talk

April 24, 2022

[1]:
```
%load_ext notexbook
%texify --code-theme github
```

[1]: `<IPython.core.display.HTML object>`

[2]:
```
from IPython.core.display import HTML
HTML("""
<style>

div.text_cell_render h1 {
text-align:center;
}

div.text_cell_render h2 {
text-align:center;
}

div.text_cell_render h3 {
text-align:center;
}

div.text_cell_render h4 {
text-align:center;
}

div.text_cell_render { /* Customize text cells */
font-size:1em;
}
</style>
""")
```

[2]: `<IPython.core.display.HTML object>`

1 Proj as a Scheme

1.0.1 Scheme

Scheme is a locally ringed space X that is locally affine, i.e. for any $x \in X$, there is an open neighourhood $x \in U$, such that $X|_U \cong \mathrm{Spec} R$ for some commutative ring R

```
structure Scheme extends to_LocallyRingedSpace : LocallyRingedSpace :=
(local_affine :   x : to_LocallyRingedSpace,
      (U : open_nhds x) (R : CommRing), nonempty
        (to_LocallyRingedSpace.restrict U.open_embedding
        Spec.to_LocallyRingedSpace.obj (op R)))
```

1.0.2 Locally ringed space

A locally ringed space is a sheafed space X such that for any $x \in X$, the stalk at x is a local ring.

```
structure LocallyRingedSpace extends SheafedSpace CommRing :=
(local_ring :   x, local_ring (presheaf.stalk x))
```

1.0.3 Sheafed space

A sheafed space is a presheafed space whose structure presheaf is a sheaf.

```
structure SheafedSpace extends PresheafedSpace C :=
(is_sheaf : presheaf.is_sheaf)
```

1.0.4 Presheafed space

A presheafed space is a topological space X with a structure presheaf on it.

```
structure PresheafedSpace :=
(carrier : Top)
(presheaf : carrier.presheaf C)
```

Given a $\mathbb{N}$-graded ring, we need a

1. a topology
2. a sheaf
3. a proof that stalks are local ring
4. an affine cover

1.1 Topology

Given an $\mathbb{N}$-graded ring A, define its projective spectrum as

$$\left\{ I : \text{homogeneous prime ideal of } A \,\middle|\, \bigoplus_{1 \le i} A_i \not\subseteq I \right\}$$

Then given any set $S \subseteq A$, the zero locus of s, $Z(s)$ is defined as

$$\{x : \mathrm{Proj} | s \subseteq x\}$$

Then - $Z(\emptyset) = \text{Proj}$ and $Z(A) = \emptyset$

- $Z(s) \cup Z(s') = Z(s \cap s')$
- $\bigcap_i Z(s_i) = Z\left(\bigcup_i s_i\right)$

Hence, by taking all the zero loci to be closed sets, we get a topology on Proj. This is the Zariski topology.

For any $a \in A$, define basic open set $D(a)$ as the complement of $Z(\{a\})$. All the basic open sets form a basis for the Zariski topology.

To prove this, we need to show that for any open set $O = Z(s)$ and $p \in O$, there is some basic open set $p \in D(a) \subseteq O$. Since $p \notin Z(s)$, $s \nsubseteq p$, so some a is in s but not in p. Then $p \in D(a)$. $D(a) \subseteq O$ if and only if $Z(s) \subseteq Z(a)$

1.2 Structure sheaf

Next we will define a sheaf of rings $\mathcal{O}$ on Proj S. For each $\mathfrak{p} \in \text{Proj } S$, we consider the ring $S_{(\mathfrak{p})}$ of elements of degree zero in the localized ring $T^{-1}S$, where T is the multiplicative system consisting of all *homogeneous* elements of S which are not in $\mathfrak{p}$. For any open subset $U \subseteq \text{Proj } S$, we define $\mathcal{O}(U)$ to be the set of functions $s: U \to \coprod S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$, and such that s is locally a quotient of elements of S: for each $\mathfrak{p} \in U$, there exists a neighborhood V of $\mathfrak{p}$ in U, and homogeneous elements a, f in S, of the same degree, such that for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$. Now it is clear that $\mathcal{O}$ is a presheaf of rings, with the natural restrictions, and it is also clear from the local nature of the definition that $\mathcal{O}$ is a sheaf.

