Problem 1

Using the definition of a convergence of a sequence, prove that

$$\lim_{n\to\infty} \frac{2n}{n+1} = 2$$

Solution:

To prove that the sequence (a_n) converges to 2, I must show that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - 2| < \varepsilon$.

I will start by finding a general N.

$$|\frac{2N}{N+1} - 2| < \varepsilon$$
$$|\frac{-2}{N+1}| < \varepsilon$$
$$\frac{2}{N+1} < \varepsilon$$
$$\frac{2}{\varepsilon} < N+1$$
$$\frac{2}{\varepsilon} - 1 < N$$

Now I will prove that such an N exists because whenever $n \geq N$ it follows that $|a_n - 2| < \varepsilon$.

$$n > \frac{2}{\varepsilon} - 1$$
$$n+1 > \frac{2}{\varepsilon}$$
$$\frac{1}{n+1} < \frac{\varepsilon}{2}$$
$$\frac{2}{n+1} < \varepsilon$$
$$|\frac{-2}{n+1}| < \varepsilon$$
$$|\frac{-2+2n-2n}{n+1}| < \varepsilon$$
$$|\frac{2n}{n+1} + \frac{-2n-2}{n+1}| < \varepsilon$$
$$|\frac{2n}{n+1} - 2\left(\frac{n+1}{n+1}\right)| < \varepsilon$$
$$|\frac{2n}{n+1} - 2| < \varepsilon$$

Thus $\lim_{n\to\infty} \frac{2n}{n+1} = 2$

Problem 2

Let $a_1 = 2$ and $a_{n+1} = \sqrt{2a_n - 1}$ for $n \in \mathbb{N}$

a) Use induction to show that $a_n > a_{n+1}$ for all $n \in \mathbb{N}$

Solution:

Base Case:
$a_1 = 2$, and $a_2 = \sqrt{3} \approx 1.732051$. Therefore $a_1 > a_2$.

Induction Hypothesis:
Assume that $a_k > a_{k+1}$ for some $k \in \mathbb{N}$.

$$\begin{aligned} a_k &> a_{k+1} \\ 2a_k &> 2a_{k+1} \\ 2a_k - 1 &> 2a_{k+1} - 1 \\ \sqrt{2a_k - 1} &> \sqrt{2a_{k+1} - 1} \\ a_{k+1} &> a_{k+2} \end{aligned}$$

Thus $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.

b) Use induction to show that $a_n > 1$ for all $n\ in\mathbb{N}$

Solution:

Base Case:
Clearly $a_1 = 2 > 1$, therefore the base case is satisfied.

Induction Hypothesis:
Now I assume that $a_n > 1$ for some $k \in \mathbb{N}$.

$$\begin{aligned} a_k &> 1 \\ 2a_k &> 2 \\ 2a_k - 1 &> 2 - 1 \\ \sqrt{2a_k - 1} &> \sqrt{1} \\ a_{k+1} &> 1 \end{aligned}$$

Thus $a_{k+1} > 1$ for all $n \in \mathbb{N}$.

c) What can you say about the convergence of (a_n)? If (a_n) converges, find the limit.

Solution:

Because of part (a) and (b) I know that (a_n) is strictly decreasing and bounded below, therefore by the Monotone Convergence Theorem, I know that (a_n) converges.

To find the limit I will let $\lim a_n = \lim a_{n+1} = a$

$$\begin{aligned}
\lim a_n &= \lim a_{n+1} \\
\lim \sqrt{2a_n - 1} &= a \\
\sqrt{2a - 1} &= a \\
2a - 1 &= a^2 \\
0 &= a^2 - 2a + 1 \\
0 &= (a-1)(a-1)
\end{aligned}$$

Therefore, $\lim a_n = 1$

Problem 3

Use the Archimedean Property of $\mathbb{R}$ to prove that $\inf\{1/n : n \in \mathbb{N}\} = 0$.

Solution:

Consider the set $A = \{y \in \mathbb{R} : y > 0\}$. I will show that $\inf A = 0$. It is easy to see that 0 is a lower bound for set A because every element of A is greater than 0.

Now if b was also a lower bound for A then $0 \geq b$, because any value greater than zero would be in the set A, thus making it no longer a lower bound. Therefore, 0 satisfies the two criteria to be an infimum, so $\inf A = 0$.

The second part of the Archimedean Property states: Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfiying $1/n < y$.

This means that there exists an an element in the set $B = \{1/n : n \in \mathbb{N}\}$ that is less than an element in set A. However, I know that $0 \notin B$.

It follows that 0 is a lower bound of B. I also know that it is the greatest lower bound of B because it was for A.
Therefore, $\inf\{1/n : n \in \mathbb{N}\} = 0$.

Problem 4

Use the Cauchy Criterion for Series, Thm 2.7.2 to prove the Comparison Thm, 2.7.4 part (i), which is:

Assume (a_k) and (b_k) are sequences satisfying $o \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

Solution:

Let $\sum_{k=1}^{\infty} b_k = \lim s_{b_k} = B$, where s_{b_k} is defined as the partial sums of b_k. Because this series converges I know through the Cauchy Criterion for Series that there exists an N such that whenever $n > m \geq N$ it follows that

$$|s_{b_n} - s_{b_m}| < \varepsilon$$

a_k is less than b_k for all $k \in \mathbb{N}$, therefore I also know the following

$$|s_{a_n} - s_{a_m}| < |s_{b_n} - s_{b_m}| < \varepsilon.$$

Since $|s_{a_n} - s_{a_m}| < \varepsilon$ it must be true that $\sum a_k$ converges. Theorem 2.7.3 lets us say that because the series covnerges then the sequence, (a_k), also converges.

Problem 5

Let $x_1 = 1/2$, $x_2 = 1/3 + 1/4, \ldots$, $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$. Prove the sequence (x_n) converges.

Solution:

I will use the monotone convergence theorem to prove that that the sequence is monotone and bounded which implies that it is convergent.

First it will be shown that it is bounded by 1 and 0. Note that x_n is a sum of decreasing postive values. There are always n number of terms in the sum.

The largest value in the sum is always the first $\frac{1}{n+1}$. If we were to add the largest value n times then we would get $\frac{n}{n+1}$. Because the denominator is larger than the numerator it can be said that $\frac{n}{n+1} < 1$. Thus it follows that $0 < x_n < \frac{n}{n+1} < 1$.

Next it will be shown that $x_n < x_{n+1}$. Consider the following:

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$$

and

$$x_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}.$$

x_{n+1} will always gain 2 terms and loose 1 term. So I must show that $\frac{1}{n+1} < \frac{1}{2n+1} + \frac{1}{2n+2}$. This will be done by stating something obviously true.

$$\begin{aligned}
2 &< 3 \\
4n+2 &< 4n+3 \\
\frac{4n+2}{4n+2} &< \frac{4n+3}{4n+2} \\
1 &< \frac{4n+3}{2(2n+1)} \\
\frac{1}{n+1} &< \frac{4n+3}{2(2n+1)(n+1)} \\
\frac{1}{n+1} &< \frac{(2n+1)+(2n+2)}{(2n+1)(2n+2)} \\
\frac{1}{n+1} &< \frac{(2n+1)}{(2n+1)(2n+2)} + \frac{(2n+2)}{(2n+1)(2n+2)} \\
\frac{1}{n+1} &< \frac{1}{2n+2} + \frac{1}{2n+1}
\end{aligned}$$

This shows that the next term in the sequence will always be larger than the previous, therefore it is monotone.
Because the sequence is bounded and monotone, (x_n) converges.