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Problem 3.4.21

Let A be a nonempty open subset of $\mathbb{R}$ and let $\mathbb{Q}$ be the set of rationals. Prove that $A \cap \mathbb{Q} \neq \varnothing$.

Theorems and Definitions

The following, from pages 135 and 137, will be used in the proof below.

Theorem 3.4.7: A is open iff every point in A is an interior point of A.

Interior Point: Let $A \subseteq \mathbb{R}$. A point x in $\mathbb{R}$ is an interior point of S if there exists a neighborhood N of x such that $N \subseteq A$.

Proof:

Let A be a nonempty open subset of $\mathbb{R}$ and let $\mathbb{Q}$ be the set of rationals.

From theorem 3.4.7 we know that every point x in set A is an interior point. For each x to be an interior point, by the definition above, the neighborhood of x, $N(x; \varepsilon > 0)$, must be a subset of A.

Since every $x \in A$ is an interior point then every x has a neighborhood $N(x; \varepsilon > 0)$, which is an open interval containing real numbers.

Because $\mathbb{Q}$ is dense in $\mathbb{R}$, we know that there is an infinite number of rational numbers within each one of those neighborhoods of x.

Since $N(x; \varepsilon > 0) \subseteq A$ it then follows that $A \cap \mathbb{Q} \neq \varnothing$.

Problem 3.5.8

If S is a compact subset of $\mathbb{R}$, and T is closed subset of S, then T is compact.

(a) Prove this using the definition of compactness.

(b) Prove this using Heine-Borel theorem.

Theorems and Definitions

The following will be used in the proof below.

Compact: A set S is said to be compact iff every open cover of S contains a finite subcover.

Heine-Borel Theorem: A subset of $\mathbb{R}$ is compact iff S is closed and bounded.

3.4.10 Theorem: The union of any collection of open sets is an open set.

Proof for part a:

Let S be a compact subset of $\mathbb{R}$ and T be a closed subset of S.

Further, let $\mathscr{F}$ be an open cover of T.

The complement of the closed set T, $(\mathbb{R} \setminus T)$ is an open set and by theorem 3.4.10 the union, $\mathscr{F} \cup (\mathbb{R} \setminus T)$ is an open set. This union is also an open cover of S because it covers all elements in S.

Since S is compact there exists a finite subcover, denoted $\mathscr{F}'$, from the open cover $\mathscr{F} \cup (\mathbb{R} \setminus T)$ such that $\mathscr{F}' \subseteq [\mathscr{F} \cup (\mathbb{R} \setminus T)]$.

$\mathscr{F}'$ covers S, but since $T \subseteq S$ it will also cover T.

Because $T \cap (\mathbb{R} \setminus T) = \varnothing$ we can remove $(\mathbb{R} \setminus T)$ from $\mathscr{F}'$ and it would still cover T. That removal would make $\mathscr{F}' \subseteq \mathscr{F}$ and it means that $\mathscr{F}'$ is a finite subcover of T.

Thus every open cover of T contains a finite subcover.

Therefore T is compact.

Proof for part b:

Let S be a compact subset of $\mathbb{R}$ and T be a closed subset of S.

Heine-Borel theorem states that if T is closed and bounded then it is compact.

T is closed so we must show that T is bounded to prove that T is compact.

To show this we conclude some properties of S.

Since S is compact, by the Hiene-Borel theorem, S is closed and bounded.

Because $T \subseteq S$, T is also a bounded subset of $\mathbb{R}$.

Thus T is bounded and closed

Therefore T is compact.