2

Geometrical Applications of Calculus

TERMINOLOGY

Anti-differentiation: The process of finding a primitive (original) function from the derivative. It is the inverse operation to differentiation

Concavity: The shape of a curve as it bends around (it can be concave up or concave down)

Differentiation: The process of finding the gradient of a tangent to a curve or the derivative

Gradient of a secant: The gradient (slope) of a line between two points that lie close together on a curve

Gradient of a tangent: The gradient (slope) of a line that is a tangent to a curve at a point on a function. It is the derivative of the function

Horizontal point of inflexion: A stationary point (where the first derivative is zero) where the concavity of the curve changes

Instantaneous rate of change: The derivative of a function

Maximum turning point: A local stationary point (where the first derivative is zero) and where the curve is concave down. The gradient of the tangent is zero

Minimum turning point: A local stationary point (where the first derivative is zero) and where the curve is concave up. The gradient of the tangent is zero

Monotonic increasing or decreasing function: A function is always increasing or decreasing

Point of inflexion: A point at which the curve is neither concave upwards nor downwards, but where the concavity changes

Primitive function: The original function found by working backwards from the derivative. Found by anti-differentiation

Rate of change: The rate at which the dependent variable changes as the independent variable changes

Stationary (turning) point: A local point at which the gradient of the tangent is zero and the tangent is horizontal. The first derivative is zero

INTRODUCTION

YOU LEARNED ABOUT **differentiation** in the Preliminary Course. This is the process of finding the gradient of a tangent to a curve. This chapter looks at how the gradient of a tangent can be used to describe the shape of a curve. Knowing this will enable us to sketch various curves and find their maximum and minimum values. The theory also allows us to solve various problems involving maximum and minimum values.

DID YOU KNOW?

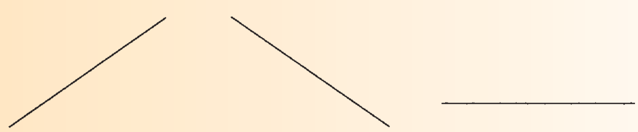
Although **Newton** and **Leibniz** are said to have discovered calculus, elements of calculus were around before then. It was Newton and Leibniz who *perfected* the processes and developed the notation associated with calculus.

Pierre de Fermat (1601–65) used coordinate geometry to find maximum and minimum values of functions. His method is very close to calculus. He also worked out a way of finding the tangent to a curve.

The 17th-century mathematicians who developed calculus knew that it worked, but it was not fully understood. Limits were not introduced into calculus until the nineteenth century.

Gradient of a Curve

To learn about the shape of a curve, we first need to revise what we know about the gradient of a tangent. The gradient (slope) of a straight line measures the rate of change of y with respect to the change in x.

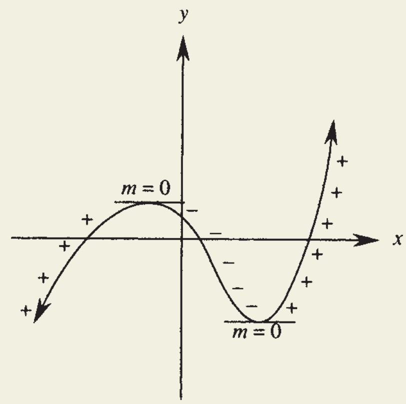


positive gradient **negative** gradient **zero** gradient

Since the gradient of a curve varies, we find the gradient of the tangent at each point along the curve.

EXAMPLES

1.



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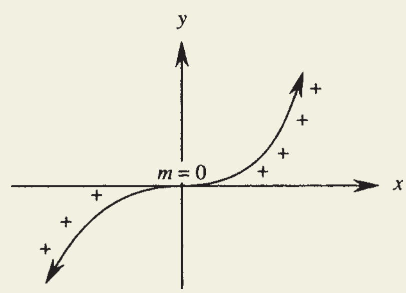

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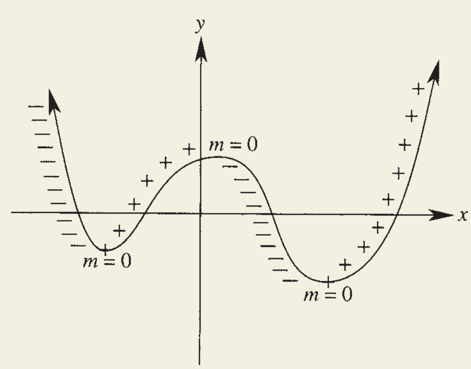

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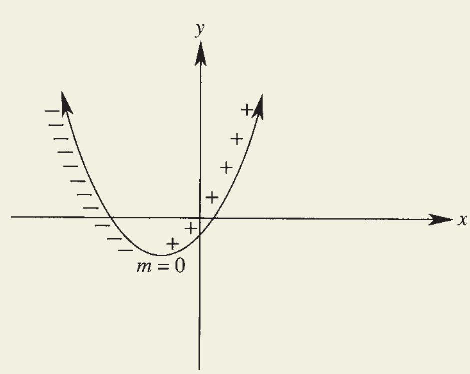


In the examples on the previous page, where the gradient is positive, the curve is going up, or increasing (reading from left to right).

Where the gradient is negative, the curve is going downwards, or decreasing.

The gradient is **zero** at particular points of the curves. At these points the curve isn't increasing or decreasing. We say the curve is *stationary* at these points.

If $f'(x) > 0$, the curve is increasing
If $f'(x) < 0$, the curve is decreasing
If $f'(x) = 0$, the curve is stationary

A curve is **monotonic** increasing or decreasing if it is *always* increasing or decreasing; that is,

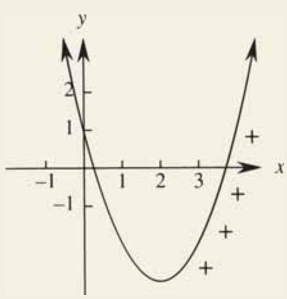
if $f'(x) > 0$ for all x (monotonic increasing)
or $f'(x) < 0$ for all x (monotonic decreasing)

EXAMPLES

1. Find all x values for which the curve $f(x) = x^2 - 4x + 1$ is increasing.

Solution

$f'(x) = 2x - 4$
$f'(x) > 0$ for increasing curve
i.e. $2x - 4 > 0$
$2x > 4$
$x > 2$
So the curve is increasing for $x > 2$.



This function is a parabola.

CONTINUED

2. Find the stationary point on the parabola $y = x^2 - 6x + 3$.

Solution

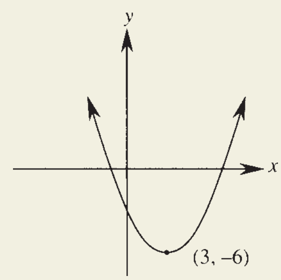
$$\frac{dy}{dx} = 2x - 6$$

For stationary points, $\frac{dy}{dx} = 0$

i.e.
$$\begin{aligned} 2x - 6 &= 0 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

When $x = 3$, $y = 3^2 - 6(3) + 3$
$= -6$

So the stationary point is $(3, -6)$.



3. Find any stationary points on the curve $y = x^3 - 48x - 7$.

Solution

$y' = 3x^2 - 48$

For stationary points, $y' = 0$

i.e.
$$\begin{aligned} 3x^2 - 48 &= 0 \\ 3x^2 &= 48 \\ x^2 &= 16 \\ \therefore \quad x &= \pm 4 \end{aligned}$$

When $x = 4$, $y = 4^3 - 48(4) - 7$
$= -135$

When $x = -4$, $y = (-4)^3 - 48(-4) - 7$
$= 121$

So the stationary points are $(4, -135)$ and $(-4, 121)$.

You will use stationary points to sketch curves later in this chapter.

PROBLEM

What is wrong with this working out?

Find the stationary point on the curve $y = 2x^2 + x - 1$.

Solution

$y' = 4x + 1$

For stationary points, $y' = 0$

i.e. $4x + 1 = 0$

$4x = -1$

$x = -0.25$

When $x = -0.25$, $y = 4(-0.25) + 1$

$= -1 + 1$

$= 0$

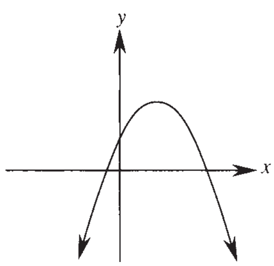
So the stationary point is $(-0.25, 0)$.

Can you find the correct answer?

2.1 Exercises

1. Find the parts of each curve where the gradient of the tangent is positive, negative or zero. Label each curve with +, – or 0.

(a)



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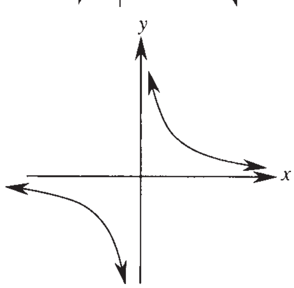

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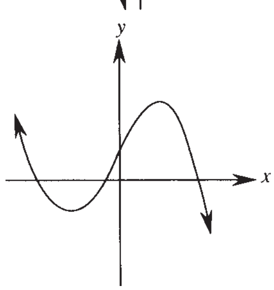

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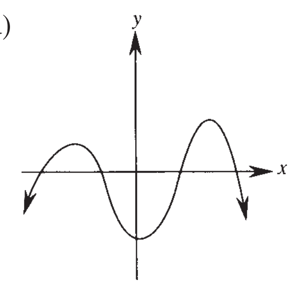


2. Find all values of x for which the curve $y = 2x^2 - x$ is decreasing.

3. Find the domain over which the function $f(x) = 4 - x^2$ is increasing.

4. Find values of x for which the curve $y = x^2 - 3x - 4$ is
   (a) decreasing
   (b) increasing
   (c) stationary.

5. Show that the function $f(x) = -2x - 7$ is always (monotonic) decreasing.

6. Prove that $y = x^3$ is monotonic increasing for all $x \neq 0$.

7. Find the stationary point on the curve $f(x) = x^3$.

8. Find all x values for which the curve $y = 2x^3 + 3x^2 - 36x + 9$ is stationary.

9. Find all stationary points on the curve
   (a) $y = x^2 - 2x - 3$
   (b) $f(x) = 9 - x^2$
   (c) $y = 2x^3 - 9x^2 + 12x - 4$
   (d) $y = x^4 - 2x^2 + 1$.

10. Find any stationary points on the curve $y = (x - 2)^4$.

11. Find all values of x for which the curve $f(x) = x^3 - 3x + 4$ is decreasing.

12. Find the domain over which the curve $y = x^3 + 12x^2 + 45x - 30$ is increasing.

13. Find any values of x for which the curve $y = 2x^3 - 21x^2 + 60x - 3$ is
   (a) stationary
   (b) decreasing
   (c) increasing.

14. The function $f(x) = 2x^2 + px + 7$ has a stationary point at $x = 3$. Evaluate p.

15. Evaluate a and b if $y = x^3 - ax^2 + bx - 3$ has stationary points at $x = -1$ and $x = 2$.

16. (a) Find the derivative of $y = x^3 - 3x^2 + 27x - 3$.
   (b) Show that the curve is monotonic increasing for all values of x.

17. Sketch a function with $f'(x) > 0$ for $x < 2$, $f'(2) = 0$ and $f'(x) < 0$ when $x > 2$.

18. Draw a sketch showing a curve with $\frac{dy}{dx} < 0$ for $x < 4$, $\frac{dy}{dx} = 0$ when $x = 4$ and $\frac{dy}{dx} > 0$ for $x > 4$.

19. Sketch a curve with $\frac{dy}{dx} > 0$ for all $x \neq 1$ and $\frac{dy}{dx} = 0$ when $x = 1$.

20. Draw a sketch of a function that has $f'(x) > 0$ for $x < -2$, $x > 5$, $f'(x) = 0$ for $x = -2$, 5 and $f'(x) < 0$ for $-2 < x < 5$.

21. A function has $f(3) = 2$ and $f'(3) < 0$. Show this information on a sketch.

22. The derivative is positive at the point $(-2, -1)$. Show this information on a graph.

23. Find the stationary points on the curve $y = (3x - 1)(x - 2)^4$.

24. Differentiate $y = x\sqrt{x + 1}$. Hence find the stationary point on the curve, giving the exact value.

25. The curve $f(x) = ax^4 - 2x^3 + 7x^2 - x + 5$ has a stationary point at $x = 1$. Find the value of a.

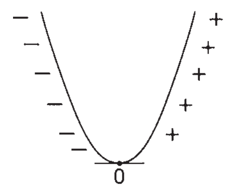
26. Show that $f(x) = \sqrt{x}$ has no stationary points.

