6

Applications of Calculus to the Physical World

TERMINOLOGY

Acceleration: The rate of change of velocity with respect to time

At rest: Stationary (zero velocity)

Displacement: The movement of an object in relation to its original position

Exponential growth: Growth, or increase in a quantity, where the rate of change of a quantity is in direct proportional to the quantity itself. The growth becomes more rapid over time

Exponential decay: Decay, or decrease in a quantity, where the rate of change of a quantity is in direct proportion to the quantity itself. The decay becomes less rapid over time

Rate of change: The change of one variable with respect to another variable over time

Velocity: The rate of change of displacement of an object with respect to time involving speed and direction

INTRODUCTION

CALCULUS IS USED IN many situations involving rates of change, such as physics or economics. This chapter looks at **rates of change of a particle in motion**, and **exponential growth and decay**. Both these types of rates of change involve calculus.

DID YOU KNOW?

Galileo (1564–1642) was very interested in the behaviour of bodies in motion. He dropped stones from the Leaning Tower of Pisa to try to prove that they would fall with equal speed. He rolled balls down slopes to prove that they move with uniform speed until friction slows them down. He showed that a body moving through the air follows a curved path at a fairly constant speed.

John Wallis (1616–1703) continued this study with his publication *Mechanica*, *sive Tractatus de Motu Geometricus*. He applied mathematical principles to the laws of motion and stimulated interest in the subject of mechanics.

Soon after Wallis' publication, **Christiaan Huygens** (1629–1695) wrote *Horologium Oscillatorium sive de Motu Pendulorum*, in which he described various mechanical principles. He invented the pendulum clock, improved the telescope and investigated circular motion and the descent of heavy bodies.

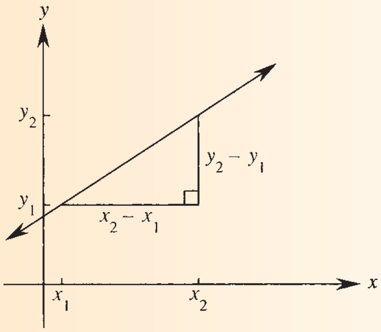
These three mathematicians provided the foundations of mechanics. **Sir Isaac Newton** (1642–1727) used calculus to increase the understanding of the laws of motion. He also used these concepts as a basis for his theories on gravity and inertia.

Galileo

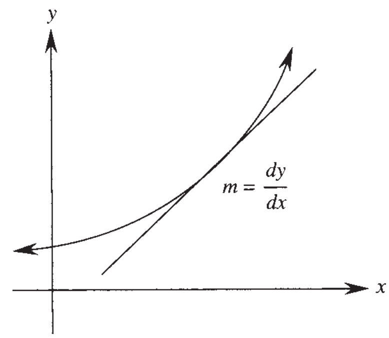
Rates of Change

Simple rates of change

The gradient of a straight line measures the rate of change of y with respect to the change in x.



With a curve, the rate of change of y with respect to x is measured by the gradient of the tangent to the curve. That is, the derivative measures the rate of change.



The rate of change of a quantity Q with respect to time t is measured by $\frac{dQ}{dt}$.

EXAMPLES

1. The number of bacteria in a culture increases according to the formula $B = 2t^4 - t^2 + 2000$, where t is time in hours. Find

(a) the number of bacteria initially

(b) the number of bacteria after 5 hours

(c) the rate at which the number of bacteria will be increasing after 5 hours.

Solution

(a)
$$B = 2t^4 - t^2 + 2000$$

Initially, $t = 0$

$$\therefore \quad B = 2(0)^4 - 0^2 + 2000 = 2000$$

So there are 2000 bacteria initially.

(b) When $t = 5$,

$$B = 2(5)^4 - 5^2 + 2000 = 3225$$

So there will be 3225 bacteria after 5 hours.

(c) The rate of change is given by the derivative

$$\frac{dB}{dt} = 8t^3 - 2t$$

When $t = 5$,

$$\frac{dB}{dt} = 8(5)^3 - 2(5) = 990$$

So the rate of increase after 5 hours will be 990 bacteria per hour.

2. The volume flow rate of water into a pond is given by $R = 4 + 3t^2$ litres per hour. If there is initially no water in the pond, find the volume of water after 12 hours.

Solution

$$R = 4 + 3t^2$$

i.e. $\dfrac{dV}{dt} = 4 + 3t^2$

$\therefore$ $V = \int (4 + 3t^2)\,dt$

$= 4t + t^3 + C$

When $t = 0, V = 0$

$\therefore$ $0 = 0 + 0 + C$

$= C$

$\therefore$ $V = 4t + t^3$

When $t = 12$

$V = 4(12) + (12)^3$

$= 1776 \text{ L}$

So there will be 1776 L of water in the pond after 12 hours.

Volume flow rate is the rate of change of volume over time.

6.1 Exercises

1. Find a formula for the rate of change in each question.
   (a) $h = 20t - 4t^2$
   (b) $D = 5t^3 + 2t^2 + 1$
   (c) $A = 16x - 2x^2$
   (d) $x = 3t^5 - x^4 + 2x - 3$
   (e) $V = e^t + 4$
   (f) $S = 3\cos 5\theta$
   (g) $S = 2\pi r + \dfrac{50}{r^2}$
   (h) $D = \sqrt{x^2 - 4}$
   (i) $S = 800r + \dfrac{400}{r}$
   (j) $V = \dfrac{4}{3}\pi r^3$

2. Find an original formula for each question given its rate of change.
   (a) $R = 4t - 12t^2$ is the rate at which the height h of an object changes over time t
   (b) $R = 8x^3 + 1$ is the rate at which the area A of a figure changes with side x
   (c) $R = 4\pi r^2$ is the rate at which the volume V changes with radius r
   (d) $R = 7\sin t$ is the rate at which the distance d of an object changes over time t
   (e) $R = 8e^{2t} - 3$ is the rate at which the speed s changes over time t.

3. If $h = t^3 - 7t + 5$, find the rate of change of h when $t = 3$.

4. Given $f(t) = \sin 2t$, find the rate of change when $t = \dfrac{\pi}{6}$.

5. A particle moves so that its distance x is given by $x = 2e^{3t}$. Find the exact rate of change when $t = 4$.

6. The volume of water flowing through a pipe is given by $V = t^2 + 3t$. Find the rate at which the water is flowing when $t = 5$.

7. The rate of change of the angle sum S of a polygon with n sides is a constant 180. If S is 360 when $n = 4$, find S when $n = 7$.

8. A particle moves so that the rate of change of distance D over time t is given by $R = 2e^t - 1$. If $D = 10$ when $t = 0$, find D when $t = 3$.

9. For a certain graph, the rate of change of y values with respect to its x values is given by $R = 3x^2 - 2x + 1$. If the graph passes through the point $(-1, 3)$, find its equation.

10. The mass in grams of a melting iceblock is given by the formula $M = t - 2t^2 + 100$, where t is time in minutes. Show that the rate at which the iceblock is melting is given by $R = 1 - 4t$ grams per minute and find the rate at which it will be melting after 5 minutes.

11. The rate of change in velocity over time is given by $\frac{dy}{dt} = 4t + t^2 - t^3$. If the initial velocity is 2 cms^{-1}, find the velocity after 15 s.

12. The rate of flow of water into a dam is given by $R = 500 + 20t$ Lh^{-1}. If there is 15 000 L of water initially in the dam, how much water will there be in the dam after 10 hours?

13. The surface area in cm^2 of a balloon being inflated is given by $S = t^3 - 2t^2 + 5t + 2$, where t is time in seconds. Find the rate of increase in the balloon's surface area after 8 s.

14. According to Boyle's Law, the pressure of a gas is given by the formula $P = \frac{k}{V}$, where k is a constant and V is the volume of the gas. If $k = 100$ for a certain gas, find the rate of change in the pressure when $V = 20$.

15. A circular disc expands as it is heated. The area, in cm^2, of the disc increases according to the formula $A = 4t^2 + t$, where t is time in minutes. Find the rate of increase in the area after 5 minutes.

16. A balloon is inflated so that its increase in volume is at a constant rate of 10 cm^{3}s^{-1}. If its volume is initially 1 cm^3, find its volume after 3 s.

17. The number of people in a certain city is given by $P = 200\,000e^{0.2t}$, where t is time in years. Find the rate at which the population of the city will be growing after 5 years.

18. A radioactive substance has a mass that decreases over time t according to the formula $M = 200e^{-0.1t}$. Find
    (a) the mass of the substance after 20 years, to the nearest gram
    (b) the rate of its decrease after 20 years.

