5

Trigonometric Functions

TERMINOLOGY

Amplitude: Maximum displacement from the mean or centre position in vibrating or oscillating motion

Circular measure: The measurement of an angle as the length of an arc cut off by an angle at the centre of a unit circle. The units are called radians

Period: One complete cycle of a wave or other recurring function

Radian: A unit of angular measurement defined as the length of arc of one unit that an angle subtended at the centre of a unit circle cuts off

Sector: Part of a circle bounded by the centre and arc of a circle cut off by two radii

Segment: Part of a circle bounded by a chord and the circumference of a circle

INTRODUCTION

IN THIS CHAPTER YOU will learn about circular measure, which uses radians rather than degrees. Circular measure is very useful in solving problems involving properties of the circle, such as arc length and areas of sectors and segments, which you will study in this chapter.

You will also study trigonometric graphs in greater detail than in the Preliminary Course. These graphs have many applications in the movement of such things as sound, light and waves.

Circular measure is also useful in calculus, and you will learn about differentiation and integration of trigonometric functions in this chapter.

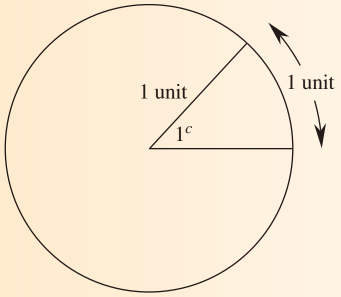
Circular Measure

Radians

We are used to degrees being used in geometry and trigonometry. However, they are limited. For example, when we try to draw the graph of a trigonometric function, it is difficult to mark degrees along the *x*- or *y*-axis as there is no concept of how large a degree is along a straight line. This is one of the reasons we use radians.

Other reasons for using radians are that they make solving circle problems simple and they are used in calculus.

A radian is based on the length of an arc in a unit circle (a circle with radius 1 unit), so we can visualise a radian as a number that can be measured along a line.



The symbol for 1 radian is 1^c but we usually don't use the symbol as radians are simply distances, or arc lengths.

One radian is the angle that an arc of 1 unit subtends at the centre of a circle of radius 1 unit.

π radians = 180°

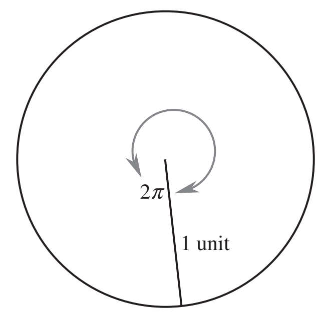
Proof

Circumference of the circle with radius 1 unit is given by:

$$
\begin{aligned}
C &= 2\pi r \\
&= 2\pi(1) \\
&= 2\pi
\end{aligned}
$$

The arc length of the whole circle is 2π.
$\therefore$ there are 2π radians in a whole circle.
But there are 360° in a whole circle (angle of revolution).

$$
\begin{aligned}
\text{So } 2\pi &= 360° \\
\pi &= 180°
\end{aligned}
$$



EXAMPLES

1. Convert $\frac{3\pi}{2}$ into degrees.

Solution

Since $\pi = 180°$,

$$
\begin{aligned}
\frac{3\pi}{2} &= \frac{3(180°)}{2} \\
&= 270°
\end{aligned}
$$

2. Change 60° to radians, leaving your answer in terms in π.

Solution

$$
\begin{aligned}
180° &= \pi \text{ radians} \\
\text{So } 1° &= \frac{\pi}{180} \text{ radians} \\
60° &= \frac{\pi}{180} \times 60 \\
&= \frac{60\pi}{180} \\
&= \frac{\pi}{3}
\end{aligned}
$$

3. Convert 50° into radians, correct to 2 decimal places.

Solution

$$180° = \pi \text{ radians}$$

$$\text{So } 1° = \frac{\pi}{180} \text{ radians}$$

$$\begin{aligned} 50° &= \frac{\pi}{180} \times 50 \\ &= \frac{50\pi}{180} \\ &\doteqdot 0.87 \end{aligned}$$

4. Change 1.145 radians into degrees, to the nearest minute.

Solution

$$\begin{aligned} \pi \text{ radians} &= 180° \\ \therefore \quad 1 \text{ radian} &= \frac{180°}{\pi} \\ 1.145 \text{ radians} &= \frac{180°}{\pi} \times 1.145 \\ &= 65.6° \\ &= 65° 36' \end{aligned}$$

Press 180 $\boxed{\div}$ $\boxed{\pi}$ $\boxed{\times}$ 1.145 $\boxed{=}$
$\boxed{\text{SHIFT}}$ $\boxed{\circ\ '\ ''}$

5. Convert 38° 41′ into radians, correct to 3 decimal places.

Solution

$$\begin{aligned} 180° &= \pi \text{ radians} \\ 1° &= \frac{\pi}{180°} \\ 38° 41' &= \frac{\pi}{180°} \times 38° 41' \\ &= 0.675 \end{aligned}$$

The calculator may give the answer in degrees and minutes. Simply change this into a decimal.

6. Evaluate cos 1.145 correct to 2 decimal places.

Solution

For cos 1.145, use radian mode on your calculator.

$$\begin{aligned} \cos 1.145 &= 0.413046135 \\ &= 0.41 \text{ correct to 2 decimal places} \end{aligned}$$

Notice from the examples that $1° = \frac{\pi}{180}$ and 1 radian $= \frac{180}{\pi}$. You can use these as conversions rather than starting each time from $\pi = 180°$.

> To change from radians to degrees: Multiply by $\frac{180}{\pi}$.
>
> To change from degrees to radians: Multiply by $\frac{\pi}{180}$.

Notice that
$$\begin{aligned} 1° &= \frac{\pi}{180} \\ &= 0.017 \text{ radians} \end{aligned}$$

Also
$$\begin{aligned} 1 \text{ radian} &= \frac{180}{\pi} \\ &= 57° 18' \end{aligned}$$

While we can convert between degrees and radians for any angle, there are some special angles that we use regularly in this course. It is easier if you know these without having to convert each time.

$$\begin{aligned} \frac{\pi}{2} &= 90° \\ \pi &= 180° \\ \frac{3\pi}{2} &= 270° \\ 2\pi &= 360° \\ \frac{\pi}{4} &= 45° \\ \frac{\pi}{3} &= 60° \\ \frac{\pi}{6} &= 30° \end{aligned}$$

5.1 Exercises

1. Change into degrees.
   (a) $\frac{\pi}{5}$
   (b) $\frac{2\pi}{3}$
   (c) $\frac{5\pi}{4}$
   (d) $\frac{7\pi}{6}$
   (e) 3π
   (f) $\frac{7\pi}{9}$
   (g) $\frac{4\pi}{3}$
   (h) $\frac{7\pi}{3}$
   (i) $\frac{\pi}{9}$
   (j) $\frac{5\pi}{18}$

2. Convert into radians in terms of π.
(a) 135°
(b) 30°
(c) 150°
(d) 240°
(e) 300°
(f) 63°
(g) 15°
(h) 450°
(i) 225°
(j) 120°

3. Change into radians, correct to 2 decimal places.
(a) 56°
(b) 68°
(c) 127°
(d) 289°
(e) 312°

4. Change into radians, to 2 decimal places.
(a) 18° 34′
(b) 35° 12′
(c) 101° 56′
(d) 88° 29′
(e) 50° 39′

5. Change these radians into degrees and minutes, to the nearest minute.
(a) 1.09
(b) 0.768
(c) 1.16
(d) 0.99
(e) 0.32
(f) 3.2
(g) 2.7
(h) 4.31
(i) 5.6
(j) 0.11

6. Find correct to 2 decimal places.
(a) sin 0.342
(b) cos 1.5
(c) tan 0.056
(d) cos 0.589
(e) tan 2.29
(f) sin 2.8
(g) tan 5.3
(h) cos 4.77
(i) cos 3.9
(j) sin 2.98

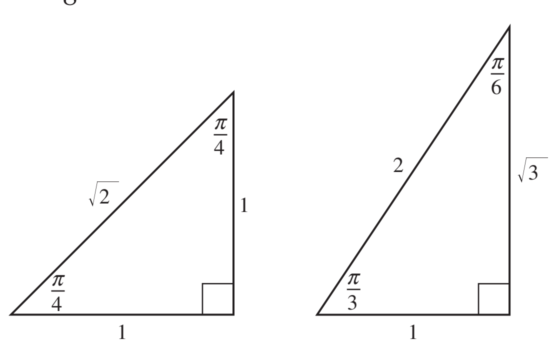
The calculator may give the answer in degrees and minutes. Simply change this into a decimal.

Trigonometric Results

All the trigonometry that you studied in the Preliminary Course can be done using radians instead of degrees.

Special triangles

The two triangles that give exact trigonometric ratios can be drawn using radians rather than degrees.



Using trigonometric ratios and these special triangles gives the results:

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\tan\frac{\pi}{4} = 1$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\cos\frac{\pi}{3} = \frac{1}{2}$$
$$\tan\frac{\pi}{3} = \sqrt{3}$$

$$\sin\frac{\pi}{6} = \frac{1}{2}$$
$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$
$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

EXAMPLE

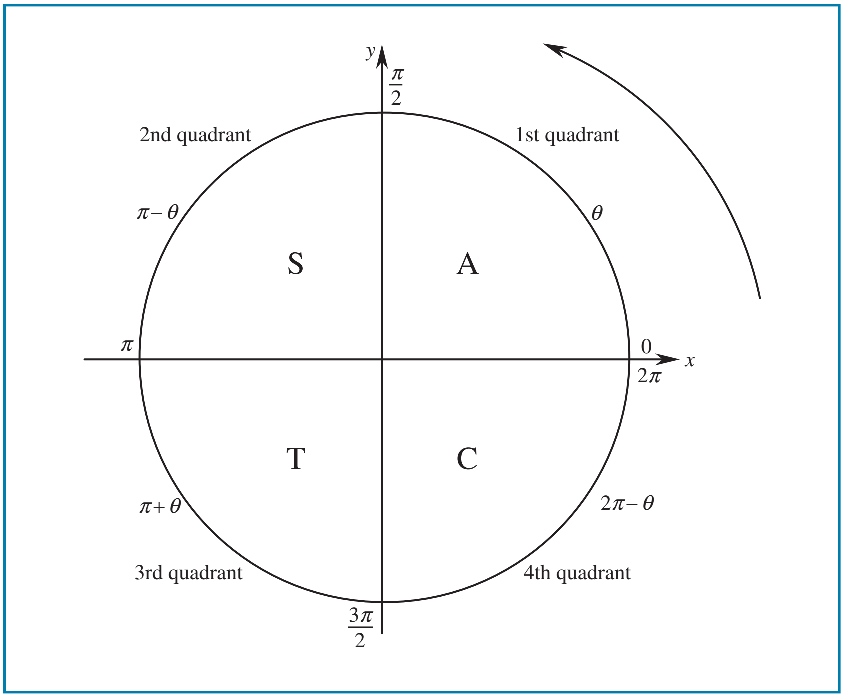
Find the exact value of $3\cos\frac{\pi}{6} - \operatorname{cosec}\frac{\pi}{4}$.

