Exercise 5.3.11a

Use the Generalized Mean Value Theorem to furnish a proof of the $0/0$ case of L'Hospital's Rule (Theorem 5.3.6)

A large portion of this work was adapted from https://www.math.hmc.edu/calculus/tutorials/lhopital/sketch_proof.html

Solution:

The Generalized Mean Value theorem states that if f and g are continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a point $c \in (a,b)$ where

$$[f(b)-f(a)]g'(c) = [g(b)-g(a)]f'(c)$$

For this proof we will assume that that if f and g are continuous on a closed interval $[a,a+h]$ and differentiable on the open interval $(a,a+h)$, where $h>0$.

It will also be assumed that $f(a)=g(a)=0$ and $g'(x)\neq 0$ for all $x\neq a$. Further, let $\lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L$

The generalized mean value theorem applies so it can be said that there exists a $c\in(a,b)$ such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Since $f(a)=g(a)=0$, it is also that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f(a+h)}{g(a+h)} = \frac{f'(c)}{g'(c)}.$$

If $h\to 0+$ from the right then the the interval would become smaller and the c will get closer to the left end of the interval, a.

$$\lim_{h\to 0^+}\frac{f'(c)}{g'(c)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}$$

Similarly it can be said that

$$\lim_{h\to 0^+}\frac{f(a)}{g(a)} = \lim_{x\to a^+}\frac{f(x)}{g(x)}$$

Now since it is true that $\frac{f(a+h)}{g(a+h)} = \frac{f'(c)}{g'(c)}$, it follows that we can equate the two equations above as follows,

$$\lim_{x\to a^+}\frac{f'(x)}{g'(x)} = \lim_{x\to a^+}\frac{f(x)}{g(x)} = L.$$

The same argument can be applied with left hand limits if the interval was $(a+h,a)$. Thus the limits together finish the proof.

Exercise 6.2.1

Let

$$f_n(x) = \frac{nx}{1+nx^2}$$

a) Find the point-wise limit of (f_n) for all $n \in (0,\infty)$.

Solution:

$$f = \lim \frac{nx}{1+nx^2} = \lim \frac{x}{x^2} = \frac{1}{x}$$

b) Is the convergence uniform on $(0,\infty)$?

Solution:

$$\begin{aligned} |f_n(x) - f(x)| &< \varepsilon \\ \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right| &< \varepsilon \\ \left| \frac{-1}{x+nx^3} \right| &< \varepsilon \\ |x+nx^3| &> 1/\varepsilon \\ n &> \frac{1}{x^2}\left(\frac{1}{\varepsilon x} - 1\right) \end{aligned}$$

Given any ε, when x is really large, then a finite N can be found. However, is x is really close to 0 then no N can be found that would make the statement true. Therefore, (f_n) does not converge uniformly on the interval $(0,\infty)$.

c) Is the convergence uniform on $(0,1)$?

Solution:

Once again note that given $N > \frac{1}{x^2}\left(\frac{1}{\varepsilon x} - 1\right)$, it would be impossible to find an N when x is close to 0. (f_n) is not uniformly convergent on the interval $(0,1)$.

d) Is the convergence uniform on $(1,\infty)$?

Solution:

When x is really large, $N > \frac{1}{x^2}\left(\frac{1}{\varepsilon x} - 1\right)$ then we see that any N will do because the right side of the inequality will becoming negative.

So the smaller the x will require a larger N. So this means that when $x = 1$ we will require an N such that $N > \left(\frac{1}{\varepsilon} - 1\right)$ for any given $\varepsilon > 0$. Thus it converges uniformly on the interval $(1,\infty)$.

Exercise 6.2.3

For each $n \in \mathbb{N}$ and $x \in [0, \infty)$. let

$$g_n(x) = \frac{x}{1+x^n} \quad \text{and} \quad h_n(x) = \begin{cases} 1\,, & \text{if } x \geq 1/n \\ nx\,, & \text{if } 0 \leq x < 1/n \end{cases}$$

Answer the following questions for the sequences (g_n) and (h_n):

a) Find the point-wise limit on $[0, \infty)$.

Solution:

$$\lim g_n(x) = \begin{cases} x\,, & \text{if } 0 \leq x < 1 \\ 1/2\,, & \text{if } x = 1 \\ 0\,, & \text{if } x > 1 \end{cases}$$

and also,

$$\lim h_n(x) = \begin{cases} 0\,, & \text{if } x = 0 \\ 1\,, & \text{if } x > 0 \end{cases}$$

b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.

Solution:

The Contiuous Limit Theorem states that given a uniformly convergent sequence of functions, if (f_n) is continuous at c, then f is continuous at c.

We see that (f_n) and (h_n) are both continuous at a point c but where f and g are not continuous at c.

