Exercise 1

Use the definition of continuity to show that $f(x) = ax + b$ is continuous at every point $c \in \mathbb{R}$

Solution:

Let $c \in \mathbb{R}$ and $\varepsilon > 0$.

$$\begin{aligned} |f(x) - f(c)| &< \varepsilon \\ |(ax+b) - (ac+b)| &< \varepsilon \\ |ax - ac| &< \varepsilon \\ |x - c| &< \varepsilon/|a| \end{aligned}$$

Now let $\delta = \varepsilon/|a| > 0$, and consider the following,

$$\begin{aligned} |x - c| &< \delta \\ |x - c| &< \varepsilon/|a| \\ |ax - ac| &< \varepsilon \\ |ax - b + b - ac| &< \varepsilon \\ |(ax+b) - (ac+b)| &< \varepsilon \\ |f(x) - f(c)| &< \varepsilon \end{aligned}$$

Thus by definition of continuity, f is continuous at $c \in \mathbb{R}$.

Exercise 2

Use the definition of compactness to show that if set $K \subseteq \mathbb{R}$ is closed and bounded, then it is compact.

Solution:

The definition of compactness states that every sequence in K must a have a subsequence that converges to a limit that is also in K.

Because K is bounded it is true that every sequence in K will also be bounded. Bolzano-Weierstrass Theorem states that every bounded sequence will have a convergent subsequence.

The only thing left to prove is that the limit of those subsequences are in K.

Because of the definition of a closed set, it is true that K contains all of its limit points. Theorem 3.2.5 defines x as a limit point of K if $x = \lim k_n$ for some sequence (k_n) contained in K.

Since K contains all of its limit points and every sequence in K will converge to a limit point, it follows that every subsequence will converge to a limit in K.

Thus K is compact.

Exercise 3

Use the definition of continuity to show that $f(x) = x^3$ is continuous for all real numbers $x \geq 0$.

Solution:

Let $\varepsilon > 0$ and $c > 0$.

$$\begin{aligned} |f(x) - f(c)| &< \varepsilon \\ |x^3 - c^3| &< \varepsilon \\ |x-c||x^2+xc+c^2| &< \varepsilon \\ |x-c| &< \frac{\varepsilon}{|x^2+xc+c^2|} \end{aligned}$$

Since f is only defined for $x \geq 0$ and c is positive it follows that,

$$|x-c| < \frac{\varepsilon}{|x^2+xc+c^2|} \leq \frac{\varepsilon}{c^2}$$

Therefore, let $\delta = \varepsilon / c^2$.

$$\begin{aligned} |x-c| &< \delta \\ |x-c| &< \frac{\varepsilon}{c^2} \\ |x-c| &< \frac{\varepsilon}{|x^2+xc+c^2|} \leq \frac{\varepsilon}{c^2} \\ |x-c||x^2+xc+c^2| &< \varepsilon \\ |x^3 - c^3| &< \varepsilon \\ |f(x) - f(c)| &< \varepsilon \end{aligned}$$

Thus f is continuous for all $x \geq 0$.

Exercise 4

Assume that $f : [0, \infty] \to \mathbb{R}$ is continuous at every point in its domain. Show that if there exists $b > 0$ such that f is uniformly continuous on the set $[b.\infty)$, then f is uniformly continuous on $[0, \infty)$

Solution:

Since $b > 0$, f is, at the moment, known to be uniform on all postive real numbers except 0 itself.

I must show that the point 0 will require the same or larger δ for any given ε defined for the set $[b, \infty)$.

f is uniform on the set $[b.\infty)$, meaning that δ is not dependent on a point c.

Since it is also known that f is defined on 0 and b is allowed to be arbitrarily close to 0. It follows that no smaller δ is needed if 0 was included in the set.

Exercise 5

Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Solution:

I must show that K is either empty, equivalent to $\mathbb{R}$, or only contains isolated points.

K will be empty if $h(X) \neq 0$. In this case K will closed.

K will be equivalent to $\mathbb{R}$ when $h(x) = 0$ for all $x \in \mathbb{R}$. K is also closed in this case.

The only other option is for K to contain only isolated points.

Since the function is continuous the Intermediate Value Theorem implies that when $L = 0$, there exists a point, c, such that $f(c) = 0$, and any other value greater or less than c will result in a value greater or less than $f(c) = 0$.

It follows that in this situation K will only contain isolated points and is closed.