Solution

$$\begin{aligned}
&3\cos\frac{\pi}{6} - \operatorname{cosec}\frac{\pi}{4} \\
&= 3\cos\frac{\pi}{6} - \frac{1}{\sin\frac{\pi}{4}} \\
&= 3\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{1} \\
&= \frac{3\sqrt{3}}{2} - \sqrt{2} \\
&= \frac{3\sqrt{3}}{2} - \frac{2\sqrt{2}}{2} \\
&= \frac{3\sqrt{3} - 2\sqrt{2}}{2}
\end{aligned}$$

Angles of any magnitude

The results for angles of any magnitude can also be looked at using radians. If we change degrees for radians, the ASTC rule looks like this:



We can summarise the trigonometric ratios for angles of any magnitude in radians as follows:

First quadrant:
Angle θ:
$\sin\theta$ is positive
$\cos\theta$ is positive
$\tan\theta$ is positive

Second quadrant:
Angle $\pi - \theta$:
$\sin(\pi - \theta) = \sin\theta$
$\cos(\pi - \theta) = -\cos\theta$
$\tan(\pi - \theta) = -\tan\theta$

Third quadrant:
Angle $\pi + \theta$:
$\sin(\pi + \theta) = -\sin\theta$
$\cos(\pi + \theta) = -\cos\theta$
$\tan(\pi + \theta) = \tan\theta$

Fourth quadrant:
Angle $2\pi - \theta$:
$\sin(2\pi - \theta) = -\sin\theta$
$\cos(2\pi - \theta) = \cos\theta$
$\tan(2\pi - \theta) = -\tan\theta$

EXAMPLES

1. Find the exact value of $\sin \frac{5\pi}{4}$.

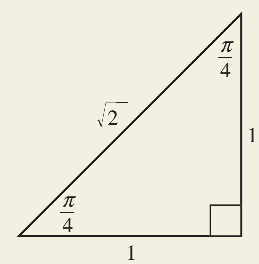
Solution

$$\pi = \frac{4\pi}{4}$$

So $$\frac{5\pi}{4} = \frac{4\pi}{4} + \frac{\pi}{4}$$
$$= \pi + \frac{\pi}{4}$$

In the 3rd quadrant, angles are in the form $\pi + \theta$, so the angle is in the 3rd quadrant, and $\sin\theta$ is negative in the 3rd quadrant.

$$\sin\left(\frac{5\pi}{4}\right) = \sin\left(\pi + \frac{\pi}{4}\right)$$
$$= -\sin\frac{\pi}{4}$$
$$= -\frac{1}{\sqrt{2}}$$



2. Find the exact value of $\cos \frac{11\pi}{6}$.

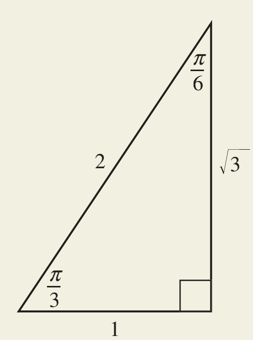
Solution

$$2\pi = \frac{12\pi}{6}$$

So $$\frac{11\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6}$$
$$= 2\pi - \frac{\pi}{6}$$

In the 4th quadrant, angles are in the form $2\pi - \theta$, so the angle is in the 4th quadrant, and $\cos\theta$ is positive.

$$\cos\left(\frac{11\pi}{6}\right) = \cos\left(2\pi - \frac{\pi}{6}\right)$$
$$= \cos\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{2}$$



3. Find the exact value of $\tan\left(-\frac{4\pi}{3}\right)$.

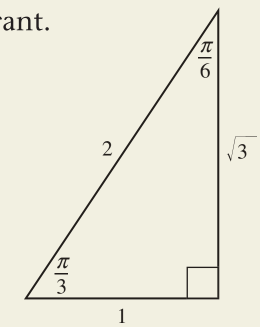
Solution

$$\pi = \frac{3\pi}{3}$$

So $$-\frac{4\pi}{3} = -\left(\frac{3\pi}{3} + \frac{\pi}{3}\right) = -\left(\pi + \frac{\pi}{3}\right)$$

In the 2nd quadrant, angles are in the form $-(\pi + \theta)$, so the angle is in the 2nd quadrant, and $\tan\theta$ is negative in the 2nd quadrant.

$$\tan\left(-\frac{4\pi}{3}\right) = -\tan\left[-\left(\pi + \frac{\pi}{3}\right)\right] = -\tan\frac{\pi}{3} = -\sqrt{3}$$



You could change the radians into degrees before finding these ratios.

We can use the ASTC rule to solve trigonometric equations. You studied these in the Preliminary Course using degrees.

You could solve all equations in degrees then convert into radians where necessary.

EXAMPLES

1. Solve $\cos x = 0.34$ for $0 \le x \le 2\pi$.

Solution

cos is positive in the 1st and 4th quadrants.
Using a calculator (with radian mode) gives $x = 1.224$.
i.e. $\cos 1.224 = 0.34$
This is an angle in the 1st quadrant since $1.224 < \frac{\pi}{2}$.
In the 4th quadrant, angles are in the form of $2\pi - \theta$.
So the angle in the 4th quadrant will be $2\pi - 1.224$.
So $x = 1.224, 2\pi - 1.224$
$= 1.224, 5.06$.

2. Solve $\sin\alpha = -\frac{1}{\sqrt{2}}$ in the domain $0 \le \alpha \le 2\pi$.

Solution

Here the sin of the angle is negative.
Since sin is positive in the 1st and 2nd quadrants, it is negative in the 3rd and 4th quadrants.

CONTINUED

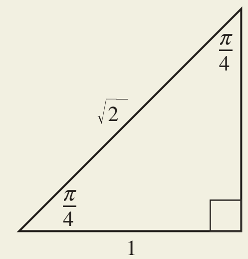
$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

In the 3rd quadrant, angles are $\pi + \theta$ and in the 4th quadrant, $2\pi - \theta$.

So $\alpha = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$= \frac{4\pi}{4} + \frac{\pi}{4}, \frac{8\pi}{4} - \frac{\pi}{4}$

$= \frac{5\pi}{4}, \frac{7\pi}{4}$



5.2 Exercises

1. Copy and complete the table, giving exact values.

	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$
sin			
cos			
tan			
cosec			
sec			
cot			

2. Find the exact value, with rational denominator where relevant.

Remember that $\tan^2\theta = (\tan\theta)^2$

(a) $\tan^2\frac{\pi}{6}$

(b) $\left(\sin\frac{\pi}{4}\right)^2$

(c) $\left(\cos\frac{\pi}{6}\right)^3$

(d) $\tan\frac{\pi}{3} + \tan\frac{\pi}{6}$

(e) $\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$

(f) $\tan\frac{\pi}{3} + \cos\frac{\pi}{3}$

(g) $\sec\frac{\pi}{4} - \tan\frac{\pi}{3}$

(h) $\cos\frac{\pi}{4} + \cot\frac{\pi}{4}$

(i) $\left(\cos\frac{\pi}{4} - \tan\frac{\pi}{4}\right)^2$

(j) $\left(\sin\frac{\pi}{6} + \cos\frac{\pi}{6}\right)^2$

3. Find the exact value, with rational denominator where relevant.

(a) $\cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{4}$

(b) $\sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

(c) $\cos\frac{\pi}{6}\cos\frac{\pi}{3} + \sin\frac{\pi}{6}\sin\frac{\pi}{3}$

(d) $\sin^2\frac{\pi}{4} + \cos^2\frac{\pi}{4}$

(e) $\sec^2\frac{\pi}{3} + \sin^2\frac{\pi}{6}$

4. Find the exact value with rational denominator of

(a) $\sin\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos\frac{\pi}{3}$

(b) $\dfrac{\tan\frac{\pi}{4} - \tan\frac{\pi}{3}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{3}}$

5. Show that

$\cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} =$

$\sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6}$.

6. (a) Show that $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$.

(b) Which quadrant is the angle $\frac{3\pi}{4}$ in?

(c) Find the exact value of $\cos\frac{3\pi}{4}$.

7. (a) Show that $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$.
(b) Which quadrant is the angle $\frac{5\pi}{6}$ in?
(c) Find the exact value of $\sin \frac{5\pi}{6}$.

8. (a) Show that $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$.
(b) Which quadrant is the angle $\frac{7\pi}{4}$ in?
(c) Find the exact value of $\tan \frac{7\pi}{4}$.

9. (a) Show that $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$.
(b) Which quadrant is the angle $\frac{4\pi}{3}$ in?
(c) Find the exact value of $\cos \frac{4\pi}{3}$.

10. (a) Show that $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$.
(b) Which quadrant is the angle $\frac{5\pi}{3}$ in?
(c) Find the exact value of $\sin \frac{5\pi}{3}$.

11. Find the exact value of each ratio.
(a) $\tan \frac{3\pi}{4}$
(b) $\cos \frac{11\pi}{6}$
(c) $\tan \frac{2\pi}{3}$
(d) $\sin \frac{5\pi}{4}$
(e) $\tan \frac{7\pi}{6}$

12. (a) (i) Show that $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$.
(ii) Which quadrant is the angle $\frac{13\pi}{6}$ in?
(iii) Find the exact value of $\cos \frac{13\pi}{6}$.
(b) Find the exact value of
(i) $\sin \frac{9\pi}{4}$
(ii) $\tan \frac{7\pi}{3}$
(iii) $\cos \frac{11\pi}{4}$
(iv) $\tan \frac{19\pi}{6}$
(v) $\sin \frac{10\pi}{3}$

13. Solve for $0 \le x \le 2\pi$
(a) $\cos x = \frac{1}{2}$
(b) $\sin x = -\frac{1}{\sqrt{2}}$
(c) $\tan x = 1$
(d) $\tan x = \sqrt{3}$
(e) $\cos x = -\frac{\sqrt{3}}{2}$

14. Simplify
(a) $\sin(\pi - \theta)$
(b) $\tan(2\pi - x)$
(c) $\cos(\pi + \alpha)$
(d) $\cos\left(\frac{\pi}{2} - x\right)$
(e) $\tan\left(\frac{\pi}{2} - \theta\right)$

15. Simplify $1 - \sin^2\left(\frac{\pi}{2} - \theta\right)$

16. If $\tan x = -\frac{3}{4}$ and $\frac{\pi}{2} < x < \pi$, find the value of $\cos x$ and $\sin x$.

17. Solve $\tan^2 x - 1 = 0$ for $0 \le x \le 2\pi$

Circle Results

The **area** and **circumference** of a circle are useful in this section.

Area of a circle

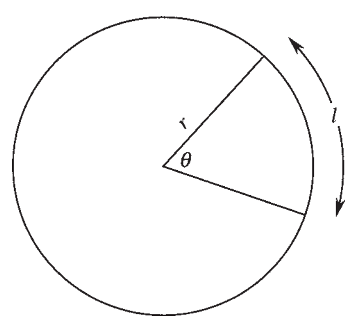
$$A = \pi r^2 \text{ where } r \text{ is the radius of the circle}$$

Circumference of a circle

$$C = 2\pi r = \pi d$$

We use equal ratios to find the other circle results. The working out is simpler when using radians instead of degrees.

Length of arc



$$l = r\theta$$
$$(\theta \text{ is in radians})$$

Proof

$$\frac{\text{arc length } l}{\text{circumference}} = \frac{\text{angle } \theta}{\text{whole revolution}}$$

$$\frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

$$\therefore \quad l = \frac{\theta 2\pi r}{2\pi}$$

$$= r\theta$$

Using 360° instead of 2π gives a different formula. Can you find it?

EXAMPLES

1. Find the length of the arc formed if an angle of $\frac{\pi}{4}$ is subtended at the centre of a circle of radius 5 m.

Solution

$$
\begin{aligned}
l &= r\theta \\
&= 5\left(\frac{\pi}{4}\right) \\
&= \frac{5\pi}{4}\text{ m}
\end{aligned}
$$

2. The area of a circle is 450 cm^2. Find, in degrees and minutes, the angle subtended at the centre of the circle by a 2.7 cm arc.

Solution

$$
\begin{aligned}
A &= \pi r^2 \\
450 &= \pi r^2 \\
\frac{450}{\pi} &= r^2 \\
\sqrt{\frac{450}{\pi}} &= r \\
11.97 &= r
\end{aligned}
$$

Now $l = r\theta$

$$
\begin{aligned}
2.7 &= 11.97\theta \\
\frac{2.7}{11.97} &= \theta \\
0.226 &= \theta
\end{aligned}
$$

$$
\begin{aligned}
\pi\text{ radians} &= 180^\circ \\
1\text{ radian} &= \frac{180^\circ}{\pi} \\
0.226\text{ radians} &= \frac{180^\circ}{\pi} \times 0.226 \\
&= 12.93^\circ \\
&= 12^\circ\, 56'
\end{aligned}
$$

So $\theta = 12^\circ\, 56'$.

5.3 Exercises

1. Find the exact arc length of a circle if
   (a) radius is 4 cm and angle subtended is π
   (b) radius is 3 m and angle subtended is $\frac{\pi}{3}$
   (c) radius is 10 cm and angle subtended is $\frac{5\pi}{6}$
   (d) radius is 3 cm and angle subtended is $30°$
   (e) radius is 7 mm and angle subtended is $45°$.

2. Find the arc length, correct to 2 decimal places, given
   (a) radius is 1.5 m and angle subtended is 0.43
   (b) radius is 3.21 cm and angle subtended is 1.22
   (c) radius is 7.2 mm and angle subtended is $55°$
   (d) radius is 5.9 cm and angle subtended is $23° 12'$
   (e) radius is 2.1 m and angle subtended is $82° 35'$.