1.2.1 Homogeneous localization

For a prime ideal $\mathfrak{p}$, the homogeneous localization of $A_{(\mathfrak{p})}$ is the subring

$$\left\{ \frac{a}{b} \in A_{(\mathfrak{p})} \,|a, b \in A_n \text{ for some } n \in \mathbb{N} \right\}.$$

We denote the homogeneously localized ring as $A^0_{(\mathfrak{p})}$ Homogeneously localized rings are local.

Implementation 1

```
def homogeneous_localization : subring (localization.at_prime  ) :=
{ carrier := {x |   (n :  ) (a b :   n) (b_not_mem : b.1    ),
    x = localization.mk a.1  b.1, b_not_mem },
  mul_mem' := _,
  one_mem' := _,
  add_mem' := _,
  zero_mem' := _,
  neg_mem' := _ }
```

1.2.2 Implementation 2

Using quotient

```
structure num_denom_same_deg :=
(deg :  )
(num denom :   deg)
(denom_not_mem : (denom : A)    )

def emb (p : num_denom_same_deg    ) : localization.at_prime   :=
localization.mk p.num  p.denom, p.denom_not_mem

def homogeneous_localization : Type* :=
quotient (setoid.ker $ emb    )
```

1.3 Structure sheaf

For an open set $U \subseteq \mathrm{Proj}$, take the sections on U as subring of dependent functions $\prod_{x \in U} A^0_{(x)}$ that are locally fractions, i.e.

$$\left\{ s \,\middle|\, \text{for all } x \in U \text{, there are some } a, b \text{ of same degree and } b \notin x \text{, such that } s(x) \text{ locally is } \frac{a}{b} \right\}$$

and restriction map as restriction of function.

This is indeed a ring because:

- $0 = \frac{0}{1}$
- $1 = \frac{1}{1}$

if $s(x) = \frac{a}{b}$ on V and $t(x) = \frac{c}{d}$ on W

- $(s + t)(x) = \frac{ad+bc}{bd}$ on $V \cap W$
- $(s \cdot t)(x) = \frac{ac}{bd}$
- $(-s)(x) = \frac{-a}{b}$

1.3.1 Implementation

- Prelocal predicate

```
variables {X : Top.{v}}
variables (T : X → Type v)

structure prelocal_predicate :=
(pred : Π {U : opens X}, (Π x : U, T x) → Prop)
(res :   {U V : opens X} (i : U   V) (f : Π x : V, T x)
    (h : pred f), pred (  x : U, f (i x)))
```

- Local predicate

```
structure local_predicate extends prelocal_predicate T :=
(locality :   {U : opens X} (f : Π x : U, T x)
```

```
  (w :   x : U,   (V : opens X) (m : x.1   V) (i : V   U),
      pred (  x : V, f (i x : U))), pred f)
```

- Sheafify

```
def prelocal_predicate.sheafify
    {T : X → Type v}
    (P : prelocal_predicate T) : local_predicate T :=
{ pred :=   U f,
      x : U,   (V : opens X) (m : x.1   V) (i : V   U),
        P.pred (  x : V, f (i x : U)),
  res := _,
  locality := _}
```

- In our case, we take T to be $\mathfrak{p} \to A^0_{(\mathfrak{p})}$ and

```
def is_fraction
    {U : opens (projective_spectrum.Top  )}
    (f : Π x : U, at x.1) : Prop :=
  (i :  ) (r s :   i),
    x : U,   (s_nin : s.1   x.1.as_homogeneous_ideal),
  (f x) = quotient.mk'  i, r, s, s_nin

def is_fraction_prelocal :
    prelocal_predicate (  (x : projective_spectrum.Top  ), at x) :=
{ pred :=   U f, is_fraction f,
  res := _ }

def is_locally_fraction :
    local_predicate (  (x : projective_spectrum.Top  ), at x) :=
(is_fraction_prelocal  ).sheafify
```

Then we can use `subsheaf_to_Types` which states the presheaf of $U \mapsto \{\prod_{x \in U} Tx\}$ satisfying P is a sheaf of sets.

```
def subsheaf_to_Types (P : local_predicate T) : sheaf (Type v) X := _
```