19. If $y = e^{4x}$, show that the rate is given by $R = 4y$.

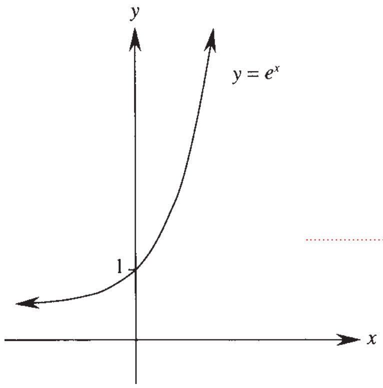
20. Given $S = 2e^{2t} + 3$, show that the rate of change is $R = 2(S - 3)$.

Exponential Growth and Decay

Exponential growth or **decay** are terms that describe a **special rate of change** that occurs in many situations. Population growth and growth of bacteria in a culture are examples of exponential growth. The decay of radioactive substances, the cooling of a substance and change in pressure are examples of exponential decay.

When a substance grows or decays exponentially, its rate of change is directly proportional to the amount of the substance itself. That is, the more of the substance there is, the faster it grows. For example, in a colony of rabbits, there is fairly slow growth in the numbers at first, but the more rabbits there are, the more rabbits will be born. In other species, such as koalas, the clearing of habitats and other factors may cause the population to decrease, or decay, exponentially.

The exponential function is a function where the rate of change (gradient) is increasing as the function increases.



This is why it is called ***exponential*** *growth.*

In mathematical symbols, exponential growth or decay can be written as $\frac{dQ}{dt} = kQ$, where k is the **growth or decay constant**.

$$\frac{dQ}{dt} = kQ \text{ is solved by the equation } Q = Ae^{kt}$$

*When **k** is positive there is exponential **growth**. When **k** is negative there is exponential **decay**.*

Proof

$$\frac{dQ}{dt} = kQ$$
$$\therefore \quad \frac{dt}{dQ} = \frac{1}{kQ}$$
$$t = \int \frac{1}{kQ}\,dQ$$
$$= \frac{1}{k}\log_e Q + k_1$$
$$kt = \log_e Q + k_2$$
$$kt - k_2 = \log_e Q$$
$$\therefore \quad Q = e^{kt - k_2}$$
$$= e^{kt} \times e^{-k_2}$$
$$= Ae^{kt}$$

$$\boxed{A \text{ is always the initial quantity}}$$

Initial means 'at first' when $t = 0$.

Proof

Given $Q = Ae^{kt}$, suppose the initial quantity is Q_0
i.e. $Q = Q_0$ when $t = 0$
Substituting into the equation gives

$$Q_0 = Ae^{k \times 0}$$
$$= Ae^0$$
$$= A$$

$\therefore$ A is the initial quantity.
Sometimes the equation is written as $Q = Q_0 e^{kt}$.

DID YOU KNOW?

Malthus was an economist.

Thomas Maltlhus (1766–1834), at the beginning of the Industrial Revolution, developed a theory about population growth that we still use today. His theory states that under ideal conditions, the birth rate is proportional to the size of the population. That is, $\frac{dN}{dt} = kN$ (Malthusian Law of Population Growth).

Malthus was concerned that the growth rate of populations would be higher than the increase in food supplies, and that people would starve.

Was he right? Is this happening? How could we prove this?

EXAMPLES

1. The population of a colony of rabbits over time t months is given by $P = 1500e^{0.046t}$. Find

(a) the initial population
(b) the population after 2 years
(c) when the population reaches 5000.

Solution

(a) Initially, $t = 0$
Substituting:

$$\begin{aligned}P &= 1500e^{0.046t}\\ &= 1500e^{0.046 \times 0}\\ &= 1500e^{0}\\ &= 1500 \qquad (e^0 = 1)\end{aligned}$$

So the initial population is 1500 rabbits.

(b) 2 years = 24 months
So $t = 24$
Substituting:

$$\begin{aligned}P &= 1500e^{0.046t}\\ &= 1500e^{0.046 \times 24}\\ &= 1500e^{1.104}\\ &= 4524.31\end{aligned}$$

So the population after 2 years is 4524 rabbits.

(c) $P = 5000$
Substituting:

$$\begin{aligned}P &= 1500e^{0.046t}\\ 5000 &= 1500e^{0.046t}\\ \frac{5000}{1500} &= e^{0.046t}\\ \ln\frac{5000}{1500} &= \ln e^{0.046t}\\ &= 0.046t \ln e\\ &= 0.046t \qquad (\text{since } \ln e = 1)\\ \frac{\ln\frac{5000}{1500}}{0.046} &= t\\ 26.2 &= t\end{aligned}$$

So the population reaches 5000 after 26.2 months.

CONTINUED

2. The number of bacteria in a culture is given by $N = Ae^{kt}$. If 6000 bacteria increase to 9000 after 8 hours, find

(a) k correct to 3 significant figures
(b) the number of bacteria after 2 days
(c) the rate at which the bacteria will be increasing after 2 days
(d) when the number of bacteria will reach 1 000 000
(e) the growth rate per hour as a percentage.

Solution

(a)
$$N = Ae^{kt}$$
When $t = 0, N = 6000$
$$\therefore \quad 6000 = Ae^0 = A$$

A gives the initial number of bacteria.

So $N = 6000e^{kt}$

When $t = 8, N = 9000$
$$9000 = 6000e^{8k}$$
$$\frac{9000}{6000} = e^{8k}$$
$$1.5 = e^{8k}$$
$$\log_e 1.5 = \log_e e^{8k} = 8k \log_e e = 8k$$
$$\frac{\log_e 1.5}{8} = k$$
$$0.0507 \doteqdot k$$

*Don't round **k** off, but put it in the calculator's memory to use again.*

So $N = 6000e^{0.0507t}$.

(b) When $t = 48$, (48 hours in 2 days)
$$N = 6000e^{0.0507 \times 48} = 68\,344$$
So there will be 68 344 bacteria after 2 days.

(c) Rate: $\dfrac{dN}{dt} = 6000(0.0507e^{0.0507t}) = 304.1e^{0.0507t}$

When $t = 48$,
$$\frac{dN}{dt} = 304.1e^{0.0507 \times 48} = 3464$$

The rate of growth is the derivative.

So after 2 days the rate of growth will be 3464 bacteria per hour.

Another method:
$$\frac{dN}{dt} = kN = 0.0507N$$
When $t = 48, N = 68\,344$
$$\frac{dN}{dt} = 0.0507 \times 68\,344 = 3464$$

(d) When $N = 1\,000\,000$

$$1\,000\,000 = 6000e^{0.0507t}$$

$$\frac{1\,000\,000}{6000} = e^{0.0507t}$$

$$166.7 = e^{0.0507t}$$

$$\log_e 166.7 = \log_e e^{0.0507t}$$

$$= 0.0507t \log_e e$$

$$= 0.0507t$$

$$\frac{\log_e 166.7}{0.0507} = t$$

$$100.9 \doteqdot t$$

So the number of bacteria will be 1 000 000 after 100.9 hours.

(e) $\frac{dN}{dt} = kN$ where k = growth constant

$$k = 0.0507$$

$$= 5.07\%$$

So the growth rate is 5.07% per hour.

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3. A 50 g mass of uranium decays to 35 g after 2 years. If the rate of decay of its mass is proportional to the mass itself, find the amount of uranium left after 25 years.

Solution

$$Q = Ae^{kt}$$

When $t = 0, Q = 50$

$$\therefore \quad 50 = Ae^0$$

$$= A$$

So $Q = 50e^{kt}$

When $t = 2, Q = 35$

$$35 = 50e^{2k}$$

$$\frac{35}{50} = e^{2k}$$

$$0.7 = e^{2k}$$

$$\log_e 0.7 = \log_e e^{2k}$$

$$= 2k \log_e e$$

$$= 2k$$

$$\frac{\log_e 0.7}{2} = k$$

$$-0.1783 \doteqdot k$$

Notice that k is negative for decay. You could use the formula $Q = Ae^{-kt}$ for decay. What would k be?

So $Q = 50e^{-0.1783t}$

When $t = 25$

$$Q = 50e^{-0.1783 \times 25}$$

$$\doteqdot 0.579$$

So there will be 0.579 grams, or 579 mg, left after 25 years.

