7

Series

TERMINOLOGY

Annuity: An annuity is a fixed sum of money invested every year that accumulates interest over a number of years

Arithmetic sequence: A set of numbers that form a pattern where each successive term is a constant amount (positive or negative) more than the previous one

Common difference: The constant amount in an arithmetic sequence that is added to each term to get the next term

Common ratio: The constant amount in a geometric sequence that is multiplied to each term to get the next term

Compound interest: Interest is added to the balance of a bank account so that the interest and balance both earn interest

Geometric sequence: A set of numbers that form a pattern where each successive term is multiplied by a constant amount to get the next term

Limiting sum: (or sum to infinity) The sum of infinite terms of a geometric series where the common ratio r obeys the condition $-1 < r < 1$

Partial sum: The sum of a certain finite number of terms of a sequence

Sigma notation: The Greek letter stands for the sum of a sequence of a certain number of terms, called a partial sum

Sequence: A set of numbers that form a pattern or obey a fixed rule

Series: The sum of a sequence of n terms

Superannuation: A sum of money that is invested every year (or more frequently) as part of a salary to provide a large amount of money when a person retires from the paid workforce

Term: A term refers to the position of the number in a sequence. For example the first term is the first number in the sequence

INTRODUCTION

THE INFINITE SUM OF a sequence of numbers (or terms) is called a **series.** Many series occur in real life—think of the way plants grow, or the way money accumulates as a certain amount earns interest in a bank.

You will look at series in general, arithmetic, geometric series and their applications.

General Series

A sequence forms a pattern. Some patterns are easy to see and some are difficult. People are sometimes asked to identify sequences and their patterns in tests of intelligence.

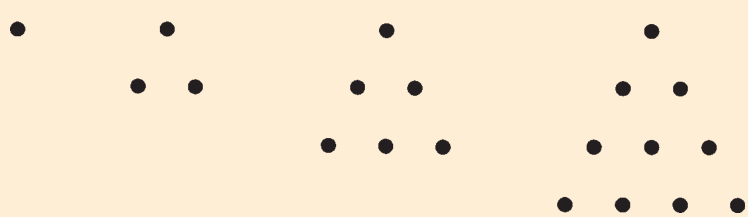
DID YOU KNOW?

Number patterns and series of numbers have been known since the very beginning of civilisation.

The *Rhind Papyrus*, an Egyptian book written about 1650 BC, is one of the oldest books on mathematics. It was discovered and deciphered by **Eisenlohr** in 1877, and it showed that the ancient Egyptians had a vast knowledge of mathematics. The *Rhind Papyrus* describes the problem of dividing 100 loaves among 5 people in a way that shows the Egyptians were exploring arithmetic series.

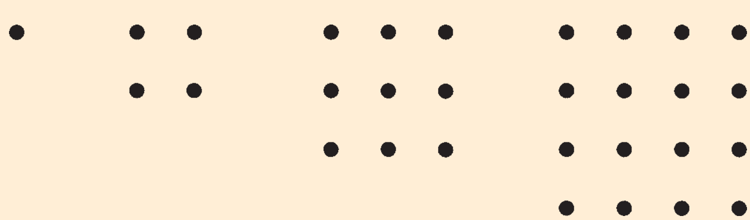
Around 500 BC the Pythagoreans explored different polygonal numbers:

- **triangular numbers:** $1 + 2 + 3 + 4 + \ldots$



1     $1 + 2$ $1 + 2 + 3$ $1 + 2 + 3 + 4$

- **square numbers:** $1 + 3 + 5 + 7 + \ldots$



1     $1 + 3$ $1 + 3 + 5$ $1 + 3 + 5 + 7$

Could you find a series for pentagonal or hexagonal numbers?

7.1 Exercises

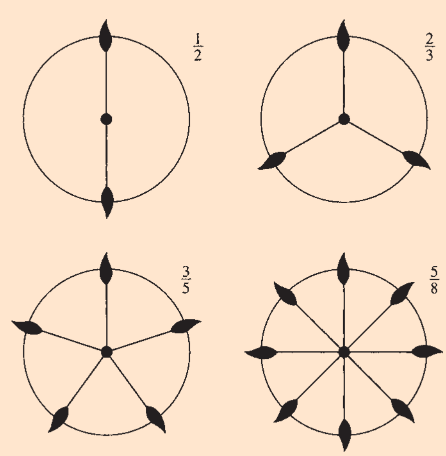
Find the next 3 terms in each sequence or series of numbers.

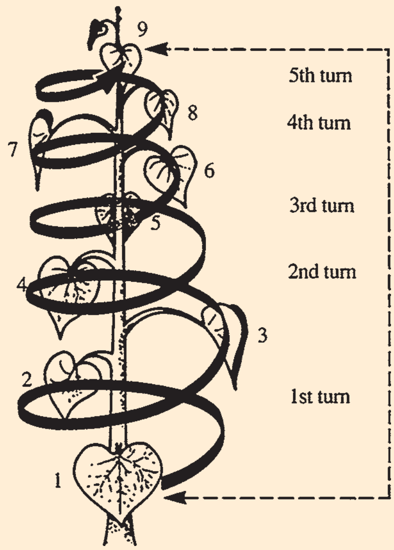
1. 5, 8, 11, ...
2. 8, 13, 18, ...
3. $11 + 22 + 33 + \ldots$
4. 100, 95, 90, ...
5. $7 + 5 + 3 + \ldots$
6. 99, 95, 91, ...
7. $\frac{1}{2}, 1, 1\frac{1}{2}, \ldots$
8. $1.3 + 1.9 + 2.5 + \ldots$
9. 2, 4, 8, ...
10. $4 + 12 + 36 + \ldots$
11. $1, -2, 4, -8, \ldots$
12. $3, -6, 12, -24, \ldots$
13. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
14. $\frac{2}{5}, \frac{4}{15}, \frac{8}{45}, \ldots$
15. $1 + 4 + 9 + 16 + 25 + \ldots$
16. 1, 8, 27, 64, ...
17. 0, 3, 8, 15, 24, ...
18. $3 + 6 + 11 + 18 + 27 + \ldots$
19. $2 + 9 + 28 + 65 + \ldots$
20. 1, 1, 2, 3, 5, 8, 13, ...

DID YOU KNOW?

The numbers 1, 1, 2, 3, 5, 8, . . . are called **Fibonacci** numbers after **Leonardo Fibonacci** (1170–1250). The Fibonacci numbers occur in natural situations.

For example, when new leaves grow on a plant's stem, they spiral around the stem. The ratio of the number of turns to the number of spaces between successive leaves gives the series of fractions $\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \frac{34}{55}, \frac{55}{89}, \frac{89}{144}, \ldots$





The Fibonacci ratio is the number of turns divided by the number of spaces.

Research Fibonacci numbers and find out where else they appear in nature.

Formula for the *n*th term of a series

When a series follows a mathematical pattern, its terms can be described by a formula. We call any general or nth term of a series T_n where n stands for the number of the term and must be a positive integer.

EXAMPLE

For the series $6 + 13 + 20 + \dots$ find

(a) T_1
(b) T_2
(c) T_3
(d) T_n

Solution

(a) The first term is 6, so when $n = 1$: $T_1 = 6$
(b) The 2nd term is 13 so $T_2 = 13$
(c) The 3rd term is 20 so $T_3 = 20$
(d) Each term is 7 more than the previous term.
The 1st term is 6
The 2nd term is $13 = 6 + 7$
The 3rd term is $20 = 6 + 7 + 7$
$= 6 + 2 \times 7$

CONTINUED

Following this pattern:

The 4th term $= 6 + 7 + 7 + 7$

$= 6 + 3 \times 7$

and so on

$T_n = 6 + (n - 1) \times 7$

$T_n = 6 + 7n - 7$

$= 7n - 1$

Notice that the product with 7 in each term involves a number one less than the number of the term.

If we are given a formula for the nth term, we can find out much more about the series.

EXAMPLES

1. The nth term of a series is given by the formula $T_n = 3n + 4$. Find the first 3 terms and hence write down the series.

Solution

$T_1 = 3(1) + 4$

$= 7$

$T_2 = 3(2) + 4$

$= 10$

$T_3 = 3(3) + 4$

$= 13$

So the series is $7 + 10 + 13 + \ldots$

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2. The nth term of a series is given by $t_n = 5n - 1$. Which term of the series is equal to 104?

Solution

$t_n = 5n - 1 = 104$

$5n = 105$

$n = 21$

So 104 is the 21st term of the series.

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3. The nth term of a series is given by $u_n = 103 - 3n$.

(a) Find the first 3 terms.

(b) Find the value of n for the first negative term in the series.

Solution

(a) $u_1 = 103 - 3(1)$
$= 100$
$u_2 = 103 - 3(2)$
$= 97$
$u_3 = 103 - 3(3)$
$= 94$
So the series is $100 + 97 + 94 + \ldots$.

(b) For the first negative term, we want
$$u_n < 0$$
i.e. $103 - 3n < 0$
$$-3n < -103$$
$$n > 34\frac{1}{3}$$
Since n is an integer, $n = 35$
$\therefore$ the 35th term is the first negative term.

4. The nth term of a series is given by the formula $t_n = 2^n - 1$. Which term of the series is equal to 4095?

Solution

$t_n = 2^n - 1$
We are given that the nth term is 4095 and need to find n.

$$4095 = 2^n - 1$$
$$4096 = 2^n$$
$$\log_{10} 4096 = \log_{10} 2^n$$

You could use base e instead of base 10.

$$= n \log_{10} 2$$
$$\frac{\log_{10} 4096}{\log_{10} 2} = n$$
$$12 = n$$

You could use trial and error to guess what power of 2 is equal to 4096 if you don't want to use logarithms.

So 4096 is the 12th term of the sequence.

7.2 Exercises

1. Find the first 3 terms of the series with nth term as follows
(a) $T_n = 8n - 5$
(b) $T_n = 2n + 3$
(c) $u_n = 6n - 1$
(d) $T_n = 8 - 5n$
(e) $t_n = 20 - n$
(f) $u_n = 3^n$
(g) $Q_n = 2^n + 7$
(h) $t_n = 4^n - 2n$
(i) $T_n = 8n^2 - n + 1$
(j) $T_n = n^3 + n$

2. Find the first 3 terms of each series.
   (a) $T_n = 3n - 2$
   (b) $t_n = 4^n$
   (c) $T_n = n^2 + n$

3. Find the 50th term of each series.
   (a) $T_n = 7n - 1$
   (b) $t_n = 2n + 5$
   (c) $T_n = 5n - 2$
   (d) $T_n = 40 - 3n$
   (e) $U_n = 8 - 7n$

4. Find the 10th term of each series.
   (a) $T_n = 2^n + 5$
   (b) $t_n = 10 - 3^n$
   (c) $T_n = 2^n - n$
   (d) $u_n = n^2 - 5n + 3$
   (e) $T_n = n^3 + 2$

5. Which of these are terms of the series $T_n = 3n + 5$?
   (a) 68
   (b) 158
   (c) 205
   (d) 266
   (e) 300

6. Which of these are terms of the series $T_n = 5^n - 1$?
   (a) 3126
   (b) 124
   (c) 15 634
   (d) 78 124
   (e) 0

7. Which term of the series with nth term $t_n = 9n - 15$ is equal to 129?

8. Is 255 a term of the series with nth term $T_n = 2^n - 1$?

9. Which term of the series with nth term $u_n = n^3 + 5$ is 348?

10. Which term of $T_n = n^2 - 3$ is equal to 526?

11. For the series $T_n = n^3$, find
    (a) the 12th term
    (b) n if the nth term is 15 625.

12. A series is given by the formula $T_n = 3 + 2n - n^2$.
    (a) Find the 25th term.
    (b) Find n if the nth term is -252.

13. Find the value of n that gives the first term of the series $T_n = 3n + 2$ greater than 100.

14. Find the value of n that gives the first term of the series $T_n = 2^n$ larger than 500.

15. Find the values of n where the series $T_n = 4n - 1$ is greater than 350.

16. Find the values of n for which the series $T_n = 400 - 5n$ is less than 200.

17. Find the value of n that gives the first negative term of the series $T_n = 1000 - 2n$.

18. Find the value of n that gives the first positive term of the series $T_n = 2n - 300$.

19. Find
    (a) the value of n that gives the first negative term of the series $u_n = 80 - 6n$
    (b) the first negative term.

20. Find the first negative term of the series $T_n = 50 - 7n$.

Sigma Notation

Series are often written in **sigma notation**. The Greek letter sigma (Σ) is like an English 'S'. Here it stands for the sum of a series.

Application

The mean of a set of scores is given by

$$\overline{x} = \frac{\Sigma fx}{\Sigma f}$$

where Σf is the sum of frequencies and Σfx is the sum of the scores $\times$ frequencies.

EXAMPLES

1. Evaluate $\sum_{r=1}^{5} r^2$

Solution

$\sum_{r=1}^{5} r^2$ means the sum of terms where the formula is r^2 with r starting at 1 and ending at 5.

So $\sum_{r=1}^{5} r^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

$= 1 + 4 + 9 + 16 + 25$

$= 55$

2. Evaluate $\sum_{3}^{7} (2n + 5)$

Solution

$\sum_{3}^{7} (2n + 5)$ means the sum of terms where the formula is $2n + 5$ with n starting at 3 and ending at 7.

Can you work out how many terms there are in this series?

$\sum_{3}^{7} (2n + 5) = (2 \times 3 + 5) + (2 \times 4 + 5) + (2 \times 5 + 5) + (2 \times 6 + 5) + (2 \times 7 + 5)$

$= 11 + 13 + 15 + 17 + 19$

$= 75$

Notice that there are 5 terms, since they include both the 3 and the 7. We work out the number of terms by 7 − 3 + 1.

CONTINUED

3. Write $7 + 11 + 15 + \ldots + (4k + 3)$ in sigma notation.

Solution

If n is not 1, we could try 2 then 3 and so on until we find the first term.

