Exercise 5.2.3

a) Use definition 5.2.1 to produce the proper formula for the derivative of $h(x) = 1/x$.

Solution:

$$
\begin{aligned}
g'(c) &= \lim_{x \to c} \frac{1/x - 1/c}{x - c} \\
&= \lim_{x \to c} \frac{\frac{c-x}{xc}}{x - c} \\
&= \lim_{x \to c} \frac{\frac{-(x-c)}{xc}}{x - c} \\
&= \lim_{x \to c} \frac{-1}{xc} \\
&= \frac{-1}{c^2}, \text{ where } x \neq 0
\end{aligned}
$$

b) Combine the result in part (a) with the Chain Rule to supply a proof for part (iv) of thm. 5.2.4

Solution:

$$
\begin{aligned}
\frac{d}{dx}(f/g) &= f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)' \\
&= \frac{f'}{g} + f\left(\frac{-1}{g^2}\right)g' \quad \text{(after applying ther chain rule)} \\
&= \frac{f'}{g} + \left(\frac{-f}{g^2}\right)g' \\
&= \frac{gf'}{g^2} + \frac{-fg'}{g^2} \\
&= \frac{gf' - fg'}{g^2}
\end{aligned}
$$

c) Supply a direct proof of thm. 5.2.4 (iv) by algebraically manipulating the difference quotient for (f/g) in a style similar to the proof of thm. 5.2.4 (iii).

Solution:

$$\begin{aligned}
(f/g)'(c) &= \frac{g(c)f'(c)-f(c)g'(c)}{g(c)^2} \\
&= \lim_{x\to c}\frac{g(c)\frac{f(x)-f(c)}{x-c}-f(c)\frac{g(x)-g(c)}{x-c}}{g(c)g(x)} \\
&= \lim_{x\to c}\frac{g(c)(f(x)-f(c))-f(c)(g(x)-g(c))}{g(c)g(x)(x-c)} \\
&= \lim_{x\to c}\frac{g(c)f(x)-g(c)f(c)-f(c)g(x)+f(c)g(c)}{g(c)g(x)(x-c)} \\
&= \lim_{x\to c}\frac{g(c)f(x)-f(c)g(x)}{g(c)g(x)(x-c)} \\
&= \lim_{x\to c}\frac{f(x)}{g(x)(x-c)}+\frac{-f(c)}{g(c)(x-c)} \\
&= \lim_{x\to c}\frac{f(x)/g(x)-f(c)/g(c)}{(x-c)}
\end{aligned}$$

The result becomes the definition of a derivative.

Exercise 5.2.5

Let

$$f(X=x)=\begin{cases} x^a\,, & \text{if } x>0 \\ 0\,, & \text{if } x\le 0\end{cases}$$

a) For which values of a is f continuous at zero?

Solution:

$a>0$

b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

Solution:

$a>1$. The derivative is continuous in this case.

c) For which values of a is f twice-differentiable?

Solution:

$a>2$

Exercise 5.2.11

Assume that g is differentiable on $[a,b]$ and satisfies $g'(a) < 0 < g'(b)$.

a) Show that there exists a point $x \in (a,b)$ where $g(a) > g(x)$, and a point $y \in (a,b)$ where $g(y) < g(b)$.

Solution:

Consider $g'(a) < 0$, this means,

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} < 0$$

Since $x \in (a,b)$ and $x > a$, this means that the denominator is positive. For the limit to be less than zero (negative) it must be true that $g(x) < g(a)$.
Now consider $g'(b) > 0$, which implies,

$$\lim_{x \to b} \frac{g(x) - g(b)}{x - b} > 0$$

Since $x \in (a,b)$ and $x < b$, this means that the denominator is negative. For the limit to be greater than zero (positive) it must be true that $g(x) < g(b)$.

b) Now complete the proof of Darboux's Theorem started earlier.

Solution:

The previous findings show that the set $[a,b]$ is closed and bounded. The Extreme Value Theorem then applies and shows that a minimum exists at some point c.

Now the interior Extremum theorem applies and it is known that the derivative of g at c is equal to zero, $g'(c) = 0$.

