Exercise 4.4.9

(Lipschitz Function) A function $f : A \to \mathbb{R}$ is called Lipschitz if there exists a bound $M > 0$ such that

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq M$$

for all $x \neq y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points in the graph of f.

a) Show that if $f : A \to \mathbb{R}$ is lipschitz, then it is uniformly continuous on A.

Solution:

Let $\varepsilon > 0$ and $\delta = \varepsilon/M$.

$$\begin{aligned}
|x-y| &< \delta \\
|x-y| &< \varepsilon/M \\
M|x-y| &< \varepsilon \\
|f(x)-f(y)| < M|x-y| &< \varepsilon \\
|f(x)-f(y)| &< \varepsilon
\end{aligned}$$

Thus f is uniformly continuous.

b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

Solution:

The converse is not true. $f(x) = \sqrt{x}$ is uniformly continuous, but it is not Lipschitz.

$$\frac{|\sqrt{x}-\sqrt{y}|}{|x-y|} = \frac{|\sqrt{x}-\sqrt{y}|}{|(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})|} = \frac{1}{|\sqrt{x}+\sqrt{y})|}$$

The above expression can be bounded by an M except when $x = y = 0$. Thus $f(x) = \sqrt{x}$ is not Lipschitz.

Exercise 4.5.2ab

Provide an example of each of the following, or explain why the request is impossible.

a) A continuous function defined on an open interval with range equal to a closed interval.

Solution:

Let $f(x) = 1$
This range will always be 1, which will be closed because its an isolated point.

b) A continuous function defined on a closed interval with a range equal to an open interval.

Solution:

If these intervals were bounded then by the intermediate value theorem this would be impossible. However, if you let intervals be unbounded, then we can consider $\mathbb{R}$ as a closed and open set. Therefore $f(x) = x$ is defined on a closed set and maps to an open set.

Exercise 4.5.3

A function f is increasing on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a,b]$ and satisfies the intermediate value property (Def 4.5.3), then f is continuous on $[a,b]$.

Solution:

Let the function be strictly increasing on the domain, and satisfy the intermediate value property. This implies that for all $f(x) < L < f(y)$ there exist a c such that $a < c < b$ and $f(c) = L$.

Characterizations of Continuity says that for all $V_\varepsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ implies $f(x) \in V_\varepsilon(f(c))$.

Because the intermediate value property is satisfied and the function is strictly increasing, then for any ε neighborhood around $f(c)$, it will always be possible to find a δ neighborhood around c. Thus the function must be continuous.

Exercise 4.5.7

Let f be a continuous function on the closed interval $[0,1]$ with range also contained in $[0,1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0,1]$

Solution:

I will show that $f(x) = x$ exists.

Since the range is contained in $[0,1]$ and by the intermediate value theorem, there exists at least one point, $c \in [0,1]$, where $f(c) = x$.

But since x is also an element of the domain, and every element in the domain is mapped to the range. There will always exists a point c in the domain such that $f(c) = c$, maps back to itself. Thus a fixed point exists.