Algorithms

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2. Growth of Functions & Divide-and-Conquer

Contents

- Asymptotic notation
  - Asymptotic notation in equations
  - Comparing functions
- Recursion-tree method
- The master method

- Problem 3:Thanksgiving trip

Growth of Functions

overview

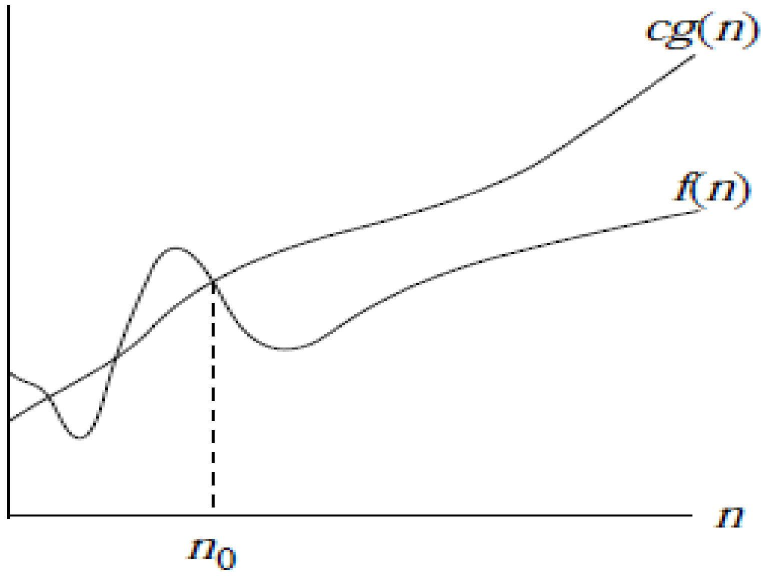
- A way to describe behavior of functions in the limit.

  We're studying asymptotic efficiency.
- Describe growth of functions.
- Focus on what's important by abstracting away low-order terms and constant factors.
- How we indicate running times of algorithms.
- A way to compare "sizes" of functions:

$$O \approx \leq$$
$$\Omega \approx \geq$$
$$\Theta \approx =$$
$$o \approx <$$
$$\omega \approx >$$

O -notation

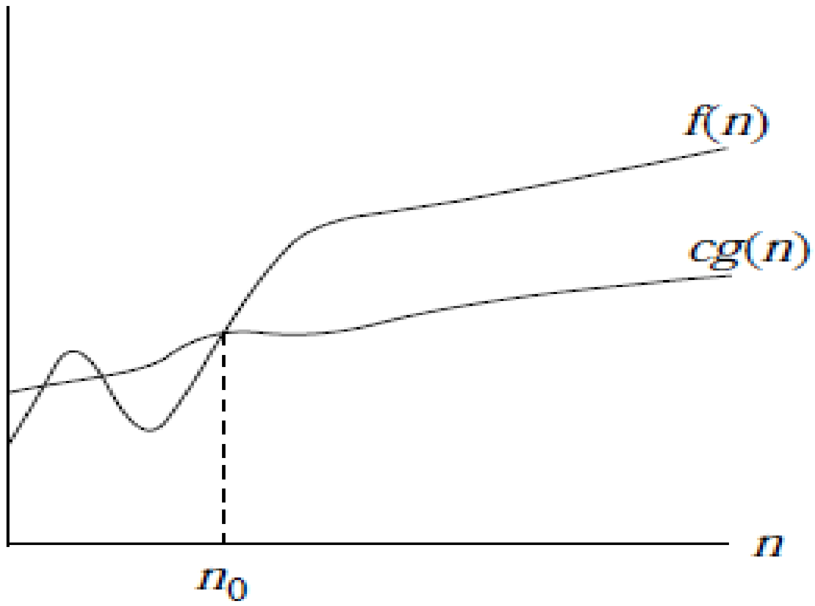
$$O(g(n)) = \{ f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le c\,g(n) \text{ for all } n \ge n_0 \}$$



g(n) is an asymptotic upper bound for f(n).

Ω - notation

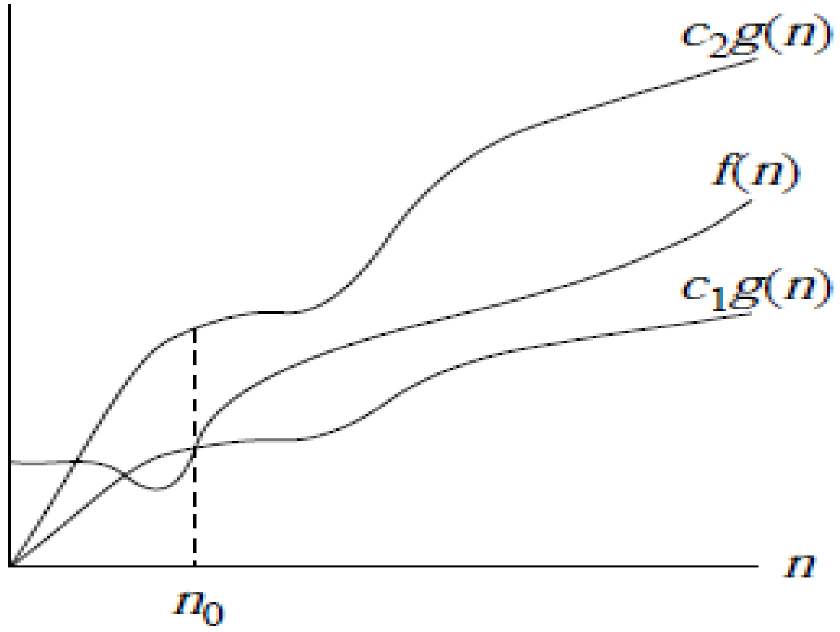
$\Omega(g(n))$ = { $f(n)$: there exist positive constants c and n_0 such that $0 \le c\,g(n) \le f(n)$ for all $n \ge n_0$}



