Exercise 5.3.11a

Use the Generalized Mean Value Theorem to furnish a proof of the $0/0$ case of L'Hospital's Rule (Theorem 5.3.6)

A large portion of this work was adapted from https://www.math.hmc.edu/calculus/tutorials/lhopital/sketch_proof.html

Solution:

The Generalized Mean Value theorem states that if f and g are continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a point $c \in (a,b)$ where

$$[f(b)-f(a)]g'(c) = [g(b)-g(a)]f'(c)$$

For this proof we will assume that that if f and g are continuous on a closed interval $[a,a+h]$ and differentiable on the open interval $(a,a+h)$, where $h>0$.

It will also be assumed that $f(a)=g(a)=0$ and $g'(x) \neq 0$ for all $x \neq a$. Further, let $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$

The generalized mean value theorem applies so it can be said that there exists a $c \in (a,b)$ such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Since $f(a)=g(a)=0$, it is also that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f(a+h)}{g(a+h)} = \frac{f'(c)}{g'(c)}.$$

If $h \to 0+$ from the right then the the interval would become smaller and the c will get closer to the left end of the interval, a.

$$\lim_{h\to 0^+} \frac{f'(c)}{g'(c)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

Similarly it can be said that

$$\lim_{h\to 0^+} \frac{f(a)}{g(a)} = \lim_{x\to a^+} \frac{f(x)}{g(x)}$$

Now since it is true that $\frac{f(a+h)}{g(a+h)} = \frac{f'(c)}{g'(c)}$, it follows that we can equate the two equations above as follows,

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = \lim_{x\to a^+} \frac{f(x)}{g(x)} = L.$$

The same argument can be applied with left hand limits if the interval was $(a+h,a)$. Thus the limits together finish the proof.

Exercise 6.2.1

Let

$$f_n(x) = \frac{nx}{1+nx^2}$$

a) Find the point-wise limit of (f_n) for all $n \in (0,\infty)$.

Solution:

$$f = \lim \frac{nx}{1+nx^2} = \lim \frac{x}{x^2} = \frac{1}{x}$$

b) Is the convergence uniform on $(0,\infty)$?

Solution:

$$\begin{aligned} |f_n(x) - f(x)| &< \varepsilon \\ \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right| &< \varepsilon \\ \left| \frac{-1}{x+nx^3} \right| &< \varepsilon \\ |x+nx^3| &> 1/\varepsilon \\ n &> \frac{1}{x^2}\left(\frac{1}{\varepsilon x} - 1\right) \end{aligned}$$

Given any ε, when x is really large, then a finite N can be found. However, is x is really close to 0 then no N can be found that would make the statement true. Therefore, (f_n) does not converge uniformly on the interval $(0,\infty)$.

c) Is the convergence uniform on $(0,1)$?

Solution:

Once again note that given $N > \frac{1}{x^2}\left(\frac{1}{\varepsilon x} - 1\right)$, it would be impossible to find an N when x is close to 0. (f_n) is not uniformly convergent on the interval $(0,1)$.

d) Is the convergence uniform on $(1,\infty)$?

Solution:

If $x = 1$, then an N can be found. When x is really large, then an N can also be found for any given ε.

Exercise 6.2.3

For each $n \in \mathbb{N}$ and $x \in [0, \infty)$. let

$$g_n(x) = \frac{x}{1+x^n} \quad \text{and} \quad h_n(x) = \begin{cases} 1\,, & \text{if } x \geq 1/n \\ nx\,, & \text{if } 0 \leq x < 1/n \end{cases}$$

Answer the following questions for the sequences (g_n) and (h_n):

a) Find the point-wise limit on $[0, \infty)$.

Solution:

x<1

b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.

Solution:

x<1

c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Solution:

x<1

Exercise 6.2.7

Let f be uniformly continuous on all of $\mathbb{R}$, and define a sequence a sequence of functions by $f_n(x) = f(x+1/n)$. Show that $f_n \to f$ uniformly. Give an example to show that this proposition fails if f is only assumed to be continuous and not uniformly continuous on $\mathbb{R}$.

Solution:

x<1

Exercise 6.3.3

Consider the sequence of functions

$$f_n(x) = \frac{x}{1+nx^2}.$$

a) find the point on $\mathbb{R}$ where each $f_n(x)$ attains its maximum and minimum value. Use this to prove (f_n) converges uniformly on $\mathbb{R}$. What is the limit function?