3. The angle subtended at the centre of a circle of radius 3.4 m is $29° 51'$. Find the length of the arc cut off by this angle, correct to 1 decimal place.

4. The arc length when a sector of a circle is subtended by an angle of $\frac{\pi}{5}$ at the centre is $\frac{3\pi}{2}$ m. Find the radius of the circle.

5. The radius of a circle is 3 cm and an arc is $\frac{2\pi}{7}$ cm long. Find the angle subtended at the centre of the circle by the arc.

6. The circumference of a circle is 300 mm. Find the length of the arc that is formed by an angle of $\frac{\pi}{6}$ subtended at the centre of the circle.

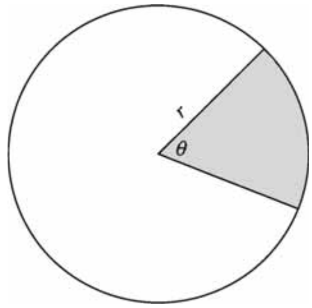
7. A circle with area 60 cm^2 has an arc 8 cm long. Find the angle that is subtended at the centre of the circle by the arc.

8. A circle with circumference 124 mm has a chord cut off it that subtends an angle of $40°$ at the centre. Find the length of the arc cut off by the chord.

9. A circle has a chord of 25 mm with an angle of $\frac{\pi}{6}$ subtended at the centre. Find, to 1 decimal place, the length of the arc cut off by the chord.

10. A sector of a circle with radius 5 cm and an angle of $\frac{\pi}{3}$ subtended at the centre is cut out of cardboard. It is then curved around to form a cone. Find its exact surface area and volume.

Area of sector



$$A = \frac{1}{2}r^2\theta$$
(θ is in radians)

Proof

$$\frac{\text{area of sector } A}{\text{area of circle}} = \frac{\text{angle } \theta}{\text{whole revolution}}$$

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\therefore \quad A = \frac{\theta \pi r^2}{2\pi}$$

$$= \frac{1}{2}r^2\theta$$

EXAMPLES

1. Find the area of the sector formed if an angle of $\frac{\pi}{4}$ is subtended at the centre of a circle of radius 5 m.

Solution

$$A = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2}(5)^2\left(\frac{\pi}{4}\right)$$

$$= \frac{25\pi}{8} \text{ m}^2$$

CONTINUED

2. The area of the sector of a circle with radius 4 cm is $\frac{6\pi}{5}$ cm^2. Find the angle, in degrees, that is subtended at the centre of the circle.

Solution

$$
\begin{aligned}
A &= \frac{1}{2}r^2\theta \\
\frac{6\pi}{5} &= \frac{1}{2}(4)^2\theta \\
&= 8\theta \\
\frac{6\pi}{40} &= \theta \\
\frac{3\pi}{20} &= \theta \\
\frac{3(180^\circ)}{20} &= \theta \\
27^\circ &= \theta
\end{aligned}
$$

Remember that $\pi = 180°$.

5.4 Exercises

1. Find the exact area of the sector of a circle if
   (a) radius is 4 cm and angle subtended is π
   (b) radius is 3 m and angle subtended is $\frac{\pi}{3}$
   (c) radius is 10 cm and angle subtended is $\frac{5\pi}{6}$
   (d) radius is 3 cm and angle subtended is 30°
   (e) radius is 7 mm and angle subtended is 45°.

2. Find the area of the sector, correct to 2 decimal places, given
   (a) radius is 1.5 m and angle subtended is 0.43
   (b) radius is 3.21 cm and angle subtended is 1.22
   (c) radius is 7.2 mm and angle subtended is 55°
   (d) radius is 5.9 cm and angle subtended is 23° 12′
   (e) radius is 2.1 m and angle subtended is 82° 35′.

3. Find the area, correct to 3 significant figures, of the sector of a circle with radius 4.3 m and an angle of 1.8 subtended at the centre.

4. The area of a sector of a circle is 20 cm^2. If the radius of the circle is 3 cm, find the angle subtended at the centre of the circle by the sector.

5. The area of the sector of a circle that is subtended by an angle of $\frac{\pi}{3}$ at the centre is 6π m^2. Find the radius of the circle.

6. Find the
   (a) arc length
   (b) area of the sector of a circle with radius 7 cm if the sector is cut off by an angle of 30° subtended at the centre of the circle.

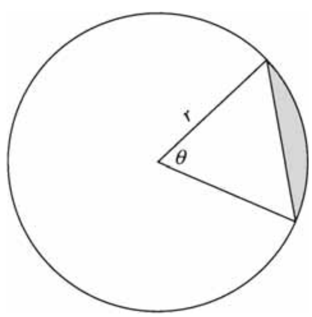
7. A circle has a circumference of 185 mm. Find the area of the sector cut off by an angle of $\frac{\pi}{5}$ subtended at the centre.

8. If the area of a circle is 200 cm^2 and a sector is cut off by an angle of $\frac{3\pi}{4}$ at the centre, find the area of the sector.

9. Find the area of the sector of a circle with radius 5.7 cm if the length of the arc formed by this sector is 4.2 cm.

10. The area of a sector is $\frac{3\pi}{10}$ cm^2 and the arc length cut off by the sector is $\frac{\pi}{5}$ cm. Find the angle subtended at the centre of the circle and find the radius of the circle.

Area of minor segment



$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$

(θ is in radians)

Proof

Area of minor segment = area of sector – area of triangle

$$= \frac{1}{2}r^2\,\theta - \frac{1}{2}r^2 \sin\,\theta$$

$$= \frac{1}{2}r^2(\theta - \sin\theta)$$

The area of the triangle is given by $A = \frac{1}{2}ab \sin C$.

EXAMPLES

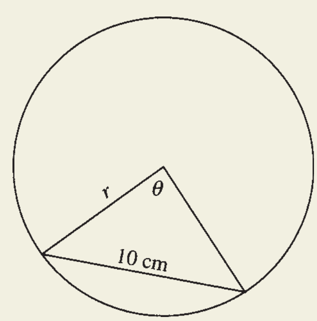
1. Find the area of the minor segment formed if an angle of $\frac{\pi}{4}$ is subtended at the centre of a circle of radius 5 m.

Solution

$$\begin{aligned}
A &= \frac{1}{2}r^2(\theta - \sin\theta)\\
&= \frac{1}{2}(5)^2\left(\frac{\pi}{4} - \sin\frac{\pi}{4}\right)\\
&= \frac{25}{2}\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)\\
&= \frac{25\pi}{8} - \frac{25}{2\sqrt{2}}\\
&= \frac{25\pi}{8} - \frac{25\sqrt{2}}{4}\\
&= \frac{25\pi - 50\sqrt{2}}{8}\text{ m}^2
\end{aligned}$$

2. An 80 cm piece of wire is bent into a circle shape, and a 10 cm piece of wire is joined to it to form a chord. Find the area of the minor segment cut off by the chord, correct to 2 decimal places.

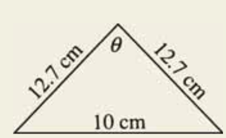
Solution



$C = 2\pi r = 80$

$$\begin{aligned}
\therefore \quad r &= \frac{80}{2\pi}\\
&= \frac{40}{\pi}\\
&\doteqdot 12.7
\end{aligned}$$

Don't round off the radius. Use the full value in your calculator's display to find θ.



$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\theta = \frac{12.7^2 + 12.7^2 - 10^2}{2(12.7)(12.7)}$$

$$= 0.69$$

$$\therefore \quad \theta = 0.807$$

$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$

$$= \frac{1}{2}(12.7)^2(0.807 - \sin 0.807)$$

$$\doteqdot 6.88 \text{ cm}^2$$

5.5 Exercises

1. Find the exact area of the minor segment of a circle if
(a) radius is 4 cm and the angle subtended is π
(b) radius is 3 m and the angle subtended is $\frac{\pi}{3}$
(c) radius is 10 cm and the angle subtended is $\frac{5\pi}{6}$
(d) radius is 3 cm and the angle subtended is 30°
(e) radius is 7 mm and the angle subtended is 45°.

2. Find the area of the minor segment correct to 2 decimal places, given
(a) radius is 1.5 m and the angle subtended is 0.43
(b) radius is 3.21 cm and the angle subtended is 1.22
(c) radius is 7.2 mm and the angle subtended is 55°
(d) radius is 5.9 cm and the angle subtended is 23° 12′
(e) radius is 2.1 m and the angle subtended is 82° 35′.

3. Find the area of the minor segment formed by an angle of 40° subtended at the centre of a circle with radius 2.82 cm, correct to 2 significant figures.

4. Find the
(a) exact arc length
(b) exact area of the sector
(c) area of the minor segment, to 2 decimal places

if an angle of $\frac{\pi}{7}$ is subtended at the centre of a circle with radius 3 cm.

5. The area of the minor segment cut off by an angle of $\frac{2\pi}{9}$ is 500 cm^2. Find the radius of the circle, correct to 1 decimal place.

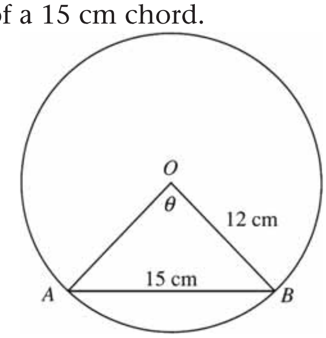
6. Find the
(a) length of the chord, to 1 decimal place
(b) length of the arc
(c) area of the minor segment, to 2 decimal places

cut off by an angle of $\frac{\pi}{6}$ subtended at the centre of a circle with radius 5 cm.

7. A chord 8 mm long is formed by an angle of 45° subtended at the centre of a circle. Find
(a) the radius of the circle
(b) the area of the minor segment cut off by the angle, correct to 1 decimal place.

8. An angle of $\frac{\pi}{8}$ is subtended at the centre of a circle. If this angle cuts off an arc of $\frac{5\pi}{4}$ cm, find
(a) the exact area of the sector
(b) the area of the minor segment formed, correct to 1 decimal place.

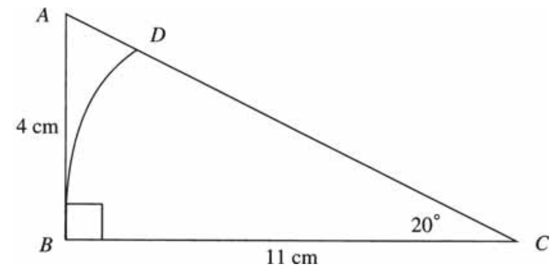
9. A triangle OAB is formed where O is the centre of a circle of radius 12 cm, and A and B are endpoints of a 15 cm chord.



(a) Find the angle (θ) subtended at the centre of the circle, in degrees and minutes.
(b) Find the area of ΔOAB, correct to 1 decimal place.
(c) Find the area of the minor segment cut off by the chord, correct to 2 decimal places.
(d) Find the area of the major segment cut off by the chord, correct to 2 decimal places.

10. The length of an arc is 8.9 cm and the area of the sector is 24.3 cm^2 when an angle of θ is subtended at the centre of a circle. Find the area of the minor segment cut off by θ, correct to 1 decimal place.

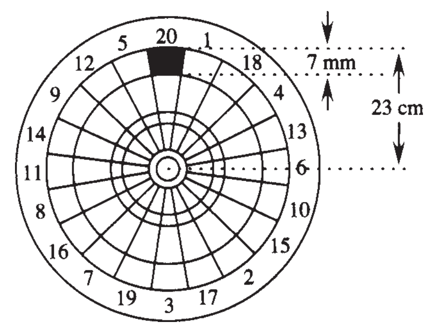
11.



An arc, centre C, cuts side AC in D.

Find the exact
(a) length of arc BD
(b) area of ABD
(c) perimeter of BDC

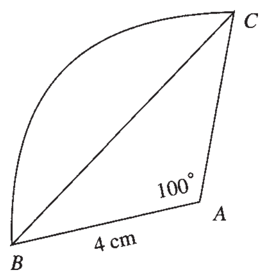
12.



The dartboard above has a radius of 23 cm. The shaded area has a height of 7 mm. If a player must hit the shaded area to win, find
(a) the area of the shaded part, to the nearest square centimetre
(b) the percentage of the shaded part in relation to the whole area of the dartboard, to 1 decimal place
(c) the perimeter of the shaded area.

