Exercise 3.3.2 abcd

Decide which of the following sets are compact. For those that are not compact, show how Definition 3.3.1 breaks down. In other words, give an example of a sequence contained in the given set that does not possess a subsequence converging to a limit in the set.

a) $\mathbb{N}$

Solution:

The set is not compact. The subsequence $\{1,3,5,...\}$ is contained in $\mathbb{N}$ but it does not converge to a limit.

b) $\mathbb{Q} \cap [0,1]$

Solution:

The set is not compact.

$$A = \{\frac{1}{6} + \frac{1}{6 \cdot 2^2} + \frac{1}{6 \cdot 3^2} + ... + \frac{1}{6 \cdot n^2} : n \in \mathbb{N}\}, \text{where}$$

$$\lim A = \frac{\pi^2}{36}$$

The limit is within the bounds but is not a rational number.

c) The Cantor set

Solution:

The set is compact.

d) $\{1 + 1/2^2 + 1/3^2 + ... + 1/n^2 : n \in \mathbb{N}\}$

Solution:

The set is not compact. Let the sequence be

$$A = \{1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} : n \in \mathbb{N}\}, \text{where}$$

Where $\lim A = \frac{\pi^2}{6}$, but the limit is not in the set.

Exercise 3.3.3

Prove the converse of theorem 3.3.4 by showing that if a set $K \subseteq \mathbb{R}$ is closed and bounded then it is compact.

Solution:

Since K is bounded every sequence in K will also be bounded. By the Bolzano Weierstass theorem, every bounded sequence will contain a convergent subsequence. Theorem 3.2.5 states that these subsequences in A converge to the limit points of A.

Those subsequence will converge to a limit within k because the set is closed and contains its limit points, thus the assumptions of compactness is met.

Exercise 3.3.5 abc

Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

a) The arbitrary intersection of compact sets is compact.

Solution:

This is true. Theorem 3.2.14 says that the intersection of an abitrary collection of closed sets is closed. Since compact sets are closed, the interesection of them would also be closed.

Campact sets are also bounded so the intersection of a bounded sets will still be bounded. Therefore the interesection will be compact.

b) The arbitrary union of compact sets is compact.

Solution:

This is false. If a had the collection of sets $\mathbb{F} = \{[-n, n] : n \in \mathbb{N}\}$. This would work for a finite union, but the union of an arbitrary collection will not be bounded, therefore, it cannot be conmpact.

c) Let A be arbitrary, and let K be compact. Then, the intersection $A \cap K$ is compact.

Solution:

This is false. Consider $\mathbb{Q} \cap [0, 1]$. A is the arbitrary set and $[0, 1]$ is the compqact set. The interesection is not compact.

Exercise 3.3.11

Consider each of the sets listed in Exercise 3.3.2. For each one that is not compact, find an open cover for which there is no finite subcover.

a) $\mathbb{N}$

Solution:

$\mathbb{F} = \{(0, n+2) : n \in \mathbb{N}\}$
For this open cover to contain $\mathbb{N}$ it must infinite.

b) $\mathbb{Q} \cap [0,1]$

Solution:

$\mathbb{F} = \{(\frac{1}{n}, 1+\frac{1}{n}) : n \in \mathbb{N}\}$

c) $\{1 + 1/2^2 + ... + 1/n^2 : n \in \mathbb{N}\}$

Solution:

$\mathbb{F} = \{(0, x') : x' = 1/2 + 1/3^2 + ... + 1/(n+1)^2\}$

Exercise 3.4.1

If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

Solution:

If the perfect set were $P = [0,5]$ and the compact set were $K = \{1,2,3,4,5\}$. The intersection of these two results in $P \cap K = \{1,2,3,4,5\}$. The interesection of any two closed sets will always be closed.

However, in this case $P \cap K$ contains isolated points. So it is not true that the intersection will always be perfect.

Exercise 3.4.4

Repeat the Cantor construction from Section 3.1 starting with the interval $[0, 1]$. This time, however, remove the open middle fourth from each of the component.

a) Is the resulting set compact? Perfect?

Solution:

x<1

b) Using the algorithms from section 3.1, compute the length and dimension of this cantor-like set.

Solution:

x<1