Exercise 3.4.9a

Let $\{r_1, r_2, r_3, ...\}$ be an enumeration of the rational numbers, and for each $n \in \mathbb{N}$ set $\varepsilon_n = 1/2^n$. Define $O = \bigcup_{n=1}^{\infty} V_{\varepsilon_n}(r_n)$, and let $F = O^c$.

a. Argue that F is a closed, nonempty set consisting only of irrational numbers.

Solution:

Demorgan's law states $(A \cup B)^c = A^c \cap B^c$. F is the complement of an union, therefore it is the intersection of complements of $V_{\varepsilon_n}(r_n)$.

$V_{\varepsilon_n}(r_n)$ is an open set, so the complement of the neighborhood would make it a closed set. Theorem 3.2.14 states that the arbitrary intersection of closed sets is closed. This proves that F is closed.

No rational number exists in the interval $V_{\varepsilon_n}(r_n)^c$, therefore F would not contain an rational numbers either.

The length that was removed is equal to the sum $2(1/2 + 1/4 + 1/8 + ...)$ (assuming no overlap of intervals), which by the geometric series is equal to $2\frac{1/2}{1-1/2} = 2$. This means that the total length that was not included in the intersection is equal to 1. Therefore F is nonempty.

Exercise 4.2.2abc

For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ε challenge.

a. $\lim_{x \to 3}(5x - 6) = 9$, where $\varepsilon = 1$.

Solution:

Given that $\varepsilon = 1$, we have:

$$\begin{aligned} |f(x) - L| &< \varepsilon \\ |5x - 6 - 9| &< 1 \\ |5x - 15| &< 1 \\ 5|x - 3| &< 1 \\ |x - 3| &< 1/5. \end{aligned}$$

It then follows that there exists a δ such that $|x - c| < \delta$. Thus I get that $|x - 3| < \delta \leq 1/5$.

Thus (3 - 1/5, 3 + 1/5) is the largest open interval possible.

b. $\lim_{x\to 4}\sqrt{x} = 2$, where $\varepsilon = 1$.

Solution:

Given that $\varepsilon = 1$, we have:

$$\begin{aligned} |f(x) - L| &< \varepsilon \\ |\sqrt{x} - 2| &< 1 \\ -1 < (\sqrt{x} - 2) &< 1 \\ 1 < \sqrt{x} &< 3 \\ 1 < x &< 9 \\ -3 < (x-4) &< 5 \\ 3 < |x-4| &< 5 \end{aligned}$$

It then follows that there exists a δ such that $|x-c| < \delta$. Thus I get that $|x-4| < \delta \leq 3$.

Thus (1, 7) is the largest open interval possible around 4.

c. $\lim_{x\to\pi}\lfloor x\rfloor = 3$, where $\varepsilon = 1$.

Solution:

Given that $\varepsilon = 1$, we have:

$$\begin{aligned} |f(x) - L| &< \varepsilon \\ |\lfloor x\rfloor - 3| &< 1 \\ -1 < (\lfloor x\rfloor - 3) &< 1 \\ 2 < \lfloor x\rfloor &< 4 \end{aligned}$$

Since $x \to \pi$ I can write $2 < \lfloor x\rfloor < 4$ as,

$$\begin{aligned} 2 < x &< 4 \\ 2-\pi < (x-\pi) &< 4-\pi \\ |2-\pi| < |(x-\pi)| &< 4-\pi \end{aligned}$$

It then follows that there exists a δ such that $|x-c| < \delta$. Thus I get that $|x-\pi| < \delta \leq |2-\pi|$.

Exercise 4.2.5abc

Use Definition 4.2.1 to supply a proper proof for the following limit statements.

a. $\lim_{x\to 2}(3x+4) = 10$

Solution:

Before the formal proof I will show how to find δ.

$$\begin{aligned} |f(x)-L| &< \varepsilon \\ |3x+4-10| &< \varepsilon \\ 3|x-2| &< \varepsilon \\ |x-2| &< \varepsilon/3 \end{aligned}$$

Now the formal proof.
For any $\varepsilon > 0$ let $\delta = \varepsilon/3$. I will show that $0 < |x-c| < \delta$ implies $|f(x)-L| < \varepsilon$.

$$\begin{aligned} |x-2| &< \varepsilon/3 \\ 3|x-2| &< \varepsilon \\ |3x-6| &< \varepsilon \\ |(3x+4)-10| &< \varepsilon \\ |f(x)-L| &< \varepsilon \end{aligned}$$

Thus $\lim_{x\to 2}(3x+4) = 10$.

b. $\lim_{x\to 0} x^3 = 0$

Solution:

Before the formal proof I will show how to find δ.

$$\begin{aligned} |f(x)-L| &< \varepsilon \\ |x^3-0| &< \varepsilon \\ -\varepsilon < x^3 &< \varepsilon \\ -(\varepsilon)^{1/3} < x &< (\varepsilon)^{1/3} \\ |x| &< (\varepsilon)^{1/3} \end{aligned}$$

