Math 301 Final Exam

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April 10, 2018

Problem 1 Part A

set	Int S	Bd S	Open/Closed/Neither	S'	Cl(S)
a	$\varnothing$	$\mathbb{N}$	closed	$\varnothing$	$\mathbb{N}$
b	$\varnothing$	$\mathbb{R}$	neither	$\mathbb{R}$	$\mathbb{R}$
c	$\varnothing$	$[-\sqrt{3}, \sqrt{3}]$	neither	$\mathbb{R}$	$\mathbb{R}$
d	$(4, 10) \cup (10, 12)$	$\{4, 10, 12\}$	neither	$[4, 12]$	$[4, 12]$
e	$(-\infty, 5)$	$\{5\}$	open	$(-\infty, 5]$	$(-\infty, 5]$

Problem 1 Part B

i.

$(\sim p \Rightarrow c), p$

p	$\sim p$	c	$\sim p \Rightarrow c$	$[\sim p \Rightarrow c] \Leftrightarrow p$
T	F	F	T	T
F	T	F	F	T

These two statements are equivalent.

ii.

$[(p \wedge \sim q) \Rightarrow r], [(p \Rightarrow \sim q) \Rightarrow r]$

p	q	~q	r	$(p \wedge \sim q)$	$[(p \wedge \sim q) \Rightarrow r]$	$(p \Rightarrow \sim q)$	$[(p \Rightarrow \sim q) \Rightarrow r]$
T	T	F	T	F	T	F	T
T	T	F	F	F	T	F	T
T	F	T	T	T	T	T	T
T	F	T	F	T	F	T	F
F	T	F	T	F	T	T	T
F	T	F	F	F	T	T	F
F	F	T	T	F	T	T	T
F	F	T	F	F	T	T	F

These two statements are **not** equivalent.

Problem 1 Part B

i.

$s_n = (-1)^n/(n+1)$

$\lim(s_n) = 0$

ii.

$s_n = 2^{3n}/3^{3n} = 8^n/9^n = (8/9)^n$

$\lim(s_n) = 0$

iii.

$s_n = \frac{(3+n-n^2)}{(1+2n)}$

$\lim \frac{(3+n-n^2)}{(1+2n)} = \lim \frac{(3/n^2+n/n^2-n^2/n^2)}{(1/n^2+2n/n^2)} = \lim \frac{(3/n^2+1/n-1)}{(1/n^2+2/n)} = -1/0$

So it diverges to $-\infty$

iv.

$s_n = n!/n^n$

Using the ratio test theorem,

$\lim \frac{s_{n+1}}{s_n} = \frac{\lim(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} = \lim(\frac{n}{n+1})^n \rightarrow L'hopitals \rightarrow \lim = e.$

Therefore s_n diverges to ∞.

v.

$s_n = (-1)^n/n$

$\lim s_n = 0$

vi. If a monotone sequence if bounded, then it converges.

True by Monotone Convergence Theorem.

vii. If a sequence is bounded, then it is convergent.

False. $s_n = (-1)^n$ is bounded but not convergent.

viii. If a convergent sequence is bounded, then it is monotone.

False. $s_n = (-1)^n/n$ is bounded and convergent but it is not monotone.

ix. If sequences s_n and t_n converge to s and t respectively, then $s_n \cdot t_n$ converges to $s \cdot t$.

True by theorem 4.2.1 in the book.

x. Given sequences s_n and t_n, if $s_n < t_n$ for all n and $\lim s_n = \infty$, then $\lim t_n = \infty$.

True. Since t_n is always larger than s_n its limit will also diverge. Theorem 4.2.12 also supports this.

Problem 3 Part A

i.

I know two types of structures for proof by contradiction.

One is $(\sim p \Rightarrow c) \Leftrightarrow p$ and the other is $[(p \wedge \sim q) \Rightarrow c] \Leftrightarrow (p \Rightarrow q)$.

In the case of the first. We assume the negation of p and show that a contradiction exists. This shows that $\sim p$ is false, thereby making p true.

In the later structure we negate $(p \Rightarrow q)$ to get $(p \wedge \sim q)$ and assume it to be true. The next step is to show that assuming this leads to a contradiction, thereby making $(p \Rightarrow q)$ true because its negation is false.

These proofs can be useful when it is easier to show the negation is false. It can be easier sometimes this way because it may only take a single counter example to show the contradiction, whereas proving the original implication might require proving infinite cases.

ii.

The goal of proof by induction is to show that $P(n)$ is true for all of $n \in \mathbb{N}$. But to do this two requirements must be met.

The first is called the basis of induction. Here you must show that the base case is true. The base case is the first n where $P(n)$ is defined, usually this means $P(1)$.

Next is called the induction step. Assuming $P(k)$ is true for all $k \in \mathbb{N}$ you must show that $P(K+1)$ is true.

Proof by induction is appropriate for statements $P(n)$ where $n \in \mathbb{N}$.

