Extreme Risk and Fractal Regularity in Finance

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Abstract. As the Great Financial Crisis reminds us, extreme movements in the level and volatility of asset prices are key features of financial markets. These phenomena are difficult to quantify using traditional approaches that specify extreme risk as a singular rare event detached from ordinary dynamics. Multifractal analysis, whose use in finance has considerably expanded over the past fifteen years, reveals that price series observed at different time horizons exhibit several major forms of scale-invariance. Building on these regularities, researchers have developed a new class of multifractal processes that permit the extrapolation from high-frequency to low-frequency events and generate accurate forecasts of asset volatility. The new models provide a structured framework for studying the likely size and price impact of events that are more extreme than the ones historically observed.

1. Introduction

Fractal modeling uses invariance principles to parsimoniously specify complex objects at multiple scales. It has proven to be of major importance in mathematics and the natural sciences, as this issue illustrates. Fractals also offer enormous benefits for the field of finance, in particular for modeling the price of traded securities, for computing the risk of financial portfolios, for managing the exposure of institutions, or for pricing derivative securities. These benefits should become more apparent as the adoption of fractal methods by the financial industry continues to gain ground. The fields of finance and economics also play a singular role in the intellectual history of fractals. Benoît Mandelbrot first discovered evidence of fractal behavior in financial returns, household income and household wealth in the late 1950's and early 1960's, and subsequently found similar patterns in coastlines, earthquakes and other natural phenomena. These observations prompted the development of the fractal and multifractal geometry of nature (**[M82]**).[1]

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[1]Benoît Mandelbrot eventually returned to finance in the mid-1990's, when he taught a *Fractals in Finance* course at Yale University. We attended this course and went on to develop with Benoît Mandelbrot the first applications of multifractals to financial series.

In financial markets, the distribution of price changes is of key importance because it determines the risk, as well as the potential gains, of a position or a portfolio of assets. Different investors may measure price changes at different horizons. For instance, a high-frequency trader may look at price changes over microseconds, while a pension fund or a university endowment may have horizons of months, years, or decades. Researchers have correspondingly investigated invariance properties in the distribution of price changes observed over different time increments. The French economist Jules Regnault (1863) may have been the first to observe that the standard deviation of a price change over a time interval of length Δt scales as the square root of Δt ([**R**]). This observation motivated Louis Bachelier ([**Ba**]) to formalize the definition of the Brownian motion and propose it as a possible model of a stock price. That is, Bachelier postulated that price changes are Gaussian, identically distributed and independent. While these assumptions have important limitations, Bachelier opened up the field of financial statistics, which has remained vibrant ever since.

In the early 1960's, Benoît Mandelbrot discovered that price changes have much thicker tails than the Gaussian distribution permits ([**M63**]). He proposed to replace the Brownian motion with another family of scale-invariant processes with independent increments – the stable processes of Paul Lévy ([**L24**]). Let $p(t)$ denote the logarithm of a stock price or an exchange rate. If $p(t)$ follows a Brownian or a Lévy process, the distributions of the returns $p(t + \Delta t) - p(t)$ observed over various horizons Δt can be obtained from each other by linear rescaling. This form of linear invariance turns out to be a rather crude approximation of financial series. Another shortcoming of Brownian motion and Lévy processes is the assumption that price changes are independent. As Benoît Mandelbrot ([**M63**]) himself pointed out,[2] the size of price changes, $|p(t + \Delta t) - p(t)|$, is persistent in financial data ([**E82**], [**Bo87**]). In fact for many series, the size of price changes is a long-memory process characterized by a hyperbolically declining autocorrelation at long lags ([**DGE**],[**D**]).

Since the mid–1990's, researchers have uncovered alternative forms of scale invariance in financial returns, based on multifractal moment-scaling. [**G**], [**CFM**], and [**CF02**] found evidence that the moments of the absolute value of price changes, $\mathbb{E}(|p(t + \Delta t) - p(t)|^q)$, scale as power functions of the horizon Δt. Multifractal moment-scaling had until then been observed in natural phenomena such as the distribution of energy in turbulent flows and the distribution of minerals in Earth's crust. These physical regularities can be modeled with multifractal measures ([**M74**]). The observation of moment-scaling in financial returns motivated researchers to construct the first family of multifractal diffusions ([**CFM**], [**CF01**], [**BDM**]). These processes are parsimonious and capture well the fat tails, long memory in volatility and moment scaling of financial series ([**CF01**], [**CF04**], [**CF**]).

The leading example considered in the survey, the Markov-Switching Multifractal (MSM), assumes that the size of a price change is driven by components that have identical distributions but different degrees of persistence. The dynamics of every component are specified by a Markov chain with its own transition probabilities. MSM thus constructs a multifractal measure stochastically over time, which improves over earlier multifractal measures with predetermined switching dates.

[2] Benoît Mandelbrot noted that: "[...] large changes tend to be followed by large changes – of either sign – and small changes tend to be followed by small changes, [...]."

The dynamic definition of MSM permits the adoption of powerful estimation and filtering methods. MSM generates accurate forecasts of the conditional distribution of returns and therefore of the upside potential and downside risk of a position.