Then we can use the fact that a presheaf of rings is a sheaf if and only if the underlying presheaf of sets is sheaf.

```
variables (G : C   D)
variables [reflects_isomorphisms G]
variables [has_limits C] [has_limits D] [preserves_limits G]

variables {X : Top.{v}} (F : presheaf C X)

lemma is_sheaf_iff_is_sheaf_comp :
  presheaf.is_sheaf F   presheaf.is_sheaf (F   G) := _
```

1.3.2 Proj as a locally ringed space

We will prove that $\mathcal{O}_{\mathrm{Proj},\mathfrak{p}} \cong A^0_{(\mathfrak{p})}$ by constructing a bijective ring homomorphism. Then we are finished since $A^0_{(\mathfrak{p})}$ is always a local ring.

- $\mathcal{O}_{\mathrm{Proj},\mathfrak{p}} \to A^0_{(\mathfrak{p})}$ is given by
$$\langle U, f\rangle \mapsto \frac{a}{b}$$
where $f(\mathfrak{p}) = \frac{a}{b}$ on some open neighbourhood of $\mathfrak{p}$. This is formally implemented using universal property of colimit by descending $f \in \mathcal{O}(U) \mapsto f(\mathfrak{p}) \in A^0_{(\mathfrak{p})}$, hence is ring homomorphism by category theory.
- $A^0_{(\mathfrak{p})} \to \mathcal{O}_{\mathrm{Proj},\mathfrak{p}}$ is given by
$$\frac{a}{b} \mapsto \left\langle D(b), x \mapsto \frac{a}{b}\right\rangle$$

We don't need to prove the latter is a ring homomorphism, all we need is that the latter function is the right inverse of the former one.

1.4 An affine cover

Proj can be covered by basic open sets $D(f)$ for $f \in A_m$ with $0 < m$.

Let $\mathfrak{p} \in \mathrm{Proj}$, then $\bigoplus_{1 \le i} A_i \not\subseteq \mathfrak{p}$, then for some $f \in \bigoplus_{1 \le i} A_i$, $f \notin \mathfrak{p}$.

Write $f = f_1 + f_2 + \cdots$ where $f_i \in A_i$, then for some $0 < m$, $f_m \notin \mathfrak{p}$. Then $\mathfrak{p} \in D(f_m)$.

We need m to be strictly positive, because we will be using $m = (m-1) + 1$ a lot.

So we will prove that for f with positive degree m, $\mathrm{Proj}\,|_{D(f)} \cong \mathrm{Spec} A^0_f$ where A^0_f is the subring of degree zero elements in the localized ring A_f, i.e.

$$A^0_f := \left\{ \frac{a}{f^i}\, | a \in A_{mi} \right\}$$

To prove that these are isomorphic as locally ringed space, we need : - a homeomorphism h : $\mathrm{Proj}\,|_{D(f)} \cong \mathrm{Spec} A^0_f$ and - an isomorphism of sheaf $h_* \mathcal{O}_{\mathrm{Proj}|_{D(f)}} \cong \mathcal{O}_{\mathrm{Spec} A^0_f}$

In the following, we will fix an $f \in A_m$ with $0 < m$.

1.4.1 $\mathrm{Proj}\,|_{D(f)} \to \mathrm{Spec} A^0_f$

Given an $\mathfrak{p} \in \mathrm{Proj}\,|_{D(f)}$, i.e. a relevant homogeneous prime ideal in A, we need a point in $\mathrm{Spec} A^0_f$, i.e. a prime ideal.

So let's define
$$h(\mathfrak{p}) := \mathrm{span}\left\{\frac{g}{1} | g \in \mathfrak{p}\right\} \cap A^0_f = \mathrm{span}\left\{\frac{g}{f^i} | g \in \mathfrak{p} \cap A_{mi}\right\}$$

Then we need to check: - $h(\mathfrak{p})$ is prime: - $h(\mathfrak{p}) \neq A^0_f$; - if $x_1 x_2 \in h(\mathfrak{p})$, then either $x_1 \in h(\mathfrak{p})$ or $x_2 \in h(\mathfrak{p})$; - h is continuous with respect to Zariski topology.

