Formalising Proj Construction in Lean

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Abstract

Many object of interest in mathematics can be studied both analytically and algebraically, while at the same time, it is known that analytic geometry and algebraic geometry generally does not behave the same. However, the famous GAGA theorem asserts that for projective varieties, analytic and algebraic geometries are closely related; proof of the Fermat last theorem, for example, use this technique to transport between the two worlds [10]. I formalise Proj construction for any $\mathbb{N}$-graded R-algebra A as a starting point to the GAGA theorem and projective n-space is constructed as $\operatorname{Proj} A[X_0, \ldots, X_n]$. This would the first family of non-affine schemes formalised in any theorem prover.

2012 ACM Subject Classification Theory of computation → Logic and verification; Mathematics of computing → Topology

Keywords and phrases Lean, formalisation, algebraic geometry, scheme, Proj construction, projective geometry

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

Supplementary Material *Software (Source Code)*: `https://github.com/leanprover-community/mathlib/pull/18138/commits/00c4b0918a2c7a8b62291581b0e1eddf2357b5be`

Funding *Jujian Zhang*: Schrödinger Scholarship Scheme

1 Introduction

Algebraic geometry concerns polynomials and analytic geometry concerns holomorphic functions. Though all polynomials are holomorphic, the converse is not true; thus many analytic objects are not algebraic, for example $\{x \in \mathbb{C} \mid \sin(x) = 0\}$ can not be defined as zero locus of a polynomial in one variable, for polynomials always have only finite number of zeros. However, for projective varieties over $\mathbb{C}$, the categories of algebraic and analytic coherent sheaves are equivalent; an almost immediate consequence for this statement is that all closed analytic subset of projective n-space $\mathbb{P}_n$ is also algebraic [10, 4]. A crucial step of proving the above statement is to consider cohomology of projective n-space $\mathbb{P}_n$ [9].

While one can define $\mathbb{P}_n$ over $\mathbb{C}$ without consideration of other projective varieties, it would be more fruitful to formalise Proj construction as a **scheme** and recover $\mathbb{P}_n$ as $\operatorname{Proj} \mathbb{C}[X_0, \ldots, X_n]$, since, among other reasons, by considering different base rings, one obtain different projective varieties, for example, for any homogeneous polynomials $f_1, \ldots, f_k$, $\operatorname{Proj}\left(\frac{\mathbb{C}[X_0,\ldots,X_n]}{(f_1,\ldots,f_k)}\right)$ defines a projective hypersurface over $\mathbb{C}$.

In this paper I describe a formal construction of $\operatorname{Proj} A$ in the Lean3 [5] theorem prover which closely follows [6, Chapter II]. The formal construction use various results from the Lean mathematical library `mathlib`, most notably the graded algebra and Spec construction; this project has been partly accepted into `mathlib` already while the remainder is still undergoing a review process. The code discussed in this paper can be found on GitHub[1]. I have freely used axiom of choice and law of excluded middle throughout the project since the rest of `mathlib` freely use classical reasoning as well; consequently, the final construction is not computable.

[1] url: `https://github.com/leanprover-community/mathlib/pull/18138/`

42nd Conference on Very Important Topics (CVIT 2016).
Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:1–23:4
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

As previously mentioned, Proj construction heavily depends on graded algebra and Spec construction. A detailed description for graded algebra in Lean and mathlib as well as comparison with graded algebra in other theorem provers can be found in [12], for my purpose, I have chosen to use internal grading for any graded ring $A \cong \bigoplus \mathcal{A}_i$ so that the result of construction is about homogeneous prime ideals of A directly instead of $\bigoplus_i \mathcal{A}_i$. The earliest Spec construction in Lean and any other theorem prover can be found in [2] where the construction followed a "sheaf-on-a-basis" approach from [11, Section 01HR], however it differs significantly from the Spec construction currently found in mathlib where proofs from [6, Chapter II] were used; for this reason I have also chosen to follow the latter reference. Some other theorem provers also have or partially have Spec construction: in Isabelle/HOL, Spec is formalised by using locales and rewriting topology and ring theory part of existing library in [1], however the category of scheme is yet to be formalized; an early formalisation of Spec in Coq can be found in [3] and a definition of scheme in general can be found in its `UniMath` library; due to homotopy type theory, only a partial formalisation of Spec construction can be found in [8]. Though some theorem provers have definition of a general scheme, I could not find any concrete construction of a scheme other that Spec of a ring[2].

After explaining the mathematical details involved in Proj construction in Section 2, Lean code will be provided and explained in Section 3. For typographical reasons, some code of formalisation will be omitted and marked as `sorry`.

2 Mathematical details

In this section, familiarity of basic ring theory, topology and category theory will be assumed. In Sections 2.1 and 2.2, definition of a scheme is explained in detail; Spec construction will also be briefly explained in order to fix the mathematical approach used in mathlib. Then by following definition of a scheme step by step, Proj construction will be explained in Section 2.3.

2.1 Sheaves and Locally Ringed Spaces

Let X be a topological space and $\mathfrak{Opens}(X)$ be the category of open subsets of X.