Solution:

A function will attain its maximum and minumum values when $f_n' = 0$.

$$f_n' = \frac{(1+nx^2)(1)-(x)(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

Now to obtain the points where each f_n reachs a maximum and minimum let $f_n' = 0$.

$$\begin{aligned} \frac{1-nx^2}{(1+nx^2)^2} &= 0 \\ 1-nx^2 &= 0 \\ x^2 &= 1/n \\ x &= \pm\sqrt{1/n} \end{aligned}$$

If $f_n'(0) = \mathbb{R}^-$. If $f_n'(a) = \mathbb{R}^+$, where $-\sqrt{1/n} < a < \sqrt{1/n}$. Thus (f_n) attains a minimum when $a = -\sqrt{1/n}$ and a maximum when $a = \sqrt{1/n}$.

As n gets large, then f_n will attain maximums and minimums when a is close to 0. However, as a gets closer 0, then $f_n(a)$ also gets closer to 0.

This means that the sequence functions are being bounded by 0, $0 < f_n(x) < 0$. Thus we see that for any given $\varepsilon > 0$ it is always possible to find an N such that $|f_n(x) - 0| < \varepsilon$, and $(f_n) \to 0$ uniformly.

b) Let $f = \lim f_n$. Compute $f_n'(x)$ and find all the values f x for which $f'(x) = \lim f_n'(x)$.

Solution:

x<1

Exercise 6.3.4

Let

$$h_n(x) = \frac{\sin(nx)}{\sqrt{n}}$$

Show that $h_n \to 0$ uniformly on $\mathbb{R}$ but that the sequence of derivatives (h'_n) diverges for every $x \in \mathbb{R}$.

Solution:

Let $\varepsilon > 0$.

$$\begin{aligned}
\left|\frac{\sin(nx)}{\sqrt{n}} - 0\right| &< \varepsilon \\
\frac{\sqrt{n}}{\sin(nx)} &> \frac{1}{\varepsilon} \\
\sqrt{n} &> \frac{\sin(nx)}{\varepsilon} \\
n &> \left(\frac{\sin(nx)}{\varepsilon}\right)^2
\end{aligned}$$

Since $\sin(nx)$ is bounded between -1 and 1, it is logical that there must exist an $N > \left(\frac{1}{\varepsilon}\right)^2$.

To prove that it converges uniformly it must be shown that whenever $n > N > \left(\frac{1}{\varepsilon}\right)^2$ then $|h_n(x) - h(x)| < \varepsilon$.

$$\begin{aligned}
n &> \left(\frac{1}{\varepsilon}\right)^2 \\
n &> \left(\frac{\sin(nx)}{\varepsilon}\right)^2 \\
\sqrt{n} &> \frac{\sin(nx)}{\varepsilon} \\
\frac{\sqrt{n}}{\sin(nx)} &> \frac{1}{\varepsilon} \\
\left|\frac{\sin(nx)}{\sqrt{n}} - 0\right| &< \varepsilon
\end{aligned}$$

Thus $(h_n) \to f$ uniformly

Exercise 6.3.5

Let

$$g_n(x) = \frac{nx + x^2}{2n},$$

and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:

a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

Solution:

$$\lim \frac{nx + x^2}{2n} = \lim \frac{x}{2} = \frac{x}{2}$$

Since $g(x) = x/2$, then $g'(x) = 1/2$.

b) Compute $g'(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives (g_n') converges uniformly on every interval $[-M, M]$. Use Theorem 6.3.3 to conclude $g'(x) = \lim g'(x)$.

Solution:

To take the derivative, the quotient rule will be applied.

$$\frac{d}{dx}\left[\frac{nx + x^2}{2n}\right] = \frac{2n(n) - (nx + x^2)(0)}{4n^2} = \frac{2n^2}{4n^2} = \frac{1}{2}$$

No limit is necessary at the end of the derivative, but we see that limit of $\lim g_n'(x) = 1/2 = g'(x)$.

c) Repeat parts (a) and (b) for the sequence $f_n(x) = (nx^2 + 1)/(2n + x)$.

Solution:

First I will find the limit of (f_n)

$$\lim \frac{nx^2 + 1}{2n + x} = \lim \frac{x^2}{2} = \frac{x^2}{2}$$

Since $f(x) = x^2/2$, then $f'(x) = x$. Now I will find $f_n'(x)$.

$$\begin{aligned} f_n'(x) = \frac{d}{dx}\left[\frac{nx^2 + 1}{2n + x}\right] &= \frac{(2n + x)(2nx) - (nx^2 + 1)(1)}{(2n + x)^2} \\ &= \frac{4xn^2 + 2nx^2 - nx^2 - 1}{4n^2 + 4nx + x^2} \\ &= \frac{4xn^2 + nx^2 - 1}{4n^2 + 4nx + x^2} \end{aligned}$$

Now I can take a limit of $f_n'(x)$ to find $f'(x)$.

$$\lim \frac{4xn^2 + nx^2 - 1}{4n^2 + 4nx + x^2} = \lim \frac{8xn + x^2}{8n + 4x} = \lim \frac{8x}{8} = x$$

We see that both methods produce the same derivative.