In this question, the formula is $4n + 3$ where the last term is at $n = k$.
We need to find the number of terms.
The first term is 7 so we guess that $n = 1$ for this term.
When $n = 1$

$$\begin{aligned} 4n + 3 &= 4 \times 1 + 3 \\ &= 7 \end{aligned}$$

So the first term is at $n = 1$ and the last term is at $n = k$.

We can write the series as $\sum_{1}^{k} (4n + 3)$.

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4. How many terms are there in the series

(a) $\sum_{1}^{100} (3n - 7)$?

(b) $\sum_{5}^{50} 2^n$?

Solution

There are 100 − 1 + 1 terms.

(a) The first term is when $n = 1$ and the last term is when $n = 100$.
There are 100 terms.

(b) The first term is when $n = 5$ and the last term is when $n = 50$.
This is harder to work out. Look at example 2 where there are 5 terms.
We can work out the number of terms by $50 - 5 + 1$.
There are 46 terms.

> The number of terms in the series $\sum_{q}^{p} f(n)$ is $p - q + 1$

7.3 Exercises

1. Evaluate

(a) $\sum_{n=1}^{4} (3^n + 2)$

(b) $\sum_{n=2}^{5} n^2$

(c) $\sum_{2}^{6} (5n - 6)$

(d) $\sum_{r=1}^{10} (3r + 1)$

(e) $\sum_{k=2}^{5} (k^3 - 1)$

(f) $\sum_{3}^{5} \frac{1}{n}$

(g) $\sum_{n=1}^{5} (n^2 - n)$

(h) $\sum_{2}^{4} |3p - 2|$

(i) $\sum_{n=1}^{6} (2^n)$

(j) $\sum_{3}^{6} (3^n - 2n - 5)$

2. How many terms are there in each series?

(a) $\sum_{n=1}^{65}(7^n)$

(b) $\sum_{n=2}^{100}(n^2+1)$

(c) $\sum_{5}^{80}(3n+4)$

(d) $\sum_{r=1}^{200}(3r^2)$

(e) $\sum_{10}^{20}(7n+1)$

(f) $\sum_{7}^{45}(n^2+3n)$

(g) $\sum_{n=12}^{108}n^3$

(h) $\sum_{n=9}^{74}4^n$

(i) $\sum_{3}^{77}(2n-3)$

(j) $\sum_{11}^{55}(7^n+n)$

3. Write these series in sigma notation.

(a) $1+3+5+7+\ldots+11$

(b) $7+14+21+\ldots+70$

(c) $1+8+27+64+125$

(d) $2+8+14+\ldots+(6n-4)$

(e) $9+16+25+36+\ldots+n^2$

(f) $-1-2-3-\ldots-50$

(g) $3+6+12+\ldots+3\times 2^n$

(h) $1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{512}$

(i) $a+(a+d)+(a+2d)+\ldots+(a+[n-1]d)$

(j) $a+ar+ar^2+ar^3+\ldots+ar^{n-1}$

Arithmetic Series

In an arithmetic series each term is a **constant amount more** than the previous term. The constant is called the **common difference**.

EXAMPLES

1. Find the common difference of the series $5+9+13+17+21+\ldots$

Solution

We can see that the common difference between the terms is 4. To check this, we notice that $9-5=4$, $13-9=4$, $17-13=4$ and $21-17=4$.

2. Find the common difference of the series $85+80+75+70+65+\ldots$

Solution

We can see that the common difference between the terms is -5. To check this, we notice that $80-85=-5$, $75-80=-5$, $70-75=-5$ and $65-70=-5$.

> If T_1, T_2 and T_3 are consecutive terms of an arithmetic series then
> $$d = T_2 - T_1 = T_3 - T_2$$

Generally, if $T_1, T_2, T_3, \ldots T_{n-1}, T_n$ are consecutive terms of an arithmetic series then $d = T_2 - T_1 = T_3 - T_2 = \ldots = T_n - T_{n-1}$.

EXAMPLES

1. If $5 + x + 31 + \ldots$ is an arithmetic series, find x.

Solution

For an arithmetic series,

$$T_2 - T_1 = T_3 - T_2$$

i.e. $x - 5 = 31 - x$

$$2x - 5 = 31$$

$$2x = 36$$

$\therefore$ $x = 18$

Notice that $x = \frac{5 + 31}{2}$ which is the average of T_1 and T_2. We call x the arithmetic mean.

2.

(a) Evaluate k if $(k + 2) + (3k + 2) + (6k - 1) + \ldots$ is an arithmetic series.
(b) Write down the first 3 terms of the series.
(c) Find the common difference d.

Solution

(a) For an arithmetic series, $T_2 - T_1 = T_3 - T_2$

So $(3k + 2) - (k + 2) = (6k - 1) - (3k + 2)$

$$3k + 2 - k - 2 = 6k - 1 - 3k - 2$$
$$2k = 3k - 3$$
$$0 = k - 3$$
$$3 = k$$

(b) The series is $(k + 2) + (3k + 2) + (6k - 1) + \ldots$

Substituting $k = 3$:

$$T_1 = k + 2 = 3 + 2 = 5$$

$$T_2 = 3k + 2 = 3 \times 3 + 2 = 11$$

$$T_3 = 6k - 1 = 6 \times 3 - 1 = 17$$

(c) The sequence is $5 + 11 + 17 + \ldots$

$11 - 5 = 17 - 11 = 6$

So common difference $d = 6$.

Terms of an arithmetic series

$a + (a + d) + (a + 2d) + (a + 3d) + \ldots + [a + (n - 1)d] + \ldots$ is an arithmetic series with first term a, common difference d and nth term given by

$$T_n = a + (n - 1)d$$

Proof

Let the first term of an arithmetic series be a and the common difference d.
Then first term is a
second term is $a + d$
third term is $a + 2d$
fourth term is $a + 3d$, and so on.
The nth term is $a + (n - 1)d$.

EXAMPLES

1. Find the 20th term of the series $3 + 10 + 17 + \ldots$.

Solution

$a = 3, d = 7, n = 20$
$T_n = a + (n - 1)d$
$T_{20} = 3 + (20 - 1)7$
$= 3 + 19 \times 7$
$= 136$

2. Find an expression for the nth term of the series $2 + 8 + 14 + \ldots$.

Solution

$a = 2, d = 6$
$T_n = a + (n - 1)d$
$= 2 + (n - 1)6$
$= 2 + 6n - 6$
$= 6n - 4$

3. Find the first positive term of the series $-50 - 47 - 44 - \ldots$.

Solution

$a = -50, d = 3$
For the first positive term,
$T_n > 0$

CONTINUED

i.e. $a + (n - 1)d > 0$

$$-50 + (n - 1)3 > 0$$
$$-50 + 3n - 3 > 0$$
$$3n - 53 > 0$$
$$3n > 53$$
$$n > 17\frac{2}{3}$$

n must be a positive integer.

$\therefore$ $n = 18$ gives the first positive term

$$T_{18} = -50 + (18 - 1)3$$
$$= -50 + 17 \times 3$$
$$= 1$$

So the first positive term is 1.

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4. The 5th term of an arithmetic series is 37 and the 8th term is 55. Find the common difference and the first term of the series.

Solution

$$T_n = a + (n - 1)d$$
$$T_5 = a + (5 - 1)d = 37$$

i.e. $a + 4d = 37$ (1)

$$T_8 = a + (8 - 1)d = 55$$

i.e. $a + 7d = 55$ (2)

(2) – (1): $a + 4d = 37$

$$3d = 18$$
$$d = 6$$

Put $d = 6$ in (1):

$$a + 4(6) = 37$$
$$a + 24 = 37$$
$$a = 13$$

So $d = 6$ and $a = 13$.

7.4 Exercises

1. The following series are arithmetic. Evaluate all pronumerals.
   (a) $5 + 9 + y + \dots$
   (b) $8 + 2 + x + \dots$
   (c) $45 + x + 99 + \dots$
   (d) $16 + b + 6 + \dots$
   (e) $x + 14 + 21 + \dots$
   (f) $32 + (x - 1) + 51 + \dots$
   (g) $3 + (2k + 3) + 21 + \dots$
   (h) $x + (x + 3) + (2x + 5) + \dots$
   (i) $(t - 5) + 3t + (3t + 1) + \dots$
   (j) $(2t - 3) + (3t + 1) + (5t + 2) + \dots$

2. Find the 15th term of each series.
(a) $4 + 7 + 10 + \ldots$
(b) $8 + 13 + 18 + \ldots$
(c) $10 + 16 + 22 + \ldots$
(d) $120 + 111 + 102 + \ldots$
(e) $-3 + 2 + 7 + \ldots$

3. Find the 100th term of each series.
(a) $-4 + 2 + 8 + \ldots$
(b) $41 + 32 + 23 + \ldots$
(c) $18 + 22 + 26 + \ldots$
(d) $125 + 140 + 155 + \ldots$
(e) $-1 - 5 - 9 - \ldots$

4. What is the 25th term of each series?
(a) $-14 - 18 - 22 - \ldots$
(b) $0.4 + 0.9 + 1.4 + \ldots$
(c) $1.3 + 0.9 + 0.5 + \ldots$
(d) $1 + 2\frac{1}{2} + 4 + \ldots$
(e) $1\frac{2}{5} + 2 + 2\frac{3}{5} + \ldots$

5. For the series $3 + 5 + 7 + \ldots$, write an expression for the nth term.

6. Write an expression for the nth term of the following series.
(a) $9 + 17 + 25 + \ldots$
(b) $100 + 102 + 104 + \ldots$
(c) $6 + 9 + 12 + \ldots$
(d) $80 + 86 + 92 + \ldots$
(e) $-21 - 17 - 13 - \ldots$
(f) $15 + 10 + 5 + \ldots$
(g) $\frac{7}{8} + 1 + 1\frac{1}{8} + \ldots$
(h) $-30 - 32 - 34 - \ldots$
(i) $3.2 + 4.4 + 5.6 + \ldots$
(j) $\frac{1}{2} + 1\frac{1}{4} + 2 + \ldots$

7. Find which term of $3 + 7 + 11 + \ldots$ is equal to 111.

8. Which term of the series $1 + 5 + 9 + \ldots$ is 213?

9. Which term of the series $15 + 24 + 33 + \ldots$ is 276?

10. Which term of the series $25 + 18 + 11 + \ldots$ is equal to -73?

11. Is zero a term of the series $48 + 45 + 42 + \ldots$?

12. Is 270 a term of the series $3 + 11 + 19 + \ldots$?

13. Is 405 a term of the series $0 + 3 + 6 + \ldots$?

14. Find the first value of n for which the terms of the series $100 + 93 + 86 + \ldots$ become less than 20.

15. Find the values of n for which the terms of the series $-86 - 83 - 80 - \ldots$ are positive.

16. Find the first negative term of $54 + 50 + 46 + \ldots$.

17. Find the first term that is greater than 100 in the series $3 + 7 + 11 + \ldots$.

18. The first term of an arithmetic series is -7 and the common difference is 8. Find the 100th term of the series.

19. The first term of an arithmetic series is 15 and the 3rd term is 31.
(a) Find the common difference
(b) Find the 10th term of the series.

20. The first term of an arithmetic series is 3 and the 5th term is 39. Find its common difference.

21. The 2nd term of an arithmetic series is 19 and the 7th term is 54. Find its first term and common difference.

22. Find the 20th term in an arithmetic series with 4th term 29 and 10th term 83.

23. The common difference of an arithmetic series is 6 and the 5th term is 29. Find the first term of the series.

24. If the 3rd term of an arithmetic series is 45 and the 9th term is 75, find the 50th term of the series.

25. The 7th term of an arithmetic series is 17 and the 10th term is 53. Find the 100th term of the series.

26. (a) Show that $\log_5 x + \log_5 x^2 + \log_5 x^3 + \dots$ is an arithmetic series.
(b) Find the 80th term.
(c) If $x = 4$, evaluate the 10th term correct to 1 decimal place.

27. (a) Show that $\sqrt{3} + \sqrt{12} + \sqrt{27} + \dots$ is an arithmetic series.
(b) Find the 50th term in simplest form.

28. Find the 25th term of $\log_2 4 + \log_2 8 + \log_2 16 + \dots$

29. Find the 40th term of $5b + 8b + 11b + \dots$

30. Which term is $213y$ of the series $28y + 33y + 38y + \dots$?

Partial sum of an arithmetic series

The sum of the first n terms of an arithmetic series (nth partial sum) is given by the formula:

$$S_n = \frac{n}{2}(a + l) \text{ where } l = \text{last or } n\text{th term}$$

Proof

Let the last or nth term be l.

$$S_n = a + (a + d) + (a + 2d) + \dots + l \qquad (1)$$
$$S_n = l + (l - d) + (l - 2d) + \dots + a \qquad (2)$$
$$2S_n = (a + l) + (a + l) + (a + l) + \dots + (a + l) \qquad (1) + (2)$$
$$= n(a + l)$$
$$\therefore \quad S_n = \frac{n}{2}(a + l)$$

In general,

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Proof

Since l = nth term,

$$l = a + (n - 1)\,d$$

$$\therefore \quad S_n = \frac{n}{2}(a + l)$$

$$= \frac{n}{2}[a + a + (n - 1)\,d]$$

$$= \frac{n}{2}[2a + (n - 1)\,d]$$

We use this formula when the nth term is unknown.

EXAMPLES

1. Evaluate 9 + 14 + 19 + … + 224.

Solution

$a = 9, d = 5$

First we find n.

$$T_n = 224$$

$$\therefore \quad T_n = a + (n - 1)\,d$$

$$224 = 9 + (n - 1)\,5$$

$$= 9 + 5n - 5$$

$$= 5n + 4$$

$$220 = 5n$$

$$44 = n$$

$$\therefore \quad S_n = \frac{n}{2}(a + l)$$

$$S_{44} = \frac{44}{2}(9 + 224)$$

$$= 22 \times 233$$

$$= 5126$$

2. For what value of n is the sum of n terms of 2 + 11 + 20 + … equal to 618?

Solution

$$a = 2,\ d = 9,\ S_n = 618$$

$$S_n = \frac{n}{2}[2a + (n - 1)\,d]$$

$$618 = \frac{n}{2}[2 \times 2 + (n - 1)\,9]$$

$$1236 = n\,(4 + 9n - 9)$$

$$= n\,(9n - 5)$$

$$= 9n^2 - 5n$$

CONTINUED

$$0 = 9n^2 - 5n - 1236$$
$$= (n-12)(9n+103)$$

n must be a positive integer.

$$\therefore \quad n = 12 \text{ or } -11.4$$

But n cannot be negative, so $n = 12$.