$h(\mathfrak{p}) \neq A_f^0$ If $h(\mathfrak{p}) = A_f^0$, then $1 \in \operatorname{span}\left\{\frac{g}{1} | g \in \mathfrak{p}\right\}$. So write

$$1 = \sum_i \frac{a_i}{f^{n_i}} \frac{g_i}{1}.$$

By multiplying a suitable power of f, we get

$$\frac{f^N}{1} = \frac{\sum_i a_i g_i f^{k_i}}{1}$$

So $f^M f^N = f^M \sum_i a_i g_i f^{k_i}$, since the right handside is in $\mathfrak{p}$, the left handside is in $\mathfrak{p}$ too. Contradiction.

if $x_1 x_2 \in h(\mathfrak{p})$, then either $x_1 \in h(\mathfrak{p})$ or $x_2 \in h(\mathfrak{p})$ Write $x_1 = \frac{a_1}{f^{n_1}}$ and $x_2 = \frac{a_2}{f^{n_2}}$, then $\frac{a_1 a_2}{f^{n_1+n_2}} \in \operatorname{span}\left\{\frac{g}{1} | g \in \mathfrak{p}\right\}$, so write

$$\frac{a_1 a_2}{f^{n_1+n_2}} = \sum_i \frac{c_i}{f^{n_i}} \frac{g_i}{1}.$$

By multiplying a suitable power of f, we get

$$\frac{a_1 a_2 f^N}{1} = \frac{\sum_i c_i g_i f^{k_i}}{1}.$$

So $a_1 a_2 f^N f^M = f^M \sum_i c_i g_i f^{k_i}$, since right handside is in $\mathfrak{p}$ and $f \notin \mathfrak{p}$, either $a_1 \in \mathfrak{p}$ or $a_2 \in \mathfrak{p}$.

continuity Since $\operatorname{Spec} A_f^0$ also has a topological basis of basic open set, we only need to consider preimage of basic open sets. Take $\frac{a}{f^n}$, then $h^{-1}\left(D\left(\frac{a}{f^n}\right)\right) = D(f) \cap D(a)$.

$D(f) \cap D(a) \subseteq h^{-1}\left(D\left(\frac{a}{f^n}\right)\right)$ because if $y \in D(f) \cap D(a)$ and $\frac{a}{f^n} \in h(y)$, then

$$\frac{a}{f^n} = \sum_i \frac{c_i}{f^{n_i}} \frac{g_i}{1}$$

Then by multiplying suitable powers of f,

$$\frac{a f^N}{1} = \frac{\sum_i c_i g_i f^{m_i}}{1}$$

then this implies $a \in y$, contradiction.

If $h(y) \in D\left(\frac{a}{f^n}\right)$ and $a \in y$, then $\frac{a}{1} \in h(y)$, contradiction because $\frac{a}{f^n} = \frac{a}{1} \frac{1}{f^n} \in h(y)$.

1.4.2 $\operatorname{Spec} A_f^0 \to \operatorname{Proj}|_{D(f)}$

In this case, given a prime ideal in A_f^0 we need to construct an relevant homogeneous prime ideal of A. We define the following

$$g : x \mapsto \left\{ a \,\middle|\, \text{for all } i \in \mathbb{N}, \frac{a_i^m}{f^i} \in x \right\}.$$

Then we need to prove that - $g(x)$ is an ideal; - $g(x)$ is homogeneous; - $g(x)$ is prime; - $g(x)$ is relevant; - g is continuous.

$g(x)$ is an ideal $0 \in g(x)$

$g(x)$ is an ideal if $a, b \in g(x)$, then $a + b \in g(x)$ because

$$\begin{aligned}\left(\frac{(a_i+b_i)^m}{f^i}\right)^2 &= \frac{(a_i+b_i)^{2m}}{f^{2i}}\\ &= \frac{\sum_{j=0}^{2m}\binom{2m}{j}a_i^j b_i^{2m-j}}{f^{2i}}\\ &= \sum_{j=0}^{2m}\binom{2m}{j}\frac{a_i^j b_i^{2m-j}}{f^{2i}}\end{aligned}$$

if $m \le j$, we write

$$\frac{a_i^j b_i^{2m-j}}{f^{2i}} = \frac{a_i^m}{f^i}\frac{a_i^{j-m}b_i^{2m-j}}{f^i};$$

otherwise, we write

$$\frac{a_i^j b_i^{2m-j}}{f^{2i}} = \frac{b_i^m}{f^i}\frac{a_i^j b_i^{m-j}}{f^i}$$

$g(x)$ is an ideal if $a, b \in A$ and $b \in g(x)$, then $ab \in g(x)$. We induction on a - if $a = 0$, then $(ab)_i = 0$; - if $a \in A_n$ and $n \le i$ then $(ab)_i = ab_{n-i}$; - if the result hold for a_1, a_2, then $(a_1 + a_2)b = a_1 b + a_2 b \in g(x)$.

