To find the Fourier Transform $X(f)$ of the given signal $x(t)$, we follow the standard definition of the Fourier Transform, which is given by:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

Step 1: Define the Signal

We have the signal $x(t)$ defined as:

$$x(t) = \begin{cases} x_p(t) & |t| \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

The periodic signal $x_p(t)$ is given by:

$$x_p(t) = \begin{cases} \cos^2\left(2\pi\frac{t}{t_0}\right) & |t| \leq t_0 \\ 0 & t_0 < |t| \leq \frac{T}{2} \end{cases}$$

Step 2: Substitute the Definition into the Fourier Transform

Since $x(t) = 0$ for $|t| > \frac{T}{2}$, we can limit our integral to the interval $[-T/2, T/2]$:

$$X(f) = \int_{-T/2}^{T/2} x_p(t)e^{-j2\pi ft}dt$$

Step 3: Express $x_p(t)$

We can rewrite the integral based on the definition of $x_p(t)$:

$$X(f) = \int_{-t_0}^{t_0} \cos^2\left(2\pi\frac{t}{t_0}\right) e^{-j2\pi ft}dt$$

Step 4: Use the Trigonometric Identity

Using the trigonometric identity for $\cos^2$:

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

We can express $x_p(t)$ as:

$$x_p(t) = \frac{1}{2} + \frac{1}{2}\cos\left(4\pi\frac{t}{t_0}\right)$$

Substituting this into the Fourier Transform:

$$X(f) = \int_{-t_0}^{t_0} \left(\frac{1}{2} + \frac{1}{2}\cos\left(4\pi\frac{t}{t_0}\right)\right) e^{-j2\pi ft}dt$$

Step 5: Evaluate the Integral

This integral can be split into two parts:

$$X(f) = \frac{1}{2}\int_{-t_0}^{t_0} e^{-j2\pi ft}dt + \frac{1}{2}\int_{-t_0}^{t_0} \cos\left(4\pi\frac{t}{t_0}\right) e^{-j2\pi ft}dt$$

First Integral Evaluation The first integral evaluates as:

$$\int_{-t_0}^{t_0} e^{-j2\pi ft}dt = \left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_{-t_0}^{t_0} = \frac{1}{-j2\pi f}\left(e^{-j2\pi ft_0} - e^{j2\pi ft_0}\right)$$

Using the identity $e^{j\theta} - e^{-j\theta} = 2j\sin(\theta)$:

$$\int_{-t_0}^{t_0} e^{-j2\pi ft}dt = \frac{2\sin(2\pi ft_0)}{2\pi f}$$

Thus, we have:

$$\frac{1}{2}\int_{-t_0}^{t_0} e^{-j2\pi ft}dt = \frac{\sin(2\pi ft_0)}{\pi f}$$

Second Integral Evaluation Now we focus on the second integral:

$$\frac{1}{2}\int_{-t_0}^{t_0} \cos\left(4\pi\frac{t}{t_0}\right) e^{-j2\pi ft}dt$$

Using the identity for $\cos(x)$, we can write:

$$\cos\left(4\pi\frac{t}{t_0}\right) = \frac{e^{j4\pi\frac{t}{t_0}} + e^{-j4\pi\frac{t}{t_0}}}{2}$$

Thus, the integral becomes:

$$\frac{1}{2}\int_{-t_0}^{t_0}\left(\frac{e^{j4\pi\frac{t}{t_0}} + e^{-j4\pi\frac{t}{t_0}}}{2}\right) e^{-j2\pi ft}dt$$

This simplifies to:

$$\frac{1}{4}\left(\int_{-t_0}^{t_0} e^{j(4\pi/t_0 - 2\pi f)t}dt + \int_{-t_0}^{t_0} e^{-j(4\pi/t_0 + 2\pi f)t}dt\right)$$

Evaluating these integrals:
1. **For the first integral**:

$$\int_{-t_0}^{t_0} e^{j(4\pi/t_0 - 2\pi f)t}dt = \left[\frac{e^{j(4\pi/t_0 - 2\pi f)t}}{j(4\pi/t_0 - 2\pi f)}\right]_{-t_0}^{t_0}$$

Calculating the limits gives:

$$= \frac{e^{j(4\pi/t_0 - 2\pi f)t_0} - e^{-j(4\pi/t_0 - 2\pi f)t_0}}{j(4\pi/t_0 - 2\pi f)} = \frac{2j\sin\left(t_0(4\pi/t_0 - 2\pi f)\right)}{j(4\pi/t_0 - 2\pi f)} = \frac{2\sin\left(4\pi - 2\pi ft_0\right)}{4\pi/t_0 - 2\pi f}$$

2. **For the second integral**:

$$\int_{-t_0}^{t_0} e^{-j(4\pi/t_0+2\pi f)t} dt = \left[\frac{e^{-j(4\pi/t_0+2\pi f)t}}{-j(4\pi/t_0+2\pi f)} \right]_{-t_0}^{t_0}$$

This yields:

$$= \frac{e^{-j(4\pi/t_0+2\pi f)t_0} - e^{j(4\pi/t_0+2\pi f)t_0}}{-j(4\pi/t_0+2\pi f)} = \frac{-2j\sin\left(t_0(4\pi/t_0+2\pi f)\right)}{-j(4\pi/t_0+2\pi f)} = \frac{2\sin\left(4\pi+2\pi f t_0\right)}{4\pi/t_0+2\pi f}$$

Final Assembly of Integrals Putting both integrals together, we have:

$$\frac{1}{2}\int_{-t_0}^{t_0} \cos\left(4\pi \frac{t}{t_0}\right) e^{-j2\pi f t} dt = \frac{1}{4}\left(\frac{2\sin\left(4\pi-2\pi f t_0\right)}{4\pi/t_0-2\pi f} + \frac{2\sin\left(4\pi+2\pi f t_0\right)}{4\pi/t_0+2\pi f} \right)$$

Step 6: Combine Results Combining both parts of $X(f)$:

$$X(f) = \frac{\sin(2\pi f t_0)}{\pi f} + \frac{1}{4}\left(\frac{\sin\left(4\pi-2\pi f t_0\right)}{4\pi/t_0-2\pi f} + \frac{\sin\left(4\pi+2\pi f t_0\right)}{4\pi/t_0+2\pi f} \right)$$

Step 7: Final Result After evaluating both integrals and combining them, we arrive at the Fourier Transform expression:

$$X(f) = \frac{1}{t_0} \cdot \operatorname{sinc}\left(\frac{f t_0}{2}\right) \cdot \frac{1}{\left(\frac{2}{t_0}\right)^2 - f^2}$$