3

Integration

TERMINOLOGY

Definite integral: The integral or primitive function restricted to a lower and upper boundary. It has the notation $\int_a^b f(x)\,dx$ and geometrically represents the area between the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$

Even function: A function where $f(-x) = f(x)$. It is symmetrical about the y-axis

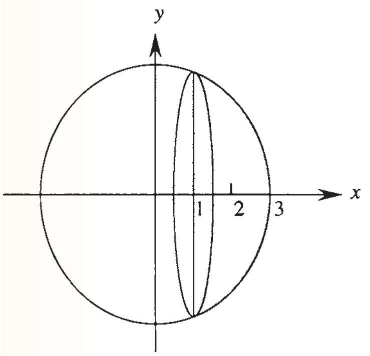
Indefinite integral: General primitive function represented by $\int f(x)\,dx$

Integration: The process of finding a primitive function

Odd function: A function where $f(-x) = -f(x)$. An odd function has rotational symmetry about the origin

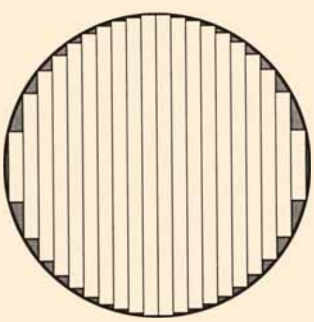
INTRODUCTION

INTEGRATION IS THE PROCESS of finding an area under a curve. It is an important process in many areas of knowledge, such as surveying, physics and the social sciences. In this chapter, you will look at approximation methods of integrating, as well as shorter methods that lead to finding areas and volumes. You will learn how integration and differentiation are related.



DID YOU KNOW?

Integration has been of interest to mathematicians since very early times. **Archimedes** (287–212 BC) found the area of enclosed curves by cutting them into very thin layers and finding their sum. He found the formula for the volume of a sphere this way. He also found an estimation of π, correct to 2 decimal places.

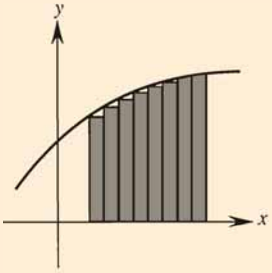


Area within curve

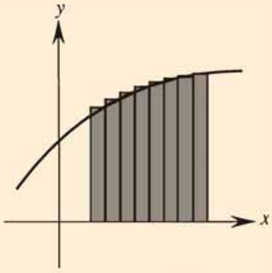
Archimedes

Approximation Methods

Mathematicians used rectangles in order to find the approximate area between a curve and the x-axis.



small rectangles



big rectangles

> The symbol $\int$ comes from S, the sum of rectangles.

NOTATION

The area of each rectangle is $f(x)\ \delta x$ where $f(x)$ is the height and δx is the width of each rectangle. As $\delta x \to 0$, sum of rectangles $\to$ exact area.

$$\text{Area} = \lim_{\delta x \to 0} \Sigma f(x)\ \delta x$$
$$= \int f(x)\ dx$$

Now there are other, more accurate ways to find the area under a curve. However, the notation is still used.

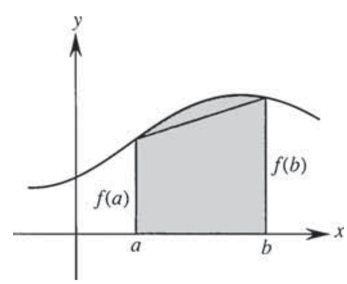
Trapezoidal rule

The **trapezoidal rule** uses a **trapezium** for the approximate area under a curve. A trapezium generally gives a better approximation to the area than a rectangle.

$$\boxed{\int_a^b f(x)\,dx \doteqdot \frac{1}{2}(b-a)\,[f(a)+f(b)]}$$

Proof

$$A = \frac{1}{2}h\,(a+b)$$
$$= \frac{1}{2}(b-a)\,[f(a)+f(b)]$$



EXAMPLES

1. Find an approximation for $\int_1^4 \frac{dx}{x}$ using the trapezoidal rule.

Solution

$$\int_1^4 \frac{dx}{x} = \int_1^4 \frac{1}{x}\,dx$$
$$\int_a^b f(x)dx \doteqdot \frac{1}{2}(b-a)\,[f(a)+f(b)]$$
$$\int_1^4 \frac{dx}{x} \doteqdot \frac{1}{2}(4-1)\,[f(1)+f(4)]$$

$= \frac{1}{2}(3)\left(\frac{1}{1}+\frac{1}{4}\right)$

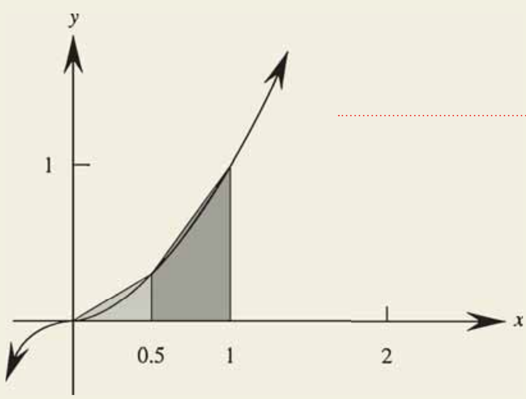
$= \frac{3}{2} \times \frac{5}{4}$

$= 1.875$

The function $f(x)$ is $\frac{1}{x}$.

2. Find an approximation for $\int_0^1 x^3\,dx$ using the trapezoidal rule with 2 subintervals.

Solution



Two subintervals mean 2 trapezia.

$$\int_a^b f(x)\,dx \doteqdot \frac{1}{2}(b-a)\,[f(a)+f(b)]$$

$$\int_0^1 x^3\,dx \doteqdot \int_0^{0.5} x^3\,dx + \int_{0.5}^1 x^3\,dx$$

$$= \frac{1}{2}(0.5-0)\,[f(0)+f(0.5)] + \frac{1}{2}(1-0.5)\,[f(0.5)+f(1)]$$

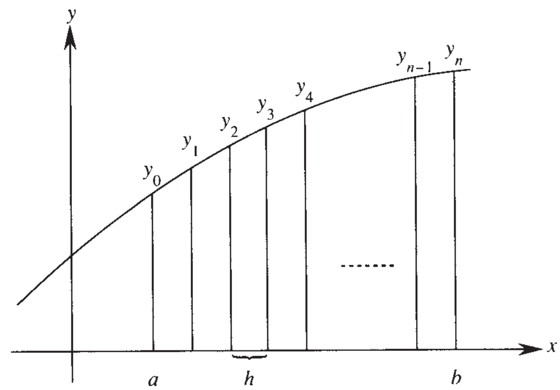
$$= \frac{1}{2}(0.5)\,(0^3+0.5^3) + \frac{1}{2}(0.5)\,(0.5^3+1^3)$$

$$= 0.25\,(0.125) + 0.25\,(1.125)$$

$$= 0.3125$$

The function is x^3.

There is a more general formula for n subintervals. Several trapezia give a more accurate area than one. However, you could use the first formula several times if you prefer using it.



$$\int_a^b f(x)\,dx \doteqdot \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+y_3+\ldots+y_{n-1})]$$

where $h = \frac{b-a}{n}$ (width of each trapezium)

Proof

$$\int_a^b f(x)\,dx \doteqdot \frac{h}{2}(y_0+y_1)+\frac{h}{2}(y_1+y_2)+\frac{h}{2}(y_2+y_3)+\ldots+\frac{h}{2}(y_{n-1}+y_n)$$

$$= \frac{h}{2}(y_0+2y_1+2y_2+2y_3+\ldots+2y_{n-1}+y_n)$$

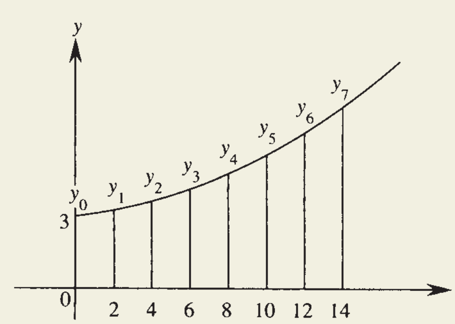
$$= \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+y_3+\ldots+y_{n-1})]$$

EXAMPLES

1. Find an approximation for $\int_0^{14}(t^2+3)\,dt$, using the trapezoidal rule with 7 subintervals.

Solution

Seven subintervals mean 7 trapezia.



$$\int_a^b f(x)\,dx \doteqdot \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1})]$$

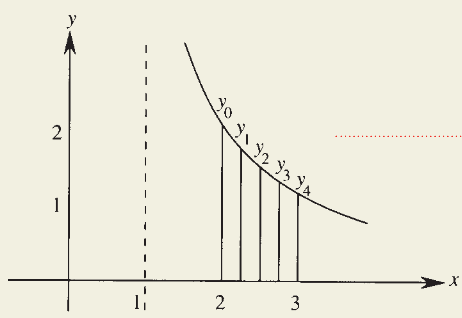
$$\int_0^{14} (t^2 + 3)\,dt \doteqdot \frac{h}{2}[(y_0 + y_7) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)]$$

where $\quad h = \dfrac{14 - 0}{7} = 2$

$$\begin{aligned}\int_0^{14} (t^2 + 3)\,dt &\doteqdot \frac{2}{2}\{[(0^2 + 3) + (14^2 + 3)] + 2[(2^2 + 3) + (4^2 + 3)\\ &\quad + (6^2 + 3) + (8^2 + 3) + (10^2 + 3) + (12^2 + 3)]\}\\ &= 966\end{aligned}$$

2. Find an approximation for $\int_2^3 \frac{2}{x-1}\,dx$, using the trapezoidal rule with 4 subintervals, correct to 3 decimal places.

Solution



There are 4 trapezia.

$$\int_a^b f(x)\,dx \doteqdot \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$\begin{aligned}\text{where} \quad h &= \frac{b-a}{n}\\ &= \frac{3-2}{4}\\ &= 0.25\end{aligned}$$

$$\begin{aligned}\int_2^3 \frac{2}{x-1}\,dx &\doteqdot \frac{0.25}{2}\left[\left(\frac{2}{2-1} + \frac{2}{3-1}\right) + 2\left(\frac{2}{2.25-1} + \frac{2}{2.5-1} + \frac{2}{2.75-1}\right)\right]\\ &\doteqdot 0.125[(2 + 1) + 2(1.6 + 1.3333 + 1.1429)]\\ &= 1.394\end{aligned}$$

Application

When surveyors need to find the area of an irregular piece of land, they measure regular strips and use an approximation method such as the trapezoidal rule.

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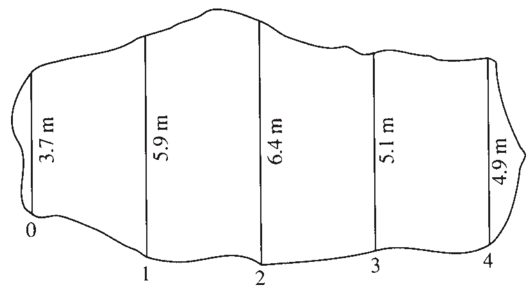
EXAMPLE



The table below gives the measurements of a certain piece of land:

x m	0	1	2	3	4
y m	3.7	5.9	6.4	5.1	4.9

Using the trapezoidal rule to find its area, correct to 2 decimal places:

$$\int_a^b f(x)\,dx \doteqdot \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$\text{Where} \quad h = \frac{b-a}{n}$$
$$= \frac{4-0}{4}$$
$$= 1$$

$$\int_0^4 f(x)\,dx \doteqdot \frac{1}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$
$$= \frac{1}{2}[(3.7 + 4.9) + 2(5.9 + 6.4 + 5.1)]$$
$$= 21.7$$

So the area of the land is approximately 21.7 m^2.

