigma}$$

  - where Z is normally distributed with **$\mu = 0$ and $\sigma^2 = 1$**



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Standard normal distribution

- Probability of $P(x_1 < X < x_2)$?

$$P(x_1 < X < x_2) = \frac{1}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx$$

$z = \frac{x-\mu}{\sigma}$, thus $z_1 = (x_1 - \mu)/\sigma$ and $z_2 = (x_2 - \mu)/\sigma$

$$P(x_1 < X < x_2) = \frac{1}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx = \frac{1}{\sqrt{2\pi}}\int_{z_1}^{z_2} e^{-\frac{1}{2}z^2}\,dz$$

$$= P(z_1 < Z < z_2)$$

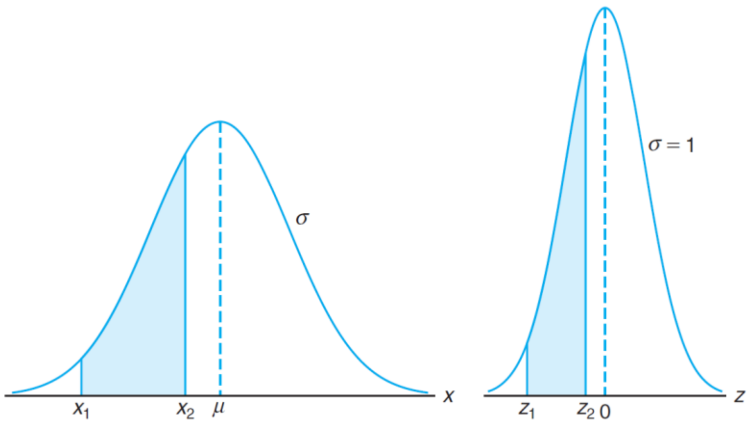

Figure 6.8: The original and transformed normal distributions.