The conclusion is false even though the hypotheis held true. This happened because the assumption of converging uniformly was not properly met.

c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Solution:

Define (g_n) for the domain $x = 1$. The limit is then equal to $1/2$. Now let $\varepsilon > 0$ and $n > N$.

$$|f_n(x) - f| = \left| \frac{x}{1+x^n} - \frac{1}{2} \right| = \left| \frac{1}{1+1^n} - \frac{1}{2} \right| = 0 < \varepsilon$$

Define (h_n) on the domain $x \geq 1$. The limit is then equal to 1. Now let $\varepsilon > 0$ and $n > N \geq 1$.

$$|h_n(x) - h| = |1 - 1| = 0 < \varepsilon$$

Exercise 6.2.7

Let f be uniformly continuous on all of $\mathbb{R}$, and define a sequence of functions by $f_n(x) = f(x+1/n)$. Show that $f_n \to f$ uniformly. Give an example to show that this proposition fails if f is only assumed to be continuous and not uniformly continuous on $\mathbb{R}$.

Solution:

Let f be uniform and $f_n(x) = f(x+1/n)$. Since f is uniform there exists an $x, y \in \mathbb{R}$ such that $|y-x| < \delta$ implies that $|f(y) - f(x)| < \varepsilon$
Let $|y-x| < \delta$ for every $\varepsilon > 0$. Further let $y = x + 1/n$.

$$|y-x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$$
$$|x + \frac{1}{n} - x| < \delta \Rightarrow |f(1+\frac{1}{n}) - f(x)| < \varepsilon$$
$$|x + \frac{1}{n} - x| < \delta \Rightarrow |f_n(x) - f(x)| < \varepsilon$$

Thus $(f_n) \to f$ uniformly.
The next step will be a proof by contradiction. Assume the function $f = 1/x$ on the domain $x > 0$ is uniformly continuous.

$$f_n(x) = f(1+1/n) = \frac{1}{x+1/n} = \frac{n}{nx+1}$$

Since the function is uniform Cauchy Criterion for Uniform Convergence applies, which implies that for every $\varepsilon > 0$ there exists an N such that whenever $m, n > N$ implies $|f_n(x) - f_m(x)| < \varepsilon$.

$$\left| \frac{n}{nx+1} - \frac{m}{mx+1} \right| < \varepsilon$$

However this statement cannot be true for all ε. No N can be found when x is arbitrarily close to 0. Therefore f cannot be uniformly convergent.

Exercise 6.3.3

Consider the sequence of functions

$$f_n(x) = \frac{x}{1+nx^2}.$$

a) find the point on $\mathbb{R}$ where each $f_n(x)$ attains its maximum and minimum value. Use this to prove (f_n) converges uniformly on $\mathbb{R}$. What is the limit function?

Solution:

A function will attain its maximum and minimum values when $f_n' = 0$.

$$f_n' = \frac{(1+nx^2)(1)-(x)(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

Now to obtain the points where each f_n reaches a maximum and minimum let $f_n' = 0$.

$$\begin{aligned}
\frac{1-nx^2}{(1+nx^2)^2} &= 0 \\
1-nx^2 &= 0 \\
x^2 &= 1/n \\
x &= \pm\sqrt{1/n}
\end{aligned}$$

The derivative obtains negative values when left of $-\sqrt{1/n}$ and to the right of $\sqrt{1/n}$. The derivative obtains positive values between $-\sqrt{1/n}$ and $\sqrt{1/n}$.Thus (f_n) attains a minimum when $a = -\sqrt{1/n}$ and a maximum when $b = \sqrt{1/n}$.

Also, note that the value of the maximum or minimum is

$$\left|f_n\left(\frac{1}{\sqrt{n}}\right)\right| = \left|\frac{\frac{1}{\sqrt{n}}}{1+n\frac{1}{\sqrt{n}^2}}\right| = \left|\frac{1}{2\sqrt{n}}\right|.$$

As n gets large, then a and b get closer to 0. However, as a, b gets closer 0, then $f_n(a), f_n(b)$ also gets closer to 0. This means that the sequence of functions are converging to 0.

To prove that the function is uniformly converging to 0, let $\varepsilon > 0$ and $n > N > \frac{1}{4\varepsilon^2}$.

$$\begin{aligned}
n &> \frac{1}{4\varepsilon^2} \\
\frac{1}{2\sqrt{n}} &< \varepsilon \\
\left|\frac{x}{1+nx^2}\right| < \left|\frac{1}{2\sqrt{n}}\right| &< |\varepsilon| \\
\left|\frac{x}{1+nx^2} - 0\right| &< |\varepsilon|
\end{aligned}$$

Thus $(f_n) \to 0$ uniformly.

b) Let $f = \lim f_n$. Compute $f_n'(x)$ and find all the values of x for which $f'(x) = \lim f_n'(x)$.