$g(x)$ is homogeneous If $a \in g(x)$ then for any $i \in \mathbb{N}$, $a_i \in g(x)$, for $(a_i)_j = a_i$ or 0.

$g(x)$ is prime For a homogeneous ideal, prime condition is equivalent to being homogeneously prime, i.e. $\mathfrak{p}$ is prime if and only if $1 \notin \mathfrak{p}$ and for any $a \in A_i$ and $b \in A_j$, $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. $1 \notin g(x)$ because the zeroth projection of 1 is 1.

Suppose $a \in A_i$ and $b \in A_j$, suppose $a, b \notin g(x)$ then $\frac{a_n^m}{f^n} \notin x$ for some $n \in \mathbb{N}$ and $\frac{b_k^m}{f^k} \notin x$ for some $k \in \mathbb{N}$. Then $n = i$ for otherwise $0 \notin x$ and similarly $k = j$. So $\frac{(ab)_{i+j}^m}{f^{i+j}} = \frac{a_i^m}{f^i}\frac{b_j^m}{f^j} \notin x$.

$g(x)$ is relevant If $\bigoplus_{1 \le i} A_i \le g(x)$ then

- $f \notin g(x)$, for otherwise $1 = \frac{f_m^m}{f^m} \in x$, but $x \ne A_f^0$
- also $f \in \bigoplus_{1 \le i} A_i$, since $f_0 = 0$. Contradicting $\bigoplus_{1 \le i} A_i \le g(x)$ and $f \notin g(x)$.

$h \circ g = 1$

- $h(g(x)) \le x$ because if $z \in h(g(x))$ then $z \in \operatorname{span}\left\{\frac{c}{f^i} | c \in g(x) \cap A_{mi}\right\}$. So we write

  $$z = \sum_i \frac{a_i}{f^{n_i}}\frac{c_i}{f^{k_i}}$$

  since $c_i \in g(x) \cap A_{mk_i}$, $\frac{c_i^m}{f^{mk_i}} \in x$, so $\left(\frac{c_i}{f^{k_i}}\right)^m \in x$, so we are done since x is prime.

- $x \le h(g(x))$ because if $\frac{a}{f^k} \in x$ for $a \in A_{mk}$, then $a \in g(x)$ for $\frac{a_i^m}{f^i} = \frac{a^m}{f^{mk}} = \left(\frac{a}{f^k}\right)^m \in x$ if $i = mk$ or 0 otherwise. Thus $\frac{a}{f^k} \in \operatorname{span}\left\{\frac{c}{1} | c \in g(x)\right\} \cap A_f^0$ since $\frac{a}{f^k} = \frac{a}{1}\frac{1}{f^k}$.

$g \circ h = 1$

- $g(h(x)) \le x$. If $z \in g(h(x))$, we need to show $z_i \in x$. Since $\frac{z_i^m}{f^i} \in h(x)$, we write

  $$\frac{z_i^m}{f^i} = \sum_j \frac{c_j}{f^{n_j}} \frac{d_j}{1},$$

  with $d_j \in x$, by multiplying a suitable power of f, we get

  $$z_i^m f^N = \sum_j c_j d_j f^{N_j}.$$

  So $z_i \in x$.

- $x \le g(h(x))$. If $z \in x$, then $z_i \in x$ because x is homogeneous. So $\frac{z_i^m}{f^i} = \frac{1}{f^i}\left(\frac{z_i}{1}\right)^m \in h(x)$ because $\frac{z_i}{1} \in h(x)$.