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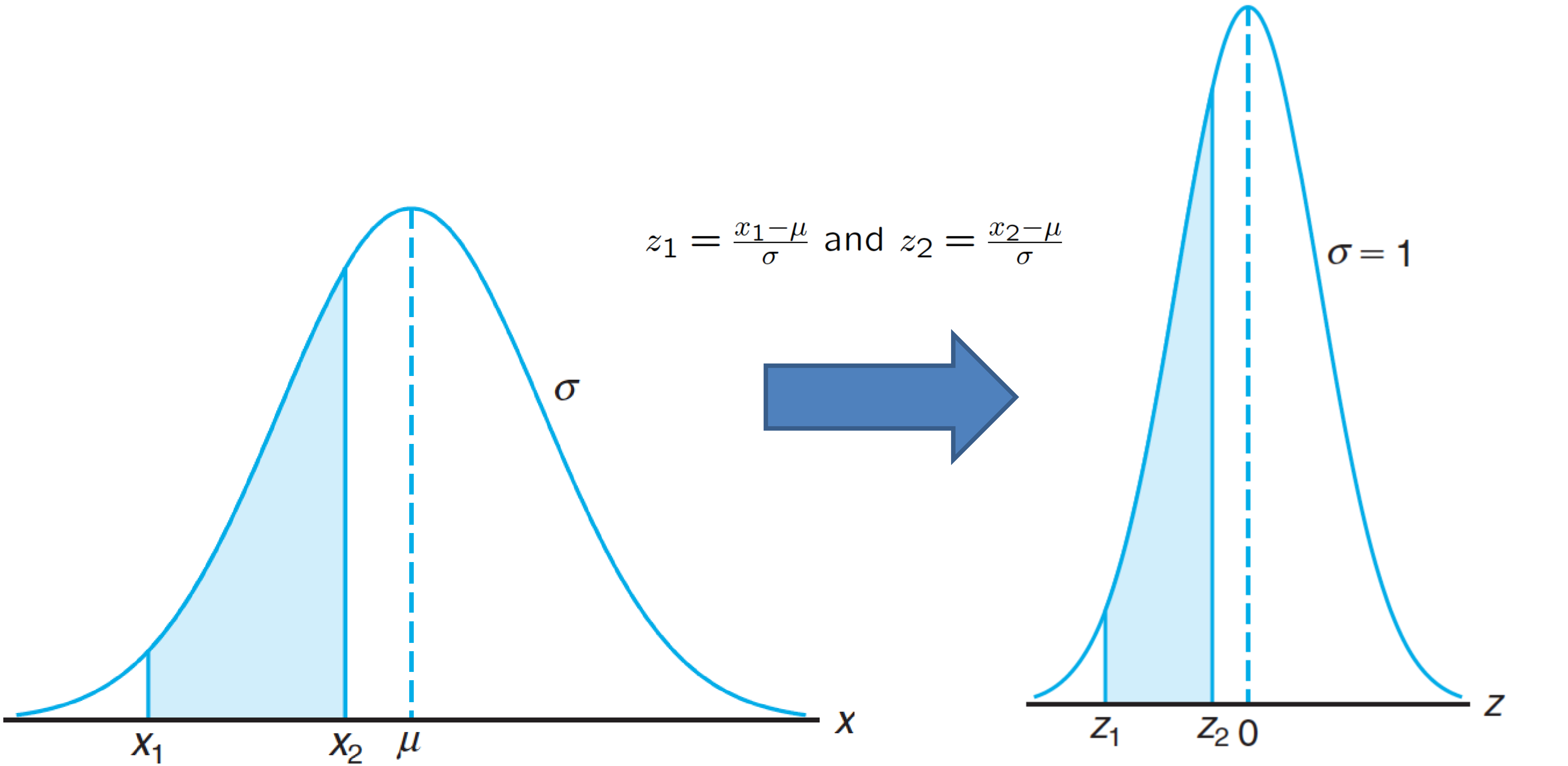
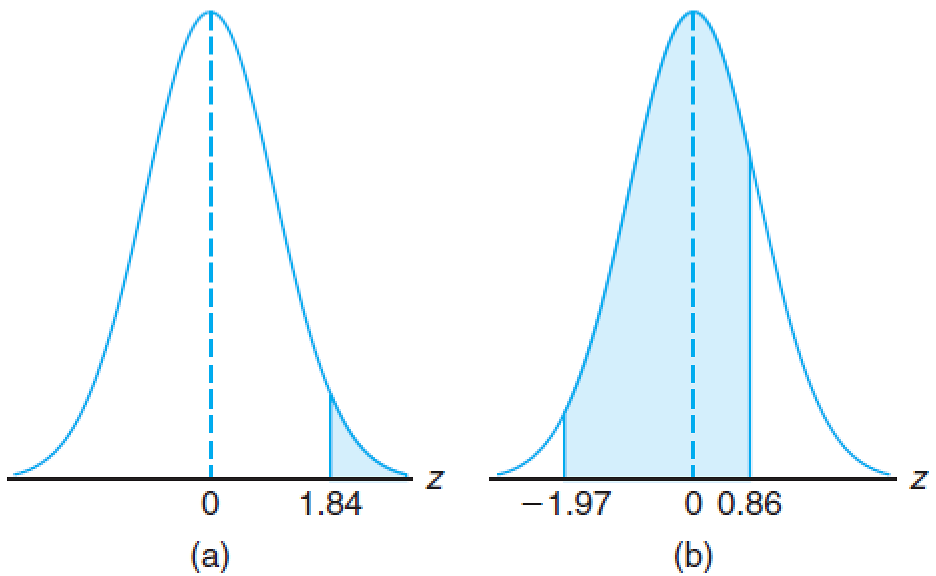


The original and transformed normal distributions.

- Example 6.2
  - Given a standard normal distribution, find the area under the curve that lies
  - (a) to the right of z = 1. 84 and
  - (b) between z = − 1. 97 and z = 0. 86



- Solution:
  - (a) 1 minus the area in Table A.3 to the left of $z = 1.84$
    - $1 - $ **0.9671** $= 0.0329$.
  - (b) The area between $z = -1.97$ and $z = 0.86$ is equal to the area to the left of $z = 0.86$ minus the area to the left of $z = -1.97$.
    - From Table A.3 we find the desired area to be $0.8051 - 0.0244 = 0.7807$.