My example is: Prove that $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{1}{4}n^2(n+1)^2$ for all $n \in \mathbb{N}$.

So our goal is to show that no matter what n is used in the statement both sides will always be the same.

Proof:

Let $P(n)$ be the statement

$$1^3 + 2^3 + 3^3 + ... + n^3 = \frac{1}{4}n^2(n+1)^2 \quad \text{where } n \in \mathbb{N}$$

To establish the basis of induction we show that $P(1)$ is true.

By plugging 1 into both sides of the statement it is easy to see that $P(1)$ is true because $1 = 1$.

For the induction step we must verify $P(k+1)$ is true, assuming that $P(k)$ is true for some $k \in \mathbb{N}$, or that $1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2((k+1)+1)^2$ is true for some $k \in \mathbb{N}$.

Assume $P(k)$

$$1^3 + 2^3 + 3^3 + ... + k^3 = \frac{1}{4}k^2(k+1)^2 \qquad \text{is true for some } k \in \mathbb{N}$$

Adding $(k+1)^3$ to both sides of the statement above, yields:

$$1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

After a lot of simplification the statement above, yields:

$$1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2((k+1)+1)^2$$

The above statement is the same as $P(k+1)$, therefore $P(k+1)$ is true.

Now because we have proven the basis of induction and the induction step we know that $P(n)$ is true for all $n \in \mathbb{N}$.

Problem 3 Part B

i. Let S and T be subsets of $\mathbb{R}$. Prove or disprove the following: $\text{cl}(S \cup T) = (\text{cl}S) \cup (\text{cl}T)$.

Direct Proof:

We wish to prove the following: $\text{cl}(S \cup T) = (\text{cl}S) \cup (\text{cl}T)$.

Let S and T be subsets of $\mathbb{R}$.

Consider only the left side of the statement above statement:

$$= \text{cl}(S \cup T).$$

Recall the definition of closure is defined by $\text{cl}S{=}S \cup S'$ where S' is the set of accumulation points. We can express the statement above as the following:

$$= (S \cup T) \cup (S \cup T)'.$$

Now consider only the right side of the statement above statement. We have

$$= (\text{cl}S) \cup (\text{cl}T).$$

By the definition of closure we can express it as the following:

$$= (S \cup S') \cup (T \cup T')$$

$$= (S \cup T) \cup (S' \cup T').$$

To prove that the two sides are equal I need to show that $(S' \cup T')$ is equal to $(S \cup T)'$.

In the case that $T = \varnothing$ the set of accumulation points of an empty set is the empty set. From there it is easy to see that $(S' \cup T')$ is equal to $(S \cup T)'$, because (S') is equal to $(S)'$.

For the case that $T \neq \varnothing$ I will refer to the definition of accumulation point.

A point $x \in \mathbb{R}$ is an accumulation point of S if every deleted neighborhood of x contains a point of S.

Let $x \in S'$. Every deleted neighborhood of x will always contain a point of $(S \cup T)$ because the set contains all of S. So for every $x \in S'$, $x \in (S \cup T)'$.

Let $r \in T'$. Every deleted neighborhood of r will always contain a point of $(S \cup T)$ because the set contains all of T. So for every $r \in T'$, $r \in (S \cup T)'$.

Thus we see that every element of S' and T' will also be an element of $(S \cup T)'$. It follows that every element of $(S \cup T)'$ is also an element of $S' \cup T'$. So $(S' \cup T') = (S \cup T)'$

Therefore $\text{cl}(S \cup T) = (\text{cl}S) \cup (\text{cl}T)$.

ii.

If S is a compact subset of $\mathbb{R}$, and T is closed subset of S, then T is compact. Prove using the definition of compactness.

The following will be used in the proof below.

Compact: A set S is said to be compact iff every open cover of S contains a finite subcover.

Heine-Borel Theorem: A subset of $\mathbb{R}$ is compact iff S is closed and bounded.

3.4.10 Theorem: The union of any collection of open sets is an open set.

Proof:

Let S be a compact subset of $\mathbb{R}$ and T be a closed subset of S.

Further, let U be an open cover of T.

The complement of the closed set T, $(\mathbb{R} \setminus T)$ is an open set and by theorem 3.4.10 the union, $U \cup (\mathbb{R} \setminus T)$ is an open set. This union is also an open cover of S because it covers all elements in S.

Since S is compact there exists a finite subcover, denoted U', from the open cover $U \cup (\mathbb{R} \setminus T)$ such that $U' \subseteq [U \cup (\mathbb{R} \setminus T)]$.

U' covers S, but since $T \subseteq S$, U' will also cover T.

Because $T \cap (\mathbb{R} \setminus T) = \varnothing$ we can remove $(\mathbb{R} \setminus T)$ from U' and it would still cover T. That removal would make $U' \subseteq U$ and it means that U' is a finite subcover of T.

Thus every open cover of T contains a finite subcover.

Therefore T is compact.