Fractals provide a natural mathematical structure for modeling large risks. A common approach in finance is to represent an asset price as the sum of an Itô diffusion and a jump process. The diffusion describes "ordinary" fluctuations, while jumps are meant to capture "rare events." Difficulties in the empirical implementation of such approaches are readily apparent. Because rare events are modeled as intrinsically different from regular variations, inference on rare events must be conducted on a small set of observations and is therefore highly imprecise. Of increasing importance, researchers would like to understand the implications for asset prices of events that have never been previously observed ("peso effects," [**R88**], [**B06**], [**G12**], [**W**], [**IM**]). Statistical inference on an empty set, however, is a notoriously challenging exercise!

Fractal modeling offers scale invariance as a solution to this quandary. It is therefore a promising tool to a profession that is becoming increasingly aware of the importance of rare events. Scale-invariant properties permit researchers to model all price variations using a single data-generating mechanism. As a consequence, models constructed using fractal principles are extremely parsimonious. A small number of well-identified parameters, combined with testable assumptions on scale-invariance, specify price dynamics at all timescales. The tight specification of rare events, even those more extreme than have been observed in existing data, is a natural outcome of a fractal approach to modeling financial prices.

The organization of the paper is as follows. Section 2 discusses early fractal models and reviews fractal regularities in financial markets. Section 3 presents the Markov-Switching multifractal model and its empirical applications. Section 4 analyzes the pricing implications of multifractal risk. Section 5 concludes.

2. Fractal Regularities in Financial Markets

2.1. Self-Similar Proposals. Let $P(t)$ denote the price at date t of a financial asset, such as a stock or a currency, and let $p(t)$ denote its logarithm. The asset's logarithmic return between dates t and $t + \Delta t$ is given by:

$$p(t + \Delta t) - p(t).$$

For over a century, one of the leading themes in finance has been to understand the dynamics of asset returns.

In his 1900 doctoral dissertation, the French mathematician Bachelier introduced an early definition of the Brownian motion as a model of the stock price ([**Ba**]). If $p(t)$ follows a Brownian motion with drift, the return $p(t+\Delta t) - p(t)$ has a Gaussian distribution with mean $\mu\Delta t$ and variance $\sigma^2\Delta t$, or more concisely

$$p(t + \Delta t) - p(t) \stackrel{d}{=} \mathcal{N}(\mu\Delta t; \sigma^2\Delta t).$$

The Brownian motion now pervades modern financial theory and notably the Black–Merton–Scholes approach to continuous–time valuation ([**BS**], [**M**]). Its lasting success arises from its tractability and consistency with the financial concepts of no-arbitrage and market efficiency. However, empirical difficulties with the Brownian motion have become apparent over time.

In the late 1950's and early 1960's, advances in computing technology made it possible to conduct more precise tests of Bachelier's hypotheses. In a series of pathbreaking papers, Benoît Mandelbrot ([**M63**], [**M67**]) uncovered major departures from the Brownian motion in commodity, stock and currency series. His main observation was that the tails of return distributions are thicker than the Brownian motion permits. Benoît Mandelbrot understood that this phenomenon was not a mere statistical curiosity, as some researchers suggested at the time, but a major failure of the Brownian paradigm. In layman's terms, extreme price changes are key features of financial markets that the Brownian motion cannot capture. Since the purpose of risk management is to weather financial institutions against storms, underestimating the size of these storms, as the thin-tailed Brownian model does, is a recipe for financial disaster, panic and bankruptcy. The Great Financial Crisis reminds us of the severity of the shocks that can be unleashed on financial institutions, especially those who took on excessive risk as a result of poor risk management models and practices.

Benoît Mandelbrot advocated that financial prices should be modeled by a broader class of stochastic processes.

Definition 1. **(Self-similar process)** The real-valued process $\{p(t); t \in \mathbb{R}_+\}$ is said to be self-similar with index H if the vector $(p(ct_1), ..., p(ct_n))$ has the same distribution as $(c^H p(t_1), ..., c^H p(t_n))$, or more concisely

$$(p(ct_1), ..., p(ct_n)) \overset{d}{=} (c^H p(t_1), ..., c^H p(t_n)), \tag{2.1}$$

for every $c > 0$, $n > 0$, and $t_1, ..., t_n \in \mathbb{R}_+$.

The Brownian motion satisfies (2.1) with the index $H = 1/2$. Two other families of self-similar processes have also been influential in finance (e.g., [**M97**]), as we now discuss.[3]

The stable processes of Paul Lévy ([**L24**]) are self-similar with $H \in (1/2, +\infty)$. Their increments are independent and have Paretian tails:

$$\mathbb{P}\{|p(t + \Delta t) - p(t)| > x\} \sim K_\alpha \, \Delta t \, x^{-\alpha} \tag{2.2}$$

as $x \to +\infty$, where $\alpha = 1/H \in (0; 2)$ and K_α is a positive constant. As (2.2) shows, Lévy processes have thicker tails than the Brownian motion and are therefore more likely to accommodate large price changes ([**M63**], [**M67**]). One difficulty with Lévy-stable processes, however, is that they have infinite variances, which is at odds with the empirical evidence available for a large number of price series ([**BG**], [**FR**], [**AB**]). Furthermore, infinite variances pose major difficulties for financial theory because much of the asset pricing literature uses a security return's variance or its covariance with other securities or factors as the main quantitative measures of risk (see, e.g, [**CV**], [**MK**], [**S**], [**T**], [**M**]).

