HW #8 .5 Hours
4.3.3, 4.3.11, 4.3.13
4.4.1, 4.4.5, 4.4.6ab

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Exercise 4.3.3

a) Supply a proof for Theorem 4.3.9 using the $\varepsilon - \delta$ characterization of continuity.

Solution:

x<1

b) Give another proof of this theorem using the sequential characterization of continuity (thm 4.3.2 iii).

Solution:

x<1

Exercise 4.3.11

(CONTRACTION MAPPING THEOREM) Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

a) Show that f is continuous on $\mathbb{R}$.

Solution:

x<1

b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.

Solution:

x<1

c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.

Solution:

x<1

d) Finally, prove that if x is any arbitrary point in $\mathbb{R}$, then the sequence $(y_1, f(y_1), f(f(y_1)), \ldots)$ converges to y defined in (b).

Solution:

x<1

Exercise 4.3.13

Let f be a function defined on all of $\mathbb{R}$ that satisfies the additive condition $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Solution:

x<1

b) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbb{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbb{Z}$. Now, prove that $f(r) = kr$ for any rational number r.

Solution:

x<1

c) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$ and conclude that $f(x) = kx$ for all $x \in \mathbb{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

Solution:

x<1

Exercise 4.4.1

a) Show that $f(x) = x^3$ is continuous on all $\mathbb{R}$.

Solution:

x<1

b) Argue, using them 4.4.5, that f is not uniformly continuous on $\mathbb{R}$.

Solution:

x<1

c) Show that f is uniformly continuous on any bounded subset of $\mathbb{R}$.

Solution:

x<1

Exercise 4.4.5

Assume that g is defined on an open interval (a,c) and it is known to be uniformly continuous on $(a,b]$ and $[b,c)$, where $a < b < c$. Prove that g is uniformly continuous on (a,c).

Solution:

x<1

Exercise 4.4.6ab

Give an example of each of the following, or state that such a request is impossible. for any that are impossible, supply a short explanation for why this is the case.

a) A continuous function $f : (0,1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Solution:

x<1

b) A uniformly continuous function $f : (0,1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Solution:

x<1