Lesson 1 Presentation

Jacob Jashinsky and Caleb Spear

Problem 1.1.1

Mark each statement True or False. Justify each answer.

(a) In order to be classified as a statement, a sentence must be true.

Answer: False, it must be either true or false.

(b) Some statements are both true and false.

Answer: False, the statement must be one or the other, and not both.

(c) When statement p is true, its negation ~p is false.

Answer: True, by definition of negation.

(d) A statement and its negation may both be false.

Answer: False, by definition of negation, they must be opposites.

(e) In mathematical logic, the word “or” has an inclusive meaning.

Answer: True, the inclusive meaning allows two statements to be one or the other or both.

Lesson 2 Presentation

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Problem 1.4.14

If $x/(x-2) \leq 3$, then $x < 2$ or $x \geq 3$, where x is a real number.

To establish the logic for the proof we start by defining are statements p and q.

$$p : \frac{x}{x-2} \leq 3$$

$$q : x \geq 3 \text{ or } x < 2$$

Converting the original statement above into short hand we get,

$$p \Rightarrow q$$

Proof:

We prove this directly by letting p be true,

$$\frac{x}{x-2} \leq 3$$

For simplicity we solve for x by multiplying $(x-2)$ on both sides of the inequality.

$$x \leq 3(x-2)$$

Distributing the 3,

$$x \leq 3x - 6$$

Subtract x and add 6 to both sides,

$$6 \leq 2x$$

Divide by 2 so that p becomes,

$$x \geq 3$$

Now consider the implication $p \Rightarrow q$

$$x \geq 3 \text{ implies } x \geq 3 \text{ or } x < 2$$

Since p is true we know that $x \geq 3$ which will also make the consequent true.

$$4x$$

Problem 2.1.6, parts d-f

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1 Problem

6. Let A and B be subsets of a universal set U. Simplify each of the following expressions.
(d) $A \cup [B \cap (U \setminus A)]$
(e) $(A \cup B) \cap [A \cup (U \setminus B)]$
(f) $(A \cap B) \cup [A \cap (U \setminus B)]$

References on page 44 of the textbook
Theorem a: $A \cup (U \setminus A) = U$
Theorem d: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Theorem e: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2 Solution

(d)

$$A \cup [B \cap (U \setminus A)] \tag{1}$$

Using theorem d we get

$$(A \cup B) \cap [A \cup (U \setminus A)] \tag{2}$$

Using theorem a we get

$$(A \cup B) \cap U \tag{3}$$

The union of A and B is a subset of U, the intersection of the two sets is just the union of A and B.

$$(A \cup B) \tag{4}$$

(e)
$$(A \cup B) \cap [A \cup (U \setminus B)] \tag{5}$$

Using theorem e we get

$$[(A \cup B) \cap A] \cup [(A \cup B) \cap (U \setminus B)] \tag{6}$$

Using theorem e again we get

$$[(A \cap A) \cup (B \cap A)] \cup [((U \setminus B) \cap A) \cup ((U \setminus B) \cap B)] \tag{7}$$

We simplify to get

$$(A \cup (B \cap A)) \cup ((U \setminus B) \cap A) \cup \emptyset \tag{8}$$

$B \cap A$ is a subset of A so the union is just A.

$$A \cup ((U \setminus B) \cap A) \tag{9}$$

$(U \setminus B) \cap A$ is a subset of A so it reduces to just A.

$$A \tag{10}$$

(f)
$$(A \cap B) \cup [A \cap (U \setminus B)] \tag{11}$$

Using theorem e we get

$$[(A \cap B) \cup A] \cap [(A \cup B) \cup (U \setminus B)] \tag{12}$$

Left of intersection reduces to just A. Right of the intersection reduces to U.

$$A \cap U \tag{13}$$

This simplifies to

$$A \tag{14}$$

Lesson 4 Presentation

Jacob Jashinsky as Scribe and Caleb Spear as Editor

Problem 2.3.2

Mark each statement as True or False. Justify each answer.

a. If $f : A \to B$ and C is nonempty subset of A, then $f(C)$ is a nonempty subset of B.

True. For f to be a function every element in A, which includes the subset C, will map to a $b \in B$. Therefore, $f(C)$ will also be a subset of B.

b. If $f : A \to B$ is surjective and $y \in B$, then $f^{-1}(y) \in A$.

True. Since the function is surjective $\text{rng} f = B$ and $f^{-1} : B \to A$.

c. If $f : A \to B$ and D is nonempty subset of B then $f^{-1}(D)$ is a nonempty subset of A.