27. Show that $f(x) = \frac{1}{x^3}$ has no stationary points.

Types of Stationary Points

There are three types of stationary points.

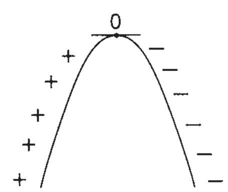
Local minimum point



The curve is decreasing on the left and increasing on the right of the minimum turning point.

x	*LHS*	*Minimum*	*RHS*
$f'(x)$	< 0	0	> 0

Local maximum point

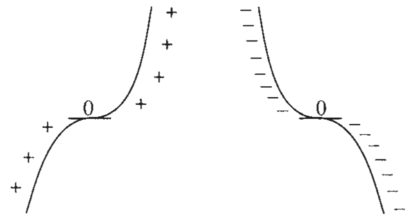


The curve is increasing on the left and decreasing on the right of the maximum turning point.

x	*LHS*	*Maximum*	*RHS*
$f'(x)$	> 0	0	< 0

Local maximum and minimum points are also called **turning points**, as the curve turns around at these points. They can also be called relative maxima or minima.

Point of horizontal inflexion



The derivative has the same sign on both sides of the inflexion.

The curve is either increasing on both sides of the inflexion or it is decreasing on both sides. It is not called a turning point as the curve does not turn around at this point.

x	*LHS*	*Inflexion*	*RHS*
$f'(x)$	> 0	0	> 0
or	< 0	0	< 0

The stationary points are important to the shape of a curve. A reasonably accurate sketch of the curve can be made by finding these points, together with the intercepts on the axes if possible.

EXAMPLES

1. Find the stationary point on the curve $y = x^3$ and determine which type it is.

Solution

$$\frac{dy}{dx} = 3x^2$$

For stationary points, $\frac{dy}{dx} = 0$

i.e. $3x^2 = 0$

$x = 0$

When $x = 0$, $y = 0^3$

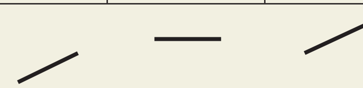
$= 0$

So the stationary point is $(0, 0)$.

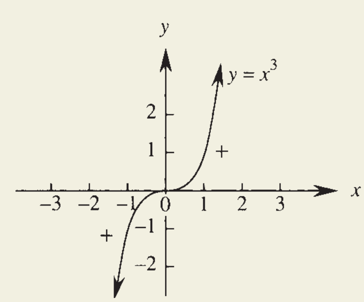
To determine its type, check the curve on the LHS and the RHS.

Substitute $x = \pm 1$ into $\frac{dy}{dx}$.

x	-1	0	1
$\frac{dy}{dx}$	3	0	3



Since the curve is increasing on both sides, $(0, 0)$ is a point of inflexion.



2. Find any stationary points on the curve $f(x) = 2x^3 - 15x^2 + 24x - 7$ and distinguish between them.

Solution

$f'(x) = 6x^2 - 30x + 24$

For stationary points, $f'(x) = 0$

i.e. $\qquad 6x^2 - 30x + 24 = 0$

$x^2 - 5x + 4 = 0$

$(x - 1)(x - 4) = 0$

$\therefore \qquad x = 1 \text{ or } 4$

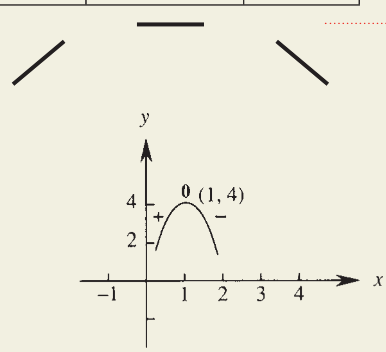
$f(1) = 2(1)^3 - 15(1)^2 + 24(1) - 7$

$= 4$

So $(1, 4)$ is a stationary point.

x	0	1	2
$f'(x)$	24	0	–12

Substitute $x = 0$ and $x = 2$ into $f'(x)$. Take care that there is no other stationary point between the x values you choose.



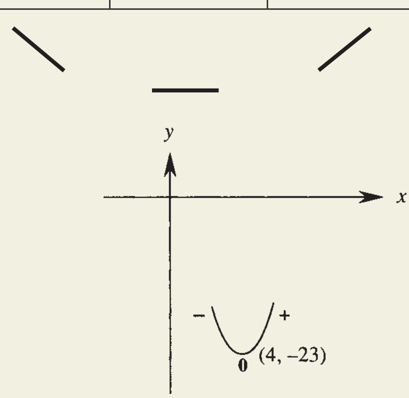
$\therefore$ $(1, 4)$ is a maximum stationary point

$f(4) = 2(4)^3 - 15(4)^2 + 24(4) - 7$

$= -23$

So $(4, -23)$ is a stationary point.

x	2	4	5
$f'(x)$	–12	0	24



$\therefore$ $(4, -23)$ is a minimum stationary point.

2.2 Exercises

1. Find the stationary point on the curve $y = x^2 - 1$ and show that it is a minimum point by checking the derivative on both sides of it.

2. Find the stationary point on the curve $y = x^4$ and determine its type.

3. Find the stationary point on the curve $y = x^3 + 2$ and determine its nature.

4. The function $f(x) = 7 - 4x - x^2$ has one stationary point. Find its coordinates and show that it is a maximum turning point.

5. Find the turning point on the curve $y = 3x^2 + 6x + 1$ and determine its nature.

6. For the curve $y = (4 - x)^2$ find the turning point and determine its nature.

7. The curve $y = x^3 - 6x^2 + 5$ has 2 turning points. Find them and use the derivative to determine their nature.

8. Show that the curve $f(x) = x^5 + 1$ has a point of inflexion at $(0, 1)$.

9. Find the turning points on the curve $y = x^3 - 3x^2 + 5$ and determine their nature.

10. Find any stationary points on the curve $f(x) = x^4 - 2x^2 - 3$. What type of stationary points are they?

11. The curve $y = x^3 - 3x + 2$ has 2 stationary points. Find their coordinates and determine their type.

12. The curve $y = x^5 + mx^3 - 7x + 5$ has a stationary point at $x = -2$. Find the value of m.

13. For a certain function, $f'(x) = 3 + x$. For what value of x does the function have a stationary point? What type of stationary point is it?

14. A curve has $f'(x) = x(x + 1)$. For what x values does the curve have stationary points? What type are they?

15. For a certain curve, $\frac{dy}{dx} = (x - 1)^2(x - 2)$. Find the x values of its stationary points and determine their nature.

16. (a) Differentiate $P = 2x + \frac{50}{x}$ with respect to x.
(b) Find any stationary points on the curve and determine their nature.

17. For the function $A = \frac{h^2 - 2h + 5}{8}$, find any stationary points and determine their nature.

18. Find any stationary points for the function $V = 40r - \pi r^3$ and determine their nature (correct to 2 decimal places).

19. Find any stationary points on the curve $S = 2\pi r + \frac{120}{r}$ (correct to 2 decimal places) and determine their nature.

20. (a) Differentiate $A = x\sqrt{3600 - x^2}$.
(b) Find any stationary points for $A = x\sqrt{3600 - x^2}$ (to 1 decimal place) and determine their nature.

Higher Derivatives

A function can be differentiated several times:

- differentiating $f(x)$ gives $f'(x)$
- differentiating $f'(x)$ gives $f''(x)$
- differentiating $f''(x)$ gives $f'''(x)$, and so on
- the other notation is $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ and so on

Only the first 2 derivatives are used in sketching graphs of curves.

EXAMPLES

1. Find the first 4 derivatives of $f(x) = x^3 - 4x^2 + 3x - 2$.

Solution

$$f'(x) = 3x^2 - 8x + 3$$
$$f''(x) = 6x - 8$$
$$f'''(x) = 6$$
$$f''''(x) = 0$$

Differentiating further will only give 0.

2. Find the second derivative of $y = (2x + 5)^7$.

Solution

$$\frac{dy}{dx} = 7(2x+5)^6 \cdot 2$$
$$= 14(2x+5)^6$$
$$\frac{d^2y}{dx^2} = 14 \cdot 6(2x+5)^5 \cdot 2$$
$$= 168(2x+5)^5$$

3. Find $f''(-1)$ if $f(x) = x^4 - 1$

Solution

$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$
$$\therefore \quad f''(-1) = 12(-1)^2$$
$$= 12$$

2.3 Exercises

1. Find the first 4 derivatives of $x^7 - 2x^5 + x^4 - x - 3$.

2. If $f(x) = x^9 - 5$, find $f''(x)$.

3. Find $f'(x)$ and $f''(x)$ if $f(x) = 2x^5 - x^3 + 1$.

4. Find $f'(1)$ and $f''(-2)$, given $f(t) = 3t^4 - 2t^3 + 5t - 4$.

5. Find the first 3 derivatives of $x^7 - 2x^6 + 4x^4 - 7$.

6. Differentiate $y = 2x^2 - 3x + 3$ twice.

7. If $f(x) = x^4 - x^3 + 2x^2 - 5x - 1$, find $f'(-1)$ and $f''(2)$.

8. Differentiate x^{-4} twice.

9. If $g(x) = \sqrt{x}$, find $g''(4)$.

10. Given $h = 5t^3 - 2t^2 + t + 5$, find $\frac{d^2h}{dt^2}$ when $t = 1$.

11. Find the value of x for which $\frac{d^2y}{dx^2} = 3$ given $y = 3x^3 - 2x^2 + 5x$.

12. Find all values of x for which $f''(x) > 0$ given that $f(x) = x^3 - x^2 + x + 9$.

13. Differentiate $(4x - 3)^5$ twice.

14. Find $f'(x)$ and $f''(x)$ if $f(x) = \sqrt{2 - x}$.

15. Find the first and second derivatives of $f(x) = \frac{x + 5}{3x - 1}$.

16. Find $\frac{d^2v}{dt^2}$ if $v = (t + 3)(2t - 1)^2$.

17. Find the value of b in $y = bx^3 - 2x^2 + 5x + 4$ if $\frac{d^2y}{dx^2} = -2$ when $x = \frac{1}{2}$.

18. Find the exact value of $f''(2)$ if $f(x) = x\sqrt{3x - 4}$.

19. Find $f''(1)$ if $f(t) = t(2t - 1)^7$.

20. Find the value of b if $f(x) = 5bx^2 - 4x^3$ and $f''(-1) = -3$.

Sign of the Second Derivative

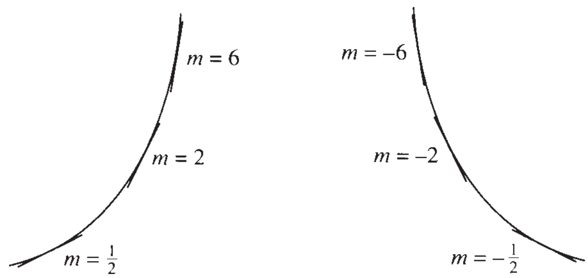
The second derivative gives extra information about a curve that helps us to find its shape.

Since $f''(x)$ is the derivative of $f'(x)$, then $f''(x)$ and $f'(x)$ have the same relationship as $f'(x)$ and $f(x)$.

That is if $f''(x) > 0$ then $f'(x)$ is increasing
if $f''(x) < 0$ then $f'(x)$ is decreasing
if $f''(x) = 0$ then $f'(x)$ is stationary

Concavity

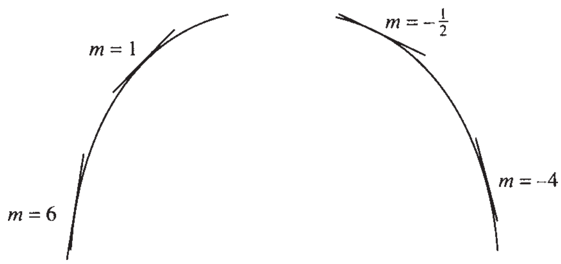
If $f''(x) > 0$ then $f'(x)$ is increasing. This means that the **gradient of the tangent** is increasing, that is, the curve is becoming steeper.



$f'(x)$ is increasing since a gradient of $-\frac{1}{2}$ is greater than a gradient of -6.

Notice the upward shape of these curves. The curve lies above the tangents. We say that the curve is **concave upwards**.

If $f''(x) < 0$ then $f'(x)$ is decreasing. This means that the gradient of the tangent is decreasing. That is, the curve is becoming less steep.

