Exercise 4.3.3

a) Supply a proof for Theorem 4.3.9 using the $\varepsilon - \delta$ characterization of continuity.

Solution:

The theorem states: If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g(f(c))$ is continuous at c.

Since g is continuous at $f(c)$ it can be said that whenever $|f(x) - f(c)| < \delta$ it follows that $|g(f(x)) - g(f(c))| < \varepsilon$. This means that $g(f(x))$ is continuous at c whenever $|x - c| < \delta$.

b) Give another proof of this theorem using the sequential characterization of continuity (thm 4.3.2 iii).

Solution:

thm 4.3.2 iii) states: For all $(x_n) \to c$ (with $x_n \in A$), it follows that $f(x_n) \to f(c)$.
Since g is continuous at $f(c)$ it can be said that, for all $(f(x_n)) \to f(c)$ (with $f(x_n) \in B$), it follows that $g(f(x_n)) \to g(f(c))$. This means that $g(f(x))$ is continuous at c.

Exercise 4.3.11

(CONTRACTION MAPPING THEOREM) Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

a) Show that f is continuous on $\mathbb{R}$.

Solution:

Let $\delta = \varepsilon / c$, where $epsilon > 0$.

Whenever $|x - y| < \delta$ it follows that

$$\begin{aligned} |x - y| &< \varepsilon / c \\ c|x - y| &< \varepsilon \\ |f(x) - f(y)| \leq c|x - y| &< \varepsilon \\ |f(x) - f(y)| &< \varepsilon \end{aligned}$$

Thus f is continuous by definition of continuity.

b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.

Solution:

To show that the sequence is Cauchy, I must show that there exist an N such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \varepsilon$.
Let $y_{n+1} = f(y_n)$ and pick some arbitrary y_1 within $\mathbb{R}$. Given $|f(x) - f(y)| \leq c|x - y|$, it follows that

$$\begin{aligned} |f(y_1) - f(y_2)| &\leq c|y_1 - y_2| \\ |y_2 - y_3| &\leq c|y_1 - y_2| \end{aligned}$$

It is also true that

$$|y_3 - y_4| \leq c|y_2 - y_3|$$

Then since $0 < c < 1$

$$|y_3 - y_4| < c|y_2 - y_3| \leq c^2|y_1 - y_2|$$

Staying with the same pattern it can be said that

$$|y_n - y_{n+1}| < c^{n-1}|y_1 - y_2| \tag{1}$$

The right side will get closer to 0.

Solution:

This is not enough to prove that the sequence is Cauchy because I need to show that I can choose any $n,m > N$.

Let $n,m > N$. Consider the following,

$$|y_N - y_m| = |y_N - y_{N+1} + y_{N+1} - \ldots - y_{m-1} + y_{m-1} - y_m|$$

Then using the inequality above (1),

$$|y_N - y_{N+1} + y_{N+1} - \ldots - y_{m-1} + y_{m-1} - y_m| \leq ||y_N - y_{N+1}| + c|y_N - y_{N+1}| + \ldots + c^{m-2}|y_N - y_{N+1}||$$

After using the triangle inequality theorem,

$$|y_N - y_m| < (|y_N - y_{N+1}| + c|y_N - y_{N+1}| + \ldots + c^{m-2}|y_N - y_{N+1}|)$$
$$|y_N - y_m| < (1 + c + c^2 + \ldots + c^{m-2})|y_N - y_{N+1}|$$

Every c can be summed using a geometric formula. But the finite sum will be less than the infinite sum so I write the above inequality as the following,

$$|y_N - y_m| < \frac{|y_N - y_{N+1}|}{1-c}$$

It can also be said that

$$|y_N - y_n| < \frac{|y_N - y_{N+1}|}{1-c}$$

Consider

$$|y_n - y_m| = |y_n - y_N + y_N - y_m| \leq |y_N - y_n| + |y_N - y_m|$$
$$|y_n - y_m| < \frac{|y_N - y_{N+1}|}{1-c} + \frac{|y_N - y_{N+1}|}{1-c}$$
$$|y_n - y_m| < \frac{2c^{N-1}|y_1 - y_2|}{1-c}$$
$$\frac{2c^{N-1}|y_1 - y_2|}{1-c} = \varepsilon$$
$$c^{N-1} = \frac{\varepsilon(1-c)}{2|y_1 - y_2|}$$

Thus,

$$N = \lfloor \frac{\ln\left(\frac{\varepsilon(1-c)}{2|y_1-y_2|}\right)}{\ln(c)} \rfloor + 1$$

Let $\varepsilon > 0$. Thus choose an N such that $m,n > N$ it follows that

$$|y_n - y_m| < \frac{2|y_N - y_{N+1}|}{1-c} < \frac{2c^{N-1}|y_1 - y_2|}{1-c} = \varepsilon$$

c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.

Solution:

Because the sequence is Cauchy we know that it converges to y. Also note that the function is continuous on its domain and is a recursive dynamical sequence. So the sequence $(y_1, f(y_1), f(f(y_1)), \ldots) = (y, f(y), f(f(y), \ldots) = (y, y, f(y), \ldots) = (y, y, y, \ldots)$ is obviously converging to y.

d) Finally, prove that if x is any arbitrary point in $\mathbb{R}$, then the sequence $(xf(x), f(f(x)), \ldots)$ converges to y defined in (b).

Solution:

Since the function is continuous on all of $\mathbb{R}$ and converges to a point, y, then it must be true that any starting point on the real number line will lead the sequence to converge to y.

Exercise 4.3.13

Let f be a function defined on all of $\mathbb{R}$ that satisfies the additive condition $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Solution:

Let $f(a) = c$

Because $f(x+y) = f(x) + f(y)$, the only way for a sum of two functions to be zero is if c is already zero, or if c was added with its inverse, $-c$.