3.1 Exercises

Use the trapezoidal rule to find an approximation for

1. $\int_1^2 x^2\,dx$ using 1 subinterval.
2. $\int_0^2 (x^3 + 1)\,dx$ using 1 subinterval.
3. $\int_1^5 \frac{dx}{x}$ using 1 subinterval.
4. $\int_1^2 \frac{dx}{x+3}$ using 1 subinterval.
5. $\int_1^3 x^3\,dx$ using
   (a) 1 subinterval
   (b) 2 subintervals.
6. $\int_2^3 \log x\,dx$ using 2 subintervals.
7. $\int_0^2 \frac{dx}{x+4}$ using 2 strips.

8. $\int_1^4 \log x \, dx$ using 3 subintervals.

9. $\int_0^2 (x^2 - x) \, dx$ using 4 trapezia.

10. $\int_0^1 \sqrt{x} \, dx$ using 5 subintervals.

11. $\int_1^5 \frac{1}{x^2} \, dx$ using 4 subintervals.

12. $\int_3^6 \frac{1}{x-1} \, dx$ using 6 trapezia.

13. $\int_1^9 f(x) \, dx$ where values of $f(x)$ are given in the table:

x	1	3	5	7	9
$f(x)$	3.2	5.9	8.4	11.6	20.1

14. $\int_1^4 f(t) \, dt$ where values of $f(t)$ are given in the table:

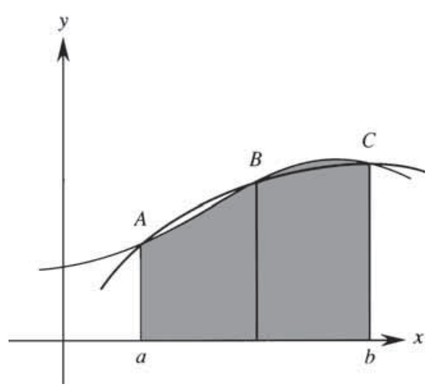
t	1	2	3	4
$f(t)$	8.9	6.5	4.1	2.9

15. $\int_2^{14} f(x) \, dx$ where values of $f(x)$ are given in the table:

x	2	4	6	8	10	12	14
$f(x)$	25.1	37.8	52.3	89.3	67.8	45.4	39.9

Simpson's rule

This is generally more accurate than the trapezoidal rule, since it makes use of parabolic arcs instead of straight lines.

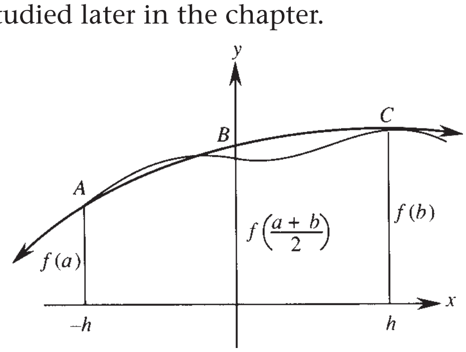


A parabola is drawn through points A, B and C to give the formula

$$\boxed{\int_a^b f(x)\,dx \doteq \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]}$$

Proof

This proof is difficult and is only included for completion of the topic. It uses results that will be studied later in the chapter.



Let $f(a) = y_0$, $f\left(\frac{a+b}{2}\right) = y_1$ and $f(b) = y_2$

Let the parabola passing through points $A(-h, y_0)$, $B(0, y_1)$ and $C(h, y_2)$ be given by $y = ax^2 + bx + c$

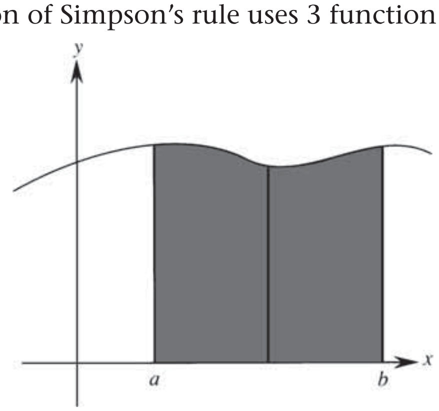
Then
$$y_0 = a(-h)^2 + b(-h) + c$$
$$y_0 = ah^2 - bh + c \qquad (1)$$
$$y_1 = a(0)^2 + b(0) + c$$
$$y_1 = c \qquad (2)$$
$$y_2 = a(h)^2 + b(h) + c$$
$$y_2 = ah^2 + bh + c \qquad (3)$$
$$y_0 + 4y_1 + y_2 = ah^2 - bh + c + 4c + ah^2 + bh + c$$
$$= 2ah^2 + 6c \qquad (4)$$
$$\int_{-h}^{h} (ax^2 + bx + c)\, dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h}$$
$$= \left(\frac{ah^3}{3} + \frac{bh^2}{2} + ch\right) - \left(\frac{-ah^3}{3} + \frac{bh^2}{2} - ch\right)$$
$$= \frac{2ah^3}{3} + 2ch$$
$$= \frac{h}{3}(2ah^2 + 6c)$$
$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

You will study this later in the chapter.

Now $h = \frac{b-a}{2}$, $y_0 = f(a)$, $y_1 = f\left(\frac{a+b}{2}\right)$ and $y_2 = f(b)$

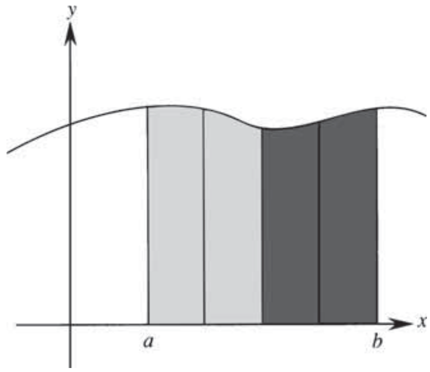
$$\therefore \quad \int_a^b f(x)\, dx = \frac{\frac{b-a}{2}}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$
$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

- One application of Simpson's rule uses 3 function values (ordinates).

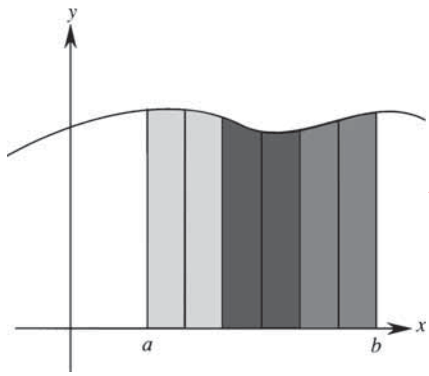


Ordinates are y-values. You first met this name in Chapter 5 of the Preliminary Course book.

- Two applications use 5 function values.



- Three applications use 7 function values.

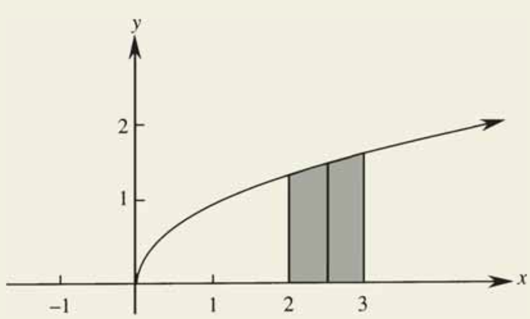


How many function values would you need for 4 applications of Simpson's rule?

EXAMPLES

1. Use Simpson's rule with 3 function values to find an approximation for $\int_2^3 \sqrt{x}\,dx$.

Solution



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$$\int_a^b f(x)\,dx \doteqdot \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$\int_2^3 \sqrt{x}\,dx \doteqdot \frac{3-2}{6}\left[f(2)+4f\left(\frac{2+3}{2}\right)+f(3)\right]$$

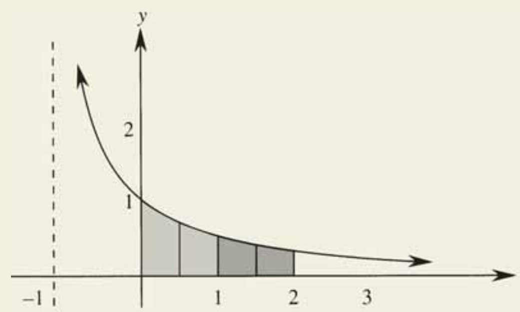
$$= \frac{1}{6}(\sqrt{2}+4\sqrt{2.5}+\sqrt{3})$$

$$\doteqdot 1.58$$

2. Use Simpson's rule with 5 function values to find an approximation for $\int_0^2 \frac{dx}{x+1}$.

Solution

Five function values mean 2 applications of Simpson's rule.



$$\int_a^b f(x)dx \doteqdot \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$\int_0^2 \frac{dx}{x+1} = \int_0^1 \frac{dx}{x+1} + \int_1^2 \frac{dx}{x+1}$$

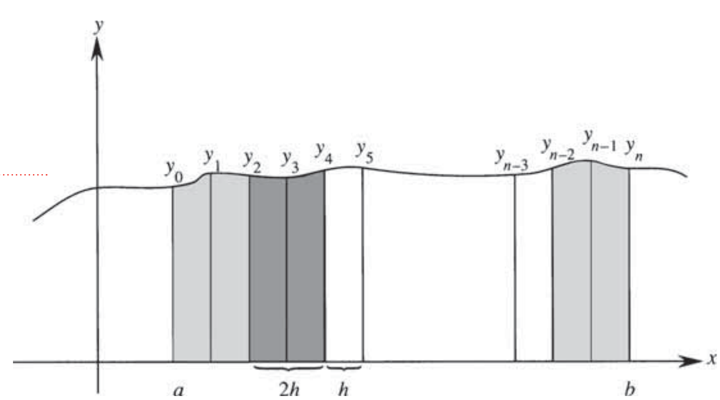
$$\doteqdot \frac{1-0}{6}\left[f(0)+4f\left(\frac{0+1}{2}\right)+f(1)\right]+\frac{2-1}{6}\left[f(1)+4f\left(\frac{1+2}{2}\right)+f(2)\right]$$

$$= \frac{1}{6}\left[\frac{1}{0+1}+4\left(\frac{1}{0.5+1}\right)+\frac{1}{1+1}\right]+\frac{1}{6}\left[\frac{1}{1+1}+4\left(\frac{1}{1.5+1}\right)+\frac{1}{2+1}\right]$$

$$= 1.1$$

There is a general formula for n equal subintervals. There are different versions of this formula. This one uses the function values y_0 to y_n.

The $n + 1$ function values give n subintervals.



$$\int_a^b f(x)dx \doteqdot \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots) + 2(y_2 + y_4 + \ldots)]$$

where $h = \dfrac{b-a}{n}$ for n subintervals ($n + 1$ function values)

Proof

$$\begin{aligned}\int_a^b f(x)\,dx &= \frac{2h}{6}(y_0 + 4y_1 + y_2) + \frac{2h}{6}(y_2 + 4y_3 + y_4) + \frac{2h}{6}(y_4 + 4y_5 + y_6)\\ &\quad + \ldots + \frac{2h}{6}(y_{n-2} + 4y_{n-1} + y_n)\\ &= \frac{h}{3}(y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + y_4 + 4y_5 + y_6 + \ldots + y_{n-2} + 4y_{n-1} + y_n)\\ &= \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots) + 2(y_2 + y_4 + \ldots)]\end{aligned}$$

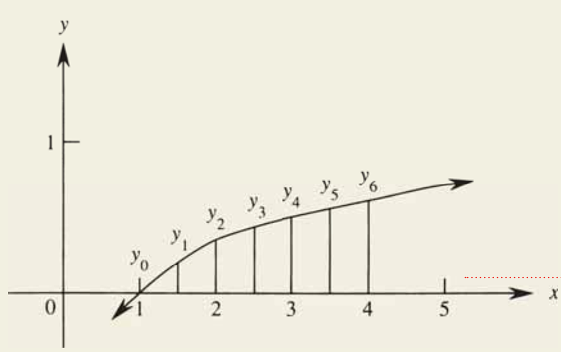
You could also remember this rule as

$\int_a^b f(x)\,dx \doteqdot \frac{h}{3}[(y_0 + y_n) + 4(\text{odds}) + 2(\text{evens})]$.