g(n) is an asymptotic lower bound for f(n).

Θ - notation

$\Theta(g(n)) = \{\, f(n) :$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1\, g(n) \le f(n) \le c_2\, g(n)$ for all $n \ge n_0 \,\}$



g(n) is an asymptotically tight bound for f(n).

Θ - notation

- ***Engineering:***
  - Drop low-order terms; ignore leading constants.
  - Example : $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

o - notation

$$\boldsymbol{o(g(n))} := \{f(n) : \text{ for all constants } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le f(n) < cg(n) \text{ for all } n \ge n_0\}.$$

Another view, probably easier to use: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$

$$n^{1.9999} = o(n^2)$$
$$n^2/\lg n = o(n^2)$$
$$n^2 \ne o(n^2) \text{ (just like } 2 \not< 2)$$
$$n^2/1000 \ne o(n^2)$$

ω - notation

$\boldsymbol{\omega}$ **(g(n))** $= \{ f(n) :$ for all constants $c > 0$, there exists a constant $n_0 > 0$ such that $0 \le cg(n) < f(n)$ for all $n \ge n_0 \}$.

Another view, again, probably easier to use: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$.

$$n^{2.0001} = \omega(n^2)$$
$$n^2 \lg n = \omega(n^2)$$
$$n^2 \neq \omega(n^2)$$

Comparing functions

Relational properties:

Transitivity:

$f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$.

Same for O, Ω, o, and ω.

Reflexivity:

$f(n) = \Theta(f(n))$.

Same for O and Ω.

Symmetry:

$f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.

Transpose symmetry:

$f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.

$f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.

Divide-and-Conquer

Three methods for solving recurrences

- Substitution method
  - guess the bound and then use mathematical induction to prove our guess correct.
- **Recursion-tree method**
  - converts the recurrence into a tree whose nodes represent the costs incurred at various levels of the recursion.
- Master method
  - provides for recurrences of the form.

$$T(n) = aT(n/b) + f(n)$$

Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
타원
- The recursion-tree method promotes intuition, however.

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$T(n)$$

Example of recursion tree

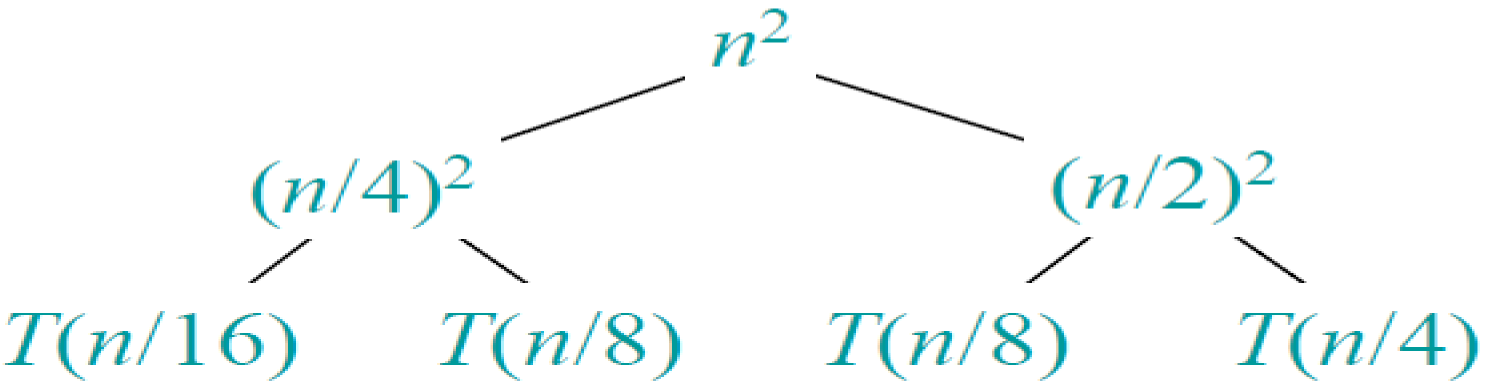
Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$n^2$$

