Summary of GAGA

Jujian Zhang

1 Affine schemes of finite type

2 $\mathbb{P}^n$ as a Group Scheme

2.1 Product of Affine Schemes

Notation. In this subsection, $(X, \mathcal{O}_X) = (\mathrm{Spec} R, \widetilde{R})$ and $(Y, \mathcal{O}_Y) = (\mathrm{Spec} S, \widetilde{S})$ are affine schemes of finite type over $\mathbb{C}$ where R and S are finitely generated $\mathbb{C}$-algebra.

Definition 1. $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ is defined as $(\mathrm{Spec}(R \otimes_{\mathbb{C}} S), \widetilde{R \otimes_{\mathbb{C}} S})$.

$(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ satisfies the universal property of product in the category of affine schemes of finite type over $\mathbb{C}$. The natural inclusions $i_1 : R \to R \otimes_{\mathbb{C}} S$ given by $i_1(r) = r \otimes 1$ and $i_2 : S \to R \otimes_{\mathbb{C}} S$ given by $i_2(s) = 1 \otimes s$ induces $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ and $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$; and for any $(Z, \mathcal{O}_Z)$, an affine schemes of finite type over $\mathbb{C}$, then any maps $(Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ and $(Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ factors uniquely through $(X \times Y, \mathcal{O}_{X \times Y})$, i.e. we have the following commutative diagram

$$
\begin{tikzcd}
 & & (X,\mathcal{O}_X) \\
(Z,\mathcal{O}_Z) \arrow[r, dashed, "\exists!"] \arrow[urr, bend left] \arrow[drr, bend right] & (X,\mathcal{O}_X)\times(Y,\mathcal{O}_Y) \arrow[ur] \arrow[dr] & \\
 & & (Y,\mathcal{O}_Y)
\end{tikzcd}
.
$$

2.2 Affine Group Scheme

Notation. In this subsection, $(G, \mathcal{O}_G) = (\mathrm{Spec} R, \widetilde{R})$ is an affine scheme of finite type over $\mathbb{C}$.

Definition 2. $(G, \mathcal{O}_G)$ is an affine group scheme if there are $\mu : (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \to (G, \mathcal{O}_G)$, $e : (\mathrm{Spec}\mathbb{C}, \widetilde{\mathbb{C}}) \to (G, \mathcal{O}_G)$ and $\iota : (G, \mathcal{O}_G) \to (G, \mathcal{O}_G)$ such that the following diagrams commute

- associativity of μ:

$$
\begin{tikzcd}
(G,\mathcal{O}_G)\times(G,\mathcal{O}_G)\times(G,\mathcal{O}_G) \arrow[r, "\mathrm{id}\times\mu"] \arrow[d, "\mu\times\mathrm{id}"] & (G,\mathcal{O}_G)\times(G,\mathcal{O}_G) \arrow[d, "\mu"] \\
(G,\mathcal{O}_G)\times(G,\mathcal{O}_G) \arrow[r, "\mu"] & (G,\mathcal{O}_G)
\end{tikzcd}
$$

- neutral element e:

$$
\begin{tikzcd}
(\mathrm{Spec}\mathbb{C},\widetilde{\mathbb{C}})\times(G,\mathcal{O}_G)\times(\mathrm{Spec}\mathbb{C},\widetilde{\mathbb{C}}) \arrow[rr, "\mathrm{id}\times\mathrm{id}\times e"] \arrow[dr, "\cong"] \arrow[dd, "e\times\mathrm{id}\times\mathrm{id}"] & & (\mathrm{Spec}\mathbb{C},\widetilde{\mathbb{C}})\times(G,\mathcal{O}_G)\times(G,\mathcal{O}_G) \arrow[d, "\cong"] \\
 & (G,\mathcal{O}_G) \arrow[dr, "\mathrm{id}"] & (G,\mathcal{O}_G)\times(G,\mathcal{O}_G) \arrow[d, "\mu"] \\
(G,\mathcal{O}_G)\times(G,\mathcal{O}_G)\times(\mathrm{Spec}\mathbb{C},\widetilde{\mathbb{C}}) \arrow[r, "\cong"] & (G,\mathcal{O}_G)\times(G,\mathcal{O}_G) \arrow[r, "\mu"] & (G,\mathcal{O}_G)
\end{tikzcd}
,
$$