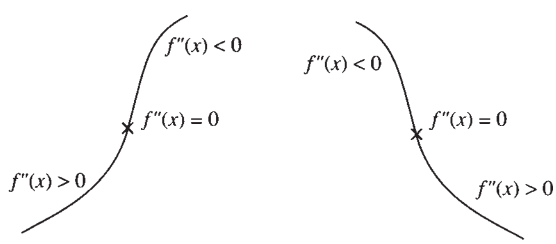


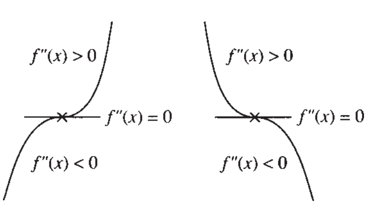
$f'(x)$ is decreasing since -4 is less then $-\frac{1}{2}$.

Notice the downward shape of these curves. The curve lies below the tangents. We say that the curve is **concave downwards**.

If $f''(x) = 0$ then $f'(x)$ is stationary. That is, it is neither increasing nor decreasing. This happens when the curve goes from being concave upwards to concave downwards, or when the curve changes from concave downwards to concave upwards. We say that the curve is **changing concavity**.

*The point where concavity changes is called an **inflexion**.*





*These points of inflexion are called **horizontal inflexions**, as the tangent is horizontal.*

The curve has a point of inflexion as long as **concavity changes**.

Class Investigation

How would you check that concavity changes?

- If $f''(x) > 0$, the curve is **concave upwards**.
- If $f''(x) < 0$, the curve is **concave downwards**.
- If $f''(x) = 0$ and concavity changes, there is a **point of inflexion**.

EXAMPLES

1. Does the curve $y = x^4$ have a point of inflexion?

Solution

$$\frac{dy}{dx} = 4x^3$$

$$\frac{d^2y}{dx^2} = 12x^2$$

For inflexions, $\frac{d^2y}{dx^2} = 0$

i.e. $12x^2 = 0$

$x = 0$

When $x = 0$, $y = 0^4$

$= 0$

So $(0, 0)$ is a possible point of inflexion.

Check that concavity changes:

The sign must change for concavity to change.

x	-1	0	1
$\frac{d^2y}{dx^2}$	12	0	12

What type of point is it?

Since concavity doesn't change, $(0, 0)$ is not a point of inflexion.

2. Find all values of x for which the curve $f(x) = 2x^3 - 7x^2 - 5x + 4$ is concave downwards.

Solution

$f'(x) = 6x^2 - 14x - 5$
$f''(x) = 12x - 14$

For concave downwards, $f''(x) < 0$

$$12x - 14 < 0$$
$$12x < 14$$
$$\therefore \quad x < 1\frac{1}{6}$$

---

3. Find the point of inflexion on the curve $y = x^3 - 6x^2 + 5x + 9$.

Solution

$y' = 3x^2 - 12x + 5$
$y'' = 6x - 12$

For inflexions, $y'' = 0$

$$6x - 12 = 0$$
$$6x = 12$$
$$\therefore \quad x = 2$$

When $x = 2$, $y = 2^3 - 6(2)^2 + 5(2) + 9$
$= 3$

$\therefore$ (2, 3) is a possible point of inflexion

Check that concavity changes:

x	1	2	3
$\frac{d^2y}{dx^2}$	-6	0	6

Since concavity changes, (2, 3) is a point of inflexion.

2.4 Exercises

1. For what values of x is the curve $y = x^3 + x^2 - 2x - 1$ concave upwards?
2. Find all values of x for which the curve $y = (x - 3)^3$ is concave downwards.
3. Prove that the curve $y = 8 - 6x - 4x^2$ is always concave downwards.
4. Show that the curve $y = x^2$ is always concave upwards.
5. Find the domain over which the curve $f(x) = x^3 - 7x^2 + 1$ is concave downwards.
6. Find any points of inflexion on the curve $g(x) = x^3 - 3x^2 + 2x + 9$.
7. Find the points of inflexion on the curve $y = x^4 - 6x^2 + 12x - 24$.

8. Find the stationary point on the curve $y = x^3 - 2$. Show that it is an inflexion.

9. Find all values of x for which the function $f(x) = x^4 + 2x^3 - 12x^2 + 12x - 1$ is concave downwards.

10. Determine whether there are any points of inflexion on the curve
    (a) $y = x^6$
    (b) $y = x^7$
    (c) $y = x^5$
    (d) $y = x^9$
    (e) $y = x^{12}$

11. Sketch a curve that is always concave up.

12. Sketch a curve where $f''(x) < 0$ for $x > 1$ and $f''(x) > 0$ for $x < 1$.

13. Find any points of inflexion on the curve $y = x^4 - 8x^3 + 24x^2 - 4x - 9$.

14. Show that $f(x) = \frac{2}{x^2}$ is concave upwards for all $x \neq 0$.

15. For the function $f(x) = 3x^5 - 10x^3 + 7$
    (a) Find any points of inflexion.
    (b) Find which of these points are horizontal points of inflexion (stationary points).

16. (a) Show that the curve $y = x^4 + 12x^2 - 20x + 3$ has no points of inflexion.
    (b) Describe the concavity of the curve.

17. If $y = ax^3 - 12x^2 + 3x - 5$ has a point of inflexion at $x = 2$, evaluate a.

18. Evaluate p if $f(x) = x^4 - 6px^2 - 20x + 11$ has a point of inflexion at $x = -2$.

19. The curve $y = 2ax^4 + 4bx^3 - 72x^2 + 4x - 3$ has points of inflexion at $x = 2$ and $x = -1$. Find the values of a and b.

20. The curve $y = x^6 - 3x^5 + 21x - 8$ has two points of inflexion.
    (a) Find these points.
    (b) Show that these points of inflexion are not stationary points.

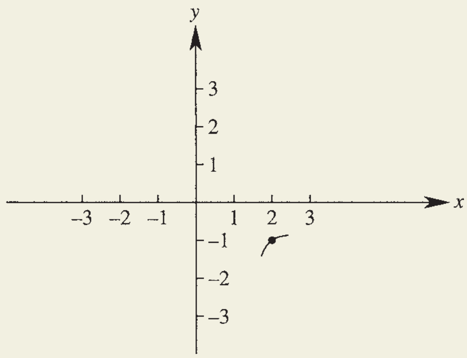
If we combine the information from the first and second derivatives, this will tell us about the shape of the curve.

EXAMPLES

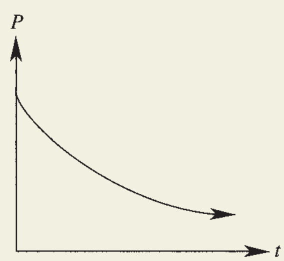
1. For a particular curve, $f(2) = -1$, $f'(2) > 0$ and $f''(2) < 0$. Sketch the curve at this point, showing its shape.

Solution

$f(2) = -1$ means that the point $(2, -1)$ lies on the curve.
If $f'(2) > 0$, the curve is increasing at this point.
If $f''(2) < 0$, the curve is concave downwards at this point.



2. The curve below shows the number of unemployed people P over time t months.



(a) Describe the sign of $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$.

(b) How is the number of unemployed people changing over time?

(c) Is the unemployment rate increasing or decreasing?

Remember that the gradient, or first derivative, measures the rate of change.

Solution

(a) The curve is decreasing, so $\frac{dP}{dt} < 0$ and the curve is concave upwards, so $\frac{d^2P}{dt^2} > 0$.

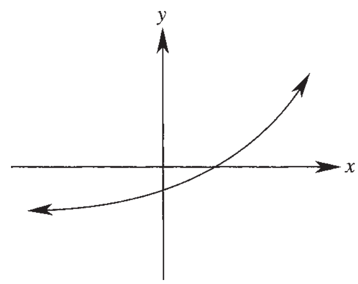
(b) As the curve is decreasing, the number of unemployed people is decreasing.

(c) Since the curve is concave upwards, the gradient is increasing. This means that the unemployment rate is increasing.

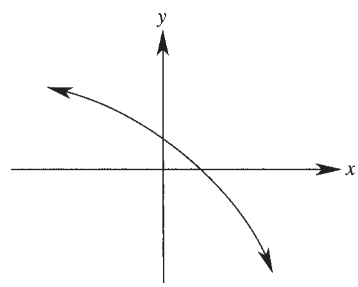
2.5 Exercises

1. For each curve, describe the sign of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

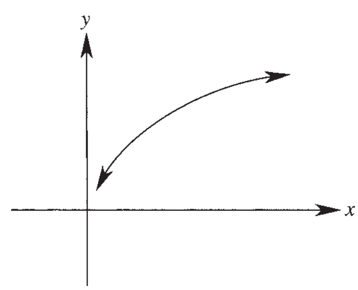
(a)



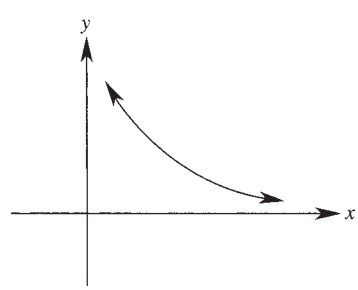
(b)



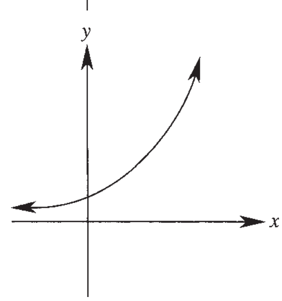
(c)



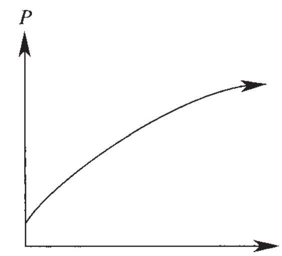
(d)



(e)



2. The curve below shows the population of a colony of seals.



(a) Describe the sign of the first and second derivatives.
(b) How is the population rate changing?

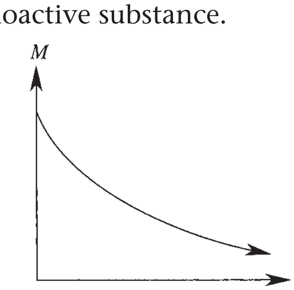
3. Inflation is increasing, but the rate of increase is slowing. Draw a graph to show this trend.

4. Draw a sketch to show the shape of each curve:
(a) $f'(x) < 0$ and $f''(x) < 0$
(b) $f'(x) > 0$ and $f''(x) < 0$
(c) $f'(x) < 0$ and $f''(x) > 0$
(d) $f'(x) > 0$ and $f''(x) > 0$

5. The size of classes at a local TAFE college is decreasing and the rate at which this is happening is decreasing. Draw a graph to show this.

6. As an iceblock melts, the rate at which it melts increases. Draw a graph to show this information.

7. The graph shows the decay of a radioactive substance.



Describe the sign of $\dfrac{dM}{dt}$ and $\dfrac{d^2M}{dt^2}$.

8. The population P of fish in a certain lake was studied over time, and at the start the number of fish was 2500.

(a) During the study, $\frac{dP}{dt} < 0$.

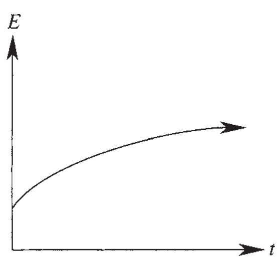
What does this say about the number of fish during the study?

(b) If at the same time, $\frac{d^2P}{d^2} > 0$,

what can you say about the population rate?

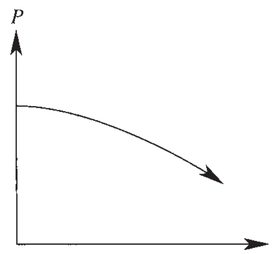
(c) Sketch the graph of the population P against t.

9. The graph shows the level of education of youths in a certain rural area over the past 100 years.



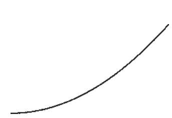
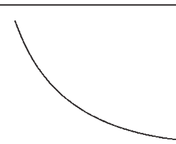
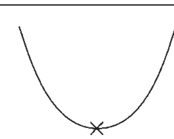
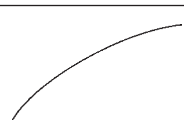
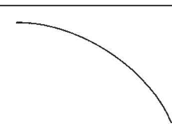
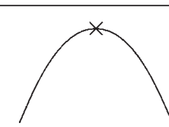
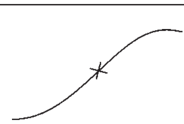
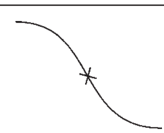
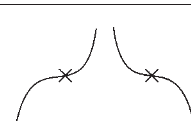
Describe how the level of education has changed over this period of time. Include mention of the rate of change.

10. The graph shows the number of students in a high school over several years.



Describe how the school population is changing over time, including the rate of change.

Here is a summary of the shape of a curve given the first and second derivatives.

