Jacob Jashinsky Math 461-01 November 27, 2018

Exercise 1

3.4

Use the definition of continuity to show that f(x) = ax + b is continuous at every point $c \in \mathbb{R}$

Solution:

Let $c \in \mathbb{R}$ and $\varepsilon > 0$.

$$|f(x) - f(c)| < \varepsilon$$

$$|(ax+b) - (ac+b)| < \varepsilon$$

$$|ax - ac| < \varepsilon$$

$$|x - c| < \varepsilon/|a|$$

Now let $\delta = \varepsilon/|a| > 0$, and consider the following,

$$|x - c| < \delta$$

$$|x - c| < \varepsilon/|a|$$

$$|ax - ac| < \varepsilon$$

$$|ax - b + b - ac| < \varepsilon$$

$$|(ax + b) - (ac + b)| < \varepsilon$$

$$|f(x) - f(c)| < \varepsilon$$

Thus by definition of continuity, f is continuous at $c \in \mathbb{R}$.

Exercise 2

Use the definition of compactness to show that if set $K \subseteq \mathbb{R}$ is closed and bounded, then it is compact.

Solution:

The definition of compactness states that every sequence in K must a have a subsequence that converges to a limit that is also in K.

Because *K* is bounded it is true that every sequence in *K* will also be bounded. Bolzano-Weierstrass Theorem states that every bounded sequence will have a convergent subsequence.

The only thing left to prove is that the limit of those subsequences are in K.

Because of the definition of a closed set, it is true that K contains all of its limit points. Theorem 3.2.5 defines x as a limit point of K if $x = \lim_{n \to \infty} k_n$ for some sequence (k_n) contained in K.

Since K contains all of its limit points and every sequence in K will converge to a limit point, it follows that every subsequence will converge to a limit in K.

Thus *K* is compact.

Exercise 3

Use the definition of continuity to show that $f(x) = x^3$ is continuous for all real numbers $x \ge 0$.

Solution:

Let $\varepsilon > 0$ and c > 0.

$$|f(x) - f(c)| < \varepsilon$$

$$|x^3 - c^3| < \varepsilon$$

$$|x - c||x^2 + xc + c^2| < \varepsilon$$

$$|x - c| < \frac{\varepsilon}{|x^2 + xc + c^2|}$$

Since f is only defined for $x \ge 0$ and c is positive it follows that,

$$|x-c| < \frac{\varepsilon}{|x^2 + xc + c^2|} \le \frac{\varepsilon}{c^2}$$

Therefore, let $\delta = \varepsilon/c^2$.

$$\begin{aligned} |x-c| &< \delta \\ |x-c| &< \frac{\varepsilon}{c^2} \\ |x-c| &< \frac{\varepsilon}{|x^2 + xc + c^2|} \leq \frac{\varepsilon}{c^2} \\ |x-c||x^2 + xc + c^2| &< \varepsilon \\ |x^3 - c^3| &< \varepsilon \\ |f(x) - f(c)| &< \varepsilon \end{aligned}$$

Thus *f* is continuous for all $x \ge 0$.

Exercise 4

Assume that $f:[0,\infty]\to\mathbb{R}$ is continuous at every point in its domain. Show that if there exists b>0 such that f is uniformly continuous on the set $[b,\infty)$, then f is uniformly continuous on $[0,\infty)$

Solution:

Since b > 0, f is, at the moment, known to be uniform on all postive real numbers except 0 itself.

I must show that the point 0 will require the same or larger δ for any given ε defined for the set $[b, \infty)$.

f is uniform on the set $[b.\infty)$, meaning that δ is not dependent on a point c.

Since it is also known that f is defined on 0 and b is allowed to be arbitrarly close to 0. It follows that no smaller δ is needed if 0 was included in the set.

Exercise 5

Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on \mathbb{R} and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Solution:

I must show that K is either empty, equivalent to \mathbb{R} , or only contains isolated points.

K will be empty if $h(X) \neq 0$. In this case *K* will closed.

K will be equivalent to \mathbb{R} when h(x) = 0 for all $x \in \mathbb{R}$. *K* is also closed in this case.

The only other option is for *K* to contain only isolated points.

Since the function is continuous the Intermediate Value Theorem implies that when L=0, there exists a point, c, such that f(c)=0, and any other value greater or less than c will result in a value greater or less than f(c)=0.

It follows that in this situation *K* will only contain isolated points and is closed.