False, D may include elements that are not in the range of f, and the inverse of f will not map elements outside of the range back to A.

d. The composition of two surjective functions is always surjective.

True, by Theorem 2.3.20 on page 72, we know that composition preserves properties of being injective and surjective.

e. If $f : A \to B$ is bijective, then $f^{-1} B \to A$ is bijective.

True. Since f is bijective each $a \in A$ maps to a unique $b \in B$. Therefore the inverse would also map every every $b \in B$ back to every element $a \in A$, which would make the inverse bijective.

f. The identity function maps $\mathbb{R}$ onto $\{ 1 \}$.

False. An identity function would map the set $\mathbb{R}$ back onto $\mathbb{R}$.

3.1.4

Prove that $1^3+2^3+3^3+...+n^3=\frac{1}{4}n^2(n+1)^2$ for all $n\in\mathbb{N}$.

Proof:

Let $P(n)$ be the statement

$$1^3+2^3+3^3+...+n^3=\frac{1}{4}n^2(n+1)^2 \quad \text{where } n\in\mathbb{N}$$

To establish the basis of induction we show that $P(1)$ is true.

First we look at the left side of $P(n)$ and let $n=1$.

$$\begin{aligned} n^3 \\ (1)^3 \\ 1 \end{aligned}$$

Then the right side of $P(n)$ with $n=1$.

$$\begin{aligned} \frac{1}{4}n^2(n+1)^2 \\ \frac{1}{4}1^2(1+1)^2 \\ \frac{1}{4}2^2 \\ \frac{4}{4} \\ 1 \end{aligned}$$

Therefore because the left side agrees with the right we now know that $P(1)$ is true.

For the induction step we must verify $P(k+1)$ is true, assuming that $P(k)$ is true for some $k\in\mathbb{N}$, or that $1^3+2^3+3^3+...+k^3+(k+1)^3=\frac{1}{4}(k+1)^2((k+1)+1)^2$ is true for some $k\in\mathbb{N}$.

Assume $P(k)$

$$1^3+2^3+3^3+...+k^3=\frac{1}{4}k^2(k+1)^2 \quad \text{is true for some } k\in\mathbb{N}$$

Adding $(k+1)^3$ to both sides of the statement above and simplifying, yeilds:

$$\begin{aligned}
1^3+2^3+3^3+\ldots+k^3+(k+1)^3 &= \frac{1}{4}k^2(k+1)^2+(k+1)^3 \\
&= \frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2+4(k+1)^3}{4} \\
&= \frac{1}{4}(k+1)^2(k^2+4(k+1)) \\
&= \frac{1}{4}(k+1)^2(k^2+4k+4) \\
&= \frac{1}{4}(k+1)^2(k+2)^2 \\
&= \frac{1}{4}(k+1)^2((k+1)+1)^2
\end{aligned}$$

The above statement is the same as $P(k+1)$, therefore $P(k+1)$ is true.

Now because we have shown that $P(1)$ and $P(k+1)$, assuming that $P(k)$ is true for some $k \in \mathbb{N}$, are true we know that $P(n)$ is true for all $n \in \mathbb{N}$.

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Problem 3.3.8

Let S and T be nonempty bounded subsets of $\mathbb{R}$ with $S \subseteq T$. Prove that $\inf T \leq \inf S \leq \sup S \leq \sup T$.

Definitions

The following definitions will be used

Maximum: If m is an upper bound of S and $m \in S$, then m is called the maximum of S.

Minimum: If m is a lower bound of S and $m \in S$, then m is called the minimum of S.

Supremum: If S is bounded above the least upper bound of S is called its Supremum iff

a. $m \geq s$ for all $s \in S$ and,

b. if $m' < m$, then there exists $s' \in S$ such that $s' > m'$.

Infimum: If S is bounded below the greatest upper bound of S is called its Infimum iff

a. $m \leq s$ for all $s \in S$ and,

b. if $m' > m$, then there exists $s' \in S$ such that $s' < m'$.

