

Exercise 5.3.11a

Use the Generalized Mean Value Theorem to furnish a proof of the 0/0 case of L'Hospital's Rule (Theorem 5.3.6)

A large portion of this work was adapted from https://www.math.hmc.edu/calculus/tutorials/lhopital/sketch_proof.html

Solution:

The Generalized Mean Value theorem states that if f and g are continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there exists a point $c \in (a, b)$ where

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

For this proof we will assume that that if f and g are continuous on a closed interval $[a, a + h]$ and differentiable on the open interval $(a, a + h)$, where $h > 0$.

It will also be assumed that $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$. Further, let $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L$

The generalized mean value theorem applies so it can be said that there exists a $c \in (a, b)$ such that

$$\frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Since $f(a) = g(a) = 0$, it is also that

$$\frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f(a+h)}{g(a+h)} = \frac{f'(c)}{g'(c)}.$$

If $h \rightarrow 0+$ from the right then the the interval would become smaller and the c will get closer to the left end of the interval, a .

$$\lim_{h \rightarrow 0^+} \frac{f'(c)}{g'(c)} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}$$

Similarly it can be said that

$$\lim_{h \rightarrow 0^+} \frac{f(a)}{g(a)} = \lim_{x \rightarrow a^+} \frac{f(x)}{g(x)}$$

Now since it is true that $\frac{f(a+h)}{g(a+h)} = \frac{f'(c)}{g'(c)}$, it follows that we can equate the two equations above as follows,

$$\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L.$$

The same argument can be applied with left hand limits if the interval was $(a + h, a)$. Thus the limits together finish the proof.

Exercise 6.2.1

Let

$$f_n(x) = \frac{nx}{1+nx^2}$$

a) Find the point-wise limit of (f_n) for all $n \in (0, \infty)$.

Solution:

$$f = \lim \frac{nx}{1+nx^2} = \lim \frac{x}{x^2} = \frac{1}{x}$$

b) Is the convergence uniform on $(0, \infty)$?

Solution:

$$\begin{aligned} |f_n(x) - f(x)| &< \varepsilon \\ \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right| &< \varepsilon \\ \left| \frac{-1}{x+nx^3} \right| &< \varepsilon \\ |x+nx^3| &> 1/\varepsilon \\ n &> \frac{1}{x^2} \left(\frac{1}{\varepsilon x} - 1 \right) \end{aligned}$$

Given any ε , when x is really large, then a finite N can be found. However, if x is really close to 0 then no N can be found that would make the statement true. Therefore, (f_n) does not converge uniformly on the interval $(0, \infty)$.

c) Is the convergence uniform on $(0, 1)$?

Solution:

Once again note that given $N > \frac{1}{x^2} \left(\frac{1}{\varepsilon x} - 1 \right)$, it would be impossible to find an N when x is close to 0. (f_n) is not uniformly convergent on the interval $(0, 1)$.

d) Is the convergence uniform on $(1, \infty)$?

Solution:

When x is really large, $N > \frac{1}{x^2} \left(\frac{1}{\varepsilon x} - 1 \right)$ then we see that any N will do because the right side of the inequality will become negative.

So the smaller the x will require a larger N . So this means that when $x = 1$ we will require an N such that $N > \left(\frac{1}{\varepsilon} - 1 \right)$ for any given $\varepsilon > 0$. Thus it converges uniformly on the interval $(1, \infty)$.

Exercise 6.2.3

For each $n \in \mathbb{N}$ and $x \in [0, \infty)$, let

$$g_n(x) = \frac{x}{1+x^n} \quad \text{and} \quad h_n(x) = \begin{cases} 1, & \text{if } x \geq 1/n \\ nx, & \text{if } 0 \leq x < 1/n \end{cases}$$

Answer the following questions for the sequences (g_n) and (h_n) :

- a) Find the point-wise limit on $[0, \infty)$.

Solution:

$x < 1$

- b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.

