4

Exponential and Logarithmic Functions

TERMINOLOGY

Exponential equation: Equation where the pronumeral is the index or exponent such as $3^x = 9$

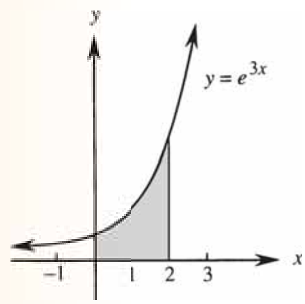
Exponential function: A function in the form $y = a^x$ where the variable x is a power or exponent

Logarithm: A logarithm is an index. The logarithm is the power or exponent of a number to a certain base i.e. $2^x = 8$ is the same as $\log_2 8 = x$

INTRODUCTION

THIS CHAPTER INTRODUCES A new irrational number, 'e', that has special properties in calculus. You will learn how to differentiate and integrate the exponential function $f(x) = e^x$.

The definition and laws of **logarithms** are also introduced in this chapter, as well as **differentiation** and **integration** involving logarithms.



DID YOU KNOW?

John Napier (1550–1617), a Scottish theologian and an amateur mathematician, was the first to invent logarithms. These 'natural', or 'Naperian', logarithms were based on **'e'**. Napier was also one of the first mathematicians to use decimals rather than fractions. He invented the notation of the decimal, using either a comma or a point. The point was used in England, but a few European countries still use a comma.

Henry Briggs (1561–1630), an Englishman who was a professor at Oxford, decided that logarithms would be more useful if they were based on **10** (our decimal system). These are called **common logarithms**. Briggs painstakingly produced a table of logarithms correct to 14 decimal places. He also produced sine tables—to 15 decimal places—and tangent tables—to 10 decimal places.

The work on logarithms was greatly appreciated by **Kepler, Galileo** and other astronomers at the time, since they allowed the computation of very large numbers.

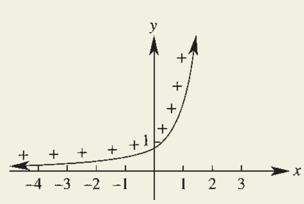
Differentiation of Exponential Functions

When differentiating exponential functions $f(x) = a^x$ from first principles, an interesting result can be seen. The derivative of any exponential function gives a constant which is multiplied by the original function.

EXAMPLE

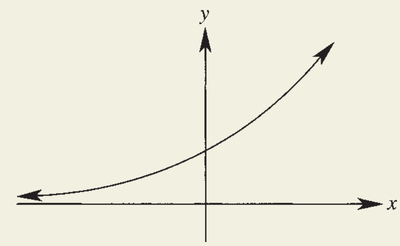
Sketch the derivative (gradient) function of $y = 10^x$.

Solution



CONTINUED

The graph of $y = 10^x$ always has a positive gradient that is becoming steeper. So the derivative function will always be positive, becoming steeper.



The derivative function of an exponential function will always have a shape similar to the original function.

We can use differentiation from first principles to find how close this derivative function is to the original function.

EXAMPLE

Differentiate $f(x) = 10^x$ from first principles.

Solution

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{10^{x+h} - 10^x}{h} \\
&= \lim_{h \to 0} \frac{10^x(10^h - 1)}{h} \\
&= 10^x \lim_{h \to 0} \frac{10^h - 1}{h}
\end{aligned}
$$

You can explore limits using a graphics package on a computer or a graphical calculator.

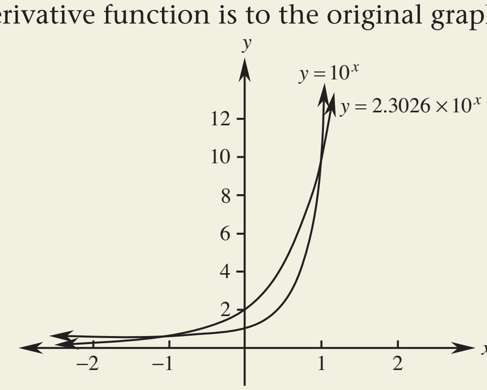
Using the $\boxed{10^x}$ key on the calculator, and finding values of $\frac{10^h - 1}{h}$ when h is small, gives the result:

$$f'(x) \doteqdot 2.3026 \times 10^x$$

or

$$\frac{d}{dx}(10^x) \doteqdot 2.3026 \times 10^x$$

Drawing the graphs of $y = 2.3026 \times 10^x$ and $y = 10^x$ together shows how close the derivative function is to the original graph.



Similar results occur for other exponential functions. In general, $\frac{d}{dx}(a^x) = ka^x$ where k is a constant.

Application

$$\text{If } y = a^x \text{ then } \frac{dy}{dx} = ka^x$$
$$= ky$$

You will study exponential growth and decay in Chapter 6.

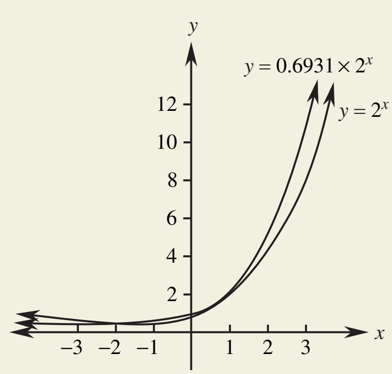
This means that the rate of change of y is proportional to y itself. That is, if y is small, its rate of change is small, but if y is large, then it is changing rapidly.

This is called **exponential growth** (or **decay**, if k is negative) and has many applications in areas such as population growth, radioactive decay, the cooling of objects, the spread of infectious diseases and the growth of technology.

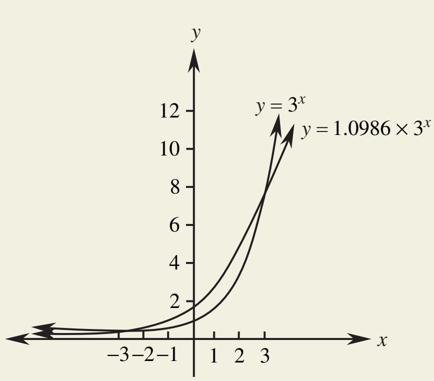
Different exponential functions have different values of k.

EXAMPLES

1. $\frac{d}{dx}(2^x) \doteqdot 0.6931 \times 2^x$.



2. $\frac{d}{dx}(3^x) \doteqdot 1.0986 \times 3^x$.



Notice that the derivative function of $y = 3^x$ is very close to the original function.

We can find a number close to 3 that gives exactly the same graph for the derivative function. This number is approximately 2.71828, and is called e.

e is an irrational number like π.

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

DID YOU KNOW?

A **transcendental number** is a number beyond ordinary numbers. Another transcendental number is π.

The number **e** was linked to logarithms before this useful result in calculus was known. It is a transcendental (irrational) number. This was proven by a French mathematician, **Hermite**, in 1873. **Leonhard Euler** (1707–83) gave e its symbol, and he gave an approximation of e to 23 decimal places. Currently, e is known to about 100 000 decimal places.

Euler studied mathematics, theology, medicine, astronomy, physics and oriental languages. He did extensive research into mathematics and wrote more than 500 books and papers.

Euler gave mathematics much of its important notation. He caused π to become standard notation and used ***i*** for the square root of –1. He first used small letters to show the sides of triangles and the corresponding capital letters for their opposite angles. Also, he introduced the symbol Σ for sums and f(x) notation.