Now we know that h and g are both bijective.

continuity $\quad \mathrm{Proj}|_{D(f)}$ has a basis of the form $D(f) \cap D(a)$, so we check preimages of these are open. We consider $h(D(f) \cap D(a)) = h\left(D(f) \cap \bigcup_i D(a_i)\right) = \bigcup_i h(D(f) \cap D(a_i))$. Each $h(D(f) \cap D(a_i))$ is open because $h(D(f) \cap D(a_i)) = D\left(\frac{a_i^m}{f^i}\right)$ in $\mathrm{Spec}A_f^0$. To prove $h(D(f) \cap D(a_i)) = D\left(\frac{a_i^m}{f^i}\right)$, it is sufficient to prove $h^{-1}(D\left(\frac{a_i^m}{f^i}\right)) = D(f) \cap D(a)$ and this proven in continuity of h.

Now we only need to prove the preimage of $D(f) \cap D(a)$ is $h(D(f) \cap D(a))$, this can be easily proved by the fact that $h \circ g$ and $g \circ h$ are both identity.

Thus we have proven that $\mathrm{Proj}|_{D(f)} \cong \mathrm{Spec}A_f^0$ as topological spaces.

1.5 $\phi : h_* \mathcal{O}_{\mathrm{Proj}|_{D(f)}} \to \mathcal{O}_{\mathrm{Spec}A_f^0}$

Fix an open set $U \subseteq \mathrm{Spec}A_f^0$, so we need a ring homomorphism $\mathcal{O}_{\mathrm{Proj}|_{D(f)}}(h^{-1}(U)) \to \mathcal{O}_{\mathrm{Spec}A_f^0}(U)$.

Fix a $s : \prod_{z \in h^{-1}(U)} A_{(z)}^0$. If $x \in U$, then $g(x) \in h^{-1}(U)$. Then we have $s(g(x)) = \frac{n}{d} \in A_{(g(x))}^0$ for some $n, d \in A_i$.

Thus define

$$\phi_U(s)(x) = \frac{\frac{nd^{m-1}}{f^i}}{\frac{d^m}{f^i}}.$$

One can check that this is well defined. Assuming that this is indeed a ring homomorphism, it is easy to check the following diagram commute.

$$\begin{array}{ccc}
\mathcal{O}_{\mathrm{Proj}|_{D(f)}}(h^{-1}(U)) & \xrightarrow{\phi_U} & \mathcal{O}_{\mathrm{Spec}A_f^0}(U) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathrm{Proj}|_{D(f)}}(h^{-1}(V)) & \xrightarrow{\phi_V} & \mathcal{O}_{\mathrm{Spec}A_f^0}(V)
\end{array}$$

1.5.1 $\phi_U(1)(x) = 1$

Suppose $1 = s(g(1)) = \frac{n}{d}$ where $n, d \in A_i$, so for some $c \notin g(1)$, $nc = dc$. Since $c \notin g(1)$, there is some $j \in \mathbb{N}$, $\frac{c_j^m}{f^j} \notin x$. Then $(nc)_{i+j} = (dc)_{i+j} = nc_j = dc_j$. Then

$$\frac{c_j^m}{f^j}\frac{nd^{m-1}}{f^i} = \frac{c_j^m nd^{m-1}}{f^{i+j}} = \frac{(dc_j)^m}{f^{i+j}} = \frac{c_j^m}{f^j}\frac{d^m}{f^i},$$

i.e. $\phi_U(1)(x) = 1$.

Similarly one can prove that $\phi_U(0)(x) = 0$.

1.5.2 $\phi_U(s_1 + s_2)(x) = \phi_U(s_1)(x) + \phi_U(s_2)(x)$

Let's say $s_1(g(x)) = \frac{n_1}{d_1}$ with $n_1, d_1 \in A_{i_1}$, $s_2(g(x)) = \frac{n_2}{d_2}$ with $n_2, d_2 \in A_{i_2}$ and $(s_1 + s_2)(g(x)) = \frac{n_{12}}{d_{12}} = \frac{n_1}{d_1} + \frac{n_2}{d_2}$ with $n_{12}, d_{12} \in A_{i_{12}}$. So our goal is to check

$$\frac{\frac{n_{12}d_{12}^{m-1}}{f^{i_{12}}}}{\frac{d_{12}^m}{f^{i_{12}}}} = \frac{\frac{n_1 d_1^{m-1}}{f^{i_1}}}{\frac{d_1^m}{f^{i_1}}} + \frac{\frac{n_2 d_2^{m-1}}{f^{i_2}}}{\frac{d_2^m}{f^{i_2}}}. \tag{1}$$

