Problem 3.2.2

Let

$$A = \left\{(-1)^n + \frac{2}{n} : n = 1,2,3,...\right\} \text{ and } B = \{x \in \mathbb{Q} : 0 < x < 1\}$$

Answer the following questions for each set:

a) what are the limit points?

Solution:

The limits of set A are $\{-1, 1\}$.

The limits of set B are $[0, 1]$.

b) Is the set open? Closed?

Solution:

Set A is not closed because it does not contain all of its limit points. It is also not open because it is possible to find a neighborhood around a point within A that is not a subset of A.

Set B is not closed because it does not contain its limit points. It is also not open because no neighborhood around a point within B is a subset of B because the neighborhood contains irrational numbers.

c) Does the set contain any isolated points?

Solution:

Set A does and set B does not.

d) Find the closure of the set.

Solution:

$\bar{A} = \left\{(-1)^n + \frac{2}{n} : n = 1,2,3,...\right\} \cup \{-1, 1\}$

$\bar{B} = [0, 1]$

Problem 3.2.2

Decide whether the following sets are open, closed, or neither. If the set is not open, find a point in the set for which there is no ε-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

a) $\mathbb{Q}$

Solution:

It is not open or closed
$1/2 \in \mathbb{Q}$, however, any neighborhood around $1/2$ would contain irrational numbers which would make it not a subset of $\mathbb{Q}$
$\sqrt{2}$ is a limit point but $\sqrt{2} \notin \mathbb{Q}$.

b) $\mathbb{N}$

Solution:

It is closed but not open. $1 \in \mathbb{N}$ but $V_{1/2}(1) \nsubseteq \mathbb{N}$.

c) $\{x \in \mathbb{R} : x \neq 0\}$

Solution:

The set is not closed because it doesn't contain all of its limit point which 0 is included as a limit point.

d) $\{1 + 1/4 + 1/9 + ... + 1/n^2 : n \in \mathbb{N}\}$

Solution:

The set is not closed because 0 is a limit point but it is not included in the set. It is not open because I can find a nieghborhood around 1/4 that is not a subset of the set.

Problem 3.2.4

Let A be nonempty and bounded above so that $s = \sup A$ exists.

a) Show that $s \in \bar{A}$.

Solution:

For $s \in \bar{A}$ then it must be either in the closure of A, in A or both. Since $s = \sup A$ then any element less than than s is inside the set A.
Therefore every neighborhood of s will intersect the set A, meaning that it is a limit point. Thus $s \in \bar{A}$.

b) Can an open set contains its supremum?

Solution:

No. Any element greater than the supremum is an upper bound of the set, meaning that it is outside of the set. Therefore, any neighborhood around the supremum will not be a subset of the set because it will contain upper bounds.