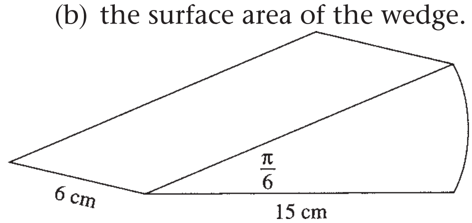
13. The hour hand of a clock is 12 cm long. Find the exact
(a) length of the arc through which the hand would turn in 5 hours
(b) area through which the hand would pass in 2 hours.

14.



Arc *BC* subtends an angle of 100° at the centre *A* of a circle with radius 4 cm. Find
(a) the exact perimeter of sector *ABC*
(b) the approximate ratio of the area of the minor segment to the area of the sector.

15. A wedge is cut so that its cross-sectional area is a sector of a circle, radius 15 cm and subtending an angle of $\frac{\pi}{6}$ at the centre. Find
(a) the volume of the wedge
(b) the surface area of the wedge.



Small Angles

When we use **radians** of small angles, there are some interesting results.

EXAMPLES

1. Find sin 0.0023.

Solution

Make sure your calculator is in radian mode
$\sin 0.0023 = 0.002299997972$

2. Find tan 0.0023.

Solution

$\tan 0.0023 = 0.002300004056$

3. Find cos 0.0023.

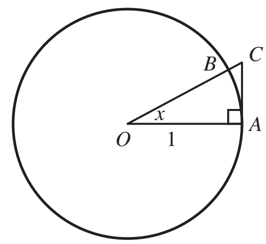
Solution

$\cos 0.0023 = 0.999997355$

Use radians for the small angles. You could use a computer spreadsheet to find these trigonometric ratios.

Class Investigation

Use the calculator to explore other small angles and their trigonometric ratios. What do you notice?



Can you see why?

In ΔOAC, $OC \doteqdot 1$ when x is small.

Arc $AB = x$ (definition of a radian)

$\therefore \quad AC \doteqdot x$ when x is small

$$\begin{aligned} \sin x &= \frac{AC}{OC} \\ &\doteqdot \frac{x}{1} \\ &= x \end{aligned}$$

$$\begin{aligned} \cos x &= \frac{OA}{OC} \\ &\doteqdot \frac{1}{1} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \tan x &= \frac{AC}{OA} \\ &\doteqdot \frac{x}{1} \\ &= x \end{aligned}$$

If x (radians) is a small angle then

$$\sin x \doteqdot x$$
$$\tan x \doteqdot x$$
$$\cos x \doteqdot 1$$
and $\sin x < x < \tan x$

Proof

$$\begin{aligned}
\text{Area } \Delta OAB &= \frac{1}{2}ab\sin C \\
&= \frac{1}{2} \times 1 \times 1 \times \sin x \\
&= \frac{1}{2}\sin x \\
\text{Area sector } OAB &= \frac{1}{2}r^2 \\
&= \frac{1}{2} \times 1^2 \times x \\
&= \frac{1}{2}x \\
\text{Area } \Delta OAC &= \frac{1}{2}bh \\
&= \frac{1}{2} \times 1 \times AC \\
&= \frac{1}{2} \times 1 \times \tan x \qquad \left(\tan x = \frac{AC}{1}\right) \\
&= \frac{1}{2}\tan x
\end{aligned}$$

Area $\Delta OAB <$ area sector $OAB <$ area ΔOAC

$\therefore \frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x$

$\therefore \quad \sin x < x < \tan x$

Class Investigation

Check that $\sin x < x < \tan x$ for $0 < x < \frac{\pi}{2}$ by using your calculator.

Remember that x is in radians. Does this work for $x > \frac{\pi}{2}$?

These results give the following result:

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

Also $\lim_{x \to 0} \frac{\tan x}{x} = 1$. Can you see why?

Proof

As $x \to 0$, $\sin x \to x$

$\therefore \frac{\sin x}{x} \to 1$

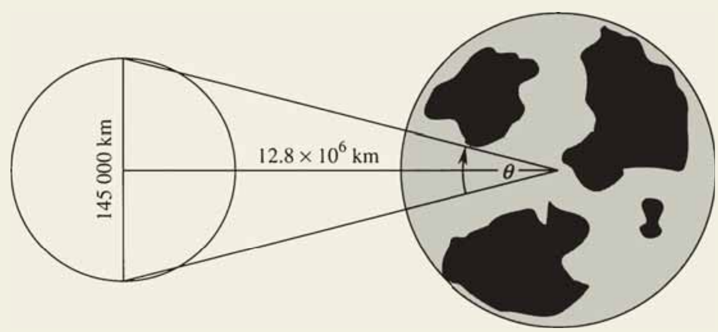
EXAMPLES

1. Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 7x}{x}$.

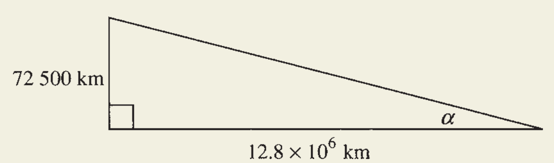
Solution

$$
\begin{aligned}
\lim_{x \to 0} \frac{\sin 7x}{x} &= \lim_{x \to 0} \frac{7(\sin 7x)}{7x} \\
&= 7 \lim_{x \to 0} \frac{\sin 7x}{7x} \\
&= 7(1) \\
&= 7
\end{aligned}
$$

2. The planet of Alpha Zeta in a faraway galaxy has a moon with a diameter of 145 000 km. If the distance between the centre of Alpha Zeta and the centre of its moon is 12.8×10^6 km, find the angle subtended by the moon at the centre of the planet, in seconds.



Solution



$$
\begin{aligned}
r &= \frac{1}{2} \times 145\,000 \\
&= 72\,500
\end{aligned}
$$

The radius of the moon is 72 500 km.

$$
\begin{aligned}
\alpha &= \frac{1}{2}\theta \\
\tan \alpha &= \frac{72\,500}{12.8 \times 10^6} \\
&= 0.00566 \\
\therefore \quad \alpha &= 0.00566 \\
\pi &= 180^\circ \\
\therefore \quad 1 &= \frac{180^\circ}{\pi} \\
0.00566 &= \frac{180^\circ}{\pi} \times 0.00566 \\
&= 0.3245^\circ \\
\therefore \quad \theta &= 0.649^\circ \\
&= 39'
\end{aligned}
$$

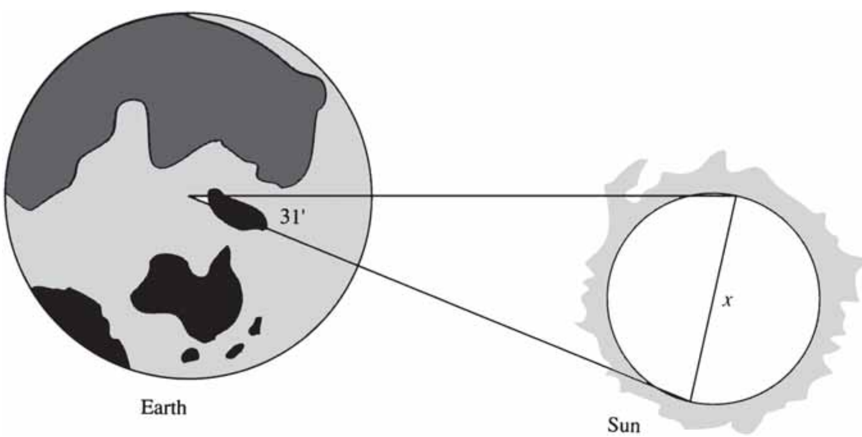
5.6 Exercises

1. Evaluate, correct to 3 decimal places.
   (a) sin 0.045
   (b) tan 0.003
   (c) cos 0.042
   (d) sin 0.065
   (e) tan 0.005

2. Evaluate $\lim_{x \to 0} \frac{\sin x}{4x}$.

3. Find $\lim_{\theta \to 0} \frac{\tan \frac{\theta}{3}}{\theta}$.

4. Find the diameter of the sun to the nearest kilometre if its distance from the Earth is 149000000 km and it subtends an angle of 31′ at the Earth.



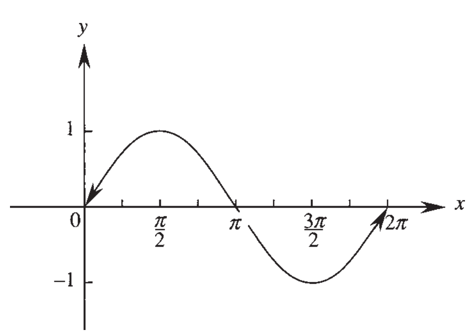
5. Given that the wingspan of an aeroplane is 30 m, find the plane's altitude to the nearest metre if the wingspan subtends an angle of 14′ when it is directly overhead.

Trigonometric Graphs

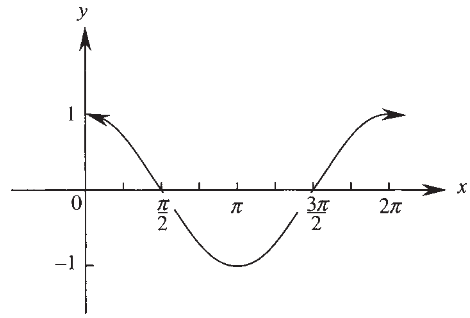
You drew the graphs of trigonometric functions in the Preliminary Course, using degrees.

You can also draw these graphs using radians.

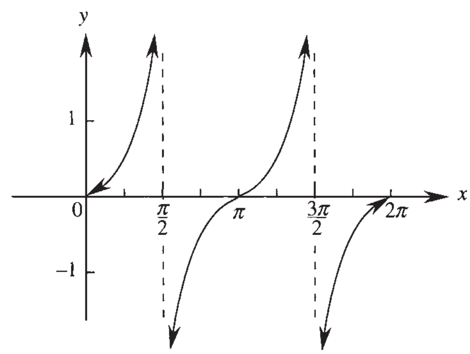
y* = sin *x



$y = \cos x$



$y = \tan x$



$y = \tan x$ has asymptotes at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ since $\tan x$ is undefined at those points. By finding values for x on each side of the asymptotes, we can see where the curve goes.

We can also sketch the graphs of the reciprocal trigonometric functions, by finding the reciprocals of $\sin x$, $\cos x$ and $\tan x$ at important points on the graphs.

Investigation

1. Use $\operatorname{cosec} x = \frac{1}{\sin x}$, $\sec x = \frac{1}{\cos x}$ and $\cot x = \frac{1}{\tan x}$ to complete the table below.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
cosec x					
sec x					
cot x					

e.g.
$$\begin{aligned}\operatorname{cosec} \frac{\pi}{2} &= \frac{1}{\sin \frac{\pi}{2}} \\ &= \frac{1}{1} \\ &= 1\end{aligned}$$

2. Find any asymptotes

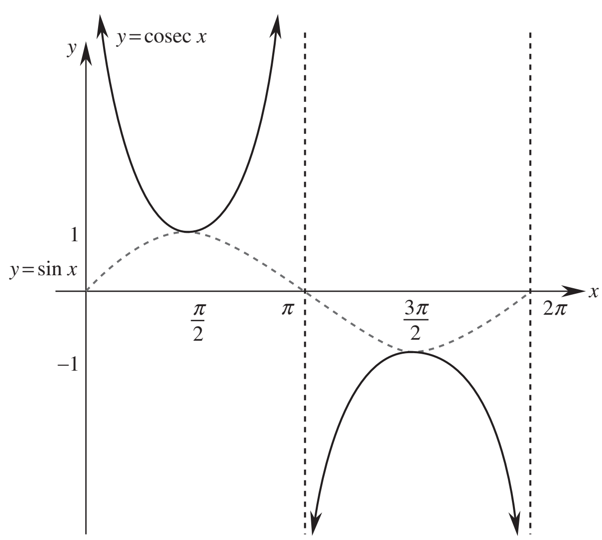
e.g. $\sec\frac{\pi}{2} = \frac{1}{\cos\frac{\pi}{2}}$

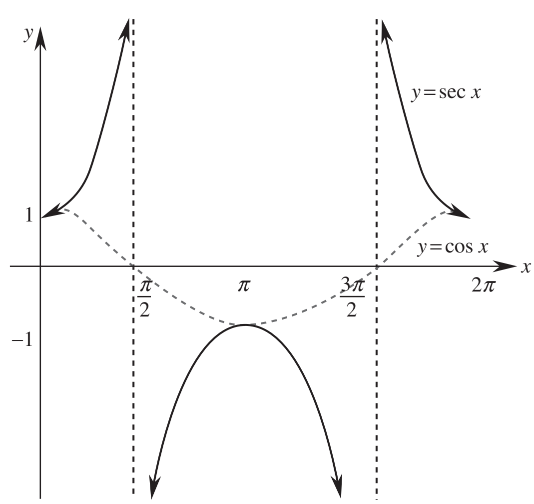
$= \frac{1}{0}$ (undefined)

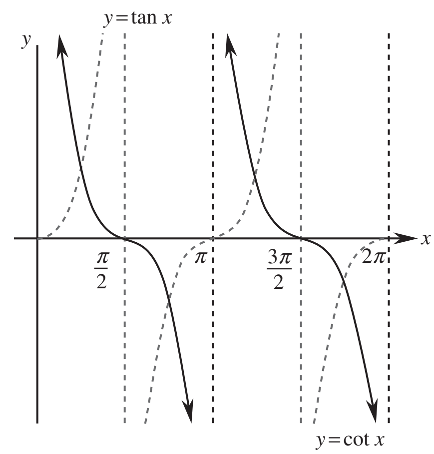
Discover what the values are on either side of the asymptotes.