Fractional Brownian motions represent another important class of self-similar processes ([**K**], [**M65**], [**MV68**]). A fractional Brownian motion with initial value $B_H(0) = 0$ can be defined as:

$$B_H(t) = \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \left\{ \int_{-\infty}^{0} [(t-s)^{H-1/2} - (-s)^{H-1/2}] dZ_s + \int_0^t (t-s)^{H-1/2} dZ_s \right\},$$

[3]We refer the reader to [**ST**] for a detailed treatment of self-similar processes.

where Γ denotes Euler's gamma function, Z is a standard Brownian motion, and the index H is between 0 and unity. A fractional Brownian motion is therefore constructed by assigning hyperbolic weights to the increments of a standard Brownian motion, which generates strong persistence.

Let

$$r_t = p(t) - p(t-1) \tag{2.3}$$

denote the return on a time interval of unit length, such as one day. If the price $p(t)$ follows a fractional Brownian motion with self-similarity index $H \neq 1/2$, the return autocorrelation declines at the hyperbolic rate:

$$Corr(r_t; r_{t+n}) \sim H(2H-1)n^{2H-2} \text{ as } n \to \infty. \tag{2.4}$$

The strong dependence of returns implied by the fractional Brownian motion is at odds with empirical evidence. Indeed, a large body of research (e.g., [**K53**], [**GM63**], [**F65**]) shows that over a wide range of sampling frequencies, asset returns exhibit either zero or weak autocorrelation: $Corr(r_t, r_{t+n}) \approx 0$ for all $n \neq 0$, as the simplest forms of market efficiency suggest. Furthermore, long memory in returns (2.4) is theoretically inconsistent with arbitrage-pricing in continuous time ([**MS**]), which makes it an unappealing model of financial prices. Fractional integration can be, however, useful for modeling persistence in the size of price changes (e.g., [**BBM**], [**HMS**]), as will be further discussed below.

Besides the aforementioned shortcomings of stable processes and fractional Brownian motions, self-similar processes with stationary increments face another, common difficulty. By (2.1), the returns at various horizons should have identical distributions up to a scalar renormalization:

$$p(t+\Delta t) - p(t) \overset{d}{=} (\Delta t)^H p(1). \tag{2.5}$$

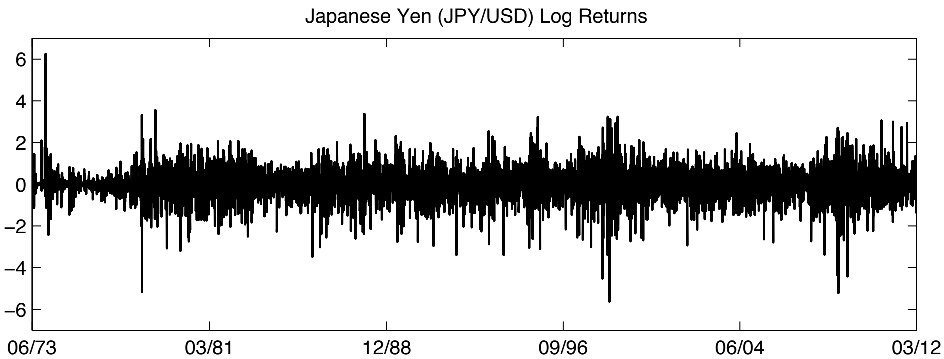
Most financial series, however, are *not* exactly self-similar, but have thicker tails and are more peaked in the bell at shorter horizons than the self-similarity condition (2.5) predicts. This empirical observation is consistent with the economic intuition that higher frequency returns are either large if new information has arrived, or close to zero otherwise. For this reason, self-similar processes cannot be fully satisfactory models of asset returns.

2.2. Empirical Evidence on Fat Tails and Long Memory. Following [**M63**] and [**M67**], a number of researchers have measured the return tail indexes α and α' defined by

$$\mathbb{P}(r_t > x) \sim Kx^{-\alpha},$$
$$\mathbb{P}(r_t < -x) \sim K'x^{-\alpha'}$$

as $x \to +\infty$, where K and K' are fixed elements of $\mathbb{R}_+$. Early studies on financial return tails were mainly parametric ([**F63**], [**F65**], [**BG**], [**FR**], [**AB**]). In the 1970's, statisticians developed precise techniques for the nonparametric estimation of the tail indexes of a distribution ([**Hi**], [**CDM**]), and subsequent empirical analyses confirmed that tail indexes are finite in financial series (e.g., [**KK**], [**KSV**], [**PMM**], [**JD**], [**LP**], [**G09**]). In most studies, α and α' are also measured to be larger than 2. Asset returns therefore have a finite variance, consistent with the assumptions of financial theory.

