Chapter 4

Mathematical Expectation – part 1

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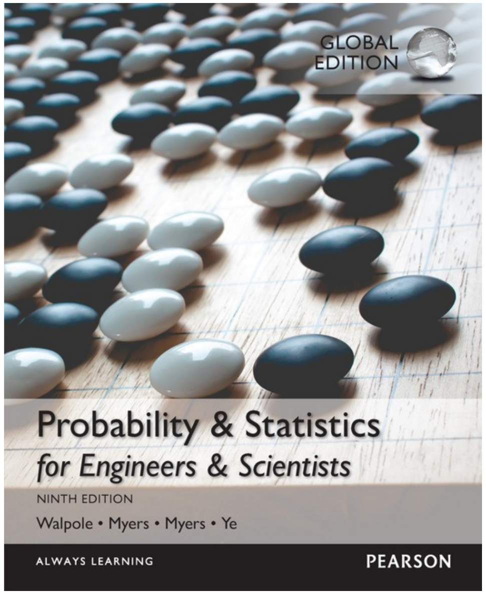

Outline

- Mean of a Random Variable
- Variance and Covariance of Random Variables
- Means and Variances of Linear Combinations of Random Variables
- Chebyshev's Theorem

4.1 Mean of a Random Variable

Question !

- Consider a casino game in which the probability of losing \$1 per game is 0.8 and the probability 0.2 win \$2 per game.
- The gain or loss of a gambler who plays this game only a few times depends on his "luck" more than anything else.
- But, if a gambler decides to play the game a large number of times, his loss or gain depends more on "**the number of plays** " than on his luck.
- "Expected" total gain when playing this game ***n*** times = ??

$$(0.8)n \cdot (-1) + (0.2)n \cdot 2 \; = (-0.4)n.$$

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Average ?

- In Ch. 1, we discussed sample mean; arithmetic mean of the data
- If two coins are tossed 16 times and X is the number of heads that occur per toss, then the values of X are 0, 1, and 2.
- Suppose that the experiment yields no heads, one head, and two heads a total of 4, 7, and 5 times, respectively. The average number of heads per toss of the two coins is then

$$\frac{(0)(4) + (1)(7) + (2)(5)}{16} = 1.06.$$

- Let us now restructure our computation for the average number of heads so as to have the following equivalent form:

$$(0)\left(\frac{4}{16}\right)+(1)\left(\frac{7}{16}\right)+(2)\left(\frac{5}{16}\right)=1.06.$$

mean or expectation of X (μ_x)

the fractions of the total tosses resulting
in 0, 1, and 2 heads, respectively

Mean and Expectation*

- This average value is called **mean of the random variable X**, or the **mean of the probability distribution of X**, and write it as μ_x or simply as μ when it is clear to which random variable we refer
- It is also common to refer to this mean as the **mathematical expectation**, or **expected value** of random variable X, and denote it as **E(X)**.

*기대값

Assuming that 1 fair coin was tossed twice, we find that the sample space for our experiment is

$$S = \{HH, HT, TH, TT\}.$$

Since the 4 sample points are all equally likely, it follows that

$$P(X = 0) = P(TT) = \frac{1}{4}, \quad P(X = 1) = P(TH) + P(HT) = \frac{1}{2},$$

and

$$P(X = 2) = P(HH) = \frac{1}{4},$$

where a typical element, say TH, indicates that the first toss resulted in a tail followed by a head on the second toss. Now, these probabilities are just the relative frequencies for the given events in the long run. Therefore,

$$\boxed{\mu = E(X) = (0)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{4}\right) = 1.}$$

This result means that a person who tosses 2 coins over and over again will, on the average, get 1 head per toss.

$$\boxed{\mu = E(X) = \sum_x x f(x)}$$

Expectation

- Definition 4.1:

Let X be a random variable with probability distribution $f(x)$. The **mean**, or **expected value**, of X is

$$\mu = E(X) = \sum_x x f(x)$$

if X is discrete, and

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

if X is continuous.

In the case of continuous random variables, the definition of an expected value is essentially the same with summations replaced by integrations.

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- **Example 4.1**
  - A box containing 7 components is sampled by a quality inspector; the box contains 4 good component and 3 defective components.
  - A sample of 3 is taken by the inspector. Find the **expected value of the number of good components** in this sample.