$$T(n/4) \qquad\qquad T(n/2)$$

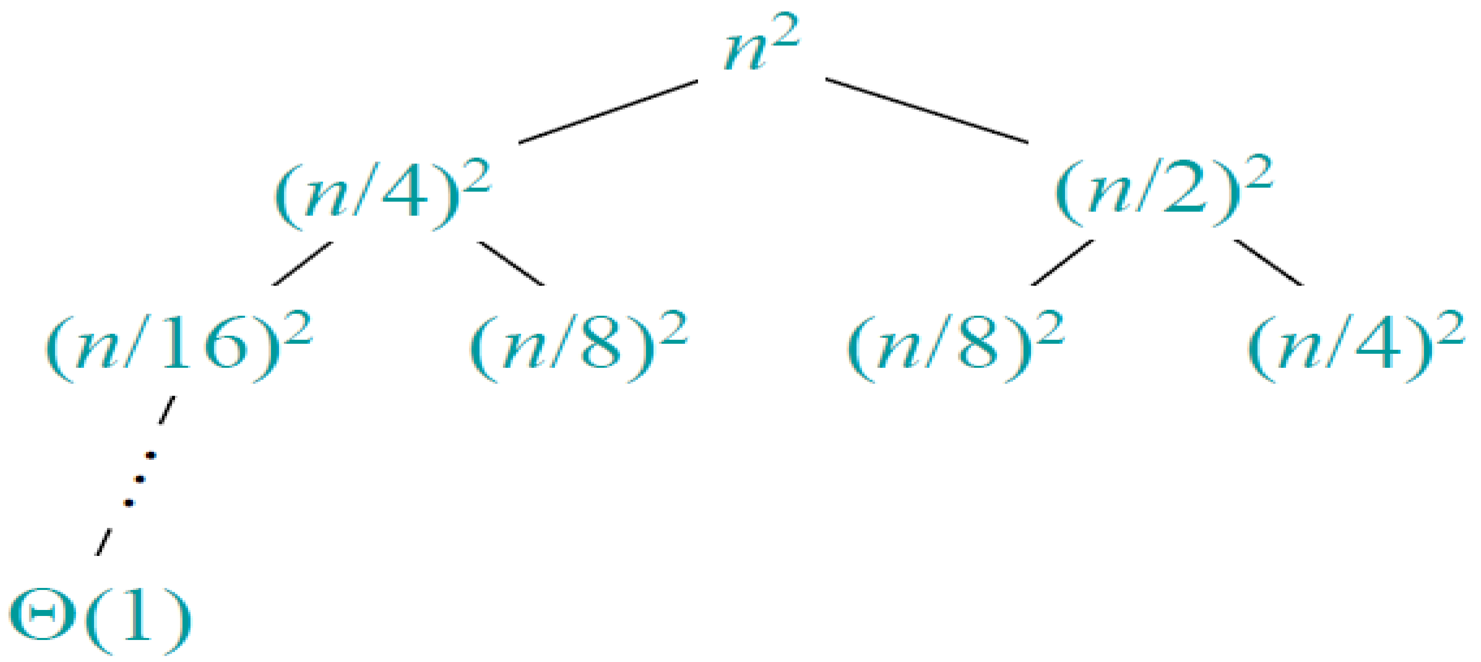
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



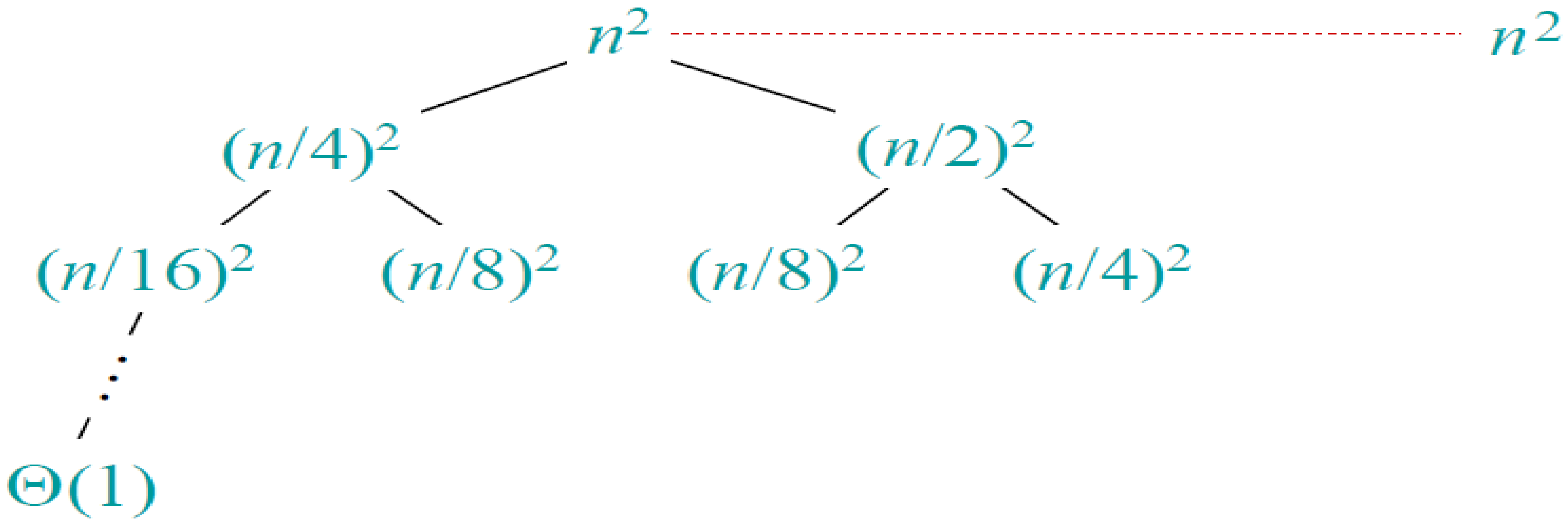
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Example of recursion tree