---

3. The 6th term of an arithmetic series is 23 and the sum of the first 10 terms is 210. Find the sum of 20 terms.

Solution

$$T_n = a + (n-1)d$$
$$T_6 = a + (6-1)d = 23$$
$$a + 5d = 23 \qquad (1)$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$S_{10} = \frac{10}{2}[2a + (10-1)d] = 210$$
$$5(2a + 9d) = 210$$
$$2a + 9d = 42 \qquad (2)$$
$$(1) \times (2): \quad 2a + 10d = 46 \qquad (3)$$
$$(2) - (3): \quad -d = -4$$
$$d = 4$$

Substitute $d = 4$ in (1):

$$a + 5(4) = 23$$
$$a + 20 = 23$$
$$a = 3$$

$$S_{20} = \frac{20}{2}[2(3) + (20-1)4]$$
$$= 10[6 + 19(4)]$$
$$= 10 \times 82$$
$$= 820$$

---

4. Evaluate $\sum_{r=1}^{50} 3r + 2$.

Solution

It would take a long time to add these up!

$$\sum_{r=1}^{50} 3r + 2 = (3 \times 1 + 2) + (3 \times 2 + 2) + (3 \times 3 + 2) + \ldots + (3 \times 50 + 2)$$
$$= 5 + 8 + 11 + \ldots + 152$$

Arithmetic series with $a = 5, d = 3, l = 152, n = 50$

$$S_n = \frac{n}{2}(a + l)$$
$$S_{50} = \frac{50}{2}(5 + 152)$$
$$= 25 \times 157$$
$$= 3925$$

7.5 Exercises

1. Find the sum of 15 terms of the series.
   (a) $4 + 7 + 10 + \ldots$
   (b) $2 + 7 + 12 + \ldots$
   (c) $60 + 56 + 52 + \ldots$

2. Find the sum of 30 terms of the series.
   (a) $1 + 7 + 13 + \ldots$
   (b) $15 + 24 + 33 + \ldots$
   (c) $95 + 89 + 83 + \ldots$

3. Find the sum of 25 terms of the series.
   (a) $-2 + 5 + 12 + \ldots$
   (b) $5 - 4 - 13 - \ldots$

4. Find the sum of 50 terms of the series.
   (a) $50 + 44 + 38 + \ldots$
   (b) $11 + 14 + 17 + \ldots$

5. Evaluate
   (a) $15 + 20 + 25 + \ldots + 535$
   (b) $9 + 17 + 25 + \ldots + 225$
   (c) $5 + 2 - 1 - \ldots - 91$
   (d) $81 + 92 + 103 + \ldots + 378$
   (e) $229 + 225 + 221 + \ldots + 25$
   (f) $-2 + 6 + 14 + \ldots + 94$
   (g) $0 - 9 - 18 - \ldots - 216$
   (h) $79 + 81 + 83 + \ldots + 229$
   (i) $14 + 11 + 8 + \ldots - 43$
   (j) $1\frac{1}{2} + 1\frac{3}{4} + 2 + \ldots + 25\frac{1}{4}$

6. Evaluate
   (a) $\sum_{n=1}^{20} 4n - 7$
   (b) $\sum_{r=1}^{15} 5 - 3r$
   (c) $\sum_{r=3}^{20} 4 - 6r$
   (d) $\sum_{n=1}^{50} 5n + 3$
   (e) $\sum_{5}^{40} 4 - 3n$

7. How many terms of the series $45 + 47 + 49 + \ldots$ give a sum of 1365?

8. For what value of n is the sum of the arithmetic series $5 + 9 + 13 + \ldots$ equal to 152?

9. How many terms of the series $80 + 73 + 66 + \ldots$ give a sum of 495?

10. The sum of the first 5 terms of an arithmetic series is 110 and the sum of the first 10 terms is 320. Find the first term and the common difference.

11. The sum of the first 5 terms of an arithmetic series is 35 and the sum of the next 5 terms is 160. Find the first term and the common difference.

12. Find S_{25}, given an arithmetic series whose 8th term is 16 and whose 13th term is 81.

13. The sum of 12 terms of an arithmetic series is 186 and the 20th term is 83. Find the sum of 40 terms.

14. How many terms of the series $20 + 18 + 16 + \ldots$ give a sum of 104?

15. The sum of the first 4 terms of an arithmetic series is 42 and the sum of the 3rd and 7th term is 46. Find the sum of the first 20 terms.

16. (a) Show that $(x + 1) + (2x + 4) + (3x + 7) + \ldots$ is an arithmetic series.
    (b) Find the sum of the first 50 terms of the series.

17. The 20th term of an arithmetic series is 131 and the sum of the 6th to 10th terms inclusive is 235. Find the sum of the first 20 terms.

18. The sum of 50 terms of an arithmetic series is 249 and the sum of 49 terms of the series is 233. Find the 50th term of the series.

19. Prove that $T_n = S_n - S_{n-1}$ for any series.

20. Find the sum of all integers between 1 and 100 that are not multiples of 6.

Class Investigation

Look at the working out of question 14 in the previous set of exercises. Why are there two values for n?

DID YOU KNOW?

Here is an interesting series:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots.$$

Gottfried Wilhelm Leibniz (1646–1716) discovered this result. It is interesting that while π is an irrational number, it can be written as the sum of rational numbers.

Geometric Series

In a geometric series each term is formed by multiplying the preceding term by a constant. The constant is called the **common ratio.**

EXAMPLE

Find the common ratio of the series $3 + 6 + 12 + \ldots$

Solution

By looking at this sequence, each term is multiplied by 2 to get the next term.

If you can't see this, we divide the terms as follows:

$$\frac{6}{3} = \frac{12}{6} = 2$$

If $T_1 + T_2 + T_3 + \ldots$ is a geometric series then

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$$

In general, if $T_1 + T_2 + T_3 + T_4 + \ldots + T_{n-1} + T_n$ is a geometric series then

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = \ldots = \frac{T_n}{T_{n-1}}.$$

EXAMPLES

1. Find x if $5 + x + 45 + \ldots$ is a geometric series.

Solution

For a geometric series $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$$\frac{x}{5} = \frac{45}{x}$$

$$x^2 = 225$$

$$x = \pm\sqrt{225}$$

$$= \pm 15$$

x is called the geometric mean.

If $x = 15$ the series is $5 + 15 + 45 + \ldots (r = 3)$.

If $x = -15$, the series is $5 - 15 + 45 - \ldots (r = -3)$.

When r is negative, the signs alternate.

---

2. Is $\frac{1}{4} + \frac{1}{6} + \frac{1}{18} + \ldots$ a geometric series?

Solution

$$\frac{T_2}{T_1} = \frac{1}{6} \div \frac{1}{4}$$

$$= \frac{1}{6} \times \frac{4}{1}$$

$$= \frac{2}{3}$$

$$\frac{T_3}{T_2} = \frac{1}{18} \div \frac{1}{6}$$

$$= \frac{1}{18} \times \frac{6}{1}$$

$$= \frac{1}{3}$$

$$\neq \frac{T_2}{T_1}$$

$\therefore$ the series is not geometric.

Terms of a geometric series

$a + ar + ar^2 + ar^3 + \dots + ar^{n-1} + \dots$ is a geometric series with first term a, common ratio r and nth term given by

$$T_n = ar^{n-1}$$

Proof

Let the first term of a geometric series be a and the common ratio be r.
Then first term is a
second term is ar
third term is ar^2
fourth term is ar^3, and so on
nth term is ar^{n-1}

EXAMPLES

1.

(a) Find the 10th term of the series $3 + 6 + 12 + \dots$

(b) Write an expression for the nth term of the series.

Solution

(a) This is a geometric series with $a = 3$ and $r = 2$.
We want the 10th term, so $n = 10$.

$$\begin{aligned} T_n &= ar^{n-1} \\ T_{10} &= 3(2)^{10-1} \\ &= 3 \times 2^9 \\ &= 1536 \end{aligned}$$

(b)
$$\begin{aligned} T_n &= ar^{n-1} \\ &= 3(2)^{n-1} \end{aligned}$$

2. Find the common ratio of $\frac{2}{3} + \frac{4}{15} + \frac{8}{75} + \dots$ and hence find the 8th term in index form.

Solution

$$\begin{aligned} r &= \frac{4}{15} \div \frac{2}{3} \left(= \frac{8}{75} \div \frac{4}{15} \right) \\ &= \frac{4}{15} \times \frac{3}{2} \\ &= \frac{2}{5} \end{aligned}$$

$$
\begin{aligned}
T_n &= ar^{n-1}\\
T_8 &= \frac{2}{3}\left(\frac{2}{5}\right)^{8-1}\\
&= \frac{2}{3}\left(\frac{2}{5}\right)^{7}\\
&= \frac{2^8}{3 \times 5^7}
\end{aligned}
$$

Terms can become very large in geometric series.

3. Find the 10th term of the series $-5 + 10 - 20 + \ldots$

Solution

$$
\begin{aligned}
a = -5,\ r &= -2,\ n = 10\\
T_n &= ar^{n-1}\\
T_{10} &= -5(-2)^{10-1}\\
&= -5(-2)^9\\
&= -5(-512)\\
&= 2560
\end{aligned}
$$

4. Which term of the series $4 + 12 + 36 + \ldots$ is equal to 78 732?

Solution

This is a geometric series with $a = 4$ and $r = 3$.
The nth term is 78 732.

$$
\begin{aligned}
T_n &= ar^{n-1}\\
78\,732 &= 4(3)^{n-1}\\
19\,683 &= 3^{n-1}\\
\log_{10} 19\,683 &= \log_{10} 3^{n-1}\\
&= (n-1)\log 3\\
\frac{\log_{10} 19\,683}{\log_{10} 3} &= n - 1\\
9 &= n - 1\\
10 &= n
\end{aligned}
$$

So the 10th term is 78 732.

You could use logarithms to the base e or ln instead of logarithms to the base 10.

CONTINUED

5. The third term of a geometric series is 18 and the 7th term is 1458. Find the first term and the common ratio.

Solution

$$T_n = ar^{n-1}$$
$$T_3 = ar^{3-1} = 18$$
$$ar^2 = 18 \qquad (1)$$
$$T_7 = ar^{7-1} = 1458$$
$$ar^6 = 1458 \qquad (2)$$

Always divide the equations in geometric series to eliminate a.

$$(2) \div (1): r^4 = 81$$
$$r = \pm\sqrt[4]{81}$$
$$= \pm 3$$

Substitute $r = \pm 3$ into (1):

$$a(\pm 3)^2 = 18$$
$$9a = 18$$
$$a = 2$$

---

6. Find the first value of n for which the terms of the series $\frac{1}{5} + 1 + 5 + \dots$ exceed 3000.

Solution

$a = \frac{1}{5}, r = 5$

For terms to exceed 3000,

$$T_n > 3000$$
$$ar^{n-1} > 3000$$
$$\frac{1}{5}(5^{n-1}) > 3000$$
$$5^{n-1} > 15\,000$$
$$\log 5^{n-1} > \log 15\,000$$
$$(n-1)\log 5 > \log 15\,000$$
$$n - 1 > \frac{\log 15\,000}{\log 5}$$
$$n > \frac{\log 15\,000}{\log 5} + 1$$
$$n > 6.97$$

n must be an integer.

$$\therefore \quad n = 7$$

So the 7th term is the first term to exceed 3000.

7.6 Exercises

1. Are the following series geometric? If they are, find the common ratio.
   (a) $5 + 20 + 60 + \dots$
   (b) $-4 + 3 - 2\frac{1}{4} + \dots$
   (c) $\frac{3}{4} + \frac{3}{14} + \frac{3}{49} + \dots$
   (d) $7 + 5\frac{5}{6} + 3\frac{1}{3} + \dots$
   (e) $-14 + 42 - 168 + \dots$
   (f) $1\frac{1}{3} + \frac{8}{9} + \frac{8}{27} + \dots$
   (g) $5.7 + 1.71 + 0.513 + \dots$
   (h) $2\frac{1}{4} - 1\frac{7}{20} + \frac{81}{100} + \dots$
   (i) $63 + 9 + 1\frac{7}{8} + \dots$
   (j) $-1\frac{7}{8} + 15 - 120 + \dots$

2. Evaluate all pronumerals in these geometric series.
   (a) $4 + 28 + x + \dots$
   (b) $-3 + 12 + y + \dots$
   (c) $2 + a + 72 + \dots$
   (d) $y + 2 + 6 + \dots$
   (e) $x + 8 + 32 + \dots$
   (f) $5 + p + 20 + \dots$
   (g) $7 + y + 63 + \dots$
   (h) $-3 + m - 12 + \dots$
   (i) $3 + (x - 4) + 15 + \dots$
   (j) $3 + (k - 1) + 21 + \dots$
   (k) $\frac{1}{4} + t + \frac{1}{9} +, \dots$
   (l) $\frac{1}{3} + t + \frac{4}{3} +, \dots$

3. Write an expression for the nth term of the following series.
   (a) $1 + 5 + 25 + \dots$
   (b) $1 + 1.02 + 1.0404 + \dots$
   (c) $1 + 9 + 81 + \dots$
   (d) $2 + 10 + 50 + \dots$
   (e) $6 + 18 + 54 + \dots$
   (f) $8 + 16 + 32 + \dots$
   (g) $\frac{1}{4} + 1 + 4 + \dots$
   (h) $1000 - 100 + 10 - \dots$
   (i) $-3 + 9 - 27 + \dots$
   (j) $\frac{1}{3} + \frac{2}{15} + \frac{4}{75} + \dots$