EXAMPLES

1. Use Simpson's rule with 7 ordinates to find an approximation for $\int_1^4 \log x\,dx$.

Solution



Seven ordinates or function values give 6 subintervals.

$$\int_a^b f(x)\,dx \doteqdot \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$\begin{aligned}\text{where}\quad h &= \frac{b-a}{n}\\ &= \frac{4-1}{6}\\ &= 0.5\end{aligned}$$

Use the log key on your calculator.

$$\begin{aligned}\int_1^4 \log x\,dx &\doteqdot \frac{0.5}{3}[\log 1 + \log 4 + 4(\log 1.5 + \log 2.5 + \log 3.5)\\ &\qquad + 2(\log 2 + \log 3)]\\ &\doteqdot 1.105\end{aligned}$$

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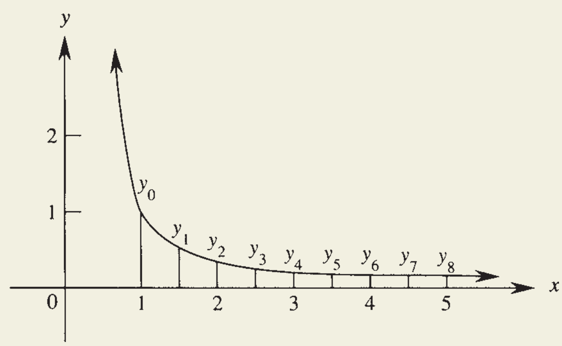
2. Use Simpson's rule with 9 function values to find an approximation for $\int_1^5 \frac{dx}{x^2}$.

Solution



$$\int_a^b f(x)\,dx \doteqdot \frac{h}{3}[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)]$$

where $h = \frac{b-a}{n}$

$= \frac{5-1}{8}$

$= 0.5$

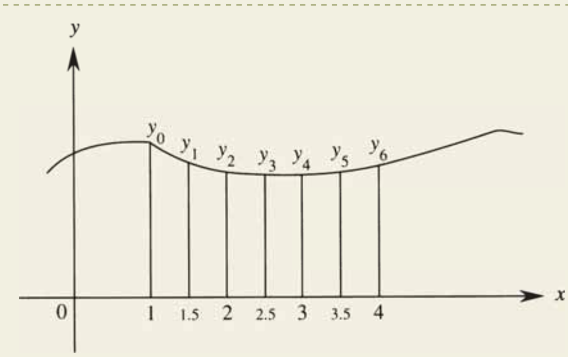
$$\int_1^5 \frac{1}{x^2}\,dx \doteqdot \frac{0.5}{3}\left[\frac{1}{1^2}+\frac{1}{5^2}+4\left(\frac{1}{1.5^2}+\frac{1}{2.5^2}+\frac{1}{3.5^2}+\frac{1}{4.5^2}\right)+2\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\right)\right]$$

$\doteqdot 0.8$

3. Use Simpson's rule to find an approximation for $\int_1^4 f(x)\,dx$ using the values in the table below:

x	1	1.5	2	2.5	3	3.5	4
$f(x)$	8.6	11.9	23.7	39.8	56.7	71.4	93.2

Solution



$$\int_1^4 f(x)\,dx \doteqdot \frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$$

where $h = \frac{4-1}{6}$

$= 0.5$

$$\int_1^4 f(x)\,dx \doteqdot \frac{0.5}{3}[(8.6+93.2)+4(11.9+39.8+71.4)+2(23.7+56.7)]$$

$\doteqdot 125.83$

3.2 Exercises

Use Simpson's rule to find an approximation for

1. $\int_1^3 x^4\,dx$ using 3 function values.
2. $\int_3^5 (x^2 - 1)\,dx$ using 3 ordinates.
3. $\int_1^3 \frac{dx}{x}$ using 2 subintervals.
4. $\int_0^1 \frac{dx}{x+2}$ using 3 function values.
5. $\int_2^4 \sqrt{x}\,dx$ using
   (a) 3 function values
   (b) 5 function values.
6. $\int_3^7 \log x\,dx$ using 5 function values.
7. $\int_2^5 \frac{dx}{x+1}$ using 6 subintervals.
8. $\int_3^6 \log x\,dx$ using 7 function values.
9. $\int_0^4 (x^3 + x)\,dx$ using 9 ordinates.
10. $\int_0^4 \sqrt{x}\,dx$ using 5 function values.
11. $\int_1^7 \frac{1}{x^3}\,dx$ using 7 function values.
12. $\int_2^5 \frac{1}{x^2 - 1}\,dx$ using 6 subintervals.
13. $\int_0^4 f(x)\,dx$ where values of $f(x)$ are given in the table:

x	0	1	2	3	4
$f(x)$	0.7	1.3	5.4	−0.5	−3.8

14. $\int_2^4 f(t)\,dt$ where values of $f(t)$ are given in the table:

t	2	2.5	3	3.5	4
$f(t)$	3.7	1.2	9.8	4.1	2.7

15. $\int_0^3 f(x)\,dx$ where values of $f(x)$ are given in the table:

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	15.3	29.2	38.1	56.2	69.9	94.8	102.5

Computer Application

There are computer application packages that can calculate the area under curves by using rectangles, the trapezoidal rule or Simpson's rule.

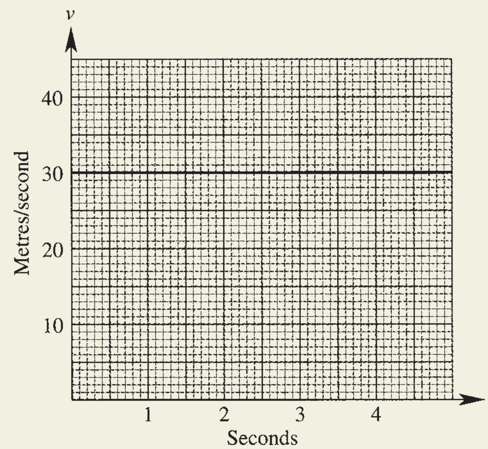
A graphics calculator will also do this.

Integration and the Primitive Function

Mathematicians found a link between finding areas under a curve and the primitive function. This made possible a simple method for finding exact areas.

EXAMPLE

The graph below shows the speed at which an object travels over time.



(a) Find the distance it travels in
  (i) 1 s
  (ii) 2 s
  (iii) 3 s

(b) Find the area under the line between
  (i) $t = 0$ and 1
  (ii) $t = 0$ and 2
  (iii) $t = 0$ and 3

Solution

(a) The speed is a constant 30 metres per second (ms^{-1})
  (i) The object travels 30 metres in 1 s
  (ii) The object travels 60 metres in 2 s
  (iii) The object travels 90 metres in 3 s

(b) (i) The area is $30 \times 1 = 30$ units^2
  (ii) The area is $30 \times 2 = 60$ units^2
  (iii) The area is $30 \times 3 = 90$ units^2

Notice that the area gives the distances.

In this example, the line graph gives a rate of change. The area under the curve gives the information about the original data for this rate of change.

In the same way, the area under a rate of change curve will give the original data. This original data is the primitive function of the curve.

DID YOU KNOW?

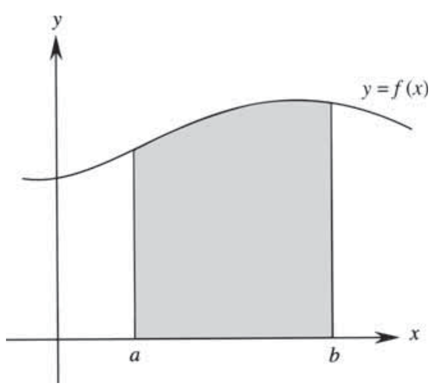
Many mathematicians in the 17th century were interested in the problem of finding areas under a curve. **Isacc Barrow** (1630–77) is said to be the first to discover that differentiation and integration are inverse operations. This discovery is called the **fundamental theorem of calculus**.

Barrow was an Englishman who was an outstanding Greek scholar as well as making contributions in the areas of mathematics, theology, astronomy and physics. However, when he was a schoolboy, he was so often in trouble that his father was overheard saying to God in his prayers that if He decided to take one of his children, he could best spare Isaac.

Sir Isaac Newton (1643–1727), another English mathematician, and scientist and astronomer, helped to discover calculus. He was not interested in his school work, but spent most of his time inventing things, such as a water clock and sundial.

He left school at 14 to manage the estate after his stepfather died. However, he spent so much time reading that he was sent back to school. He went on to university and developed his theories in mathematics and science which have made him famous today.

Fundamental theorem of calculus

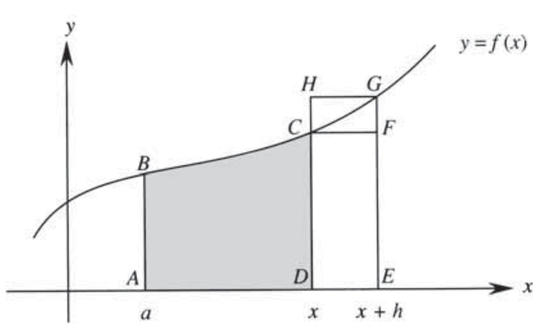


The area enclosed by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ where } F(x) \text{ is the primitive of function } f(x)$$

Proof

Consider a continuous curve $y = f(x)$ for all values of $x > a$



Let area $ABCD$ be $A(x)$
Let area $ABGE$ be $A(x+h)$
Then area $DCGE$ is $A(x+h) - A(x)$

Area $DCFE$ < area $DCGE$ < area $DHGE$

i.e. $f(x) \cdot h < A(x+h) - A(x) < f(x+h) \cdot h$

$$f(x) < \frac{A(x+h) - A(x)}{h} < f(x+h)$$

This is the formula for differentiation from first principles. You first saw this in Chapter 8 of The Preliminary Course book.

$$\lim_{h \to 0} f(x) < \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} < \lim_{h \to 0} f(x+h)$$

$$f(x) < A'(x) < f(x)$$

$$\therefore \quad A'(x) = f(x)$$

i.e. $A(x)$ is a primitive function of $f(x)$

$A(x) = F(x) + C$ where $F(x)$ is the primitive function of $f(x)$ (1)

Now $A(x)$ is the area between a and x

$\therefore \quad A(a) = 0$

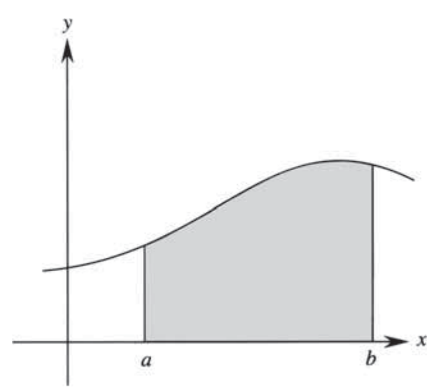
Substitute in (1):

$$\begin{aligned} A(a) &= F(a) + C \\ 0 &= F(a) + C \\ -F(a) &= C \end{aligned}$$

$\therefore \quad A(x) = F(x) - F(a)$

If $x = b$ where $b > a$,

$A(b) = F(b) - F(a)$



Definite Integrals

By the fundamental theorem of calculus,

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ where } F(x) \text{ is the primitive function of } f(x).$$

The primitive function of x^n is $\dfrac{x^{n+1}}{n+1} + C$.