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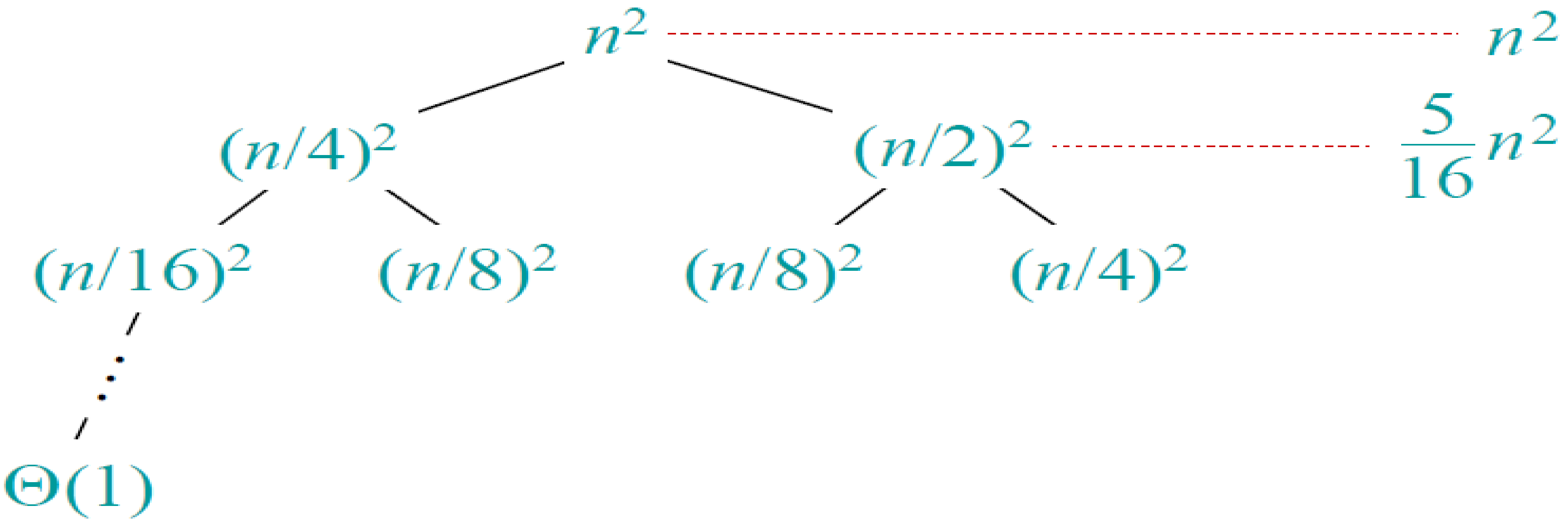
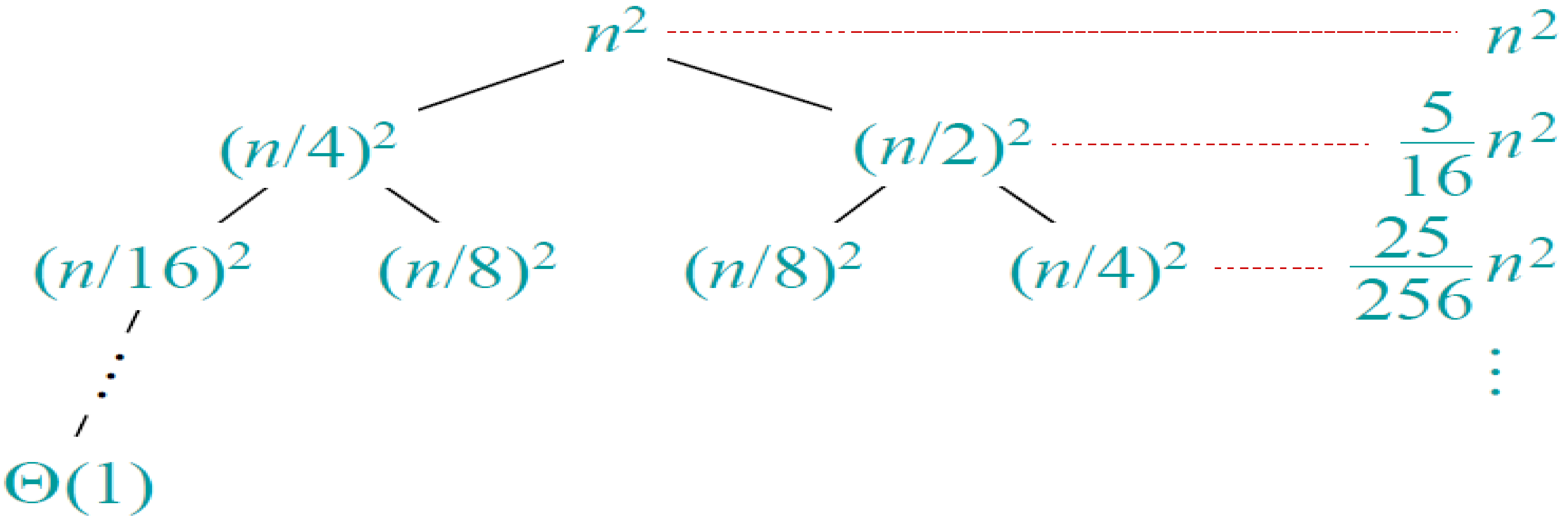


