Summary of GAGA

Jujian Zhang

1 Affine schemes of finite type

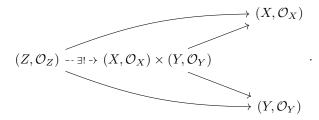
2 \mathbb{P}^n as a Group Scheme

2.1 Product of Affine Schemes

Notation. In this subsection, $(X, \mathcal{O}_X) = (\operatorname{Spec} R, \widetilde{R})$ and $(Y, \mathcal{O}_Y) = (\operatorname{Spec} S, \widetilde{S})$ are affine schemes of finite type over $\mathbb C$ where R and S are finitely generated $\mathbb C$ -algebra.

Definition 1. $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ is defined as $(\operatorname{Spec}(R \otimes_{\mathbb{C}} S), \widetilde{R \otimes_{\mathbb{C}} S})$.

 $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ satisfies the universal property of product in the category of affine schemes of finite type over \mathbb{C} . The natural inclusions $i_1: R \to R \otimes_{\mathbb{C}} S$ given by $i_1(r) = r \otimes 1$ and $i_2: S \to R \otimes_{\mathbb{C}} S$ given by $i_2(s) = 1 \otimes s$ induces $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ and $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$; and for any (Z, \mathcal{O}_Z) , an affine schemes of finite type over \mathbb{C} , then any maps $(Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ and $(Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ factors uniquely through $(X \times Y, \mathcal{O}_{X \times Y})$, i.e. we have the following commutative diagram



2.2 Affine Group Scheme

Notation. In this subsection, $(G, \mathcal{O}_G) = (\operatorname{Spec} R, \widetilde{R})$ is an affine scheme of finite type over \mathbb{C} .

Definition 2. (G, \mathcal{O}_G) is an affine group scheme if there are $\mu: (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \to (G, \mathcal{O}_G)$, $e: (\operatorname{Spec}\mathbb{C}, \widetilde{\mathbb{C}}) \to (G, \mathcal{O}_G)$ and $\iota: (G, \mathcal{O}_G) \to (G, \mathcal{O}_G)$ such that the following diagrams commute

• associativity of μ :

$$(G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \xrightarrow{\mathrm{id} \times \mu} (G, \mathcal{O}_G) \times (G, \mathcal{O}_G)$$

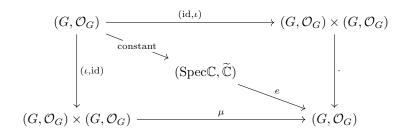
$$\downarrow^{\mu \times \mathrm{id}} \qquad \qquad \downarrow^{\mu}$$

$$(G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \xrightarrow{\mu} (G, \mathcal{O}_G)$$

• neutral element e:

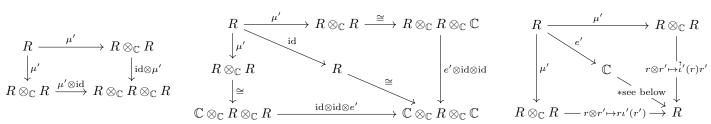
$$(\operatorname{Spec}\mathbb{C},\widetilde{\mathbb{C}})\times (G,\mathcal{O}_G)\times (\operatorname{Spec}\mathbb{C},\widetilde{\mathbb{C}}) \xrightarrow{\operatorname{id}\times\operatorname{id}\times e} (\operatorname{Spec}\mathbb{C},\widetilde{\mathbb{C}})\times (G,\mathcal{O}_G)\times (G,\mathcal{O}_G) \\ \stackrel{=}{\swarrow} (G,\mathcal{O}_G) \times (G,\mathcal{O}_G) \times (G,\mathcal{O}_G) \\ \downarrow^{e\times\operatorname{id}\times\operatorname{id}} (G,\mathcal{O}_G)\times (G,\mathcal{O}_G) \times (G,\mathcal{O}_G) \xrightarrow{\operatorname{id}} (G,\mathcal{O}_G) \times (G,\mathcal{O}_G) \\ \downarrow^{\mu} (G,\mathcal{O}_G)\times (G,\mathcal{O}_G)\times (\operatorname{Spec}\mathbb{C},\widetilde{\mathbb{C}}) \xrightarrow{\cong} (G,\mathcal{O}_G)\times (G,\mathcal{O}_G) \xrightarrow{\mu} (G,\mathcal{O}_G)$$

• inverse i:



Notation. μ is often written as \cdot to indicate multiplication and ι as $^{-1}$ to indicate inverse.

In definition 2, since $(G, \mathcal{O}_G) \times (G, \mathcal{O}_G) = (\operatorname{Spec}(R \otimes_{\mathbb{C}} R), \widetilde{R \otimes_{\mathbb{C}} R})$, the maps μ $(e, \iota \text{ resp.})$ gives a map $\mu' : R \to R \otimes_{\mathbb{C}} R$ $(e' : R \to \mathbb{C}, \iota' : \mathbb{R} \to \mathbb{R} \text{ resp.})$ such that the following diagrams commute



where (*) is the map $\mathbb{C} \to R$ giving R the \mathbb{C} -algebra structure.