- inverse i:

$$
\begin{tikzcd}
(G,\mathcal{O}_G) \arrow[rr, "(\mathrm{id},\iota)"] \arrow[dr, "\text{constant}"] \arrow[dd, "(\iota,\mathrm{id})"] & & (G,\mathcal{O}_G)\times(G,\mathcal{O}_G) \arrow[dd] \\
 & (\mathrm{Spec}\mathbb{C},\widetilde{\mathbb{C}}) \arrow[dr, "e"] & \\
(G,\mathcal{O}_G)\times(G,\mathcal{O}_G) \arrow[rr, "\mu"] & & (G,\mathcal{O}_G)
\end{tikzcd}
.
$$

Notation. μ is often written as $\cdot$ to indicate multiplication and ι as $^{-1}$ to indicate inverse.

In definition 2, since $(G, \mathcal{O}_G) \times (G, \mathcal{O}_G) = (\mathrm{Spec}(R \otimes_{\mathbb{C}} R), \widetilde{R \otimes_{\mathbb{C}} R})$, the maps μ (e, ι resp.) gives a map $\mu' : R \to R \otimes_{\mathbb{C}} R$ ($e' : R \to \mathbb{C}$, $\iota' : \mathbb{R} \to \mathbb{R}$ resp.) such that the following diagrams commute

$$\begin{tikzcd}
R \arrow[r, "\mu'"] \arrow[d, "\mu'"] & R \otimes_{\mathbb{C}} R \arrow[d, "\mathrm{id}\otimes\mu'"] \\
R \otimes_{\mathbb{C}} R \arrow[r, "\mu'\otimes\mathrm{id}"] & R \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} R
\end{tikzcd}
\quad
\begin{tikzcd}
R \arrow[r, "\mu'"] \arrow[d, "\mu'"] \arrow[dr, "\mathrm{id}"] & R \otimes_{\mathbb{C}} R \arrow[r, "\cong"] & R \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} \mathbb{C} \arrow[dd, "e'\otimes\mathrm{id}\otimes\mathrm{id}"] \\
R \otimes_{\mathbb{C}} R \arrow[d, "\cong"] & R \arrow[dr, "\cong"] & \\
\mathbb{C} \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} R \arrow[rr, "\mathrm{id}\otimes\mathrm{id}\otimes e'"] & & \mathbb{C} \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} \mathbb{C}
\end{tikzcd}
\quad
\begin{tikzcd}
R \arrow[rr, "\mu'"] \arrow[dr, "e'"] \arrow[dd, "\mu'"] & & R \otimes_{\mathbb{C}} R \arrow[dd, "r\otimes r' \mapsto \iota'(r)r'"] \\
& \mathbb{C} \arrow[dr, "*\text{see below}"] & \\
R \otimes_{\mathbb{C}} R \arrow[rr, "r\otimes r' \mapsto r\iota'(r')"] & & R
\end{tikzcd}$$

where (*) is the map $\mathbb{C} \to R$ giving R the $\mathbb{C}$-algebra structure.

2.2.1 Trivial Group Scheme

Consider $\mathbb{C}$ as the trivial $\mathbb{C}$-algebra. Let $\mu' : \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ be $\mathrm{id} : \mathbb{C} \to \mathbb{C}$ after identifying $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ and $e', \iota' : \mathbb{C} \to \mathbb{C}$ all be identity on $\mathbb{C}$. In this case $(G, \mathcal{O}_G) = (\mathrm{Spec}\mathbb{C}, \widetilde{\mathbb{C}})$ is a space with only one point.

2.2.2 A Non-trivial Group Scheme

Notation. In this subsubsection, R is $\mathbb{C}[x]$ and localization maps are denoted by α with subscript, for example $\alpha_x : R \to R\left[\frac{1}{x}\right]$ localises x.