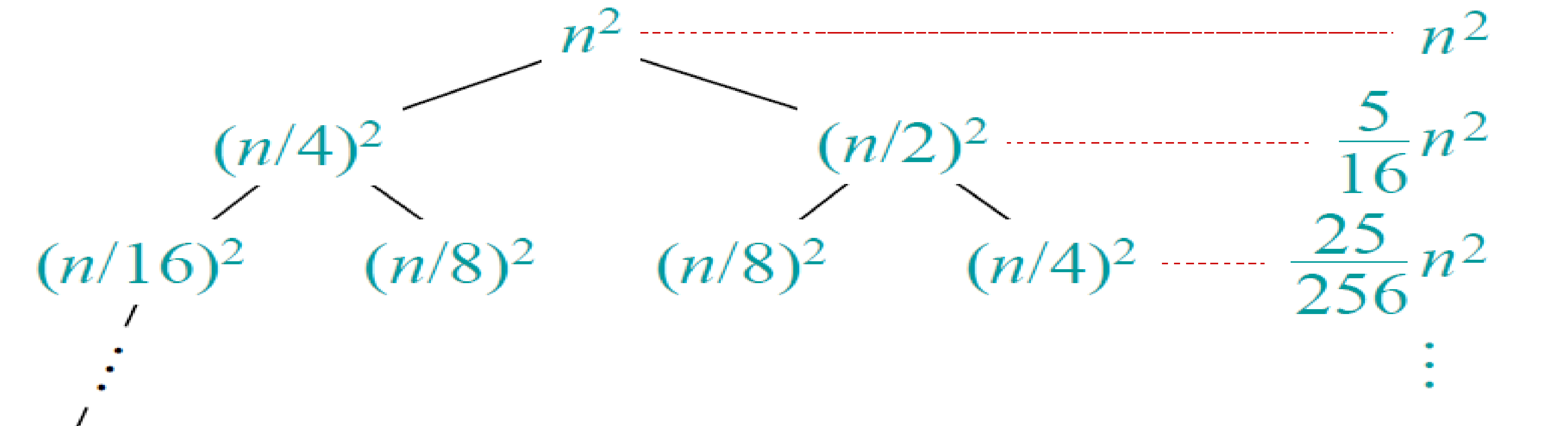
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



$$\text{Total} < n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$$

$$= \Theta(n^2)$$ *geometric series*

The master method

재발

- The master method applies to recurrences of the form

$$T(n) = aT(n/b) + f(n),$$

where $a \geq 1$, $b > 1$, *and f is asymptotically* positive.

점근적으로

Three common cases

Compare $f(n)$ with $\boxed{n^{\log_b a}}$: → *# of leaves*

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.

2. $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

Three common cases (cont'd)

Compare $f(n)$ with $n^{\log_b a}$:

0에 가까운 수

3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor),

   and $f(n)$ satisfies the ***regularity condition*** that $af(n/b) \le cf(n)$ for some constant $c < 1$.

   Solution: $T(n) = \Theta(f(n))$.

Examples

Ex. $T(n) = 4T(n/2) + n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n.$

CASE 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^2).$

Ex. $T(n) = 4T(n/2) + n^2$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2.$

CASE 2: $f(n) = \Theta(n^2 \lg^0 n)$, that is, $k = 0$.