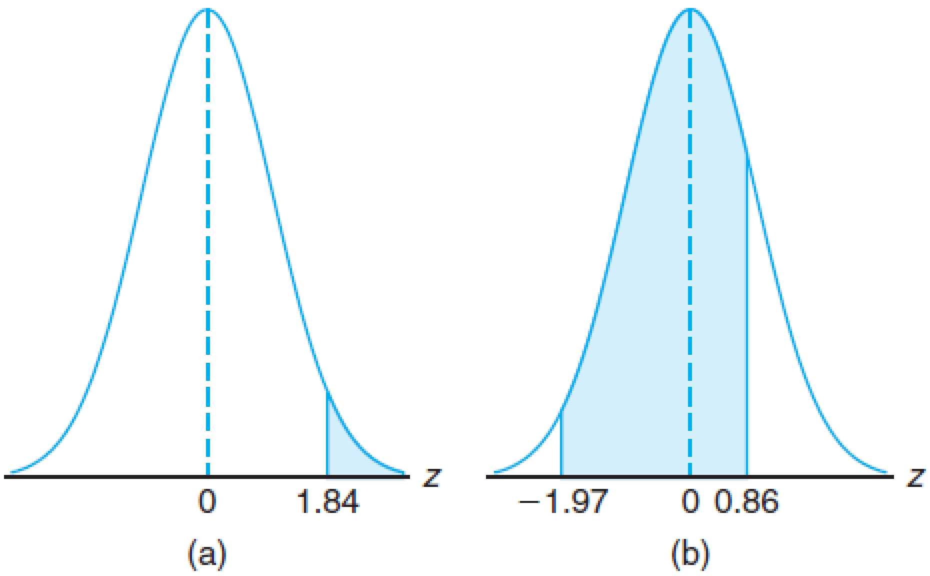


Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

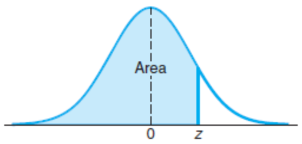


Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

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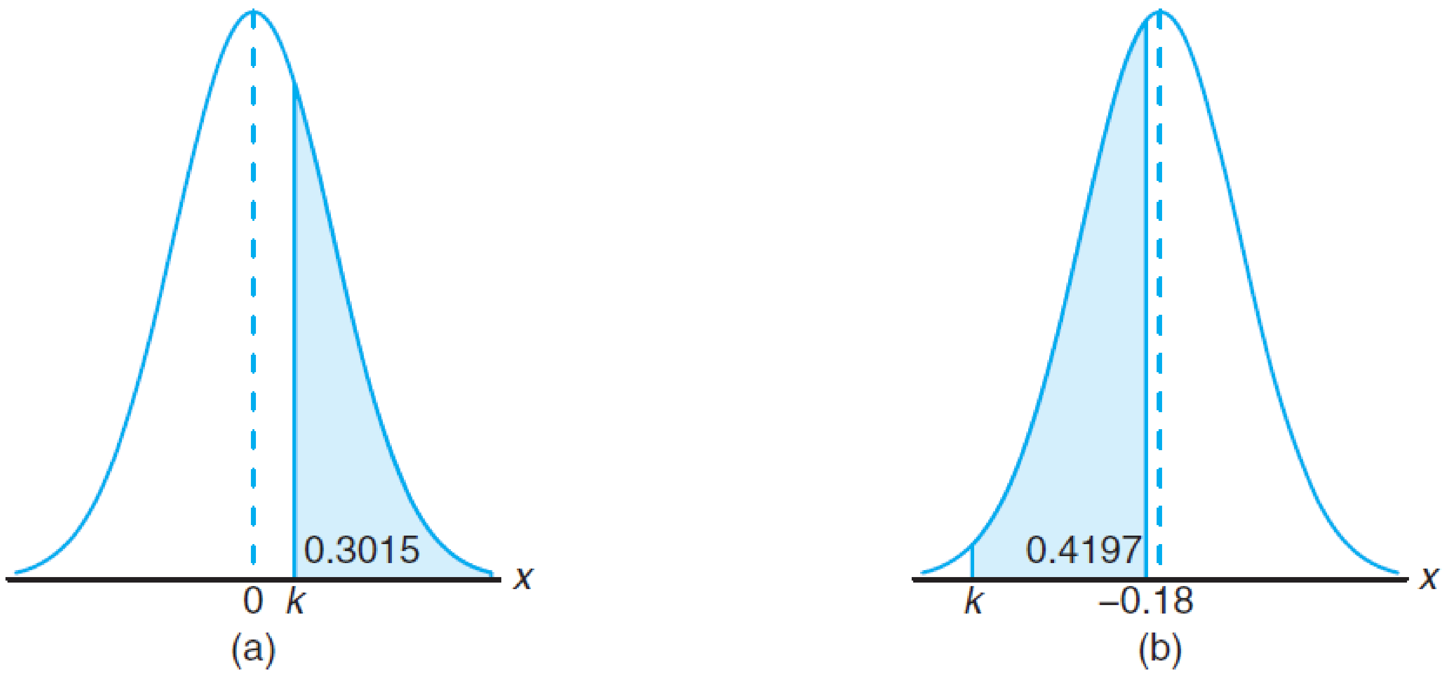
- Example 6.3
    - Given a standard normal distribution, find the value of k such that
    - (a) $P(Z > k) = 0.3015$ and
    - (b) $P(k < Z < -0.18) = 0.4197$.