In the early 1990's, researchers uncovered evidence of strong persistence in the absolute value of returns ([**D**], [**DGE**]). Long memory is often defined by a



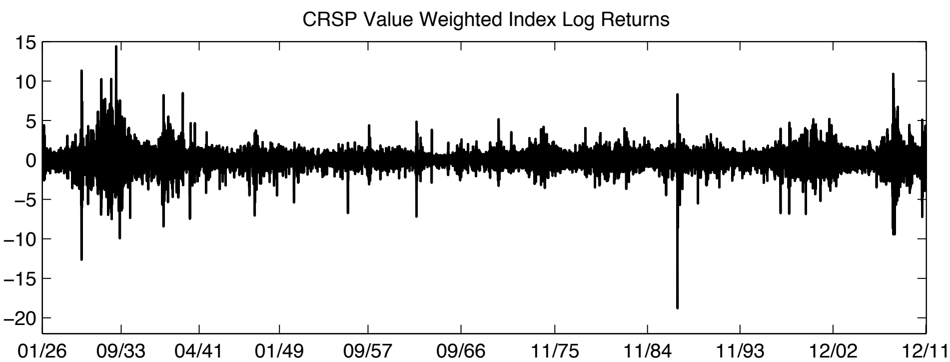


FIGURE 1. Financial Return Series. This figure shows daily logarithmic returns for the Japanese yen / U.S. dollar exchange rate series, and for the value weighted U.S. stock index compiled by the Center for Research in Securities Prices (CRSP). The yen series begins on June 1, 1973 and ends on March 30, 2012. The stock series begins on January 2, 1926 and ends on December 30, 2011.

hyperbolic decline in the autocorrelation function as the lag goes to infinity. For every moment $q \geq 0$ and every integer n, let

$$\rho_q(n) = Corr(|r_t|^q, |r_{t+n}|^q) \tag{2.6}$$

denote the autocorrelation in levels. We say that the asset exhibits long memory in the size of returns if $\rho_q(n)$ is hyperbolic in n :

$$\rho_q(n) \propto c_q n^{-\delta(q)} \tag{2.7}$$

as $n \to +\infty$.

These important features of financial data can be seen by casual observation of standard asset returns. Figure 1, Panel A, shows the Japanese Yen / U.S. dollar exchange rate series from 1973, following the demise of the Bretton-Woods system of fixed exchange rates, to the present day. The yen series contains 9751 return observations. The series shows both fat tails and volatility clustering at different time scales, including over periods as long as several years, as occurs in the presence

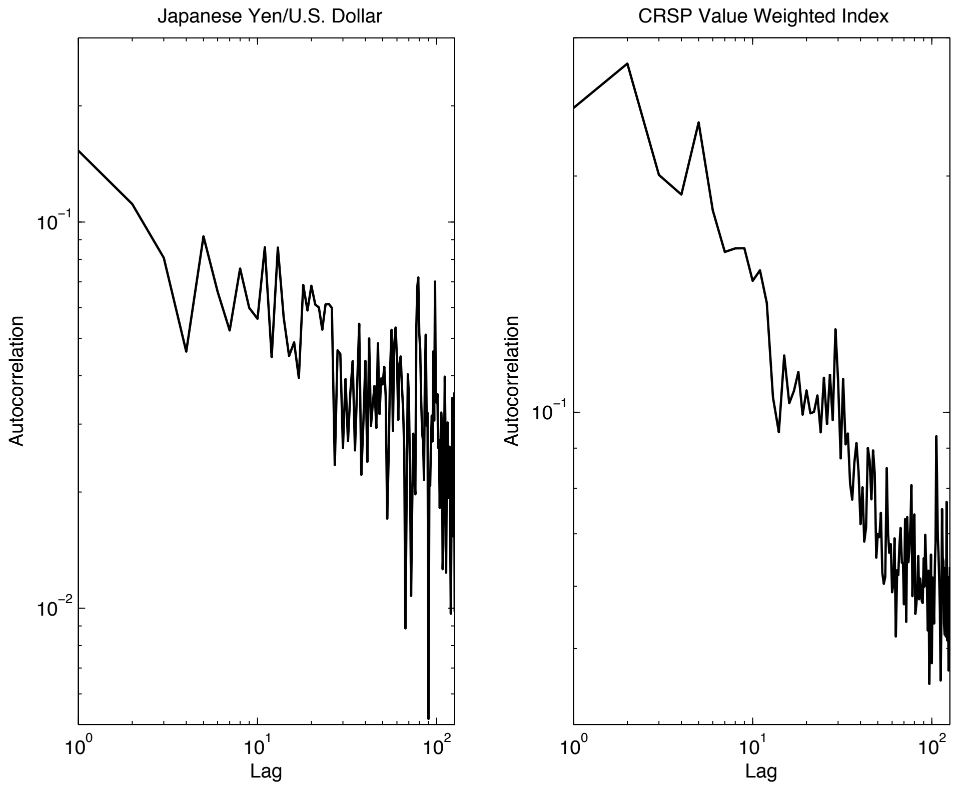


FIGURE 2. Long Memory in Squared Returns. This figure illustrates the autocorrelations of squared logarithmic returns for the yen (Panel A) and the CRSP stock index return series (Panel B). The autocorrelations are plotted on a log-log scale, so that a hyperbolic decay in autocorrelations, as occurs under long-memory, will appear as an approximately straight line in the figure.

of long memory. Panel B shows the same features in a long time series of 22,780 daily U.S. stock index returns obtained from the University of Chicago's Center for Research in Security Prices (CRSP).