6.2 Exercises

1. The number of birds in a colony is given by $N = 80e^{0.02t}$ where t is in days.
   (a) How many birds are there in the colony initially?
   (b) How many birds will there be after 30 days?
   (c) After how many days will there be 500 birds?
   (d) Sketch the curve of the population over time.

2. The number of bacteria in a culture is given by $N = N_0 e^{0.32t}$, where t is time in hours.
   (a) If there are initially 20 000 bacteria, how many will there be after 5 hours?
   (b) How many hours, to the nearest hour, would it take for the number of bacteria to reach 200 000?

3. The decay of radium is proportional to its mass. If 100 kg of radium takes 5 years to decay to 95 kg
   (a) show that the mass of radium is given by $M = 100\, e^{-0.01t}$
   (b) find its mass after 10 years
   (c) find its half-life (the time taken for the radium to halve its mass).

4. A chemical reaction causes the amount of chlorine to be reduced at a rate proportional to the amount of chlorine present at any one time. If the amount of chlorine is given by the formula $A = A_0 e^{-kt}$, and 100 L reduces to 65 L after 5 minutes, find
   (a) the amount of chlorine after 12 minutes
   (b) how long it will take for the chlorine to reduce to 10 L.

5. The production output in a factory increases according to the equation $P = P_0 e^{kt}$, where t is in years.
   (a) Find P_0 if the initial output is 5000 units.
   (b) The factory produces 8000 units after 3 years. Find the value of k, to 3 decimal places.
   (c) How many units will the factory produce after 6 years?
   (d) The factory needs to produce 20 000 units to make a maximum profit. After how many years, correct to 1 decimal place, will this happen?

6. The rate of depletion of rainforests can be estimated as proportional to the amount of rainforests. If 3 million m^2 of rainforest is reduced to 2.7 million m^2 after 20 years, find how much rainforest there will be after 50 years.

7. The annual population of a country is increasing at a rate of 6.9%; that is, $\frac{dP}{dt} = 0.069P$. If the population is 50000 in 2015, find
(a) a formula for the population growth
(b) the population in the year 2020
(c) the rate at which the population will be growing in the year 2020
(d) in which year the population will reach 300000.

8. An object is cooling down according to the formula $T = T_0e^{-kt}$, where T is temperature in degrees Celsius and t is time in minutes. If the temperature is initially 90°C and the object cools down to 81°C after 10 minutes, find
(a) its temperature after half an hour
(b) how long (in hours and minutes) it will take to cool down to 30°C.

9. In the process of the inversion of sugar, the amount of sugar present is given by the formula $S = Ae^{kt}$. If 150 kg of sugar is reduced to 125 kg after 3 hours, find
(a) the amount of sugar after 8 hours, to the nearest kilogram
(b) the rate at which the sugar will be reducing after 8 hours
(c) how long it will take to reduce to 50 kg.

10. The mass, in grams, of a radioactive substance is given by $M = M_0e^{-kt}$, where t is time in years. Find
(a) M_0 and k if a mass of 200 kg decays to 195 kg after 10 years
(b) mass after 15 years
(c) the rate of decay after 15 years
(d) the half-life of the substance (time taken to decay to half its mass).

11. The number of bacteria in a culture increases from 15000 to 25000 in 7 hours. If the rate of bacterial growth is proportional to the number of bacteria at that time, find
(a) a formula for the number of bacteria
(b) the number of bacteria after 12 hours
(c) how long it will take for the culture to produce 5000000 bacteria.

12. A population in a certain city is growing at a rate proportional to the population itself. After 3 years the population increases by 20%. How long will it take for the population to double?

13. The half-life of radium is 1600 years.
(a) Find the percentage of radium that will be decayed after 500 years.
(b) Find the number of years that it will take for 75% of the radium to decay.

14. The population of a city is $P(t)$ at any one time. The rate of decline in population is proportional to the population $P(t)$, that is, $\frac{dP(t)}{dt} = -kP(t)$.
(a) Show that $P(t) = P(t_0)e^{-kt}$ is a solution of the differential equation $\frac{dP(t)}{dt} = -kP(t)$.
(b) What percentage decline in population will there be after 10 years, given a 10% decline in 4 years?
(c) What will the percentage rate of decline in population be after 10 years?
(d) When will the population fall by 20%?

15. The rate of leakage of water out of a container is proportional to the amount of water in the container at any one time. If the container is 60% empty after 5 minutes, find how long it will take for the container to be 90% empty.

16. Numbers of sheep in a certain district are dropping exponentially due to drought. A survey found that numbers had declined by 15% after 3 years. If the drought continues, how long would it take to halve the number of sheep in that district?

17. Anthony has a blood alcohol level of 150 mg/dL. The amount of alcohol in the bloodstream decays exponentially. If it decreases by 20% in the first hour, find
(a) the amount of alcohol in Anthony's blood after 3 hours
(b) when the blood alcohol level reaches 20 mg/dL.

18. The current C flowing in a conductor dissipates according to the formula $\frac{dC}{dt} = -kC$. If it dissipates by 40% in 5 seconds, how long will it take to dissipate to 20% of the original current?

19. Pollution levels in a city have been rising exponentially with a 10% increase in pollution levels in the past two years. At this rate, how long will it take for pollution levels to increase by 50%?

20. If $\frac{dQ}{dt} = kQ$, prove that $Q = Ae^{kt}$ satisfies this equation by
(a) differentiating $Q = Ae^{kt}$
(b) integrating $\frac{dQ}{dt} = kQ$.

Class Investigation

Research some environmental issues. For example:

1. It is estimated that some animals, such as pandas and koalas, will be extinct soon. How soon will pandas be extinct? Can we do anything to stop this extinction?
2. How do the greenhouse effect and global warming affect the earth? How soon will they have a noticeable effect on us?
3. How long do radioactive substances such as radium and plutonium take to decay? What are some of the issues concerning the storage of radioactive waste?
4. The erosion and salination of Australian soil are problems that may affect our farming in the future. Find out about this issue, and some possible solutions.
5. The effect of blue-green algae in some of our rivers is becoming a major problem. What steps have been taken to remedy this situation?

Look at the mathematical aspects of these issues. For example, what formulae are used to make predictions? What sorts of time scales are involved in these issues?

Research other issues, relating to growth and decay, that affect our environment.

Motion of a Particle in a Straight Line

In this study of the motion of a particle moving along a straight line, we ignore friction, gravity and other influences on the motion. We can find the particle's displacement, velocity and acceleration at any one time. The term 'particle', or 'body', can refer to something quite large (e.g. a car). It describes any moving object.

DID YOU KNOW?

Calculus was developed in the 17th century as a solution to problems about the **study of motion**. Some problems of the time included finding the speed and acceleration of planets, calculating the lengths of their orbits, finding maximum and minimum values of functions, finding the direction in which an object is moving at any one time and calculating the areas and volumes of certain figures.

Displacement

Displacement (x) measures the distance of a particle from a fixed point (origin). Unlike distance, displacement can be positive or negative, according to which side of the origin it is on. Usually, on the **right-hand side** it will be **positive** and on the **left** it will be **negative**.

When the particle is at the **origin**, its **displacement** is **zero**. That is, $\boldsymbol{x = 0}$.

Velocity

Velocity (v) is a measure of the rate of change of displacement with respect to time. Since the rate of change is the gradient of the graph of displacement against time,

$$v = \frac{dx}{dt}.$$

Velocity can also be written as $\dot{x}$.

Velocity (unlike speed) can be positive or negative, according to which direction the object is travelling in. If the particle is moving to the **right**, velocity is **positive**. If it is moving to the **left**, velocity is **negative**.

When the object is not moving, we say that it is **at rest**. That is, $\boldsymbol{v = 0}$.

Acceleration

Acceleration is the measure of the **rate of change of velocity** with respect to **time**. This gives the gradient of the graph of velocity against time. That is,

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}.$$

Acceleration can also be written as $\ddot{x}$.

Acceleration can be positive or negative, according to which direction it is in. If the acceleration is to the **right**, it is **positive**. If the acceleration is to the **left**, it is **negative**.

If the acceleration is in the **same direction** as the velocity, the particle is **speeding up** (accelerating). If the acceleration is in the **opposite direction** from the velocity, the particle is **slowing down** (decelerating).

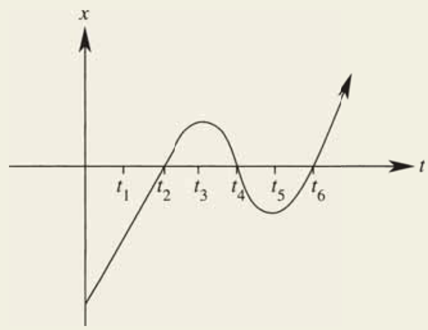
If the object is travelling at a **constant velocity**, there is **no acceleration.** That is, $\boldsymbol{a = 0}$.

