

Exercise 3.4.9a

Let $\{r_1, r_2, r_3, \dots\}$ be an enumeration of the rational numbers, and for each $n \in \mathbb{N}$ set $\varepsilon_n = 1/2^n$. Define $O = \bigcup_{n=1}^{\infty} V_{\varepsilon_n}(r_n)$, and let $F = O^c$.

- a. Argue that F is a closed, nonempty set consisting only of irrational numbers.

Solution:

Demorgan's law states $(A \cup B)^c = A^c \cap B^c$. F is the complement of an union, therefore it is the intersection of complements of $V_{\varepsilon_n}(r_n)$.

$V_{\varepsilon_n}(r_n)$ is an open set, so the complement of the neighborhood would make it a closed set. Theorem 3.2.14 states that the arbitrary intersection of closed sets is closed. This proves that F is closed.

No rational number exists in the interval $V_{\varepsilon_n}(r_n)^c$, therefore F would not contain an rational numbers either.

The length that was removed is equal to the sum $2(1/2 + 1/4 + 1/8 + \dots)$ (assuming no overlap of intervals), which by the geometric series is equal to $2 \frac{1/2}{1-1/2} = 2$. This means that the total length that was not included in the intersection is equal to 1. Therefore F is nonempty.

Exercise 4.2.2abc

For each stated limit, find the largest possible δ -neighborhood that is a proper response to the given ε challenge.

- a. $\lim_{x \rightarrow 3} (5x - 6) = 9$, where $\varepsilon = 1$.

Solution:

Given that $\varepsilon = 1$, we have:

$$\begin{aligned} |f(x) - L| &< \varepsilon \\ |5x - 6 - 9| &< 1 \\ |5x - 15| &< 1 \\ 5|x - 3| &< 1 \\ |x - 3| &< 1/5. \end{aligned}$$

It then follows that there exists a δ such that $|x - c| < \delta$. Thus I get that $|x - 3| < \delta \leq 1/5$.

Thus $(3 - 1/5, 3 + 1/5)$ is the largest open interval possible.

b. $\lim_{x \rightarrow 4} \sqrt{x} = 2$, where $\varepsilon = 1$.

Solution:

Given that $\varepsilon = 1$, we have:

$$\begin{aligned} |f(x) - L| &< \varepsilon \\ |\sqrt{x} - 2| &< 1 \\ -1 &< (\sqrt{x} - 2) < 1 \\ 1 &< \sqrt{x} < 3 \\ 1 &< x < 9 \\ -3 &< (x - 4) < 5 \\ 3 &< |x - 4| < 5 \end{aligned}$$

It then follows that there exists a δ such that $|x - c| < \delta$. Thus I get that $|x - 4| < \delta \leq 3$.

Thus $(1, 7)$ is the largest open interval possible around 4.

c. $\lim_{x \rightarrow \pi} \lfloor x \rfloor = 3$, where $\varepsilon = 1$.

Solution:

Given that $\varepsilon = 1$, we have:

$$\begin{aligned} |f(x) - L| &< \varepsilon \\ |\lfloor x \rfloor - 3| &< 1 \\ -1 &< (\lfloor x \rfloor - 3) < 1 \\ 2 &< \lfloor x \rfloor < 4 \end{aligned}$$

Since $x \rightarrow \pi$ I can write $2 < \lfloor x \rfloor < 4$ as,

$$\begin{aligned} 2 &< x < 4 \\ 2 - \pi &< (x - \pi) < 4 - \pi \\ |2 - \pi| &< |(x - \pi)| < 4 - \pi \end{aligned}$$

It then follows that there exists a δ such that $|x - c| < \delta$. Thus I get that $|x - \pi| < \delta \leq |2 - \pi|$.

Exercise 4.2.5abc

Use Definition 4.2.1 to supply a proper proof for the following limit statements.

a. $\lim_{x \rightarrow 2} (3x + 4) = 10$

Solution:

Before the formal proof I will show how to find δ .

$$\begin{aligned} |f(x) - L| &< \varepsilon \\ |3x + 4 - 10| &< \varepsilon \\ 3|x - 2| &< \varepsilon \\ |x - 2| &< \varepsilon/3 \end{aligned}$$

Now the formal proof.

For any $\varepsilon > 0$ let $\delta = \varepsilon/3$. I will show that $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$.

$$\begin{aligned} |x - 2| &< \varepsilon/3 \\ 3|x - 2| &< \varepsilon \\ |3x - 6| &< \varepsilon \\ |(3x + 4) - 10| &< \varepsilon \\ |f(x) - L| &< \varepsilon \end{aligned}$$

Thus $\lim_{x \rightarrow 2} (3x + 4) = 10$.

b. $\lim_{x \rightarrow 0} x^3 = 0$

Solution:

Before the formal proof I will show how to find δ .

$$\begin{aligned} |f(x) - L| &< \varepsilon \\ |x^3 - 0| &< \varepsilon \\ -\varepsilon &< x^3 < \varepsilon \\ -(\varepsilon)^{1/3} &< x < (\varepsilon)^{1/3} \\ |x| &< (\varepsilon)^{1/3} \end{aligned}$$

Now the formal proof.