4. Find the 6th term of each series.
   (a) $8 + 24 + 72 + \dots$
   (b) $9 + 36 + 144 + \dots$
   (c) $8 - 32 + 128 + \dots$
   (d) $-1 + 5 - 25 + \dots$
   (e) $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \dots$

5. What is the 9th term of each series?
   (a) $1 + 2 + 4 + \dots$
   (b) $4 + 12 + 36 + \dots$
   (c) $1 + 1.04 + 1.0816 + \dots$
   (d) $-3 + 6 - 12 + \dots$
   (e) $\frac{3}{4} - \frac{3}{8} + \frac{3}{16} - \dots$

6. Find the 8th term of each series.
   (a) $3 + 15 + 75 + \dots$
   (b) $2.1 + 4.2 + 8.4 + \dots$
   (c) $5 - 20 + 80 - \dots$
   (d) $-\frac{1}{2} + \frac{3}{10} - \frac{9}{50} + \dots$
   (e) $1\frac{47}{81} + 2\frac{10}{27} + 3\frac{5}{9} + \dots$

7. Find the 20th term of each series, leaving the answer in index form.
   (a) $3 + 6 + 12 + \dots$
   (b) $1 + 7 + 14 + \dots$
   (c) $1.04 + 1.04^2 + 1.04^3 + \dots$
   (d) $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$
   (e) $\frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \dots$

8. Find the 50th term of $1 + 11 + 121 + \ldots$ in index form.

9. Which term of the series $4 + 20 + 100 + \ldots$ is equal to 12 500?

10. Which term of $6 + 36 + 216 + \ldots$ is equal to 7776?

11. Is 1200 a term of the series $2 + 16 + 128 + \ldots$?

12. Which term of $3 + 21 + 147 + \ldots$ is equal to 352 947?

13. Which term of the series $8 - 4 + 2 - \ldots$ is $\dfrac{1}{128}$?

14. Which term of $54 + 18 + 6 + \ldots$ is $\dfrac{2}{243}$?

15. Find the value of n if the nth term of the series $-2 + 1\frac{1}{2} - 1\frac{1}{8} + \ldots$ is $-\dfrac{81}{128}$.

16. The first term of a geometric series is 7 and the 6th term is 1701. Find the common ratio.

17. The 4th term of a geometric series is –648 and the 5th term is 3888.
(a) Find the common ratio.
(b) Find the 2nd term.

18. The 3rd term of a geometric series is $\dfrac{2}{5}$ and the 5th term is $1\frac{3}{5}$. Find its first term and common ratio.

19. Find the value of n for the first term of the series $5000 + 1000 + 200 + \ldots$ that is less than 1.

20. Find the first term of the series $\dfrac{2}{7} + \dfrac{6}{7} + 2\dfrac{4}{7} + \ldots$ that is greater than 100.

Partial sum of a geometric series

The sum of the first n terms of a geometric series (nth partial sum) is given by the formula:

$$S_n = \frac{a(r^n - 1)}{r - 1} \text{ for } |r| > 1$$
$$S_n = \frac{a(1 - r^n)}{1 - r} \text{ for } |r| < 1$$

You studied absolute values in the Preliminary Course $|r| < 1$ means $-1 < r < 1$. What does $|r| > 1$ mean?

Proof

The sum of a geometric series can be written

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \quad (1)$$
$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n \quad (2)$$

(1) – (2):
$$\begin{aligned} S_n(1 - r) &= a - ar^n \\ &= a(1 - r^n) \\ S_n &= \frac{a(1 - r^n)}{1 - r} \end{aligned}$$

(2) – (1) gives the formula

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

The two formulae are the same, just rearranged.

EXAMPLES

1. Find the sum of the first 10 terms of the series 3 + 12 + 48 + …

Solution

This is a geometric series with $a = 3$, $r = 4$ and we want $n = 10$.
Since $r > 1$ we use the first formula.

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{3(4^{10} - 1)}{4 - 1} \\ &= \frac{3(4^{10} - 1)}{3} \\ &= 4^{10} - 1 \\ &= 1\,048\,575 \end{aligned}$$

2. Evaluate $\sum_{3}^{11} 2^n$.

Solution

$$\begin{aligned} \sum_{3}^{11} 2^n &= 2^3 + 2^4 + 2^5 + \ldots + 2^{11} \\ &= 8 + 16 + 32 + \ldots + 2048 \end{aligned}$$

Geometric series with $a = 8$, $r = 2$, $n = 9$

Can you see why n = 9? Count the terms carefully.

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ S_9 &= \frac{8(2^9 - 1)}{2 - 1} \\ &= \frac{8(512 - 1)}{1} \\ &= 8 \times 511 \\ &= 4088 \end{aligned}$$

3. Evaluate $60 + 20 + 6\frac{2}{3} + \ldots + \frac{20}{81}$.

Solution

$a = 60$, $r = \frac{1}{3}$, $T_n = \frac{20}{81}$

First find n.

$T_n = ar^{n-1} = \frac{20}{81}$

CONTINUED

$$60\left(\frac{1}{3}\right)^{n-1} = \frac{20}{81}$$
$$\left(\frac{1}{3}\right)^{n-1} = \frac{1}{243}$$
$$\frac{1}{3^{n-1}} = \frac{1}{243}$$
$$\frac{1}{3^{n-1}} = \frac{1}{3^5}$$
$$\therefore n - 1 = 5$$
$$n = 6$$
$$S_n = \frac{a(1-r^n)}{1-r}$$
$$S_6 = \frac{60\left[1-\left(\frac{1}{3}\right)^6\right]}{1-\frac{1}{3}}$$
$$= \frac{60\left(1-\frac{1}{729}\right)}{\frac{2}{3}}$$
$$= 60\left(\frac{728}{729}\right) \times \frac{3}{2}$$
$$= 89\frac{71}{81}$$

---

4. The sum of n terms of $1 + 4 + 16 + \ldots$ is 21 845. Find the value of n.

Solution

$$a = 1, r = 4, S_n = 21\,845$$
$$S_n = \frac{a(r^n - 1)}{r-1}$$
$$21\,845 = \frac{1(4^n - 1)}{4-1}$$
$$= \frac{4^n - 1}{3}$$
$$65\,535 = 4^n - 1$$
$$65\,536 = 4^n$$
$$4^8 = 4^n$$
$$\therefore \quad n = 8$$

So 8 terms gives a sum of 21 845.

7.7 Exercises

1. Find the sum of 10 terms of the series.
   (a) $6 + 24 + 96 + \ldots$
   (b) $3 + 15 + 75 + \ldots$

2. Find the sum of 8 terms of the series.
   (a) $-1 + 7 - 49 + \ldots$
   (b) $8 + 24 + 72 + \ldots$

3. Find the sum of 15 terms of the series.
   (a) $4 + 8 + 16 + \ldots$
   (b) $\frac{3}{4} - \frac{3}{8} + \frac{3}{16} - \ldots$

4. Evaluate
   (a) $2 + 10 + 50 + \ldots + 6250$
   (b) $18 + 9 + 4\frac{1}{2} + \ldots + \frac{9}{64}$
   (c) $3 + 21 + 147 + \ldots + 7203$
   (d) $\frac{3}{4} + 2\frac{1}{4} + 6\frac{3}{4} + \ldots + 182\frac{1}{4}$
   (e) $-3 + 6 - 12 + \ldots + 384$

5. Evaluate
   (a) $\sum_{n=1}^{8} 2^{n-1}$
   (b) $\sum_{n=1}^{6} \frac{1}{3^n}$
   (c) $\sum_{n=2}^{10} 5^{n-3}$
   (d) $\sum_{r=1}^{8} \frac{1}{2^{r-1}}$
   (e) $\sum_{k=1}^{7} 4^{k+1}$

6. Find the
   (a) 9th term
   (b) sum of 9 terms of the
   series $7 + 14 + 28 + \ldots$

7. Find the sum of 30 terms of the series $1.09 + 1.09^2 + 1.09^3 + \ldots$, correct to 2 decimal places.

8. Find the sum of 25 terms of the series $1 + 1.12 + 1.12^2 + \ldots$, correct to 2 decimal places.

9. Find the value of n if the sum of n terms of the series $11 + 33 + 99 + \ldots$ is equal to 108 251.

10. How many terms of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ give a sum of $\frac{1023}{1024}$?

11. The common ratio of a geometric series is 4 and the sum of the first 5 terms is 3069. Find the first term.

12. Find the number of terms needed to be added for the sum to exceed 1 000 000 in the series $4 + 16 + 64 + \ldots$

13. Lucia currently earns \$25 000. Her wage increases by 5% each year. Find
   (a) her wage after 6 years
   (b) her total earnings (before tax) in 6 years.

14. Write down an expression for the series $2 - 10 + 50 - \ldots + 2(-5^{k-1})$
   (a) in sigma notation
   (b) as a sum of n terms.

15. Find the sum of the first 10 terms of the series $3 + 7 + 13 + \ldots + [2^n + (2n - 1)] + \ldots$

PUZZLES

1. A poor girl saved a rich king from drowning one day. The king offered the girl a reward of sums of money in 30 daily payments. He gave the girl a choice of payments:
   Choice 1: \$1 the first day, \$2 the second day, \$3 the third day and so on.
   Choice 2: 1 cent the first day, 2 cents the second day, 4 cents the third day and so on, the payment doubling each day.
   How much money would the girl receive for each choice? Which plan would give the girl more money?

This is a hard one!

2. Can you solve **Fibonacci's Problem?**
   A man entered an orchard through 7 guarded gates and gathered a certain number of apples. As he left the orchard he gave the guard at the first gate half the apples he had and 1 apple more. He repeated this process for each of the remaining 6 guards and eventually left the orchard with 1 apple. How many apples did he gather? (He did not give away any half apples.)

Limiting sum (sum to infinity)

In some geometric series the sum becomes very large as n increases.
For example,

$$2 + 4 + 8 + 16 + 32 + 64 + 128 + \ldots$$

We say that this series diverges, or it has an infinite sum.

In some geometric series, the sum does not increase greatly after a few terms. We say this series converges and it has a limiting sum (the sum is limited to a finite number).

EXAMPLE

Can you estimate the sum of this series?

For the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$, notice that after a while the terms are becoming closer and closer to zero and so will not add much to the sum of the whole series.

(a) Evaluate, correct to 4 decimal places, the sum of
  (i) 10 terms
  (ii) 20 terms.

(b) Estimate its limiting sum.

Solution

(a) $a = 2, \quad r = \frac{1}{2}$

(i) $n = 10$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$= \frac{2\left(1 - \left(\frac{1}{2}\right)^{10}\right)}{1 - \frac{1}{2}}$$

$$= \frac{2\left(1 - \frac{1}{2^{10}}\right)}{\frac{1}{2}}$$

$$= 3.9961$$

(ii) $n = 20$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$= \frac{2\left(1 - \left(\frac{1}{2}\right)^{20}\right)}{1 - \frac{1}{2}}$$

$$= \frac{2\left(1 - \frac{1}{2^{20}}\right)}{\frac{1}{2}}$$

$$= 4.0000$$

(b) The limiting sum is 4.

Can you see why the series $2 + 4 + 8 + 16 + \ldots$ diverges and the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots$ converges?

The difference is in the common ratio. Only geometric series with common ratios $|r| < 1$ will converge and have a limiting sum.

$$S_\infty = \frac{a}{1 - r}$$

$|r| < 1$ is the necessary condition for the limiting sum to exist.

Proof

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

For $|r| < 1$, as n increases, r^n decreases and approaches zero

e.g. When $r = \frac{1}{2}$:

$$\left(\frac{1}{2}\right)^{10} = \frac{1}{1024} \text{ and } \left(\frac{1}{2}\right)^{20} = \frac{1}{1\,048\,576}$$

$$\therefore \; S_\infty = \frac{a(1 - 0)}{1 - r}$$
$$= \frac{a}{1 - r}$$

EXAMPLES

1. Find the limiting sum of $2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots$

Solution

In the previous example, we guessed that the limiting sum was 4. Here we will use the formula to find the limiting sum.

$a = 2,\ r = \frac{1}{2}$

$$S_\infty = \frac{a}{1 - r}$$
$$= \frac{2}{1 - \frac{1}{2}}$$
$$= \frac{2}{\frac{1}{2}}$$
$$= 2 \times \frac{2}{1}$$
$$= 4$$

---

2. Find the sum of the series $6 + 2 + \frac{2}{3} + \dots$

Solution

$$a = 6,\ r = \frac{2}{6} = \frac{\frac{2}{3}}{2} = \frac{1}{3}$$

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{6}{1-\frac{1}{3}}$$
$$= \frac{6}{\frac{2}{3}}$$
$$= 6 \times \frac{3}{2}$$
$$= 9$$

3. Which of the following series have a limiting sum?

(a) $\frac{3}{4} + \frac{15}{16} + 1\frac{11}{64} + \ldots$

(b) $100 + 50 + 25 + \ldots$

(c) $3 + 2\frac{1}{4} + 1\frac{11}{16} + \ldots$

Solution

(a) $r = \frac{\frac{15}{16}}{\frac{3}{4}} = \frac{1\frac{11}{64}}{\frac{15}{16}}$

$= 1\frac{1}{4}$

> 1

So this series does not have a limiting sum.

(b) $r = \frac{50}{100} = \frac{25}{50}$

$= \frac{1}{2}$

< 1

So this series does have a limiting sum.

(c) $r = \frac{2\frac{1}{4}}{3} = \frac{1\frac{11}{16}}{2\frac{1}{4}}$

$= \frac{3}{4}$

< 1

So this series does have a limiting sum.