$\boxed{e^x}$ KEY

Use this key to find powers of e.

For example, to find e^2:

Press $\boxed{\text{SHIFT}}$ $\boxed{e^x}$ 2 $\boxed{=}$ $\boxed{e^2 \quad 7.389056099}$

To find e:

Press $\boxed{\text{SHIFT}}$ $\boxed{e^x}$ 1 $\boxed{=}$ $\boxed{e^1 \quad 2.718281828}$

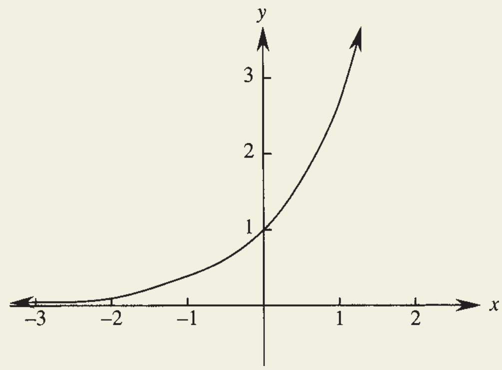
EXAMPLES

1. Sketch the curve $y = e^x$.

Solution

Use e^x on your calculator to draw up a table of values:

x	–3	–2	–1	0	1	2	3
y	0.05	0.1	0.4	1	2.7	7.4	20.1



2. Differentiate $5e^x$.

Solution

$$\frac{d}{dx}(e^x) = e^x$$

$$\therefore \frac{d}{dx}(5e^x) = 5\frac{d}{dx}(e^x)$$
$$= 5e^x$$

3. Find the equation of the tangent to the curve $y = 3e^x$ at the point (0, 3).

Solution

$$\frac{dy}{dx} = 3e^x$$

At (0, 3), $\frac{dy}{dx} = 3e^0$
$= 3$

$\therefore$ $m = 3$

$\frac{dy}{dx}$ gives the gradient of the tangent.

Equation $y - y_1 = m(x - x_1)$
$y - 3 = 3(x - 0)$
$= 3x$
$y = 3x + 3$

CONTINUED

4. Differentiate $\frac{2x+3}{e^x}$.

Solution

This is the quotient rule from Chapter 8 of the Preliminary Course book.

$$\frac{dy}{dx} = \frac{u'v - v'u}{v^2}$$
$$= \frac{2 \cdot e^x - e^x(2x+3)}{(e^x)^2}$$
$$= \frac{2e^x - 2xe^x - 3e^x}{e^{2x}}$$
$$= \frac{-e^x - 2xe^x}{e^{2x}}$$
$$= \frac{-e^x(1+2x)}{e^{2x}}$$
$$= \frac{-(1+2x)}{e^x}$$

4.1 Exercises

1. Find, correct to 2 decimal places, the value of
   (a) $e^{1.5}$
   (b) e^{-2}
   (c) $2e^{0.3}$
   (d) $\frac{1}{e^3}$
   (e) $-3e^{-3.1}$

2. Sketch the curve
   (a) $y = 2e^x$
   (b) $y = e^{-x}$
   (c) $y = -e^x$

3. Differentiate
   (a) $9e^x$
   (b) $-e^x$
   (c) $e^x + x^2$
   (d) $2x^3 - 3x^2 + 5x - e^x$
   (e) $(e^x + 1)^3$
   (f) $(e^x + 5)^7$
   (g) $(2e^x - 3)^2$
   (h) xe^x
   (i) $\frac{e^x}{x}$
   (j) x^2e^x
   (k) $(2x+1)e^x$
   (l) $\frac{e^x}{7x-3}$
   (m) $\frac{5x}{e^x}$

4. If $f(x) = x^3 + 3x - e^x$, find $f'(1)$ and $f''(1)$ in terms of e.

5. Find the exact gradient of the tangent to the curve $y = e^x$ at the point $(1, e)$.

6. Find the exact gradient of the normal to the curve $y = e^x$ at the point where $x = 5$.

7. Find the gradient of the tangent to the curve $y = 4e^x$ at the point where $x = 1.6$, correct to 2 decimal places.

8. Find the equation of the tangent to the curve $y = -e^x$ at the point $(1, -e)$.

9. Find the equation of the normal to the curve $y = e^x$ at the point where $x = 3$, in exact form.

10. Find the stationary point on the curve $y = xe^x$ and determine its nature. Hence sketch the curve.

11. Find the first and second derivatives of $y = 7e^x$. Hence show that $\frac{d^2y}{dx^2} = y$.

12. If $y = 2e^x + 1$, show that $\frac{d^2y}{dx^2} = y - 1$.

Function of a function rule

Remember that the function of a function rule uses the result

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

You studied this in Chapter 8 of the Preliminary Course book.

EXAMPLE

Differentiate e^{x^2+5x-3}.

Solution

Let $u = x^2 + 5x - 3$

Then $y = e^u$

$$\frac{du}{dx} = 2x + 5 \quad \text{and} \quad \frac{dy}{du} = e^u$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u (2x + 5) \\ &= e^{x^2+5x-3}(2x + 5) \\ &= (2x + 5)\, e^{x^2+5x-3}\end{aligned}$$

Can you see a quick way to do this?

$$\boxed{\text{If } y = e^{f(x)} \text{ then } \frac{dy}{dx} = f'(x)\, e^{f(x)}}$$

Proof

Let $u = f(x)$

Then $y = e^u$

$$\frac{dy}{du} = e^u \quad \text{and} \quad \frac{du}{dx} = f'(x)$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u f'(x) \\ &= f'(x)\, e^{f(x)}\end{aligned}$$

EXAMPLES

1. Differentiate e^{5x-2}

Solution

$$y' = f'(x)e^{f(x)}$$
$$= 5e^{5x-2}$$

2. Differentiate $x^2 e^{3x}$.

Solution

This is the product rule from Chapter 8 of the Preliminary Course book.

$$\frac{dy}{dx} = u'v + v'u$$
$$= 2x \cdot e^{3x} + 3e^{3x} \cdot x^2$$
$$= xe^{3x}(2 + 3x)$$

3. Given $y = 2e^{3x} + 1$, show that $\frac{d^2y}{dx^2} = 9(y - 1)$.

Solution

$$y = 2e^{3x} + 1$$
$$\frac{dy}{dx} = 6e^{3x}$$
$$\frac{d^2y}{dx^2} = 18e^{3x}$$
$$= 9(2e^{3x})$$
$$= 9(2e^{3x} + 1 - 1)$$
$$= 9(y - 1)$$

4.2 Exercises

1. Differentiate
   (a) e^{7x}
   (b) e^{-x}
   (c) e^{6x-2}
   (d) e^{x^2+1}
   (e) e^{x^3+5x+7}
   (f) e^{5x}
   (g) e^{-2x}
   (h) e^{10x}
   (i) $e^{2x} + x$
   (j) $x^2 + 2x + e^{1-x}$
   (k) $(x + e^{4x})^5$
   (l) xe^{2x}
   (m) $\frac{e^{3x}}{x^2}$
   (n) $x^3 e^{5x}$
   (o) $\frac{e^{2x+1}}{2x+5}$