2.2.1 Trivial Group Scheme

Consider \mathbb{C} as the trivial \mathbb{C} -algebra. Let $\mu': \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ be id: $\mathbb{C} \to \mathbb{C}$ after identifying $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ and $e', \iota': \mathbb{C} \to \mathbb{C}$ all be identity on \mathbb{C} . In this case $(G, \mathcal{O}_G) = (\operatorname{Spec}\mathbb{C}, \mathbb{C})$ is a space with only one point.

2.2.2 A Non-trivial Group Scheme

Notation. In this subsubsection, R is $\mathbb{C}[x]$ and localization maps are denoted by α with subscript, for example $\alpha_x : R \to R\left[\frac{1}{x}\right]$ localises x.

Consider the ring of Laurent polynomial $\mathbb{C}[x,x^{-1}]\cong R\left[\frac{1}{x}\right]$ as a \mathbb{C} -algebra. Let the map $\nu':R\to R\otimes_{\mathbb{C}}R$ be given by $x\mapsto x\otimes x$. Then

$$R \xrightarrow{\quad \nu' \quad} R \otimes_{\mathbb{C}} R \xrightarrow{\quad \phi \otimes \phi \quad} \mathbb{C}[x,x^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x,x^{-1}]$$

send x to $x \otimes x$ which is invertible, hence by universal property of localisation, there is a unique map μ' such that

$$R \xrightarrow{\nu'} R \otimes_{\mathbb{C}} R$$

$$\downarrow^{\alpha_x} \qquad \qquad \downarrow^{\phi \otimes \phi}$$

$$R \left[\frac{1}{x}\right] \xrightarrow{--}^{\exists !} \mathbb{C}[x, x^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x, x^{-1}]$$

$$\stackrel{\cong}{\mathbb{C}}[x, x^{-1}]$$

The map $e': \mathbb{C}[x,x^{-1}] \to \mathbb{C}[x,x^{-1}]$ is given by evaluating at x=1 and $\iota': \mathbb{C}[x,x^{-1}] \to \mathbb{C}[x,x^{-1}]$ is given by $\iota'(x)=x^{-1}$ and $\iota'(x^{-1})=x$.

Associativity of μ' Need to check $x \in \mathbb{C}[x, x^{-1}]$ and $x^{-1} \in \mathbb{C}[x, x^{-1}]$. For x, the first composition

$$x \in \mathbb{C}[x,x^{-1}] \xrightarrow{\mu'} x \otimes x$$

$$\downarrow^{\operatorname{id} \otimes \mu'}$$

$$x \otimes x \otimes x$$

equals to the second composition

$$\begin{aligned} x &\in \mathbb{C}[x, x^{-1}] \\ & & \downarrow^{\mu'} \\ & x \otimes x & \longmapsto^{\mu' \otimes \mathrm{id}} & x \otimes x \otimes x. \end{aligned}$$

For x^{-1} , just replace every x by x^{-1} , and get the same thing.

Neutral element e' For $x \in \mathbb{C}[x, x^{-1}]$, the first composition is

the second composition is

$$\begin{array}{c}
x \\
\downarrow^{\mu'} \\
x \otimes x \\
\downarrow^{\cong} \\
1 \otimes x \otimes x \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes e'} 1 \otimes x \otimes 1;
\end{array}$$

and the diagonal composition is $x \stackrel{\text{id}}{\longmapsto} x \stackrel{\cong}{\longmapsto} 1 \otimes x \otimes 1$. The case for x^{-1} is the same as above because x^{-1} evaluate at 1 is also 1.

Inverse ι' The first composition is

$$x \stackrel{\mu'}{\longmapsto} x \otimes x$$

$$\downarrow^{r \otimes r' \mapsto \iota'(r)r'}$$

$$x^{-1}x = 1,$$

and the second composition is

$$\begin{array}{c} x \\ \downarrow^{\mu'} \\ x \otimes x \xrightarrow{r \otimes r' \mapsto r\iota'(r')} xx^{-1} = 1, \end{array}$$

and the third composition is $x \xrightarrow{e'} 1 \longrightarrow 1$. The case for x^{-1} is similar.

2.3 Affine group scheme acting on affine group scheme

Notation. In this subsection, assume $(G, \mathcal{O}_G) = (\operatorname{Spec} R, \widetilde{R}), (X, \mathcal{O}_X) = (\operatorname{Spec} S, \widetilde{S})$ and $(Y, \mathcal{O}_Y) = (\operatorname{Spec}(S'), \widetilde{S}')$ are affine schemes finite type over $\mathbb C$ and that (G, \mathcal{O}_G) is a group scheme.