Putting these pieces of information together, we can find areas under simple curves.

$$\int_a^b x^n = \left[\frac{x^{n+1}}{n+1}\right]_a^b$$
$$= \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

EXAMPLES

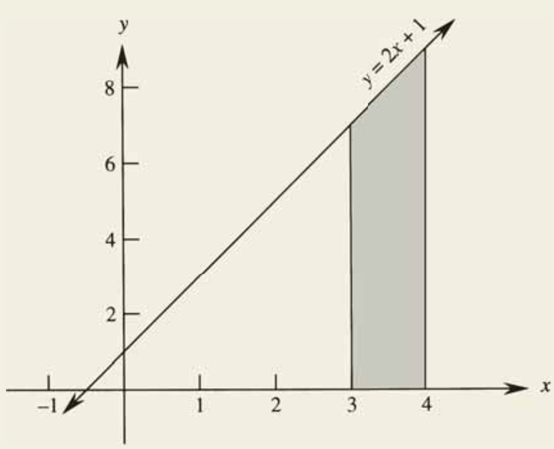
Evaluate

1. $\int_3^4 (2x + 1)\, dx$

Solution

$$\int_3^4 (2x + 1)\, dx = \left[x^2 + x\right]_3^4$$
$$= (4^2 + 4) - (3^2 + 3)$$
$$= 20 - 12$$
$$= 8$$

Constant C will cancel out. That is, $(4^2 + 4 + C) - (3^2 + 3 + C)$ $= 20 - 12$ $= 8$.



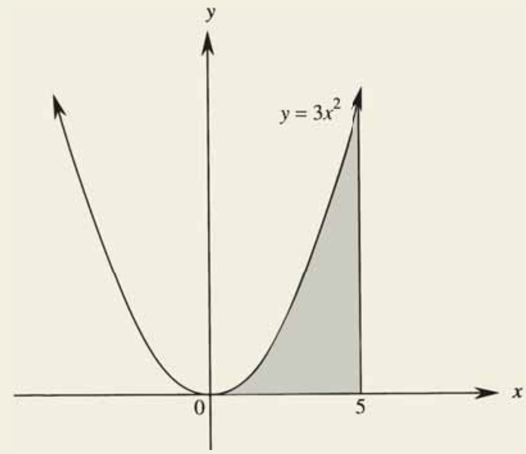
The graph shows the area that the definite integral calculates.

2. $\int_0^5 3x^2\, dx$

Solution

$$\int_0^5 3x^2 dx = \left[x^3\right]_0^5$$
$$= (5^3) - (0^3)$$
$$= 125$$

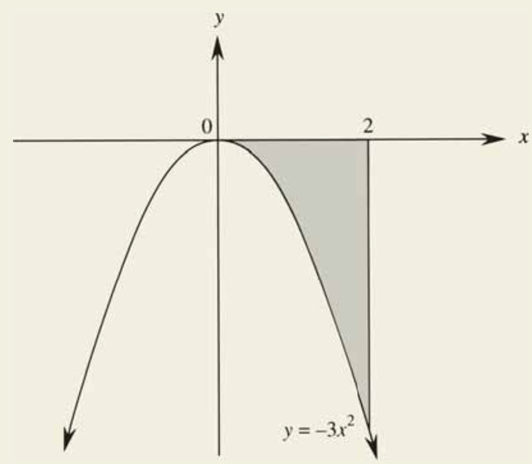
CONTINUED



3. $\int_0^2 -3x^2\, dx$

Solution

$$\int_0^2 -3x^2\, dx = \left[-x^3\right]_0^2$$
$$= (-2^3) - (-0^3)$$
$$= -8$$

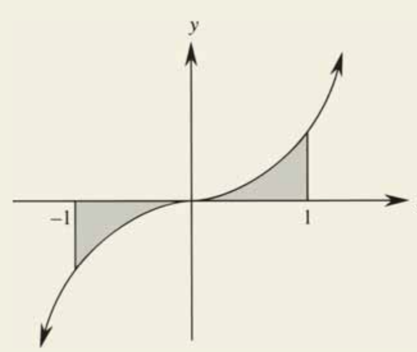


Can you see why there is a negative solution?

4. $\int_{-1}^1 x^3\, dx$

Solution

$$\int_{-1}^1 x^3\, dx = \left[\frac{x^4}{4}\right]_{-1}^1$$
$$= \frac{1^4}{4} - \frac{(-1)^4}{4}$$
$$= \frac{1}{4} - \frac{1}{4}$$
$$= 0$$



The definite integral gives an area of zero. Can you see why?

3.3 Exercises

Evaluate

1. $\int_0^2 4x\,dx$
2. $\int_1^3 (2x+1)\,dx$
3. $\int_0^5 3x^2\,dx$
4. $\int_1^2 (4t-7)\,dt$
5. $\int_{-1}^1 (6y+5)\,dy$
6. $\int_0^3 6x^2\,dx$
7. $\int_1^2 (x^2+1)\,dx$
8. $\int_0^2 4x^3\,dx$
9. $\int_{-1}^4 (3x^2-2x)\,dx$
10. $\int_1^3 (4x^2+6x-3)\,dx$
11. $\int_{-1}^1 x^2\,dx$
12. $\int_{-2}^3 (x^3+1)\,dx$
13. $\int_{-2}^2 x^5\,dx$
14. $\int_1^4 \sqrt{x}\,dx$
15. $\int_0^1 (x^3-3x^2+4x)\,dx$
16. $\int_1^2 (2x-1)^2\,dx$
17. $\int_{-1}^1 (y^3+y)\,dy$
18. $\int_3^4 (2-x)^2\,dx$
19. $\int_{-2}^2 4t^3\,dt$
20. $\int_2^4 \frac{x^2}{3}\,dx$
21. $\int_0^3 \frac{5x^4}{x}\,dx$
22. $\int_{-2}^1 \frac{x^4-3x}{x}\,dx$
23. $\int_0^2 \frac{4x^3+x^2+5x}{x}\,dx$
24. $\int_0^1 \frac{x^3-2x^2+3x}{x}\,dx$
25. $\int_3^4 \frac{x^2+x+3}{3x^5}\,dx$

Simplify first, then find the definite integral.

Class Investigation

Look at the results of definite integrals in the examples and exercises. Sketch the graphs where possible and shade in the areas found.

- Can you see why the definite integral sometimes gives a negative answer?
- Can you see why it will sometimes be zero?

Indefinite Integrals

Sometimes it is necessary to find a **general** or **indefinite integral** (primitive function).

$$\int f(x)\,dx = F(x) + C \text{ where } F(x) \text{ is a primitive function of } f(x)$$

In Chapter 2 you learned the result for the primitive function of x^n. This result is the same as the integral of x^n.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

EXAMPLES

Find each indefinite integral (primitive function)

1. $\int (x^4 - 3x^2 + 4x - 7)\, dx$

Solution

$$\begin{aligned}\int (x^4 - 3x^2 + 4x - 7)\, dx &= \frac{x^5}{5} - 3 \cdot \frac{x^3}{3} + 4 \cdot \frac{x^2}{2} - 7x + C \\ &= \frac{x^5}{5} - x^3 + 2x^2 - 7x + C\end{aligned}$$

2. $\int 5x^9\, dx$

Solution

$$\begin{aligned}\int 5x^9\, dx &= 5\left(\frac{x^{10}}{10}\right) + C \\ &= \frac{x^{10}}{2} + C\end{aligned}$$

$\int 5x^9\, dx$ is the same as $5\int x^9\, dx$.

3. $\int \left(\frac{2x^5 + 7x^2 - 3x}{x}\right) dx$

Solution

$$\begin{aligned}\int \left(\frac{2x^5}{x} + \frac{7x^2}{x} - \frac{3x}{x}\right) dx &= \int (2x^4 + 7x - 3)\, dx \\ &= \frac{2x^5}{5} + \frac{7x^2}{2} - 3x + C\end{aligned}$$

4. $\int \left(\frac{1}{x^3} + \sqrt{x}\right) dx$

Solution

$$\begin{aligned}\int \left(\frac{1}{x^3} + \sqrt{x}\right) dx &= \int (x^{-3} + x^{\frac{1}{2}})\, dx \\ &= \frac{x^{-2}}{-2} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C \\ &= -\frac{1}{2x^2} + \frac{2\sqrt{x^3}}{3} + C\end{aligned}$$

DID YOU KNOW?

John Wallis (1616–1703) found that the area under the curve $y = 1 + x + x^2 + x^3 + \ldots$ is given by

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots .$$

He found this result independently of the fundamental theorem of calculus.

3.4 Exercises

Find each indefinite integral (primitive function)

1. $\int x^2\,dx$
2. $\int 3x^5\,dx$
3. $\int 2x^4\,dx$
4. $\int (m+1)\,dm$
5. $\int (t^2 - 7)\,dt$
6. $\int (h^2 + 5)\,dh$
7. $\int (y - 3)\,dy$
8. $\int (2x + 4)\,dx$
9. $\int (b^2 + b)\,db$
10. $\int (a^3 - a - 1)\,da$
11. $\int (x^2 + 2x + 5)\,dx$
12. $\int (4x^3 - 3x^2 + 8x - 1)\,dx$
13. $\int (6x^5 + x^4 + 2x^3)\,dx$
14. $\int (x^7 - 3x^6 - 9)\,dx$
15. $\int (2x^3 + x^2 - x - 2)\,dx$
16. $\int (x^5 + x^3 + 4)\,dx$
17. $\int (4x^2 - 5x - 8)\,dx$
18. $\int (3x^4 - 2x^3 + x)\,dx$
19. $\int (6x^3 + 5x^2 - 4)\,dx$
20. $\int (3x^{-4} + x^{-3} + 2x^{-2})\,dx$
21. $\int \frac{dx}{x^8}$
22. $\int x^{\frac{1}{3}}\,dx$
23. $\int \frac{x^6 - 3x^5 + 2x^4}{x^3}\,dx$
24. $\int (1 - 2x)^2\,dx$
25. $\int (x - 2)(x + 5)\,dx$
26. $\int \frac{3}{x^2}\,dx$
27. $\int \frac{dx}{x^3}$
28. $\int \frac{4x^3 - x^5 - 3x^2 + 7}{x^5}\,dx$
29. $\int (y^2 - y^{-7} + 5)\,dy$
30. $\int (t^2 - 4)(t - 1)\,dt$
31. $\int \sqrt{x}\,dx$
32. $\int \frac{2}{t^5}\,dt$
33. $\int \sqrt[3]{x}\,dx$
34. $\int x\sqrt{x}\,dx$
35. $\int \sqrt{x}\left(1 + \frac{1}{\sqrt{x}}\right)dx$

Function of a function rule

EXAMPLE

Differentiate $(2x+3)^5$.

Hence find $\int 10(2x+3)^4\,dx$.

Find $\int (2x+3)^4\,dx$.