CONTINUED

4. Evaluate $\sum_{n=2}^{\infty} 2\left(\frac{2}{3}\right)^n$

Solution

$$\begin{aligned}
\sum_{n=2}^{\infty} 2\left(\frac{2}{3}\right)^n &= 2\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right)^3 + 2\left(\frac{2}{3}\right)^4 + \dots \\
&= 2\left(\frac{4}{9}\right) + 2\left(\frac{8}{27}\right) + 2\left(\frac{16}{81}\right) + \dots \\
&= \frac{8}{9} + \frac{16}{27} + \frac{32}{81} + \dots
\end{aligned}$$

$$a = \frac{8}{9},\ r = \frac{2}{3}$$

$$\begin{aligned}
S_\infty &= \frac{a}{1-r} \\
&= \frac{\frac{8}{9}}{1-\frac{2}{3}} \\
&= \frac{\frac{8}{9}}{\frac{1}{3}} \\
&= \frac{8}{9} \times \frac{3}{1} \\
&= 2\frac{2}{3}
\end{aligned}$$

7.8 Exercises

1. Which of the following series have a limiting sum? Find its limiting sum where it exists.
   (a) $9 + 3 + 1 + \dots$
   (b) $\frac{1}{4} + \frac{1}{2} + 1 + \dots$
   (c) $16 - 4 + 1 - \dots$
   (d) $\frac{2}{3} + \frac{7}{9} + \frac{49}{54} + \dots$
   (e) $1 + \frac{2}{3} + \frac{4}{9} + \dots$
   (f) $\frac{5}{8} + \frac{1}{8} + \frac{1}{40} + \dots$
   (g) $-6 + 36 - 216 + \dots$
   (h) $-2\frac{1}{4} + 1\frac{7}{8} - 1\frac{27}{48} + \dots$
   (i) $\frac{1}{9} + \frac{1}{6} + \frac{1}{4} + \dots$
   (j) $2 - \frac{4}{5} + \frac{8}{25} - \dots$

2. Find the limiting sum of each series.
   (a) $40 + 20 + 10 + \dots$
   (b) $320 + 80 + 20 + \dots$
   (c) $100 - 50 + 25 - \dots$
   (d) $6 + 3 + 1\frac{1}{2} + \dots$
   (e) $\frac{2}{5} + \frac{6}{35} + \frac{18}{245} + \dots$
   (f) $72 - 24 + 8 - \dots$
   (g) $-12 + 2 - \frac{1}{3} + \dots$
   (h) $\frac{3}{4} - \frac{1}{2} + \frac{1}{3} - \dots$
   (i) $12 + 9 + 6\frac{3}{4} + \dots$
   (j) $-\frac{2}{3} + \frac{5}{12} - \frac{25}{96} + \dots$

3. Find the difference between the limiting sum and the sum of 6 terms of
(a) $56 - 28 + 14 - \ldots$
(b) $72 + 24 + 8 + \ldots$
(c) $1 + \frac{1}{5} + \frac{1}{25} + \ldots$
(d) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$
(e) $1\frac{1}{4} + \frac{15}{16} + \frac{45}{64} + \ldots$

4. Evaluate
(a) $\sum_{r=1}^{\infty}\left(\frac{1}{5}\right)^{r-1}$
(b) $\sum_{n=1}^{\infty}\left(\frac{2}{7}\right)^{n}$
(c) $\sum_{3}^{\infty}\left(\frac{1}{4}\right)^{k}$
(d) $\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^{n-1}$
(e) $\sum_{p=2}^{\infty}\left(\frac{2}{3}\right)^{p-2}$
(f) $\sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^{n}$
(g) $\sum_{1}^{\infty} 2\left(\frac{1}{6}\right)^{n}$
(h) $\sum_{n=3}^{\infty} -4\left(\frac{2}{3}\right)^{n-1}$
(i) $\sum_{2}^{\infty} 2\left(-\frac{1}{9}\right)^{n-2}$
(j) $\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{5}\right)^{n}$

5. A geometric series has limiting sum 6 and common ratio $\frac{1}{3}$. Evaluate the first term of the series.

6. A geometric series has a limiting sum of 5 and first term 3. Find the common ratio.

7. The limiting sum of a geometric series is $9\frac{1}{3}$ and the common ratio is $\frac{2}{5}$. Find the first term of the series.

8. A geometric series has limiting sum 40 and its first term is 5. Find the common ratio of the series.

9. A geometric series has limiting sum $-6\frac{2}{5}$ and a first term of -8. Find its common ratio.

10. The limiting sum of a geometric series is $-\frac{3}{10}$ and its first term is $-\frac{1}{2}$. Find the common ratio of the series.

11. The second term of a geometric series is 2 and its limiting sum is 9. Find the values of first term a and common ratio r.

12. A geometric series has 3rd term 12 and 4th term–3. Find a, r and the limiting sum.

13. A geometric series has 2nd term $\frac{2}{3}$ and 4th term $\frac{8}{27}$. Find a, r and its limiting sum.

14. The 3rd term of a geometric series is 54 and the 6th term is $11\frac{83}{125}$. Evaluate a, r and the limiting sum.

15. The 2nd term of a geometric series is $\frac{4}{15}$ and the 5th term is $\frac{32}{405}$. Find the values of a and r and its limiting sum.

16. The limiting sum of a geometric series is 5 and the 2nd term is $1\frac{1}{5}$. Find the first term and the common ratio.

17. The series $x + \frac{x}{4} + \frac{x}{16} + \ldots$ has a limiting sum of $\frac{7}{8}$. Evaluate x.

18. (a) For what values of k does the limiting sum exist for the series $k + k^2 + k^3 + \ldots$?

(b) Find the limiting sum of the series when $k = -\frac{2}{3}$.

(c) Evaluate k if the limiting sum of the series is 3.

19. (a) For what values of p will the limiting sum exist for the series $1 - 2p + 4p^2 - \ldots$?

(b) Find the limiting sum when $p = \frac{1}{5}$.

(c) Evaluate p if the limiting sum of the series is $\frac{7}{8}$.

20. Show that in any geometric series the difference between the limiting sum and the sum of n terms is $\frac{ar^n}{1-r}$.

Applications of Series

General applications

Arithmetic and geometric series are useful in solving real life problems involving patterns. It is important to choose the correct type of series and know whether you are asked to find a term or a sum of terms.

EXAMPLES

1. A stack of cans on a display at a supermarket has 5 cans on the top row. The next row down has 2 more cans and the next one has 2 more cans and so on.

(a) Calculate the number of cans in the 11th row down.

(b) If there are 320 cans in the display altogether, how many rows are there?

Solution

The first row has 5 cans, the 2nd row has 7 cans, the 3rd row 9 cans and so on. This forms an arithmetic series with $a = 5$ and $d = 2$.

(a) For the 11th row, we want $n = 11$.

$$\begin{aligned} T_n &= a + (n-1)d \\ T_{11} &= 5 + (11-1) \times 2 \\ &= 5 + 10 \times 2 \\ &= 25 \end{aligned}$$

So there are 25 cans in the 11th row.

(b) If there are 320 cans altogether, this is the sum of cans in all rows.

So $S_n = 320$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$320 = \frac{n}{2}[2 \times 5 + (n-1) \times 2]$$
$$= \frac{n}{2}[10 + 2n - 2]$$
$$= \frac{n}{2}[2n + 8]$$
$$= n^2 + 4n$$
$$0 = n^2 + 4n - 320$$
$$= (n-16)(n+20)$$
$$n - 16 = 0,\ n + 20 = 0$$
$$n = 16,\ n = -20$$

Since n must be a positive integer, then $n = 16$.

There are 16 rows of cans.

---

2. A layer of tinting for a car window lets in 95% of light.

(a) What percentage of light is let in by

(i) 2 layers

(ii) 3 layers

(iii) 10 layers

of tinting?

(b) How many layers will let in 40% of light?

Solution

(a) (i) 1 layer lets in 95% of light.

So 2 layers let in 95% × 95% of light.

$$95\% \times 95\% = 0.95 \times 0.95$$
$$= 0.9025$$
$$= 90.25\%$$

So 2 layers let in 90.25% of light.

To convert a percentage to a decimal, divide by 100 and to convert a decimal to a percentage, multiply by 100.

(ii) 1 layer lets in 95% or 0.95 of light.

2 layers let in 0.95×0.95 or 0.95^2 of light.

3 layers let in $0.95^2 \times 0.95$ or 0.95^3 of light.

$$0.95^3 = 0.857$$
$$= 85.7\%$$

So 3 layers let in 85.7% of light.

CONTINUED

(iii) The number of layers forms the geometric series
$0.95 + 0.95^2 + 0.95^3 + \ldots$ with $a = 0.95$, $r = 0.95$.
For 10 layers, $n = 10$

$$\begin{aligned}T_n &= ar^{n-1}\\ T_{10} &= 0.95(0.95)^{10-1}\\ &= 0.95(0.95)^9\\ &= 0.95^{10}\\ &= 0.5987\\ &= 59.87\%\end{aligned}$$

So 10 layers let in 59.87% of light.

(b) We want to find n when the nth term is 40% or 0.4.

$$\begin{aligned}T_n &= ar^{n-1}\\ 0.4 &= 0.95(0.95)^{n-1}\\ &= 0.95^n\\ \log 0.4 &= \log 0.95^n\\ &= n \log 0.95\\ \frac{\log 0.4}{\log 0.95} &= n\\ 17.9 &= n\end{aligned}$$

So around 18 layers of tinting will let in 40% of light.

The limiting sum can also be used to solve various problems.

EXAMPLES

$0.\dot{5}$ is called a recurring decimal.

1. Write $0.\dot{5}$ as a fraction.

Solution

$$\begin{aligned}0.\dot{5} &= 0.5555555\ldots\\ &= \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \ldots\end{aligned}$$

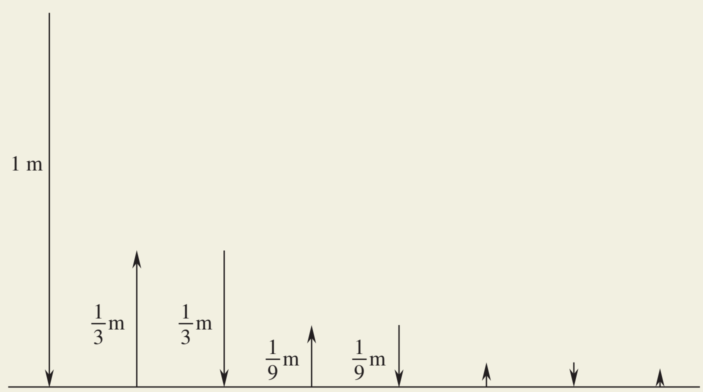
This is a geometric series with $a = \frac{5}{10} = \frac{1}{2}$ and $r = \frac{1}{10}$.

$$\begin{aligned}S_\infty &= \frac{a}{1-r}\\ &= \frac{\frac{1}{2}}{1 - \frac{1}{10}}\\ &= \frac{\frac{1}{2}}{\frac{9}{10}}\end{aligned}$$

$$= \frac{1}{2} \times \frac{10}{9}$$
$$= \frac{5}{9}$$

2. A ball is dropped from a height of 1 metre and bounces up to $\frac{1}{3}$ of its height. It continues bouncing, rising $\frac{1}{3}$ of its height on each bounce until it eventually reaches the ground. What is the total distance through which it travels?

Solution



Notice that there is a series for the ball coming downwards and another series upwards.
There is more than one way of calculating the total distance. Here is one way of solving it.

Can you find the total distance a different way?

$$\text{Total distance} = 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{9} + \frac{1}{9} + \frac{1}{27} + \frac{1}{27} + \ldots$$
$$= 1 + 2\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots\right)$$

$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ is a geometric series with $a = \frac{1}{3}$ and $r = \frac{1}{3}$

$$S_\infty = \frac{a}{1 - r}$$
$$= \frac{\frac{1}{3}}{1 - \frac{1}{3}}$$
$$= \frac{\frac{1}{3}}{\frac{2}{3}}$$
$$= \frac{1}{3} \times \frac{3}{2}$$
$$= \frac{1}{2}$$

$$\text{Total distance} = 1 + 2\left(\frac{1}{2}\right)$$
$$= 1 + 1$$
$$= 2$$

So the ball travels 2 metres altogether.

Investigation

1. In the second example above, in theory will the ball ever stop?

2. Kim owes \$1000 on her credit card. If she pays back 10% of the amount owing each month, she will never finish paying it off. Is this true or false?

7.9 Exercises

1. A market gardener plants daffodil bulbs in rows, starting with a row of 45 bulbs. Each successive row has 5 more bulbs.
   (a) Calculate the number of bulbs in the 34th row.
   (b) Which row would be the first to have more than 100 bulbs in it?
   (c) The market gardener plants 10 545 bulbs. How many rows are there?

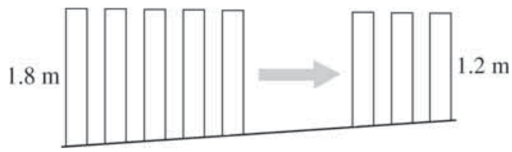
2. A stack of logs has 1 on the top, then 3 on the next row down, and each successive row has 2 more logs than the one on top of it.
   (a) How many logs are in the 20th row?
   (b) Which row has 57 logs?
   (c) If there are 1024 logs altogether, how many rows are in the stack?

3. A set of books is stacked in layers, where each layer contains three books fewer than the layer below. There are 6 books in the top layer, 9 in the next layer and so on. There are n layers altogether.
   (a) Write down the number of books in the bottom layer.
   (b) Show that there are $\frac{3}{2}n(n+3)$ books in the stack altogether.

4. A painting appreciates (increases its value) by 16% p.a. It is currently worth \$20 000.
   (a) How much will it be worth in
      (i) 1 year?
      (ii) 2 years?
      (iii) 3 years?
   (b) How much will it be worth in 11 years?
   (c) How long will it take for it to be worth \$50 000?

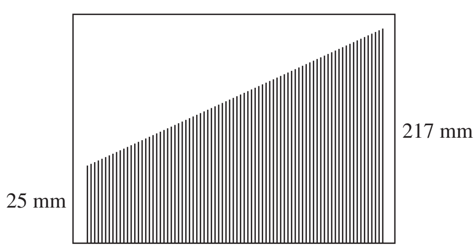
5. Water evaporates from a pond at an average rate of 7% each week.
   (a) What percentage of water is left in the pond after
      (i) 1 week?
      (ii) 2 weeks?
      (iii) 3 weeks?
   (b) What percentage is left after 15 weeks?
   (c) If there was no rain, approximately how long would it take for the pond to only have 25% of its water left?