2. Find the second derivative of $(e^{2x} + 1)^7$.

3. If $f(x) = e^{3x-2}$, find the exact value of $f'(1)$ and $f''(0)$.

4. Find the gradient of the tangent to the curve $y = e^{5x}$ at the point where $x = 0$.

5. Find the equation of the tangent to the curve $y = e^{2x} - 3x$ at the point (0, 1).

6. Find the exact gradient of the normal to the curve $y = e^{3x}$ at the point where $x = 1$.

7. Find the equation of the tangent to the curve $y = e^{x^2}$ at the point $(1, e)$.

8. If $f(x) = 4x^3 + 3x^2 - e^{-2x}$, find $f''(-1)$ in terms of e.

9. Find any stationary points on the curve $y = x^2 e^{2x}$ and sketch the curve.

10. If $y = e^{4x} + e^{-4x}$, show that $\frac{d^2y}{dx^2} = 16y$.

11. Prove $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$, given $y = 3e^{2x}$.

12. Show $\frac{d^2y}{dx^2} = b^2 y$ for $y = ae^{bx}$.

13. Find the value of n if $y = e^{3x}$ satisfies the equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + ny = 0$.

14. Sketch the curve $y = e^{x^2+x-2}$, showing any stationary points and inflexions.

15. Sketch the curve $y = \frac{x^2+1}{e^x}$, showing any stationary points and inflexions.

Integration of Exponential Functions

Since $\frac{d}{dx}(e^x) = e^x$, then the reverse must be true.

$$\boxed{\int e^x\,dx = e^x + C}$$

To find the **indefinite integral** (primitive function) when the function of a function rule is involved, look at the derivative first.

EXAMPLE

Differentiate e^{2x+1}.
Hence find $\int 2e^{2x+1}\,dx$.
Find $\int e^{2x+1}\,dx$.

Solution

$$\frac{d}{dx}(e^{2x+1}) = 2e^{2x+1}$$

$$\therefore \int 2e^{2x+1}\,dx = e^{2x+1} + C$$

Integration is the inverse of differentiation.

$$\int e^{2x+1}\,dx = \frac{1}{2}\int 2e^{2x+1}\,dx$$

$$= \frac{1}{2}e^{2x+1} + C$$

In general

$$\boxed{\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C}$$

Proof

$$\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}$$

$$\therefore \int ae^{ax+b}\,dx = e^{ax+b} + C$$

$$\int e^{ax+b}\,dx = \frac{1}{a}\int ae^{ax+b}\,dx$$

$$= \frac{1}{a}e^{ax+b} + C$$

EXAMPLES

1. Find $\int (e^{2x} - e^{-x})\,dx$.

Solution

$$\int (e^{2x} - e^{-x})\,dx = \frac{1}{2}e^{2x} - \frac{1}{(-1)}e^{-x} + C$$

$$= \frac{1}{2}e^{2x} + e^{-x} + C$$

2. Find the exact area enclosed between the curve $y = e^{3x}$, the x-axis and the lines $x = 0$ and $x = 2$.

Solution

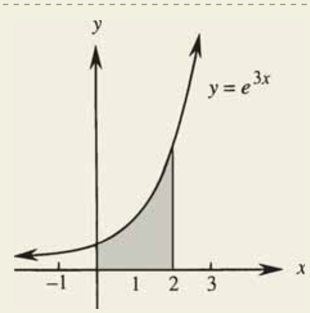
$$\text{Area} = \int_0^2 e^{3x}\,dx$$

$$= \left[\frac{1}{3}e^{3x}\right]_0^2$$

$$= \frac{1}{3}e^6 - \frac{1}{3}e^0$$

$$= \frac{1}{3}(e^6 - e^0)$$

$$= \frac{1}{3}(e^6 - 1)\text{ units}^2$$



3. Find the volume of the solid of revolution formed when the curve $y = e^x$ is rotated about the x-axis from $x = 0$ to $x = 2$.

Solution

Use index laws to simplify $(e^x)^2$.

$$y = e^x$$

$$\therefore\ y^2 = (e^x)^2$$

$$= e^{2x}$$

$$
\begin{aligned}
V &= \pi\int_a^b y^2\,dx \\
&= \pi\int_0^2 e^{2x}\,dx \\
&= \pi\left[\frac{1}{2}e^{2x}\right]_0^2 \\
&= \pi\left(\frac{1}{2}e^4 - \frac{1}{2}e^0\right) \\
&= \pi\left(\frac{1}{2}e^4 - \frac{1}{2}\right) \\
&= \frac{\pi}{2}(e^4 - 1)\text{ units}^3
\end{aligned}
$$

4.3 Exercises

1. Find these indefinite integrals.
   (a) $\int e^{2x}\,dx$
   (b) $\int e^{4x}\,dx$
   (c) $\int e^{-x}\,dx$
   (d) $\int e^{5x}\,dx$
   (e) $\int e^{-2x}\,dx$
   (f) $\int e^{4x+1}\,dx$
   (g) $\int -3e^{5x}\,dx$
   (h) $\int e^{2t}\,dt$
   (i) $\int (e^{7x} - 2)\,dx$
   (j) $\int (e^{x-3} + x)\,dx$

2. Evaluate in exact form.
   (a) $\int_0^1 e^{5x}\,dx$
   (b) $\int_0^2 -e^{-x}\,dx$
   (c) $\int_1^4 2e^{3x+4}\,dx$
   (d) $\int_2^3 (3x^2 - e^{2x})\,dx$
   (e) $\int_0^2 (e^{2x} + 1)\,dx$
   (f) $\int_1^2 (e^x - x)\,dx$
   (g) $\int_0^3 (e^{2x} - e^{-x})\,dx$

3. Evaluate correct to 2 decimal places.
   (a) $\int_1^3 e^{-x}\,dx$
   (b) $\int_0^2 2e^{3y}\,dy$
   (c) $\int_5^6 (e^{x+5} + 2x - 3)\,dx$
   (d) $\int_0^1 (e^{3t+4} - t)\,dt$
   (e) $\int_1^2 (e^{4x} + e^{2x})\,dx$

4. Find the exact area enclosed by the curve $y = 2e^{2x}$, the x-axis and the lines $x = 1$ and $x = 2$.

5. Find the exact area bounded by the curve $y = e^{4x-3}$, the x-axis and the lines $x = 0$ and $x = 1$.

6. Find the area enclosed by the curve $y = x + e^{-x}$, the x-axis and the lines $x = 0$ and $x = 2$, correct to 2 decimal places.

7. Find the area bounded by the curve $y = e^{5x}$, the x-axis and the lines $x = 0$ and $x = 1$, correct to 3 significant figures.

8. Find the exact volume of the solid of revolution formed when the curve $y = e^x$ is rotated about the x-axis from $x = 0$ to $x = 3$.

9. Find the volume of the solid formed when the curve $y = e^{-x} + 1$ is rotated about the x-axis from $x = 1$ to $x = 2$, correct to 1 decimal place.

10. Use Simpson's rule with 3 function values to find an approximation to $\int_1^2 xe^x\,dx$, correct to 1 decimal place.

11. (a) Differentiate x^2e^x.
    (b) Hence find $\int x(2+x)e^x\,dx$.

12. The curve $y = \sqrt{e^x + 1}$ is rotated about the x-axis from $x = 0$ to $x = 1$. Find the exact volume of the solid formed.