Motion graphs

You can describe the velocity of a particle by looking at the gradient function of a displacement graph. The acceleration is the gradient function of a velocity graph.

EXAMPLES

1. The graph below shows the displacement of a particle from the origin as it moves in a straight line.



(a) When is the particle
  (i) at rest?
  (ii) at the origin?

(b) When is the particle moving at its greatest velocity?

Solution

(a) (i) When the particle is at rest, velocity is zero. i.e. $\frac{dx}{dt} = 0$.

So the particle is at rest at the stationary points t_3 and t_5.

(ii) The particle is at the origin when $x = 0$, i.e. on the t-axis. So the particle is at the origin at t_2, t_4 and t_6.

(b) The greatest velocity is at t_4 (the curve is at its steepest). Notice that this is where there is a point of inflexion.

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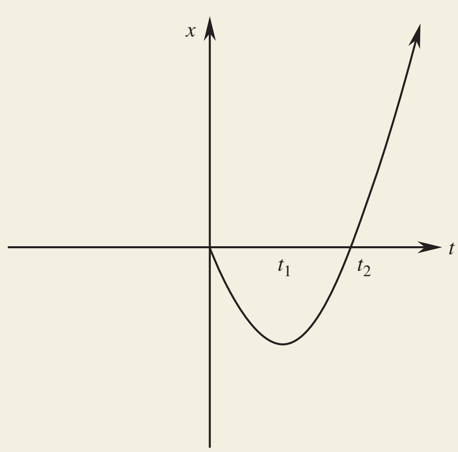
2. The graph below shows the displacement x of a particle over time t.



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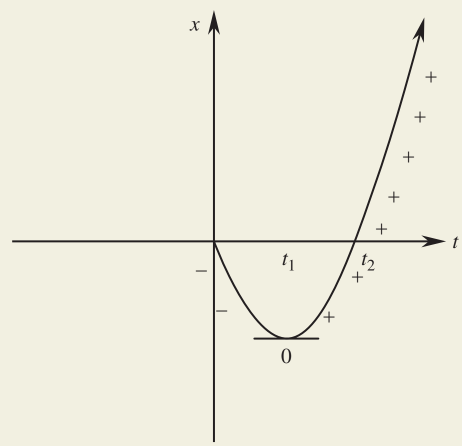
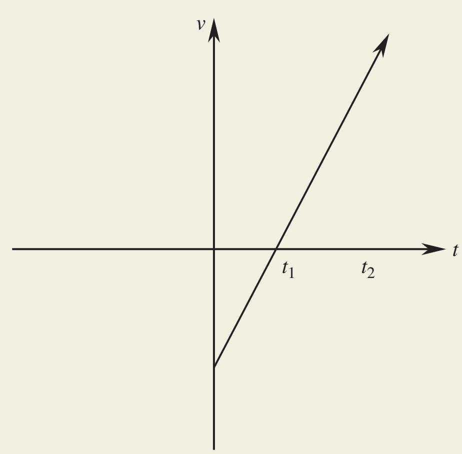
(a) Draw a sketch of its velocity
(b) Sketch its acceleration
(c) Find values of t for which the particle is
  (i) at the origin and
  (ii) at rest.

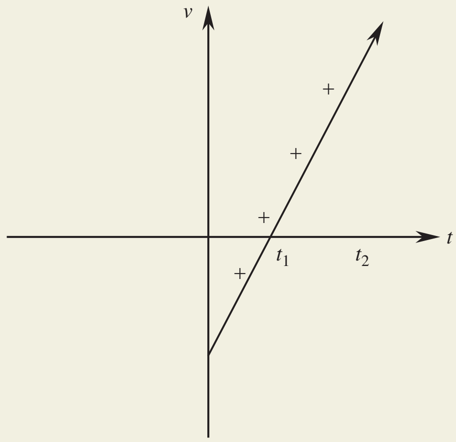
Solution

(a) The velocity is the rate of change of displacement, or $\frac{dx}{dt}$. By noting where the gradient of the tangent is positive, negative and zero, we draw the velocity graph.

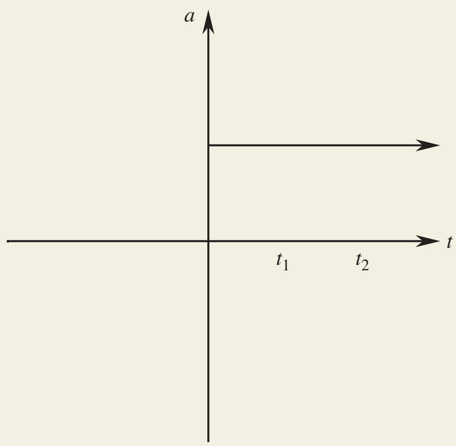




(b) Acceleration is the rate of change of velocity, or $\frac{d^2x}{dt^2}$. By noting where the gradient of the tangent is positive, negative and zero, we draw the acceleration graph.

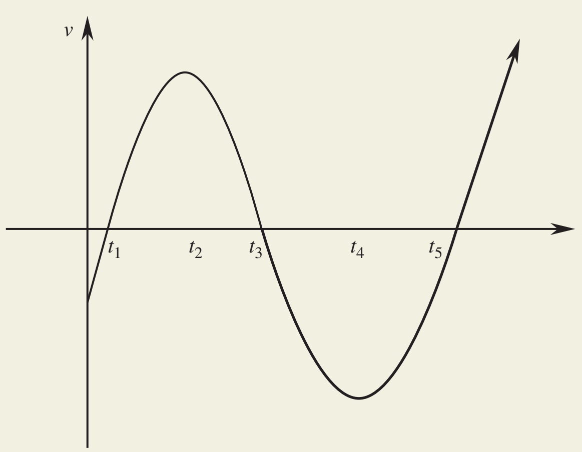


The gradient is a constant, so acceleration is constant.



(c) (i) The particle is at the origin when $x = 0$.
This is at 0 and t_2.
(ii) The particle is at rest when $v = 0$.
This is at t_1 (on the t-axis on the velocity graph or the stationary point on the displacement graph).

3. The graph below is the velocity of a particle.



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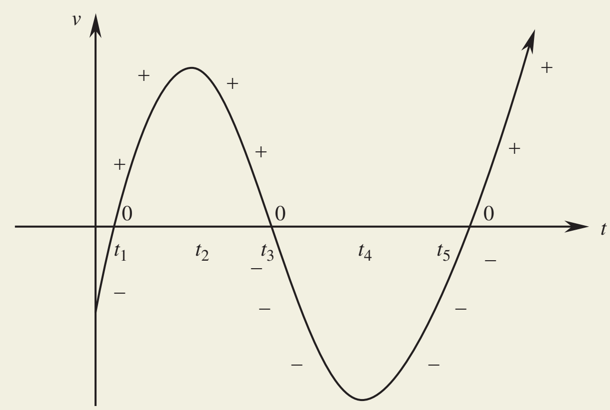
(a) Draw a sketch showing the displacement of the particle.
(b) Find the times when the particle is
   (i) at rest
   (ii) at maximum velocity.
(c) Explain why you can't state accurately when the particle is at the origin.

Solution

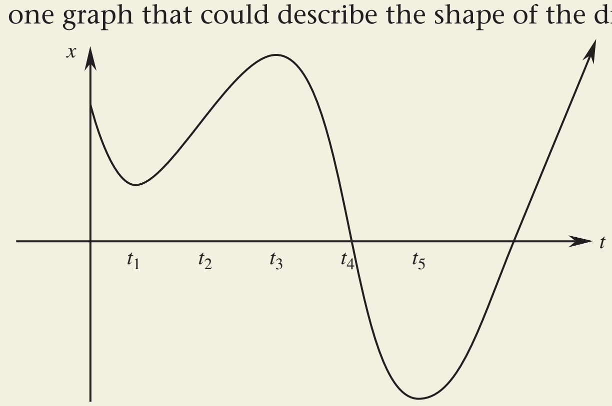
(a) For $0 - t_1$ and between t_3 and t_5, the velocity is negative (below the t-axis), so the displacement has $\frac{dx}{dt} < 0$.

Between t_1 and t_3, and all points $> t_5$, the velocity is positive (above the t-axis), so the displacement has $\frac{dx}{dt} > 0$.