Assumptions

We define the statement above as $p \Rightarrow q$, where p and q are,

$$p : S \text{ and } T \text{ be nonempty bounded subsets of } \mathbb{R} \text{ with } S \subseteq T$$

$$q : (\inf T \leq \inf S) \wedge (\inf S \leq \sup S) \wedge (\sup S \leq \sup T)$$

We also define $\sim q$ as,

$$\sim q : (\inf T > \inf S) \vee (\inf S > \sup S) \vee (\sup S > \sup T)$$

The structure of the proof will be as follows

$$(p \Rightarrow q) \Leftrightarrow [(p \wedge \sim q) \Rightarrow c]$$

Because of the nature of $\sim q$, this proof will be done in the following three cases,

$$\begin{array}{ll} (1) & p \wedge (\inf T > \inf S) \Rightarrow c \\ (2) & p \wedge (\inf S > \sup S) \Rightarrow c \\ (3) & p \wedge (\sup S > \sup T) \Rightarrow c \end{array}$$

Proof:

Statement 1

$$(1) \quad p \wedge (\inf T > \inf S) \Rightarrow c$$

Here the contradiction will lie within the hypothesis (see page 30).

Let $\inf T = m$, $\inf S = q$ and assume $[p \wedge (\inf T > \inf S)] \Rightarrow c$ to be true.

Part (b) of the definition of infimum for $\inf S$, states that if $m' > m$, then there exists $s' \in S$ such that $s' < m'$. So any element in $s' \in S$ must be larger than m. The same logic applies to T, in that any $t' \in T$ must be greater than q.

Now because we assumed that $(\inf T > \inf S)$ to be true or that $(q > m)$ the logic above implies that there exists an $s \in S$ such that $s < t \in T$.

Further, this means that some element in S does not exist in T.

In conjunction with p, this is a contradiction, as T must contain all of the elements in S for $S \subseteq T$ to be true as was assumed.

Statement 2

$$(2) \quad p \wedge (\inf S > \sup S) \Rightarrow c$$

The contradiction lies in $(\inf S > \sup S)$, because by definition $(\inf S \leq \sup S)$ is true.

Statement 3

$$(3) \quad p \wedge (\sup S > \sup T) \Rightarrow c$$

Similar to the logic of the first statement the contradiction lies in the hypothesis.

Let $\sup T = m$, $\sup S = q$ and assume $[p \wedge (\sup S > \sup T)] \Rightarrow c$ to be true.

Part (b) of the definition of infimum for $\sup S$, states that if $m' < m$, then there exists $s' \in S$ such that $s' > m'$. So any element in $s' \in S$ must be smaller than m. The same logic applies to T, in that any $t' \in T$ must be smaller than q.

Now because we assumed that $(\sup S > \sup T)$ to be true or that $(m > q)$ the logic above implies that there exists an $s \in S$ such that $s > t \in T$.

Further, this means that some element in S does not exist in T.

In conjunction with p, this is a contradiction, as T must contain all of the elements in S for $S \subseteq T$ to be true as was assumed.

Conclusion

Thus by the contradiction of all three cases, our original statement, $(p \Rightarrow q)$ is true.

Problem 3.4.21

Let A be a nonempty open subset of $\mathbb{R}$ and let $\mathbb{Q}$ be the set of rationals. Prove that $A \cap \mathbb{Q} \neq \varnothing$.

Theorems and Definitions

The following, from pages 135 and 137, will be used in the proof below.

Theorem 3.4.7: A is open iff every point in A is an interior point of A.

Interior Point: Let $A \subseteq \mathbb{R}$. A point x in $\mathbb{R}$ is an interior point of S if there exists a neighborhood N of x such that $N \subseteq A$.

Proof:

Let A be a nonempty open subset of $\mathbb{R}$ and let $\mathbb{Q}$ be the set of rationals.

From theorem 3.4.7 we know that every point x in set A is an interior point. For each x to be an interior point, by the definition above, the neighborhood of x, $N(x;\varepsilon > 0)$, must be a subset of A.

Since every $x \in A$ is an interior point then every x has a neighborhood $N(x;\varepsilon > 0)$, which is an open interval containing real numbers.

Because $\mathbb{Q}$ is dense in $\mathbb{R}$, we know that there is an infinite number of rational numbers within each one of those neighborhoods of x.

Since $N(x;\varepsilon > 0) \subseteq A$ it then follows that $A \cap \mathbb{Q} \neq \varnothing$.

Problem 3.5.8

If S is a compact subset of $\mathbb{R}$, and T is closed subset of S, then T is compact.

(a) Prove this using the definition of compactness.

(b) Prove this using Heine-Borel theorem.

Theorems and Definitions

The following will be used in the proof below.

Compact: A set S is said to be compact iff every open cover of S contains a finite subcover.

Heine-Borel Theorem: A subset of $\mathbb{R}$ is compact iff S is closed and bounded.

3.4.10 Theorem: The union of any collection of open sets is an open set.

Proof for part a:

Let S be a compact subset of $\mathbb{R}$ and T be a closed subset of S.