13. Find the exact area enclosed between the curve $y = e^{2x}$ and the lines $y = 1$ and $x = 2$.

Application

The exponential function occurs in many fields, such as science and economics.
$P = P_0 e^{kt}$ is a general formula that describes **exponential growth**.
$P = P_0 e^{-kt}$ is a general formula that describes **exponential decay**.

You will study these formulae in Chapter 6.

Logarithms

'Logarithm' is another name for the **index** or **power** of a number. Logarithms are related to exponential functions, and allow us to solve equations like $2^x = 5$. They also allow us to change the subject of exponential equations such as $y = e^x$ to x.

Definition

If $\boldsymbol{y = a^x}$, then $\boldsymbol{x}$ is called the **logarithm of $\boldsymbol{y}$ to the base $\boldsymbol{a}$**.

$$\boxed{\text{If } y = a^x, \text{ then } x = \log_a y}$$

Logarithm keys

$\boxed{\text{log}}$ is used for $\log_{10} x$

$\boxed{\text{ln}}$ is used for $\log_e x$

EXAMPLES

1. Find $\log_{10} 5.3$ correct to 1 decimal place.

Solution

$$\begin{aligned}\log_{10} 5.3 &= 0.724275869 \\ &= 0.7 \text{ correct to 1 decimal place}\end{aligned}$$

Use the $\boxed{\log}$ key.

2. Evaluate $\log_e 80$ correct to 3 significant figures.

Solution

$$\begin{aligned}\log_e 80 &= 4.382026634 \\ &= 4.38 \text{ correct to 3 significant figures}\end{aligned}$$

Use the $\boxed{\ln}$ key.

3. Evaluate $\log_3 81$.

Solution

Let $\log_3 81 = x$

Then $3^x = 81$ (by definition)

i.e. $3^x = 3^4$

$\therefore$ $x = 4$

So $\log_3 81 = 4$.

4. Find the value of $\log_2 \frac{1}{4}$.

Solution

Let $\log_2 \frac{1}{4} = x$

Then
$$\begin{aligned}2^x &= \frac{1}{4} \\ &= \frac{1}{2^2} \\ &= 2^{-2}\end{aligned}$$

$\therefore$ $x = -2$

So $\log_2 \frac{1}{4} = -2$.

Class Investigation

1. Sketch the graph of $y = \log_2 x$.
   There is no calculator key for logarithms to the base 2. Use the definition of a logarithm to change the equation into index form, and the table of values:

x							
y	–3	–2	–1	0	1	2	3

2. On the same set of axes, sketch the curve $y = 2^x$ and the line $y = x$. What do you notice?

4.4 Exercises

1. Evaluate
(a) $\log_2 16$
(b) $\log_4 16$
(c) $\log_5 125$
(d) $\log_3 3$
(e) $\log_7 49$
(f) $\log_7 7$
(g) $\log_5 1$
(h) $\log_2 128$

2. Evaluate
(a) $3 \log_2 8$
(b) $\log_5 25 + 1$
(c) $3 - \log_3 81$
(d) $4 \log_3 27$
(e) $2 \log_{10} 10\,000$
(f) $1 + \log_4 64$
(g) $3 \log_4 64 + 5$
(h) $2 + 4 \log_6 216$
(i) $\dfrac{\log_3 9}{2}$
(j) $\dfrac{\log_8 64 + 4}{\log_2 8}$

3. Evaluate
(a) $\log_2 \dfrac{1}{2}$
(b) $\log_3 \sqrt{3}$
(c) $\log_4 2$
(d) $\log_5 \dfrac{1}{25}$
(e) $\log_7 \sqrt[4]{7}$
(f) $\log_3 \dfrac{1}{\sqrt[3]{3}}$
(g) $\log_4 \dfrac{1}{2}$
(h) $\log_8 2$
(i) $\log_6 6\sqrt{6}$
(j) $\log_2 \dfrac{\sqrt{2}}{4}$

4. Evaluate correct to 2 decimal places.
(a) $\log_{10} 1200$
(b) $\log_{10} 875$
(c) $\log_e 25$
(d) $\ln 140$
(e) $5 \ln 8$
(f) $\log_{10} 350 + 4.5$
(g) $\dfrac{\log_{10} 15}{2}$
(h) $\ln 9.8 + \log_{10} 17$
(i) $\dfrac{\log_{10} 30}{\log_e 30}$
(j) $4 \ln 10 - 7$

5. Write in logarithmic form.
   (a) $3^x = y$
   (b) $5^x = z$
   (c) $x^2 = y$
   (d) $2^b = a$
   (e) $b^3 = d$
   (f) $y = 8^x$
   (g) $y = 6^x$
   (h) $y = e^x$
   (i) $y = a^x$
   (j) $Q = e^x$

6. Write in index form.
   (a) $\log_3 5 = x$
   (b) $\log_a 7 = x$
   (c) $\log_3 a = b$
   (d) $\log_x y = 9$
   (e) $\log_a b = y$
   (f) $y = \log_2 6$
   (g) $y = \log_3 x$
   (h) $y = \log_{10} 9$
   (i) $y = \ln 4$
   (j) $y = \log_7 x$

7. Solve for x, correct to 1 decimal place where necessary.
   (a) $\log_{10} x = 6$
   (b) $\log_3 x = 5$
   (c) $\log_x 343 = 3$
   (d) $\log_x 64 = 6$
   (e) $\log_5 \frac{1}{5} = x$
   (f) $\log_x \sqrt{3} = \frac{1}{2}$
   (g) $\ln x = 3.8$
   (h) $3 \log_{10} x - 2 = 10$
   (i) $\log_4 x = \frac{3}{2}$
   (j) $\log_x 4 = \frac{1}{3}$

8. Evaluate y given that $\log_y 125 = 3$.

9. If $\log_{10} x = 1.65$, evaluate x correct to 1 decimal place.

10. Evaluate b to 3 significant figures if $\log_e b = 0.894$.

11. Find the value of $\log_2 1$. What is the value of $\log_a 1$?

12. Evaluate $\log_5 5$. What is the value of $\log_a a$?

13. (a) Evaluate $\ln e$ without a calculator.
    (b) Using a calculator, evaluate
       (i) $\log_e e^3$
       (ii) $\log_e e^2$
       (iii) $\log_e e^5$
       (iv) $\log_e \sqrt{e}$
       (v) $\log_e \frac{1}{e}$
       (vi) $e^{\ln 2}$
       (vii) $e^{\ln 3}$
       (viii) $e^{\ln 5}$
       (ix) $e^{\ln 7}$
       (x) $e^{\ln 1}$
       (xi) $e^{\ln e}$

14. Sketch the graph of $y = \log_e x$. What is its domain and range?

15. Sketch $y = 10^x$, $y = \log_{10} x$ and $y = x$ on the same number plane. What do you notice about the relationship of the curves to the line?

16. Change the subject of $y = \log_e x$ to x.

Class Discussion

1. Investigate these questions on the calculator. Can you see some patterns?
   (a) $\log_e e$
   (b) $\log_e e^2$
   (c) $\log_e e^3$
   (d) $\log_e e^4$
   (e) $\log_e e^5$
   Can you write a rule for $\log_e e^x$?

2. Evaluate using a calculator. Can you write a rule to show this pattern?
   (a) $e^{\ln 1}$
   (b) $e^{\ln 2}$
   (c) $e^{\ln 3}$
   (d) $e^{\ln 4}$
   (e) $e^{\ln 5}$
   Can you write a rule for $e^{\ln x}$?