In Figure 2, we display the autocorrelation of squared returns $\rho_2(n)$ on the vertical axis versus the lag length n on the horizontal axis. For both series the plots are approximately linear on a double logarithmic scale, indicating that $\rho_2(n)$ is hyperbolic in n. The yen/dollar exchange rate and the U.S. stock index series thus both exhibit long memory in the size of price changes.

2.3. Multifractal scaling. In the mid-1990's, the observation that asset returns exhibit both fat tails and long memory in volatility led researchers to consider that asset prices may exhibit multifractal moment-scaling:

$$\mathbb{E}\left(|p(t+\Delta t)-p(t)|^q\right)=c_q(\Delta t)^{\tau(q)+1} \tag{2.8}$$

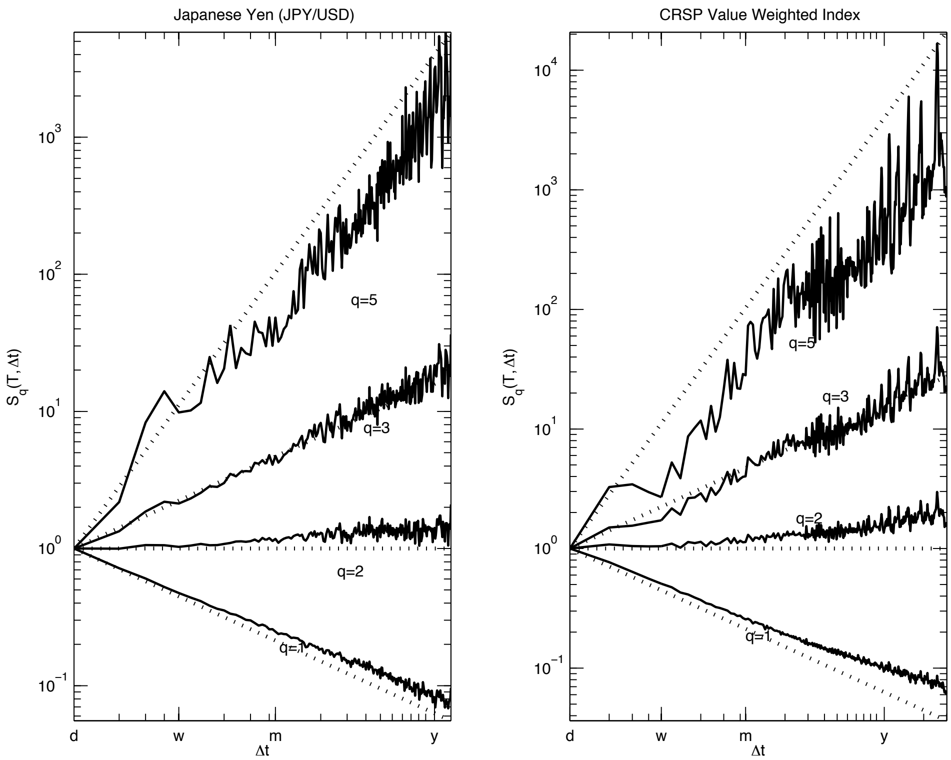


FIGURE 3. Moment Scaling of Financial Returns. For every sampling interval Δt, we partition the total observation period $[0, T]$ into $N = T/\Delta t$ subintervals and compute the partition function $S_q(T, \Delta t) \equiv \sum_{i=0}^{N-1} |p(i\Delta t + \Delta t) - X(i\Delta t)|^q$. The figure provides log-log plots of $S_q(T, \Delta t)$ against values of Δt ranging from 1 day ("d") to 1 year ("y"). If $p(t)$ is multifractal: $\mathbb{E}\left[|p(\Delta t)|^q\right] = c_q(\Delta t)^{\tau(q)+1}$, then $\log \mathbb{E}[S_q(T, \Delta t)] = \tau(q) \log(\Delta t) + \log(Tc_q)$ and the plots should be approximately linear.

for every (finite) moment q and time interval Δt. A self-similar process satisfies (2.8), with $\tau(q) = Hq - 1$. The process p is said to be strictly multifractal if (2.8) holds for a strictly concave function $\tau(q)$.

Strict multifractality has been observed in fields as diverse as fluid mechanics, geology and astronomy. We now also have strong evidence of strict multifractal moment-scaling in a variety of financial series, including currencies and equities ([**CF02**], [**CFM**], [**G**], [**VA**]). As an example, we illustrate in Figure 3 the moment-scaling properties of the Yen / U.S. dollar exchange rate and the CRSP stock index. The panels of the figure plot the partition interval Δt on the horizontal axis versus an empirical estimate of $\mathbb{E}\left(|p(t + \Delta t) - p(t)|^q\right)$ on the vertical axis. The empirical estimate is obtained by taking the sample analogue of (2.8), as explained in the figure caption, for a variety of moments q. The dotted lines in the figure represent the scaling implied by Brownian motion, which satisfies self-similarity with $H = 1/2$. The panels both show evidence of moment-scaling that is linear in

Δt, but the scaling coefficients $\tau(q)$ cannot be captured as a linear function of a single index H. These empirical facts are characteristics of multifractal scaling.