	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$
$\frac{d^2y}{dx^2} > 0$			
$\frac{d^2y}{dx^2} < 0$			
$\frac{d^2y}{dx^2} = 0$			

Class Investigation

There are three mistakes in this argument. Can you find all of them?

$$y = 2\sqrt{x} = 2x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{1}{2\sqrt{x}}$$

For stationary points, $\frac{dy}{dx} = 0$

i.e. $\frac{1}{2\sqrt{x}} = 0$

$1 = \sqrt{x}$

$\therefore \pm 1 = x$

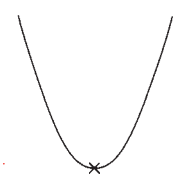
So there are stationary points at $x = 1$ and $x = -1$.

Determining Types of Stationary Points

We can determine the type of a stationary point by looking at the first and second derivatives together.

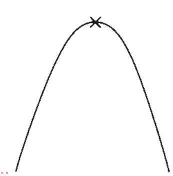
- If $f'(x) = 0$ and $f''(x) > 0$, there is a **minimum** turning point.

The curve is concave upwards at this point.

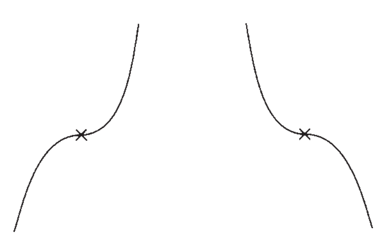


- If $f'(x) = 0$ and $f''(x) < 0$, there is a **maximum** turning point.

The curve is concave downwards at this point.



- If $f'(x) = 0$, $f''(x) = 0$ and concavity changes, then there is a **horizontal point of inflexion**.



- **Minimum** turning point: $f'(x) = 0$, $f''(x) > 0$.
- **Maximum** turning point: $f'(x) = 0$, $f''(x) < 0$.
- **Horizontal inflexion:** $f'(x) = 0$, $f''(x) = 0$ and concavity changes.

EXAMPLES

1. Find the stationary points on the curve $f(x) = 2x^3 - 3x^2 - 12x + 7$ and distinguish between them.

Solution

$f'(x) = 6x^2 - 6x - 12$

For stationary points, $f'(x) = 0$

i.e. $6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$\therefore \quad x = 2$ or -1

$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 7$

$= -13$

So $(2, -13)$ is a stationary point.

$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 7$

$= 14$

So $(-1, 14)$ is a stationary point.

Now $f''(x) = 12x - 6$

$f''(2) = 12(2) - 6$

$= 18$

> 0 (concave upwards)

So $(2, -13)$ is a minimum turning point.

$f''(-1) = 12(-1) - 6$

$= -18$

< 0 (concave downwards)

So $(-1, 14)$ is a maximum turning point.

---

2. Find the stationary point on the curve $y = 2x^5 - 3$ and determine its nature.

Solution

$y' = 10x^4$

For stationary points, $y' = 0$

i.e. $10x^4 = 0$

$\therefore \quad x = 0$

CONTINUED

When $x = 0$, $y = 2(0)^5 - 3$
$= -3$

So $(0, -3)$ is a stationary point.

Now $y'' = 40x^3$

When $x = 0, y'' = 40(0)^3$
$= 0$

So $(0, -3)$ is a possible point of inflexion.

Check concavity on the LHS and RHS:

x	-1	0	1
$\frac{d^2y}{dx^2}$	-40	0	40

Since concavity changes, $(0, -3)$ is a horizontal point of inflexion.

2.6 Exercises

1. Find the stationary point on the curve $y = x^2 - 2x + 1$ and determine its nature.

2. Find the stationary point on the curve $y = 3x^4 + 1$ and determine what type of point it is.

3. Find the stationary point on the curve $y = 3x^2 - 12x + 7$ and show that it is a minimum point.

4. Determine the stationary point on the curve $y = x - x^2$ and show that it is a maximum turning point.

5. Show that the curve $f(x) = 2x^3 - 5$ has an inflexion and find its coordinates.

6. Does the function $f(x) = 2x^5 + 3$ have a stationary point? If it does, determine its nature.

7. Find any stationary points on the curve $f(x) = 2x^3 + 15x^2 + 36x - 50$ and determine their nature.

8. Find the stationary points on the curve $y = 3x^4 - 4x^3 - 12x^2 + 1$ and determine whether they are maximum or minimum turning points.

9. Find any stationary points on the curve $y = (4x^2 - 1)^4$ and determine their nature.

10. (a) Find any stationary points on the curve $y = 2x^3 - 27x^2 + 120x$ and distinguish between them.
(b) Find any points of inflexion on the curve.

11. Find any stationary points on the curve $y = (x - 3)\sqrt{4 - x}$ and determine their nature.

12. Find any stationary points on the curve $y = x^4 + 8x^3 + 16x^2 - 1$ and determine their nature.

13. The curve $y = ax^2 - 4x + 1$ has a stationary point where $x = -3$.
(a) Find the value of a.
(b) Hence, or otherwise, determine the nature of the stationary point.

14. The curve $y = x^3 - mx^2 + 8x - 7$ has a stationary point where $x = -1$. Find the value of m.

15. The curve $y = ax^3 + bx^2 - x + 5$ has a point of inflexion at $(1, -2)$. Find the values of a and b.

Curve Sketching

We can sketch curves by finding all of their important features, such as stationary points, points of inflexion and intercepts. Here is a summary of strategies for sketching a curve.

1. Find **stationary points** $(y' = 0)$.
2. Find **points of inflexion** $(y'' = 0)$.
3. Find intercepts on axes.
   - For x-intercept, $y = 0$
   - For y-intercept, $x = 0$
4. Find **domain** and **range**.
5. Find any **asymptotes** or **limits**.
6. Use **symmetry, odd** or **even functions**.
7. Draw up a **table of values**.

We cannot always find the x-intercept.

EXAMPLES

1. Find any stationary points and points of inflexion on the curve $f(x) = x^3 - 3x^2 - 9x + 1$ and hence sketch the curve.

Solution

$f'(x) = 3x^2 - 6x - 9$
and $f''(x) = 6x - 6$
First, find the stationary points.
For stationary points, $f'(x) = 0$

$$\begin{aligned} \text{i.e.} \quad 3x^2 - 6x - 9 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ \therefore \quad x &= -1 \text{ or } 3 \end{aligned}$$

CONTINUED

$$f(3) = (3)^3 - 3(3)^2 - 9(3) + 1$$
$$= -26$$

So $(3, -26)$ is a stationary point.

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 1$$
$$= 6$$

So $(-1, 6)$ is a stationary point.

We use the second derivative to determine their type.

$$f''(3) = 6(3) - 6$$
$$= 12$$
$$> 0 \qquad \text{(concave upwards)}$$

$\therefore \quad (3, -26)$ is a minimum turning point

$$f''(-1) = 6(-1) - 6$$
$$= -12$$
$$< 0 \qquad \text{(concave downwards)}$$

$\therefore \quad (-1, 6)$ is a maximum turning point

Next, find any points of inflexion.
For inflexions, $f''(x) = 0$

i.e. $\quad 6x - 6 = 0$
$$6x = 6$$
$$x = 1$$

$$f(1) = 1^3 - 3(1)^2 - 9(1) + 1$$
$$= -10$$

x	0	1	2
$f''(x)$	-6	0	6

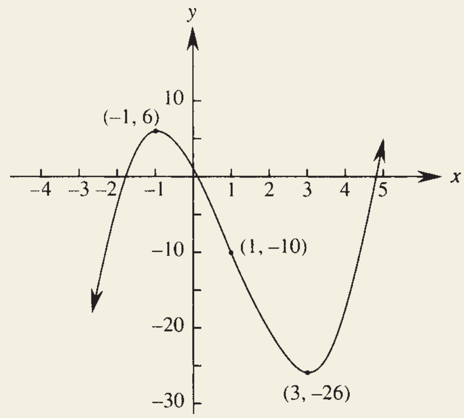
Since concavity changes, $(1, -10)$ is a point of inflexion.
Next, try to find intercepts on the axes.

For y-intercept, $x = 0$:
$$f(0) = 0^3 - 3(0)^2 - 9(0) + 1$$
$$= 1$$

For x-intercept, $y = 0$:

i.e. $x^3 - 3x^2 - 9x + 1 = 0$
This is too hard to solve.

Now sketch the graph using an appropriate scale so that all stationary points and points of inflexion fit on the graph.



2. Sketch the curve $y = 2x^3 + 1$, showing any important features.

Solution

$\frac{dy}{dx} = 6x^2$

For stationary points, $\frac{dy}{dx} = 0$

i.e. $6x^2 = 0$

$x = 0$

When $x = 0$, $y = 2(0)^3 + 1$

$= 1$

So (0, 1) is a stationary point.

$\frac{d^2y}{dx^2} = 12x$

At (0, 1) $\frac{d^2y}{dx^2} = 12(0)$

$= 0$

So (0, 1) is a possible point of inflexion.

x	−1	0	1
$\frac{d^2y}{dx^2}$	−12	0	12

Since concavity changes, (0, 1) is a horizontal inflexion.

For x-intercept, $y = 0$

i.e. $2x^3 + 1 = 0$

$2x^3 = -1$

$x^3 = -0.5$

$\therefore \quad x = -0.8$

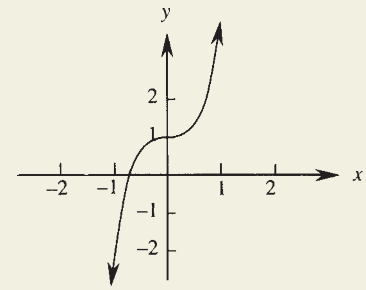
For y-intercept, $x = 0$

CONTINUED

We already know this point. It is (0, 1), the inflexion. We can also find a point on either side of the inflexion.

When $x = -1$, $y = 2(-1)^3 + 1$
$= -1$

When $x = 1$, $y = 2(1)^3 + 1$
$= 3$



2.7 Exercises

1. Find the stationary point on the curve $f(x) = x^2 - 3x - 4$ and determine its type. Find the intercepts on the axes and sketch the curve.

2. Sketch $y = 6 - 2x - x^2$, showing the stationary point.

3. Find the stationary point on the curve $y = (x - 1)^3$ and determine its nature. Hence sketch the curve.

4. Sketch $y = x^4 + 3$, showing any stationary points.

5. Find the stationary point on the curve $y = x^5$ and show that it is a point of inflexion. Hence sketch the curve.

6. Sketch $f(x) = x^7$.

7. Find any stationary points on the curve $y = 2x^3 - 9x^2 - 24x + 30$ and sketch its graph.

8. (a) Determine any stationary points on the curve $y = x^3 + 6x^2 - 7$.
   (b) Find any points of inflexion on the curve.
   (c) Sketch the curve.

9. Find any stationary points and inflexions on the curve $y = x^3 - 6x^2 + 3$ and hence sketch the curve.

10. Find any stationary points and inflexions on the curve $y = 2 + 9x - 3x^2 - x^3$. Hence sketch the curve.

11. Sketch the function $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$, showing all stationary points.

12. Find the stationary points on the curve $y = (x - 4)(x + 2)^2$ and hence sketch the curve.

13. Find all stationary points and inflexions on the curve $y = (2x + 1)(x - 2)^4$. Sketch the curve.

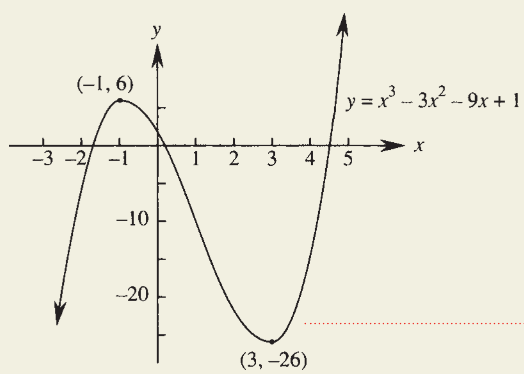
14. Show that the curve $y = \frac{2}{1 + x}$ has no stationary points. By considering the domain and range of the function, sketch the curve.

15. Find any stationary points on the curve $y = \frac{x^2}{x - 2}$. By also considering the domain of the curve, sketch its graph.

Maximum and Minimum Values

A curve may have maximum and minimum turning points, but they are not necessarily the maximum or minimum values of the function.

EXAMPLE



You sketched this curve on page 55.

This curve has a maximum turning point at $(-1, 6)$. We call it a **relative maximum point**, since it does not give the maximum value of the graph. Similarly, the curve has a **relative minimum** turning point at $(3, -26)$. The curve does not have any absolute maximum or minimum value since it increases to infinity and decreases to negative infinity.