3. Do these rules work if x is negative?

Logarithm laws

Because logarithms are closely related to indices there are logarithm laws that correspond to the index laws.

This corresponds to the law $a^m \times a^n = a^{m+n}$ from Chapter 1 of the Preliminary Course book.

$$\log_a(xy) = \log_a x + \log_a y$$

Proof

Let $\quad x = a^m$ and $y = a^n$

Then $\quad m = \log_a x$ and $n = \log_a y$

$$xy = a^m \times a^n$$
$$= a^{m+n}$$

$\therefore \log_a(xy) = m + n \qquad$ (by definition)

$$= \log_a x + \log_a y$$

This corresponds to the law $a^m \div a^n = a^{m-n}$.

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

Proof

Let $x = a^m$ and $y = a^n$

Then $m = \log_a x$ and $n = \log_a y$

$$\frac{x}{y} = a^m \div a^n$$
$$= a^{m-n}$$

$\therefore \log_a \left(\frac{x}{y}\right) = m - n$ (by definition)

$= \log_a x - \log_a y$

$$\boxed{\log_a x^n = n \log_a x}$$

This corresponds to the law $(a^m)^n = a^{mn}$.

Proof

Let $x = a^m$

Then $m = \log_a x$

$$x^n = (a^m)^n$$
$$= a^{mn}$$

$\therefore \log_a x^n = mn$ (by definition)

$= n \log_a x$

EXAMPLES

1. Evaluate $\log_6 3 + \log_6 12$.

Solution

$$\log_6 3 + \log_6 12 = \log_6 (3 \times 12)$$
$$= \log_6 36$$
$$= 2$$

$\log_6 36 = 2$, since $6^2 = 36$.

2. Given $\log_5 3 = 0.68$ and $\log_5 4 = 0.86$, find

(a) $\log_5 12$

(b) $\log_5 0.75$

(c) $\log_5 9$

(d) $\log_5 20$

CONTINUED

Solution

(a) $\log_5 12 = \log_5 (3 \times 4)$
$= \log_5 3 + \log_5 4$
$= 0.68 + 0.86$
$= 1.54$

(b) $\log_5 0.75 = \log_5 \frac{3}{4}$
$= \log_5 3 - \log_5 4$
$= 0.68 - 0.86$
$= -0.18$

(c) $\log_5 9 = \log_5 3^2$
$= 2 \log_5 3$
$= 2 \times 0.68$
$= 1.36$

(d) $\log_5 20 = \log_5 (5 \times 4)$
$= \log_5 5 + \log_5 4$
$= 1 + 0.86$
$= 1.86$

$\log_5 5 = 1$, since $5^1 = 5$.

3. Solve $\log_2 12 = \log_2 3 + \log_2 x$.

Solution

$\log_2 12 = \log_2 3 + \log_2 x$
$= \log_2 3x$
So $12 = 3x$
$4 = x$

4.5 Exercises

1. Use the logarithm laws to simplify
   (a) $\log_a 4 + \log_a y$
   (b) $\log_a 4 + \log_a 5$
   (c) $\log_a 12 - \log_a 3$
   (d) $\log_a b - \log_a 5$
   (e) $3 \log_x y + \log_x z$
   (f) $2 \log_k 3 + 3 \log_k y$
   (g) $5 \log_a x - 2 \log_a y$
   (h) $\log_a x + \log_a y - \log_a z$
   (i) $\log_{10} a + 4 \log_{10} b + 3 \log_{10} c$
   (j) $3 \log_3 p + \log_3 q - 2 \log_3 r$

2. Given $\log_7 2 = 0.36$ and $\log_7 5 = 0.83$, find
   (a) $\log_7 10$
   (b) $\log_7 0.4$
   (c) $\log_7 20$
   (d) $\log_7 25$
   (e) $\log_7 8$
   (f) $\log_7 14$
   (g) $\log_7 50$
   (h) $\log_7 35$
   (i) $\log_7 98$
   (j) $\log_7 70$

3. Use the logarithm laws to evaluate
   (a) $\log_5 50 - \log_5 2$
   (b) $\log_2 16 + \log_2 4$
   (c) $\log_4 2 + \log_4 8$
   (d) $\log_5 500 - \log_5 4$
   (e) $\log_9 117 - \log_9 13$
   (f) $\log_8 32 + \log_8 16$
   (g) $3 \log_2 2 + 2 \log_2 4$
   (h) $2 \log_4 6 - (2 \log_4 3 + \log_4 2)$
   (i) $\log_6 4 - 2 \log_6 12$
   (j) $2 \log_3 6 + \log_3 18 - 3 \log_3 2$

4. If $\log_a 3 = x$ and $\log_a 5 = y$, find an expression in terms of x and y for
   (a) $\log_a 15$
   (b) $\log_a 0.6$
   (c) $\log_a 27$
   (d) $\log_a 25$
   (e) $\log_a 9$
   (f) $\log_a 75$
   (g) $\log_a 3a$
   (h) $\log_a \dfrac{a}{5}$
   (i) $\log_a 9a$
   (j) $\log_a \dfrac{125}{a}$

5. If $\log_a x = p$ and $\log_a y = q$, find, in terms of p and q.
   (a) $\log_a xy$
   (b) $\log_a y^3$
   (c) $\log_a \dfrac{y}{x}$
   (d) $\log_a x^2$
   (e) $\log_a xy^5$
   (f) $\log_a \dfrac{x^2}{y}$
   (g) $\log_a ax$
   (h) $\log_a \dfrac{a}{y^2}$
   (i) $\log_a a^3y$
   (j) $\log_a \dfrac{x}{ay}$

6. If $\log_a b = 3.4$ and $\log_a c = 4.7$, evaluate
   (a) $\log_a \dfrac{c}{b}$
   (b) $\log_a bc^2$
   (c) $\log_a (bc)^2$
   (d) $\log_a abc$
   (e) $\log_a a^2c$
   (f) $\log_a b^7$
   (g) $\log_a \dfrac{a}{c}$
   (h) $\log_a a^3$
   (i) $\log_a bc^4$
   (j) $\log_a b^4c^2$

7. Solve
   (a) $\log_4 12 = \log_4 x + \log_4 3$
   (b) $\log_3 4 = \log_3 y - \log_3 7$
   (c) $\log_a 6 = \log_a x - 3 \log_a 2$
   (d) $\log_2 81 = 4 \log_2 x$
   (e) $\log_x 54 = \log_x k + 2 \log_x 3$

Change of base

Sometimes we need to evaluate logarithms such as $\log_2 7$. We use a change of base formula.

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}}$$

Proof

Let $\quad y = \log_a x$
Then $\quad x = a^y$
Take logarithms to the base b of both sides of the equation:

$$\begin{aligned} \log_b x &= \log_b a^y \\ &= y \log_b a \\ \therefore \quad \frac{\log_b x}{\log_b a} &= y \\ &= \log_a x \end{aligned}$$

You can use the change of base formula to find the logarithm of any number, such as $\log_5 2$. You change it to either $\log_{10} x$ or $\log_e x$, and use a calculator.

EXAMPLE

Find the value of $\log_5 2$, correct to 2 decimal places.

Solution

$$\begin{aligned} \log_5 2 &= \frac{\log 2}{\log 5} \qquad \text{(by change of base)} \\ &\doteqdot 0.430676558 \\ &= 0.43 \end{aligned}$$

You can use either log or ln

Exponential equations

You can also use the change of base formula to solve exponential equations such as $5^x = 7$.

