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MATH 301

Problem 4.1.12

a. Suppose that $\lim s_n = 0$. If (t_n) is a bounded sequence, prove that $\lim(s_n t_n) = 0$.

b. Show by an example that the boundedness of (t_n) is a necessary condition in part (a).

Part (a)

Proof:

To prove this directly let s_n and $s_n t_n$ be a sequences and $\lim s_n = 0$.

Now suppose that (t_n) is bounded.

Recall the definition of bounded is that (t_n) is said to be bounded if the range $\{t_n : n \in \mathbb{N}\}$ is a bounded set, that is, if there exists a real number $M \geq 0$ such that $|t_n| \leq M$ for all $n \in \mathbb{N}$.

Using theorem 4.1.8, let a_n be a sequence whos limit is 0. If for some $k > 0$ and some $m \in \mathbb{N}$ we have

$$|s_n t_n - s| \leq k|a_n|, \text{ for all } n \geq m,$$

and if $\lim a_n = 0$, then it follows that $\lim s_n t_n = s$.

To fit the theorem we let $a_n = s_n$ because $\lim s_n = 0$. Further, let $k = M + 1$ and $m = 1$. By substitution the above becomes,

$$|s_n t_n| \leq (M+1)|s_n|, \text{ for all } n \geq 1.$$

Because we know that $|t_n| \leq M$ for all $n \in \mathbb{N}$, we know that $(M+1)|s_n| \geq |s_n t_n|$ for all $n \in \mathbb{N}$.

Thus it follows that $\lim s_n t_n = 0$.

Part (b)

If $(t_n) = n^2$, which is not bounded, then (t_n) could be any value from the interval $(-\infty, \infty)$.

This is problematic because we would not be able to find a k sufficiently large enough such that $k|s_n| > |s_n t_n|$.