Solution:

Let $f = \lim f_n = 0$.

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2} = \frac{1-nx^2}{(1+2nx^2+2n^2x^4)}$$

$$\lim f_n'(x) = \lim \frac{1-nx^2}{(1+2nx^2+2n^2x^4)} = \lim \frac{x^2}{(2x^2+4nx^4)} = 0$$

The derivative of the sequene function will converge to 0 for an x for which the function is defined.

Exercise 6.3.4

Let

$$h_n(x) = \frac{\sin(nx)}{\sqrt{n}}$$

Show that $h_n \to 0$ uniformly on $\mathbb{R}$ but that the sequence of derivatives (h_n') diverges for every $x \in \mathbb{R}$.

Solution:

To prove that h_n converges uniformly it must be shown that whenever $n > N > \left(\frac{1}{\varepsilon}\right)^2$ then $|h_n(x) - h(x)| < \varepsilon$.

$$n > \left(\frac{1}{\varepsilon}\right)^2$$
$$n > \left(\frac{1}{\varepsilon}\right)^2 \geq \left(\frac{\sin(nx)}{\varepsilon}\right)^2$$
$$\sqrt{n} > \frac{|\sin(nx)|}{\varepsilon}$$
$$\frac{\sqrt{n}}{|\sin(nx)|} > \frac{1}{\varepsilon}$$
$$\left|\frac{\sin(nx)}{\sqrt{n}} - 0\right| < \varepsilon$$

Thus $(h_n) \to 0$ uniformly.
The derivative of (h_n) is the equal to the following,

$$h_n'(x) = \frac{n\cos(nx)}{\sqrt{n}}$$

Now we will see that limit of (h_n') diverges for all $x \in \mathbb{R}$.

$$\lim_{n\to\infty} \frac{n\cos(nx)}{\sqrt{n}} = \lim_{n\to\infty} \sqrt{n}\cos(nx) = \infty$$

This is true because $\cos(nx)$ is bounded by $[-1, 1]$, but $\sqrt{n}$ will grow. If nx is a multiple of $\pi/2, 3\pi/2$ then $\cos(nx) = 0$, however nx is never always a multiple of $\pi/2, 3\pi/2$. The function will then oscillate in those cases, but this is also diverging.

Therefore as $n \to \infty$, then $(h_n') \to \infty$.

Exercise 6.3.5

Let

$$g_n(x) = \frac{nx + x^2}{2n},$$

and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:

a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

Solution:

$$\lim \frac{nx + x^2}{2n} = \lim \frac{x}{2} = \frac{x}{2}$$

Since $g(x) = x/2$, then $g'(x) = 1/2$.

b) Compute $g'(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives (g_n') converges uniformly on every interval $[-M, M]$. Use Theorem 6.3.3 to conclude $g'(x) = \lim g'(x)$.

Solution:

$$\frac{d}{dx}\left[\frac{nx+x^2}{2n}\right] = \frac{1}{2n}\cdot\frac{d}{dx}\left[nx+x^2\right] = \frac{n+2x}{2n}$$

Now I will show that $(f_n') \to 1/2$ uniformly on the closed interval $[-M,M]$. Let $n > N > |M|/\varepsilon$.

$$n > \frac{|M|}{\varepsilon}$$
$$\left|\frac{M}{n}\right| < \varepsilon$$
$$\left|\frac{x}{n}\right| \leq \left|\frac{M}{n}\right| < \varepsilon$$
$$\left|\frac{n+2x-n}{2n}\right| < \varepsilon$$
$$\left|\frac{n+2x}{2n} - \frac{n}{2n}\right| < \varepsilon$$
$$\left|\frac{n+2x}{2n} - \frac{1}{2}\right| < \varepsilon$$

Thus $(g_n') \to 1/2$ uniformly on the closed interval $[0M,M]$.
Now assume a point x_0 such that $x_0 \in [-M,M]$. Below we see that if we fix x at the point x_0 the sequence of function is still uniformly convergent.

$$n > \frac{|M|}{\varepsilon}$$
$$\left|\frac{x_0}{n}\right| \leq \left|\frac{M}{n}\right| < \varepsilon$$
$$\left|\frac{n+2x-n}{2n}\right| < \varepsilon$$
$$\left|\frac{n+2x_0}{2n} - \frac{1}{2}\right| < \varepsilon$$

Theorem 6.3.3 can now apply because (f_n) is differentiable, defined on the closed interval $[-M,M]$ and (g_n') was proven to be converging uniformly. Also, there exists a point $x_0 \in [-M,M]$ for which $g_n(x_0$ is convergent.

Thus by the theorem 6.3.3, g is differentiable. Therefore $g' = 1/2$ and $\lim g_n'(x) = 1/2$, which implies $(g_n') \to g'$.