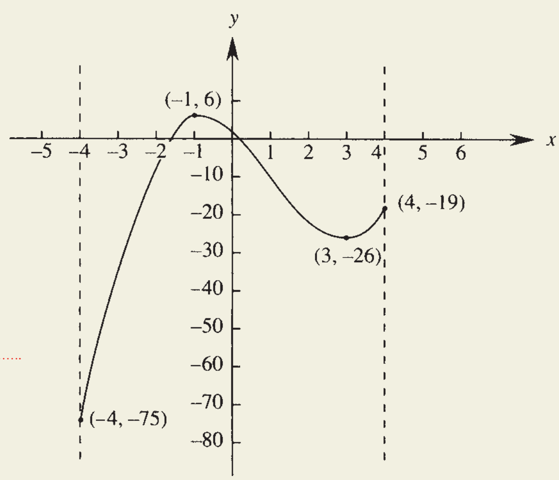
The relative maximum or minimum point is also called a **local maximum** or **minimum** point.

If we restrict the domain of the curve to, say, $-4 \le x \le 4$, then the curve will have an absolute maximum and minimum value. The **absolute maximum** is the greatest value of the curve in the domain. The **absolute minimum** is the least value of the curve in the domain.

In order to find the maximum or minimum value of a curve, find the values of the function at the endpoints of the domain as well as its turning points.

CONTINUED

When $x = -4$, $y = (-4)^3 - 3(-4)^2 - 9(-4) + 1$
$= -75$
When $x = 4$, $y = 4^3 - 3(4)^2 - 9(4) + 1$
$= -19$



The highest part of the curve is at $y = 6$ and the lowest part is at $y = -75$.

By restricting the curve to $-4 \le x \le 4$, the maximum value is 6 and the minimum value is –75.

EXAMPLES

1. Find the maximum and minimum values of y for the function $f(x) = x^2 - 4x + 3$ in the domain $-4 \le x \le 3$.

Solution

$f'(x) = 2x - 4$

For stationary points, $f'(x) = 0$

i.e. $2x - 4 = 0$
$2x = 4$
$x = 2$

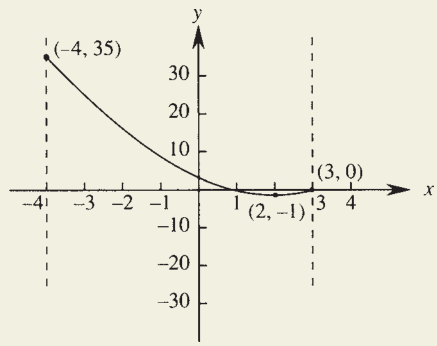
$f(2) = 2^2 - 4(2) + 3$
$= -1$

So $(2, -1)$ is a stationary point.

$f''(x) = 2$
> 0 (concave upwards)

$\therefore (2, -1)$ is a minimum turning point

Endpoints: $f(-4) = (-4)^2 - 4(-4) + 3$
$= 35$
$f(3) = 3^2 - 4(3) + 3$
$= 0$



The maximum value of y is 35 and the minimum value is –1 in the domain $-4 \le x \le 3$.

---

2. Find the maximum and minimum values of the curve $y = x^4 - 2x^2 + 1$ for $-2 \le x \le 3$.

Solution

$y' = 4x^3 - 4x$

For stationary points, $y' = 0$

i.e. $4x^3 - 4x = 0$

$4x(x^2 - 1) = 0$

$4x(x + 1)(x - 1) = 0$

$\therefore \quad x = 0 \text{ or } \pm 1$

When $x = 1$, $y = 1^4 - 2(1)^2 + 1$
$= 0$

So (1, 0) is a stationary point.

When $x = -1$, $y = (-1)^4 - 2(-1)^2 + 1$
$= 0$

So $(-1, 0)$ is a stationary point.

When $x = 0$, $y = (0)^4 - 2(0)^2 + 1$
$= 1$

So (0, 1) is a stationary point.

Now $y'' = 12x^2 - 4$

When $x = 1$, $y'' = 12(1)^2 - 4$
$= 8$
> 0 (concave upwards)

CONTINUED

So (1, 0) is a minimum turning point.

When $x = -1$, $y'' = 12(-1)^2 - 4$

$= 8$

> 0 (concave upwards)

So (−1, 0) is a minimum turning point.

When $x = 0$, $y'' = 12(0)^2 - 4$

$= -4$

< 0 (concave downwards)

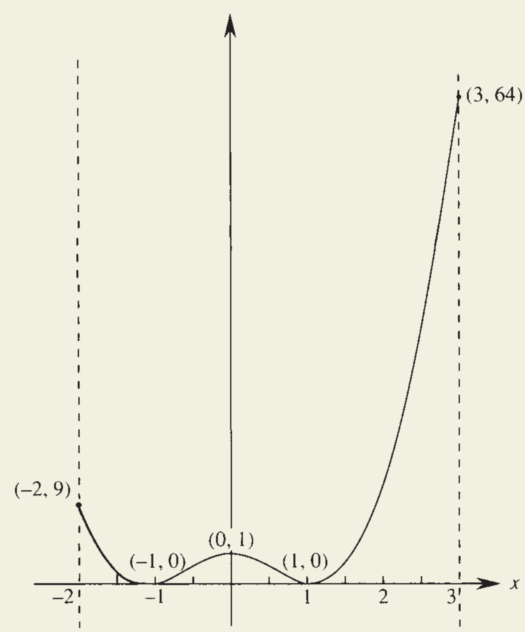
So (0, 1) is a maximum turning point.

Endpoints: When $x = -2$, $y = (-2)^4 - 2(-2)^2 + 1$

$= 9$

When $x = 3$, $y = 3^4 - 2(3)^2 + 1$

$= 64$



The maximum value is 64 and the minimum value is 0 in the domain $-2 \le x \le 3$.

2.8 Exercises

1. Sketch $y = x^2 + x - 2$ for $-2 \le x \le 2$ and find the maximum value of y.

2. Sketch $f(x) = 9 - x^2$ over the domain $-4 \le x \le 2$. Hence find the maximum and minimum values of the curve over this domain.

3. Find the maximum value of the curve $y = x^2 - 4x + 4$ for $-3 \le x \le 3$.

4. Sketch $f(x) = 2x^3 + 3x^2 - 36x + 5$ for $-3 \le x \le 3$, showing any stationary points. Find the maximum and minimum values of the function for $-3 \le x \le 3$.

5. Find the maximum value of y for the curve $y = x^5 - 3$ for $-2 \le x \le 1$.

6. Sketch the curve $f(x) = 3x^2 - 16x + 5$ for $0 \le x \le 4$ and find the maximum and minimum values of the function over this domain.

7. Find the relative and absolute maximum and minimum values of the function $f(x) = 3x^4 + 4x^3 - 12x^2 - 3$ for $-2 \le x \le 2$.

8. Sketch $y = x^3 + 2$ over the domain $-3 \le x \le 3$ and find its minimum and maximum values in that domain.

9. Sketch $y = \sqrt{x + 5}$ for $-4 \le x \le 4$ and find its maximum and minimum values.

10. Show that $y = \dfrac{1}{x - 2}$ has no stationary points. Find the maximum and minimum values of the curve for $-3 \le x \le 3$.

Problems Involving Maxima and Minima

Often a problem involves finding maximum or minimum values. For example, a salesperson wants to maximise profit; a warehouse manager wants to maximise storage; a driver wants to minimise petrol consumption; a farmer wants to maximise paddock size.

PROBLEM

One disc 20 cm in diameter and one 10 cm in diameter are cut from a disc of cardboard 30 cm in diameter. Can you find the largest disc that can be cut from the remainder of the cardboard?

EXAMPLES

1. The equation for the expense per year (in units of ten thousand dollars) of running a certain business is given by $E = x^2 - 6x + 12$, where x is the number (in hundreds) of items manufactured.

(a) Find the expense of running the business if no items are manufactured.

(b) Find the number of items needed to minimise the expense of the business.

(c) Find the minimum expense of the business.

CONTINUED

E is the expense in units of ten thousand dollars.

This will give any maximum or minimum values of E.

x is the number of items in hundreds.

Solution

(a) When $x = 0$, $E = 0^2 - 6(0) + 12$

$= 12$

So the expense of running the business when no items are manufactured is $12 \times \$10\,000$, or $\$120\,000$ per year.

(b) $\dfrac{dE}{dx} = 2x - 6$

For stationary points, $\dfrac{dE}{dx} = 0$

i.e.

$$
\begin{aligned}
2x - 6 &= 0 \\
2x &= 6 \\
x &= 3 \\
\frac{d^2E}{dx^2} &= 2 > 0 \quad \text{(concave upwards)}
\end{aligned}
$$

$\therefore x = 3$ gives a minimum value

So 300 items manufactured each year will give the minimum expenses.

(c) When $x = 3$, $E = 3^2 - 6(3) + 12$

$= 3$

So the minimum expense per year is $\$30\,000$.

---

2. The formula for the volume of a pond is given by $V = 2h^3 - 12h^2 + 18h + 50$, where h is the depth of the pond in metres. Find the maximum volume of the pond.

Solution

$V' = 6h^2 - 24h + 18$

For stationary points, $V' = 0$

i.e.

$$
\begin{aligned}
6h^2 - 24h + 18 &= 0 \\
h^2 - 4h + 3 &= 0 \\
(h - 1)(h - 3) &= 0
\end{aligned}
$$

So $h = 1$ or 3

$V'' = 12h - 24$

When $h = 1$, $V'' = 12(1) - 24$

$= -12$

< 0 (concave downwards)

So $h = 1$ gives a maximum V.

When $h = 3$, $V'' = 12(3) - 24$

$= 12$

> 0 (concave upwards)

So $h = 3$ gives a minimum V.

When $h = 1$, $V = 2(1)^3 - 12(1)^2 + 18(1) + 50$
$= 58$

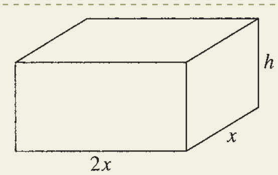
So the maximum volume is 58 m^3.

In the above examples, the equation is given as part of the question. However, we often need to work out an equation before we can start answering the question. Working out the equation is often the hardest part!

EXAMPLES

1. A rectangular prism has a base with length twice its breadth. The volume is to be 300 cm^3. Show that the surface area is given by $S = 4x^2 + \frac{900}{x}$.

Solution



Volume: $V = lbh$

$= 2x \times x \times h$

$300 = 2x^2 h$

$\therefore \quad \frac{300}{2x^2} = h \qquad (1)$

Surface area:

$S = 2(lb + bh + lh)$

$= 2(2x^2 + xh + 2xh)$

$= 2(2x^2 + 3xh)$

$= 4x^2 + 6xh$

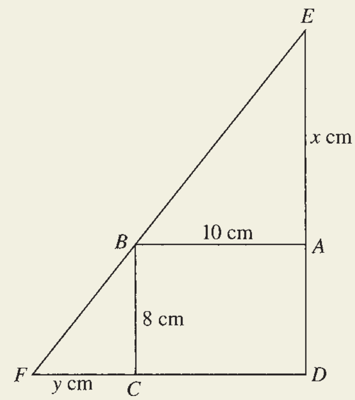
Now we substitute (1) into this equation.

$S = 4x^2 + 6x \cdot \frac{300}{2x^2}$

$= 4x^2 + \frac{900}{x}$

2. $ABCD$ is a rectangle with $AB = 10$ cm and $BC = 8$ cm. Length $AE = x$ cm and $CF = y$ cm.

CONTINUED



(a) Show that triangles *AEB* and *CBF* are similar.
(b) Show that $xy = 80$.
(c) Show that triangle *EDF* has area given by $A = 80 + 5x + \dfrac{320}{x}$.

Solution

(a) $\angle EAB = \angle BCF = 90°$ (given)
$\angle BFC = \angle EBA$ (corresponding $\angle$s, $AB \parallel DC$)
$\angle BEA = \angle FBC$ (similarly, $AD \parallel BC$)
$\therefore\ \Delta AEB$ and ΔCBF are similar (AAA)

(b) $\therefore\ \dfrac{10}{y} = \dfrac{x}{8}$ (similar triangles have sides in proportion)

$xy = 80$

(c) Side $FD = y + 10$ and side $ED = x + 8$

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}(y + 10)(x + 8)$$
$$= \frac{1}{2}(xy + 8y + 10x + 80)$$

We need to eliminate y from this equation.