The mathematical modeling of multifractal objects first focused on random measures, constructed by iterative reallocation of mass over a domain (e.g., [**M74**]). One of the simplest examples is the binomial measure[4] on the unit interval $[0,1]$, which we derive as the limit of a *multiplicative cascade*. Consider a fixed $m_0 \in [1/2,1]$ and the Bernoulli (also called binomial) distribution taking the high value m_0 or the low value $1-m_0$ with equal probability. In the first step of the cascade, we draw two independent values M_0 and M_1 from the binomial distribution. We define a measure μ_1 by uniformly spreading the mass M_0 on the *left* subinterval $[0,1/2]$, and the mass M_1 on the *right* subinterval $[1/2,1]$. The density of μ_1 is a step function.

In the second stage of the cascade, we draw four independent binomials $M_{0,0}$, $M_{0,1}$, $M_{1,0}$ and $M_{1,1}$. We split the interval $[0,1/2]$ into two subintervals of equal length; the left subinterval $[0,1/4]$ is allocated a fraction $M_{0,0}$ of $\mu_1[0,1/2]$, while the right subinterval $[1/4,1/2]$ receives a fraction $M_{0,1}$. Applying a similar procedure to $[1/2,1]$, we obtain a measure μ_2 such that:

$$\begin{aligned} &\mu_2[0,1/4] = M_0 M_{0,0}, \qquad &&\mu_2[1/4,1/2] = M_0 M_{0,1}, \\ &\mu_2[1/2,3/4] = M_1 M_{1,0}, &&\mu_2[3/4,1] = M_1 M_{1,1}. \end{aligned}$$

Iteration of this procedure generates an infinite sequence of random measures (μ_k) that weakly converges to the binomial measure μ.

Consider a dyadic interval[5] $[t, t+2^{-k}]$, where $t = \sum_{i=1}^{k} \eta_i 2^{-i}$ and $\eta_1, ..., \eta_k \in \{0,1\}$. The measure of the interval is

$$\mu[t, t+2^{-k}] = M_{\eta_1} M_{\eta_1,\eta_2} \dots M_{\eta_1,\dots,\eta_k} \Omega,$$

where Ω is a random variable determined by the change in the mass generated by stages $k+1, \dots, \infty$ of the cascade. Equation (2.3) implies that

$$\mathbb{E}(\mu[t, t+2^{-k}]^q) = c_q \, [\mathbb{E}(M^q)]^k = c_q \, (\Delta t)^{\tau_\mu(q)+1},$$

where $c_q = \mathbb{E}(\Omega^q)$, $\Delta t = 2^{-k}$, and $\tau_\mu(q) = -\log_2[\mathbb{E}(M^q)] - 1$. The moments of the limiting measure of a dyadic interval is therefore a power of its length Δt, similar to the scaling relation (2.8).

The extension of multifractality from random measures to stochastic processes was first achieved in a framework called the Multifractal Model of Asset Returns ("MMAR", [**CFM**], [**CF02**]). The MMAR provides a class of diffusions consistent with the multifractal scaling relation (2.8). In the MMAR, an asset price is specified by compounding a Brownian motion with an independent random time-deformation:

$$p(t) = p(0) + B[\theta(t)],$$

where θ is the cumulative distribution of the multifractal measure μ. The defining feature of the MMAR is the use of a multifractal time deformation. The MMAR is thus related to subordination, a concept introduced in harmonic analysis by Bochner ([**B55**]) and used for the first time by Clark ([**Cl**]) in the finance literature. In the original formulation of [**B55**] and [**Cl**], a subordinator $\theta(t)$ is a right-continuous increasing process that has independent and homogenous increments.

[4]The binomial is sometimes called the Bernoulli or Besicovitch measure.

[5]A number $t \in [0,1]$ is called *dyadic* if $t = 1$ or $t = \eta_1 2^{-1} + ... + \eta_k 2^{-k}$ for a finite k and $\eta_1, .., \eta_k \in \{0,1\}$. A dyadic interval has dyadic endpoints.

While the original assumptions of independence and homogeneity have proven too restrictive for financial applications, stochastic time changes are generally appealing for modeling financial prices (see, eg., [**AG**]). By specifying the evolution of random trading time as multifractal, the MMAR provides an empirically realistic class of models that parsimoniously captures the fat tails and long-memory volatility dependence of financial series.

The MMAR price process inherits the moment-scaling properties of the measure, in the sense that $\mathbb{E}(|p(t+\Delta t)-p(t)|^q) = (\Delta t)^{\tau_\mu(q/2)+1}$ on any dyadic interval $[t, t+\Delta t]$. These moment restrictions represent the basis of estimation and testing ([**CFM**], [**CF02**], [**L08**]). The MMAR provides a well-defined stochastic framework for the analysis of moment-scaling. In [**CF02**], we have verified that the moment-scaling properties of financial returns, such as the ones exhibited in Figure 3, are consistent with the range of variations predicted by the MMAR. Consistent with its ability to explain return moments at various frequencies, the MMAR captures nonlinear variations in the unconditional density of returns observed at various time horizons ([**L01**]).