At t_1, t_3 and t_5, the velocity is zero, so $\frac{dx}{dt} = 0$ and there is a stationary point.



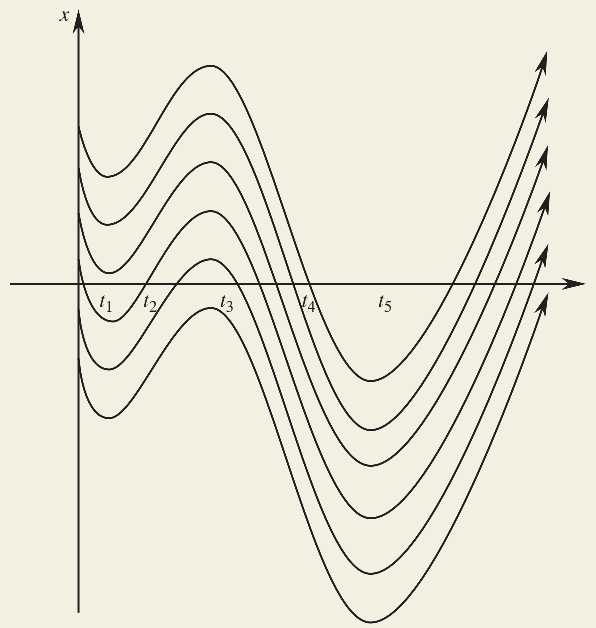
Notice that at t_1 and t_5, LHS < 0 and RHS > 0 so they are minimum turning points.
At t_3, LHS > 0 and RHS < 0 so it is a maximum turning point.
Here is one graph that could describe the shape of the displacement.



(b) (i) At rest, $v = 0$.
This occurs at t_1, t_3 and t_5.

(ii) Maximum velocity is at t_4 since it is furthest from the t-axis. Notice also that there is a point of inflexion at t_4 on the displacement graph.

(c) You can't tell when the particle is at the origin as the graph is not accurate. For example, the graph in (a) is at the origin at t_4 and another point to the right of t_5. However, when sketching displacement from the velocity (or the original function given the gradient or derivative function) there are many possible graphs, forming a family of graphs.

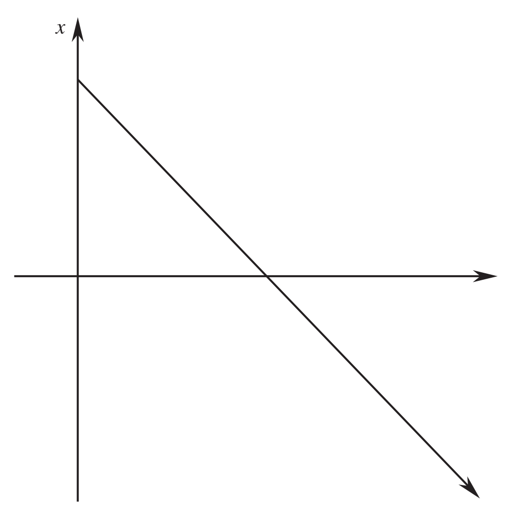


Notice that these cross the t-axis at various places, so we can't tell where the displacement graph shows the particle at the origin.

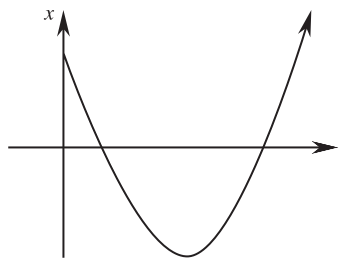
6.3 Exercises

1. The graphs at right and overleaf show the displacement of an object. Sketch the graphs for velocity and acceleration.

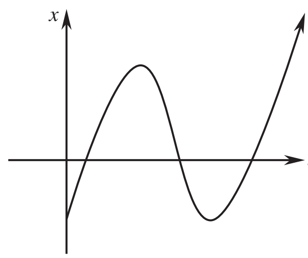
(a)



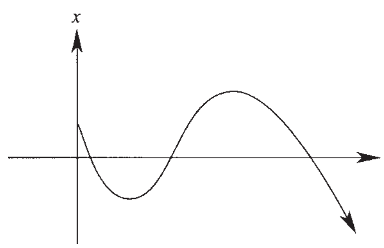
(b)



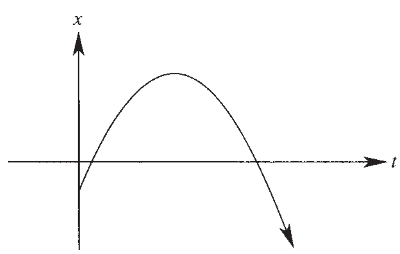
(c)



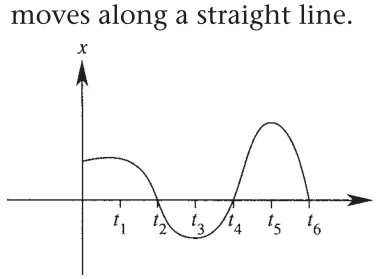
(d)



(e)

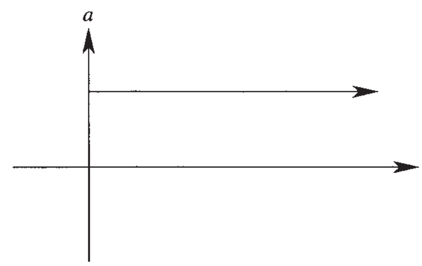


2. The graph below shows the displacement of a particle as it moves along a straight line.



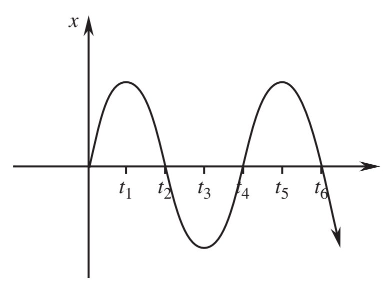
When is the particle
(a) at the origin?
(b) at rest?
(c) travelling with constant acceleration?
(d) furthest from the origin?

3. The graph below shows a particle travelling at a constant acceleration.



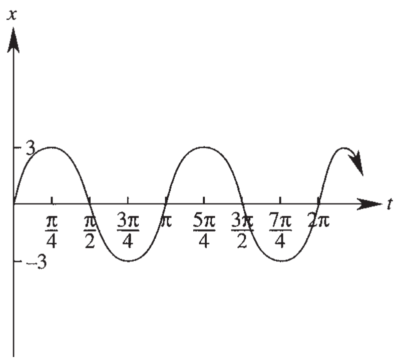
Sketch the graphs that could represent
(a) its velocity and
(b) its displacement.

4. The graph below shows the velocity of a particle.



(a) When is the particle at rest?
(b) When is the acceleration zero?
(c) When is the velocity the greatest?
(d) Describe the motion of the particle at (i) t_2 and (ii) t_3.

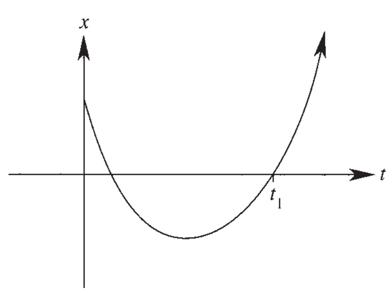
5. The graph below shows the displacement of a pendulum.



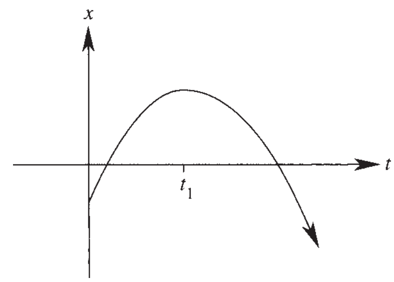
(a) When is the pendulum at rest?
(b) When is the pendulum in its equilibrium position (at the origin)?