3. Sketch each graph of the reciprocal trigonometric functions.

Here are the graphs of the reciprocal ratios.





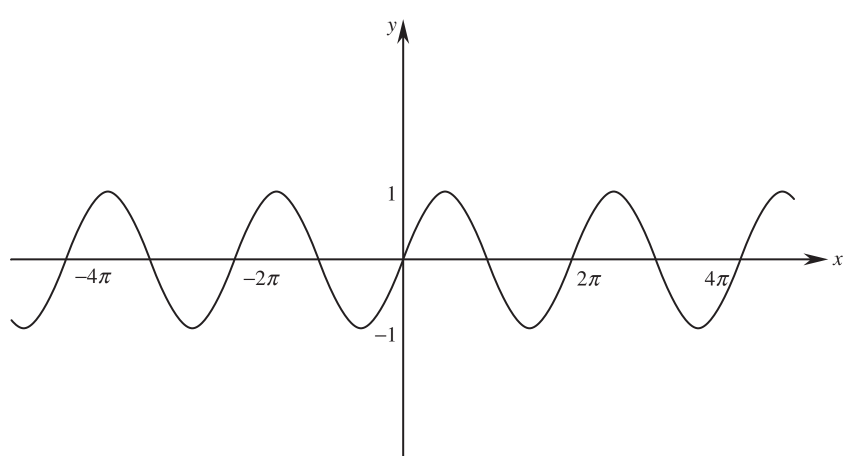


We have sketched these functions in the domain $0 \le x \le 2\pi$ which gives all the angles in one revolution of a circle. However, we could turn around the circle again and have angles greater than 2π, as well as negative angles.

These graphs all repeat at regular intervals. They are called periodic functions.

Investigation

Here is a general sine function. Notice that the shape that occurs between 0 and 2π repeats as shown.



1. Draw a general cosine curve. How are the sine and cosine curves related? Do these curves have symmetry?
2. Draw a general tangent curve. Does it repeat at the same intervals as the sine and cosine curves?
3. Look at the reciprocal trigonometric curves and see if they repeat in a similar way.

4. Use a graphics calculator or a graphing computer program to draw the graphs of trigonometric functions. Choose different values of k to sketch these families of trig functions. (Don't forget to look at values where k is a fraction or negative.)
   (a) $f(x) = k \sin x$
   (b) $f(x) = k \cos x$
   (c) $f(x) = k \tan x$
   (d) $f(x) = \sin kx$
   (e) $f(x) = \cos kx$
   (f) $f(x) = \tan kx$
   (g) $f(x) = \sin x + k$
   (h) $f(x) = \cos x + k$
   (i) $f(x) = \tan x + k$
   (j) $f(x) = \sin (x + k)$
   (k) $f(x) = \cos (x + k)$
   (l) $f(x) = \tan (x + k)$

Can you see patterns in these families?
Could you predict what the graph of $f(x) = a \sin bx$ looks like?

The trig functions have period and amplitude.

The **period** is the distance over which the curve moves along the x-axis before it repeats.

The **amplitude** is the maximum distance that the graph stretches out from the centre of the graph on the y-axis.

$y = \sin x$ has amplitude 1 and period 2π
$y = \cos x$ has amplitude 1 and period 2π
$y = \tan x$ has no amplitude and period π

$y = a \sin bx$ has amplitude a and period $\dfrac{2\pi}{b}$

$y = a \cos bx$ has amplitude a and period $\dfrac{2\pi}{b}$

$y = a \tan bx$ has no amplitude and period $\dfrac{\pi}{b}$

EXAMPLES

1. Sketch in the domain $0 \le x \le 2\pi$.

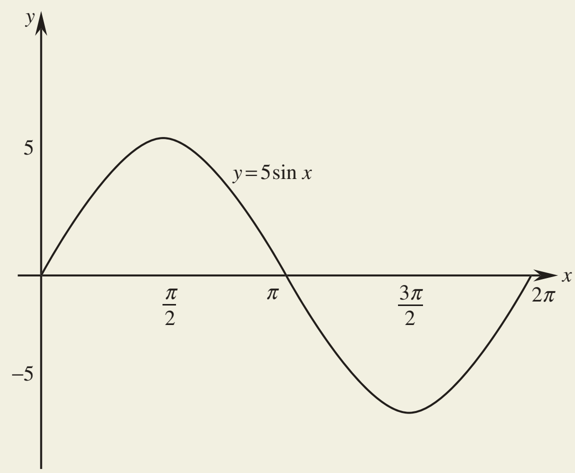
(a) $y = 5 \sin x$ for

(b) $y = \sin 4x$ for

(c) $y = 5 \sin 4x$ for

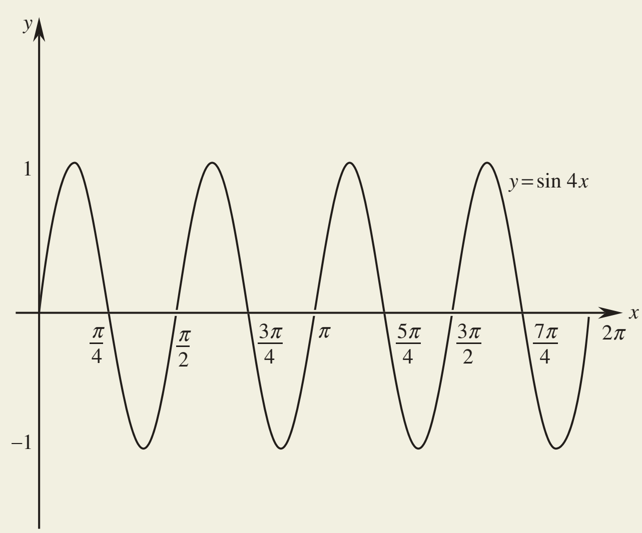
Solution

(a) The graph of $y = 5 \sin x$ has amplitude 5 and period 2π.

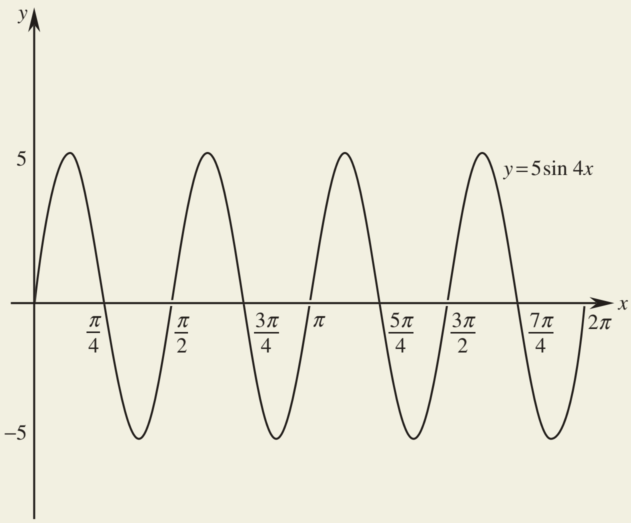


(b) The graph $y = \sin 4x$ has amplitude 1 and period $\frac{2\pi}{4}$ or $\frac{\pi}{2}$.

This means that the curve repeats every $\frac{\pi}{2}$, so in the domain $0 \le x \le 2\pi$ there will be 4 repetitions.



(c) The graph $y = 5\sin 4x$ has amplitude 5 and period $\frac{\pi}{2}$.



2. Sketch $f(x) = \sin\left(x + \frac{\pi}{2}\right)$ for $0 \le x \le 2\pi$.

Solution

Amplitude = 1

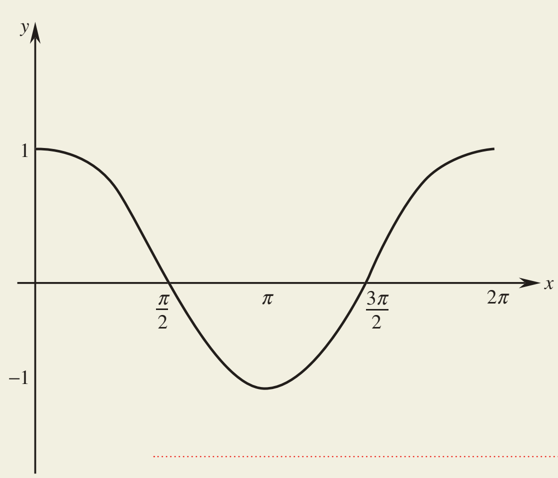
$$\text{Period} = \frac{2\pi}{1} = 2\pi$$

$f(x) = \sin(x + k)$ translates the curve k units to the left (see Investigation on page 194–5).

So $f(x) = \sin\left(x + \frac{\pi}{2}\right)$ will be moved $\frac{\pi}{2}$ units to the left. The graph is the same as $f(x) = \sin x$ but starts in a different position.

If you are not sure where the curve goes, you can draw a table of values.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	1	0	–1	0	1



This is the same as the graph $y = \cos x$.

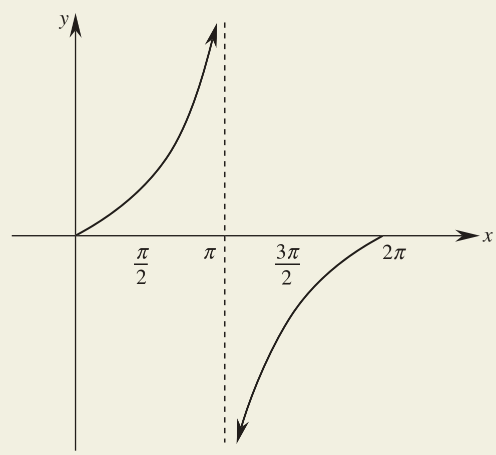
CONTINUED

3. Sketch $y = 2 \tan \frac{x}{2}$ for $0 \le x \le 2\pi$.

Solution

There is no amplitude.

$$\text{Period} = \frac{\pi}{\frac{1}{2}}$$
$$= 2\pi$$



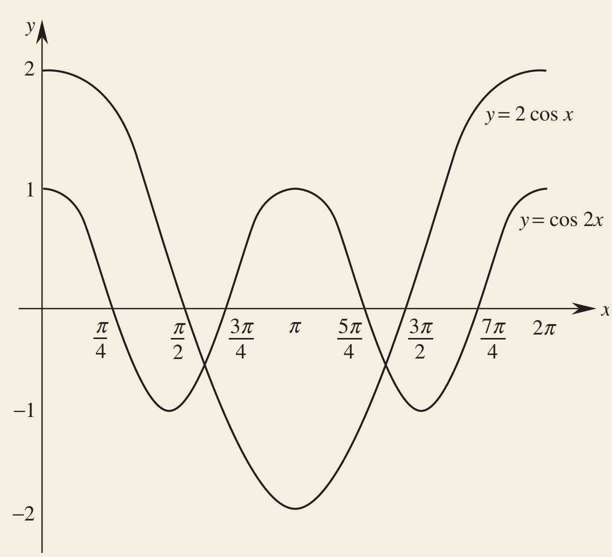
4. (a) Sketch $y = 2 \cos x$ and $y = \cos 2x$ on the same set of axes for $0 \le x \le 2\pi$.