Consider the ring of Laurent polynomial $\mathbb{C}[x, x^{-1}] \cong R\left[\frac{1}{x}\right]$ as a $\mathbb{C}$-algebra. Let the map $\nu' : R \to R \otimes_{\mathbb{C}} R$ be given by $x \mapsto x \otimes x$. Then

$$R \xrightarrow{\nu'} R \otimes_{\mathbb{C}} R \xrightarrow[\phi \text{ is the inclusion}]{\phi\otimes\phi} \mathbb{C}[x, x^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x, x^{-1}]$$

send x to $x \otimes x$ which is invertible, hence by universal property of localisation, there is a unique map μ' such that

$$\begin{tikzcd}
R \arrow[r, "\nu'"] \arrow[d, "\alpha_x"] & R \otimes_{\mathbb{C}} R \arrow[d, "\phi\otimes\phi"] \\
R\left[\frac{1}{x}\right] \arrow[r, dashed, "\exists!"] \arrow[d, "\cong"] & \mathbb{C}[x, x^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x, x^{-1}] \\
\mathbb{C}[x, x^{-1}] \arrow[ur, dashed, "\mu'"'] &
\end{tikzcd}.$$

The map $e' : \mathbb{C}[x, x^{-1}] \to \mathbb{C}[x, x^{-1}]$ is given by evaluating at $x = 1$ and $\iota' : \mathbb{C}[x, x^{-1}] \to \mathbb{C}[x, x^{-1}]$ is given by $\iota'(x) = x^{-1}$ and $\iota'(x^{-1}) = x$.

Associativity of μ' Need to check $x \in \mathbb{C}[x, x^{-1}]$ and $x^{-1} \in \mathbb{C}[x, x^{-1}]$. For x, the first composition

$$\begin{tikzcd}
x \in \mathbb{C}[x, x^{-1}] \arrow[r, mapsto, "\mu'"] & x \otimes x \arrow[d, mapsto, "\mathrm{id}\otimes\mu'"] \\
& x \otimes x \otimes x
\end{tikzcd}$$

equals to the second composition

$$\begin{tikzcd}
x \in \mathbb{C}[x, x^{-1}] \arrow[d, mapsto, "\mu'"] & \\
x \otimes x \arrow[r, mapsto, "\mu'\otimes\mathrm{id}"] & x \otimes x \otimes x.
\end{tikzcd}$$

For x^{-1}, just replace every x by x^{-1}, and get the same thing.

Neutral element e' For $x \in \mathbb{C}[x, x^{-1}]$, the first composition is

$$\begin{tikzcd}
x \arrow[r, mapsto, "\mu'"] & x \otimes x \arrow[r, "\cong"] & x \otimes x \otimes 1 \arrow[d, mapsto, "e'\otimes\mathrm{id}\otimes\mathrm{id}"] \\
& & 1 \otimes x \otimes 1;
\end{tikzcd}$$

the second composition is

$$\begin{tikzcd}
x \arrow[d, mapsto, "\mu'"] & \\
x \otimes x \arrow[d, mapsto, "\cong"] & \\
1 \otimes x \otimes x \arrow[r, mapsto, "\mathrm{id}\otimes\mathrm{id}\otimes e'"] & 1 \otimes x \otimes 1;
\end{tikzcd}$$

and the diagonal composition is $x \xmapsto{\mathrm{id}} x \xmapsto{\cong} 1 \otimes x \otimes 1$. The case for x^{-1} is the same as above because x^{-1} evaluate at 1 is also 1.

Inverse ι' The first composition is

$$\begin{tikzcd}
x \arrow[r, mapsto, "\mu'"] & x \otimes x \arrow[d, mapsto, "r\otimes r' \mapsto \iota'(r)r'"] \\
 & x^{-1}x = 1,
\end{tikzcd}$$

and the second composition is

$$\begin{tikzcd}
x \arrow[d, mapsto, "\mu'"] & \\
x \otimes x \arrow[r, "r\otimes r' \mapsto r\iota'(r')"] & xx^{-1} = 1,
\end{tikzcd}$$

and the third composition is $x \xrightarrow{e'} 1 \longrightarrow 1$. The case for x^{-1} is similar.

2.3 Affine group scheme acting on affine group scheme

Notation. In this subsection, assume $(G, \mathcal{O}_G) = (\mathrm{Spec}R, \widetilde{R})$, $(X, \mathcal{O}_X) = (\mathrm{Spec}S, \widetilde{S})$ and $(Y, \mathcal{O}_Y) = (\mathrm{Spec}(S'), \widetilde{S'})$ are affine schemes finite type over $\mathbb{C}$ and that $(G, \mathcal{O}_G)$ is a group scheme.

