3.1.4

Prove that $1^3+2^3+3^3+...+n^3 = \frac{1}{4}n^2(n+1)^2$ for all $n \in \mathbb{N}$.

Proof:

Let $P(n)$ be the statement

$$1^3+2^3+3^3+...+n^3 = \frac{1}{4}n^2(n+1)^2 \quad \text{where } n \in \mathbb{N}$$

To establish the basis of induction we show that $P(1)$ is true.

First we look at the left side of $P(n)$ and let $n=1$.

$$n^3$$
$$(1)^3$$
$$1$$

Then the right side of $P(n)$ with $n=1$.

$$\frac{1}{4}n^2(n+1)^2$$
$$\frac{1}{4}1^2(1+1)^2$$
$$\frac{1}{4}2^2$$
$$\frac{4}{4}$$
$$1$$

Therefore because the left side agrees with the right we now know that $P(1)$ is true.

For the induction step we must verify $P(k+1)$ is true, assuming that $P(k)$ is true for some $k \in \mathbb{N}$, or that $1^3+2^3+3^3+...+k^3+(k+1)^3 = \frac{1}{4}(k+1)^2((k+1)+1)^2$ is true for some $k \in \mathbb{N}$.

Assume $P(k)$

$$1^3+2^3+3^3+...+k^3 = \frac{1}{4}k^2(k+1)^2 \quad \text{is true for some } k \in \mathbb{N}$$

Adding $(k+1)^3$ to both sides of the statement above and simplifying, yeilds:

$$
\begin{aligned}
1^3+2^3+3^3+\ldots+k^3+(k+1)^3 &= \frac{1}{4}k^2(k+1)^2+(k+1)^3 \\
&= \frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2+4(k+1)^3}{4} \\
&= \frac{1}{4}(k+1)^2(k^2+4(k+1)) \\
&= \frac{1}{4}(k+1)^2(k^2+4k+4) \\
&= \frac{1}{4}(k+1)^2(k+2)^2 \\
&= \frac{1}{4}(k+1)^2((k+1)+1)^2
\end{aligned}
$$

The above statement is the same as $P(k+1)$, therefore $P(k+1)$ is true.

Now because we have shown that $P(1)$ and $P(k+1)$, assuming that $P(k)$ is true for some $k \in \mathbb{N}$, are true we know that $P(n)$ is true for all $n \in \mathbb{N}$.