▶ **Definition 1** (Presheaves [7]). *Let C be a category, a C-valued presheaf $\mathcal{F}$ on X is a functor $\mathfrak{Opens}(X)^{\mathsf{op}} \Longrightarrow C$. Morphisms between C-valued functor $\mathcal{F}, \mathcal{G}$ are natural transformation. The category thus formed is denoted as $\mathfrak{PSh}(X, C)$.*

In this paper, the category of interest is category of presheaves of rings $\mathfrak{Psh}(X, \mathfrak{Ring})$. More explicitly, a presheaf of ring $\mathcal{F}$ assigns each open subset $U \subseteq X$ with a ring $\mathcal{F}(U)$ called sections on U and for any open subsets $U \subseteq V \subseteq X$ a ring homomorphism $\mathcal{F}(V) \to \mathcal{F}(U)$ often denoted as res_U^V or simply with a vertical bar $s|_U$ (a section s on V restricted to U). Examples of presheaf of rings are abundant: considering open subsets of $\mathbb{C}$, $U \mapsto \{$(continuous, holomorphic) functions on $U\}$ with the natural restriction map defines presheaf of rings. In these examples, compatible sections on different open subsets can be glued together to form bigger sections on union of the said open subsets; this property can be generalized to arbitrary category:

[2] In this paper, all rings are assumed to be unital and commutative.

▶ **Definition 2** (Sheaves [7, 11]). *A presheaf $\mathcal{F} \in \mathfrak{Psh}(X, C)$ is said to be a sheaf if for any open covering of open set $U = \bigcup_i U_i \subseteq X$, the following diagram is an equalizer*

$$\mathcal{F}(U) \xrightarrow{(\mathrm{res}^U_{U_i})} \prod_i \mathcal{F}(U_i) \overset{\left(\mathrm{res}^{U_i}_{U_i \cap U_j}\right)}{\underset{\left(\mathrm{res}^{U_j}_{U_i \cap U_j}\right)}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j).$$

The category of sheaves $\mathfrak{Sh}(X, C)$ is the full subcategory of the category of presheaves.

▶ **Definition 3** (Locally Ringed Space [11, 6]). *If $\mathcal{O}_X$ is a sheaf on X, then the pair $(X, \mathcal{O}_X)$ is called a ringed space; a morphism between two ringed space $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a pair (f, ϕ) such that $f : X \to Y$ is continuous and $\phi : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves where $f_*\mathcal{O}_X \in \mathfrak{Sh}(Y)$ assigns $V \subseteq Y$ to $\mathcal{O}_X(f^{-1}(V))$. A locally ringed space $(X, \mathcal{O}_X)$ is ringed space such that for any $x \in X$, its stalk $\mathcal{O}_{X,x}$ is a local ring where $\mathcal{O}_{X,x} = \mathrm{colim}_{x \in U \in \mathfrak{Opens} X}\, \mathcal{O}_X(U)$; a morphism between two locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a morphism (f, ϕ) of ringed space such that for any $x \in X$ the ring homomorphism induced on stalk $\phi_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is local.*

Following from the previous definitions, if $\mathcal{O}_X$ is a presheaf and $U \subseteq X$ is an open subset, then there is a presheaf $\mathcal{O}_X|_U$ on U by assigning every open subset V of U to $\mathcal{O}_X(V)$, this is called restricting a presheaf; sheaves, ringed spaces and locally ringed spaces can also be similarly restricted.

2.2 Definition of Affine Scheme and Scheme

Spec construction

Let R be a ring and $\mathrm{Spec}\, R$ denote the set of prime ideals of R. Then for any subset $s \subseteq R$, its zero locus is defined as $\{\mathfrak{p} \mid s \subseteq \mathfrak{p}\}$. These zero loci can be considered as closed subsets of $\mathrm{Spec}\, R$, the topology thus formed is called the Zariski topology. Then a sheaf of rings on $\mathrm{Spec}\, R$ can be defined by assign $U \subseteq \mathrm{Spec}\, R$ to the ring

$$\left\{ s : \prod_{x \in U} R_x \mid s \text{ is locally a fraction} \right\},$$

where s is locally a fraction if and only if for any prime ideal $x \in U$, there is always an open subset $x \in V \subseteq U$ and $a, b \in R$ such that for any prime ideal $y \in V$, $b \notin y$ and $s(y) = \frac{a}{b}$. This sheaf $\mathcal{O}$ is called the structure sheaf of $\mathrm{Spec}\, R$. $(\mathrm{Spec}\, R, \mathcal{O})$ is a locally ringed space because for any prime ideal $x \subseteq R$, $\mathcal{O}_x \cong A_x$ [6].

Scheme

▶ **Definition 4** (Scheme). *A locally ringed space $(X, \mathcal{O}_X)$ is said to be a scheme if for any $x \in X$, there is always some ring R and some open subset $x \in U \subseteq X$ such that $(U, \mathcal{O}_X|_U) \cong (\mathrm{Spec}\, R, \mathcal{O}_{\mathrm{Spec}\, R})$ as locally ringed spaces. The category of schemes is the full subcategory of locally ringed spaces.*

2.3 Proj Construction

3 Formalisation details

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