Further, let $\mathscr{F}$ be an open cover of T.

The complement of the closed set T, $(\mathbb{R} \setminus T)$ is an open set and by theorem 3.4.10 the union, $\mathscr{F} \cup (\mathbb{R} \setminus T)$ is an open set. This union is also an open cover of S because it covers all elements in S.

Since S is compact there exists a finite subcover, denoted $\mathscr{F}'$, from the open cover $\mathscr{F} \cup (\mathbb{R} \setminus T)$ such that $\mathscr{F}' \subseteq [\mathscr{F} \cup (\mathbb{R} \setminus T)]$.

$\mathscr{F}'$ covers S, but since $T \subseteq S$ it will also cover T.

Because $T \cap (\mathbb{R} \setminus T) = \varnothing$ we can remove $(\mathbb{R} \setminus T)$ from $\mathscr{F}'$ and it would still cover T. That removal would make $\mathscr{F}' \subseteq \mathscr{F}$ and it means that $\mathscr{F}'$ is a finite subcover of T.

Thus every open cover of T contains a finite subcover.

Therefore T is compact.

Proof for part b:

Let S be a compact subset of $\mathbb{R}$ and T be a closed subset of S.

Heine-Borel theorem states that if T is closed and bounded then it is compact.

T is closed so we must show that T is bounded to prove that T is compact.

To show this we conclude some properties of S.

Since S is compact, by the Hiene-Borel theorem, S is closed and bounded.

Because $T \subseteq S$, T is also a bounded subset of $\mathbb{R}$.

Thus T is bounded and closed

Therefore T is compact.

Problem 4.1.12

a. Suppose that $\lim s_n = 0$. If (t_n) is a bounded sequence, prove that $\lim(s_n t_n) = 0$.

b. Show by an example that the boundedness of (t_n) is a necessary condition in part (a).

Part (a)

Proof:

To prove this directly let s_n and $s_n t_n$ be a sequences and $\lim s_n = 0$.

Now suppose that (t_n) is bounded.

Recall the definition of bounded is that (t_n) is said to be bounded if the range $\{t_n : n \in \mathbb{N}\}$ is a bounded set, that is, if there exists a real number $M \geq 0$ such that $|t_n| \leq M$ for all $n \in \mathbb{N}$.

Using theorem 4.1.8, let a_n be a sequence whos limit is 0. If for some $k > 0$ and some $m \in \mathbb{N}$ we have

$$|s_n t_n - s| \leq k|a_n|, \text{ for all } n \geq m,$$

and if $\lim a_n = 0$, then it follows that $\lim s_n t_n = s$.

To fit the theorem we let $a_n = s_n$ because $\lim s_n = 0$. Further, let $k = M + 1$ and $m = 1$. By substitution the above becomes,

$$|s_n t_n| \leq (M+1)|s_n|, \text{ for all } n \geq 1.$$

Because we know that $|t_n| \leq M$ for all $n \in \mathbb{N}$, we know that $(M+1)|s_n| \geq |s_n t_n|$ for all $n \in \mathbb{N}$.

Thus it follows that $\lim s_n t_n = 0$.

Part (b)

If $(t_n) = n^2$, which is not bounded, then (t_n) could be any value from the interval $(-\infty, \infty)$.

This is problematic because we would not be able to find a k sufficiently large enough such that $k|s_n| > |s_n t_n|$.

Unit 3 Presentation 4.3.11

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1 4.3.11

Prove theorem 4.3.3 for a bounded decreasing sequence.

1.1 Theorem 4.3.3

A monotone sequence is convergent if and only if it is bounded.

1.2 Definition 4.3.1

A sequence s_n is increasing if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$. A sequence is decreasing if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$. A sequence is monotone if it is increasing or decreasing.

1.3 Completeness axiom

Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. It follows that every subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound., that is to say $\inf S$ exists and is a real number.

1.4 Proof

Let s_n be a bounded decreasing sequence and let S denote the nonempty bounded set $\{s_n : n \in \mathbb{N}\}$ By the completeness axiom, S has a greatest lower bound, and we let $s = \inf S$ and claim that $\lim s_n = s$. Given any $\epsilon > 0, s + \epsilon$ is not a lower bound of S, thus there exists a natural number N such that $s_N < s + \epsilon$. Furthermore since s_n is decreasing and s is a lower bound of S we have

$$s + \epsilon > s_N \geq s_n \geq s$$

for all $n \geq N$. Hence s_n converges to s.