Solution:

- (a) we see that the *k* value leaving an area of 0.3015 to the right must then leave an area of **0.6985** to the left. From Table A.3 it follows that *k* = 0.52.

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389

Solution:

- (b) From Table A.3 we note that the total area to the left of −0.18 is equal to 0.4286. In the Figure (b), we see that the area between *k* and −0.18 is 0.4197, so the area to the left of *k* must be 0.4286 − 0.4197 = 0.0089. Hence, from Table A.3, we have *k* = −2.37.

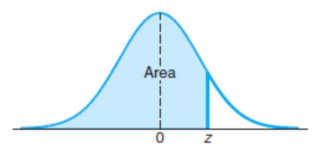


Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183

Example 6.4

Given a random variable X having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.

***Solution*:** The z values corresponding to $x_1 = 45$ and $x_2 = 62$ are

$$z_1 = \frac{45 - 50}{10} = -0.5 \text{ and } z_2 = \frac{62 - 50}{10} = 1.2.$$

Therefore,

$$P(45 < X < 62) = P(-0.5 < Z < 1.2).$$

$P(-0.5 < Z < 1.2)$ is shown by the area of the shaded region in Figure 6.11. This area may be found by subtracting the area to the left of the ordinate $z = -0.5$ from the entire area to the left of $z = 1.2$. Using Table A.3, we have

$$\begin{aligned} P(45 < X < 62) &= P(-0.5 < Z < 1.2) = P(Z < 1.2) - P(Z < -0.5) \\ &= 0.8849 - 0.3085 = 0.5764. \end{aligned}$$

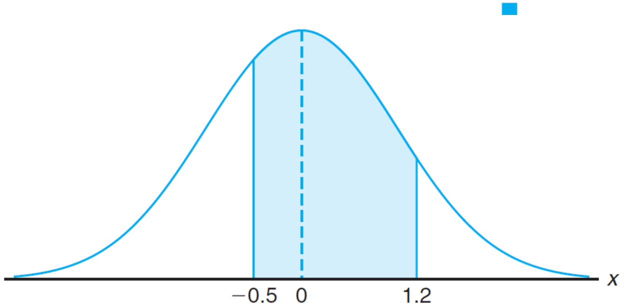


Figure 6.11: Area for Example 6.4.

Example 6.5

Given that X has a normal distribution with $\mu = 300$ and $\sigma = 50$, find the probability that X assumes a value greater than 362.