Thus if $f(a) = c$ then it must be true that $f(-a) = -c$ because $f(a-a) = c + (-c) = 0 = f(0)$.

b) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbb{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbb{Z}$. Now, prove that $f(r) = kr$ for any rational number r.

Solution:

I know that $f(1) = k$ and its also true that $f(1) + f(1) = k + k = 2k$.

If $f(n) = nk$ then it follows that

$$f(n) + f(1) = nk + k$$

Then using the function's additive property,

$$f(n+1) = (n+1)k$$

Thus through induction, $f(n) = nk$ will be true for all $n \in \mathbb{N}$.

The exact same logic works for $f(z) = kz$. The only thing left to prove is that this works for the negative integers.

However, we know that $f(-n) = -nk$, thus f can map to $\mathbb{Z}$.

To prove that $f(r) = kr$ consider the following,

$$\begin{aligned}
f(1) = f(\sum^{n}(1/n)) &= k \\
\sum^{n} f(1/n) &= k \\
nf(1/n) &= k \\
f(1/n) &= k/n \\
f(m/n) &= \frac{mk}{n} \\
f(r) &= rk
\end{aligned}$$

c) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$ and conclude that $f(x) = kx$ for all $x \in \mathbb{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

Solution:

Thm. 4.3.2 (iii) states that if $(x_n) \to c$ then it follows that $f(x_n) \to f(c)$.

We know that $f(r) = rk$, and we can find a sequence of rational numbers that converges to a particular irrational number. This means that since $(r_n) \to c \Rightarrow f(x_n) \to f(c)$. Thus our function is defined for all integers, rational and irrational numbers.

Let $f(x) = kx$ and $\delta = \varepsilon/k > 0$. We already know that When ever $|x-c| < \delta$ it follows that

$$\begin{aligned} |x-c| &< \varepsilon/k \\ |kx-kc| &< \varepsilon \\ |f(x)-f(c)| &< \varepsilon \end{aligned}$$

$f(x) = kx$ is a linear equation with varying slope.

Exercise 4.4.1

a) Show that $f(x) = x^3$ is continuous on all $\mathbb{R}$.

Solution:

I know that x is continuous on all $\mathbb{R}$. I can use the algebraic continuity theorem (iii) to show that $x \cdot x \cdot x = x^3$ is also continuous.

b) Argue, using them 4.4.5, that f is not uniformly continuous on $\mathbb{R}$.

Solution:

Let $x_n = n$ and $y_n = \frac{n^2+1}{n}$.

$$\lim |n - \frac{n^2+1}{n}|$$
$$\lim |\frac{n^2}{n} - \frac{n^2+1}{n}|$$
$$\lim \left(\frac{1}{n}\right) = 0$$

However,

$$\begin{aligned}
|f(x_n) - f(y_n)| = \\
&= |n^3 - \left(\frac{n^2+1}{n}\right)^3 | \\
&= |\frac{n^6}{n^3} - \frac{n^6+3n^4+3n^2+1}{n^3}| \\
&= |\frac{-3n^4-3n^2-1}{n^3}| \\
&= 3n + 3 + 1/n^3 > \varepsilon_0
\end{aligned}$$

As we can see, that the difference between the two functions will always result in a value greater than epsilon that we could set it at.

c) Show that f is uniformly continuous on any bounded subset of $\mathbb{R}$.

Solution:

If $x, c \in A \subseteq \mathbb{R}$, where A is a bounded. You will notice that $|x^2 + xc + c^2| \leq |3c^2|$ if $c \geq x$. This will serve as our upper bound. Let $\delta = \frac{\varepsilon}{|3c^3|}$

$$\begin{aligned} |x-c| &< \delta \\ |x-c| &< \frac{\varepsilon}{|3c^3|} \\ |3c^2||x-c| &< \varepsilon \\ |x^2+xc+c^2||x-c| \leq |3c^2||x-c| &< \varepsilon \\ |x^3-c^3| &< \varepsilon \end{aligned}$$

Thus for any bounded interval you can find a delta that works for all $x \in A$ for any chosen epsilon.

Exercise 4.4.5

Assume that g is defined on an open interval (a,c) and it is known to be uniformly continuous on $(a,b]$ and $[b,c)$, where $a<b<c$. Prove that g is uniformly continuous on (a,c).

Solution:

g is uniformly continuous given that $x,y \in (a,b]$. g is also uniformly continuous given that $x,y \in [b,c)$. But we don't know that g is uniformly continuous that when $x \in (a,b]$ and $y \in [b,c)$.

Let $\varepsilon/2 > 0$. Let $|g(x)-g(b)| < \varepsilon/2$, given that $x \in (a,b]$. Further, let $|g(y)-g(b)| < \varepsilon/2$, given that $y \in [b,c)$.

$$\begin{aligned} |g(x)-g(b)|+|g(y)-g(b)| &< \varepsilon/2+\varepsilon/2 \\ |g(x)-g(y)| \leq |g(x)-g(b)|+|g(y)-g(b)| &< \varepsilon \\ |g(x)-g(y)| &< \varepsilon \end{aligned}$$

Exercise 4.4.6ab

Give an example of each of the following, or state that such a request is impossible. for any that are impossible, supply a short explanation for why this is the case.

a) A continuous function $f : (0,1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Solution:

True. Consider $y = \cot(\pi x)$ will map $(0,1) \to \mathbb{R}$, and if $x_n = \frac{1}{2n}$.

Here we see that x_n is Cauchy, however, as the sequence gets closer to zero then $f(x_n)$ will get infinitely large, meaning it diverges.

b) A uniformly continuous function $f : (0,1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Solution:

This is false. No continuous function that maps $(0,1) \to \mathbb{R}$ is uniformly continuous.