If $xy = 80$,

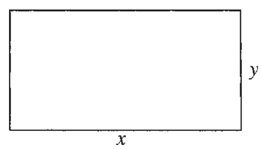
then $y = \dfrac{80}{x}$

Substituting $xy = 80$ and $y = \dfrac{80}{x}$ into the area equation gives:

$$A = \frac{1}{2}(xy + 8y + 10x + 80)$$
$$= \frac{1}{2}\left(80 + 8 \cdot \frac{80}{x} + 10x + 80\right)$$
$$= \frac{1}{2}\left(160 + \frac{640}{x} + 10x\right)$$
$$= 80 + \frac{320}{x} + 5x$$

2.9 Exercises

1. The area of a rectangle is to be 50 m^2.

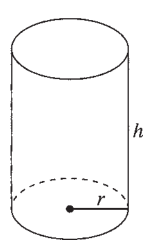


Show that the equation of its perimeter is given by $P = 2x + \frac{100}{x}$.

2. A rectangular paddock on a farm is to have a fence with a 120 m perimeter. Show that the area of the paddock is given by $A = 60x - x^2$.

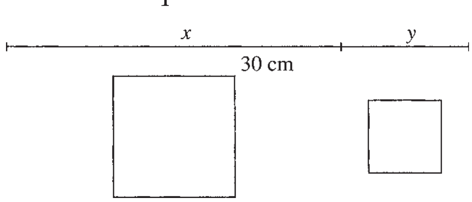
3. The product of two numbers is 20. Show that the sum of the numbers is $S = x + \frac{20}{x}$.

4. A closed cylinder is to have a volume of 400 cm^3.



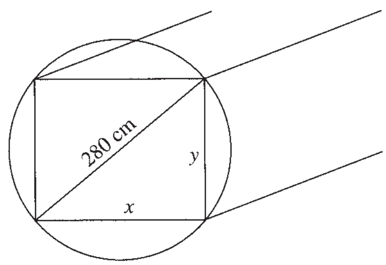
Show that its surface area is $S = 2\pi r^2 + \frac{800}{r}$.

5. A 30 cm length of wire is cut into 2 pieces and each piece bent to form a square.



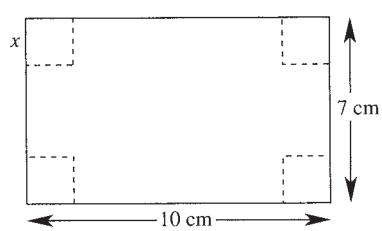
(a) Show that $y = 30 - x$.
(b) Show that the total area of the 2 squares is given by $A = \frac{x^2 - 30x + 450}{8}$.

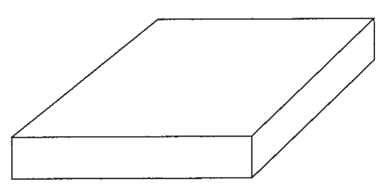
6. A timber post with a rectangular cross-sectional area is to be cut out of a log with a diameter of 280 mm as shown.



(a) Show that $y = \sqrt{78400 - x^2}$.
(b) Show that the cross-sectional area is given by $A = x\sqrt{78400 - x^2}$.

7. A 10 cm × 7 cm rectangular piece of cardboard has equal square corners with side x cut out and folded up to make an open box.

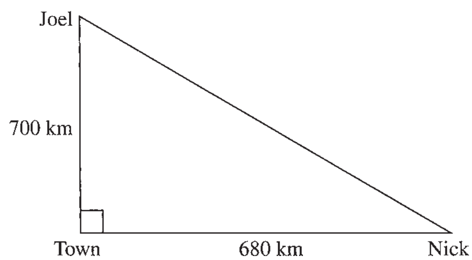




Show that the volume of the box is $V = 70x - 34x^2 + 4x^3$.

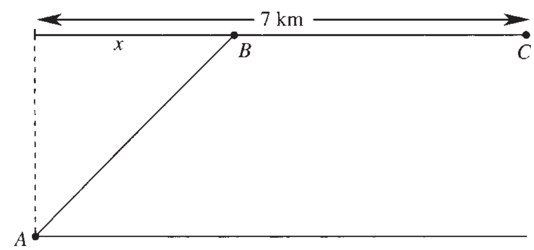
8. A travel agency calculates the expense E of organising a holiday per person in a group of x people as $E = 200 + 400x$. The cost C for each person taking a holiday is $C = 900 - 100x$. Show that the profit to the travel agency on a holiday with a group of x people is given by $P = 700x - 500x^2$.

9. Joel is 700 km north of a town, travelling towards it at an average speed of 75 km/h. Nick is 680 km east of the town, travelling towards it at 80 km/h.



Show that after t hours, the distance between Joel and Nick is given by $d = \sqrt{952400 - 213800t + 12025t^2}$.

10. Sean wants to swim from point A to point B across a 500 m wide river, then walk along the river bank to point C. The distance along the river bank is 7 km.



If he swims at 5 km/h and walks at 4 km/h, show that the time taken to reach point C is given by $t = \frac{\sqrt{x^2 + 0.25}}{5} + \frac{7 - x}{4}$.

When you have found the equation, you can use calculus to find the maximum or minimum value. The process is the same as for finding stationary points on curves.

Always check that an answer gives a maximum or minimum value. Use the second derivative to find its concavity, or if the second derivative is too hard to find, check the first derivative either side of this value.

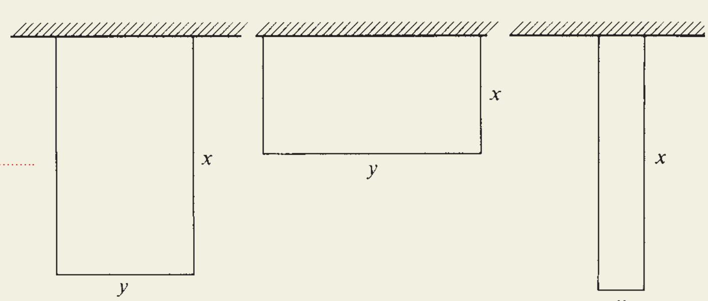
EXAMPLES

1. The council wanted to make a rectangular swimming area at the beach using a straight cliff on one side and a length of 300 m of sharkproof netting for the other three sides. What are the dimensions of the rectangle that encloses the greatest area?

Solution

Let the length of the rectangle be y and the width be x.

There are many differently shaped rectangles that could have a perimeter of 300 m.



Perimeter: $2x + y = 300\text{m}$

$\therefore \quad y = 300 - 2x \qquad (1)$

$$\begin{aligned} \text{Area } A &= xy \\ &= x(300 - 2x) \qquad [\text{substituting } (1)] \\ &= 300x - 2x^2 \\ \frac{dA}{dx} &= 300 - 4x \end{aligned}$$

For stationary points, $\frac{dA}{dx} = 0$

i.e. $\quad 300 - 4x = 0$

$300 = 4x$

$75 = x$

$\frac{d^2A}{dx^2} = -4 < 0 \qquad$ (concave downwards)

So $x = 75$ gives maximum A.

When $x = 75$, $y = 300 - 2(75)$

$= 150$

So the dimensions that give the maximum area are 150 m × 75 m.

---

2. Trinh and Soi set out from two towns. They travel on roads that meet at right angles, and they walk towards the intersection. Trinh is initially 15 km from the intersection and walks at 3 km/h. Soi is initially 10 km from the intersection and walks at 4 km/h.

(a) Show that their distance apart after t hours is given by $D^2 = 25t^2 - 170t + 325$.

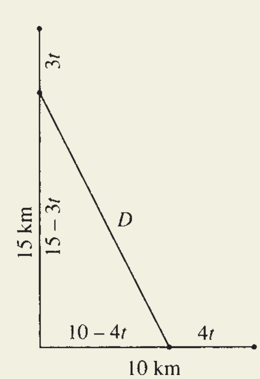
(b) Hence find how long it takes them to reach their minimum distance apart.

(c) Find their minimum distance apart.

Solution

(a) After t hours, Trinh has walked $3t$ km. She is now $15 - 3t$ km from the intersection.

After t hours, Soi has walked $4t$ km. He is now $10 - 4t$ km from the intersection.



CONTINUED

By Pythagoras' theorem:

$$\begin{aligned} D^2 &= (15-3t)^2 + (10-4t)^2 \\ &= 225 - 90t + 9t^2 + 100 - 80t + 16t^2 \\ &= 25t^2 - 170t + 325 \end{aligned}$$

(b) $\dfrac{dD^2}{Dt} = 50t - 170$

For stationary points, $\dfrac{dD^2}{dt} = 0$

i.e. $\quad 50t - 170 = 0$

$50t = 170$

$t = 3.4$

$\dfrac{d^2D^2}{dt^2} = 50 > 0$ (concave upwards)

So $t = 3.4$ gives minimum D^2.

Now 0.4×60 minutes $= 24$ minutes

$\therefore$ 3.4 hours = 3 hours and 24 minutes

So Trinh and Soi are a minimum distance apart after 3 hours and 24 minutes.

(c) When $t = 3.4$, $D^2 = 25(3.4)^2 - 170(3.4) + 325$

$= 36$

$D = \sqrt{36}$

$= 6$

So the minimum distance apart is 6 km.

2.10 Exercises

1. The height, in metres, of a ball is given by the equation $h = 16t - 4t^2$, where t is time in seconds. Find when the ball will reach its maximum height, and what the maximum height will be.

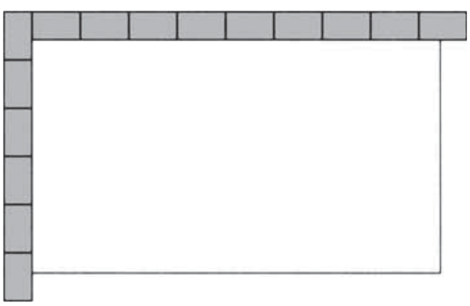
2. The cost per hour of a bike ride is given by the formula $C = x^2 - 15x + 70$, where x is the distance travelled in km. Find the distance that gives the minimum cost.

3. The perimeter of a rectangle is 60 m and its length is x m. Show that the area of the rectangle is given by the equation $A = 30x - x^2$. Hence find the maximum area of the rectangle.

4. A farmer wants to make a rectangular paddock with an area of 4000 m². However, fencing costs are high and she wants the paddock to have a minimum perimeter.

(a) Show that the perimeter is given by the equation $P = 2x + \dfrac{8000}{x}$.
(b) Find the dimensions of the rectangle that will give the minimum perimeter, correct to 1 decimal place.
(c) Calculate the cost of fencing the paddock, at $48.75 per metre.

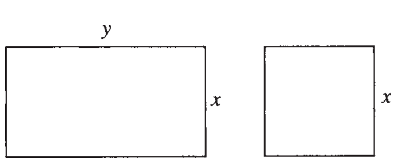
5. Bill wants to put a small rectangular vegetable garden in his backyard using two existing walls as part of its border. He has 8 m of garden edging for the border on the other two sides. Find the dimensions of the garden bed that will give the greatest area.



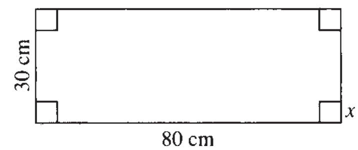
6. Find two numbers whose sum is 28 and whose product is a maximum.

7. The difference of two numbers is 5. Find these numbers if their product is to be minimum.

8. A piece of wire 10 m long is broken into two parts, which are bent into the shape of a rectangle and a square as shown. Find the dimensions x and y that make the total area a maximum.



9. A box is made from an 80 cm by 30 cm rectangle of cardboard by cutting out 4 equal squares of side x cm from each corner. The edges are turned up to make an open box.



(a) Show that the volume of the box is given by the equation $V = 4x^3 - 220x^2 + 2400x$.
(b) Find the value of x that gives the box its greatest volume.
(c) Find the maximum volume of the box.

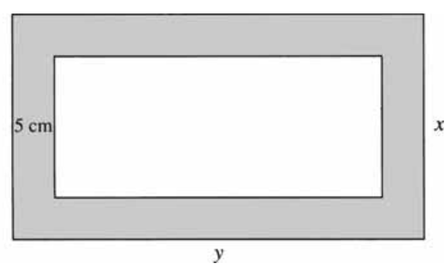
10. The formula for the surface area of a cylinder is given by $S = 2\pi r(r + h)$, where r is the radius of its base and h is its height. Show that if the cylinder holds a volume of $54\pi \text{m}^3$, the surface area is given by the equation $S = 2\pi r^2 + \dfrac{108\pi}{r}$.
Hence find the radius that gives the minimum surface area.