The moment-scaling properties of the MMAR have generated extensive interest in econophysics (for example, [**LB**]). They are also related to recent econometric research on power variation, which interprets return moments at various frequencies in the context of traditional jump-diffusions (for example, [**ABDL**], [**BNS**]). Furthermore, recent research confirms that the arrival of transactions in financial markets is well described by a multifractal driving process ([**CDS**]), which confirms the economic motivation of the time deformation $\theta(t)$ as a multifractal "trading time."

Despite its appealing properties, the MMAR is unwieldy for econometric applications because of two features of the underlying measure: (a) the recursive reallocation of mass on an entire time-interval does not fit well with standard time series tools; and (b) the limiting measure contains a residual grid of instants that makes it non-stationary. A solution to these problems is proposed in the next section.

3. The Markov-Switching Multifractal (MSM)

The Markov-switching multifractal (MSM) is a fully stationary multifractal diffusion ([**CF01**], [**CF04**], [**CF**]), which parsimoniously incorporates arbitrarily many components of heterogeneous durations. MSM builds a bridge between multifractality and Markov-switching and therefore permits the application of powerful statistical methods.

3.1. Definition in Discrete Time. We assume that time is defined on the grid $t = 0, 1, \ldots, \infty$. We consider:

- a first-order Markov state vector $M_t = (M_{k,t})_{1\leq k\leq \bar{k}} \in \mathbb{R}_+^{\bar{k}}$, and
- a random variable $M \geq 0$ with a unit mean: $\mathbb{E}(M) = 1$.

In this survey, we consider for simplicity that M has a Bernoulli distribution taking either a high value m_0 or a low value $2 - m_0$ with equal probability, where m_0 is fixed element of the interval $[1, 2]$. We also assume that the components $M_{1,t}, M_{2,t}, \ldots, M_{\bar{k},t}$ are mutually independent across k.

Each component $\{M_{k,t}\}_{t\geq 0}$ is a Markov process in its own right, which is constructed through time as follows. Given $M_{k,t-1}$, the next-period component $M_{k,t}$

is

drawn from the distribution of M	with probability γ_k
set equal to its current level $M_{k,t-1}$	with probability $1-\gamma_k$.

The transition probabilities are tightly specified by

$$\gamma_k = 1 - (1-\gamma_1)^{\left(b^{k-1}\right)}, \tag{3.1}$$

where $\gamma_1 \in (0,1)$ and $b \in (1,\infty)$. The definition implies that $\gamma_1 < ... < \gamma_{\bar{k}}$, so that components with a low index k are more persistent than higher-k components. If the parameter γ_1 is small compared to unity, the transition probabilities $\gamma_k \sim \gamma_1 b^{k-1}$ grow approximately at geometric rate b for low values of k; the growth rate of γ_k eventually slows down for high values of k so that γ_k remains lower than unity.

We assume that returns $r_t = p_t - p_{t-1}$ are given by

$$r_t = \mu + \sigma(M_t)\varepsilon_t, \tag{3.2}$$

where $\mu \in \mathbb{R}$ and $\bar{\sigma} \in \mathbb{R}_{++}$ are constants, $\{\varepsilon_t\}_{t\geq 0}$ are independent standard Gaussians, and volatility at date t is

$$\sigma(M_t) = \bar{\sigma}\left(\prod_{k=1}^{\bar{k}} M_{k,t}\right)^{1/2}. \tag{3.3}$$

We call this construct the Markov-Switching Multifractal (MSM). We observe that the MSM process r_t is stationary, with unconditional mean $\mathbb{E}(r_t) = \mu$ and unconditional standard deviation $\{\mathbb{E}[(r_t-\mu)^2]\}^{1/2} = \bar{\sigma}$.

The multiplicative structure (3.3) is appealing to model the high variability and high volatility persistence exhibited by financial time series. The components have the same marginal distribution M but differ in their transition probabilities γ_k. When a low–k multiplier changes, volatility varies discontinuously and has strong persistence. In addition, high frequency multipliers produce substantial outliers.

Figure 4 illustrates the construction of binomial MSM. The top three panels represent the sample path of the volatility components $M_{k,t}$ for k varying from 1 to 3. We see that the number of switches increases with k, as implied by (3.1). The fourth panel represents the variance $\sigma^2(M_t) \equiv \bar{\sigma}^2 M_{1,t}...M_{\bar{k},t}$, where $\bar{k} = 8$ and $\bar{\sigma} = 1$. The construction generates cycles of different frequencies, consistent with the empirical observation that there are volatile decades and less volatile decades, volatile years and less volatile years, and so on. MSM thus provides a tight model for the behavior of financial returns at various horizons documented in **[DG]** and **[LZ]**. The panel also shows pronounced peaks and intermittent bursts of volatility, which produce fat tails in returns. The last panel illustrates the impact of the various frequencies on the return series.