6. Describe the motion of the particle at t_1 for each displacement graph.

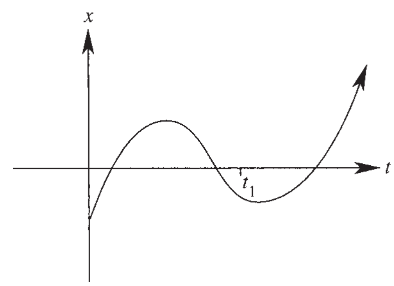
(a)



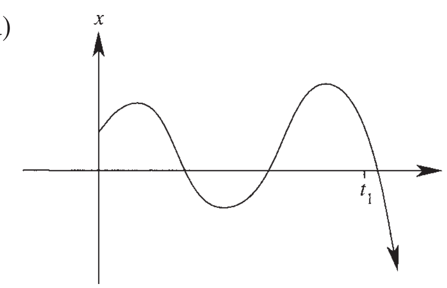
(b)



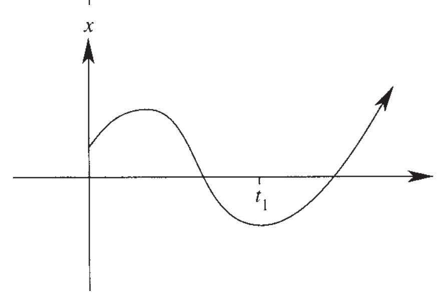
(c)



(d)



(e)



Motion and Differentiation

If displacement is given, then the velocity is the first derivative and the acceleration is the second derivative.

Displacement x

Velocity $\dot{x} = \dfrac{dx}{dt}$

Acceleration $\ddot{x} = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

EXAMPLES

1. The displacement x cm of a particle over time t seconds is given by $x = 4t - t^2$.

(a) Find any times when the particle is at rest.

(b) How far does the particle move in the first 3 seconds?

Solution

(a) At rest means that the velocity $\left(\frac{dx}{dt}\right)$ is zero.

$$\frac{dx}{dt} = 4 - 2t$$

When $\frac{dx}{dt} = 0$

$$\begin{aligned} 4 - 2t &= 0 \\ 4 &= 2t \\ 2 &= t \end{aligned}$$

So the particle is at rest after 2 seconds.

Notice that there is a stationary point at $\frac{dx}{dt} = 0$, so the particle turns around at this time.

(b) Initially, when $t = 0$:

$$\begin{aligned} x &= 4(0) - 0^2 \\ &= 0 \end{aligned}$$

So the particle is at the origin.

After 3 seconds, $t = 3$:

$$\begin{aligned} x &= 4(3) - 3^2 \\ &= 3 \end{aligned}$$

So the particle is 3 cm from the origin.

However, after 2 seconds, the particle turns around, so it hasn't simply moved from the origin to 3 cm away. We need to find where it is when it turns around.

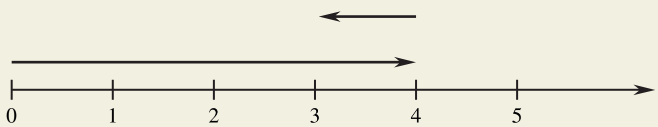
When $t = 2$:

$$\begin{aligned} x &= 4(2) - 2^2 \\ &= 4 \end{aligned}$$

So the particle is 4 cm from the origin.

The particle moves from the origin ($x = 0$) to 4 cm away in the first 2 seconds, so it has travelled 4 cm.

It then turns and goes back to 3 cm from the origin. So in the 3rd second it travels 1 cm back.



Total distance travelled in the first 3 seconds is 4 + 1 or 5 cm.

2. The displacement of a particle is given by $x = -t^2 + 2t + 3$ cm, where t is in seconds.

(a) Find the initial velocity of the particle.
(b) Show that the particle has constant acceleration.
(c) Find when the particle will be at the origin.
(d) Find the particle's maximum displacement from the origin.
(e) Sketch the graph showing the particle's motion.

Solution

(a)
$$v = \frac{dx}{dt}$$
$$= -2t + 2$$

Initially, $t = 0$

$\therefore \quad v = -2(0) + 2$
$$= 2$$

So the initial velocity is 2 cms^{-1}.

(b) $a = \dfrac{d^2x}{dt^2}$
$$= -2$$

$\therefore$ the particle has a constant acceleration of -2 cms^{-2}.

(c) At the origin $x = 0$

i.e. $-t^2 + 2t + 3 = 0$

$-(t+1)(t-3) = 0$

$\therefore \quad t = -1$ or 3

So the particle will be at the origin after 3 s.

A time of –1 does not exist.

(d) For maximum displacement,
$$\frac{dx}{dt} = 0$$
$$-2t + 2 = 0$$
$$2 = 2t$$
$$1 = t$$

(Maximum displacement occurs when $t = 1$.)

When $t = 1$,
$$x = -(1)^2 + 2(1) + 3$$
$$= 4$$

So maximum displacement is 4 cm.

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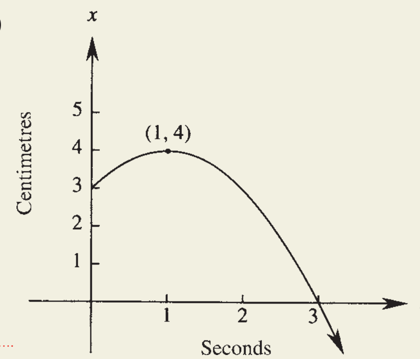
(e)



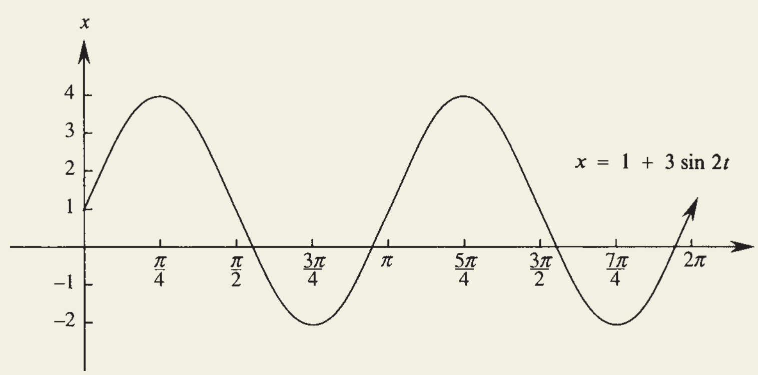
*This is the graph of a parabola. Notice that **x** can be positive or negative, but **t** is never negative.*

3. The displacement of a particle is given by the formula $x = 1 + 3\sin 2t$, where x is in metres and t is in seconds.

(a) Sketch the graph of x as a function of t.
(b) Find the times when the particle will be at rest.
(c) Find the position of the particle at those times.

Solution

(a) Graph has period π and amplitude 3.



(b) $v = \dfrac{ds}{dt}$

$= 6\cos 2t$

When the particle is at rest

$$v = 0$$

i.e. $6\cos 2t = 0$

$$\cos 2t = 0$$

$$2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\therefore \quad t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$$

(c) $x = 1 + 3\sin 2t$

When $t = \frac{\pi}{4}$

$$\begin{aligned} x &= 1 + 3\sin\frac{\pi}{2} \\ &= 1 + 3(1) \\ &= 4 \end{aligned}$$

When $t = \frac{3\pi}{4}$

$$\begin{aligned} x &= 1 + 3\sin\frac{3\pi}{2} \\ &= 1 + 3(-1) \\ &= -2 \end{aligned}$$

Similarly, when $t = \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$, x moves between 4 m and –2 m from the origin.

Investigation

In physics, equations for displacement and velocity are given by $\boldsymbol{s = ut + \frac{1}{2}at^2}$ and $\boldsymbol{v = u + at}$, where $\boldsymbol{u}$ is the initial velocity and $\boldsymbol{a}$ is the acceleration. Differentiate the displacement formula to show the formula for velocity. What happens when you differentiate again?

6.4 Exercises

1. The displacement of a particle is given by $x = t^3 - 9t$ cm, where t is time in seconds.
   (a) Find the velocity of the particle after 3 s.
   (b) Find the acceleration after 2 s.
   (c) Show that the particle is initially at the origin, and find any other times that the particle will be at the origin.
   (d) Find after what time the acceleration will be 30 cms^{-2}.

2. A particle is moving such that its displacement is given by $s = 2t^2 - 8t + 3$, where s is in metres and t is in seconds.
   (a) Find the initial velocity.
   (b) Show that acceleration is constant and find its value.
   (c) Find the displacement after 5 s.
   (d) Find when the particle will be at rest.
   (e) What will the particle's displacement be at that time?
   (f) Sketch the graph of the displacement against time.

3. A projectile is fired into the air and its height in metres is given by $h = 40t - 5t^2 + 4$, where t is in seconds.
   (a) Find the initial height.
   (b) Find the initial velocity.
   (c) Find the height after 1 s.
   (d) What is the maximum height of the projectile?
   (e) Sketch the graph of the height against time.

