Problem 1.4.1

a) Show that if $a, b \in \mathbb{Q}$, then ab and $a+b$ are elements of $\mathbb{Q}$ as well.

Solution:

Let $a = m/n$ and $b = r/q$, where $m, n, r, q \in \mathbb{Z}$ and $n, q \neq 0$, thus making a and b rational numbers.
First I will prove that $ab \in \mathbb{Q}$

$$\begin{aligned} ab &= \frac{m}{n} \cdot \frac{r}{q} \\ &= \frac{mr}{nq} \end{aligned}$$

We know that $mr, nq \in \mathbb{Z}$ because integers are closed under multiplication. Thus, $\frac{mr}{nq}$ is a fraction of two integers, making $ab \in \mathbb{Q}$.
Next I will prove that $a+b \in \mathbb{Q}$.

$$\begin{aligned} a+b &= \frac{m}{n} + \frac{r}{q} \\ &= \frac{mq+rn}{nq} \end{aligned}$$

We know that $mq, rn, nq \in \mathbb{Z}$ because integers are closed under addition and multiplication. Thus, $\frac{mq+rn}{nq}$ is a fraction of two integers, making $a+b \in \mathbb{Q}$.

b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a+t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

Solution:

We will prove this by contradiction.
First will assume that $a+t \in \mathbb{Q}$. Because of part a) I know that a rational sum can be expressed two rational numbers.

$$\begin{aligned} a+t &= \frac{q}{p} \\ \frac{m}{n}+t &= \frac{q}{p} \\ t &= \frac{q}{p}+\left(-\frac{n}{m}\right) \end{aligned}$$

This shows that t is the sum of two rational numbers which is also rational number, therefore I have a contradiction.
It will be the same process for at. I assume that $at \in \mathbb{Q}$. So the product can be expressed as a rational number.

$$\begin{aligned} at &= \frac{q}{p} \\ \frac{m}{n}t &= \frac{q}{p} \\ t &= \frac{nq}{mp} \end{aligned}$$

This leads to a contradiction because t now appears to be rational number.

c) Is $\mathbb{I}$ closed under addition and multiplication? Give two irrational numbers s and t, what can we say about $s+t$ and st?

Solution:

Irrational numbers are closed under addition. Let $s = t = \sqrt{2}$.
$s+t = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$, which is still irrational. However if we look at st.
$st = (\sqrt{2})(\sqrt{2}) = (\sqrt{2})^2 = 2$, which is an integer.

Problem 1.4.8

Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

a) Two sets A and B with $A \cap B = \varnothing$, $\sup A = \sup B$, $\sup A \notin A$ and $\sup B \notin B$.

Solution:

$$A = \{1 - \frac{1}{(n+1)^2 - 1} \mid n \in \mathbb{N}\}$$

and

$$B = \{1 - \frac{1}{(n+1)^2 + 1} \mid n \in \mathbb{N}\}$$

b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

Solution:

Let $J_n = \left(\frac{-1}{2n}, \frac{1}{2n}\right)$, where $n \in \mathbb{N}$.
This means that as n approaches infinity the interval will get smaller and smaller. The only real number that is within every set is 0. Thus the intersection is finite.

c) A sequence of nested and unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \ldots$ with $\bigcap_{n=1}^{\infty} L_n = \varnothing$.

Solution:

Let $L_n = \{(n, \infty) \mid n \in \mathbb{N}\}$
The intersection is empty because you can always find an interval that will exclude the number you claim is in the intersection.

d) A sequence of closed bounded (not necessarily nested) intervals $I_a, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \varnothing$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

Solution:

This is impossible. If $\bigcap_{n=1}^{\infty} I_n = \varnothing$ is true, then we know that there exists at least two sets where the intersection is empty, $I_k \cap I_m = \varnothing$.
However, since $k, m < N \in \mathbb{N}$ then these two sets are part of a finite family of set. Thus $\bigcap_{n=1}^{N} I_n \neq \varnothing$ is false.

Problem 1.5.4a

a) Show $(a,b) \sim \mathbb{R}$ for any interval (a,b).

Solution:

https://math.stackexchange.com/questions/1434479/prove-any-open-interval-has-the-same-cardinality-of-bbb-r-without-using-tri
The equation was taken from this website above, and I take no credit for its creation.