Definition 3. An action of $(G, \mathcal{O}_G)$ on $(X, \mathcal{O}_X)$ is given by a homomorphism $a' : S \to R \otimes_{\mathbb{C}} S$ such that

$$\begin{tikzcd}
S \arrow[r, "a'"] \arrow[d, "a'"] & R \otimes_{\mathbb{C}} S \arrow[d, "\mathrm{id}\otimes a'"] \\
R \otimes_{\mathbb{C}} S \arrow[r, "\mu'\otimes\mathrm{id}"] & R \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} S,
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & R \otimes_{\mathbb{C}} S \arrow[d, "e'\otimes\mathrm{id}"] \\
S \arrow[ur, "a'"] \arrow[dr, "\mathrm{id}"] & \mathbb{C} \otimes_{\mathbb{C}} S \arrow[d, dash, "\cong"] \\
 & S
\end{tikzcd}$$

Definition 4. Suppose $(G, \mathcal{O}_G)$ acts on $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. A morphism $\beta : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of schemes of finite type over $\mathbb{C}$ is called a G-morphism if the following diagram commutes

$$\begin{tikzcd}
(G, \mathcal{O}_G) \times (X, \mathcal{O}_X) \arrow[r, "a_X"] \arrow[d, "\mathrm{id}\times\beta"] & (X, \mathcal{O}_X) \arrow[d, "\beta"] \\
(G, \mathcal{O}_G) \times (Y, \mathcal{O}_Y) \arrow[r, "a_Y"] & (Y, \mathcal{O}_Y).
\end{tikzcd}$$

We can rephrase definition 4 in terms of ring/algebra homomorphism: β is $(\mathrm{Spec}(\beta'), \widetilde{\beta'})$ for a unique $\mathbb{C}$-algebra homomorphism $\beta' : S' \to S$, the commutative squre becomes

$$\begin{tikzcd}
S' \arrow[r, "a_{S'}"] \arrow[d, "\beta'"] & R \otimes_{\mathbb{C}} S' \arrow[d, "\mathrm{id}\otimes\beta'"] \\
S \arrow[r, "a_S"] & R \otimes_{\mathbb{C}} S.
\end{tikzcd}$$

Notation. We say that S is a G-ring if the group scheme $(G, \mathcal{O}_G)$ acts on $(\mathrm{Spec}S, \widetilde{S})$. And β' as above is a G-homomorphism of G-rings.

Theorem 1. Let S be a G-ring and $f \in S$ such that $f \neq 0$ and $a'(f) = r \otimes f$. Then $r \in R$ is invertible.

Proof. Since $(G, \mathcal{O}_G) = (\mathrm{Spec}R, \widetilde{R})$ is a group scheme, we have the following commutative diagram:

$$\begin{tikzcd} R \arrow[r, "e'"] \arrow[d, "\mu'"] & \mathbb{C} \arrow[d, "\rho"] \\ R \otimes_{\mathbb{C}} R \arrow[r, "r\otimes r' \mapsto r\iota'(r')"] & R, \end{tikzcd}$$

where $\rho : \mathbb{C} \to R$ is the algebra map. We tensor this diagram with S:

$$\begin{tikzcd} R \otimes_{\mathbb{C}} S \arrow[r, "e'\otimes \mathrm{id}"] \arrow[d, "\mu\otimes \mathrm{id}'"] & \mathbb{C} \otimes_{\mathbb{C}} S \arrow[d, "\rho\otimes \mathrm{id}"] \\ R \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} S \arrow[r, "(r\otimes r' \mapsto r\iota'(r'))\otimes \mathrm{id}"] & R \otimes_{\mathbb{C}} S, \end{tikzcd}$$

so by defniition of action, we get the following commutative diagram:

$$\begin{tikzcd} & S \arrow[dr, dash, "\cong"] & \\ S \arrow[ur, "\mathrm{id}"] \arrow[r, "a'"] \arrow[d, "a'"] & R \otimes_{\mathbb{C}} S \arrow[r, "e'\otimes \mathrm{id}"] \arrow[d, "\mu\otimes \mathrm{id}'"] & \mathbb{C} \otimes_{\mathbb{C}} S \arrow[d, "\rho\otimes \mathrm{id}"] \\ R \otimes_{\mathbb{C}} S \arrow[r, "\mathrm{id}\otimes a'"] & R \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} S \arrow[r, "(r\otimes r' \mapsto r\iota'(r'))\otimes \mathrm{id}"] & R \otimes_{\mathbb{C}} S. \end{tikzcd}$$

