Homwework 1

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Problem 1.2.8

a)

$f : \mathbb{N} \to \mathbb{N}$ that is 1-1 but not onto

answer:

$f(n) = 5n$, where $n \in \mathbb{N}$

to obtain $f(x) = 11$, x be equal to 5.5 which is not an element of the naturals. Therefore it is not onto.

b)

$f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1-1

answer:

$f(n) = (n-1)(n-3)(n-6)$, where $n \in \mathbb{N}$

This is not 1-1 because $f(1) = 0$ and $f(6) = 0$ but $1 \neq 6$

b)

$f : \mathbb{N} \to \mathbb{Z}$ that is onto and 1-1

answer:

$$P(n) = \begin{cases} \text{if } n = 1, & 0 \\ \text{if } n \text{ is even,} & n/2 \\ \text{if } n \text{ is odd,} & -(n-1)/2 \end{cases}$$

Problem 1.2.12

Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$

a)

Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$

answer:

Base Case:

It is clear that by definition, $y_1 > -6$

Induction Hypothesis:

Now I will show that if $y_k > -6$ then, $y_{k+1} > -6$ for all $k \in \mathbb{N}$.

I start by letting $y_k > -6$ by true for some $k \in \mathbb{N}$

$$\begin{aligned} y_k &> -6 \\ 2y_k &> -12 \\ 2y_k - 6 &> -18 \\ (2y_k - 6)/3 &> -6 \\ y_{k+1} &> -6 \end{aligned}$$

Since we have showed that $y_{k+1} > -6$ and $y_2 > -6$, by induction $y_n > -6$ for all $n \in \mathbb{N}$.

b)

Use another induction argument to show the sequence $(y_1, y_2, y_3, ...)$ is decreasing

answer:

I must show that $y_1 > y_2 > y_3 > y_{n+1}$ for all $n \in \mathbb{N}$.

Base Case:

This is easily shown by finding y_2.

$$\begin{aligned} y_2 &= (2(y_1) - 6)/3 \\ y_2 &= (2(6) - 6)/3 \\ y_2 &= 2 \end{aligned}$$

Therefore $y_1 > y_2$

Induction Hypothesis:

Next I show that if $y_k > y_{k+1}$, then $y_{k+1} > y_{k+2}$ for all $k \in \mathbb{N}$.

$$\begin{gathered} y_k > y_{k+1} \\ 2y_k - 6 > 2y_{k+1} - 6 \\ (2y_k - 6)/3 > (2y_{k+1} - 6)/3 \\ y_{k+1} > y_{k+2} \end{gathered}$$

Therefore, because the base case and the induction step are proven we know that the sequence is decreasing for all $n \in \mathbb{N}$.

Problem 1.3.7

Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

answer:

For a to be a supremum it must be an upper bound of A and if b is also an upper bound of A, then $a \leq b$.

It is given that the first part is true so now I only need to prove that if b is also an upper bound of A, then $a \leq b$.

I start with letting b be an upper bound of A. By definition of an upper bound, this means $x \leq b$ for all $x \in A$.

We know that $a \in A$ so it must be true that $a \leq b$. Thus $a = \sup A$.

Problem 1.3.9

a)

If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

answer:

By definition of supremums and upper bounds I know that there exists a $b \in B$ where $b \leq \sup B$, and I know that there exists an $a \in A$ where $a \leq \sup A$.

But since $\sup B$ is larger than $\sup A$ it follows that there exists a $b > a$, and this would mean that b is an upper bound for the set A.

b)

Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

answer:

If set A was the interval $(0, 4)$ and set B was the interval $[0, 4]$, then $\sup A \leq \sup B$ would not be true.

Problem 1.3.11

Decide if the following statements are true or false. Give a short proof for those that are true. For any that are false, supply an example that makes it false.

a)

If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

answer:

True.

A is a subset of B so if $x \in A$ then $x \in B$. I also know that the sets are bounded, therefore it is impossible for A to have any elements there are larger then those contained in B. Therefore $\sup A \leq \sup B$ will always hold true.

b)

If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

answer:

True.

By the definition of supremums I know that $\sup A \geq a$ for all $a \in A$. Like wise, I know that $\sup B \leq b$ for all $b \in B$. Further, $\sup A < \inf B$ so I know there exists a real number between the two which I will call c. Therefore, $a < c < b$ is true for all $a \in A$ and $b \in B$.

c)

If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

answer:

False. If A was the set $(-4, 0)$, and if B was $(0, 4)$, there 0 greater than set A and less than set B and $\sup A = \inf B$.