- **Solution:**
  - Let X be represent the number of good components in the sample. The probability distribution of X is

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2, 3.$$

$$\mu = E(X) = \sum_x x f(x) = 0 \cdot \frac{1}{35} + 1 \cdot \frac{12}{35} + 2 \cdot \frac{18}{35} + 3 \cdot \frac{4}{35} = \frac{12}{7}.$$

- Meaning:
    - Thus, if a sample of size 3 is selected at random over and over again from a box of 4 good components and 3 defective components, it would contain, on average, 12/7 ($\approx$ 1.7) good components.

Question!

- **In a gambling game a man is paid $5 if he gets all heads or all tails when three coin are tossed, and he will pay out $3 if either one or two heads show. What is his expected gain?**

- Solution:
  - The sample space for the possible outcomes when three coins are tossed simultaneously is
  - S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}.
  - The random variable of interest is Y, the amount the gambler can win; and the possible values of Y are
    - \$5 if event E_1 = {HHH, TTT} occurs and
    - \$-3 if event E_2 = {HHT, HTH, THH, HTT, THT, TTH} occurs,
  - that is, the probability function of Y is given by

$$f(y) = P(X = y) = \begin{cases} \frac{1}{4}, & y = 5; \\ \frac{3}{4}, & y = -3; \\ 0, & \text{elsewhere.} \end{cases} \qquad \mu = E(Y) = 5 \cdot \frac{1}{4} + (-3) \cdot \frac{3}{4} = -1.$$

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- Example 4.3
  - Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \frac{20{,}000}{x^3}, & x > 100, \\ 0, & \text{elsewhere.} \end{cases}$$

  - Find the expected life of this type of device.

- **Solution:** $\boxed{\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx}$

$$\mu = E(X) = \int_{100}^{\infty} x \frac{20,000}{x^3}\, dx = \int_{100}^{\infty} \frac{20,000}{x^2}\, dx = 200.$$

  - Therefore, we can expect this type of device to last, *on average*, 200 hours.

Expectation of Function

- For the following probability mass function for X, calculate $E[X^2]$

$$p(0) = .2, \quad p(1) = .5, \quad p(2) = .3$$

Let a random variable $Y = X^2$,

$$p_Y(0) = P\{Y = 0^2\} = .2$$

$$p_Y(1) = P\{Y = 1^2\} = .5$$

$$p_Y(4) = P\{Y = 2^2\} = .3$$

$$\boxed{\mu = E(X) = \sum_x x f(x)}$$

$$E[X^2] = E[Y] = 0(.2) + 1(.5) + 4(.3) = 1.7$$

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Theorem 4.1

Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x)$$

if X is discrete, and

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

if X is continuous.

Expectation of Function: Example

- For the following probability mass function for X, calculate $E[X^2]$

$$p(0) = .2, \quad p(1) = .5, \quad p(2) = .3$$

  - Using the Theorem 4.1, $\boxed{\mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x)}$

$$E[X^2] = 0^2(0.2) + (1^2)(0.5) + (2^2)(0.3) = 1.7$$

- Example 4.4
  - Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability dist.:

x	4	5	6	7	8	9
$P(X = x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

  - Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager.
  - Find the attendant's expected earnings for this particular time period.

x	4	5	6	7	8	9
$P(X=x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

- Solution:
  - By theorem, the attendant can expect to receive

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x)$$

$$E[g(X)] = E(2X-1) = \sum_{x=4}^{9}(2x-1)f(x)$$

$$= (7)\left(\frac{1}{12}\right) + (9)\left(\frac{1}{12}\right) + (11)\left(\frac{1}{4}\right) + (13)\left(\frac{1}{4}\right) + (15)\left(\frac{1}{6}\right) + (17)\left(\frac{1}{6}\right) = \$12.67.$$

- Example 4.5:
  - Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

  - Find the expected value of $g(X) = 4X + 3$.

- Solution:

By Theorem 4.1, we have $\boxed{\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx}$

$$E(4X+3) = \int_{-1}^{2} \frac{(4x+3)x^2}{3}\, dx = \frac{1}{3}\int_{-1}^{2} (4x^3 + 3x^2)\, dx = 8.$$

• Definition 4.2:

- For joint probability distribution **f**(**x, y**),

Let X and Y be random variables with joint probability distribution $f(x, y)$. The mean, or expected value, of the random variable $g(X, Y)$ is

$$\mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y)$$

if X and Y are discrete, and

$$\mu_{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y) \; dx \; dy$$

if X and Y are continuous.