6. A timber fence is to be built on sloping land, with the shortest piece of timber 1.2 m and the longest 1.8 m. There are 61 pieces of timber in the fence.



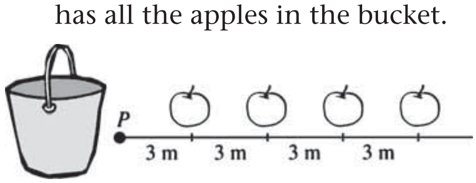
(a) What is the difference in height between each piece of timber?
(b) What length of timber is needed for the fence altogether?

7. A logo is made with vertical lines equally spaced as shown. The shortest line is 25 mm, the longest is 217 mm and the sum of the lengths of all the lines is 5929 mm.



(a) How many lines are in the logo?
(b) Find the difference in length between adjacent lines.

8. In a game, a child starts at point P and runs and picks up an apple 3 m away. She then runs back to P and puts the apple in a bucket. The child then runs to get the next apple 6 m away, and runs back to P to place it in the bucket. This continues until she has all the apples in the bucket.



(a) How far does the child run to get to the kth apple?
(b) How far does the child run to fetch all k apples, including return trips to P?
(c) The child runs 270 m to fetch all apples and return them to the bucket. How many apples are there?

9. The price of shares in a particular company is falling by an average of 2% each day.
(a) What percentage of their value are they after 2 days?
(b) How many days will it take for the shares to halve in value?
(c) After how many days will the shares be worth 25% of their value?

10. A southern brown bandicoot population in WA is decreasing by 5% each year.
(a) What percentage of the population is left after 5 years?
(b) After how many years will the population be only 50%?
(c) How many years will it take for the population to decrease by 80%?

11. Write each recurring decimal as a fraction.
   (a) $0.\dot{4}$
   (b) $0.\dot{7}$
   (c) $1.\dot{2}$
   (d) $0.\dot{2}\dot{5}$
   (e) $2.\dot{8}\dot{1}$
   (f) $0.2\dot{3}$
   (g) $1.4\dot{7}$
   (h) $1.01\dot{5}$
   (i) $0.1\dot{3}\dot{2}$
   (j) $2.\dot{3}\dot{6}\dot{1}$

12. A frog jumps 0.5 metres. It then jumps 0.1 m and on each subsequent jump, travels 0.2 m of the previous distance. Find the total distance through which the frog jumps.

13. A tree 3 m high grows by another $2\frac{2}{5}$ m after 5 years, then by another $1\frac{23}{25}$ m after 5 more years and so on, each year growing $\frac{4}{5}$ more than the previous year's growth. Find the ultimate height of the tree.

14. An 8 cm seedling grows by $4\frac{4}{5}$ cm in the first week, and then keeps growing by $\frac{3}{5}$ of its previous week's growth. How tall will it grow?

15. An object rolls 0.5 m in the first second. Then each second after, it rolls by $\frac{5}{6}$ of its previous roll. Find how far it will roll altogether.

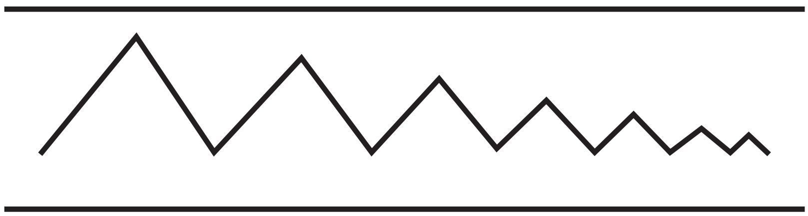
16. A lamb grows by $\frac{2}{5}$ of its previous growth each month. If a lamb is 45 cm tall,
   (a) how tall will it be after 6 months?
   (b) what will its final height be?

17. A 100 m cliff erodes by $\frac{2}{7}$ of its height each year.
   (a) What will the height of the cliff be after 10 years?
   (b) After how many years will the cliff be less than 50 m high?

18. An elastic string drops down 60 cm and then bounces back to $\frac{2}{3}$ of its initial height. It keeps bouncing, each time rising back to $\frac{2}{3}$ of its previous height. What is the total distance through which the string travels?

19. Mary bounces a ball, dropping it from 1.5 m on its first bounce. It then rises up to $\frac{2}{5}$ of its height on each bounce. Find the distance through which the ball travels.

20. A roadside wall has a zig zag pattern on it as shown.



The two longest lines are each 2 m long, then the next two lines are $1\frac{3}{4}$ m long and each subsequent pair of lines are $\frac{7}{8}$ of the length of the previous lines. Find the total distance of the lines.

21. Kate receives an email chain letter that she is asked to send to 8 friends. If Kate forwards this email on to 8 friends and each of them sends it on to 8 friends, and so on,
(a) describe the number of people receiving the email as a sequence (including Kate's email)
(b) how many people would receive the email when it is sent for the 9th time?
(c) how many people would have received the email altogether if it is sent 9 times?

Compound interest

Compound interest refers to when the interest earned on an investment is then added to the amount of the investment. This means that each amount of interest earned is calculated on the previous interest as well as the investment.

Money that earns compound interest grows according to a geometric series.

EXAMPLE

An amount of \$2000 is invested at the rate of 6% p.a. Find the amount in the bank at the end of n years.

You may remember that p.a. means per annum or annually.

Solution

After 1 year: The amount in the bank is the original amount plus the interest of 6% earned on the amount.

$$\begin{aligned}\text{Amount } 1 &= \$2000 + 6\% \text{ of } \$2000\\ &= \$2000(1 + 6\%)\\ &= \$2000(1 + 0.06)\\ &= \$2000(1.06)\end{aligned}$$

CONTINUED

After 2 years: The amount in the bank is the previous amount plus the interest of 6% earned on this amount.

Factorise using a common factor of \$2000(1.06).

$$\begin{aligned}\text{Amount } 2 &= \text{Amount } 1 + 6\% \text{ of amount } 1\\ &= \$2000(1.06) + 6\% \text{ of } \$2000(1.06)\\ &= \$2000(1.06)(1 + 6\%)\\ &= \$2000(1.06)(1 + 0.06)\\ &= \$2000(1.06)(1.06)\\ &= \$2000(1.06)^2\end{aligned}$$

Similarly after 3 years the amount is $\$2000(1.06)^3$ and so on.
The amount after n years will be $\$2000(1.06)^n$.

In general the compound interest formula is

$$\begin{aligned}A &= P(1 + r)^n \text{ where}\\ P &= \text{principal (initial amount)}\\ r &= \text{interest rate as a decimal}\\ n &= \text{number of time periods}\end{aligned}$$

Proof

Let $\$P$ be invested at the rate of r p.a. compound interest for n years where r is a decimal.

After 1 year:

$$\begin{aligned}\text{Amount} &= P + r \text{ of } P\\ &= P(1 + r)\end{aligned}$$

After 2 years:

Factorise using a common factor of $P(1 + r)$.

$$\begin{aligned}\text{Amount} &= P(1 + r) + r \text{ of } P(1 + r)\\ &= P(1 + r)(1 + r)\\ &= P(1 + r)^2\end{aligned}$$

After 3 years:

$$\begin{aligned}\text{Amount} &= P(1 + r)^2 + r \text{ of } P(1 + r)^2\\ &= P(1 + r)^2(1 + r)\\ &= P(1 + r)^3\end{aligned}$$

Following this pattern, the amount after n years will be $P(1 + r)^n$.

EXAMPLES

1. Find the amount that will be in the bank after 6 years if \$2000 is invested at 12% p.a. with interest paid

(a) yearly
(b) quarterly
(c) monthly.

Solution

$P = 2000$

(a)
$$\begin{aligned} r &= 12\% \\ &= 0.12 \\ n &= 6 \\ A &= P(1+r)^n \\ &= 2000(1+0.12)^6 \\ &= 2000(1.12)^6 \\ &= 3947.65 \end{aligned}$$

So the amount is \$3947.65.

Always round money off to 2 decimal places (to the nearest cent).

(b) For quarterly interest, it is divided into 4 amounts each year.

$$\begin{aligned} r &= 0.12 \text{ p.a.} \\ &= 0.12 \div 4 \text{ quarterly} \\ &= 0.03 \end{aligned}$$

Also the interest is paid in 4 times a year.

$$\begin{aligned} n &= 6 \times 4 \\ &= 24 \text{ quarters} \\ A &= P(1+r)^n \\ &= 2000(1+0.03)^{24} \\ &= 2000(1.03)^{24} \\ &= 4065.59 \end{aligned}$$

So the amount is \$4065.59.

(c) For monthly interest, it is divided into 12 amounts each year.

$$\begin{aligned} r &= 0.12 \text{ p.a.} \\ &= 0.12 \div 12 \text{ monthly} \\ &= 0.01 \end{aligned}$$

Also the interest is paid in 12 times a year.

$$\begin{aligned} n &= 6 \times 12 \\ &= 72 \text{ months} \\ A &= P(1+r)^n \\ &= 2000(1+0.01)^{72} \\ &= 2000(1.01)^{72} \\ &= 4094.20 \end{aligned}$$

So the amount is \$4049.20.

CONTINUED

2. Geoff wants to invest enough money to pay for a \$10 000 holiday in 5 years' time. If interest is 8% p.a., how much does Geoff need to invest now?

Solution

$A = 10\,000$, $r = 8\% = 0.08$ and $n = 5$

We want to find the principal P.

$$\begin{aligned} A &= P(1+r)^n \\ 10\,000 &= P(1+0.08)^5 \\ &= P(1.08)^5 \\ \frac{10\,000}{1.08^5} &= P \\ 6805.83 &= P \end{aligned}$$

So Geoff will need to invest \$6805.83 now.

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3. An amount of \$1800 was invested at 6% p.a. and is now worth \$2722.66. For how many years was the money invested if interest is paid twice a year?

Solution

$P = 1800$, $A = 2722.66$

Interest is paid twice a year.

$$\begin{aligned} r &= 6\% = 0.06 \text{ p.a.} \\ &= 0.06 \div 2 \\ &= 0.03 \text{ twice a year} \end{aligned}$$

We want to find n

$$\begin{aligned} A &= P(1+r)^n \\ 2722.66 &= 1800(1+0.03)^n \\ &= 1800(1.03)^n \\ \frac{2722.66}{1800} &= 1.03^n \\ 1.5126 &= 1.03^n \\ \log 1.5126 &= \log 1.03^n \\ &= n \log 1.03 \\ \frac{\log 1.5126}{\log 1.03} &= n \\ 14 &= n \end{aligned}$$

Since interest is paid in twice a year, the number of years will be $14 \div 2$.

So the money was invested for 7 years.

7.10 Exercises

1. Find the amount of money in the bank after 10 years if
   (a) \$500 is invested at 4% p.a.
   (b) \$7500 is invested at 7% p.a.
   (c) \$8000 is invested at 8% p.a.
   (d) \$5000 is invested at 6.5% p.a.
   (e) \$2500 is invested at 7.8% p.a.

2. Sam banks \$1500 where it earns interest at the rate of 6% p.a. Find the amount after 5 years if interest is paid
   (a) annually
   (b) twice a year
   (c) quarterly.

3. Chad banks \$3000 in an account that earns 5% p.a. Find the amount in the bank after 10 years if interest is paid
   (a) quarterly
   (b) monthly.

4. I put \$350 in the bank where it earns interest of 8% p.a. with interest paid annually. Find the amount there will be in the account after 2 years.

5. How much money will there be in an investment account after 3 years if interest of 4.5% p.a. is paid twice a year on \$850?

6. Find the amount of money there will be in a bank after 8 years if \$1000 earns interest of 7% p.a. with interest paid twice a year.

7. Abdul left \$2500 in a building society account for 4 years, with interest of 5.5% p.a. paid yearly. How much money did he have in the account at the end of that time?

8. Find the amount of money there will be after 15 years if \$6000 earns 9% p.a. interest, paid quarterly.

9. How much money will be in a bank account after 5 years if \$500 earns 6.5% p.a. with interest paid monthly?

10. Find the amount of interest earned over 4 years if \$1400 earns 6% p.a. paid quarterly.

11. How much money will be in a credit union account after 8 years if \$8000 earns 7.5% p.a. interest paid monthly?

12. Elva wins a lottery and invests \$500 000 in an account that earns 8% p.a. with interest paid monthly. How much will be in the account after 12 years?

13. Calculate the amount deposited 4 years ago at 5% p.a. if the current amount in the account is
    (a) \$5000
    (b) \$675
    (c) \$12 000
    (d) \$289.50
    (e) \$12 800.

14. How much was banked 3 years ago if the amount in the bank now is \$5400 and interest is 5.8% p.a. paid quarterly?

15. How many years ago was an investment made if \$5000 was invested at 6% p.a. paid monthly and is now worth \$6352.45?

16. Find the number of years that \$10 000 was invested at 8% p.a. with interest paid twice a year if there is now \$18 729.81 in the bank.

17. Jude invested \$4500 five years ago at x% p.a. Evaluate x if Jude now has an amount of
    (a) \$6311.48
    (b) \$5743.27
    (c) \$6611.98
    (d) \$6165.39
    (e) \$6766.46 in the bank.

18. Hamish is given the choice of a bank account in which interest is paid annually or quarterly. If he deposits \$1200, find the difference in the amount of interest paid over 3 years if interest is 7% p.a.

19. Kate has \$4000 in a bank account that pays 5% p.a. with interest paid annually and Rachel has \$4000 in a different account paying 4% quarterly. Which person will receive the most interest over 5 years and by how much?

20. A bank offers investment account A at 8% p.a. with interest paid twice a year and account B with interest paid at 6% p.a. at monthly intervals. If Georgie invests \$5000 over 6 years, which account pays the most interest? How much more does it pay?

Annuities

An annuity is a fund where a certain amount of money is invested regularly (often annually, which is where the name comes from) for a number of years. Scholarship and trust funds, certain types of insurance and superannuation are examples of annuities.

While superannuation and other investments are affected by changes in the economy, due to their long-term nature, the amount of interest they earn balances out over the years. In this course, we average out these changes, and use a constant amount of interest annually.