You studied exponential equations such as $2^x = 8$ in the Preliminary Course. Exponential equations such as $2^x = 9$ can be solved by taking logarithms of both sides, or by using the definition of a logarithm and the change of base formula.

EXAMPLES

1. Solve $5^x = 7$ correct to 1 decimal place.

Solution

$$5^x = 7$$

Using the definition of a logarithm, this means:

$$\log_5 7 = x$$

$$\frac{\log 7}{\log 5} = x \qquad \text{(using change of base formula)}$$

$$1.2 = x$$

You can use either log or ln.

If you do not like to solve the equation this way, you can use the logarithm laws instead.

Taking logs of both sides:

$$\log 5^x = \log 7$$

$$x \log 5 = \log 7$$

Use $\log_a x^n = n \log_a x$

$$\therefore \quad x = \frac{\log 7}{\log 5}$$

$$= 1.2 \text{ correct to 1 decimal place}$$

2. Solve $4^{y-3} = 9$ correct to 2 decimal places.

Solution

$$4^{y-3} = 9$$

Using the logarithm definition and change of base:

$$\log_4 9 = y - 3$$

$$\frac{\log 9}{\log 4} = y - 3$$

$$\frac{\log 9}{\log 4} + 3 = y$$

$$4.58 = y$$

Using the logarithm laws:

Taking logs of both sides:

$$\log 4^{y-3} = \log 9$$

$$(y - 3)\log 4 = \log 9$$

$$y - 3 = \frac{\log 9}{\log 4}$$

$$y = \frac{\log 9}{\log 4} + 3$$

$$= 4.58 \text{ correct to 2 decimal place}$$

4.6 Exercises

1. Use the change of base formula to evaluate to 2 decimal places.
   (a) $\log_4 9$
   (b) $\log_6 25$
   (c) $\log_9 200$
   (d) $\log_2 12$
   (e) $\log_3 23$
   (f) $\log_8 250$
   (g) $\log_5 9.5$
   (h) $2 \log_4 23.4$
   (i) $7 - \log_7 108$
   (j) $3 \log_{11} 340$

2. By writing each equation as a logarithm and changing the base, solve the equation correct to 2 significant figures.
   (a) $4^x = 9$
   (b) $3^x = 5$
   (c) $7^x = 14$
   (d) $2^x = 15$
   (e) $5^x = 34$
   (f) $6^x = 60$
   (g) $2^x = 76$
   (h) $4^x = 50$
   (i) $3^x = 23$
   (j) $9^x = 210$

3. Solve, correct to 2 decimal places.
   (a) $2^x = 6$
   (b) $5^y = 15$
   (c) $3^x = 20$
   (d) $7^m = 32$
   (e) $4^k = 50$
   (f) $3^t = 4$
   (g) $8^x = 11$
   (h) $2^p = 57$
   (i) $4^x = 81.3$
   (j) $6^n = 102.6$

4. Solve, to 1 decimal place.
   (a) $3^{x+1} = 8$
   (b) $5^{3n} = 71$
   (c) $2^{x-3} = 12$
   (d) $4^{2n-1} = 7$
   (e) $7^{5x+2} = 11$
   (f) $8^{3-n} = 5.7$
   (g) $2^{x+2} = 18.3$
   (h) $3^{7k-3} = 32.9$
   (i) $9^{\frac{x}{2}} = 50$
   (j) $6^{2y+1} = 61.3$

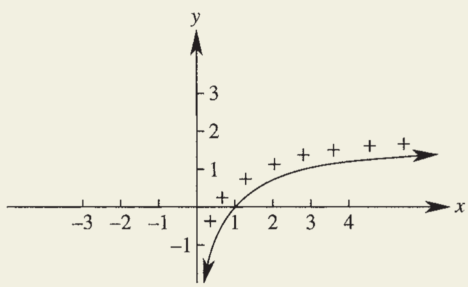
5. Solve each equation correct to 3 significant figures.
   (a) $e^x = 200$
   (b) $e^{3t} = 5$
   (c) $2e^t = 75$
   (d) $45 = e^x$
   (e) $3000 = 100e^n$
   (f) $100 = 20e^{3t}$
   (g) $2000 = 50e^{0.15t}$
   (h) $15\,000 = 2000e^{0.03k}$
   (i) $3Q = Qe^{0.02t}$
   (j) $0.5M = Me^{0.016k}$

Derivative of the Logarithmic Function

Drawing the derivative (gradient) function of a logarithm function gives a hyperbola.

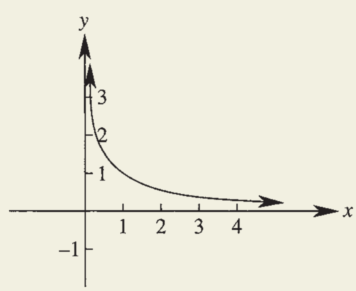
EXAMPLE

Sketch the derivative function of $y = \log_2 x$.



Solution

The gradient is always positive but is decreasing.



$$\boxed{\text{If } y = \log_e x \text{ then } \frac{dy}{dx} = \frac{1}{x}}$$

Proof

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ is a special result that can be proved by differentiating from first principles.

Given $y = \log_e x$

Then $x = e^y$

$$\frac{dx}{dy} = e^y$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{e^y}$$

$$= \frac{1}{x}$$

Function of a function rule

If $y = \log_e f(x)$, then $\dfrac{dy}{dx} = f'(x) \cdot \dfrac{1}{f(x)} = \dfrac{f'(x)}{f(x)}$

Proof

Let $u = f(x)$

Then $y = \log_e u$

$\therefore \quad \dfrac{dy}{du} = \dfrac{1}{u}$

Also $\dfrac{du}{dx} = f'(x)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= \frac{1}{u} \cdot f'(x)\\ &= \frac{1}{f(x)} \cdot f'(x)\end{aligned}$$

EXAMPLES

1. Differentiate $\log_e(x^2 - 3x + 1)$.

Solution

$$\frac{d}{dx}[\log_e(x^2 - 3x + 1)] = \frac{2x - 3}{x^2 - 3x + 1}$$

2. Differentiate $\log_e \dfrac{x+1}{3x-4}$.

Solution

Let $y = \log_e \dfrac{x+1}{3x-4}$

$$\begin{aligned}&= \log_e(x+1) - \log_e(3x-4)\\ \frac{dy}{dx} &= \frac{1}{x+1} - \frac{3}{3x-4}\\ &= \frac{1(3x-4) - 3(x+1)}{(x+1)(3x-4)}\\ &= \frac{3x - 4 - 3x - 3}{(x+1)(3x-4)}\\ &= \frac{-7}{(x+1)(3x-4)}\end{aligned}$$

3. Find the gradient of the normal to the curve $y = \log_e (x^3 - 5)$ at the point where $x = 2$.

Solution

$$\frac{dy}{dx} = \frac{3x^2}{x^3 - 5}$$

When $x = 2$,

$$\frac{dy}{dx} = \frac{3(2)^2}{2^3 - 5}$$

$$m_1 = 4$$

The normal is perpendicular to the tangent

i.e. $m_1 m_2 = -1$

This result comes from the Preliminary Course.

