Summary of GAGA

1 $\mathbb{P}^n$ as a Group Scheme

1.1 Product of Affine Schemes

Notation. In this subsection, $(X, \mathcal{O}_X) = (\mathrm{Spec}R, \widetilde{R})$ and $(Y, \mathcal{O}_Y) = (\mathrm{Spec}S, \widetilde{S})$ are affine schemes of finite type over $\mathbb{C}$ where R and S are finitely generated $\mathbb{C}$-algebra.

Definition 1. $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ is defined as $(\mathrm{Spec}(R \otimes_{\mathbb{C}} S), \widetilde{R \otimes_{\mathbb{C}} S})$.

$(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ satisfies the universal property of product in the category of affine schemes of finite type over $\mathbb{C}$. The natural inclusions $i_1 : R \to R \otimes_{\mathbb{C}} S$ given by $i_1(r) = r \otimes 1$ and $i_2 : S \to R \otimes_{\mathbb{C}} S$ given by $i_2(s) = 1 \otimes s$ induces $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ and $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$; and for any $(Z, \mathcal{O}_Z)$, an affine schemes of finite type over $\mathbb{C}$, then any maps $(Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ and $(Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ factors uniquely through $(X \times Y, \mathcal{O}_{X \times Y})$, i.e. we have the following commutative diagram

$$\begin{tikzcd}
 & & (X, \mathcal{O}_X) \\
(Z, \mathcal{O}_Z) \arrow[r, dashed, "\exists!"] \arrow[urr, bend left] \arrow[drr, bend right] & (X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y) \arrow[ur] \arrow[dr] & \\
 & & (Y, \mathcal{O}_Y)
\end{tikzcd}.$$

1.2 Affine Group Scheme

Notation. In this subsection, $(G, \mathcal{O}_G) = (\mathrm{Spec}R, \widetilde{R})$ is an affine scheme of finite type over $\mathbb{C}$.

Definition 2. $(G, \mathcal{O}_G)$ is an affine group scheme if there are $\mu : (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \to (G, \mathcal{O}_G)$, $e : (\mathrm{Spec}\mathbb{C}, \widetilde{\mathbb{C}}) \to (G, \mathcal{O}_G)$ and $\iota : (G, \mathcal{O}_G) \to (G, \mathcal{O}_G)$ such that the following diagrams commute

- associativity of μ:

$$\begin{tikzcd}
(G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \arrow[r, "\mathrm{id}\times\mu"] \arrow[d, "\mu\times\mathrm{id}"] & (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \arrow[d, "\mu"] \\
(G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \arrow[r, "\mu"] & (G, \mathcal{O}_G)
\end{tikzcd}$$

- neutral element e:

$$\begin{tikzcd}
(\mathrm{Spec}\mathbb{C}, \widetilde{\mathbb{C}}) \times (G, \mathcal{O}_G) \times (\mathrm{Spec}\mathbb{C}, \widetilde{\mathbb{C}}) \arrow[rr, "\mathrm{id}\times\mathrm{id}\times e"] \arrow[dr, "\cong"] \arrow[dd, "e\times\mathrm{id}\times\mathrm{id}"] & & (\mathrm{Spec}\mathbb{C}, \widetilde{\mathbb{C}}) \times (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \arrow[d, "\cong"] \\
 & (G, \mathcal{O}_G) \arrow[dr, "\mathrm{id}"] & (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \arrow[d, "\mu"] \\
(G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \times (\mathrm{Spec}\mathbb{C}, \widetilde{\mathbb{C}}) \arrow[r, "\cong"] & (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \arrow[r, "\mu"] & (G, \mathcal{O}_G)
\end{tikzcd},$$

- inverse i:

$$\begin{tikzcd}
(G, \mathcal{O}_G) \arrow[rr, "(\mathrm{id},\iota)"] \arrow[dr, "\mathrm{constant}"] \arrow[dd, "(\iota,\mathrm{id})"] & & (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \arrow[dd] \\
 & (\mathrm{Spec}\mathbb{C}, \widetilde{\mathbb{C}}) \arrow[dr, "e"] & \\
(G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \arrow[rr, "\mu"] & & (G, \mathcal{O}_G)
\end{tikzcd}.$$

Notation. μ is often written as $\cdot$ to indicate multiplication and ι as $^{-1}$ to indicate inverse.

In definition 2, since $(G, \mathcal{O}_G) \times (G, \mathcal{O}_G) = (\mathrm{Spec}(R \otimes_{\mathbb{C}} R), \widetilde{R \otimes_{\mathbb{C}} R})$, the maps μ (e, ι resp.) gives a map $\mu' : R \to R \otimes_{\mathbb{C}} R$ ($e' : R \to \mathbb{C}$, $\iota' : \mathbb{R} \to \mathbb{R}$ resp.) such that the following diagrams commute

$$\begin{tikzcd}
R \arrow[r, "\mu'"] \arrow[d, "\mu'"] & R \otimes_{\mathbb{C}} R \arrow[d, "\mathrm{id}\otimes\mu'"] \\
R \otimes_{\mathbb{C}} R \arrow[r, "\mu'\otimes\mathrm{id}"] & R \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\mu'"] \arrow[d, "\mu'"] \arrow[dr, "\mathrm{id}"] & R \otimes_{\mathbb{C}} R \arrow[r, "\cong"] & R \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} \mathbb{C} \arrow[dd, "e'\otimes\mathrm{id}\otimes\mathrm{id}"] \\
R \otimes_{\mathbb{C}} R \arrow[d, "\cong"] & R \arrow[dr, "\cong"] & \\
\mathbb{C} \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} R \arrow[rr, "\mathrm{id}\otimes\mathrm{id}\otimes e'"] & & \mathbb{C} \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[rr, "\mu'"] \arrow[dr, "e'"] \arrow[dd, "\mu'"] & & R \otimes_{\mathbb{C}} R \arrow[dd, "r\otimes r'\mapsto\iota'(r)r'"] \\
& \mathbb{C} \arrow[dr, "*\text{see below}"] & \\
R \otimes_{\mathbb{C}} R \arrow[rr, "r\otimes r'\mapsto r\iota'(r')"] & & R
\end{tikzcd}$$

where (*) is the map $\mathbb{C} \to R$ giving R the $\mathbb{C}$-algebra structure.