EXAMPLES

1. A sum of \$1500 is invested at the beginning of each year in a superannuation fund. If interest is paid at 6% p.a., how much money will be available at the end of 25 years?

Solution

It is easier to keep track of each annual amount separately.
The first amount earns interest for 25 years, the 2nd amount earns interest for 24 years, the 3rd amount for 23 years and so on.

The 25th amount earns interest for 1 year.

$P = 1500$ and $r = 6\% = 0.06$

$$\begin{aligned} A &= P(1+r)^n \\ &= 1500(1+0.06)^n \\ &= 1500(1.06)^n \end{aligned}$$

$$\begin{aligned} A_1 &= 1500(1.06)^{25} \\ A_2 &= 1500(1.06)^{24} \\ A_3 &= 1500(1.06)^{23} \\ &\ \ \vdots \\ A_{25} &= 1500(1.06)^1 \end{aligned}$$

Since the last amount is deposited at the beginning of the year, it earns interest for 1 year.

Total superannuation amount

$$\begin{aligned} T &= A_1 + A_2 + A_3 + \ldots + A_{25} \\ &= 1500(1.06)^{25} + 1500(1.06)^{24} + 1500(1.06)^{23} + \ldots + 1500(1.06)^1 \\ &= 1500(1.06^{25} + 1.06^{24} + 1.06^{23} + \ldots + 1.06^1) \\ &= 1500(1.06 + 1.06^2 + 1.06^3 + \ldots + 1.06^{25}) \end{aligned}$$

$1.06 + 1.06^2 + 1.06^3 + \ldots + 1.06^{25}$ is a geometric series with

$a = 1.06$, $r = 1.06$ and $n = 25$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\begin{aligned} S_{25} &= \frac{1.06(1.06^{25} - 1)}{1.06 - 1} \\ &= 58.16 \end{aligned}$$

$$\begin{aligned} T &= 1500 \times 58.16 \\ &= 87\,234.57 \end{aligned}$$

So the total amount of superannuation after 25 years is \$87 234.57.

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2. An amount of \$50 is put into an investment account at the end of each month. If interest is paid at 12% p.a. paid monthly, how much is in the account at the end of 10 years?

Solution

$P = 50$

$$\begin{aligned} r &= 12\% \\ &= 0.12 \end{aligned}$$

The monthly interest rate is

$0.12 \div 12 = 0.01$.

Monthly interest over 10 years gives

$$\begin{aligned} n &= 10 \times 12 \\ &= 120 \end{aligned}$$

CONTINUED

The first amount earns interest for 119 months, since it is deposited at the end of the month.
The 2nd amount earns interest for 118 months, the 3rd amount for 117 months and so on.
The 120th amount earns no interest as it goes in at the end of the last month.

$$\begin{aligned}
A &= P(1+r)^n \\
&= 50(1+0.01)^n \\
&= 50(1.01)^n \\
A_1 &= 50(1.01)^{119} \\
A_2 &= 50(1.01)^{118} \\
A_3 &= 50(1.01)^{117} \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
A_{120} &= 50(1.01)^0
\end{aligned}$$

Total amount

$$\begin{aligned}
T &= A_1 + A_2 + A_3 + \ldots + A_{120} \\
&= 50(1.01)^{119} + 50(1.01)^{118} + 50(1.01)^{117} + \ldots + 50(1.01)^0 \\
&= 50(1.01^{119} + 1.01^{118} + 1.01^{117} + \ldots + 1.01^0) \\
&= 50(1.01^0 + 1.01^1 + 1.01^2 + \ldots + 1.01^{119}) \\
&= 50(1 + 1.01^1 + 1.01^2 + \ldots + 1.01^{119})
\end{aligned}$$

$1 + 1.01^1 + 1.01^2 + \ldots + 1.01^{119}$ is a geometric series with $a = 1$, $r = 1.01$ and $n = 120$

$$\begin{aligned}
S_n &= \frac{a(r^n - 1)}{r - 1} \\
S_{120} &= \frac{1(1.01^{120} - 1)}{1.01 - 1} \\
&= 230.04 \\
T &= 50 \times 230.04 \\
&= 11\,501.93
\end{aligned}$$

So the total amount after 10 years is \$11 501.93.

7.11 Exercises

1. Find the amount of superannuation available at the end of 20 years if \$500 is invested at the beginning of each year and earns 9% p.a.

2. Michael works for a company that puts aside \$1200 at the beginning of each year for his superannuation. If the money earns interest at the rate of 5% p.a., find the amount of superannuation that Michael will have at the end of 30 years.

3. How much superannuation will there be at the end of 20 years if \$800 is invested at 10% p.a. at the beginning of each year?

4. Rachel starts working for a business at the beginning of 2005. If she retires at the end of 2034, how much superannuation will she have if \$1000 is invested at the beginning of each year at 9.5% p.a.?

5. A sum of \$1500 is invested at the end of each year for 15 years at 8% p.a. Find the amount of superannuation available at the end of the 15 years.

6. How much superannuation will there be at the end of 18 years if \$690 is invested at 8.5% p.a. at the beginning of each year?

7. If Phan pays \$750 into a superannuation fund at the beginning of each year, how much will she have at the end of 29 years if the interest is 6.8% p.a.?

8. A sum of \$1000 is invested at the end of each year for 22 years, at 9% p.a. Find the amount of superannuation available at the end of the 22 years.

9. Matthew starts work at the beginning of 2010. If he retires at the end of 2037, how much superannuation will he have if he invests \$700 at the beginning of each year at 12.5% p.a.?

10. An apprentice starts work for a small business. If she invests \$400 at the beginning of each year, how much superannuation will she have at the end of 25 years if the money earns 15.5% p.a.?

11. Lliam wants to save up \$15 000 for a car in 5 years' time. He invests \$2000 at the end of each year in an account that pays 7.5% p.a. interest. How much more will Lliam have to pay at the end of 5 years to make up the \$15 000?

12. A school invests \$5000 at the end of each year at 6% p.a. towards a new library. How much will the school have after 10 years?

13. Jacques puts aside \$500 at the end of each year for 5 years. If the money is invested at 6.5% p.a., how much will Jacques have at the end of the 5 years?

14. An employee contributes \$200 into a superannuation fund at the end of each year. If the interest rate on this fund is 11.5% p.a., how much will the employee have at the end of 20 years?

15. Mohammed's mother invests \$200 for him each birthday up to and including his 18th birthday. The money earns 6% p.a. How much money will Mohammed have on his 18th birthday?

16. Xuan is saving up for a holiday. She invests \$800 at the end of each year at 7.5% p.a. How much will she have for her holiday after 5 years' time?

17. A couple saves \$3000 at the end of each year towards a deposit on a house. If the interest rate is 5% p.a., how much will the couple have saved after 6 years?

18. Lucia saves up \$2000 each year and at the end of the year she invests it at 6% p.a.
(a) She does this for 10 years. What is her investment worth?
(b) Lucia continues investing \$2000 a year for 5 more years. What is the future value of her investment?

19. Jodie starts work in 2012 and puts \$1000 in a superannuation fund at the end of the year. She keeps putting in this same amount at the end of every year until she retires at the end of 2029. If interest is paid at 10% p.a., calculate how much Jodie will have when she retires.

20. Michelle invests \$1000 at the end of each year. The interest rate is 8% p.a.
(a) How much will her investment be worth after 6 years?
(b) How much more would Michelle's investment be worth after 6 years if she had invested \$1200 each year?

21. Jack cannot decide whether to invest \$1000 at the end of each year for 15 years or \$500 for 30 years in a superannuation fund. If the interest rate is 5% p.a., which would be the better investment for Jack?

22. Kate is saving up to go overseas in 8 years' time. She invests \$1000 at the end of each year at 7% p.a. and estimates that the trip will cost her around \$10 000. Will she have enough? If so, how much over will it be? If she doesn't earn enough, how much will she need to add to this money to make it up to the \$10 000?

23. Farmer Brown puts aside part of the farm's earnings at the beginning of each month to buy a new truck in 10 years' time when the old one wears out. He invests \$400 each month at 9% p.a. and estimates the cost of a new truck at \$80 000. Will the investment earn enough to buy the new truck? What is the difference?

24. Marcus and Rachel want to save up \$25 000 for a deposit on an apartment in 6 years' time. They aim to pay around half the deposit each. Marcus invests an inheritance of \$9000 in a bank account where it earns 8% p.a. Rachel invests \$100 at the beginning of each month where it earns interest of 9% p.a.
(a) What is the future value of Marcus's investment after 6 years?
(b) How much will Rachel's investment be worth after 6 years?
(c) Will they be able to pay the deposit in 6 years' time?

25. Jenny puts aside \$20 at the end of each month for 3 years. How much will she have then if the investment earns 8.2% p.a., paid monthly?

Loan repayments

The formula for compound interest and the geometric series can help with working out regular loan repayments.

EXAMPLES

1. A sum of $20 000 is borrowed at 12% p.a. and paid back at regular monthly intervals over 4 years. Find the amount of each payment.

Solution

Let M stand for the monthly repayment.
The number of payments will be 4×12 or 48.
Monthly interest will be $12\% \div 12$ or 1% (0.01).
Each month, interest for that month is added to the loan and the repayment amount is taken off.
The interest added to the first month will be $20\,000(1+0.01)^1$ or $20\,000(1.01)^1$.
The interest added to other months will be $A(1+0.01)^1$ or $A(1.01)^1$.
The amount owing for the first month:

$$\begin{aligned} A_1 &= 20\,000(1+0.01)^1 - M \\ &= 20\,000(1.01)^1 - M \end{aligned}$$

The $20\,000(1.01)^1$ is the interest added for the month and the M is the repayment subtracted from the balance.

The amount owing for the 2nd month is what was owing from the 1st month, together with that month's interest, minus the repayment.

$$\begin{aligned} A_2 &= A_1(1+0.01)^1 - M \\ &= A_1(1.01)^1 - M \\ &= [20\,000(1.01)^1 - M](1.01)^1 - M \text{ (substituting in } A_1) \\ &= 20\,000(1.01)^2 - M(1.01)^1 - M \\ &= 20\,000(1.01)^2 - M(1.01^1 + 1) \end{aligned}$$

Similarly,

$$\begin{aligned} A_3 &= A_2(1+0.01)^1 - M \\ &= [20\,000(1.01)^2 - M(1.01^1+1)](1.01)^1 - M \\ &= 20\,000(1.01)^3 - M(1.01^1+1)(1.01^1) - M \\ &= 20\,000(1.01)^3 - M(1.01^2 + 1.01^1) - M \\ &= 20\,000(1.01)^3 - M(1.01^2 + 1.01^1 + 1) \end{aligned}$$

CONTINUED

Continuing this pattern, after 4 years (48 months) the amount owing will be
$A_{48} = 20\,000(1.01)^{48} - M(1.01^{47} + 1.01^{46} + \ldots + 1.01^2 + 1.01^1 + 1)$.

But the loan is paid out after 48 months.
So $A_{48} = 0$

$$0 = 20\,000(1.01)^{48} - M(1.01^{47} + 1.01^{46} + \ldots + 1.01^2 + 1.01^1 + 1)$$

$$M(1.01^{47} + 1.01^{46} + \ldots + 1.01^2 + 1.01^1 + 1) = 20\,000(1.01)^{48}$$

$$M = \frac{20\,000(1.01)^{48}}{1.01^{47} + 1.01^{46} + \ldots + 1.01^2 + 1.01^1 + 1}$$

$$1.01^{47} + 1.01^{46} + \ldots + 1.01^2 + 1.01^1 + 1$$
$$= 1 + 1.01^1 + 1.01^2 + \ldots + 1.01^{46} + 1.01^{47}$$

This is a geometric series with $a = 1$, $r = 1.01$ and $n = 48$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{48} = \frac{1(1.01^{48} - 1)}{1.01 - 1}$$
$$= 61.223$$

$$\therefore \quad M = \frac{20\,000(1.01)^{48}}{61.223}$$
$$= 526.68$$

So the monthly repayment is \$526.68.

---

2. A store charges 10% p.a. for loans and repayments do not have to be made until the 4th month. Ivan buys \$8000 worth of furniture and pays it off over 3 years.

(a) How much does Ivan owe after 3 months?
(b) What are his monthly repayments?
(c) How much does Ivan pay altogether?

Solution

Let P stand for the payments each month.

Number of payments $= 3 \times 12 - 3$ (3 months of no repayments)
$= 33$ months

Monthly interest rate
$r = 10\% \div 12$
$= 0.1 \div 12$
$= 0.008\dot{3}$

(a) After 3 months, the amount owing is

$$A = P(1 + r)^n$$
$$= 8000(1 + 0.008\dot{3})^3$$
$$= 8000(1.008\dot{3})^3$$
$$= 8201.67$$

So the amount owing after 3 months is \$8201.67.

(b) The first repayment is in the 4th month.

$A_1 = 8000(1.008\dot{3})^1$

$A_2 = 8000(1.008\dot{3})^2$

$A_3 = 8000(1.008\dot{3})^3$

$A_4 = 8000(1.008\dot{3})^4 - P$

The first repayment is in the 4th month.

$A_5 = \left[8000(1.008\dot{3})^4 - P\right](1.008\dot{3})^1 - P$

$= 8000(1.008\dot{3})^5 - P(1.008\dot{3})^1 - P$

$= 8000(1.008\dot{3})^5 - P(1.008\dot{3}^1 + 1)$

$A_6 = \left[8000(1.008\dot{3})^5 - P(1.008\dot{3}^1 + 1)\right](1.008\dot{3})^1 - P$

$= 8000(1.008\dot{3})^6 - P(1.008\dot{3}^1 + 1)(1.008\dot{3})^1 - P$

$= 8000(1.008\dot{3})^6 - P(1.008\dot{3}^2 + 1.008\dot{3}^1) - P$

$= 8000(1.008\dot{3})^6 - P(1.008\dot{3}^2 + 1.008\dot{3}^1 + 1)$

Continuing this pattern,

$A_{36} = 8000(1.008\dot{3})^{36} - P(1.008\dot{3}^{32} + 1.008\dot{3}^{31} + \ldots + 1.008\dot{3}^1 + 1)$

The loan is paid out after 36 months.