$$4m_2 = -1$$

$$\therefore \quad m_2 = -\frac{1}{4}$$

4. Differentiate $y = \log_2 x$.

Solution

$$\begin{aligned} y &= \log_2 x \\ &= \frac{\log_e x}{\log_e 2} \\ &= \frac{1}{\log_e 2} \times \log_e x \\ \frac{dy}{dx} &= \frac{1}{\log_e 2} \times \frac{1}{x} \\ &= \frac{1}{x \log_e 2} \end{aligned}$$

5. Find the derivative of 2^x.

Solution

$$\begin{aligned} 2 &= e^{\ln 2} \\ \therefore \quad 2^x &= (e^{\ln 2})^x \\ &= e^{x \ln 2} \\ \frac{dy}{dx} &= \ln 2 e^{x \ln 2} \\ &= \ln 2 \times 2^x \\ &= 2^x \ln 2 \end{aligned}$$

4.7 Exercises

1. Differentiate
   (a) $x + \log_e x$
   (b) $1 - \log_e 3x$
   (c) $\ln(3x + 1)$
   (d) $\log_e(x^2 - 4)$
   (e) $\ln(5x^3 + 3x - 9)$
   (f) $\log_e(5x + 1) + x^2$
   (g) $3x^2 + 5x - 5 + \ln 4x$
   (h) $\log_e(8x - 9) + 2$
   (i) $\log_e(2x + 4)(3x - 1)$
   (j) $\log_e \dfrac{4x + 1}{2x - 7}$
   (k) $(1 + \log_e x)^5$
   (l) $(\ln x - x)^9$
   (m) $(\log_e x)^4$
   (n) $(x^2 + \log_e x)^6$
   (o) $x \log_e x$
   (p) $\dfrac{\log_e x}{x}$
   (q) $(2x + 1)\log_e x$
   (r) $x^3 \log_e(x + 1)$
   (s) $\log_e(\log_e x)$
   (t) $\dfrac{\ln x}{x - 2}$
   (u) $\dfrac{e^{2x}}{\log_e x}$
   (v) $e^x \ln x$
   (w) $5(\log_e x)^2$

2. If $f(x) = \log_e \sqrt{2 - x}$, find $f'(1)$.

3. Find the derivative of $\log_{10} x$.

4. Find the equation of the tangent to the curve $y = \log_e x$ at the point $(2, \log_e 2)$.

5. Find the equation of the tangent to the curve $y = \log_e(x - 1)$ at the point where $x = 2$.

6. Find the gradient of the normal to the curve $y = \log_e(x^4 + x)$ at the point $(1, \log_e 2)$.

7. Find the exact equation of the normal to the curve $y = \log_e x$ at the point where $x = 5$.

8. Find the equation of the tangent to the curve $y = \log_e(5x + 4)$ at the point where $x = 3$.

9. Find the point of inflexion on the curve $y = x\log_e x - x^2$.

10. Find the stationary point on the curve $y = \dfrac{\ln x}{x}$ and determine its nature.

11. Sketch, showing any stationary points and inflexions.
    (a) $y = x - \log_e x$
    (b) $y = (\log_e x - 1)^3$
    (c) $y = x \ln x$

12. Find the derivative of $\log_3(2x + 5)$.

13. Differentiate
    (a) 3^x
    (b) 10^x
    (c) 2^{3x-4}

14. Find the equation of the tangent to the curve $y = 4^{x+1}$ at the point $(0, 4)$.

15. Find the equation of the normal to the curve $y = \log_3 x$ at the point where $x = 3$.

Integration and the Logarithmic Function

$$\int \frac{dx}{x} = \int \frac{1}{x}\, dx = \log_e x + C$$

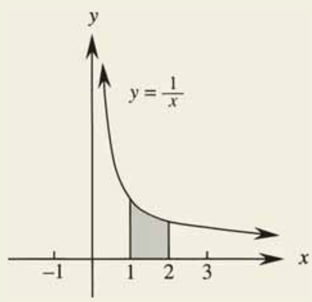
$$\int \frac{f'(x)}{f(x)}\, dx = \log_e f(x) + C$$

Integration is the inverse of differentiation.

EXAMPLES

1. Find the area enclosed between the hyperbola $y = \frac{1}{x}$, the x-axis and the lines $x = 1$ and $x = 2$, giving the exact value.

Solution



$$\begin{aligned} A &= \int_1^2 \frac{1}{x}\, dx \\ &= [\log_e x]_1^2 \\ &= \log_e 2 - \log_e 1 \\ &= \log_e 2 \end{aligned}$$

So area is $\log_e 2$ units^2.

2. Find $\int \frac{x^2}{x^3 + 7}\, dx$.

Solution

$$\begin{aligned} \int \frac{x^2}{x^3 + 7}\, dx &= \frac{1}{3}\int \frac{3x^2}{x^3 + 7}\, dx \\ &= \frac{1}{3}\log_e(x^3 + 7) + C \end{aligned}$$

CONTINUED

3. Find $\int \frac{x+1}{x^2+x+4}\,dx$.

Solution

$$\begin{aligned}\int \frac{x+1}{x^2+2x+4}\,dx &= \frac{1}{2}\int \frac{2(x+1)}{x^2+2x+4}\,dx\\ &= \frac{1}{2}\int \frac{2x+2}{x^2+2x+4}\,dx\\ &= \frac{1}{2}\log_e(x^2+2x+4)+C\end{aligned}$$

4.8 Exercises

1. Find the indefinite integral (primitive function) of
   (a) $\frac{2}{2x+5}$
   (b) $\frac{4x}{2x^2+1}$
   (c) $\frac{5x^4}{x^5-2}$
   (d) $\frac{1}{2x}$
   (e) $\frac{2}{x}$
   (f) $\frac{5}{3x}$
   (g) $\frac{2x-3}{x^2-3x}$
   (h) $\frac{x}{x^2+2}$
   (i) $\frac{3x}{x^2+7}$
   (j) $\frac{x+1}{x^2+2x-5}$

2. Find
   (a) $\int \frac{4}{4x-1}\,dx$
   (b) $\int \frac{dx}{x+3}$
   (c) $\int \frac{x^2}{2x^3-7}\,dx$
   (d) $\int \frac{x^5}{2x^6+5}\,dx$
   (e) $\int \frac{x+3}{x^2+6x+2}\,dx$

3. Evaluate correct to 1 decimal place.
   (a) $\int_1^3 \frac{2}{2x+5}\,dx$
   (b) $\int_2^5 \frac{dx}{x+1}$
   (c) $\int_1^7 \frac{x^2}{x^3+2}\,dx$
   (d) $\int_0^3 \frac{4x+1}{2x^2+x+1}\,dx$
   (e) $\int_3^4 \frac{x-1}{x^2-2x}\,dx$

4. Find the exact area between the curve $y=\frac{1}{x}$, the x-axis and the lines $x=2$ and $x=3$.

5. Find the exact area bounded by the curve $y=\frac{1}{x-1}$, the x-axis and the lines $x=4$ and $x=7$.

6. Find the exact area between the curve $y=\frac{1}{x}$, the x-axis and the lines $y=x$ and $x=2$ in the first quadrant.

7. Find the area bounded by the curve $y=\frac{x}{x^2+1}$, the x-axis and the lines $x=2$ and $x=4$, correct to 2 decimal places.