So for $f \in S$, the first composition from top left to bottom right is equal to
$$\begin{tikzcd} f \arrow[d, mapsto] & & \\ r \otimes f \arrow[r, mapsto] & r \otimes r \otimes f \arrow[r, mapsto] & r \cdot \iota'(r) \otimes f, \end{tikzcd}$$

and the second composition is equal to
$$\begin{tikzcd} & f \arrow[dr, mapsto] & \\ f \arrow[ur, mapsto, "\mathrm{id}"] \arrow[r, mapsto] & r \otimes f & 1 \otimes f \arrow[d, mapsto] \\ & & 1 \otimes f, \end{tikzcd}$$
hence $r \cdot \iota'(r) \otimes f = 1 \otimes f$, since $f \neq 0$, $r \cdot \iota'(r) = 1$,

i.e. r is invertible in R. ◻

2.3.1 Example of action by group scheme

We consider the example in section 2.2.2. So $R = \mathbb{C}[t, t^{-1}]$ and $S = \mathbb{C}[x_0, x_1, \cdots, x_n]$. Consider $(\mathrm{Spec}R, \widetilde{R})$ acting on $(\mathrm{Spec}S, \widetilde{S})$ by $a'(x_i) = t^{-1} \otimes x_i$. Check commutativity:
$$\begin{tikzcd} x_i \arrow[r, mapsto, "a'"] & t^{-1} \otimes x_i \arrow[d, mapsto, "\mathrm{id}\otimes a'"] \\ & t^{-1} \otimes t^{-1} \otimes x_i \end{tikzcd}$$
and
$$\begin{tikzcd} x_i \arrow[d, mapsto, "a'"] & \\ t^{-1} \otimes x_i \arrow[r, mapsto, "\mu'\otimes \mathrm{id}"] & t^{-1} \otimes t^{-1} \otimes x_i \end{tikzcd}$$
;

$$\begin{tikzcd} & t^{-1} \otimes x_i \arrow[d, mapsto, "e'\otimes \mathrm{id}"] \\ x_i \arrow[ur, "a'"] & 1 \otimes x_i \arrow[d, mapsto] \\ & x + i \end{tikzcd}$$
obvisouly commutes with id.

2.3.2 Examples of G-homomorphism

Suppose S is a G-ring and $f \in S$ is such that $a'_S(f) = r \otimes f$. So $r \in R$ is invertible. Hence the composite $S \xrightarrow{a'_S} R \otimes_{\mathbb{C}} S \xrightarrow{\mathrm{id}\otimes \alpha_f} R \otimes_{\mathbb{C}} S\left[\frac{1}{f}\right]$ takes $f \in S$ to an invertible $r \otimes f \in R \otimes_{\mathbb{C}} S\left[\frac{1}{f}\right]$. Thus this composite factors through $\alpha_f : S \to S\left[\frac{1}{f}\right]$, i.e.
$$\begin{tikzcd} S \arrow[r, "\alpha'_S"] \arrow[d, "\alpha_f"] & R \otimes_{\mathbb{C}} S \arrow[d, "\mathrm{id}\otimes \alpha_f"] \\ S\left[\frac{1}{f}\right] \arrow[r, dashed, "\exists \alpha'_{S\left[\frac{1}{f}\right]}"'] & R \otimes_{\mathbb{C}} S\left[\frac{1}{f}\right]. \end{tikzcd}$$
Then the following holds:

1. $S\left[\frac{1}{f}\right]$ is a G-ring.

Proof. check commutativity of

$$\begin{tikzcd} S\left[\frac{1}{f}\right] \arrow[r, "a'"] \arrow[d, "a'"] & R \otimes_{\mathbb{C}} S\left[\frac{1}{f}\right] \arrow[d, "\mathrm{id}\otimes a'"] \\ R \otimes_{\mathbb{C}} S\left[\frac{1}{f}\right] \arrow[r, "\mu'\otimes\mathrm{id}"] & R \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} S\left[\frac{1}{f}\right] \end{tikzcd} :$$

□