Solution

$$\begin{aligned}
\frac{d}{dx}(2x+3)^5 &= 5(2x+3)^4 \cdot 2\\
&= 10(2x+3)^4\\
\therefore \quad \int 10(2x+3)^4\,dx &= (2x+3)^5 + C\\
\int (2x+3)^4\,dx &= \frac{1}{10}\int 10(2x+3)^4\,dx\\
&= \frac{1}{10}(2x+3)^5 + C
\end{aligned}$$

Remember: Integration is the reverse of differentiation.

$$\boxed{\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C}$$

Proof

$$\begin{aligned}
\frac{d}{dx}(ax+b)^{n+1} &= (n+1)(ax+b)^n \cdot a\\
&= a(n+1)(ax+b)^n\\
\therefore \quad \int a(n+1)(ax+b)^n\,dx &= (ax+b)^{n+1} + C\\
\int (ax+b)^n\,dx &= \frac{1}{a(n+1)}\int a(n+1)(ax+b)^n\,dx\\
&= \frac{1}{a(n+1)}(ax+b)^{n+1} + C\\
&= \frac{(ax+b)^{n+1}}{a(n+1)} + C
\end{aligned}$$

EXAMPLES

Find

1. $\int (5x-9)^3\,dx$

Solution

$$\int (5x-9)^3\,dx = \frac{(5x-9)^4}{5\times 4}+C$$
$$= \frac{(5x-9)^4}{20}+C$$

2. $\int (3-x)^8\,dx$

Solution

$$\int (3-x)^8\,dx = \frac{(3-x)^9}{-1\times 9}+C$$
$$= -\frac{(3-x)^9}{9}+C$$

3. $\int \sqrt{4x+3}\,dx$

Solution

$$\int (4x+3)^{\frac{1}{2}}\,dx = \frac{(4x+3)^{\frac{3}{2}}}{4\times\frac{3}{2}}+C$$
$$= \frac{\sqrt{(4x+3)^3}}{6}+C$$

3.5 Exercises

Find each indefinite integral (primitive function)

1. (a) $\int (3x-4)^2\,dx$ by
   (i) expanding
   (ii) the function of a function rule
   (b) $\int (x+1)^4\,dx$
   (c) $\int (5x-1)^9\,dx$
   (d) $\int (3y-2)^7\,dy$
   (e) $\int (4+3x)^4\,dx$
   (f) $\int (7x+8)^{12}\,dx$
   (g) $\int (1-x)^6\,dx$
   (h) $\int \sqrt{2x-5}\,dx$
   (i) $\int 2(3x+1)^{-4}\,dx$
   (j) $\int 3(x+7)^{-2}\,dx$
   (k) $\int \frac{1}{2(4x-5)^3}\,dx$
   (l) $\int \sqrt[3]{4x+3}\,dx$

(m) $\int (2-x)^{-\frac{1}{2}} dx$

(n) $\int \sqrt{(t+3)^3}\, dt$

(o) $\int \sqrt{(5x+2)^5}\, dx$

2. Evaluate

(a) $\int_1^2 (2x+1)^4\, dx$

(b) $\int_0^1 (3y-2)^3\, dy$

(c) $\int_1^2 (1-x)^7\, dx$

(d) $\int_0^2 (3-2x)^5\, dx$

(e) $\int_0^1 \frac{(3x-1)^2}{6} dx$

(f) $\int_4^5 (5-x)^6\, dx$

(g) $\int_3^6 \sqrt{x-2}\, dx$

(h) $\int_0^1 \frac{dx}{(3x-2)^4}$

(i) $\int_0^2 \frac{5}{(2n+1)^3} dn$

(j) $\int_1^4 \frac{2}{\sqrt{(5x-4)^3}} dx$

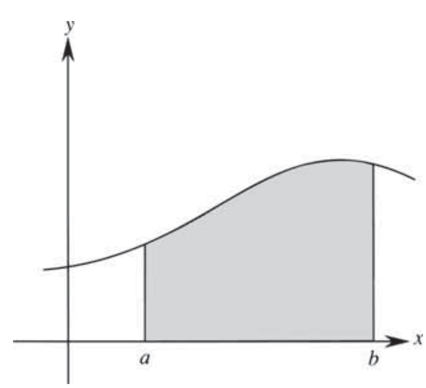
Class Exercise

Differentiate $(x+1)\sqrt{2x-3}$. Hence find $\int \frac{3x-2}{\sqrt{2x-3}} dx$.

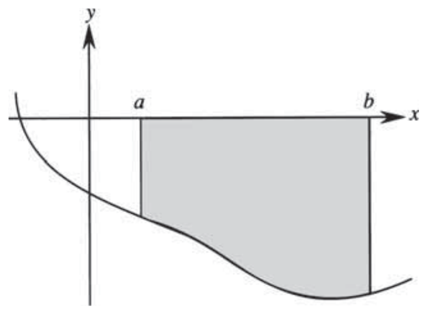
Areas Enclosed by the *x*-axis

The definite integral gives the **signed area** under a curve.

Areas **above** the *x*-axis give a **positive definite** integral.



Areas **below** the x-axis give a **negative definite** integral.



We normally think of areas as positive. So to find areas below the x-axis, take the absolute value of the definite integral. That is,

$$\text{Area} = \left| \int_a^b f(x)\, dx \right|$$

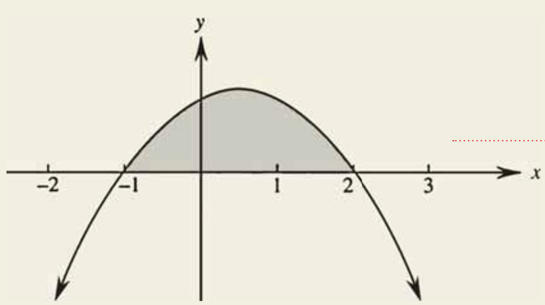
Always sketch the area to be found.

EXAMPLES

1. Find the area enclosed by the curve $y = 2 + x - x^2$ and the x-axis.

Solution

Sketch $y = 2 + x - x^2$.



The integral will be positive as the area is above the x-axis.

$$\begin{aligned}
\text{Area} &= \int_{-1}^{2} (2 + x - x^2)\, dx \\
&= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} \\
&= \left(2(2) + \frac{2^2}{2} - \frac{2^3}{3}\right) - \left(2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3}\right) \\
&= \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) \\
&= 4\frac{1}{2}
\end{aligned}$$

So the area is $4\frac{1}{2}$ units^2.

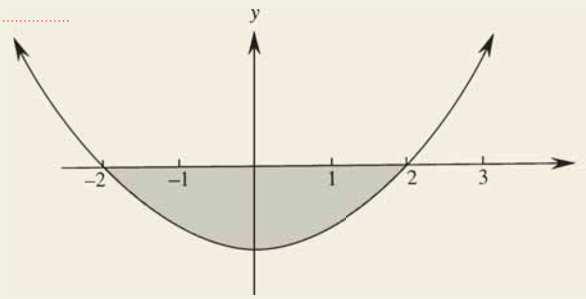
CONTINUED

The integral will be negative as the area is below the x-axis.

2. Find the area bounded by the curve $y = x^2 - 4$ and the x-axis.

Solution

Sketch $y = x^2 - 4$.



$$\begin{aligned}\int_{-2}^{2}(x^2-4)\,dx &= \left[\frac{x^3}{3}-4x\right]_{-2}^{2}\\ &= \left(\frac{2^3}{3}-4(2)\right)-\left(\frac{(-2)^3}{3}-4(-2)\right)\\ &= \left(\frac{8}{3}-8\right)-\left(\frac{-8}{3}+8\right)\\ &= -10\frac{2}{3}\end{aligned}$$

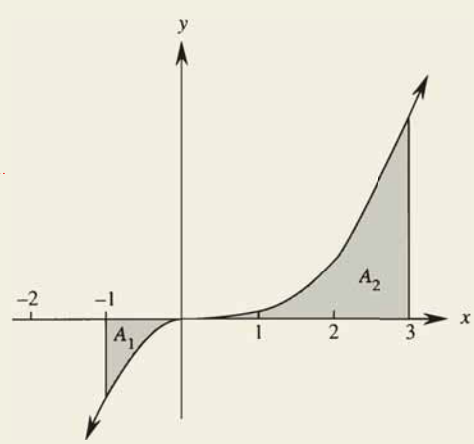
So the area is $10\frac{2}{3}$ units^2.

Some of the area is below the x-axis, so its integral will be negative.

3. Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = -1$ and $x = 3$.

Solution

Sketch $y = x^3$.



We need to find the sum of the areas between $x = -1$ and 0, and $x = 0$ and 3.

$$\int_{-1}^{0} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{0}$$
$$= \frac{0^4}{4} - \frac{(-1)^4}{4}$$
$$= -\frac{1}{4}$$

So $A_1 = \frac{1}{4}$ units^2.

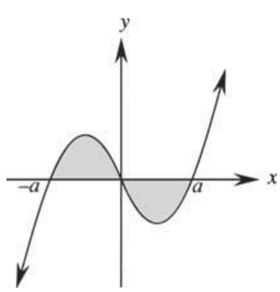
$$\int_{0}^{3} x^3\,dx = \left[\frac{x^4}{4}\right]_{0}^{3}$$
$$= \frac{3^4}{4} - \frac{0^4}{4}$$
$$= \frac{81}{4}$$

So $A_2 = \frac{81}{4}$ units^2.

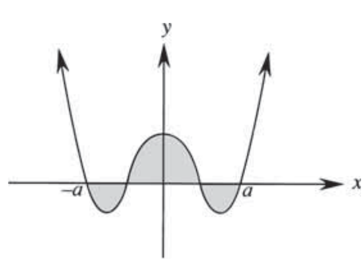
$$A_1 + A_2 = \frac{1}{4} + \frac{81}{4}$$
$$= 20\frac{1}{2} \text{ units}^2$$

Even and odd functions

Some functions have special properties.



For odd functions,
$$\int_{-a}^{a} f(x)\,dx = 0$$



For even functions,
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

EXAMPLES

1. Find the area between the curve $y = x^3$, the x-axis and the lines $x = -2$ and $x = 2$.

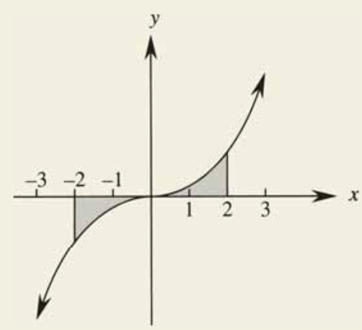
Solution

$$\int_{-2}^{2} x^3\,dx = 0$$

$y = x^3$ is an odd function since $f(-x) = -f(x)$.

$$\begin{aligned}\text{Area} &= 2\int_0^2 x^3\,dx\\ &= 2\left[\frac{x^4}{4}\right]_0^2\\ &= 2\left(\frac{2^4}{4} - \frac{0^4}{4}\right)\\ &= 8\end{aligned}$$

So the area is 8 units^2.



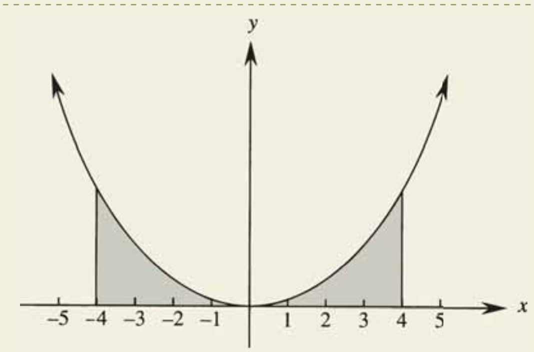
2. Find the area between the curve $y = x^2$, the x-axis and the lines $x = -4$ and $x = 4$.