11. A silo in the shape of a cylinder is required to hold 8600 m^3 of wheat.

(a) Find an equation for the surface area of the silo in terms of the base radius.
(b) Find the minimum surface area required to hold this amount of wheat, to the nearest square metre.

12. A rectangle is cut from a circular disc of radius 6 cm. Find the area of the largest rectangle that can be produced.

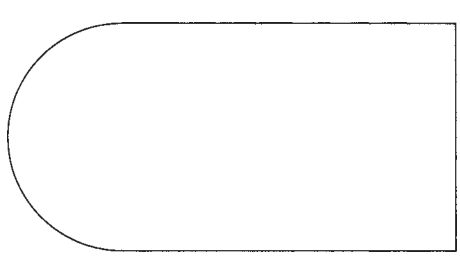
13. A poster consists of a photograph bordered by a 5 cm margin. The area of the poster is to be 400 cm^2.



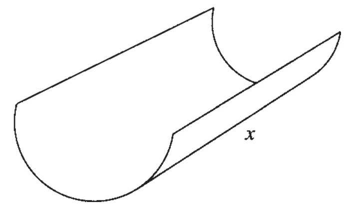
(a) Show that the area of the photograph is given by the equation $A = 500 - 10x - \frac{4000}{x}$
(b) Find the maximum area possible for the photograph.

14. The sum of the dimensions of a box with a square base is 60 cm. Find the dimensions that will give the box a maximum volume.

15. A surfboard is in the shape of a rectangle and semicircle, as shown. The perimeter is to be 4 m. Find the maximum area of the surfboard, correct to 2 decimal places.



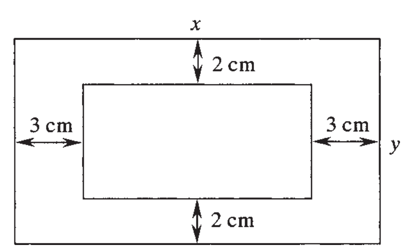
16. A half-pipe is to be made from a rectangular piece of metal of length x m. The perimeter of the rectangle is 30 m.



(a) Find the dimensions of the rectangle that will give the maximum surface area.
(b) Find the height from the ground up to the top of the half-pipe with this maximum area, correct to 1 decimal place.

17. Find the least surface area, to the nearest cm^2, of a closed cylinder that will hold a volume of 400 cm^3.

18. The picture frame shown below has a border of 2 cm at the top and bottom and 3 cm at the sides. If the total area of the border is to be 100 cm^2, find the maximum area of the frame.

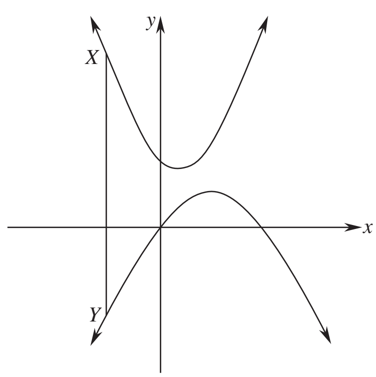


19. A 3 m piece of wire is cut into two pieces and bent around to form a square and a circle. Find the size of the two lengths, correct to 2 decimal places, that will make the total area of the square and circle a minimum.

20. Two cars are travelling along roads that intersect at right angles to one another. One starts 200 km away and travels towards the intersection at 80 kmh^{-1}, while the other starts at 120 km away and travels towards the intersection at 60 kmh^{-1}.

Show that their distance apart after t hours is given by $d^2 = 10\,000t^2 - 46\,400t + 54\,400$ and hence find their minimum distance apart.

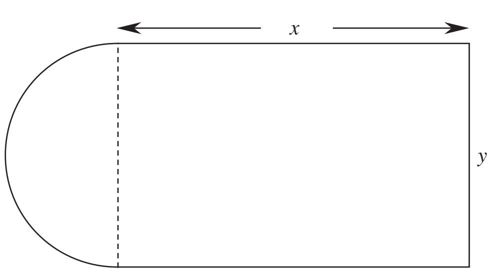
21. X is a point on the curve $y = x^2 - 2x + 5$. Point Y lies directly below X and is on the curve $y = 4x - x^2$.



(a) Show that the distance d between X and Y is given by $d = 2x^2 - 6x + 5$.

(b) Find the minimum distance between X and Y.

22. A park is to have a dog-walking enclosure in the shape of a rectangle with a semi-circle at the end as shown, and a perimeter of 1200 m.

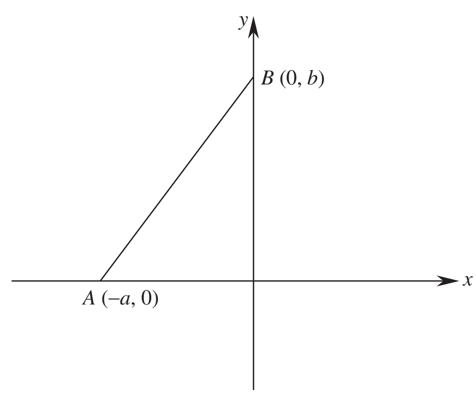


(a) Show that $x = \dfrac{2400 - 2y - \pi y}{4}$.

(b) Show that the area of the enclosure is given by $A = \dfrac{4800y - 4y^2 - \pi y^2}{8}$.

(c) Find the dimensions x and y (to the nearest metre) that maximises the area.

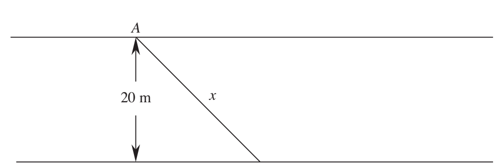
23. Points $A(-a, 0)$, $B(0, b)$ and $O(0, 0)$ form a triangle as shown and AB always passes through the point $(-1, 2)$.



(a) Show that $b = \dfrac{2a}{a - 1}$.

(b) Find values of a and b that give the minimum area of triangle OAB.

24. Grant is at point A on one side of a 20 m wide river and needs to get to point B on the other side 80 m along the bank as shown.



Grant swims to any point on the other bank and then runs along the side of the river to point B. If he can swim at 7 km/h and run at 11 km/h, find the distance he swims (x) to minimise the time taken to reach point B. Answer to the nearest metre.

25. A truck travels 1500 km at an hourly cost given by $s^2 + 9000$ cents where s is the average speed of the truck.
(a) Show that the cost for the trip is given by $C = 1500\left(s + \frac{9000}{s}\right)$.
(b) Find the speed that minimises the cost of the trip.
(c) Find the cost of the trip to the nearest dollar.

Class Challenge

Can you solve either of these problems?

1. **Heron's problem**
One boundary of a farm is a straight river bank, and on the farm stands a house and some distance away, a shed; each is sited away from the river bank. Each morning the farmer takes a bucket from his house to the river, fills it with water, and carries the water to the shed.
Find the position on the river bank that will allow him to walk the shortest distance from house to river to shed. Further, describe how the farmer could solve the problem on the ground with the aid of a few stakes for sighting.

2. **Lewis Carroll's problem**
After a battle at least 95% of the combatants had lost a tooth, at least 90% had lost an eye, at least 80% had lost an arm, and at least 75% had lost a leg. At least how many had lost all four?

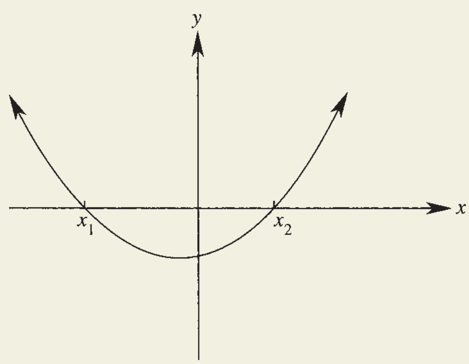
Primitive Functions

This chapter uses differentiation to find the gradient of tangents and stationary points of functions.

Sometimes you may know $f'(x)$ and need to find the original function, $f(x)$. This process is called **anti-differentiation**, and the original function is called the **primitive function**.

EXAMPLE

Sketch the primitive function (the original function) given the derivative function below.



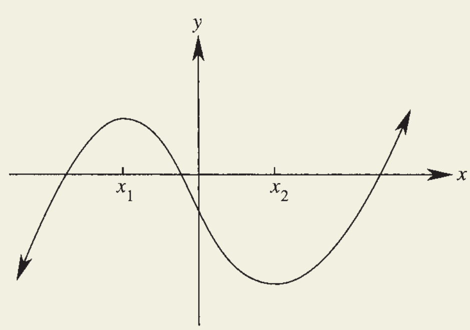
Solution

Reversing what you would do to sketch the derivative function, the parts at the x-axis have a zero gradient, so show stationary points on the original function. There are stationary points at x_1 and x_2.

The parts of the graph above the x-axis show a positive gradient, so the original function is increasing. This happens to the left of x_1 and to the right of x_2.

The parts of the graph below the x-axis show a negative gradient, so the original function is decreasing. This happens between x_1 and x_2.

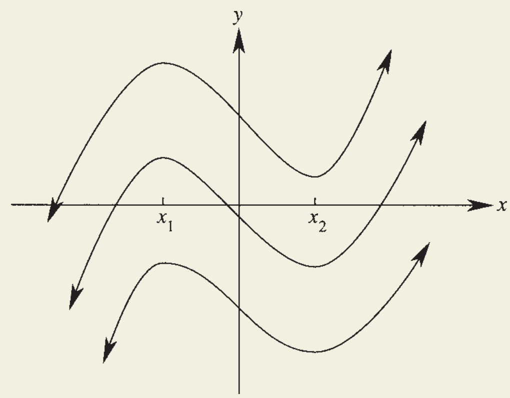
Sketching this information gives:



CONTINUED

The only problem is, we do not have enough information to know where the curve turns around. There are many choices of where we could put the graph:

The primitive function gives a family of curves.



The examples below use what we know about differentiation to find out how to reverse this to find the primitive function.

EXAMPLES

1. Differentiate x^2. Hence find a primitive function of $2x$.

Solution

The derivative of x^2 is $2x$
$\therefore$ a primitive function of $2x$ is x^2

---

2. Differentiate $x^2 + 5$. Hence find a primitive function of $2x$.

Solution

The derivative of $x^2 + 5$ is $2x$
$\therefore$ a primitive function of $2x$ is $x^2 + 5$

---

3. Differentiate $x^2 - 3$. Hence find a primitive function of $2x$.

Solution

The derivative of $x^2 - 3$ is $2x$
$\therefore$ a primitive function of $2x$ is $x^2 - 3$

Thus $2x$ has many different primitive functions. In general, the primitive of $2x$ is $x^2 + C$, where C is a constant (real number).

EXAMPLES

1. Find the primitive of x.

Solution

The derivative of x^2 is $2x$.

The derivative of $\frac{x^2}{2}$ is x. So the primitive of x is $\frac{x^2}{2} + C$.

2. Find the primitive of x^2.

Solution

The derivative of x^3 is $3x^2$.

The derivative of $\frac{x^3}{3}$ is x^2.

So the primitive of x^2 is $\frac{x^3}{3} + C$.

Continuing this pattern gives the general primitive function of x^n.

If $\frac{dy}{dx} = x^n$ then $y = \frac{x^{n+1}}{n+1} + C$

where C is a constant

Proof

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) &= \frac{(n+1)x^{n+1-1}}{n+1} \\ &= x^n\end{aligned}$$

EXAMPLES

1. The gradient of a curve is given by $\frac{dy}{dx} = 6x^2 + 8x$. If the curve passes through the point $(1, -3)$, find the equation of the curve.

Solution

$$\begin{aligned}\frac{dy}{dx} &= 6x^2 + 8x \\ y &= 6\left(\frac{x^3}{3}\right) + 8\left(\frac{x^2}{2}\right) + C \\ \therefore \quad y &= 2x^3 + 4x^2 + C\end{aligned}$$

CONTINUED

The curve passes through $(1, -3)$

$$\therefore\ -3 = 2(1)^3 + 4(1)^2 + C$$
$$= 2 + 4 + C$$
$$-9 = C$$

Equation is $y = 2x^3 + 4x^2 - 9$.