(b) Hence or otherwise, sketch $y = \cos 2x + 2 \cos x$ for $0 \le x \le 2\pi$.

Solution

(a) $y = 2 \cos x$ has amplitude 2 and period 2π

$y = \cos 2x$ has amplitude 1 and period $\frac{2\pi}{2}$ or π



(b) Method 1: Use table

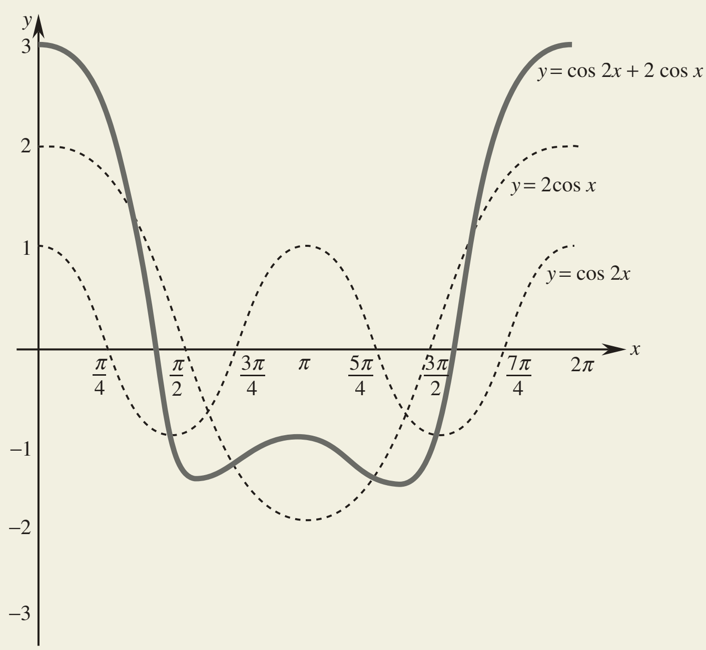
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos 2x$	1	0	-1	0	1	0	-1	0	1
$2\cos x$	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2
$\cos 2x + 2\cos x$	3	$\sqrt{2}$	-1	$-\sqrt{2}$	-1	$-\sqrt{2}$	-1	$\sqrt{2}$	3

Notice that the period of this graph is 2π.

Method 2: Add y values together on the graph itself

e.g. when $x = 0$:

$$y = 1 + 2$$
$$= 3$$



These graphs can be sketched more accurately on a graphics calculator or in a computer program such as Autograph.

5.7 Exercises

1. Sketch for $0 \le x \le 2\pi$
   (a) $y = \cos x$
   (b) $f(x) = 2\sin x$
   (c) $y = 1 + \sin x$
   (d) $y = 2 - \sin x$
   (e) $f(x) = -3\cos x$
   (f) $y = 4\sin x$
   (g) $f(x) = \cos x + 3$
   (h) $y = 5\tan x$
   (i) $f(x) = \tan x + 3$
   (j) $y = 1 - 2\tan x$

2. Sketch for $0 \le x \le 2\pi$
   (a) $y = \cos 2x$
   (b) $y = \tan 2x$
   (c) $y = \sin 3x$
   (d) $f(x) = 3\cos 4x$
   (e) $y = 6\cos 3x$
   (f) $y = \tan \frac{x}{2}$
   (g) $f(x) = 2\tan 3x$
   (h) $y = 3\cos \frac{x}{2}$
   (i) $y = 2\sin \frac{x}{2}$
   (j) $f(x) = 4\cos \frac{x}{4}$

3. Sketch for $-\pi \le x \le \pi$
(a) $y = \sin 2x$
(b) $y = 7\cos 4x$
(c) $f(x) = \tan 4x$
(d) $y = 5\sin 4x$
(e) $f(x) = 2\cos 2x$

4. Sketch $y = 8\sin\frac{x}{2}$ in the domain $0 \le x \le 4\pi$.

5. Sketch for $0 \le x \le 2\pi$
(a) $y = \sin(x + \pi)$
(b) $y = \tan\left(x + \frac{\pi}{2}\right)$
(c) $f(x) = \cos(x - \pi)$
(d) $y = 3\sin\left(x - \frac{\pi}{2}\right)$
(e) $f(x) = 2\cos\left(x + \frac{\pi}{2}\right)$
(f) $y = 4\sin\left(2x + \frac{\pi}{2}\right)$
(g) $y = \cos\left(x - \frac{\pi}{4}\right)$
(h) $y = \tan\left(x + \frac{\pi}{4}\right)$
(i) $f(x) = 2\cos\left(x + \frac{\pi}{2}\right) + 1$
(j) $y = 2 - 3\sin\left(x - \frac{\pi}{2}\right)$

6. Sketch for $-2 \le x \le 2$
(a) $y = \sin \pi x$
(b) $y = 3\cos 2\pi x$

7. Sketch in the domain $0 \le x \le 2\pi$
(a) $y = \sin x$ and $y = \sin 2x$ on the same set of axes
(b) $y = \sin x + \sin 2x$

8. Sketch for $0 \le x \le 2\pi$
(a) $y = 2\cos x$ and $y = 3\sin x$ on the same set of axes
(b) $y = 2\cos x + 3\sin x$

9. By sketching $y = \cos x$ and $y = \cos 2x$ on the same set of axes for $0 \le x \le 2\pi$, sketch the graph of $y = \cos 2x - \cos x$.

10. Sketch the graph of
(a) $y = \cos x + \sin x$
(b) $y = \sin 2x - \sin x$
(c) $y = \sin x + 2\cos 2x$
(d) $y = 3\cos x - \cos 2x$
(e) $y = \sin x - \sin\frac{x}{2}$

Applications

The sine and cosine curves are used in many applications including the study of waves. There are many different types of waves, including water, light and sound waves. Oscilloscopes display patterns of electrical waves on the screen of a cathode-ray tube.

Search waves and the oscilloscope on the Internet for further information.

Simple harmonic motion (such as the movement of a pendulum) is a wave-like or oscillatory motion when graphed against time. In 1581, when he was 17 years old, Gallileo noticed a lamp swinging backwards and forwards in Pisa cathedral. He found that the lamp took the same time to swing to and fro, no matter how much weight it had on it. This led him to discover the pendulum.

Gallileo also invented the telescope. Find out more information about Gallileo's life and his other discoveries.

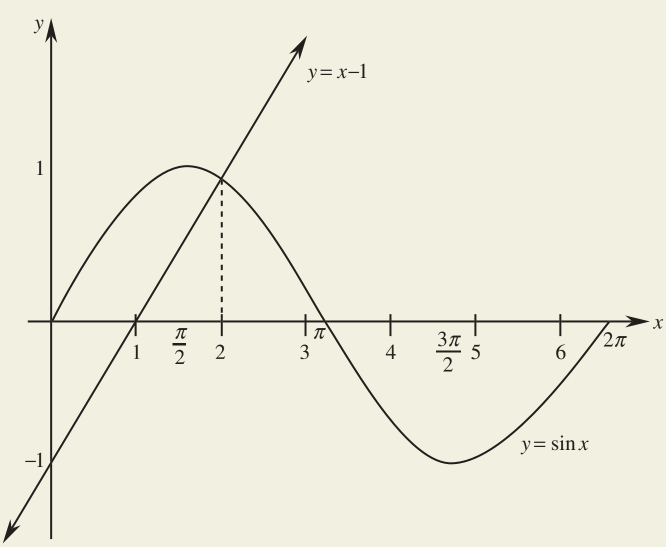
Some trigonometric equations are difficult to solve algebraically, but can be solved graphically by finding the points of intersection between two graphs.

EXAMPLES

1. Find approximate solutions to the equation $\sin x = x - 1$ by sketching the graphs $y = \sin x$ and $y = x - 1$ on a Cartesian plane.

Solution

When sketching the two graphs together, we use $\pi \approx 3.14$, $2\pi \approx 6.28$ and so on to label the x-axis as shown.
To sketch $y = x - 1$, find the gradient and y-intercept or find the x- and y-intercepts.
x-intercept (where $y = 0$) is 1 and y-intercept (where $x = 0$) is –1.



The solution to $\sin x = x - 1$ is at the point of intersection of the two graphs. $\therefore\ x \approx 2$.

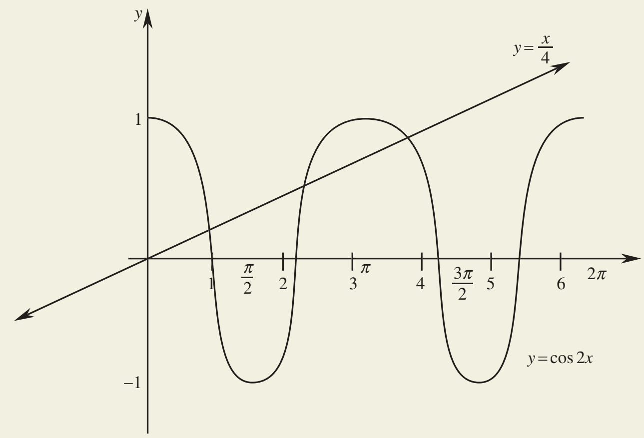
2. How many solutions are there for $\cos 2x = \dfrac{x}{4}$ in the domain $0 \le x \le 2\pi$?

Solution

Sketch $y = \cos 2x$ and $y = \dfrac{x}{4}$ on the same set of axes.
$y = \cos 2x$ has amplitude 1 and period π
$y = \dfrac{x}{4}$ has x-intercept 0 and y-intercept 0. We can find another point on the line e.g. When $x = 4$

$$y = \frac{4}{4}$$
$$= 1$$

CONTINUED



There are 3 points of intersection of the graphs, so the equation $\cos\ 2x = \frac{x}{4}$ has 3 solutions.

We can also look at applications of trigonometric graphs in real life situations.

EXAMPLE

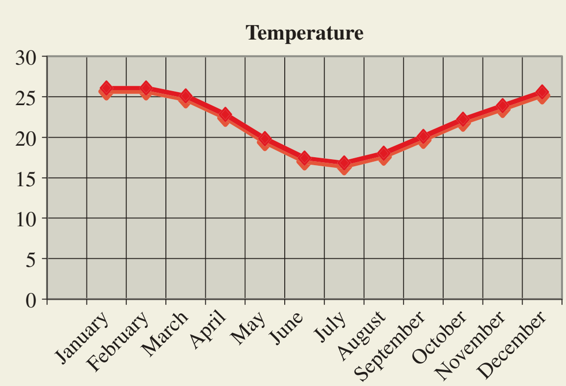
The table shows the highest average monthly temperatures in Sydney.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
°C	26.1	26.1	25.1	22.8	19.8	17.4	16.8	18.0	20.1	22.2	23.9	25.6

(a) Draw a graph of this data, by hand or on a graphics calculator or computer.
(b) Is it periodic? Why would you expect it to be periodic?
(c) What is the period and amplitude?

Solution

(a)



(b) The graph looks like it is periodic, and we would expect it to be, since the temperature varies with the seasons. It goes up and down, and reaches a maximum in summer and a minimum in winter.
(c) This curve is approximately a cosine curve with one full period, so the period is 12 months.

The maximum temperature is around 26° and the minimum is around 18°, so the centre of the graph is 22° with 4° either side. So the amplitude is 4.

To find the centre, find the average of 18 and 26 which is $\frac{18 + 26}{2}$.

5.8 Exercises

1. Show graphically that $\sin x = \frac{x}{2}$ has
   (a) 2 solutions for $0 \le x \le 2\pi$
   (b) 3 solutions for $-\pi \le x \le \pi$.

2. Solve $\sin x = x$ for $0 \le x \le 2\pi$ graphically by sketching $y = \sin x$ and $y = x$ on the same number plane.

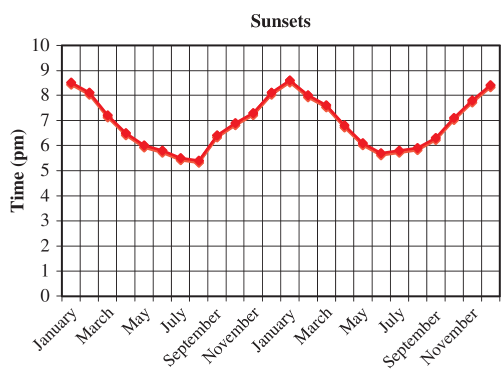
3. Solve $\cos x = 2x - 3$ for $0 \le x \le 2\pi$ by finding the points of intersection of the graphs $y = \cos x$ and $y = 2x - 3$.