Solution:

$x < 1$

- c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Solution:

$x < 1$

Exercise 6.2.7

Let f be uniformly continuous on all of \mathbb{R} , and define a sequence of functions by $f_n(x) = f(x + 1/n)$. Show that $f_n \rightarrow f$ uniformly. Give an example to show that this proposition fails if f is only assumed to be continuous and not uniformly continuous on \mathbb{R} .

Solution:

$x < 1$

Exercise 6.3.3

Consider the sequence of functions

$$f_n(x) = \frac{x}{1+nx^2}.$$

- a) find the point on \mathbb{R} where each $f_n(x)$ attains its maximum and minimum value. Use this to prove (f_n) converges uniformly on \mathbb{R} . What is the limit function?

Solution:

A function will attain its maximum and minimum values when $f'_n = 0$.

$$f'_n = \frac{(1+nx^2)(1) - (x)(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

Now to obtain the points where each f_n reaches a maximum and minimum let $f'_n = 0$.

$$\begin{aligned}\frac{1-nx^2}{(1+nx^2)^2} &= 0 \\ 1-nx^2 &= 0 \\ x^2 &= 1/n \\ x &= \pm\sqrt{1/n}\end{aligned}$$

The derivative obtains negative values when left of $-\sqrt{1/n}$ and to the right of $\sqrt{1/n}$. The derivative obtains positive values between $-\sqrt{1/n}$ and $\sqrt{1/n}$. Thus (f_n) attains a minimum when $a = -\sqrt{1/n}$ and a maximum when $b = \sqrt{1/n}$.

Also, note that the value of the maximum or minimum is

$$\left| f_n\left(\frac{1}{\sqrt{n}}\right) \right| = \left| \frac{\frac{1}{\sqrt{n}}}{1+n\frac{1}{\sqrt{n}^2}} \right| = \left| \frac{1}{2\sqrt{n}} \right|.$$

As n gets large, then a and b get closer to 0. However, as a, b gets closer 0, then $f_n(a), f_n(b)$ also gets closer to 0. This means that the sequence of functions are converging to 0.

To prove that the function is uniformly converging to 0, let $\varepsilon > 0$ and $n > N > \frac{1}{4\varepsilon^2}$.

$$\begin{aligned}n &> \frac{1}{4\varepsilon^2} \\ \frac{1}{2\sqrt{n}} &< \varepsilon \\ \left| \frac{x}{1+nx^2} \right| &< \left| \frac{1}{2\sqrt{n}} \right| < |\varepsilon| \\ \left| \frac{x}{1+nx^2} - 0 \right| &< |\varepsilon|\end{aligned}$$

Thus $(f_n) \rightarrow 0$ uniformly.

b) Let $f = \lim f_n$. Compute $f'_n(x)$ and find all the values of x for which $f'(x) = \lim f'_n(x)$.

Solution:

Let $f = \lim f_n = 0$.

$$f'_n(x) = \frac{1 - nx^2}{(1 + nx^2)^2} = \frac{1 - nx^2}{(1 + 2nx^2 + 2n^2x^4)}$$

$$\lim f'_n(x) = \lim \frac{1 - nx^2}{(1 + 2nx^2 + 2n^2x^4)} = \lim \frac{x^2}{(2x^2 + 4nx^4)} = 0$$

The derivative of the sequence function will converge to 0 for an x for which the function is defined.

Exercise 6.3.4

Let

$$h_n(x) = \frac{\sin(nx)}{\sqrt{n}}$$

Show that $h_n \rightarrow 0$ uniformly on \mathbb{R} but that the sequence of derivatives (h'_n) diverges for every $x \in \mathbb{R}$.