Definition 3. An action of (G, \mathcal{O}_G) on (X, \mathcal{O}_X) is given by a homomorphism $a': S \to R \otimes_{\mathbb{C}} S$ such that

$$S \xrightarrow{a'} R \otimes_{\mathbb{C}} S$$

$$\downarrow^{a'} \qquad \downarrow^{\operatorname{id} \otimes a'} \qquad \text{and} \qquad S \xrightarrow{a'} \bigvee^{e' \otimes \operatorname{id}} e' \otimes \operatorname{id}$$

$$R \otimes_{\mathbb{C}} S \xrightarrow{\mu' \otimes \operatorname{id}} R \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} S,$$

$$\downarrow^{a'} \qquad \downarrow^{\operatorname{id} \otimes a'} \otimes_{\mathbb{C}} S$$

Definition 4. Suppose (G, \mathcal{O}_G) acts on (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) . A morphism $\beta : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of schemes of finite type over $\mathbb C$ is called a G-morphism if the following diagram commutes

$$(G, \mathcal{O}_G) \times (X, \mathcal{O}_X) \xrightarrow{a_X} (X, \mathcal{O}_X)$$

$$\downarrow^{\mathrm{id} \times \beta} \qquad \qquad \downarrow^{\beta}$$

$$(G, \mathcal{O}_G) \times (Y, \mathcal{O}_Y) \xrightarrow{a_Y} (Y, \mathcal{O}_Y).$$

We can rephrase definition 4 in terms of ring/algebra homomorphism: β is $(\operatorname{Spec}(\beta'), \widetilde{\beta'})$ for a unique \mathbb{C} -algebra homomorphism $\beta': S' \to S$, the commutative squre becomes

$$S' \xrightarrow{a_{S'}} R \otimes_{\mathbb{C}} S'$$

$$\downarrow^{\beta'} \qquad \downarrow^{\operatorname{id} \otimes \beta'}$$

$$S \xrightarrow{a_S} R \otimes_{\mathbb{C}} S.$$

Notation. We say that S is a G-ring if the group scheme (G, \mathcal{O}_G) acts on $(\operatorname{Spec} S, \widetilde{S})$. And β' as above is a G-homomorphism of G-rings.

Theorem 1. Let S be a G-ring and $f \in S$ such that $f \neq 0$ and $a'(f) = r \otimes f$. Then $r \in R$ is invertible.

Proof. Since $(G, \mathcal{O}_G) = (\operatorname{Spec} R, \widetilde{R})$ is a group scheme, we have the following commutative diagram:

$$R \xrightarrow{e'} \mathbb{C}$$

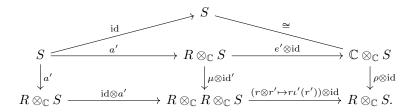
$$\downarrow^{\mu'} \qquad \downarrow^{\rho}$$

$$R \otimes_{\mathbb{C}} R \xrightarrow{r \otimes r' \mapsto r\iota'(r')} R,$$

where $\rho: \mathbb{C} \to R$ is the algebra map. We tensor this diagram with S:

$$\begin{array}{ccc} R \otimes_{\mathbb{C}} S & \xrightarrow{e' \otimes \mathrm{id}} & \mathbb{C} \otimes_{\mathbb{C}} S \\ & \downarrow^{\mu \otimes \mathrm{id'}} & & \downarrow^{\rho \otimes \mathrm{id}} \\ R \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} S & \xrightarrow{(r \otimes r' \mapsto r\iota'(r')) \otimes \mathrm{id}} R \otimes_{\mathbb{C}} S, \end{array}$$

so by defniition of action, we get the following commutative diagram:



$$\downarrow \\
r \otimes f \longmapsto r \otimes r \otimes f \longmapsto r \cdot \iota'(r) \otimes f,$$

and the second composition is equal to $f \xrightarrow{\operatorname{id}} r \otimes f$ hence $r \cdot \iota'(r) \otimes f = 1 \otimes f$, since $f \neq 0$, $r \cdot \iota'(r) = 1$,

i.e. r is invertible in R.

2.3.1 Example of action by group scheme

We consider the example in section 2.2.2. So $R = \mathbb{C}[t, t^{-1}]$ and $S = \mathbb{C}[x_0, x_1, \cdots, x_n]$. Consider (Spec R, \widetilde{R}) acting on $x_{i} \xrightarrow{a'} t^{-1} \otimes x_{i} \qquad x_{i}$ $\downarrow_{\mathrm{id} \otimes a'} \quad \mathrm{and} \quad \downarrow_{a'} \qquad ;$ $t^{-1} \otimes t^{-1} \otimes x_{i} \qquad t^{-1} \otimes x_{i} \xrightarrow{\mu' \otimes \mathrm{id}} t^{-1} \otimes t^{-1} \otimes x_{i}$ $(\operatorname{Spec} S, \widetilde{S})$ by $a'(x_i) = t^{-1} \otimes x_i$. Check commutativity:

$$x_i \xrightarrow{a'} \begin{cases} t^{-1} \otimes x_i \\ \downarrow e' \otimes \mathrm{id} \\ 1 \otimes x_i \end{cases}$$
 obvisouly commutes with id.
$$\downarrow \\ x+i \end{cases}$$