The normal probability distribution with the desired area shaded is shown in Figure 6.12. To find $P(X > 362)$, we need to evaluate the area under the normal curve to the right of $x = 362$. This can be done by transforming $x = 362$ to the corresponding z value, obtaining the area to the left of z from Table A.3, and then subtracting this area from 1. We find that

$$z = \frac{362 - 300}{50} = 1.24.$$

Hence,

$$P(X > 362) = P(Z > 1.24) = 1 - P(Z < 1.24) = 1 - 0.8925 = 0.1075.$$

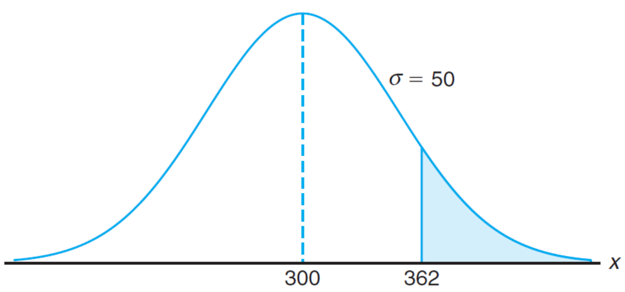


Figure 6.12: Area for Example 6.5.

Using the Normal Curve in Reverse

- Sometimes, we are required to find the value of z corresponding to a specified probability that falls between values listed in Table A.3.
  - For convenience, we shall always choose the z value corresponding to the tabular probability that comes closest to the specified probability.
- We reverse the process and begin with a known area or probability, find the z value, and then determine x by rearranging the formula

$$\boxed{z = \frac{x-\mu}{\sigma} \quad \text{to give} \quad x = \sigma z + \mu.}$$

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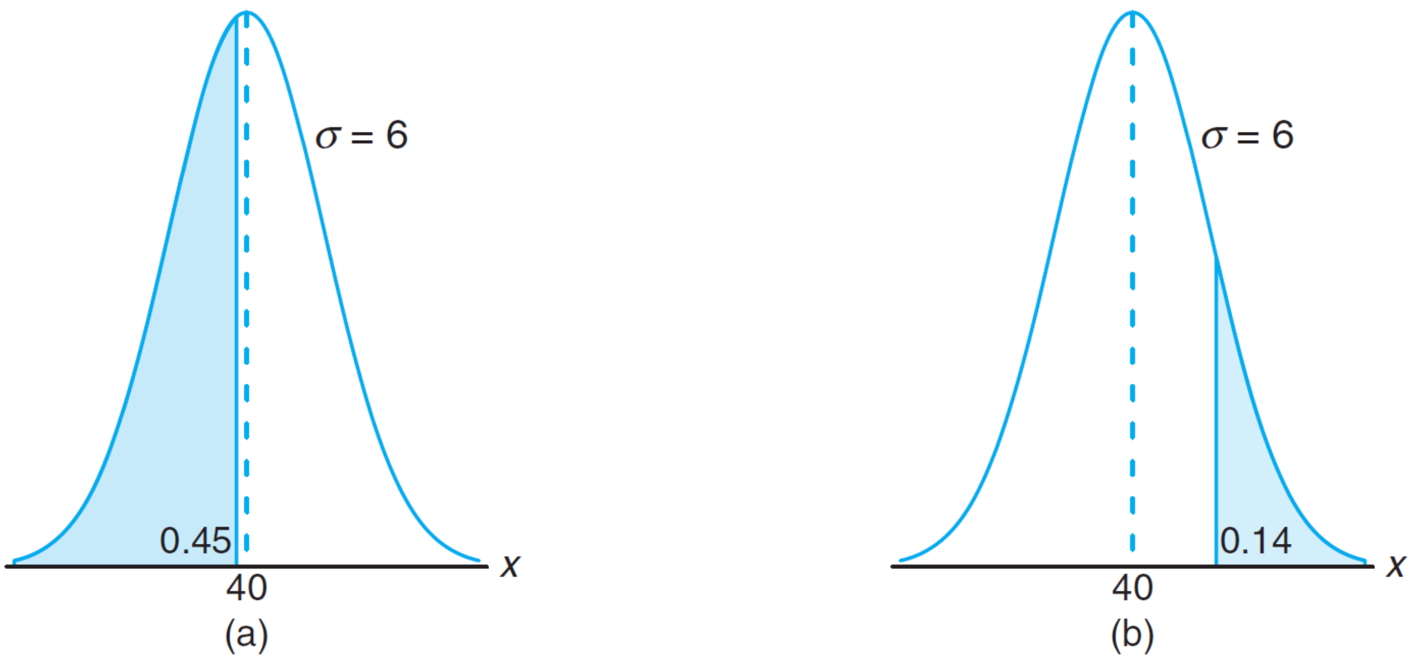
- Example
  - Given a normal distribution with $\mu = 40$ and $\sigma = 6$, find the value of x that has
    - (a) 45% of the area to the left and
    - (b) 14% of the area to the right.