$g'(x) = f'(x) - \alpha = 0$, thus $f'(x) = \alpha$.

Exercise 5.3.1

Recall from Exercise 4.4.9 that a function $f : A \to \mathbb{R}$ is Lipschitz on A if there exists an $m > 0$ such that

$$\left|\frac{f(x)-f(y)}{x-y}\right| < M$$

for all $x \neq y$ in A.

a) Show that if f is differentiable on a closed interval $[a,b]$ and if f' is continuous on $[a,b]$, then f is Lipschitz on $[a,b]$.

Solution:

Since f is continuous and differentiable on the closed interval, then $\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)|$, where $c \in [a,b]$. So if $|f'(c)| \leq M$ for all $c \in [a,b]$, then the function is Lipschitz

f' is continuous on a compact set, so the Extreme Value Theorem implies that f' attains a minimum and maximum value on the interval. Let the min and max be x and y respectively.

Let $M = \max\{|x|, |y|\}$. There is no value of f' that will exceed M. Therefore, $\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)| \leq M$ for all $c \in [a,b]$ and f is Lipschitz.

b) Review the definition of a contractive function in Exercise 4.3.11. If we add the assumption that $|f'(x) < 1|$ on $[a,b]$, does if follow that f is contractive on this set?

Solution:

By definition, f is a contractive function if there exists a c such that $0 < c < 1$ and

$$|f(x)-f(y)| \leq c|x-y|$$

for all $x, y \in \mathbb{R}$

Let the assumption be true.

$$|f'(x)| < 1$$
$$\left|\frac{f(x)-f(y)}{x-y}\right| < 1$$
$$|f(x)-f(y)| < |x-y|$$

Since the left is strictly less than the right, then we know that there exists a $0 < c < 1$ that will make the function contractive.

Exercise 5.3.3

Let h be a differentiable function defined on the interval $[0,3]$, and assume that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

a) Argue that there exists a point $d \in [0,3]$ where $h(d) = d$

Solution:

Let $g(d) = h(d) - d$. This implies the following:

$$g(0) = h(0) - 0 = 1$$
$$g(1) = h(1) - 1 = 1$$
$$g(3) = h(3) - 3 = -1$$

The Intermediate Value Theorem can apply because g is continuous by the Algebraic continuity Theorem.

This means that there exists a point d such that $g(d) = 0$.

Since $g(d) = h(d) - d$ this implies $0 = h(d) - d$ and finally, $h(d) = d$.

b) Argue that at some point c we have $h'(c) = 1/3$.

Solution:

The Mean Value Theorem can apply here because the function is differentiable and continuous on the domain.

Consider the following,

$$h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{2 - 1}{3 - 0} = \frac{1}{3} = h'(c)$$

c) Argue that $h'(x) = 1/4$ at some point in the domain.

Solution:

Using the Mean Value Theorem again, consider the following,

$$h'(c) = \frac{h(1) - h(0)}{1 - 0} = \frac{2 - 1}{1 - 0} = 1 = h'(c)$$

and

$$h'(c) = \frac{h(3) - h(1)}{3 - 1} = \frac{2 - 2}{3 - 1} = 0 = h'(c)$$

The slope of the function at some point $c \in (0,1)$ is 1 but the slope of the function at some point $d \in (1,3)$ is 0.

Because the function is differentiable and continuous we know by Darboux Theorem that there must be a point between c and d such that $h'(x) = 1/4$.

Exercise 5.3.7

A fixed point of a function f is a value x where $f(x) = x$. show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Solution:

We will start by assuming there are two or more fixed points at a and b. Let f be differentiable on an interval and $f'(x) \neq 1$.

Because f is differentiable it is also continuous. The Mean Value Theorem then applies.

$$\begin{aligned} \left| \frac{f(b) - f(a)}{b - a} \right| &= f'(c) \\ \left| \frac{b - a}{b - a} \right| &= f'(c) \\ 1 &= f'(c) \end{aligned}$$

However it was assumed from the beginning that $f'(x) \neq 1$. This is a contradiction. Therefore, the function can have at most 1 fixed point.