Solution

$y = x^2$ is an even function since $f(-x) = f(x)$.

$$\begin{aligned}\int_{-4}^{4} x^2\,dx &= 2\int_0^4 x^2\,dx\\ &= 2\left[\frac{x^3}{3}\right]_0^4\\ &= 2\left(\frac{4^3}{3} - \frac{0^3}{3}\right)\\ &= 42\frac{2}{3}\end{aligned}$$

So the area is $42\frac{2}{3}$ units^2.



3.6 Exercises

1. Find the area enclosed between the curve $y = 1 - x^2$ and the x-axis.
2. Find the area bounded by the curve $y = x^2 - 9$ and the x-axis.
3. Find the area enclosed between the curve $y = x^2 + 5x + 4$ and the x-axis.
4. Find the area enclosed between the curve $y = x^2 - 2x - 3$ and the x-axis.
5. Find the area bounded by the curve $y = -x^2 + 9x - 20$ and the x-axis.
6. Find the area enclosed between the curve $y = -2x^2 - 5x + 3$ and the x-axis.
7. Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = 0$ and $x = 2$.

8. Find the area enclosed between the curve $y = x^4$, the x-axis and the lines $x = -1$ and $x = 1$.

9. Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = -2$ and $x = 2$.

10. Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = -3$ and $x = 2$.

11. Find the area bounded between the curve $y = 3x^2$, the x-axis and the lines $x = -1$ and $x = 1$.

12. Find the area enclosed between the curve $y = x^2 + 1$, the x-axis and the lines $x = -2$ and $x = 2$.

13. Find the area enclosed between the curve $y = x^2$, the x-axis and the lines $x = -3$ and $x = 2$.

14. Find the area enclosed between the curve $y = x^2 + x$, and the x-axis.

15. Find the area enclosed between the curve $y = \frac{1}{x^2}$, the x-axis and the lines $x = 1$ and $x = 3$.

16. Find the area enclosed between the curve $y = \frac{2}{(x-3)^2}$, the x-axis and the lines $x = 0$ and $x = 1$.

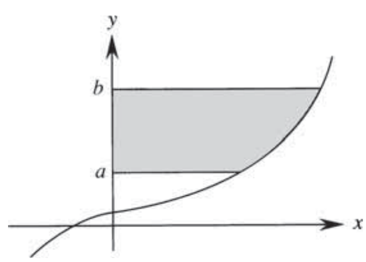
17. Find the area bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$.

18. Find the area bounded by the curve $y = \sqrt{x+2}$, the x-axis and the line $x = 7$.

19. Find the area bounded by the curve $y = \sqrt{4 - x^2}$, the x-axis and the y-axis in the first quadrant.

20. Find the area bounded by the x-axis, the curve $y = x^3$ and the lines $x = -a$ and $x = a$.

Areas Enclosed by the *y*-axis



To find the area between a curve and the y-axis, we change the subject of the equation of the curve to x. That is,

$$x = f(y).$$

The definite integral is given by

$$\boxed{\int_a^b f(y)\,dy \text{ or } \int_a^b x\,dy}$$

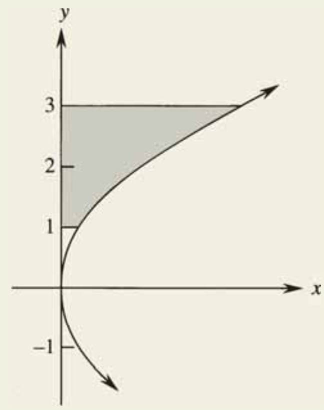
Since x is **positive** on the **right-hand side** of the y-axis, the definite integral is **positive**. Since x is **negative** on the **left-hand side** of the y-axis, the definite integral is **negative**.

EXAMPLES

1. Find the area enclosed by the curve $x = y^2$, the y-axis and the lines $y = 1$ and $y = 3$.

Solution

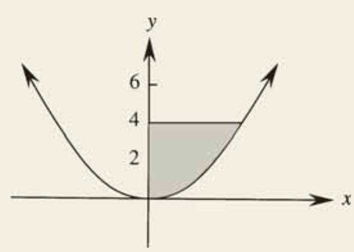
$$\text{Area} = \int_1^3 y^2\,dy$$
$$= \left[\frac{y^3}{3}\right]_1^3$$
$$= \frac{3^3}{3} - \frac{1^3}{3}$$
$$= 8\frac{2}{3}\text{ units}^2$$



The integral is positive since the area is to the right of the y-axis.

2. Find the area enclosed by the curve $y = x^2$, the y-axis and the lines $y = 0$ and $y = 4$ in the first quadrant.

Solution



Change the subject of the equation to x.

$$y = x^2$$
$$\therefore\ \pm\sqrt{y} = x$$

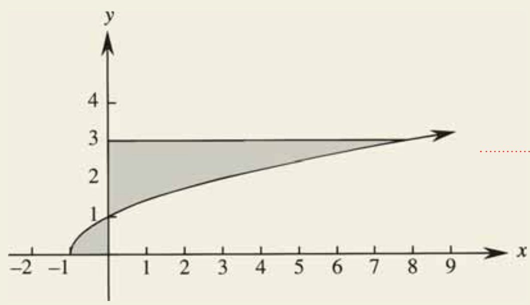
In the first quadrant,

$$\sqrt{y} = x$$
$$\text{i.e.}\quad y^{\frac{1}{2}} = x$$

$$\text{Area} = \int_0^4 y^{\frac{1}{2}}\,dy$$
$$= \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$
$$= \left[\frac{2\sqrt{y^3}}{3}\right]_0^4$$
$$= \frac{2\sqrt{4^3}}{3} - \frac{2\sqrt{0^3}}{3}$$
$$= 5\frac{1}{3}\text{ units}^2$$

3. Find the area enclosed between the curve $y = \sqrt{x + 1}$, the y-axis and the lines $y = 0$ and $y = 3$.

Solution



Some of the area is on the left of the y-axis.

$$y = \sqrt{x + 1}$$
$$y^2 = x + 1$$
$$y^2 - 1 = x$$

$$\begin{aligned}
\text{Area} &= \left|\int_0^1 (y^2 - 1)\,dy\right| + \int_1^3 (y^2 - 1)\,dy \\
&= \left|\left[\frac{y^3}{3} - y\right]_0^1\right| + \left[\frac{y^3}{3} - y\right]_1^3 \\
&= \left|\left(\frac{1^3}{3} - 1\right) - \left(\frac{0^3}{3} - 0\right)\right| + \left(\frac{3^3}{3} - 3\right) - \left(\frac{1^3}{3} - 1\right) \\
&= \left|-\frac{2}{3}\right| + 6 + \frac{2}{3} \\
&= \frac{2}{3} + 6\frac{2}{3} \\
&= 7\frac{1}{3} \text{ units}^2
\end{aligned}$$

3.7 Exercises

1. Find the area bounded by the y-axis, the curve $x = y^2$ and the lines $y = 0$ and $y = 4$.
2. Find the area enclosed between the curve $x = y^3$, the y-axis and the lines $y = 1$ and $y = 3$.
3. Find the area enclosed between the curve $y = x^2$, the y-axis and the lines $y = 1$ and $y = 4$ in the first quadrant.
4. Find the area between the lines $y = x - 1$, $y = 0$ and $y = 1$ and the y-axis.
5. Find the area bounded by the line $y = 2x + 1$, the y-axis and the lines $y = 3$ and $y = 4$.
6. Find the area bounded by the curve $y = \sqrt{x}$, the y-axis and the lines $y = 1$ and $y = 2$.

7. Find the area bounded by the curve $x = y^2 - 2y - 3$ and the y-axis.

8. Find the area bounded by the curve $x = -y^2 - 5y - 6$ and the y-axis.

9. Find the area enclosed by the curve $y = \sqrt{3x - 5}$, the y-axis and the lines $y = 2$ and $y = 3$.

10. Find the area enclosed between the curve $y = \frac{1}{x^2}$, the y-axis and the lines $y = 1$ and $y = 4$ in the first quadrant.

11. Find the area enclosed between the curve $y = x^3$, the y-axis and the lines $y = 1$ and $y = 8$.

12. Find the area enclosed between the curve $y = x^3 - 2$ and the y-axis between $y = -1$ and $y = 25$.

13. Find the area enclosed between the lines $y = 4$ and $y = 1 - x$ in the second quadrant.

14. Find the area enclosed between the y-axis and the curve $x = y(y - 2)$.

15. Find the area bounded by the curve $y = x^4 + 1$, the y-axis and the lines $y = 1$ and $y = 3$ in the first quadrant, correct to 2 significant figures.

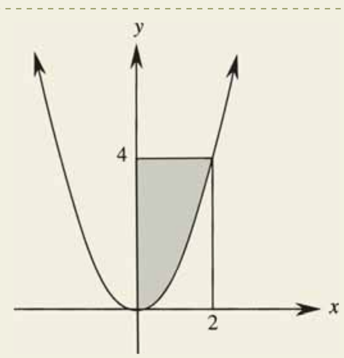
Sums and Differences of Areas

EXAMPLES

1. Find the area enclosed between the curve $y = x^2$, the y-axis and the lines $y = 0$ and $y = 4$ in the first quadrant.

Solution

$$A = \text{Area of rectangle} - \int_0^2 x^2\,dx$$
$$= (4 \times 2) - \left[\frac{x^3}{3}\right]_0^2$$
$$= 8 - \left(\frac{8}{3} - \frac{0}{3}\right)$$
$$= 5\frac{1}{3} \text{ units}^2$$



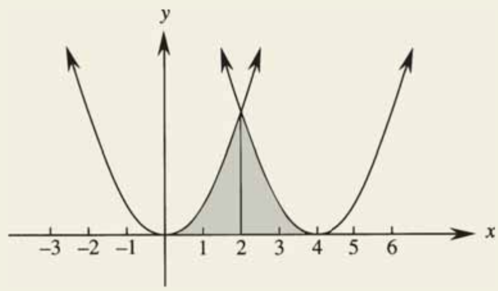
This example was done before by finding the area between the curve and the y-axis.