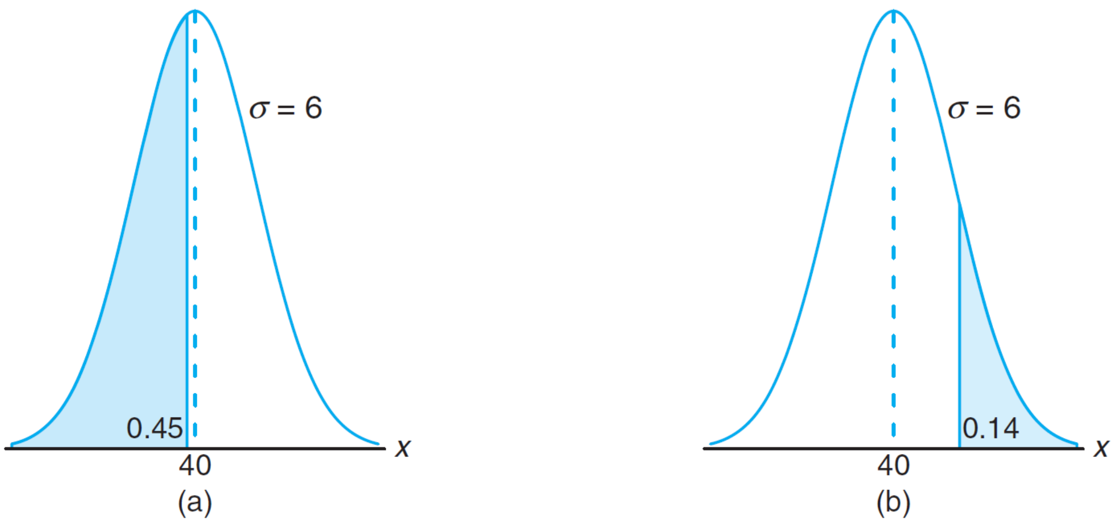
- Solution

(a) From Table A.3 we find $P(Z < -0.13) = 0.45$, so the desired z value is -0.13.

$$x = (6)(-0.13) + 40 = 39.22.$$

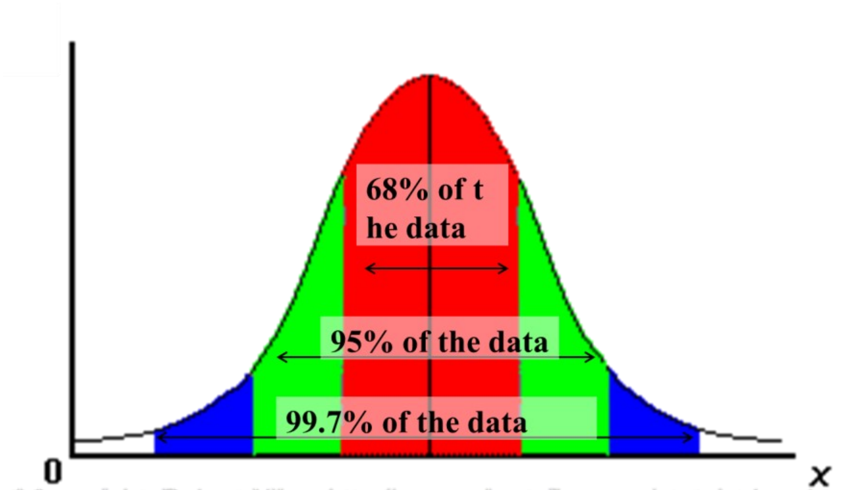
(b) we shade an area equal to 0.14 to the right of the desired x value. This time we require a z value that leaves 0.14 of the area to the right and hence an area of 0.86 to the left. Again, from Table A.3, we find $P(Z < 1.08) = 0.86$, so the desired z value is 1.08 and