Example 4.7: Find $E(Y/X)$ for the density function

$$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

***Solution*:** We have

$$E\left(\frac{Y}{X}\right) = \int_0^1 \int_0^2 \frac{y(1+3y^2)}{4}\, dxdy = \int_0^1 \frac{y+3y^3}{2}\, dy = \frac{5}{8}.$$

Outline

- Mean of a Random Variable
- **Variance and Covariance of Random Variables**
- Means and Variances of Linear Combinations of Random Variables
- Chebyshev's Theorem

4.2 Variance and Covariance of Random Variables (1)

Quantifying Variability : Variance

- The **variance** of a random variable is a measure of its statistical dispersion (분산, 이산), indicating how its possible values are spread around the expected value.
  - the expected value shows **the location** of the distribution,
  - the variance indicates **the variability of the values**.

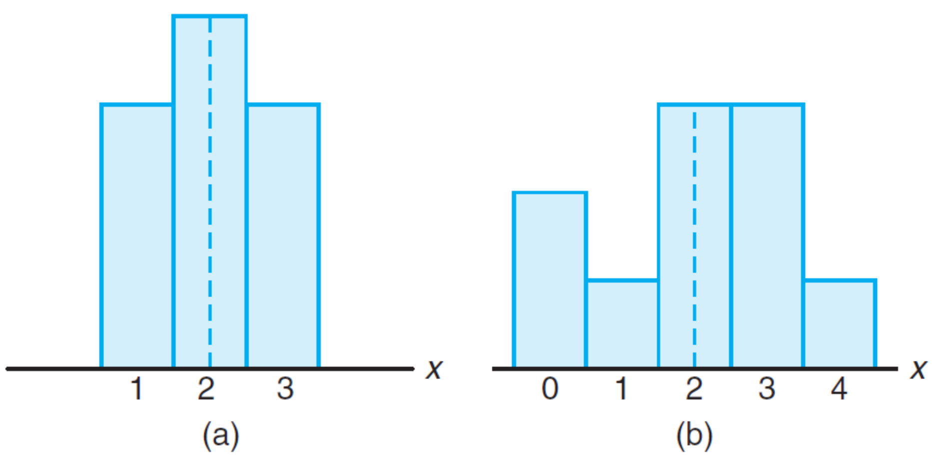

Figure 4.1: Distributions with equal means and unequal dispersions.

Variance

$E(X) = \sum_x x f(x)$

- **Definition 4.3**: variance of random variable X

Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is

$$\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x), \qquad \text{if } X \text{ is discrete, and}$$

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx, \qquad \text{if } X \text{ is continuous.}$$

The positive square root of the variance, σ, is called the **standard deviation** of X.

- Variation : σ^2 Var(X)
- The positive square root of the variance, σ, is called the standard deviation of X.

$$\sigma = \sqrt{\mathrm{Var}(X)}.$$

- **Example 4.8**
  - Let the random variable X represent the number of automobiles that are used for official business purposes on any given workday. The probability distribution for company A is

x	1	2	3
$f(x)$	0.3	0.4	0.3

  - and that for company B is

x	0	1	2	3	4
$f(x)$	0.2	0.1	0.3	0.3	0.1

- Show that the variance of the probability distribution for company B is greater than that for company A.

company A

x	1	2	3
$f(x)$	0.3	0.4	0.3

company B

x	0	1	2	3	4
$f(x)$	0.2	0.1	0.3	0.3	0.1

- *Solution:*

$$\boxed{\mu = E(X) = \sum_x x f(x)}$$

$$\mu_A = 1 \cdot 0.3 + 2 \cdot 0.4 + 3 \cdot 0.3 = 2;$$

$$\mu_B = 0 \cdot 0.2 + 1 \cdot 0.1 + 2 \cdot 0.3 + 3 \cdot 0.3 + 4 \cdot 0.1 = 2;$$

$$\boxed{\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x)}$$

$$\sigma_A^2 = \sum_{x=1}^{2} (x-2)^2 f_A(x) = 0.6;$$

$$\sigma_B^2 = \sum_{x=0}^{4} (x-2)^2 f_B(x) = 1.6.$$

Remember!