2. Find the area enclosed between the curves $y = x^2$, $y = (x-4)^2$ and the x-axis.

Solution

Solve simultaneous equations to find the intersection of the curves.

$$y = x^2 \quad (1)$$
$$y = (x-4)^2 \quad (2)$$
$$\therefore \quad x^2 = (x-4)^2$$
$$= x^2 - 8x + 16$$
$$0 = -8x + 16$$
$$8x = 16$$
$$x = 2$$



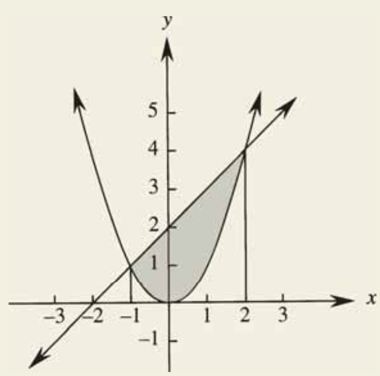
$$\text{Area} = \int_0^2 x^2\,dx + \int_2^4 (x-4)^2\,dx$$
$$= \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{(x-4)^3}{3}\right]_2^4$$
$$= \left(\frac{8}{3} - \frac{0}{3}\right) + \left(\frac{0}{3} - \frac{-8}{3}\right)$$
$$= 5\frac{1}{3} \text{ units}^2$$

3. Find the area enclosed between the curve $y = x^2$ and the line $y = x + 2$.

Solution

Solve simultaneous equations to find the intersection of the curve and line.

$$y = x^2 \quad (1)$$
$$y = x + 2 \quad (2)$$
$$\therefore \quad x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2 \text{ or } -1$$



$$\text{Area} = \int_{-1}^2 (x+2)\,dx - \int_{-1}^2 x^2\,dx$$
$$= \int_{-1}^2 (x + 2 - x^2)\,dx$$
$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^2$$
$$= \left(\frac{2^2}{2} + 2(2) - \frac{2^3}{3}\right) - \left(\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right)$$
$$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$
$$= 4\frac{1}{2} \text{ units}^2$$

3.8 Exercises

1. Find the area bounded by the line $y = 1$ and the curve $y = x^2$.

2. Find the area enclosed between the line $y = 2$ and the curve $y = x^2 + 1$.

3. Find the area enclosed by the curve $y = x^2$ and the line $y = x$.

4. Find the area bounded by the curve $y = 9 - x^2$ and the line $y = 5$.

5. Find the area enclosed between the curve $y = x^2$ and the line $y = x + 6$.

6. Find the area bounded by the curve $y = x^3$ and the line $y = 4x$.

7. Find the area enclosed between the curves $y = (x - 1)^2$ and $y = (x + 1)^2$.

8. Find the area enclosed between the curve $y = x^2$ and the line $y = -6x + 16$.

9. Find the area enclosed between the curve $y = x^3$, the x-axis and the line $y = -3x + 4$.

10. Find the area enclosed by the curves $y = (x - 2)^2$ and $y = (x - 4)^2$.

11. Find the area enclosed between the curves $y = x^2$ and $y = x^3$.

12. Find the area enclosed by the curves $y = x^2$ and $x = y^2$.

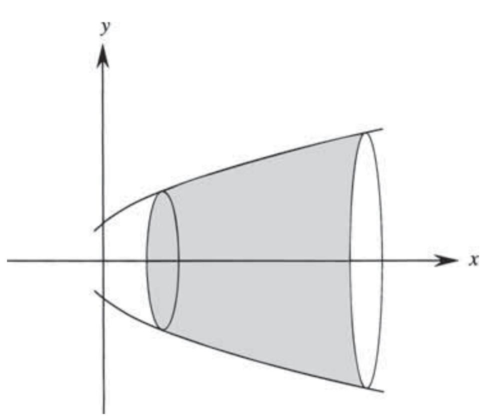
13. Find the area bounded by the curve $y = x^2 + 2x - 8$ and the line $y = 2x + 1$.

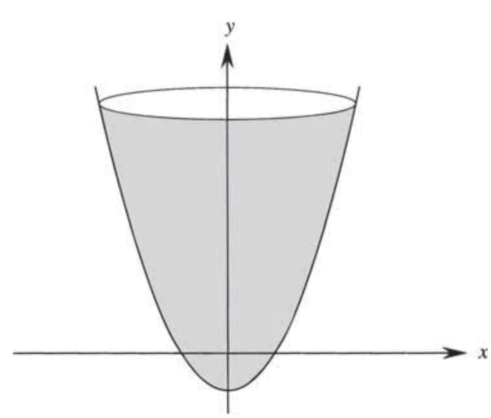
14. Find the area bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$.

15. Find the exact area enclosed between the curve $y = \sqrt{4 - x^2}$ and the line $x - y + 2 = 0$.

Volumes

The volume of a solid can be found by rotating an area under a curve about the x-axis or the y-axis.

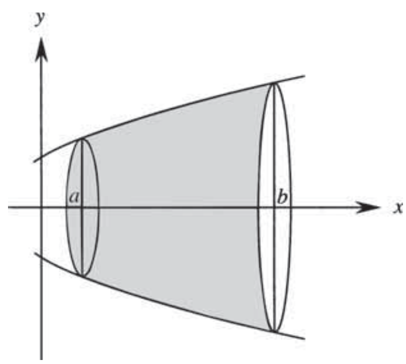




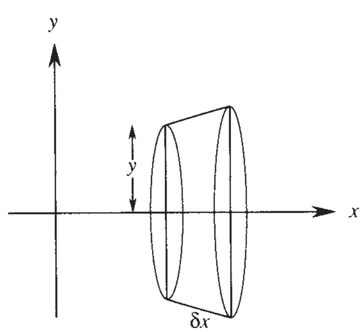
Volumes about the x-axis

$$\boxed{V = \pi\int_a^b y^2\,dx}$$

Proof



Take a disc of the solid with width δx.



The disc is approximately a cylinder with radius y and height δx.

$$\begin{aligned}\text{Volume} &= \pi r^2 h\\ &= \pi y^2\,\delta x\end{aligned}$$

Taking the sum of an infinite number of these discs,

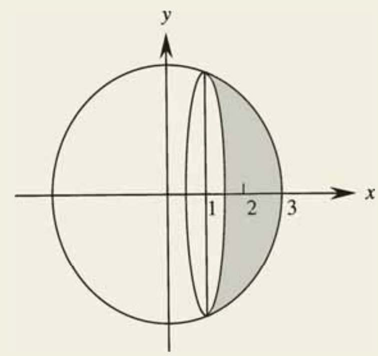
$$\begin{aligned}\text{Volume} &= \lim_{\delta x \to 0} \pi y^2\,\delta x\\ &= \int_a^b \pi y^2\,dx\\ &= \pi\int_a^b y^2\,dx\end{aligned}$$

EXAMPLES

1. Find the volume of the solid of revolution formed when the curve $x^2 + y^2 = 9$ is rotated about the x-axis between $x = 1$ and $x = 3$.

Solution

$$x^2 + y^2 = 9$$
$$\therefore \quad y^2 = 9 - x^2$$
$$V = \pi\int_a^b y^2\,dx$$
$$= \pi\int_1^3 (9 - x^2)\,dx$$
$$= \pi\left[9x - \frac{x^3}{3}\right]_1^3$$
$$= \pi\left((27 - 9) - (9 - \tfrac{1}{3})\right)$$
$$= \frac{28\pi}{3}\text{ units}^3$$

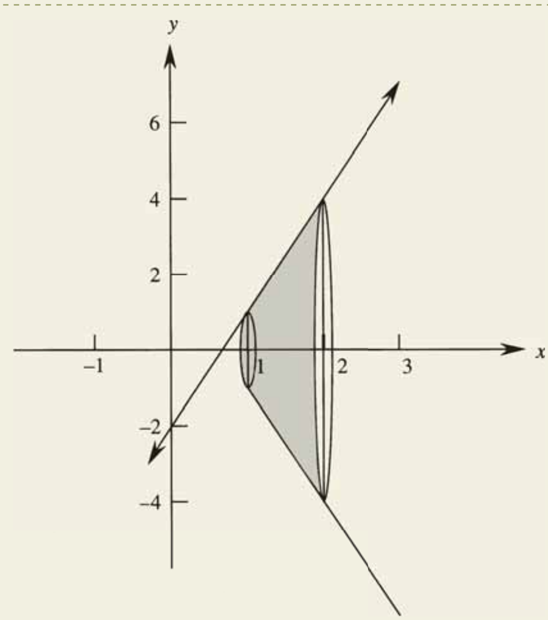


2. Find the volume of the solid formed when the line $y = 3x - 2$ is rotated about the x-axis from $x = 1$ to $x = 2$.

Solution

$$y = 3x - 2$$
$$y^2 = (3x - 2)^2$$
$$V = \pi\int_a^b y^2\,dx$$
$$= \pi\int_1^2 (3x - 2)^2\,dx$$
$$= \pi\left[\frac{(3x-2)^3}{3 \times 3}\right]_1^2$$
$$= \pi\left(\frac{4^3}{9} - \frac{1^3}{9}\right)$$
$$= \frac{63\pi}{9}$$
$$= 7\pi\text{ units}^3$$

Use the function of a function rule



PROBLEM

What is wrong with the working out of this volume?

Find the volume of the solid of revolution formed when the curve $y = x^2 + 1$ is rotated about the x-axis between $x = 0$ and $x = 2$.

Solution

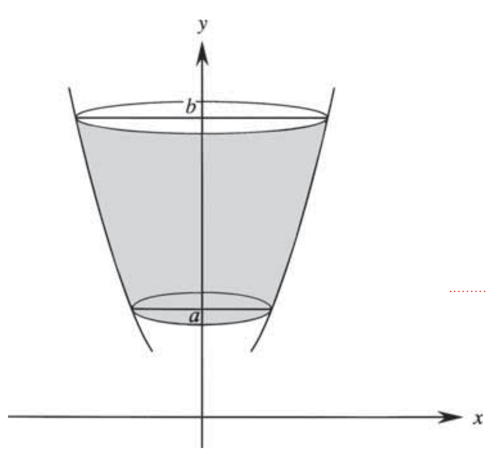
$$\begin{aligned}
y &= x^2 + 1\\
\therefore\quad y^2 &= (x^2 + 1)^2\\
V &= \pi\int_a^b y^2\,dx\\
&= \pi\int_0^2 (x^2 + 1)^2\,dx\\
&= \pi\left[\frac{(x^2 + 1)^3}{3}\right]_0^2\\
&= \pi\left(\frac{(2^2 + 1)^3}{3} - \frac{(0^2 + 1)^3}{3}\right)\\
&= \pi\left(\frac{5^3}{3} - \frac{1^3}{3}\right)\\
&= \frac{124\pi}{3}\text{ units}^3
\end{aligned}$$

What is the correct volume?

Volumes about the y-axis

The formula for rotations about the y-axis is similar to the formula for the x-axis.

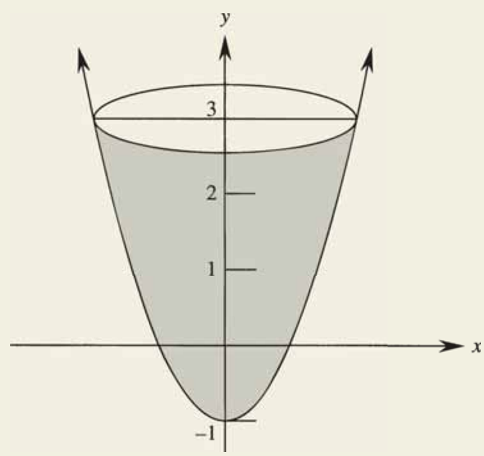
The proof of this result is similar to that for volumes about the x-axis.



$$\boxed{V = \pi\int_a^b x^2\,dy}$$

EXAMPLE

Find the volume of the solid formed when the curve $y = x^2 - 1$ is rotated about the y-axis from $y = -1$ to $y = 3$.



Solution

$$y = x^2 - 1$$
$$\therefore \quad y + 1 = x^2$$
$$\begin{aligned} V &= \pi \int_a^b x^2\,dy \\ &= \pi \int_{-1}^{3} (y+1)\,dy \\ &= \pi \left[\frac{y^2}{2} + y\right]_{-1}^{3} \\ &= \pi\left[\left(\frac{9}{2} + 3\right) - \left(\frac{1}{2} - 1\right)\right] \\ &= 8\pi \text{ units}^3 \end{aligned}$$

3.9 Exercises

1. Find the volume of the solid of revolution formed when the curve $y = x^2$ is rotated about the x-axis from $x = 0$ to $x = 3$.

2. Find the volume of the solid formed when the line $y = x + 1$ is rotated about the x-axis between $x = 2$ and $x = 7$.

3. Find the volume of the solid of revolution that is formed when the curve $y = x^2 + 2$ is rotated about the x-axis from $x = 0$ to $x = 2$.