So $A_{36} = 0$

$0 = 8000(1.008\dot{3})^{36} - P(1.008\dot{3}^{32} + 1.008\dot{3}^{31} + \ldots + 1.008\dot{3}^1 + 1)$

$P(1.008\dot{3}^{32} + 1.008\dot{3}^{31} + \ldots + 1.008\dot{3}^1 + 1) = 8000(1.008\dot{3})^{36}$

$$P = \frac{8000(1.008\dot{3})^{36}}{1.008\dot{3}^{32} + 1.008\dot{3}^{31} + \ldots + 1.008\dot{3}^2 + 1.008\dot{3}^1 + 1}$$

$1.008\dot{3}^{32} + 1.008\dot{3}^{31} + \ldots + 1.008\dot{3}^1 + 1$

$= 1 + 1.008\dot{3}^1 + \ldots + 1.008\dot{3}^{31} + 1.008\dot{3}^{32}$

This is a geometric series with $a = 1$, $r = 1.008\dot{3}$ and $n = 33$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{33} = \frac{1(1.008\dot{3}^{33} - 1)}{1.008\dot{3} - 1}$$

$$= 37.804$$

$$\therefore \quad P = \frac{8000(1.008\dot{3})^{36}}{37.804}$$

$$= 285.30$$

So the monthly repayment is \$285.30.

(c) Ivan pays \$285.30 × 33 or \$9414.94 altogether.

If you round off the repayment, your answer will be slightly different.

7.12 Exercises

1. An amount of \$3000 is borrowed at 22% p.a. and repayments made yearly for 5 years. How much will each repayment be?

2. The sum of \$20 000 is borrowed at 18% p.a. interest over 8 years. How much will the repayments be if they are made monthly?

3. David borrows \$5000 from the bank and pays back the loan in monthly instalments over 4 years. If the loan incurs interest of 15% p.a., find the amount of each instalment.

4. Mr and Mrs Nguyen mortgage their house for \$150 000.
(a) Find the amount of the monthly repayments they will have to make if the mortgage is over 25 years with interest at 6% p.a.
(b) If the Nguyens want to pay their mortgage out after 15 years, what monthly repayments would they need to make?

5. A loan of \$6000 is paid back in equal annual instalments over 3 years. If the interest is 12.5% p.a., find the amount of each annual instalment.

6. The Smith family buys a car for \$38 000, paying a 10% deposit and taking out a loan for the balance. If the loan is over 5 years with interest of 1.5% monthly, find
(a) the amount of each monthly loan repayment
(b) the total amount that the Smith family paid for the car.

7. A \$2000 loan is offered at 18% p.a. with interest charged monthly, over 3 years.
(a) If no repayment need be paid for the first 2 months, find the amount of each repayment.
(b) How much will be paid back altogether?

8. Bill thinks he can afford a mortgage payment of \$800 each month. How much can he borrow, to the nearest \$100, over 25 years at 11.5% p.a.?

9. Get Rich Bank offers a mortgage at $7\frac{1}{2}$% p.a. over 10 years and Capital Bank offers a mortgage at $5\frac{1}{2}$% p.a. over 25 years.
(a) Find the amount of the monthly repayments for each bank on a loan of \$80 000.
(b) Find the difference in the total amount paid on each mortgage.

10. Majed buys a \$35 000 car. He puts down a 5% deposit and pays the balance back in monthly instalments over 4 years at 12% p.a. Find the total amount that Majed pays for the car.

11. Amy borrowed money over 7 years at 15.5% p.a. and she pays \$1200 a month. How much did she borrow?

12. NSW Bank offers loans at 9% p.a. with an interest-free period of 3 months, while Sydney Bank offers loans at 7% p.a. Compare these loans on an amount of \$5000 over 3 years and state which bank offers the best loan and why.

13. Danny buys a plasma TV for \$10 000. He pays a \$1500 deposit and borrows the balance at 18% p.a. over 4 years.
(a) Find the amount of each monthly repayment.
(b) How much did Danny pay altogether?

14. A store offers furniture on hire purchase at 20% p.a. over 5 years, with no repayments for 6 months. Ali buys furniture worth \$12 000.
(a) How much does Ali owe after 6 months?
(b) What are the monthly repayments?
(c) How much does Ali pay for the furniture altogether?

15. A loan of \$6000 over 5 years at 15% p.a. interest, charged monthly, is paid back in 5 annual instalments.
(a) What is the amount of each instalment?
(b) How much is paid back altogether?

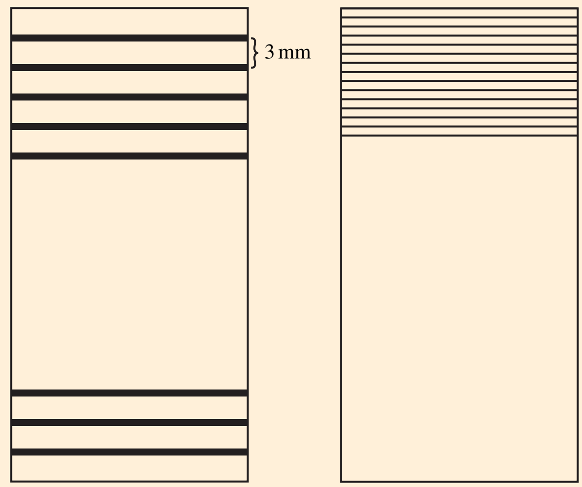
Test Yourself 7

1. Find a formula for the nth term of each sequence.
(a) 9, 13, 17, …
(b) 7, 0, −7, …
(c) 2, 6, 18, …
(d) 200, 50, $12\frac{1}{2}$, …
(e) −2, 4, −8, …

2. For the series 156 + 145 + 134 + …
(a) Find the 15th term.
(b) Find the sum of 15 terms.
(c) Find the sum of 14 terms.
(d) Write a relationship between T_{15}, S_{15} and S_{14}.
(e) Find the value of n for the first negative term.

3. Evaluate
(a) $\sum_{r=1}^{50}(9r-7)$
(b) $\sum_{3}^{7}\left(\frac{1}{n}\right)$ as a fraction
(c) $\sum_{2}^{12}5(3^n)$
(d) $\sum_{1}^{100}(7n-6)$
(e) $\sum_{n=3}^{\infty}2\left(\frac{1}{5}\right)^{n-2}$

4. A bamboo blind has 30 slats. When the blind is down, each gap between slats, and between the top and bottom of the window, is 3 mm. When the blind is up, the slats have no gaps between them.
(a) Show that when the blind is up, the bottom slat rises 90 mm.
(b) How far does the next slat rise?
(c) Explain briefly why the distances the slats rise when the blind is up form an arithmetic series.
(d) Find the distance the 17th slat from the bottom rises.
(e) What is the sum of the distances that all slats rise?



5. Find the amount invested in a bank account at 9.5% p.a. if the balance in the account is \$5860.91 after 6 years.

6. State whether each series is
(i) arithmetic
(ii) geometric
(iii) neither.
(a) 97 + 93 + 89 + …
(b) $\frac{2}{3}+\frac{1}{2}+\frac{3}{8}+\dots$
(c) $\sqrt{5}+\sqrt{20}+\sqrt{45}+\dots$
(d) −1.6 − 0.4 + 0.6 − …
(e) 3.4 + 7.5 + 11.6 + …
(f) 48 + 24 + 12 + …
(g) $-\frac{1}{5}+1-5+\dots$
(h) 105 + 100 + 95 + …
(i) $1\frac{1}{2}+1\frac{1}{4}+1+\dots$
(j) $\log_{10}x+\log_{10}x^2+\log_{10}x^3+\dots$

7. The nth term of the series 8 + 13 + 18 + … is 543. Evaluate n.

8. Amanda earned \$20 000 in one year. At the beginning of the next year she received a salary increase of \$450.

She now receives the same increase each year.
(a) What will her salary be after 10 years?
(b) How much will Amanda earn altogether over the 10 years?

9. The 11th term of an arithmetic series is 97 and the 6th term is 32. Find the first term and common difference.

10. A series has nth term given by $T_n = n^3 - 5$. Find
(a) the 4th term
(b) the sum of 4 terms
(c) which term is 5827.

11. A series has terms $5 + x + 45 + \ldots$ Evaluate x if the series is
(a) arithmetic
(b) geometric.

12. Convert each recurring decimal to a fraction.
(a) $0.\dot{4}$
(b) $0.7\dot{2}$
(c) $1.\dot{5}\dot{7}$

13. If x, $2x + 3$ and $5x$ are the first 3 terms of an arithmetic series, calculate the value of x.

14. Find the 20th term of
(a) 3, 10, 17, …
(b) 101, 98, 95, …
(c) 0.3, 0.6, 0.9, …

15. Find the limiting sum (sum to infinity) of $81 + 27 + 9 + \ldots$

16. Karl puts \$300 aside as an annuity for his son at the beginning of each year. If interest is 7% p.a., how much will his son receive at the end of 15 years?

17. For each series, write an expression for the sum of n terms.
(a) $5 + 9 + 13 + \ldots$
(b) $1 + 1.07 + 1.07^2 + \ldots$

18. (a) For what values of x does the geometric series $1 + x + x^2 + \ldots$ have a limiting sum?
(b) Find the limiting sum when $x = \frac{3}{5}$.
(c) Evaluate x when the limiting sum is $1\frac{1}{2}$.

19. The first term of an arithmetic series is 4 and the sum of 10 terms is 265. Find the common difference.

20. Every week during a typing course, Tony improves his typing speed by 3 words per minute until he reaches 60 words per minute by the end of the course.
(a) If he can type 18 words per minute in the first week of the course, how many words per minute can he type by week 8?
(b) How many weeks does the course run for?

21. A farmer borrows \$50 000 for farm machinery at 18% p.a. over 5 years and makes equal yearly repayments on the loan at the end of each year.
(a) How much does he owe at the end of the first year, just before he makes the first repayment?
(b) How much is each yearly repayment?

22. A ball drops from a height of 1.2 metres then bounces back to $\frac{3}{5}$ of this height. On the next bounce, it bounces up to $\frac{3}{5}$ of this height and so on. Through what distance will the ball travel?

23. If $x + 2$, $7x - 2$ and $15x + 6$ are consecutive terms in a geometric series, evaluate x.

24. (a) If \$2000 is invested at 4.5% p.a., how much will it be worth after 4 years?
(b) If interest is paid quarterly, how much would the investment be worth after 4 years?

25. Evaluate $8 + 14 + 20 + \ldots + 122$.

26. (a) Calculate the sum of all the multiples of 7 between 1 and 100.
(b) Calculate the sum of all numbers between 1 and 100 that are not multiples of 7.

27. Scott borrows \$200 000 to buy a house. If the interest is 6% p.a. and the loan is over 20 years,
(a) how much is each monthly repayment?
(b) how much does Scott pay altogether?

28. The sum of n terms of the series $214 + 206 + 198 + \ldots$ is 2760. Evaluate n.

29. Evaluate n if the nth term of the series $4 + 12 + 36 + \ldots$ is 354 292.

Challenge Exercise 7

1. The nth term of a sequence is given by $T_n = \dfrac{n^2}{n+1}$.
(a) What is the 9th term of the sequence?
(b) Which term is equal to $18\frac{1}{20}$?

2. For the sequence $\dfrac{3\pi}{4}, \pi, \dfrac{5\pi}{4}, \ldots$ evaluate
(a) the common difference
(b) the 7th term
(c) the sum of 6 terms.

Hint:
$3 = 2 + 1$, $5 = 4 + 1$ and so on.

3. Evaluate the sum of the first 20 terms of the series
(a) $3 + 5 + 9 + 17 + 33 + 65 + \ldots$
(b) $5 - 2 + 10 - 8 + 15 - 32 + \ldots$

4. A factory sells shoes at \$60 a pair. For 10 pairs of shoes there is a discount, whereby each pair costs \$58. For 20 pairs, the cost is \$56 a pair and so on. Find
(a) the price of each pair of shoes on an order of 100 pairs of shoes.
(b) the total price on an order of 60 pairs of shoes.

5. Which term of the sequence $\dfrac{7}{9}, \dfrac{14}{45}, \dfrac{28}{225}, \ldots$ is equal to $\dfrac{224}{28125}$?

6. Find the sum of all integers between 1 and 200 that are not multiples of 9.

7. Find the least number of terms for which the sum of the series $20 + 4 + \dfrac{4}{5} + \ldots$ is greater than 24.99.

8. Find the values of n for which the sequence given by $T_n = n^2 - 4n$ is negative.

9. The sum of the first 5 terms of a geometric sequence is 77 and the sum of the next 5 terms is -2464. Find the 4th term of the sequence.

10. Jane's mother puts \$300 into an account at the beginning of each year to pay for Jane's education in 5 years' time. If 6% p.a. interest is paid quarterly, how much money will Jane's mother have at the end of the 5 years?

11. The geometric series $2x + 4x^2 + 8x^3 + \ldots$ has a limiting sum of 3. Evaluate x.

12. Find the amount of money in a bank account if \$5000 earns 8.5% p.a. for 4 years, then 6.5% p.a. for 3 years, with interest paid monthly for all 7 years.

13. Find the value of k for which 81 920 is the 8th term of the geometric sequence $5 + k + 80 + \ldots$

14. Kim borrows \$10 000 over 3 years at a rate of 1% interest compounded each month. If she pays off the loan in three equal annual instalments, find
(a) the amount Kim owes after one month
(b) the amount she owes after the first year, just before she pays the first instalment
(c) the amount of each instalment
(d) the total amount of interest Kim pays.

15. (a) Find the limiting sum of the series $1 + \cos^2 x + \cos^4 x + \ldots$ in simplest form.
(b) Why does this series have a limiting sum?

16. A superannuation fund paid 6% p.a. for the first 10 years and then 10% p.a. after that. If Thanh put \$5000 into this fund at the end of each year, how much would she have at the end of 25 years?