Exercise 1

Let f, g and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some domain A. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ at some point $c \in A$, use the $\varepsilon - \delta$ definition of functional limits to show that $\lim_{x\to c} g(x) = L$.

Solution:

Let $\varepsilon > 0$. Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$, there exists a $\delta > 0$ such that whenever $|x-c| < \delta$ implies that $|f(x) - L| < \varepsilon$ and $|h(x) - L| < \varepsilon$.

Now consider the following

$$\begin{gathered} f(x) \leq g(x) \leq h(x) \\ f(x) - L \leq g(x) - L \leq h(x) - L \\ |f(x) - L| \leq |g(x) - L| \leq |h(x) - L| \end{gathered}$$

It is also true

$$\begin{aligned} |f(x) - L| \leq |g(x) - L| \leq |h(x) - L| &< \varepsilon \\ |g(x) - L| &< \varepsilon \end{aligned}$$

Therefore $\lim_{x\to c} g(x) = L$.

Exercise 2

Let f be uniformly continuous on $\mathbb{R}$, and define a sequence of functions by $f_n(x) = f(1 + 1/n)$. Show that $f_n \to f$ uniformly on $\mathbb{R}$.

Solution:

Let f be uniform and $f_n(x) = f(x + 1/n)$. Since f is uniform there exists an $x, y \in \mathbb{R}$ such that $|y - x| < \delta$ implies that $|f(y) - f(x)| < \varepsilon$

Let $|y - x| < \delta$ for every $\varepsilon > 0$. Further let $y = x + 1/n$. Since f is continuous we can say that

$$\begin{gathered} |y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon \\ |x + \frac{1}{n} - x| < \delta \Rightarrow |f(1 + \frac{1}{n}) - f(x)| < \varepsilon \\ |x + \frac{1}{n} - x| < \delta \Rightarrow |f_n(x) - f(x)| < \varepsilon \end{gathered}$$

Thus $(f_n) \to f$ uniformly.

Exercise 3

In section 5.4 the function

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

was shown to be nowhere differentiable (pg 163). Use the Weierstrass M-Test to show that $g(x)$ is continuous on $\mathbb{R}$

Solution:

Using page 163 as a reference, define $h(x) = |x|$, and let $h(x) = h(x+2)$. This implies that $h(x)$ is periodic, repeating every 2 units.

This also means that there is a max of $h(x)$, which is at $x = 1$, and $h(1) = 1$.

Now, let $M_n = \frac{1}{2^n}$. Note that $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a converging geometric series because $r = 1/2 < 1$. It is also true that $\frac{1}{2^n} h(2^n x) < M_n$ for all $n \in \mathbb{N}$.

The assumptions for the M-Test are now met and this implies that $\frac{1}{2^n} h(2^n x)$ converges to $g(x)$ uniformly. Theorem 6.4.2 states that because the sequence of functions is continuous and converges to g uniformly, then $g(x)$ is continuous on $\mathbb{R}$.

Exercise 4

Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Let $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to \infty$.

Solution:

Since f is differentiable and continuous for every $x > 0$, the mean value theorem states that for a closed interval on the domain $[a, b]$, there exists a point $c \in [a, b]$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Let $a = x$ and $b = x + 1$. It follows that

$$f'(c) = \frac{f(x+1) - f(x)}{(x+1) - x}.$$
$$f'(c) = f(x+1) - f(x)$$

However, $g(x) = f(x+1) - f(x)$, so we know that $g(x) = f'(c)$.

Since $\lim_{x\to\infty} f'(x) = 0$, then $\lim_{x\to\infty} f'(c) = 0$, thus $g(x) \to 0$.

Exercise 5

Let f be defined for all $x \in \mathbb{R}$, and suppose that $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

Solution:

Let $y = c \in \mathbb{R}$, therefore we obtain the following,

$$|f(x) - f(c)| \leq (x-c)^2$$
$$\left| \frac{f(x) - f(c)}{x - c} \right| \leq |(x-c)|$$

Since f be defined for all $x \in \mathbb{R}$ the definition of differentiability (5.2.1) states that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

It follows that

$$|f'(c)| = \lim_{x \to c} \left| \frac{f(x) - f(c)}{x - c} \right| \leq \lim_{x \to c} |x - c|$$
$$|f'(c)| = 0$$

The derivative of f at any point $c \in \mathbb{R}$ is zero. The derivative of a constant function is also equal to zero. Therefore, f must be constant.