4. Determine the volume of the solid formed when the curve $y = x^3$ is rotated about the x-axis from $x = 0$ to $x = 1$.

5. Find the volume of the solid formed when the curve $x = y^2 - 5$ is rotated about the x-axis between $x = 0$ and $x = 3$.

6. Find the volume of the solid of revolution formed by rotating the line $y = 4x - 1$ about the x-axis from $x = 2$ to $x = 4$.

7. Find the volume of the hemisphere formed when the curve $x^2 + y^2 = 1$ is rotated about the x-axis between $x = 0$ and $x = 1$.

8. The parabola $y = (x + 2)^2$ is rotated about the x-axis from $x = 0$ to $x = 2$. Find the volume of the solid formed.

9. Find the volume of the spherical cap formed when the curve $x^2 + y^2 = 4$ is rotated about the y-axis from $y = 1$ to $y = 2$.

10. Find the volume of the paraboloid formed when $y = x^2$ is rotated about the y-axis from $y = 0$ to $y = 3$.

11. Find the volume of the solid formed when $y = x^2 - 2$ is rotated about the y-axis from $y = 1$ to $y = 4$.

12. The line $y = x + 2$ is rotated about the y-axis from $y = -2$ to $y = 2$. Find the volume of the solid formed.

13. Determine the volume of the solid formed when the curve $y = x^3$ is rotated about the x-axis from $x = -1$ to $x = 4$.

14. Find the volume of the solid formed when the curve $y = \sqrt{x}$ is rotated about the x-axis between $x = 0$ and $x = 5$.

15. Find the volume of the solid formed when the curve $y = \sqrt{x + 3}$ is rotated about the x-axis from $x = 1$ to $x = 6$.

16. Find the volume of the solid formed when the curve $y = \sqrt{x}$ is rotated about the y-axis between $y = 1$ and $y = 4$.

17. Find the volume of the solid formed when the curve $y = \sqrt{4 - x^2}$ is rotated about the y-axis from $y = 1$ to $y = 2$.

18. Find the volume of the solid formed when the line $x + 3y - 1 = 0$ is rotated about the y-axis from $y = 1$ to $y = 2$.

19. Find the volume of the solid formed when the line $x + 3y - 1 = 0$ is rotated about the x-axis from $x = 0$ to $x = 8$.

20. The curve $y = x^3$ is rotated about the y-axis from $y = 0$ to $y = 1$. Find the volume of the solid formed.

21. Find the volume of the solid of revolution formed if the area enclosed between the curves $y = x^2$ and $y = (x - 2)^2$ is rotated about the x-axis.

22. The area bounded by the curve $y = x^2$ and the line $y = x + 2$ is rotated about the x-axis. Find the exact volume of the solid formed.

23. Show that the volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$ by rotating the semicircle $y = \sqrt{r^2 - x^2}$ about the x-axis.

Application

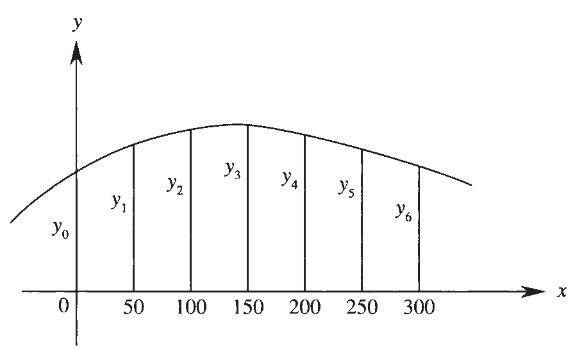
The volume of water in a small lake is to be measured by finding the depth of water at regular intervals across the lake.

What could the volume of this lake be?

The table below shows the results of the measurements, where x stands for the regular intervals chosen, in metres, and y is the depth in metres.

x (m)	0	50	100	150	200	250	300
y (m)	10.2	39.1	56.9	43.2	28.5	15.7	9.8

Using Simpson's rule to find an approximation for the volume of the lake, correct to 2 significant figures:



$$V = \pi \int_a^b y^2\, dx$$
$$= \pi \int_0^{300} y^2 dx$$

$$\int_a^b f(x)\, dx \doteqdot \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

where $h = \dfrac{b-a}{6}$

$$= \frac{300-0}{6}$$
$$= 50$$

$$\int_0^{300} y^2\,dx \doteqdot \frac{50}{3}\left[\left(10.2^2 + 9.8^2\right) + 4\left(39.1^2 + 43.2^2 + 15.7^2\right) + 2\left(56.9^2 + 28.5^2\right)\right]$$

$$= 381\,099.3$$

$$\therefore \quad V = \pi\int_0^{300} y^2\,dx$$

$$= \pi \times 381\,099.3$$

$$\doteqdot 1\,197\,258.866$$

$$= 1\,200\,000 \quad \text{correct to 2 significant figures}$$

So the volume of the lake is approximately 1 200 000 m^3 of water.

Test Yourself 3

1. (a) Use the trapezoidal rule with 2 subintervals to find an approximation to $\int_1^2 \frac{dx}{x^2}$.

(b) Use integration to find the exact value of $\int_1^2 \frac{dx}{x^2}$.

2. Find the indefinite integral (primitive function) of

(a) $3x + 1$

(b) $\frac{5x^2 - x}{x}$

(c) $\sqrt{x}$

(d) $(2x + 5)^7$

3. In an experiment, the velocity of a particle over time was found and the results placed in the table below.

t	0	1	2	3	4	5	6
v	1.3	1.7	2.1	3.5	2.8	2.6	2.2

Use Simpson's rule to find the approximate value of $\int_0^6 f(t)\,dt$

4. Evaluate

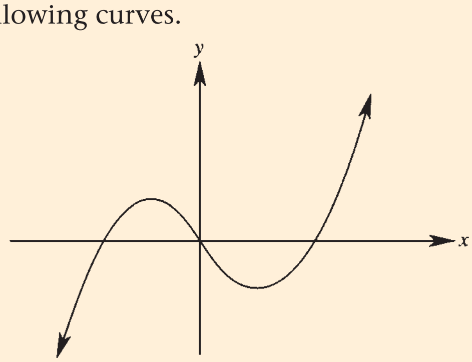
(a) $\int_0^2 (x^3 - 1)\,dx$

(b) $\int_{-1}^1 x^5\,dx$

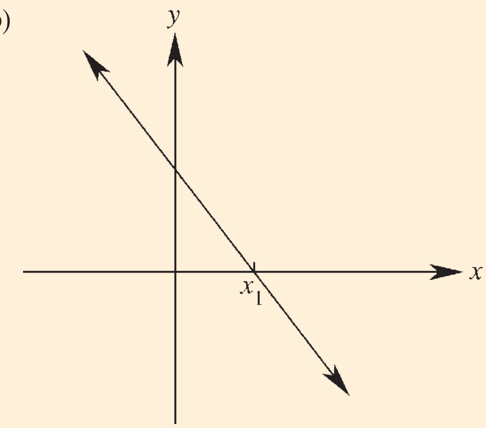
(c) $\int_0^1 (3x - 1)^4\,dx$

5. Draw a primitive function of the following curves.

(a)



(b)



6. Find the area bounded by the curve $y = x^2$, the x-axis and the lines $x = -1$ and $x = 2$.

7. Use Simpson's rule with 5 function values to find the approximate area enclosed between the curve $y = \frac{1}{x}$, the x-axis and the lines $x = 1$ and $x = 3$.

8. Find the area enclosed between the curves $y = x^2$ and $y = 2 - x^2$.

9. The line $y = 2x - 3$ is rotated about the x-axis from $x = 0$ to $x = 3$. Find the volume of the solid of revolution.

10. Evaluate $\int_1^2 \frac{3x^4 - 2x^3 + x^2 - 1}{x^2}\,dx$.

11. Find the area bounded by the curve $y = x^3$, the y-axis and the lines $y = 0$ and $y = 1$.

12. Find the indefinite integral (primitive function) of $(7x + 3)^{11}$.

13. Find the area bounded by the curve $y = x^2 - x - 2$, the x-axis and the lines $x = 1$ and $x = 3$.

14. Find the volume of the solid formed if the curve $y = x^2 + 1$ is rotated about the

(a) x-axis from $x = 0$ to $x = 2$

(b) y-axis from $y = 1$ to $y = 2$

15. (a) Change the subject of $y = (x + 3)^2$ to x.
(b) Find the area bounded by the curve $y = (x + 3)^2$, the y-axis and the lines $y = 9$ and $y = 16$ in the first quadrant.
(c) Find the volume of the solid formed if this area is rotated about the y-axis.

16. Evaluate $\int_0^4 (3t^2 - 6t + 5)\,dt$.

17. Find the area bounded by the curve $y = x^2 + 2x - 15$ and the x-axis.

18. Find the volume of the solid formed if the curve $y = x^3$ is rotated about the y-axis from $y = 0$ to $y = 1$.

19. Find
(a) $\int 5(2x - 1)^4\,dx$
(b) $\int \frac{3x^5}{4}\,dx$

Challenge Exercise 3

1. (a) Find the area enclosed between the curves $y = x^3$ and $y = x^2$.
(b) Find the volume of the solid formed if this area is rotated about the x-axis.

2. (a) Show that $f(x) = x^3 + x$ is an odd function.
(b) Hence find the value of $\int_{-2}^{2} f(x)\,dx$.
(c) Find the total area between $f(x)$, the x-axis and the lines $x = -2$ and $x = 2$.

3. The curve $y = 2^x$ is rotated about the x-axis between $x = 1$ and $x = 2$. Use Simpson's rule with 3 function values to find an approximation of the volume of the solid formed, correct to 3 significant figures.

4. Find the area enclosed between the curves $y = (x - 1)^2$ and $y = 5 - x^2$.

5. (a) Differentiate $(x^4 - 1)^9$.
(b) Hence find $\int x^3(x^4 - 1)^8\,dx$.

6. (a) Differentiate $\frac{2x^2 + 1}{3x^2 - 4}$.
(b) Hence evaluate $\int_0^1 \frac{x}{(3x^2 - 4)^2}\,dx$.

7. Find the area enclosed between the curve $y = \frac{3}{x - 2}$, the y-axis and the lines $y = 1$ and $y = 3$, using the trapezoidal rule with 4 subintervals.

8. Find the exact volume of the solid formed by rotating the curve $y = \frac{1}{x}$ about the x-axis between $x = 1$ and $x = 3$.

9. Show that the area enclosed between the curve $y = \frac{1}{x}$, the x-axis and the lines $x = 0$ and $x = 1$ is infinite by using Simpson's rule with 3 function values.

10. (a) Sketch the curve $y = x(x - 1)(x + 2)$.
(b) Find the total area enclosed between the curve and the x-axis.

11. Find the exact area bounded by the parabola $y = x^2$ and the line $y = 4 - x$.

12. Find the volume of the solid formed when the curve $y = (x + 5)^2$ is rotated about the y-axis from $y = 1$ to $y = 4$.

13. (a) Find the derivative of $x\sqrt{x+3}$.

(b) Hence find $\int \frac{x+2}{\sqrt{x+3}}\,dx$.

14. (a) Evaluate $\int_1^3 (x^2 - x + 1)\,dx$.

(b) Use Simpson's rule with 3 function values to evaluate $\int_1^3 (x^2 - x + 1)\,dx$ and show that this gives the exact value.

15. Find the area enclosed between the curves $y = \sqrt{x}$ and $y = x^3$.

16. For the shaded region below, find

(a) the area

(b) the volume when this area is rotated about the y-axis.