- **Theorem 4.2**
  - The variance of a random variable *X* is

$$\sigma^2 = E(X^2) - \mu^2.$$

Prove it !

proof

- Proof:
  - For the discrete case we can write:

- **Definition 4.3** $\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x)$

$$\sigma^2 = \sum_x (x-\mu)^2 f(x) = \sum_x (x^2 - 2\mu x + \mu^2) f(x)$$
$$= \sum_x x^2 f(x) - 2\mu \sum_x x f(x) + \mu^2 \sum_x f(x).$$
$$= \sum_x x^2 f(x) - 2\mu \cdot \mu + \mu^2 \cdot 1.$$
$$= \sum_x x^2 f(x) - \mu^2$$
$$= E(X^2) - \mu^2.$$

- Definition 4.1 $\mu = E(X) = \sum_x x f(x)$

Theorem 4.1

$\mu_{g(X)} = E[g(X)] = \sum_x g(x) f(x)$

* For the continuous case the proof is step by step the same, with summations(∑) replaced by integrations(∫).

- Example 4.9
  - Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested. The following is the probability distribution of X.

x	0	1	2	3
$f(x)$	0.51	0.38	0.10	0.01

- Solution: $\boxed{\sigma^2 = E(X^2) - \mu^2.}$

$$\begin{aligned} \mu &= 0 \cdot 0.51 + 1 \cdot 0.38 + 2 \cdot 0.10 + 3 \cdot 0.01 = 0.61; \\ E(X^2) &= 0 \cdot 0.51 + 1 \cdot 0.38 + 4 \cdot 0.10 + 9 \cdot 0.01 = 0.87; \\ \sigma^2 &= E(X^2) - \mu^2 = 0.87 - 0.61^2 = 0.4979. \end{aligned}$$

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- Example 4.10
  - The weekly demand for a drinking-water product, in thousands of liters, from a local chain of efficiency stores is a continuous random variable X having the probability density

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

- Find the mean and variance of X

$$\mu = E(X) = 2\int_1^2 x(x-1)\;dx = \frac{5}{3}$$

and

$$\boxed{\sigma^2 = E(X^2) - \mu^2.}$$

$$E(X^2) = 2\int_1^2 x^2(x-1)\;dx = \frac{17}{6}.$$

Therefore,

$$\sigma^2 = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{1}{18}.$$

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Next important Theorem!

- Let X be a random variable with probability function $f(x)$.
- The variance of the random variable $g(X)$ is

- Theorem 4.3

Let X be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$ is

$$\sigma^2_{g(X)} = E\{[g(X) - \mu_{g(X)}]^2\} = \sum_x [g(x) - \mu_{g(X)}]^2 f(x)$$

if X is discrete, and

$$\sigma^2_{g(X)} = E\{[g(X) - \mu_{g(X)}]^2\} = \int_{-\infty}^{\infty} [g(x) - \mu_{g(X)}]^2 f(x)\, dx$$

if X is continuous.

- Example 4.11
  - Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

- Solution:

Theorem 4.1

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x)$$

Theorem 4.3

$$\sigma^2_{g(X)} = E\{[g(X) - \mu_{g(X)}]^2\}$$

$$\mu_{2X+3} = E(2X+3) = \sum_{x=0}^{3}(2x+3)f(x) = 6.$$

$$\sigma^2_{2X+3} = E\{[(2X+3) - \mu_{2x+3}]^2\} = E[(2X+3-6)^2]$$

$$= E(4X^2 - 12X + 9) = \sum_{x=0}^{3}(4x^2 - 12x + 9)f(x) = 4.$$

We will study a method to solve this in an easier way in Sec. 4.3.

- Example 4.12
  - Let X be a random variable having the density function given in Example 4.5 on page 135. Find the variance of the random variable g(X) = 4X + 3.
- Solution:

$$\boxed{\sigma_{g(X)}^2 = E\{[g(X) - \mu_{g(X)}]^2\} = \int_{-\infty}^{\infty} [g(x) - \mu_{g(X)}]^2 f(x)\, dx}$$

In Example 4.5, we found that $\mu_{4X+3} = 8$. Now, using Theorem 4.3,

$$\begin{aligned}\sigma_{4X+3}^2 &= E\{[(4X+3) - 8]^2\} = E[(4X-5)^2] \\ &= \int_{-1}^{2} (4x-5)^2 \frac{x^2}{3}\, dx = \frac{1}{3}\int_{-1}^{2} (16x^4 - 40x^3 + 25x^2)\, dx = \frac{51}{5}.\end{aligned}$$

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