4. Solve $\tan x = x$ graphically in the domain $0 \le x \le 2\pi$.

5. Solve by graphical means $\sin 2x = x$ for $0 \le x \le 2\pi$.

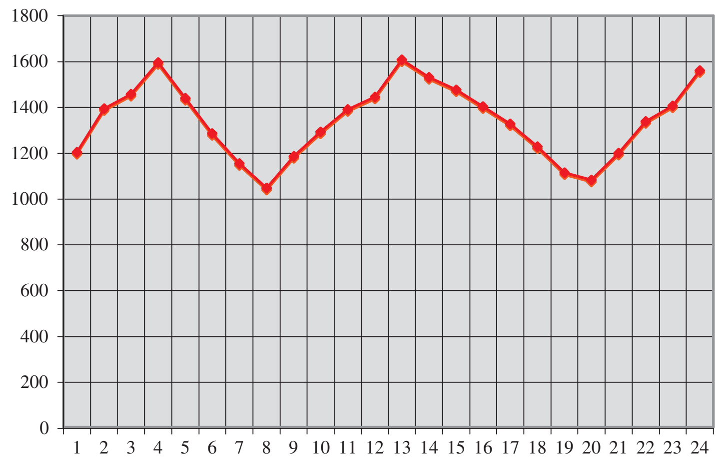
6. Draw the graphs of $y = \sin x$ and $y = \cos x$ for $0 \le x \le 2\pi$ on the same set of axes. Use your graphs to solve the equation $\sin x = \cos x$ for $0 \le x \le 2\pi$.

7. The graph below show the times of sunsets in a city over a period of 2 years.



   (a) Find the period and amplitude of the graph.
   (b) At approximately what time would you expect the sun to set in July?

8. The graph shows the incidence of crimes committed over 24 years in Gotham City.



(a) Approximately how many crimes were committed in the 10th year?
(b) What was the
 (i) highest and
 (ii) lowest number of crimes?
(c) Find the amplitude and the period of the graph.

9. Below is a table showing the average daylight hours over several months.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug
Daylight hours	15.3	14.7	13.2	13.1	12.7	12.2	12.5	13.8

(a) Draw a graph to show this data.
(b) Is it periodic? If so, what is the period?
(c) Find the amplitude.

10. The table below shows the high and low tides over a three-day period.

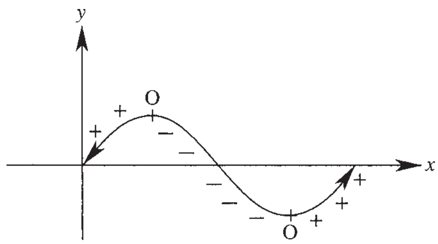
Day	Friday				Saturday				Sunday			
Time	6.20 am	11.55 am	6.15 pm	11.48 pm	6.20 am	11.55 am	6.15 pm	11.48 pm	6.20 am	11.55 am	6.15 pm	11.48 pm
Tide (m)	3.2	1.1	3.4	1.3	3.2	1.2	3.5	1.1	3.4	1.2	3.5	1.3

(a) Draw a graph showing the tides.
(b) Find the period and amplitude.
(c) Estimate the height of the tide at around 8 am on Friday.

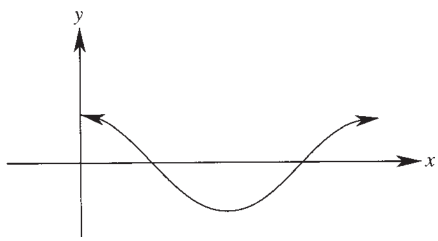
Differentiation of Trigonometric Functions

Derivative of sin *x*

We can sketch the gradient (tangent) function of $y = \sin x$.



The gradient function is $y = \cos x$.



$$\frac{d}{dx}(\sin x) = \cos x$$

Proof

Let $f(x) = \sin x$

Then $f(x + h) = \sin(x + h)$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{(\sin x\cos h+\cos x\sin h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}\\
&= \lim_{h\to 0}\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\\
&= \lim_{h\to 0}\sin x(0)+\cos x(1)\left(\text{since}\cos h\to 1 \text{ as } h\to 0 \text{ and } \lim_{h\to 0}\frac{\sin h}{h}=1\right)\\
&= \cos x.
\end{aligned}$$

This result is a formula in the Extension 1 Course and you do not need to know it.

Notice that the result $\lim_{h\to 0}\frac{\sin h}{h} = 1$ only works for radians. We always use radians when using calculus.

FUNCTION OF A FUNCTION RULE

$$\frac{d}{dx}[\sin f(x)] = f'(x) \cos f(x)$$

Proof

Let $u = f(x)$

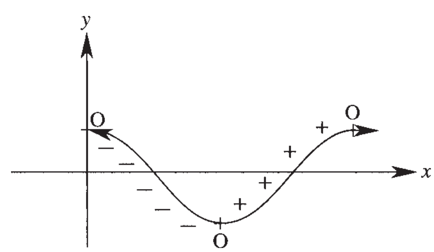
Then $y = \sin u$

$$\frac{du}{dx} = f'(x) \text{ and } \frac{dy}{du} = \cos u$$

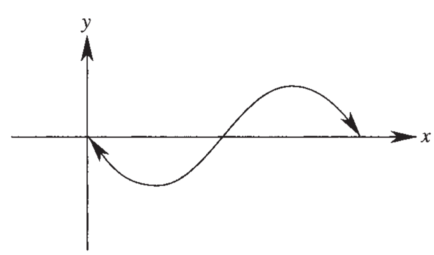
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \cos u \cdot f'(x) \\ &= f'(x) \cos f(x)\end{aligned}$$

Derivative of cos *x*

We can sketch the gradient function of $y = \cos x$.



The gradient function is $y = -\sin x$.



$$\frac{d}{dx}(\cos x) = -\sin x$$

Proof

Since $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

$$\begin{aligned}\frac{d}{dx}(\cos x) &= \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} - x\right)\right] \\ &= -1 \times \cos\left(\frac{\pi}{2} - x\right) \\ &= -\sin x\end{aligned}$$

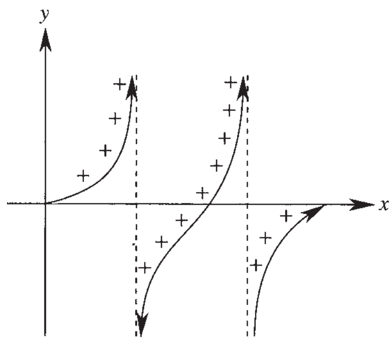
FUNCTION OF A FUNCTION RULE

$$\frac{d}{dx}[\cos f(x)] = -f'(x)\sin f(x)$$

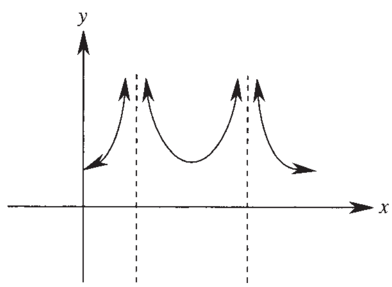
This result can be proven the same way as for sin f(x).

Derivative of tan x

The gradient function is harder to sketch for $y = \tan x$.



The gradient function is $y = \sec^2 x$.



It is easier to see the rule for differentiating $y = \tan x$ by using $\tan x = \frac{\sin x}{\cos x}$ and the quotient rule.

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Proof

$$\tan x = \frac{\sin x}{\cos x}$$

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{u'v - v'u}{v^2}\\ &= \frac{\cos x(\cos x) - (-\sin x)\sin x}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x\end{aligned}$$

FUNCTION OF A FUNCTION RULE

This result can be proven the same way as for sin f(x).

$$\frac{d}{dx}[\tan f(x)] = f'(x)\sec^2 f(x).$$

EXAMPLES

1. Differentiate $\sin(5x)$.

Solution

$$\frac{d}{dx}[\sin f(x)] = f'(x)\cos f(x)$$

$$\therefore \frac{d}{dx}[\sin(5x)] = 5\cos(5x)$$

2. Differentiate $\sin x°$.

Solution

$$\begin{aligned}\frac{d}{dx}(\sin x°) &= \frac{d}{dx}\left(\sin\frac{\pi x}{180}\right)\\ &= \frac{\pi}{180}\cos\frac{\pi x}{180}\\ &= \frac{\pi}{180}\cos x°\end{aligned}$$

3. Find the exact value of the gradient of the tangent to the curve $y = x^2\sin x$ at the point where $x = \frac{\pi}{4}$.

Solution

$$\begin{aligned}\frac{dy}{dx} &= u'v + v'u\\ &= \cos x\,(x^2) + 2x\,(\sin x)\\ &= x^2\cos x + 2x\sin x\end{aligned}$$

When $x = \frac{\pi}{4}$,

$$\begin{aligned}\frac{dy}{dx} &= \left(\frac{\pi}{4}\right)^2\cos\frac{\pi}{4} + 2\left(\frac{\pi}{4}\right)\sin\frac{\pi}{4}\\ &= \frac{\pi^2}{16}\times\frac{1}{\sqrt{2}} + \frac{\pi}{2}\times\frac{1}{\sqrt{2}}\\ &= \frac{\pi^2}{16\sqrt{2}} + \frac{\pi}{2\sqrt{2}}\\ &= \frac{\pi^2 + 8\pi}{16\sqrt{2}}\\ &= \frac{\pi\sqrt{2}\,(\pi + 8)}{32}\end{aligned}$$

5.9 Exercises

1. Differentiate
   (a) $\sin 4x$
   (b) $\cos 3x$
   (c) $\tan 5x$
   (d) $\tan(3x+1)$
   (e) $\cos(-x)$
   (f) $3\sin x$
   (g) $4\cos(5x-3)$
   (h) $2\cos(x^3)$
   (i) $7\tan(x^2+5)$
   (j) $\sin 3x + \cos 8x$
   (k) $\tan(\pi + x) + x^2$
   (l) $x\tan x$
   (m) $\sin 2x \tan 3x$
   (n) $\dfrac{\sin x}{2x}$
   (o) $\dfrac{3x+4}{\sin 5x}$
   (p) $(2x + \tan 7x)^9$
   (q) $\sin^2 x$
   (r) $3\cos^3 5x$
   (s) $e^x - \cos 2x$
   (t) $\sin(1 - \log_e x)$
   (u) $\sin(e^x + x)$
   (v) $\log_e(\sin x)$
   (w) $e^{3x}\cos 2x$
   (x) $\dfrac{e^{2x}}{\tan 7x}$
2. Find the derivative of $\cos x \sin^4 x$.
3. Find the gradient of the tangent to the curve $y = \tan 3x$ at the point where $x = \dfrac{\pi}{9}$.
4. Find the equation of the tangent to the curve $y = \sin(\pi - x)$ at the point $\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right)$, in exact form.
5. Differentiate $\log_e(\cos x)$.
6. Find the exact gradient of the normal to the curve $y = \sin 3x$ at the point where $x = \dfrac{\pi}{18}$.
7. Differentiate $e^{\tan x}$.
8. Find the equation of the normal to the curve $y = 3\sin 2x$ at the point where $x = \dfrac{\pi}{8}$, in exact form.
9. Show that $\dfrac{d^2y}{dx^2} = -25y$ if $y = 2\cos 5x$.
10. Given $f(x) = -2\sin x$, show that $f''(x) = -f(x)$.
11. Show that $\dfrac{d}{dx}[\log_e(\tan x)] = \tan x + \cot x$.
12. Find the coordinates of the stationary points on the curve $y = 2\sin x - x$ for $0 \le x \le 2\pi$.
13. Differentiate
    (a) $\tan x°$
    (b) $3\cos x°$
    (c) $\dfrac{\sin x°}{5}$
14. If $y = 2\sin 3x - 5\cos 3x$, show that $\dfrac{d^2y}{dx^2} = -9y$.
15. Find values of a and b if $\dfrac{d^2y}{dx^2} = ae^{3x}\cos 4x + be^{3x}\sin 4x$, given $y = e^{3x}\cos 4x$.

