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Problem 3.3.8

Let S and T be nonempty bounded subsets of $\mathbb{R}$ with $S \subseteq T$. Prove that $\inf T \leq \inf S \leq \sup S \leq \sup T$.

Definitions

The following definitions will be used

Maximum: If m is an upper bound of S and $m \in S$, then m is called the maximum of S.

Minimum: If m is a lower bound of S and $m \in S$, then m is called the minimum of S.

Supremum: If S is bounded above the least upper bound of S is called its Supremum iff

a. $m \geq s$ for all $s \in S$ and,
b. if $m' < m$, then there exists $s' \in S$ such that $s' > m'$.

Infimum: If S is bounded below the greatest upper bound of S is called its Infimum iff

a. $m \leq s$ for all $s \in S$ and,
b. if $m' > m$, then there exists $s' \in S$ such that $s' < m'$.

Assumptions

We define the statement above as $p \Rightarrow q$, where p and q are,

$$p : S \text{ and } T \text{ be nonempty bounded subsets of } \mathbb{R} \text{ with } S \subseteq T$$

$$q : (\inf T \leq \inf S) \wedge (\inf S \leq \sup S) \wedge (\sup S \leq \sup T)$$

We also define $\sim q$ as,

$$\sim q : (\inf T > \inf S) \vee (\inf S > \sup S) \vee (\sup S > \sup T)$$

The structure of the proof will be as follows

$$(p \Rightarrow q) \Leftrightarrow [(p \wedge \sim q) \Rightarrow c]$$

Because of the nature of $\sim q$, this proof will be done in the following three cases,

$$\begin{array}{ll} (1) & p \wedge (\inf T > \inf S) \Rightarrow c \\ (2) & p \wedge (\inf S > \sup S) \Rightarrow c \\ (3) & p \wedge (\sup S > \sup T) \Rightarrow c \end{array}$$

Proof:

Statement 1

$$(1) \quad p \wedge (\inf T > \inf S) \Rightarrow c$$

Here the contradiction will lie within the hypothesis (see page 30).

Let $\inf T = m$, $\inf S = q$ and assume $[p \wedge (\inf T > \inf S)] \Rightarrow c$ to be true.

Part (b) of the definition of infimum for $\inf S$, states that if $m' > m$, then there exists $s' \in S$ such that $s' < m'$. So any element in $s' \in S$ must be larger than m. The same logic applies to T, in that any $t' \in T$ must be greater than q.

Now because we assumed that $(\inf T > \inf S)$ to be true or that $(q > m)$ the logic above implies that there exists an $s \in S$ such that $s < t \in T$.

Further, this means that some element in S does not exist in T.

In conjunction with p, this is a contradiction, as T must contain all of the elements in S for $S \subseteq T$ to be true as was assumed.

Statement 2

$$(2) \quad p \wedge (\inf S > \sup S) \Rightarrow c$$

The contradiction lies in $(\inf S > \sup S)$, because by definition $(\inf S \leq \sup S)$ is true.

Statement 3

$$(3) \quad p \wedge (\sup S > \sup T) \Rightarrow c$$

Similar to the logic of the first statement the contradiction lies in the hypothesis.

Let $\sup T = m$, $\sup S = q$ and assume $[p \wedge (\sup S > \sup T)] \Rightarrow c$ to be true.

Part (b) of the definition of infimum for $\sup S$, states that if $m' < m$, then there exists $s' \in S$ such that $s' > m'$. So any element in $s' \in S$ must be smaller than m. The same logic applies to T, in that any $t' \in T$ must be smaller than q.

Now because we assumed that $(\sup S > \sup T)$ to be true or that $(m > q)$ the logic above implies that there exists an $s \in S$ such that $s > t \in T$.

Further, this means that some element in S does not exist in T.

In conjunction with p, this is a contradiction, as T must contain all of the elements in S for $S \subseteq T$ to be true as was assumed.

Conclusion

Thus by the contradiction of all three cases, our original statement, $(p \Rightarrow q)$ is true.