$$x = (6)(1.08) + 40 = 46.48.$$



**The beauty of the normal curve:

No matter what μ and σ are, the area between $\mu-\sigma$ and $\mu+\sigma$ is about 68%; the area between $\mu-2\sigma$ and $\mu+2\sigma$ is about 95%; and the area between $\mu-3\sigma$ an d $\mu+3\sigma$ is about 99.7%. Almost all values fall within 3 standard deviations.



6.4 Applications of the Normal Distribution

Example 6.7

- A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that the battery lives are normally distributed, find the probability that a given battery will last less than 2.3 years.

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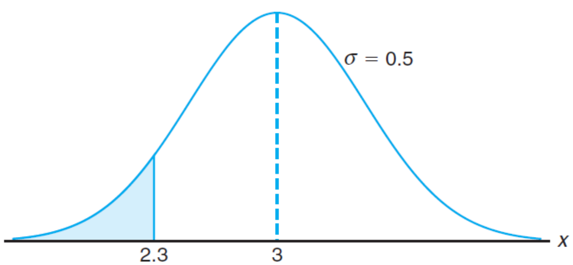
- Solution:
  - First, construct a diagram, showing the given distribution of battery lives and the desired area. To find $P(X < 2.3)$, we need to evaluate the area under the normal curve to the left of 2.3. This is accomplished by finding the area to the left of the corresponding z value. Hence, we find that



$$z = \frac{2.3 - 3}{0.5} = -1.4,$$

$$\boxed{P(X < 2.3) = P(Z < -1.4) = 0.0808.}$$

Example 6.8

- An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours.

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Solution:

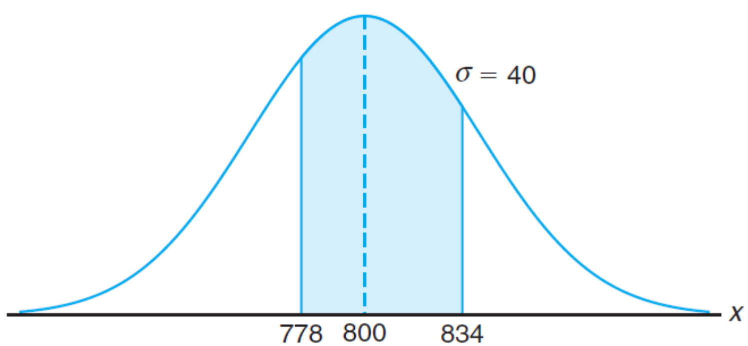
- The z values corresponding to $x_1 = 778$ and $x_2 = 834$ are

$$z_1 = \frac{778 - 800}{40} = -0.55 \text{ and } z_2 = \frac{834 - 800}{40} = 0.85.$$

- Hence,

$$P(778 < X < 834) = P(-0.55 < Z < 0.85) = P(Z < 0.85) - P(Z < -0.55)$$

$$= 0.8023 - 0.2912 = 0.5111.$$



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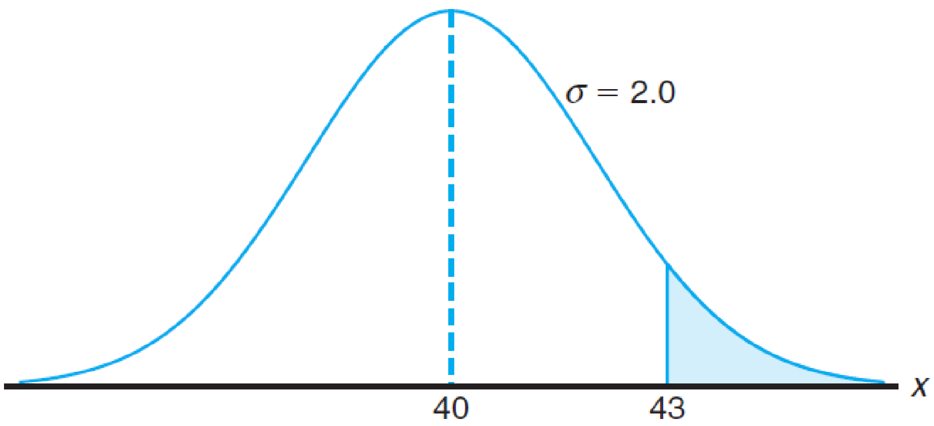
Example 6.11