---

2. If $f''(x) = 6x + 2$ and $f'(1) = f(-2) = 0$, find $f(3)$.

Solution

$$f''(x) = 6x + 2$$
$$f'(x) = 6\left(\frac{x^2}{2}\right) + 2x + C$$
$$= 3x^2 + 2x + C$$

Now $f'(1) = 0$

So $0 = 3(1)^2 + 2(1) + C$

$-5 = C$

$\therefore\ f'(x) = 3x^2 + 2x - 5$

$$f(x) = 3\left(\frac{x^3}{3}\right) + 2\left(\frac{x^2}{2}\right) - 5x + C$$
$$= x^3 + x^2 - 5x + C$$

Now $f(-2) = 0$

So $0 = (-2)^3 + (-2)^2 - 5(-2) + C$

$= -8 + 4 + 10 + C$

$-6 = C$

$\therefore\ f(x) = x^3 + x^2 - 5x - 6$

$$f(3) = 3^3 + 3^2 - 5(3) - 6$$
$$= 27 + 9 - 15 - 6$$
$$= 15$$

2.11 Exercises

1. Find the primitive function of
   (a) $2x - 3$
   (b) $x^2 + 8x + 1$
   (c) $x^5 - 4x^3$
   (d) $(x - 1)^2$
   (e) 6

2. Find $f(x)$ if
   (a) $f'(x) = 6x^2 - x$
   (b) $f'(x) = x^4 - 3x^2 + 7$
   (c) $f'(x) = x - 2$
   (d) $f'(x) = (x + 1)(x - 3)$
   (e) $f'(x) = x^{\frac{1}{2}}$

3. Express y in terms of x if
   (a) $\frac{dy}{dx} = 5x^4 - 9$
   (b) $\frac{dy}{dx} = x^{-4} - 2x^{-2}$
   (c) $\frac{dy}{dx} = \frac{x^3}{5} - x^2$
   (d) $\frac{dy}{dx} = \frac{2}{x^2}$
   (e) $\frac{dy}{dx} = x^3 - \frac{2x}{3} + 1$

4. Find the primitive function of
   (a) $\sqrt{x}$
   (b) x^{-3}
   (c) $\frac{1}{x^8}$
   (d) $x^{-\frac{1}{2}} + 2x^{-\frac{2}{3}}$
   (e) $x^{-7} - 2x^{-2}$

5. If $\frac{dy}{dx} = x^3 - 3x^2 + 5$ and $y = 4$ when $x = 1$, find an equation for y in terms of x.

6. If $f'(x) = 4x - 7$ and $f(2) = 5$, find $f(x)$.

7. Given $f'(x) = 3x^2 + 4x - 2$ and $f(-3) = 4$, find $f(1)$.

8. Given that the gradient of the tangent to a curve is given by $\frac{dy}{dx} = 2 - 6x$ and the curve passes through $(-2, 3)$, find the equation of the curve.

9. If $\frac{dx}{dt} = (t - 3)^2$ and $x = 7$ when $t = 0$, find x when $t = 4$.

10. Given $\frac{d^2y}{dx^2} = 8$ and $\frac{dy}{dx} = 0$ and $y = 3$ when $x = 1$, find the equation of y in terms of x.

11. If $\frac{d^2y}{dx^2} = 12x + 6$ and $\frac{dy}{dx} = 1$ at the point $(-1, -2)$, find the equation of the curve.

12. If $f''(x) = 6x - 2$ and $f'(2) = f(2) = 7$, find the function $f(x)$.

13. Given $f''(x) = 5x^4$, $f'(0) = 3$ and $f(-1) = 1$, find $f(2)$.

14. If $\frac{d^2y}{dx^2} = 2x + 1$ and there is a stationary point at $(3, 2)$, find the equation of the curve.

15. A curve has $\frac{d^2y}{dx^2} = 8x$ and the tangent at $(-2, 5)$ makes an angle of $45°$ with the x-axis. Find the equation of the curve.

16. The tangent to a curve with $\frac{d^2y}{dx^2} = 2x - 4$ makes an angle of $135°$ with the x-axis in the positive direction at the point $(2, -4)$. Find its equation.

17. A function has a tangent parallel to the line $4x - y - 2 = 0$ at the point $(0, -2)$ and $f''(x) = 12x^2 - 6x + 4$. Find the equation of the function.

18. A curve has $\frac{d^2y}{dx^2} = 6$ and the tangent at $(-1, 3)$ is perpendicular to the line $2x + 4y - 3 = 0$. Find the equation of the curve.

19. A function has $f'(1) = 3$ and $f(1) = 5$. Evaluate $f(-2)$ given $f''(x) = 6x + 18$.

20. A curve has $\frac{d^2y}{dx^2} = 12x - 24$ and a stationary point at $(1, 4)$. Evaluate y when $x = 2$.

Test Yourself 2

1. Find the stationary points on the curve $y = x^3 + 6x^2 + 9x - 11$ and determine their nature.

2. Find all x-values for which the curve $y = 2x^3 - 7x^2 - 3x + 1$ is concave upwards.

3. A curve has $\frac{dy}{dx} = 6x^2 + 12x - 5$. If the curve passes through the point $(2, -3)$, find the equation of the curve.

4. If $f(x) = 3x^5 - 2x^4 + x^3 - 2$, find
   (a) $f(-1)$
   (b) $f'(-1)$
   (c) $f''(-1)$

5. The height in metres of an object thrown up into the air is given by $h = 20t - 2t^2$ where t is time in seconds. Find the maximum height that the object reaches.

6. Find the stationary point on the curve $y = 2x^2$ and determine its nature.

7. Find the domain over which the curve $y = 5 - 6x - 3x^2$ is decreasing.

8. For the curve $y = 3x^4 + 8x^3 - 48x^2 + 1$
   (a) find any stationary points and determine their nature
   (b) sketch the curve.

9. Find the point of inflexion on the curve $y = 2x^3 - 3x^2 + 3x - 2$.

10. If $f''(x) = 15x + 12$, and $f(2) = f'(2) = 5$, find $f(x)$.

11. A soft drink manufacturer wants to minimise the amount of aluminium in its cans while still holding 375 mL of soft drink. Given that 375 mL has a volume of 375 cm^3,
    (a) show that the surface area of a can is given by $S = 2\pi r^2 + \frac{750}{r}$
    (b) find the radius of the can that gives the minimum surface area.

12. For the curve $y = 3x^4 + 8x^3 + 6x^2$,
    (a) find any stationary points
    (b) determine their nature
    (c) sketch the curve for $-3 \le x \le 3$.

13. A rectangular prism with a square base is to have a surface area of 250 cm^2.
    (a) Show that the volume is given by $V = \frac{125x - x^3}{2}$.
    (b) Find the dimensions that will give the maximum volume.

14. Find all x-values for which the curve $y = 3x^2 - 18x + 4$ is decreasing.

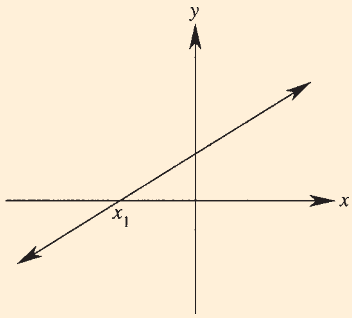
15. If $\frac{d^2y}{dx^2} = 6x + 6$, and there is a stationary point at $(0, 3)$, find the equation of the curve.

16. The cost to a business of manufacturing x products a week is given by $C = x^2 - 300x + 9000$. Find the number of products that will give the minimum cost each week.

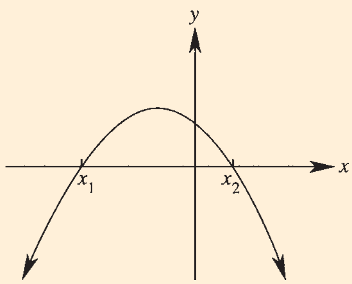
17. Show that $y = -x^3$ is monotonic decreasing for all $x \neq 0$.

18. Sketch a primitive function for each graph.

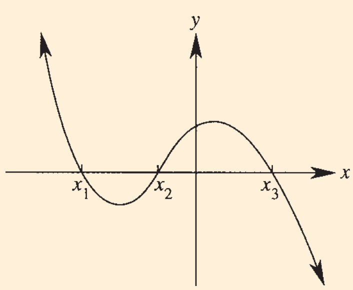
(a)



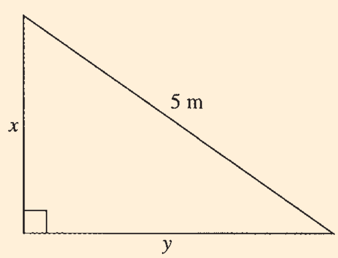
(b)



(c)



19. A 5 m length of timber is used to border a triangular garden bed, with the other sides of the garden against the house walls.



(a) Show that the area of the garden is $A = \frac{1}{2}x\sqrt{25 - x^2}$.

(b) Find the greatest possible area of the garden bed.

20. For the curve $y = x^3 + 3x^2 - 72x + 5$,
(a) find any stationary points on the curve and determine their nature
(b) find any points of inflexion
(c) sketch the curve.

21. Find the domain over which the curve $y = x^3 - 3x^2 + 7x - 5$ is concave downwards.

22. A function has $f'(3) = 5$ and $f(3) = 2$. If $f''(x) = 12x - 6$, find the equation of the function.

23. Find any points of inflexion on the curve $y = x^4 - 6x^3 + 2x + 1$.

24. Find the maximum value of the curve $y = x^3 + 3x^2 - 24x - 1$ in the domain $-5 \le x \le 6$.

25. A function has $f'(2) < 0$ and $f''(2) < 0$. Sketch the shape of the function near $x = 2$.

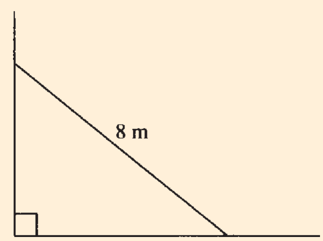
Challenge Exercise 2

1. Find the first and second derivatives of $\frac{5-x}{(4x^2+1)^3}$.

2. Sketch the curve $y = x(x-2)^3$, showing any stationary points and inflexions.

3. Find all values of x for which the curve $y = 4x^3 - 21x^2 - 24x + 5$ is increasing.

4. Find the maximum possible area if a straight 8 m length of fencing is placed across a corner to enclose a triangular space.



5. Find the greatest and least values of the function $f(x) = 4x^3 - 3x^2 - 18x$ in the domain $-2 \le x \le 3$.

6. Show that the function $f(x) = 2(5x-3)^3$ has a horizontal point of inflexion at $(0.6, 0)$.

7. Two circles have radii r and s such that $r + s = 25$. Show that the sum of areas of the circles is least when $r = s$.

8. The rate of change of V with respect to t is given by $\frac{dV}{dt} = (2t-1)^2$. If $V = 5$ when $t = \frac{1}{2}$, find V when $t = 3$.

9. (a) Show that the curve $y = \sqrt{x-1}$ has no stationary points.
(b) Find the domain and range of the curve.
(c) Hence sketch the curve.

10. Find the radius and height, correct to 2 decimal places, of a cylinder that holds 200 cm^3, if its surface area is to be a minimum.

11. A curve passes through the point $(0, -1)$ and the gradient at any point is given by $(x+3)(x-5)$. Find the equation of the curve.

12. The cost of running a car at an average speed of v km/h is given by $c = 150 + \frac{v^2}{80}$ cents per hour. Find the average speed, to the nearest km/h, at which the cost of a 500 km trip is a minimum.

13. Find the equation of a curve that is always concave upwards with a stationary point at $(-1, 2)$ and y-intercept 3.

14. Given $f'(x) = x(x-3)(x+1)$ and $f(0) = 0$
(a) find the equation for $f(x)$
(b) find any stationary points on the function
(c) sketch the function.

15. The volume of air in a cubic room on a submarine is given by the formula $V = -4x^3 + 27x^2 - 24x + 2$, where x is the side of the room in metres. Find the dimensions of the room that will give the maximum volume of air.

16. If $f''(x) = x - 9$ and $f'(-1) = f(-2) = 7$, find $f(3)$.

17. Given $\frac{dy}{dx} = \frac{1}{(x-5)^2}$ and $y = 1$ when $x = 4$, find
(a) $\frac{d^2y}{dx^2}$ when $x = 6$
(b) y when $x = 6$.

18. Show that $y = x^n$ has a stationary point at (0, 0) where n is a positive integer.
(a) If n is even, show that (0, 0) is a minimum turning point.
(b) If n is odd, show that (0, 0) is a point of inflexion.

19. Find the maximum possible volume if a rectangular prism with a length twice its breadth has a surface area of 48 cm^2.

20. (a) Find the stationary point on the curve $y = x^k + 1$ where k is a positive integer.
(b) What values of k give a minimum turning point?

21. Find the minimum and maximum values of $y = \frac{x+3}{x^2-9}$ in the domain $-2 \le x \le 2$.

22. The cost of running a car at an average speed of V km/h is given by $c = 100 + \frac{V^2}{75}$ cents per hour. Find the average speed (to the nearest km/h) at which the cost of a 1000 km trip is a minimum.