Now the formal proof.
For any $\varepsilon > 0$ let $\delta = \varepsilon^{1/3}$. I will show that $0 < |x-c| < \delta$ implies $|f(x)-L| < \varepsilon$.

$$\begin{aligned} |x-0| &< \varepsilon^{1/3} \\ -\varepsilon^{1/3} < x &< \varepsilon^{1/3} \\ -\varepsilon < x^3 &< \varepsilon \\ |x^3-0| &< \varepsilon \\ |f(x)-L| &< \varepsilon \end{aligned}$$

Thus $\lim_{x\to 0} x^3 = 0$.

c. $\lim_{x\to 2}(x^2+x-1)=5$

Solution:

Before the formal proof I will show how to find δ.

$$
\begin{aligned}
|x^2+x-1-5| &< \varepsilon \\
|x^2+x-6| &< \varepsilon \\
|x-2||x+3| &< \varepsilon
\end{aligned}
$$

We are finding the limit as x approaches 2. I do not want $\delta > 1$, therefore if I choose $x=3$, then $|x+3| \leq |3+3| = 6 < \varepsilon$, which leads to the inequality $|x+3| < \varepsilon/6$ for all x values close to 2, or $x \in V_1(2)$.

We have limited δ by 1, but it could also be smaller, depending on ε. We can let $\delta = \min\{\varepsilon/6, 1\}$. The rest of the proof is as follows,

$$
\begin{aligned}
|x-2| &< \delta \\
|x-2| &< \varepsilon/6
\end{aligned}
$$

Since $|x+3| \leq= 6$

$$
\begin{aligned}
|x-2||x+3| &< (6)\frac{\varepsilon}{6} \\
|(x^2+x-1)-5| &< \varepsilon
\end{aligned}
$$

Thus, the limit is proved.

Exercise 4.2.6

Decide if the following are true or false, and give short justifications for each conclusion.

a. If a particular δ has been constructed as a suitable response to a particular ε challenge, then any smaller positive δ will also suffice.

Solution:

True. δ is typically the largest possible delta, so any smaller will still be within the interval set by the epsilon challenge.

b. If $\lim_{x\to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$.

Solution:

False, the function could be a piece-wise function, where $f(a)$ is defined as something other than the limit.

c. If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} 3[f(x)-2]^2 = 3(L-2)^2$.

Solution:

This is true. Since $3(L-2)^2$ is found through a limit it avoids the issue discussed in part (c). This is also supported by the algebraic limit theorem for function limits.

d. If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f.)

Solution:

True. Since the domains are the same the algebraic limit theorem for functional limit applies. In which, $\lim_{x\to a} f(x)g(x) = 0(\lim_{x\to a} g(x)) = 0$

Exercise 4.3.1

Let $g(x) = \sqrt[3]{x}$

a. Prove that g is continuous at $c = 0$.

Solution:

g is continuous at c if if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ it follows that $|f(x) - f(c)| < \varepsilon$.
I will let $\delta = \varepsilon^3$. Then it follows that

$$\begin{aligned}
|x-c| &< \delta \\
|x-0| &< \varepsilon^3 \\
|\sqrt[3]{x} - 0| &< \varepsilon \\
|\sqrt[3]{x} - \sqrt[3]{0}| &< \varepsilon
\end{aligned}$$

Thus we see that g is continuous at 0.

b. Prove that g is continuous at a point $c \neq 0$. (The Identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ will be helpful.)

Solution:

First to find an appropriate δ let, $|\sqrt[3]{x} - \sqrt[3]{c}| < \varepsilon$

$$\begin{aligned}
|\sqrt[3]{x} - \sqrt[3]{c}| \cdot |\frac{x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}}{x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}}| &< \varepsilon \\
|\frac{x-c}{x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}}| &< \varepsilon
\end{aligned}$$

Note that if $x > 0$ then we get that

$$\begin{aligned}
|\sqrt[3]{x} - \sqrt[3]{c}| = |\frac{x-c}{x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}}| < |\frac{x-c}{c^{2/3}}| &< \varepsilon \\
|\sqrt[3]{x} - \sqrt[3]{c}| < |\frac{x-c}{c^{2/3}}| &< \varepsilon \\
|x-c| &< \varepsilon |c^{2/3}|
\end{aligned}$$

Now this shows that $\delta = |c^{2/3}|$. This inequality leads to show that

$$\begin{aligned}
|x-c| &< \varepsilon |c^{2/3}| \\
|\frac{x-c}{c^{2/3}}| &< \varepsilon \\
|x^{1/3} - c^{1/3}| < |\frac{x-c}{c^{2/3}}| &< \varepsilon \\
|\sqrt[3]{x} - \sqrt[3]{c}| &< \varepsilon
\end{aligned}$$

Thus this shows that when ever $x > 0$, regardless of the c (positive or negative), then $g(x)$ is continuous. Since this is an odd function, if a negative variable was entered in the function, then the result would be negative. This is the same as using positive x values but multiplying by -1. The algebraic continuous theorem states that this product will also be continuous as well.