From $\frac{n_{12}}{d_{12}} = \frac{n_1}{d_1} + \frac{n_2}{d_2} = \frac{n_1 d_2 + n_2 d_1}{d_1 d_2}$, so we can find a $c \notin g(x)$, such that

$$n_{12}d_1 d_2 c = (n_1 d_2 + n_2 d_1)d_{12}c.$$

Since $c \notin g(x)$, there is some $j \in \mathbb{N}$ such that $\frac{c_j^m}{f^j} \notin x$. Then by taking the $i_1 + i_2 + i_{12} + j$-th projection

$$n_{12}d_1 d_2 c_j = (n_1 d_2 + n_2 d_1)d_{12}c_j.$$

Using this, one can check that by multiplying $\frac{c_j^m}{f^j}$, the desired equality can be proved.

Similarly $\phi_U(s_1 s_2)(x) = \phi_U(s_1)(x)\phi_U(s_2)(x)$.

$$\frac{\frac{n_1 d_1^{m-1} d_2^m}{f^{i_1+i_2}} + \frac{n_2 d_2^{m-1} d_1^m}{f^{i_1+i_2}}}{\frac{d_1^m d_2^m}{f^{i_1+i_2}}}$$

$$\frac{\frac{n_1 d_1^{m-1} d_2^m + n_2 d_2^{m-1} d_1^m}{f^{i_1+i_2}}}{\frac{d_1^m d_2^m}{f^{i_1+i_2}}}$$

$$\frac{\frac{n_1 d_1^{m-1} d_2^m}{f^{i_1+i_2}} + \frac{n_2 d_2^{m-1} d_1^m}{f^{i_1+i_2}}}{\frac{d_1^m d_2^m}{f^{i_1+i_2}}} = \frac{\frac{n_1 d_1^{m-1} d_2^m + n_2 d_2^{m-1} d_1^m}{f^{i_1+i_2}}}{\frac{d_1^m d_2^m}{f^{i_1+i_2}}}$$

1.5.3 $\phi_U(s)$ is locally quotient

Since s is locally quotient, for any $x \in U$, there is some open set $V \subseteq \operatorname{Proj}$ such that $g(x) \in V \subseteq h^{-1}(U)$ such that $s(y) = \frac{a}{b}$ for all $y \in V$ where $a, b \in A_n$ and $b \notin y$.

To check $\phi_U(s)$ is locally quotient, use the open subset $h(V)$ and check that for all $z \in h(V)$

$$\phi_U(s)(z) = \frac{ab^{m-1}}{b^m}.$$

1.6 $\psi : \mathcal{O}_{\mathrm{Spec}A_f^0} \to h_*\mathcal{O}_{\mathrm{Proj}|_{D(f)}}$

Fix an open set $U \subseteq \mathrm{Spec}A_f^0$, so we need a ring homomorphism $\mathcal{O}_{\mathrm{Spec}A_f^0}(U) \to \mathcal{O}_{\mathrm{Proj}|_{D(f)}}(h^{-1}(U))$.

Let $s \in \mathcal{O}_{\mathrm{Spec}A_f^0}(U)$ and $y \in h^{-1}(U)$, then $h(y) \in U$, so we can write

$$s(h(y)) = \frac{a}{b}$$

where $a, b \in A_f^0$. Then we can write $a = \frac{n_a}{f^{i_a}}$ for some $n_a \in A_{mi_a}$ and $b = \frac{n_b}{f^{i_b}}$ for some $n_b \in A_{mi_b}$.

Then we can define

$$\psi_U(s)(y) = \frac{n_a f_b^i}{n_b f^{i_a}}$$

.

One can check this is well defined. Assuming that this is a ring homomorphism, it is easy to check the following diagram commute.

$$\begin{array}{ccc}
\mathcal{O}_{\mathrm{Spec}A_f^0}(U)) & \xrightarrow{\psi_U} & \mathcal{O}_{\mathrm{Proj}|_{D(f)}}(h^{-1}(U)) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathrm{Spec}A_f^0}(V)) & \xrightarrow{\psi_V} & \mathcal{O}_{\mathrm{Proj}|_{D(f)}}(h^{-1}(V))
\end{array}$$

1.6.1 $\psi_U(1)(y) = 1$

Suppose $\frac{1}{1} = 1(h(y)) = \frac{a}{b}$ where $a, b \in A_f^0$, then for some $\frac{c}{f^l} \notin h(y)$

$$\frac{n_a c}{f^{i_a + l}} = \frac{n_b c}{f^{i_b + l}}$$

Thus, for some $n_1 \in \mathbb{N}$

$$n_a c f^{i_b + l + n_1} = n_b c f^{i_a + l + n_1}$$

Thus

$$\psi_U(1)(y) = \frac{n_a f^{b_i}}{n_b f^{i_a}} = 1.$$

Similarly $\psi_U(0)(y) = 0$.

