Exercise 6.4.2a

Decide whether the proposition is true or false, provide a short justification or counter example as appropriate.

If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.

Solution:

True. The sum of a sequence that does not converge to zero would diverge.

Exercise 6.4.3

a) Show that

$$g(x) = \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$$

is continuous on all $\mathbb{R}$.

Solution:

Consider the sequence $1/2^n$. Since $cos(x)$ is bounded by 1 it can be said that $\left|\frac{\cos(2^n x)}{2^n}\right| \leq \frac{1}{2^n}$ for all $x \in \mathbb{R}$.

Also note that series $\sum_{n=0}^{\infty} 1/2^n$ is geometric and conveges. Weierstrass M-Test now applies, therefore, $\sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$ converges uniformly on $\mathbb{R}$.

Now theorem 6.4.2 states that since g_n is continuous and the series converges uniformly, then $g(x)$ is continuous on its domain.

b) The function g was cited in section 5.4 as an example of a continuous nowhere differentiable function. What happens if we try to use Theorem 6.4.3 to explore whether g is differentiable?

Solution:

The assumptions are not met. The sum of the sequence $f_n'(x) = -\sin(2^n x)$ does not converge, therefore the theorem cannot apply.

Exercise 6.4.5

a) Prove that

$$h(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}=x+\frac{x^2}{4}+\frac{x^3}{9}+...$$

is continuous on $[-1,1]$

Solution:

Consider the sequence $1/n^2$. Since the domain of h_n is bounded above by 1, it can be said that $\left|\frac{x^n}{n^2}\right| \leq \frac{1}{2^n}$ for all $x \in [-1,1]$.

Also note that series $\sum_{n=0}^{\infty} 1/n^2$ is P-series and conveges. Weierstrass M-Test now applies, therefore, $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ converges uniformly on $[-1,1]$.

Now theorem 6.4.2 states that since h_n is continuous and the series converges uniformly, then $h(x)$ is continuous on its domain.

b) The series

$$f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}=x+\frac{x^2}{2}+\frac{x^3}{3}+...$$

converges for every x in the half-open interval $[-1,1)$ but does not converge when $x=1$. For a fixed $x_0 \in (-1,1)$, explain how we can still use the Weierstrass M-test to prove that f is continuous at x_0.

Solution:

Since we know that the series conveges for the half-open interval this can be our M_n. For every $x_0 \in (-1,1)$, choose a point $c \in [-1,1)$ such that that $\left|\frac{x_0^n}{n}\right| \leq \frac{c^n}{n}$. The sum of the sequence M_n will converge, implying that $\sum f_n(x_0)$ converges uniformly.

Theorem 6.4.2 can then be leveraged to show that since it converges uniformly it is also continuous on the appropriate domain.

Exercise 6.5.1

Consider the function g defined by the power series

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

a) Is g defined on $(-1,1)$? Is it continuous on this set? Is g defined on $(-1,1]$? Is it continuous on this set? What happens on $[-1,1]$? Can the power series for $g(x)$ possibly converge for an other points $|x| > 1$? Explain.

Solution:

The power series is defined on the set $(-1,1)$. Choose an $|x_0| < 1$. The power series now becomes an alternating series. The sequence is strictly decreasing and converges to 0, therefore the power series converges when $x = x_0$.

Theorem 6.5.1 states that since the power series converges at a some point x_0, then it converges absolutely for every $|x| < |x_0|$. Theorem 6.5.2 can now be applied which implies that the power series at point x_0 converges uniformly on the closed interval $[-|x_0|, |x_0|]$. Because the power series is uniformly convergent it is also coninuous on the interval $(-1,1)$.

The power series is also defined on all real numbers. The same arguments can be applied to all intervals greater than $(-1,1)$.

b) For what values of x is $g'(x)$ defined? Find a formula for g'.

Solution:

x<1

Exercise 6.5.3

Use the Weierstrass M-Test to prove Theorem 6.5.2

Solution:

Theorem 6.5.2: If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on the closed interval $[-c, c]$, where $c = |x_0|$.
Let $M_n = a_n |x_0|^n$ be a converging series. Choose a point x_0 and a point $|c| \leq |x_0|$, and note that

$$|c| \leq |x_0|$$
$$|c|^n \leq |x_0|^n$$
$$|a_n c^n| \leq a_n |x_0|^n$$
$$|f_n(x)| \leq |M_n|$$

The M-Test now applies, therefore, $\sum_{n=0}^{\infty} a_n x^n$ converges at all points $|c| \leq |x_0|$, or in other words, the power series converges on the interval $[-c, c]$.

Exercise 6.5.8

a) Show that the power series representations are unique. If we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in an interval $(-\mathbb{R}, \mathbb{R})$, prove that $a_n = b_n$ for all $n = 0, 1, 2, \ldots$

Solution:

x<1

b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge on $(-\mathbb{R}, \mathbb{R})$, and assume $f'(x) = f(x)$ for all $x \in (-\mathbb{R}, \mathbb{R})$ and $f(0) = 1$. Deduce the values of a_n.

Solution:

x<1