4. The displacement in cm after time t s of a particle moving in a straight line is given by $x = 2 - t - t^2$.
   (a) Find the initial displacement.
   (b) Find when the particle will be at the origin.
   (c) Find the displacement after 2 s.
   (d) How far will the particle move in the first 2 seconds?
   (e) Find its velocity after 3 s.

5. The equation for displacement of a particle is given by $x = e^{2t} + 1$ m after time t seconds.
   (a) Find the initial velocity.
   (b) Find the exact acceleration after 1 s.
   (c) Show that the acceleration is always double the velocity.
   (d) Sketch the graph of velocity over time t.

6. The displacement of a pendulum is given by $x = \cos 2t$ cm after time t seconds.
   (a) Find the equation for the velocity of the pendulum.
   (b) Find the equation for its acceleration.
   (c) Find the initial displacement.
   (d) Find the times when the pendulum will be at rest.
   (e) What will the displacement be at these times?
   (f) When will there be zero displacement?
   (g) Show that acceleration $a = -4x$.

7. An object is travelling along a straight line over time t seconds, with displacement according to the formula $x = t^3 + 6t^2 - 2t + 1$ m.
   (a) Find the equations of its velocity and acceleration.
   (b) What will its displacement be after 5 s?
   (c) What will its velocity be after 5 s?
   (d) Find its acceleration after 5 s.

8. The displacement, in centimetres, of a body is given by $x = (4t - 3)^5$, where t is time in seconds.
   (a) Find equations for velocity $\dot{x}$ and acceleration $\ddot{x}$.
   (b) Find the values of x, $\dot{x}$ and $\ddot{x}$ after 1 s.
   (c) Describe the motion of the body after 1 s.

9. The displacement of a particle, in metres, over time t seconds is $s = ut + \frac{1}{2}gt^2$, where $u = 5$ and $g = -10$.
   (a) Find the equation of the velocity of the particle.
   (b) Find the velocity after 10 s.
   (c) Show that the acceleration is equal to g.

10. The displacement in metres after t seconds is given by $s = \frac{2t - 5}{3t + 1}$. Find the equations for velocity and acceleration.

11. The displacement of a particle is given by $x = \log_e(t+1)$, where x is in centimetres and t is in seconds.
(a) Find the initial position of the particle.
(b) Find the velocity after 5 s.
(c) Find the acceleration after 5 s.
(d) Describe the motion of the particle at 5 s.
(e) Find the exact time when the particle will be 3 cm to the right of the origin.

12. Displacement of a particle is given by $x = t^3 - 4t^2 + 3t$, where s is in metres and t is in seconds.
(a) Find the initial velocity.
(b) Find the times when the particle will be at the origin.
(c) Find the acceleration after 3 s.

13. The displacement of a particle moving in simple harmonic motion is given by $x = 3\sin 2t$, where x is in centimetres and t is in seconds.
(a) Sketch the graph of the displacement of the particle from $t = 0$ to $t = 2\pi$.
(b) Find equations for the velocity $\dot{x}$ and acceleration $\ddot{x}$.
(c) Find the exact acceleration after $\frac{\pi}{3}$ s.
(d) Show that $\ddot{x} = -4x$.

14. The height of a projectile is given by $h = 7 + 6t - t^2$, where height is in metres and time is in seconds.
(a) Find the initial height.
(b) Find the maximum height reached.
(c) When will the projectile reach the ground?
(d) Sketch the graph showing the height of the projectile over time t.
(e) How far will the projectile travel in the first 4 s?

15. A ball is rolled up a slope at a distance from the base of the slope, after time t seconds, given by $x = 15t - 3t^2$ metres.
(a) How far up the slope will the ball roll before it starts to roll back down?
(b) What will its velocity be when it reaches the base of the slope?
(c) How long will the motion of the ball take altogether?

16. The displacement of a particle is given by $x = 2t^3 - 3t^2 + 42t$.
(a) Show that the particle is initially at the origin but never returns to the origin.
(b) Show that the particle is never at rest.

17. A weight on the end of a spring has a height given by $h = 3\sin 2t$ cm where t is time in seconds.
(a) Find its initial height.
(b) Find when the spring is at its maximum distance from the origin.
(c) What is the acceleration at these times?

18. A particle moves so that its displacement x cm is given by $x = e^{t^2}$ at t seconds.
(a) Find the exact velocity of the particle after 4 seconds.
(b) When is the particle at rest?
(c) Where is the particle at that time?

19. A particle is moving in a straight line so that its displacement x cm over time t seconds is given by $x = t\sqrt{49 - t^2}$.
(a) For how many seconds does the particle travel?
(b) Find the exact time at which the particle comes to rest.
(c) How far does the particle move altogether?

Motion and Integration

Displacement is the area under the velocity–time graph.

If the **first derivative of the displacement** of a particle gives its **velocity**, then the reverse is true. That is, the **primitive function** (indefinite integral) **of the velocity** gives the **displacement**. Similarly, the **integral of the acceleration** gives the **velocity**.

$$\boxed{\begin{aligned} x &= \int v\,dt \\ v &= \int a\,dt \end{aligned}}$$

EXAMPLES

1. The velocity of a particle is given by $v = 3t^2 + 2t + 1$. If initially the particle is 2 cm to the left of the origin, find the displacement after 5 s.

Solution

$$\begin{aligned} x &= \int v\,dt \\ &= \int (3t^2 + 2t + 1)\,dt \\ &= t^3 + t^2 + t + C \end{aligned}$$

Initially means $t = 0$

When $t = 0$, $x = -2$

$$\therefore\quad -2 = 0^3 + 0^2 + 0 + C$$
$$= C$$
$$\therefore\quad x = t^3 + t^2 + t - 2$$

When $t = 5$,

$$\begin{aligned} x &= 5^3 + 5^2 + 5 - 2 \\ &= 125 + 25 + 5 - 2 \\ &= 153 \end{aligned}$$

So after 5 s the particle will be 153 cm to the right of the origin.

2. The acceleration of a particle is given by $a = 6 - \dfrac{2}{(t+1)^2}$ ms^{-2}. If the particle is initially at rest 1 m to the right of the origin, find its exact displacement after 9 s.

Solution

$$
\begin{aligned}
v &= \int a\, dt \\
&= \int \left(6 - \frac{2}{(t+1)^2}\right) dt \\
&= \int [6 - 2(t+1)^{-2}]\, dt \\
&= 6t - 2\frac{(t+1)^{-1}}{-1} + C \\
&= 6t + \frac{2}{t+1} + C
\end{aligned}
$$

You must find C before going on to find the displacement.

When $t = 0$, $v = 0$

'At rest' means v = 0.

$$
\begin{aligned}
\therefore \quad 0 &= 6(0) + \frac{2}{0+1} + C \\
&= 2 + C \\
\therefore \quad C &= -2
\end{aligned}
$$

So $\quad v = 6t + \dfrac{2}{t+1} - 2$

$$
\begin{aligned}
x &= \int v\, dt \\
&= \int \left(6t + \frac{2}{t+1} - 2\right) dt \\
&= 3t^2 + 2\log_e(t+1) - 2t + C
\end{aligned}
$$

When $t = 0$, $x = 1$

$$
\begin{aligned}
\therefore \quad 1 &= 3(0)^2 + 2\log_e(0+1) - 2(0) + C \\
&= C \\
\therefore \quad x &= 3t^2 + 2\log_e(t+1) - 2t + 1
\end{aligned}
$$

When $t = 9$

$$
\begin{aligned}
x &= 3(9)^2 + 2\log_e(9+1) - 2(9) + 1 \\
&= 243 + 2\log_e 10 - 18 + 1 \\
&= 226 + 2\log_e 10 \\
&= 2(113 + \log_e 10)
\end{aligned}
$$

So after 9 s the displacement will be $2(113 + \log_e 10)$ m.

6.5 Exercises

1. The velocity of a particle is given by $v = 3t^2 - 5$ cms^{-1}. If the particle is at the origin initially, find its displacement after 3 s.

2. A particle has velocity given by $\frac{dx}{dt} = 2t - 3$ ms^{-1}. After 3 s the particle is at the origin. Find its displacement after 7 s.

3. The velocity of a particle is given by $v = 4 - 3t$ cms^{-1}. If the particle is 5 cm from the origin after 2 s, find the displacement after 7 s.

4. The velocity of a particle is given by $v = 8t^3 - 3t^2$ cms^{-1}. If the particle is initially at the origin, find
   (a) its acceleration after 5 s
   (b) its displacement after 3 s
   (c) when it will be at the origin again.