$\therefore\ T(n) = \Theta(n^2 \lg n).$

Examples

Ex. $T(n) = 4T(n/2) + n^3$

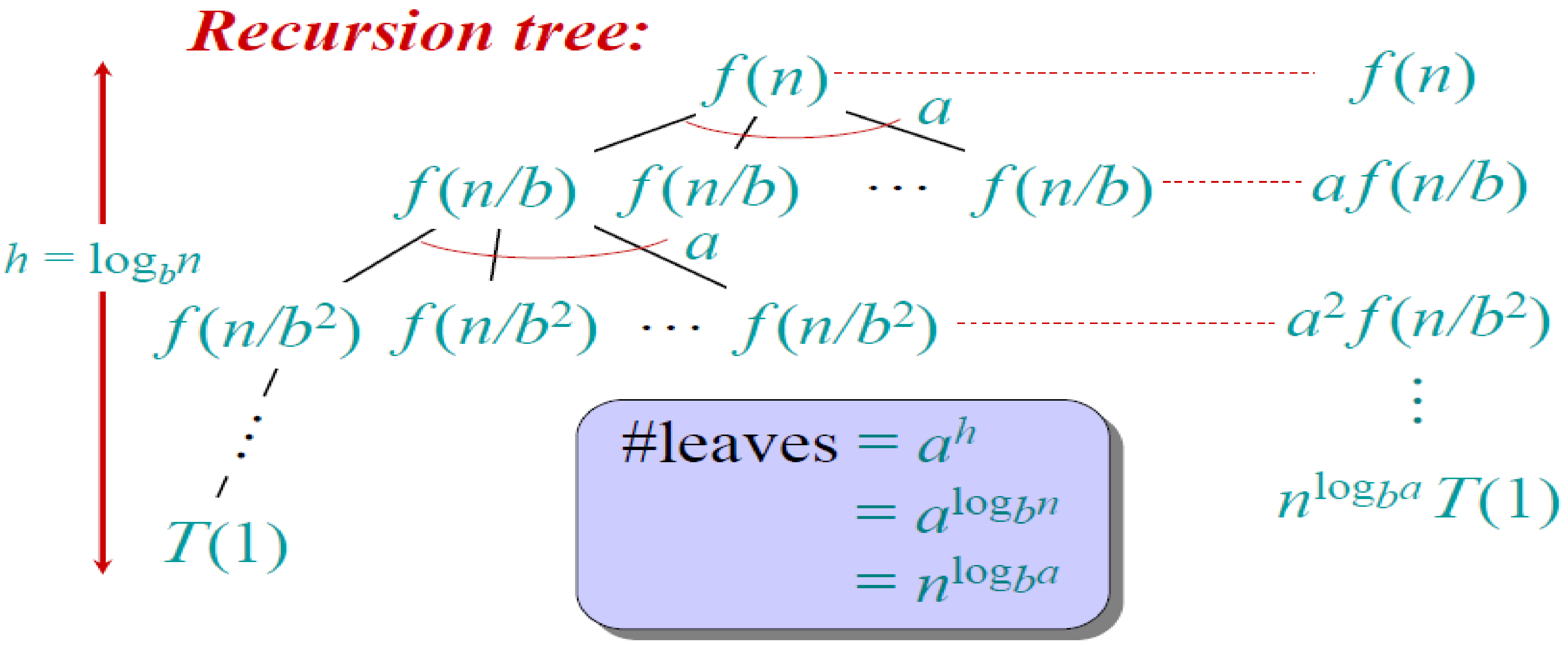
$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^3.$

Case 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

and $4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

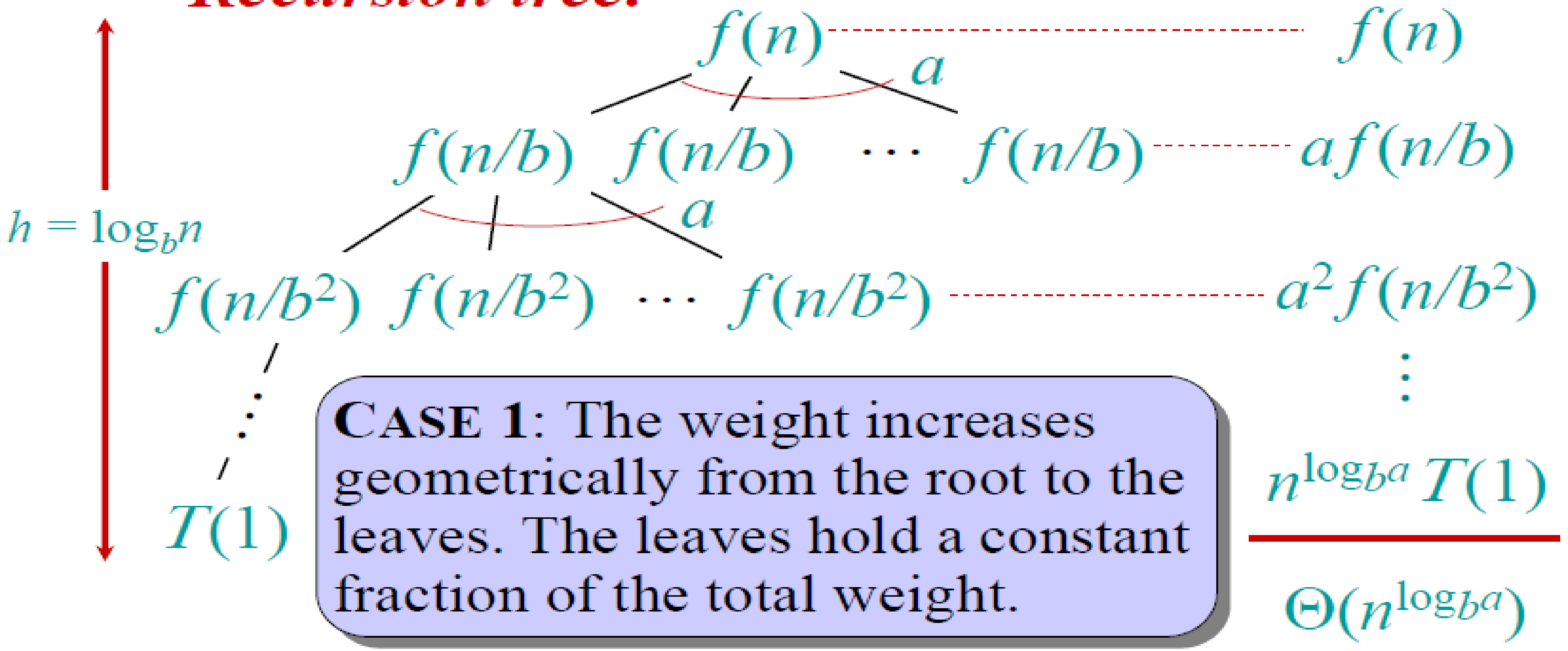
$\therefore\ T(n) = \Theta(n^3)$.