1.6.2 $\psi_U(s_1 s_2)(y) = \psi_U(s_1)(y) \cdot \psi_U(s_2)(y)$

Let's write

$$s_1(h(y)) = \frac{\frac{a_1}{f^{i_1}}}{\frac{b_1}{f^{j_1}}}, s_2(h(y)) = \frac{\frac{a_2}{f^{i_2}}}{\frac{b_2}{f^{i_2}}}, (s_1 s_2)(h(y)) = \frac{\frac{a_{12}}{f^{i_{12}}}}{\frac{b_{12}}{f^{j_{12}}}}.$$

Then $\frac{\frac{a_1 a_2}{f^{i_1+i_2}}}{\frac{b_1 b_2}{f^{j_1+j_2}}} = \frac{\frac{a_{12}}{f^{i_{12}}}}{\frac{b_{12}}{f^{j_{12}}}}$, i.e. for some $\frac{c}{f^l}$, we have

$$\frac{a_1 a_2 b_{12} c}{f^{i_1+i_2+j_{12}+l}} = \frac{a_{12} b_1 b_2 c}{f^{i_{12}+j_1+j_2+l}},$$

i.e. for some $L \in \mathbb{N}$, we have

$$a_1 a_2 b_{12} c f^{i_{12}+j_1+j_2+l+L} = a_{12} b_1 b_2 c f^{i_1+i_2+j_{12}+l+L}.$$

We can use this to prove that

$$\frac{a_{12} f^{j_{12}}}{b_{12} f^{i_{12}}} = \frac{a_1 f^{j_1}}{b_1 f^{i_1}} \cdot \frac{a_2 f^{j_2}}{b_2 f^{i_2}}.$$

Similarly, we can prove that $\psi_U(s_1 + s_2)(y) = \psi_U(s_1)(y) + \psi_U(s_2)(y)$.

1.6.3 $\psi_U(s)$ locally is fraction

Since s locally is a fraction, there are open sets $h(y) \in V \subseteq U$, such that for all $z \in V$,

$$s(z) = \frac{\frac{a}{f^{l_1}}}{\frac{b}{f^{l_2}}}.$$

Then we use $h^{-1}(V)$ and verify that locally

$$\psi_U(s)(y) = \frac{a f^{l_2}}{b f^{l_1}}.$$

1.7 $\psi \circ \phi = 1$

Let $s \in \mathcal{O}_{\mathrm{Proj}|_{D(f)}}(h^{-1}(U))$, then for $x \in h^{-1}(U)$

$$\phi_U(s) = x \mapsto \frac{\frac{nd^{m-1}}{f^i}}{\frac{d^m}{f^i}},$$

where $s(x) = \frac{n}{d}$.

Thus

$$\psi_U(\phi_U(s))(x) = \frac{nd^{m-1} f^i}{d^m f^i} = \frac{n}{d} = s(x)$$

1.8 $\phi \circ \psi = 1$

Let $s \in \mathcal{O}_{\mathrm{Spec} A_f^0}(U)$, then for $x \in U$

$$\psi_U(s) = x \mapsto \frac{n_a f^{i_b}}{n_b f^{i_a}}$$

where

$$s(x) = \frac{\frac{n_a}{f^{i_a}}}{\frac{n_b}{f^{i_b}}}.$$

Thus

$$\phi_U(\psi_U(s))(x) = \frac{\frac{n_a f^{i_b} \left(n_b f^{i_a}\right)^{m-1}}{f^j}}{\frac{(n_b f^{i_a})^m}{f^j}} = \frac{\frac{n_a}{f^{i_a}}}{\frac{n_b}{f^{i_b}}} = s(x).$$