Integration of Trigonometric Functions

The **integrals** of the trigonometric functions are their **primitive functions**.

$$\int \cos x\, dx = \sin x + C$$

Integration is the inverse of differentiation.

$$\int -\sin x \, dx = \cos x + C$$
$$\therefore \int \sin x \, dx = -\cos x + C$$

$$\boxed{\int \sin x \, dx = -\cos x + C}$$

$$\boxed{\int \sec^2 x \, dx = \tan x + C}$$

Function of a function rule

$$\boxed{\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C}$$

Proof

$$\frac{d}{dx}[\sin(ax+b)] = a\cos(ax+b)$$
$$\therefore \int a\cos(ax+b)\, dx = \sin(ax+b) + C$$
$$\int \cos(ax+b)\, dx = \frac{1}{a}\int a\cos(ax+b)\, dx$$
$$= \frac{1}{a}\sin(ax+b) + C$$

Similarly,

$$\boxed{\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C}$$

$$\boxed{\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + C}$$

EXAMPLES

1. Find $\int \sin(3x)\, dx$.

Solution

$$\int \sin(3x)\, dx = -\frac{1}{3}\cos(3x) + C$$

2. Evaluate $\int_0^{\frac{\pi}{2}} \sin x\, dx$.

Solution

$$\int_0^{\frac{\pi}{2}} \sin x\, dx = \Big[-\cos x\Big]_0^{\frac{\pi}{2}}$$
$$= -\cos\frac{\pi}{2} - (-\cos 0)$$
$$= 0 - (-1)$$
$$= 1$$

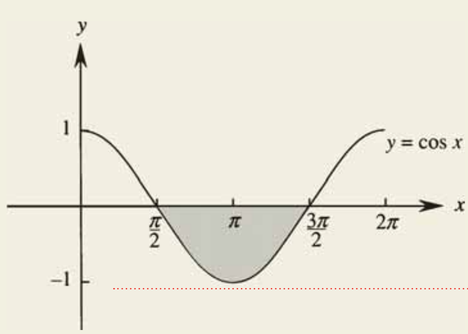
3. Find $\int \cos x°\, dx$.

Solution

$$\int \cos x°\, dx = \int \cos\frac{\pi x}{180}\, dx$$
$$= \frac{1}{\frac{\pi}{180}} \sin\frac{\pi x}{180} + C$$
$$= \frac{180}{\pi} \sin x° + C$$

4. Find the area enclosed between the curve $y = \cos x$, the x-axis and the lines $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

Solution



You studied areas in Chapter 3. The definite integral is negative as the area is below the x-axis.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\, dx = \Big[\sin x\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$
$$= \sin\frac{3\pi}{2} - \sin\frac{\pi}{2}$$
$$= -1 - 1$$
$$= -2$$

$\therefore$ area is 2 units^2.

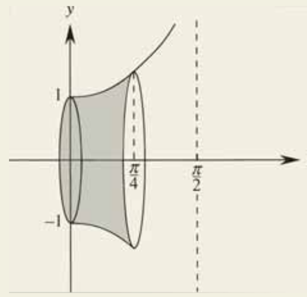
You could use

$$\text{Area} = \left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\, dx\right|$$
$$= |-2|$$
$$= 2.$$

CONTINUED

5. Find the volume of the solid formed if the curve $y = \sec x$ is rotated about the x-axis from $x = 0$ to $x = \dfrac{\pi}{4}$.

Solution



$$y = \sec x$$
$$\therefore \quad y^2 = \sec^2 x$$

$$V = \pi \int_a^b y^2\, dx$$

The volume formula comes from Chapter 3.

$$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x\, dx$$
$$= \pi \Big[\tan x\Big]_0^{\frac{\pi}{4}}$$
$$= \pi\left(\tan\frac{\pi}{4} - \tan 0\right)$$
$$= \pi(1 - 0)$$
$$= \pi$$

So the volume is π units^3.

5.10 Exercises

1. Find the indefinite integral (primitive function) of
   (a) $\cos x$
   (b) $\sin x$
   (c) $\sec^2 x$
   (d) $\dfrac{\sin x^\circ}{4}$
   (e) $\sin 3x$
   (f) $-\sin 7x$
   (g) $\sec^2 5x$
   (h) $\cos(x + 1)$
   (i) $\sin(2x - 3)$
   (j) $\cos(2x - 1)$
   (k) $\sin(\pi - x)$
   (l) $\cos(x + \pi)$
   (m) $2\sec^2 7x$
   (n) $4\sin\dfrac{x}{2}$
   (o) $3\sec^2\dfrac{x}{3}$
   (p) $-\sin(3 - x)$

2. Evaluate, giving exact answers where appropriate.
   (a) $\displaystyle\int_0^{\frac{\pi}{2}} \cos x\, dx$
   (b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x\, dx$
   (c) $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \sin\frac{x}{2}\, dx$
   (d) $\displaystyle\int_0^{\frac{\pi}{2}} \cos 3x\, dx$

(e) $\int_0^{\frac{1}{2}} \sin \pi x \, dx$

(f) $\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx$

(g) $\int_0^{\frac{\pi}{12}} 3\cos 2x \, dx$

(h) $\int_0^{\frac{\pi}{10}} -\sin 5x \, dx$

3. Find the area enclosed between the curve $y = \sin x$ and the x-axis in the domain $0 \leq x \leq 2\pi$.

4. Find the exact area bounded by the curve $y = \cos 3x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{12}$.

5. Find the area enclosed between the curve $y = \sec^2 \frac{x}{4}$, the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$, correct to 2 decimal places.

6. Find the volume, correct to 2 decimal places, of the solid formed when the curve $y = \sec \pi x$ is rotated about the x-axis from $x = 0$ to $x = 0.15$.

7. Find, in exact form, the volume of the solid of revolution formed by rotating the curve $y = \sqrt{\sin 2x}$ about the x-axis from $x = 0$ to $x = \frac{\pi}{6}$.

8. Find the exact area enclosed by the curve $y = \sin x$ and the line $y = \frac{1}{2}$ for $0 \leq x \leq 2\pi$.

9. Find the exact area bounded by the curves $y = \sin x$ and $y = \cos x$ in the domain $0 \leq x \leq 2\pi$.

10. (a) Show that the volume of the solid formed by rotating the curve $y = \sqrt{\cos x}$ about the x axis between $x = 0$ and $x = \frac{\pi}{2}$ is π units^3.

(b) Use the trapezoidal rule with 4 subintervals to find an approximation to the volume of the solid formed by rotating the curve $y = \sqrt{\cos x}$ about the x-axis from $x = 0$ to $x = \frac{\pi}{2}$.

11. A curve has $\frac{d^2y}{dx^2} = 18 \sin 3x$ and a stationary point at $\left(\frac{\pi}{6}, -2\right)$. Find the equation of the curve.

Test Yourself 5

1. A circle with radius 5 cm has an angle of $\frac{\pi}{6}$ subtended at the centre. Find
   (a) the exact arc length
   (b) the exact area of the sector
   (c) the area of the minor segment to 3 significant figures.

2. Find the exact value of
   (a) $\tan \frac{\pi}{3}$
   (b) $\cos \frac{\pi}{6}$
   (c) $\sin \frac{2\pi}{3}$
   (d) $\cos \frac{3\pi}{4}$

3. Solve for $0 \le x \le 2\pi$
   (a) $\tan x = -1$
   (b) $2 \sin x = 1$

4. Sketch for $0 \le x \le 2\pi$
   (a) $y = 3\cos 2x$
   (b) $y = 7\sin \frac{x}{2}$

5. Differentiate
   (a) $\cos x$
   (b) $2 \sin x$
   (c) $\tan x + 1$
   (d) $x \sin x$
   (e) $\frac{\tan x}{x}$
   (f) $\cos 3x$
   (g) $\tan 5x$

6. Find the indefinite integral (primitive function) of
   (a) $\sin 2x$
   (b) $3 \cos x$
   (c) $\sec^2 5x$
   (d) $1 + \sin x$

7. Evaluate
   (a) $\int_0^{\frac{\pi}{4}} \cos x \, dx$
   (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \, dx$

8. Find the equation of the tangent to the curve $y = \sin 3x$ at the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

9. If $x = \cos 2t$, show that $\frac{d^2 x}{dt^2} = -4x$.

10. Find the exact area bounded by the curve $y = \sin x$, the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.

11. Find the volume of the solid formed if the curve $y = \sec x$ is rotated about the x-axis from $x = 0$ to $x = \frac{\pi}{6}$.

12. Simplify
    (a) $\lim_{x \to 0} \frac{\sin 5x}{x}$
    (b) $\lim_{\theta \to 0} \frac{2\tan\theta}{\theta}$

13. Find the gradient of the tangent to the curve $y = 3\cos 2x$ at the point where $x = \frac{\pi}{6}$.

14. A circle has a circumference of 8π cm. If an angle of $\frac{\pi}{7}$ is subtended at the centre of the circle, find
    (a) the exact area of the sector
    (b) the area of the minor segment, to 2 decimal places.

15. (a) Sketch $y = \cos 2x$ and $y = \frac{2x}{3}$ on the same set of axes for $0 \le x \le 2\pi$.
    (b) Solve $\cos 2x = \frac{2x}{3}$ for $0 \le x \le 2\pi$.

16. Find the exact area with rational denominator bounded by the curve $y = \sin x$, the x-axis and the lines $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$.

17. Find the area bounded by the curve $y = \cos 2x$, the x-axis and the lines $x = 0$ to $x = \pi$.

18. Find the equation of the normal to the curve $y = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$.

19. A curve has $\frac{dy}{dx} = 6 \sin 2x$, and passes through the point $\left(\frac{\pi}{2}, 3\right)$. Find the equation of the curve.

20. (a) Sketch $y = 7 \sin 3x$ and $y = 2x - 1$ on the same number plane for $0 \leq x \leq 2\pi$.
(b) Solve $7 \sin 3x = 2x - 1$ for $0 \leq x \leq 2\pi$.

Challenge Exercise 5

1. Use Simpson's rule with 5 function values to find an approximation to $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \tan x \, dx$, correct to 2 decimal places.

2. Evaluate $\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2 2x \, dx$, in exact form.

3. The area of the sector of a circle is 4π units^2 and the length of the arc bounded by this sector is $\frac{\pi}{8}$ units. Find the radius of the circle and the angle that is subtended at the centre.

4. If $f(x) = 3 \cos \pi x$
(a) find the period and amplitude of the function
(b) sketch $f(x)$ for $0 \leq x \leq 4$.

5. Given $\frac{d^2y}{dx^2} = 9 \sin 3x$
(a) find y if there is a stationary point at $\left(\frac{\pi}{2}, 1\right)$
(b) Show that $\frac{d^2y}{dx^2} + 9y = 0$.

6. Sketch $y = 5 \sin (x + \pi)$ for $0 \leq x \leq 2\pi$.

7. Find the derivative of $\tan x°$.

8. (a) Show that $\sec x \operatorname{cosec} x = \frac{\sec^2 x}{\tan x}$.
(b) Hence, or otherwise, find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec} x \sec x \, dx$.

9. Differentiate $e^{x \sin 2x}$.

10. (a) Find the stationary points on the curve $y = \sin 2x + 3$ over the domain $0 \leq x \leq \pi$.
(b) What is the maximum value of the curve?
(c) What is the amplitude?

11. The area of a sector in a circle of radius 4 cm is $\frac{8\pi}{3}$ cm^2. Find the area of the minor segment, in exact form.

12. Find $\int \sin x° dx$.

13. Find the gradient of the normal to the curve $y = x^2 + \cos \pi x$ at the point where $x = 1$.

14. Use the trapezoidal rule with 4 subintervals to find, correct to 3 decimal places, an approximation to the volume of the solid formed by rotating the curve $y = \sin x$ about the x-axis from $x = 0.2$ to $x = 0.6$.

15. Differentiate $\log_e(\sin x + \cos x)$.

16. Find the exact area of the minor segment cut off by a quarter of a circle with radius 3 cm.

17. Sketch $y = \tan\left(x - \frac{\pi}{4}\right)$ for $0 \le x \le 2\pi$.

18. Find all the points of inflexion on the curve $y = 3\cos\left(2x + \frac{\pi}{4}\right)$ for $0 \le x \le 2\pi$.

19. Find the exact area bounded by the curve $y = \cos x$, the x-axis and the lines $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$.

20. If $f(x) = 2\cos 3x$, show that $f''(x) = -9f(x)$.