Solution:

To prove that h_n converges uniformly it must be shown that whenever $n > N > \left(\frac{1}{\varepsilon}\right)^2$ then $|h_n(x) - h(x)| < \varepsilon$.

$$\begin{aligned} n &> \left(\frac{1}{\varepsilon}\right)^2 \\ n &> \left(\frac{1}{\varepsilon}\right)^2 \geq \left(\frac{\sin(nx)}{\varepsilon}\right)^2 \\ \sqrt{n} &> \frac{|\sin(nx)|}{\varepsilon} \\ \frac{\sqrt{n}}{|\sin(nx)|} &> \frac{1}{\varepsilon} \\ \left|\frac{\sin(nx)}{\sqrt{n}} - 0\right| &< \varepsilon \end{aligned}$$

Thus $(h_n) \rightarrow 0$ uniformly.

The derivative of (h_n) is the equal to the following,

$$h'_n(x) = \frac{n \cos(nx)}{\sqrt{n}}$$

Now we will see that limit of (h'_n) diverges for all $x \in \mathbb{R}$.

$$\lim_{n \rightarrow \infty} \frac{n \cos(nx)}{\sqrt{n}} = \lim_{n \rightarrow \infty} \sqrt{n} \cos(nx) = \infty$$

This is true because $\cos(nx)$ is bounded by $[-1, 1]$, but \sqrt{n} will grow. If nx is a multiple of $\pi/2, 3\pi/2$ then $\cos(nx) = 0$, however nx is never always a multiple of $\pi/2, 3\pi/2$. The function will then oscillate in those cases, but this is also diverging.

Therefore as $n \rightarrow \infty$, then $(h'_n) \rightarrow \infty$.

Exercise 6.3.5

Let

$$g_n(x) = \frac{nx + x^2}{2n},$$

and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:

- a) Compute $g(x)$ by algebraically taking the limit as $n \rightarrow \infty$ and then find $g'(x)$.

Solution:

$$\lim \frac{nx + x^2}{2n} = \lim \frac{x}{2} = \frac{x}{2}$$

Since $g(x) = x/2$, then $g'(x) = 1/2$.

- b) Compute $g'_n(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives (g'_n) converges uniformly on every interval $[-M, M]$. Use Theorem 6.3.3 to conclude $g'(x) = \lim g'_n(x)$.

Solution:

$$\frac{d}{dx} \left[\frac{nx + x^2}{2n} \right] = \frac{1}{2n} \cdot \frac{d}{dx} [nx + x^2] = \frac{n + 2x}{2n}$$

Now I will show that $(f'_n) \rightarrow 1/2$ uniformly on the closed interval $[-M, M]$. Let $n > N > |M|/\varepsilon$.

$$\begin{aligned} n &> \frac{|M|}{\varepsilon} \\ \left| \frac{M}{n} \right| &< \varepsilon \\ \left| \frac{x}{n} \right| &\leq \left| \frac{M}{n} \right| < \varepsilon \\ \left| \frac{n + 2x - n}{2n} \right| &< \varepsilon \\ \left| \frac{n + 2x}{2n} - \frac{n}{2n} \right| &< \varepsilon \\ \left| \frac{n + 2x}{2n} - \frac{1}{2} \right| &< \varepsilon \end{aligned}$$

Thus $(g'_n) \rightarrow 1/2$ uniformly on the closed interval $[0M, M]$.

Now assume a point x_0 such that $x_0 \in [-M, M]$. Below we see that if we fix x at the point x_0 the sequence of function is still uniformly convergent.

$$\begin{aligned} n &> \frac{|M|}{\varepsilon} \\ \left| \frac{x_0}{n} \right| &\leq \left| \frac{M}{n} \right| < \varepsilon \\ \left| \frac{n + 2x_0 - n}{2n} \right| &< \varepsilon \\ \left| \frac{n + 2x_0}{2n} - \frac{1}{2} \right| &< \varepsilon \end{aligned}$$

Theorem 6.3.3 can now apply because (f_n) is differentiable, defined on the closed interval $[-M, M]$ and (g'_n) was proven to be converging uniformly. Also, there exists a point $x_0 \in [-M, M]$ for which $g_n(x_0)$ is convergent.

Thus by the theorem 6.3.3, g is differentiable. Therefore $g' = 1/2$ and $\lim g'_n(x) = 1/2$, which implies $(g'_n) \rightarrow g'$.