5. The velocity of a particle is given by $\dot{x} = 3e^{3t}$ cms^{-1}. Given that the particle is initially 2 cm to the right of the origin, find its exact displacement after 1 s.

6. A particle has a constant acceleration of 12 ms^{-2}. If the particle has a velocity of 2 ms^{-1} and is 3 m from the origin after 5 s, find its displacement after 10 s.

7. The acceleration of an object is given by $6t + 4$ cms^{-2}. The particle is initially at rest at the origin. Find
   (a) its velocity after 5 s
   (b) its displacement after 5 s.

8. A projectile is accelerating at a constant rate of -9.8 ms^{-2}. If it is initially 2 m high and has a velocity of 4 ms^{-1}, find the equation of the height of the projectile.

9. A particle is accelerating according to the equation $a = (3t + 1)^2$ ms^{-2}. If the particle is initially at rest 2 m to the left of 0, find its displacement after 4 s.

10. The velocity of a particle is given by $v = e^t - 1$ ms^{-1}. If the particle is initially 3 m to the right of the origin, find its exact position after 5 s.

11. A particle accelerates according to the equation $a = -e^{2t}$. If the particle has initial velocity of zero cms^{-1} and initial displacement of -1 cm, find its displacement after 4 s, to the nearest centimetre.

12. The acceleration of a particle is given by $a = -9\sin 3t$ cms^{-2}. If the initial velocity is 5 cms^{-1} and the particle is 3 cm to the left of the origin, find the exact displacement after π s.

13. The velocity of a particle is given by $\dot{x} = \frac{t}{t^2 + 3}$ ms^{-1}. If the particle is initially at the origin, find its displacement after 10 s, correct to 2 decimal places.

14. The acceleration of an object is given by $a = e^{3t}$ ms^{-2}. If initially the object is at the origin with velocity -2 ms^{-1}, find its displacement after 3 seconds, correct to 3 significant figures.

15. The velocity of a particle is given by $v = 4\cos 2t\ \text{ms}^{-1}$. If the particle is 3 m to the right of the origin after π s, find the exact

(a) displacement after $\frac{\pi}{6}$ s

(b) acceleration after $\frac{\pi}{6}$ s.

16. The acceleration of a particle is given by $a = \frac{2}{(t+3)^2}\ \text{ms}^{-2}$ and the particle is initially at rest 2 ln 3 m to the left of the origin.

(a) Evaluate its velocity after 2 seconds.

(b) Find its displacement after n seconds.

(c) Show that $0 \le v < 1$ for all t.

17. The acceleration of a particle is $\ddot{x} = 25\, e^{5t}\ \text{ms}^{-2}$ and its velocity is 5 ms^{-1} initially. Its displacement is initially 1 m.

(a) Find its velocity after 9 seconds.

(b) Find its displacement after 6 seconds.

(c) Show that the acceleration $\ddot{x} = 25x$.

(d) Find the acceleration when the particle is 2 m to the right of the origin.

Test Yourself 6

1. A particle moves so that its displacement after t seconds is $x = 4t^2 - 5t^3$ metres. Find
   (a) its initial displacement, velocity and acceleration
   (b) when the particle is at the origin
   (c) its maximum displacement.

2. The velocity of a particle is given by $V = 6t - 12t^2$ ms^{-1}. Find its displacement and acceleration after 3 seconds if it is initially 5 m to the right of the origin.

3. A city doubles its population in 25 years. If it is growing exponentially, when will it triple its population?

4. The displacement of a particle after t seconds is $x = 2e^{3t}$ cm.
   (a) Find its initial velocity.
   (b) Find its acceleration after 3 seconds, to 1 decimal place.
   (c) Show that $\ddot{x} = 9x$.

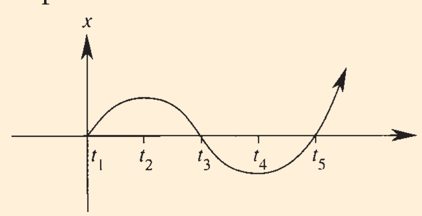
5. A particle accelerates at $\ddot{x} = 8\cos 2t$ ms^{-2}. The particle is initially at rest at the origin. Find its displacement after $\frac{\pi}{6}$ seconds.

6. If a particle has displacement $x = 2\sin 3t$, show that its acceleration is given by $\ddot{x} = -9x$.

7. A particle has displacement $x = t^3 - 12t^2 + 36t - 9$ cm at time t seconds.
   (a) When is the particle at rest?
   (b) What is the
      (i) displacement
      (ii) velocity
      (iii) acceleration after 1 s?
   (c) Describe the motion of the particle after 1 s.

8. The acceleration of a particle is $\ddot{x} = 6t - 12$ ms^{-2}. The particle is initially at rest 3 metres to the left of the origin.
   (a) Find the
      (i) acceleration
      (ii) velocity
      (iii) displacement after 5 seconds.
   (b) Describe the motion of the particle after 5 seconds.

9. The graph below shows the displacement of a particle.



   (a) When is the particle at the origin?
   (b) When is it at rest?
   (c) When is it travelling at its greatest speed?
   (d) Sketch the graph of its
      (i) velocity
      (ii) acceleration.

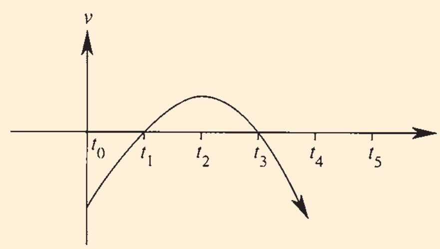
10. A radioactive substance decays by 10% after 80 years.
   (a) By how much will it decay after 500 years?
   (b) When will it decay to a quarter of its mass?

11. The equation for displacement of a particle over time t seconds is $x = \sin t$ metres.
   (a) When is the displacement 0.5 m?
   (b) When is the velocity 0.5 ms^{-1}?
   (c) Find the acceleration after $\frac{\pi}{4}$ s.

12. The height of a ball is $h = 20t - 5t^2$ metres after t seconds.
(a) Find the height after 1 s.
(b) What is the maximum height of the ball?
(c) What is the time of flight of the ball?

13. A bird population of 8500 increases to 12 000 after 5 years. Find
(a) the population after 10 years
(b) the rate at which the population is increasing after 10 years
(c) when the population reaches 30 000.

14. The graph below shows the velocity of a particle.



(a) Sketch a graph that shows
(i) displacement
(ii) acceleration
(b) When is the particle at rest?

15. A particle is moving with acceleration $a = 10e^{2t}$ ms^{-2}. If it is initially 4 m to the right of the origin and has a velocity of -2 ms^{-1}, find its displacement after 5 seconds, to the nearest metre.

Challenge Exercise 6

1. (a) Find an equation for the displacement of a particle from the origin if its acceleration is given by $\frac{dv}{dt} = 6 \cos 3t$ cms^{-2} and initially the particle is at rest 2 cm to the right of the origin.
(b) Write the acceleration of the particle in terms of x.

2. The displacement of a particle is given by $x = (t^3 + 1)^6$, where x is in metres and t is in seconds.
(a) Find its initial displacement and velocity.
(b) Find its acceleration after 2 s in scientific notation, correct to 3 significant figures.
(c) Show that the particle is never at the origin.

3. (a) Find an equation for the displacement of a particle from the origin if its acceleration is given by $\frac{dv}{dt} = -16 \cos 4t$ cms^{-2} and initially the particle is at rest 1 cm to the right of the origin.
(b) Find the exact velocity of the particle when it is 0.5 cm from the origin.

4. The velocity, in ms^{-1}, of an object is given by $v = 5 \cos 5t$.
(a) Show that if the object is initially at the origin, its acceleration will always be -25 times its displacement.
(b) Find the maximum acceleration of the object.
(c) Find the acceleration when the object is 0.3 m to the right of the origin.

5. The population of a flock of birds over t years is given by the formula $P = P_0 e^{0.0151t}$.
(a) How long will it take, correct to 1 decimal place, to increase the population by 35%?
(b) What will be the percentage increase in population after 10 years, to the nearest per cent?

6. The Logistic Law of Population Growth, first proposed by **Verhulst** in 1837, is given by $\frac{dN}{dt} = kN - bN^2$, where k and b are constants. Show that the equation $N = \frac{kN_0}{bN_0 + (k - bN_0)e^{-kt}}$ is a solution of this differential equation (N_0 is a constant).

7. A particle moves so that its velocity is given by $\dot{x} = te^{t^2}$ cms^{-1}.
Find its exact acceleration after 1 s.