Idea of master method



Idea of master method

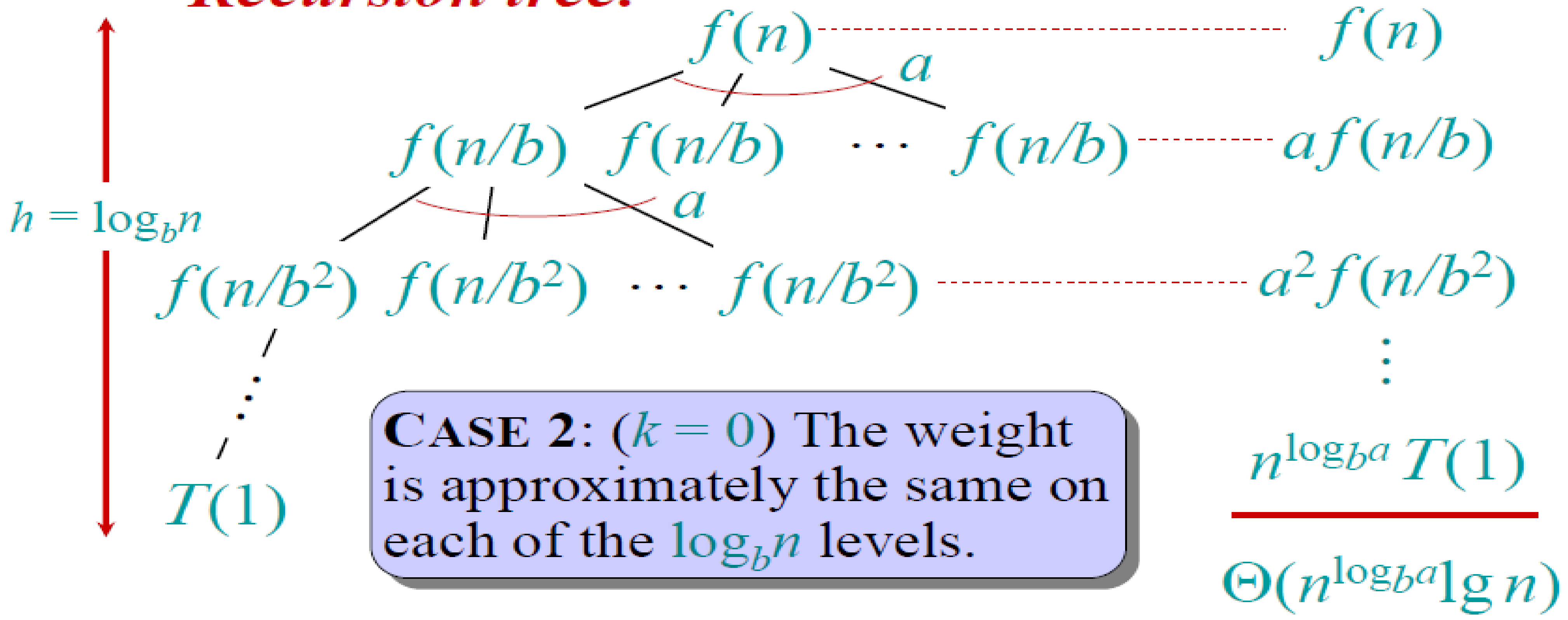
Recursion tree:



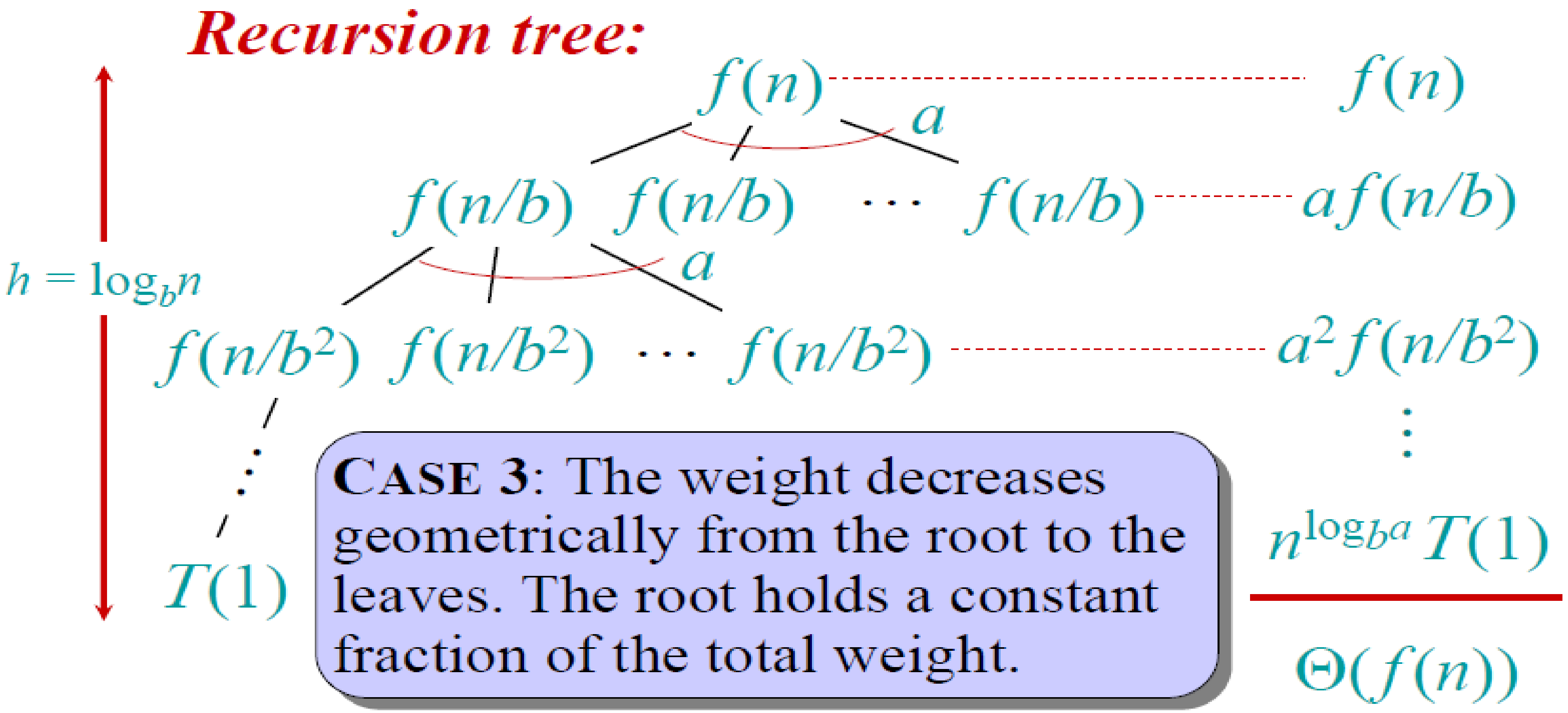
CASE 1: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

Idea of master method

Recursion tree:



Idea of master method



Appendix: geometric series

$1 \sim n-1$

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{for } x \neq 1$$

$$\frac{a(1-r^n)}{1-r}$$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x} \quad \text{for } |x| < 1$$

Implementation

Implementation of quick sort

QUICKSORT(A, p, r)
  if $p < r$
    $q = $ PARTITION(A, p, r)
    QUICKSORT$(A, p, q - 1)$
    QUICKSORT$(A, q + 1, r)$

Initial call is QUICKSORT$(A, 1, n)$.

PARTITION(A, p, r)
  $x = A[r]$
  $i = p - 1$
  for $j = p$ **to** $r - 1$
    if $A[j] \leq x$
      $i = i + 1$
      exchange $A[i]$ with $A[j]$
  exchange $A[i + 1]$ with $A[r]$
  return $i + 1$

```python
# Definition of quick sort
def quickSort(A, p:int, r:int):
  if p < r:
    q = partition(A, p, r)
    quickSort(A, p, q-1)
    quickSort(A, q+1, r)

def partition(A, p:int, r:int):
  x = A[r]
  i = p-1
  for j in range(p, r):
    if A[j] < x:
      i += 1
      A[i], A[j] = A[j], A[i]
  A[i+1], A[r] = A[r], A[i+1]
  return i+1
```

Example code test

- Code test: https://www.acmicpc.net/problem/11004
- Solving the problem using quick sort
- Example result of submission

제출 번호	아이디	문제	결과	메모리	시간	언어	코드 길이	제출한 시간
48606321	aikiho	11004	맞았습니다!!	646904 KB	1936 ms	PyPy3 / 수정	1207 B	1분 전

THANK YOU