- A certain machine makes electrical resistors having a mean resistance of 40 ohms and a standard deviation of 2 ohms. Assuming that the resistance follows a normal distribution and can be measured to any degree of accuracy, what percentage of resistors will have a resistance exceeding 43 ohms?



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- Solution:

A percentage is found by multiplying the relative frequency by 100%. Since the relative frequency for an interval is equal to the probability of a value falling in the interval, we must find the area to the right of $x = 43$ in Figure 6.18. This can be done by transforming $x = 43$ to the corresponding z value, obtaining the area to the left of z from Table A.3, and then subtracting this area from 1. We find

$$z = \frac{43 - 40}{2} = 1.5.$$

Therefore,

$$P(X > 43) = P(Z > 1.5) = 1 - P(Z < 1.5) = 1 - 0.9332 = 0.0668.$$

Hence, 6.68% of the resistors will have a resistance exceeding 43 ohms.

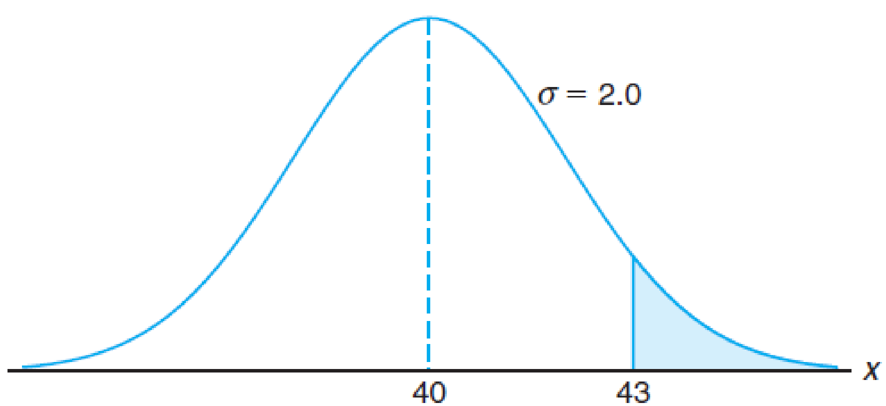

Figure 6.18: Area for Example 6.11.

Example 6.13

- The average grade for a midterm exam is 74, and the standard deviation is 7. If 12% of the class is given A's, and the grades are curved to follow a normal distribution what is the lowest possible A and the highest possible B?

- Solution:
  - Find ***z*** value from x, determine x from z
    - $z = (x - \mu)/\sigma$ or $x = \sigma z + \mu$
  - An area of 0.12, corresponding to the fraction of students receiving ***A***s, → we require a value for the area of 0.88 to the left.
    - $P(Z < 1.18)$ has the closest value to 0.88
    - → z ~= 1.18
  - So,
    - $x = (7)(1.18) + 74 = 82.26.$
  - Therefore, the lowest *A* is 83 and the highest *B* is 82.

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Appendix

- Proof of Variance of Normal Distribution

Proof of Variance of Normal Distribution

The variance of the normal distribution is given by

$$E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-\frac{1}{2}[(x-\mu)/\sigma]^2}\,dx.$$

Again setting $z=(x-\mu)/\sigma$ and $dx=\sigma\,dz$, we obtain

$$E[(X-\mu)^2] = \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}}\,dz.$$

Integrating by parts with $u=z$ and $dv=ze^{-z^2/2}\,dz$ so that $du=dz$ and $v=-e^{-z^2/2}$, we find that

$$E[(X-\mu)^2] = \frac{\sigma^2}{\sqrt{2\pi}}\left(-ze^{-z^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-z^2/2}\,dz\right) = \sigma^2(0+1) = \sigma^2.$$