If I let $f(x) = \ln\left(\frac{1}{x-a} - \frac{1}{b-a}\right)$, where the domain is the open interval (a,b). Then we can see that the domain maps to all of $\mathbb{R}$.

$$\lim_{x \to a} \ln\left(\frac{1}{x-a} - \frac{1}{b-a}\right) = \infty$$

and

$$\lim_{x \to b} \ln\left(\frac{1}{x-a} - \frac{1}{b-a}\right) = -\infty$$

Because $f(x)$ is also function that is one-to-one and is onto we know by definition 1.5.2, that (a,b) has the same cardinality as $\mathbb{R}$.

Problem 2.2.2ab

Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$

Solution:

Before I begin a formal proof I will first find an N in relation to ε.

$$|\frac{2N+1}{5N+4} - \frac{2}{5}| < |\varepsilon|$$
$$|\frac{10N+5-10N-8}{5(5N+4)}| < |\varepsilon|$$
$$|\frac{-3}{5(5N+4)}| < |\varepsilon|$$
$$|\frac{1}{5N+4}| < |\frac{5\varepsilon}{-3}|$$
$$|5N+4| > |\frac{-3}{5\varepsilon}|$$
$$N > \frac{3}{25\varepsilon} - 4/5$$

I will show that any $n > N > \frac{3}{25\varepsilon} - 4/5$ implies $|\frac{2N+1}{5N+4} - \frac{2}{5}| < |\varepsilon|$.

$$n > \frac{3}{25\varepsilon} - \frac{4}{5}$$
$$\frac{25}{3}\left(n + \frac{4}{5}\right) > \frac{1}{\varepsilon}$$
$$\frac{3}{25}\left(\frac{5}{5n+4}\right) < \varepsilon$$
$$\frac{3}{5(5n+4)} < \varepsilon$$
$$\frac{3}{5(5n+4)} < \varepsilon$$
$$|\frac{10n-10n+5-8}{5(5n+4)}| < |\varepsilon|$$
$$|\frac{10n+5}{5(5n+4)} + \frac{-10n-8}{5(5n+4)}| < \varepsilon$$
$$|\frac{5n+1}{(5n+4)} - \frac{2}{5}| < \varepsilon$$

b) $\lim \frac{2n^2}{n^3+3} = 0$

Problem 2.2.3

Describe what we would have to demonstrate in order to disprove each of the following statements.

a) At every college in the United States, there is a student who is at least seven feet tall.

Solution:

Find one college that does not have any students over seven feet tall.

b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

Solution:

Find one college where every professor does not give every student either an A or B.

c) There exists a college in the US where every student is at least six feet tall.

Solution:

Show that every college in the US has a student who's height is under 6 feet.

Problem 2.2.4

Give an example of each or state that the request is impossible. for any that are impossible give a compelling argument for why that is the case.

a) A sequence with an infinite number of ones that does not converge to one.

Solution:

$$s(n) = (-1)^n = \{-1, 1, -1, 1, \dots\}$$

b) A sequence with an infinite number of ones that converges to a limit not equal to one.

Solution:

It is impossible. You cannot have an infinite number of ones in a sequence yet get infinity close to the limit that does not equal 1.

c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution:

$$s(n) = \begin{cases} 1\,, & n \leq N \\ n!\,, & n > N \end{cases}$$

Problem 2.2.6

Prove Theorem 2.2.7. The limit of a sequence, when it exists must be unique.

Solution:

First I assume that $\lim a_n = a$ and also that $\lim a_n = b$. Now I will show that $a = b$.

I know that for any $\varepsilon > 0$ there exists an $N_a \in \mathbb{N}$ implies $|a_n - a| < \varepsilon/2$. and that for any $\varepsilon > 0$ there exists an $N_b \in \mathbb{N}$ implies $|a_n - b| < \varepsilon/2$. Since I claim that $a = b$, then it should be true that $|a - b| < \varepsilon$.

$$\begin{aligned} |a-b| &< \varepsilon \\ |a - a_n + a_n - b| &< \varepsilon \\ |(a - a_n) + (a_n - b)| &< \varepsilon \end{aligned}$$

Now through the Triangle Inequality (1.2.5) and properties of absolute values I can express the previous inequality as the following,

$$\begin{aligned} |a-b| = |(a-a_n)+(a_n-b)| < |a-a_n| + |a_n-b| &< \varepsilon \\ |a-b| < |a-a_n| + |a_n-b| &< \varepsilon/2 + \varepsilon/2 = \varepsilon \\ |a_n-a| + |a_n-b| &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

Here I know that the last inequality is true because it was assumed to be true from the beginning. Therefore, $a = b$, and every limit is unique.