

Let  $U = \{U_0, U_1, U_2, U_3\}$  be an open covering indexed by  $I$  and  $V = \{V_0, V_1\}$  be another open covering indexed by  $J$ . Say  $U$  refines  $V$  by

$$\begin{aligned} U_0 &\subseteq V_1, & U_1 &\subseteq V_0 \\ U_2 &\subseteq V_1, & U_3 &\subseteq V_0, \end{aligned}$$

call this refinement  $\tau$ .

Consider the diagram

$$\begin{array}{ccc} C(U, 0) & \xrightarrow{d} & C(U, 1) \\ \tau \uparrow & & \tau \uparrow \\ C(V, 0) & \xrightarrow{d} & C(V, 1), \end{array}$$

where  $\tau$  stands for the refinement map

$$\tau(f)(i_0, \dots, i_p) = f(\tau(i_0), \dots, \tau(i_p)) \mid_{U_{i_0} \cap \dots \cap U_{i_p}}.$$

Let  $f \in C(V, 0)$ , then  $\tau df \in C(U, 1)$ , then

$$\begin{aligned} \tau df\{0, 3\} &= df\{0, 1\} \mid_{U_0 \cap U_3} \\ &= (f\{1\} \mid_{V_0 \cap V_1}) \mid_{U_0 \cap U_3} - (f\{0\} \mid_{V_0 \cap V_1}) \mid_{U_0 \cap U_3} \\ &= f\{1\} \mid_{U_0 \cap U_3} - f\{0\} \mid_{U_0 \cap U_3}, \\ d\tau f\{0, 3\} &= \tau f\{3\} \mid_{V_0 \cap V_3} - \tau f\{0\} \mid_{V_0 \cap V_3} \\ &= (f\{0\} \mid_{U_0}) \mid_{V_0 \cap V_3} - (f\{1\} \mid_{U_0}) \mid_{V_0 \cap V_3}. \end{aligned}$$

They have different signs!