For any $\varepsilon > 0$ let $\delta = \varepsilon^{1/3}$. I will show that $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$.

$$\begin{aligned} |x - 0| &< \varepsilon^{1/3} \\ -\varepsilon^{1/3} &< x < \varepsilon^{1/3} \\ -\varepsilon &< x^3 < \varepsilon \\ |x^3 - 0| &< \varepsilon \\ |f(x) - L| &< \varepsilon \end{aligned}$$

Thus $\lim_{x \rightarrow 0} x^3 = 0$.

c. $\lim_{x \rightarrow 2}(x^2 + x - 1) = 5$

Solution:

Before the formal proof I will show how to find δ .

$$\begin{aligned} |x^2 + x - 1 - 5| &< \varepsilon \\ -\varepsilon &< (x^2 + x - 6) < \varepsilon \\ -\varepsilon &< ((x + 1/2)^2 - 6 - 1/4) < \varepsilon \\ 25/4 - \varepsilon &< ((x + 1/2)^2) < \varepsilon + 25/4 \\ \sqrt{25/4 - \varepsilon} &< (x + 1/2) < \sqrt{\varepsilon + 25/4} \\ \sqrt{25/4 - \varepsilon} - 5/2 &< (x - 2) < \sqrt{\varepsilon + 25/4} - 5/2 \\ -\delta &< (x - 2) < \delta \end{aligned}$$

Let $\delta = \min\{(-\sqrt{25/4 - \varepsilon} + 5/2), (\sqrt{\varepsilon + 25/4} - 5/2)\}$.

I will show that $\sqrt{\varepsilon + 25/4} - 5/2$ is the larger one.

I will show that $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$.

$$x < 1$$

Exercise 4.2.6

Decide if the following are true or false, and give short justifications for each conclusion.

- a. If a particular δ has been constructed as a suitable response to a particular ϵ challenge, then any smaller positive δ will also suffice.

Solution:

True. δ is typically the largest possible delta, so any smaller will still be within the interval set by the epsilon challenge.

- b. If $\lim_{x \rightarrow a} f(x) = L$ and a happens to be in the domain of f , then $L = f(a)$.

Solution:

False, the function could be a piece-wise function, where $f(a)$ is defined as something other than the limit.

- c. If $\lim_{x \rightarrow a} f(x) = L$, then $\lim_{x \rightarrow a} 3[f(x) - 2]^2 = 3(L - 2)^2$.

Solution:

This is true. Since $3(L - 2)^2$ is found through a limit it avoids the issue discussed in part (c). This is also supported by the algebraic limit theorem for function limits.

- d. If $\lim_{x \rightarrow a} f(x) = 0$, then $\lim_{x \rightarrow a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f .)

Solution:

True. Since the domains are the same the algebraic limit theorem for functional limit applies. In which, $\lim_{x \rightarrow a} f(x)g(x) = 0(\lim_{x \rightarrow a} g(x)) = 0$

Exercise 4.3.1

Let $g(x) = \sqrt[3]{x}$

- a. Prove that g is continuous at $c = 0$.

Solution:

g is continuous at c if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ it follows that $|f(x) - f(c)| < \varepsilon$.

I will let $\delta = \varepsilon^3$. Then it follows that

$$\begin{aligned} |x - c| &< \delta \\ |x - 0| &< \varepsilon^3 \\ |\sqrt[3]{x} - 0| &< \varepsilon \\ |\sqrt[3]{x} - \sqrt[3]{0}| &< \varepsilon \end{aligned}$$

Thus we see that g is continuous at 0.

- b. Prove that g is continuous at a point $c \neq 0$. (The Identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ will be helpful.)

Solution:

Consider the function $f : A \rightarrow \mathbb{R}$ where $f(x) = x$ is continuous at a point $c \neq 0$, and $c \in \mathbb{R}$

Let $\delta = \varepsilon$. Then it follows that

$$\begin{aligned} |x - c| &< \delta \\ |x - c| &< \varepsilon \\ |f(x) - f(c)| &< \varepsilon \end{aligned}$$

Thus we see that x is continuous. Note that $x = \sqrt[3]{x} \cdot \sqrt[3]{x} \cdot \sqrt[3]{x}$

Then by the Algebraic Continuity Theorem (iii) we see that x is a continuous at point c ; therefore, $\sqrt[3]{x}$ must be continuous at c .