1.2.1 Trivial Group Scheme

Consider $\mathbb{C}$ as the trivial $\mathbb{C}$-algebra. Let $\mu' : \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ be $\mathrm{id} : \mathbb{C} \to \mathbb{C}$ after identifying $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ and $e', \iota' : \mathbb{C} \to \mathbb{C}$ all be identity on $\mathbb{C}$. In this case $(G, \mathcal{O}_G) = (\mathrm{Spec}\mathbb{C}, \widetilde{\mathbb{C}})$ is a space with only one point.

1.2.2 A Non-trivial Group Scheme

Notation. In this subsubsection, R is $\mathbb{C}[x]$ and localization maps are denoted by α with subscript, for example $\alpha_x : R \to R\left[\frac{1}{x}\right]$ localises x.

Consider the ring of Laurent polynomial $\mathbb{C}[x, x^{-1}] \cong R\left[\frac{1}{x}\right]$ as a $\mathbb{C}$-algebra. Let the map $\nu' : R \to R \otimes_{\mathbb{C}} R$ be given by $x \mapsto x \otimes x$. Then

$$R \xrightarrow{\nu'} R \otimes_{\mathbb{C}} R \xrightarrow[\phi \text{ is the inclusion}]{\phi\otimes\phi} \mathbb{C}[x, x^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x, x^{-1}]$$

send x to $x \otimes x$ which is invertible, hence by universal property of localisation, there is a unique map μ' such that

$$\begin{tikzcd}
R \arrow[r, "\nu'"] \arrow[d, "\alpha_x"] & R \otimes_{\mathbb{C}} R \arrow[d, "\phi\otimes\phi"] \\
R\left[\frac{1}{x}\right] \arrow[r, dashed, "\exists!"] \arrow[d, "\cong"] & \mathbb{C}[x, x^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x, x^{-1}] \\
\mathbb{C}[x, x^{-1}] \arrow[ur, dashed, "\mu'"'] &
\end{tikzcd}.$$

The map $e' : \mathbb{C}[x, x^{-1}] \to \mathbb{C}[x, x^{-1}]$ is given by evaluating at $x = 1$ and $\iota' : \mathbb{C}[x, x^{-1}] \to \mathbb{C}[x, x^{-1}]$ is given by $\iota'(x) = x^{-1}$ and $\iota'(x^{-1}) = x$.

Associativity of μ' Need to check $x \in \mathbb{C}[x, x^{-1}]$ and $x^{-1} \in \mathbb{C}[x, x^{-1}]$. For x, the first composition

$$\begin{tikzcd}
x \in \mathbb{C}[x, x^{-1}] \arrow[r, mapsto, "\mu'"] & x \otimes x \arrow[d, mapsto, "\mathrm{id}\otimes\mu'"] \\
& x \otimes x \otimes x
\end{tikzcd}$$

equals to the second composition

$$\begin{tikzcd}
x \in \mathbb{C}[x, x^{-1}] \arrow[d, mapsto, "\mu'"] & \\
x \otimes x \arrow[r, mapsto, "\mu'\otimes\mathrm{id}"] & x \otimes x \otimes x.
\end{tikzcd}$$

For x^{-1}, just replace every x by x^{-1}, and get the same thing.

Neutral element e' For $x \in \mathbb{C}[x, x^{-1}]$, the first composition is

$$\begin{tikzcd}
x \arrow[r, mapsto, "\mu'"] & x \otimes x \arrow[r, "\cong"] & x \otimes x \otimes 1 \arrow[d, mapsto, "e'\otimes\mathrm{id}\otimes\mathrm{id}"] \\
& & 1 \otimes x \otimes 1;
\end{tikzcd}$$

the second composition is

$$\begin{tikzcd} x \arrow[d, mapsto, "\mu'"] \\ x \otimes x \arrow[d, mapsto, "\cong"] \\ 1 \otimes x \otimes x \arrow[r, mapsto, "\mathrm{id}\otimes\mathrm{id}\otimes e'"] & 1 \otimes x \otimes 1; \end{tikzcd}$$

and the diagonal composition is $x \xmapsto{\mathrm{id}} x \xmapsto{\cong} 1 \otimes x \otimes 1$. The case for x^{-1} is the same as above because x^{-1} evaluate at 1 is also 1.

Inverse ι' The first composition is

$$\begin{tikzcd} x \arrow[r, mapsto, "\mu'"] & x \otimes x \arrow[d, mapsto, "r\otimes r'\mapsto\iota'(r)r'"] \\ & x^{-1}x = 1, \end{tikzcd}$$

and the second composition is

$$\begin{tikzcd} x \arrow[d, "\mu'"] \\ x \otimes x \arrow[r, "r\otimes r'\mapsto r\iota'(r')"] & xx^{-1} = 1, \end{tikzcd}$$

and the third composition is $x \xrightarrow{e'} 1 \longrightarrow 1$. The case for x^{-1} is similar.