8. Find the exact volume of the solid formed when the curve $y = \frac{1}{\sqrt{x}}$ is rotated about the x-axis from $x = 1$ to $x = 3$.

9. Find the volume of the solid formed when the curve $y = \frac{2}{\sqrt{2x - 1}}$ is rotated about the x-axis from $x = 1$ to $x = 5$, giving an exact answer.

10. Find the area between the curve $y = \ln x$, the y-axis and the lines $y = 2$ and $y = 4$, correct to 3 significant figures.

11. Find the exact volume of the solid formed when the curve $y = \log_e x$ is rotated about the y-axis from $y = 1$ to $y = 3$.

12. (a) Show that $\frac{3x + 3}{x^2 - 9} = \frac{1}{x + 3} + \frac{2}{x - 3}$.

(b) Hence find $\int \frac{3x + 3}{x^2 - 9}\,dx$.

13. (a) Show that $\frac{x - 6}{x - 1} = 1 - \frac{5}{x - 1}$.

(b) Hence find $\int \frac{x - 6}{x - 1}\,dx$.

14. Find the indefinite integral (primitive function) of 3^{2x-1}.

15. Find, correct to 2 decimal places, the area enclosed by the curve $y = \log_2 x$, the x-axis and the lines $x = 1$ and $x = 3$ by using Simpson's rule with 3 function values.

Test Yourself 4

1. Evaluate to 3 significant figures.
(a) $e^2 - 1$
(b) $\log_{10} 95$
(c) $\log_e 26$
(d) $\log_4 7$
(e) $\log_4 3$
(f) $\ln 50$
(g) $e + 3$
(h) $\dfrac{5e^3}{\ln 4}$
(i) $e^{\ln 6}$
(j) $e^{\ln 2}$

2. Differentiate
(a) e^{5x}
(b) $2e^{1-x}$
(c) $\log_e 4x$
(d) $\ln(4x + 5)$
(e) xe^x
(f) $\dfrac{\ln x}{x}$
(g) $(e^x + 1)^{10}$

3. Find the indefinite integral (primitive function) of
(a) e^{4x}
(b) $\dfrac{x}{x^2 - 9}$
(c) e^{-x}
(d) $\dfrac{1}{x + 4}$

4. Find the equation of the tangent to the curve $y = 2 + e^{3x}$ at the point where $x = 0$.

5. Find the exact gradient of the normal to the curve $y = x - e^{-x}$ at the point where $x = 2$.

6. Find the exact area bounded by the curve $y = e^{2x}$, the x-axis and the lines $x = 2$ and $x = 5$.

7. Find the volume of the solid formed if the area bounded by $y = e^{3x}$, the x-axis and the lines $x = 1$ and $x = 2$ is rotated about the x-axis.

8. If $\log_7 2 = 0.36$ and $\log_7 3 = 0.56$, find the value of
(a) $\log_7 6$
(b) $\log_7 8$
(c) $\log_7 1.5$
(d) $\log_7 14$
(e) $\log_7 3.5$

9. Find the area enclosed between the curve $y = \ln x$, the y-axis and the lines $y = 1$ and $y = 3$.

10. (a) Use Simpson's rule with 3 function values to find the area bounded by the curve $y = \ln x$, the x-axis and the lines $x = 2$ and $x = 4$.
(b) Change the subject of $y = \ln x$ to x.
(c) Hence find the exact area in part (a).

11. Solve
(a) $3^x = 8$
(b) $2^{3x-4} = 3$
(c) $\log_x 81 = 4$
(d) $\log_6 x = 2$
(e) $12 = 10e^{0.01t}$

12. Evaluate
(a) $\displaystyle\int_0^1 3e^{2x}\,dx$
(b) $\displaystyle\int_1^4 \frac{dx}{3x - 2}$
(c) $\displaystyle\int_1^2 \frac{2x^3 - x^2 + 5x + 3}{x}\,dx$

13. Find the equation of the tangent to the curve $y = e^x$ at the point $(4, e^4)$.

14. Evaluate $\log_9 8$ to 1 decimal place.

15. (a) Find the area bounded by the curve $y = e^x$, the x-axis and the lines $x = 1$ and $x = 2$.
(b) This area is rotated about the x-axis. Find the volume of the solid of revolution formed.

16. Simplify
(a) $5 \log_a x + 3 \log_a y$
(b) $2 \log_x k - \log_x 3 + \log_x p$

17. Find the equation of the normal to the curve $y = \ln x$ at the point $(2, \ln 2)$.

18. Find the stationary points on the curve $y = x^3 e^x$ and determine their nature.

19. Use the trapezoidal rule with 4 strips to find the area bounded by the curve $y = \ln(x^2 - 1)$, the x-axis and the lines $x = 3$ and $x = 5$.

20. Evaluate to 2 significant figures
(a) $\log_{10} 4.5$
(b) $\ln 3.7$

Challenge Exercise 4

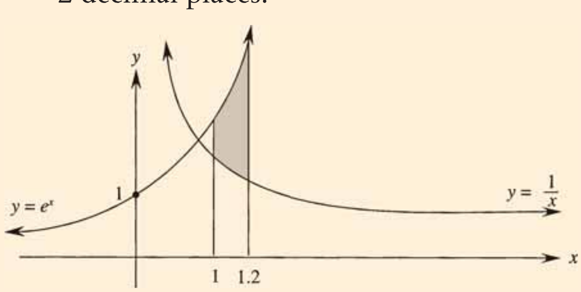
1. Differentiate $\dfrac{\log_e x}{e^{2x} + x}$.

2. Find the exact gradient of the tangent to the curve $y = e^{x + \log_e x}$ at the point where $x = 1$.

3. If $\log_b 2 = 0.6$ and $\log_b 3 = 1.2$, find
(a) $\log_b 6b$
(b) $\log_b 8$
(c) $\log_b 1.5b^2$

4. Differentiate $(e^{4x} + \log_e x)^9$.

5. Find the shaded area, correct to 2 decimal places.



6. Find the derivative of $\log_e \dfrac{1}{2x - 3}$.

7. Use Simpson's rule with 5 function values to find the volume of the solid formed when the curve $y = e^x$ is rotated about the y-axis from $y = 3$ to $y = 5$, correct to 2 significant figures.

8. Differentiate 5^x.

9. Show that $\dfrac{d}{dx}(x^2 \log_e x) = x(1 + 2 \log_e x)$.

Hence evaluate $\int_1^3 2x(1 + 2 \log_e x)\, dx$, giving an exact answer.

10. Find $\int 3^x \, dx$.

11. (a) Find the point of intersection of the curves $y = \log_e x$ and $y = \log_{10} x$.
(b) Find the exact equations of the tangents to the two curves at this point of intersection.
(c) Find the exact length of the interval XY where X and Y are the y-intercepts of the tangents.

12. Use Simpson's rule with 3 function values to find the area enclosed by the curve $y = e^{2x}$, the y-axis and the line $y = 3$, correct to 3 significant figures.

13. Find the derivative of $\dfrac{x \log_e x}{e^x}$.

14. If $y = e^x + e^{-x}$, show $\dfrac{d^2y}{dx^2} = y$.

15. Prove $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 5y - 10 = 0$, given $y = 3e^{5x} - 2$.

16. Find the equation of the curve that has $f''(x) = 12e^{2x}$ and a stationary point at $(0, 3)$.

17. Sketch $y = \log_e(x - x^2)$.