In empirical applications, it is numerically convenient to estimate parameters of the same magnitude. Since $\gamma_1 < \cdots < \gamma_{\bar{k}} < 1 < b$, we choose $\gamma_{\bar{k}}$ and b to specify the set of transition probabilities. Overall, an MSM process with $\bar{k}$ components is fully parameterized by

$$\psi \equiv (m_0, \bar{\sigma}, b, \gamma_{\bar{k}}, \mu) \in [1,2] \times \mathbb{R}_{++} \times (1,+\infty) \times (0,1) \times \mathbb{R},$$

where m_0 characterizes the distribution of the multipliers, $\bar{\sigma}$ is the unconditional standard deviation of returns, b and $\gamma_{\bar{k}}$ define the set of switching probabilities, and μ the unconditional mean of returns. The number of components $\bar{k} \in \mathbb{N}^*$ can also

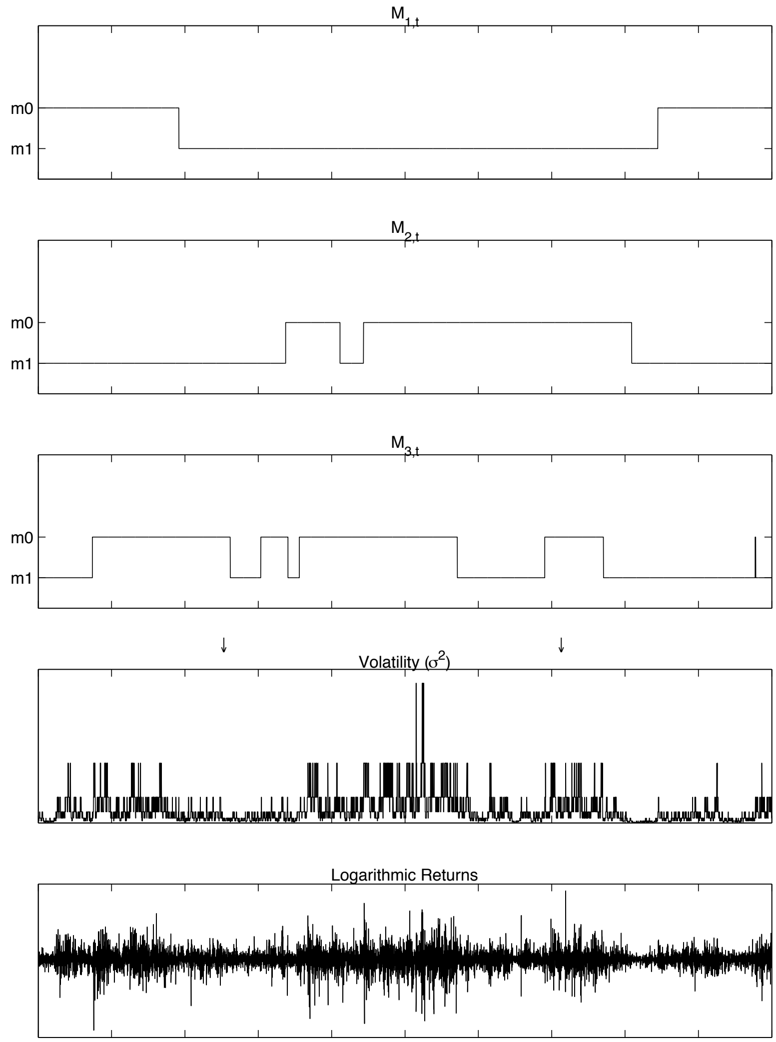


FIGURE 4. Construction of the Binomial Markov-Switching Multifractal. The figure illustrates construction of binomial MSM with $\bar{k} = 8$ components over $T = 10,000$ periods with parameters $m_0 = 1.4$, $b = 2$, and $\gamma_1 = b/T$. The first three panels show the components $M_{1,t}$, $M_{2,t}$, and $M_{3,t}$. The fourth panel shows the variance σ_t^2, which is the product of all eight multipliers. The final panel shows the return series.

be viewed as a discrete parameter of MSM, and we discuss below how to estimate it along with the continuous vector ψ.

3.2. Filtering and Estimation. Since the components $M_{k,t}$ have binomial distributions, the state vector M_t takes $d = 2^{\bar{k}}$ possible values $m^1, \ldots, m^d \in \mathbb{R}_+^{\bar{k}}$. The transition matrix of M_t is by definition the $d \times d$ matrix $A = (a_{i,j})_{1 \leq i,j \leq d}$ with components

$$a_{ij} = \mathbb{P}(\, M_{t+1} = m^j \,\big|\, M_t = m^i).$$

For a general Markov chain with d states, the transition matrix contains d^2 elements. So for example if $d = 2^8$ states, the transition matrix contains $2^{16} = 65,536$ parameters and estimation is generally unfeasible with current numerical methods. By comparison, an MSM return process with $\bar{k} = 8$ components and $2^8 = 256$ states is fully defined by only five parameters.[6] MSM thus offers a parsimonious specification of a high-dimensional state space, which paves the way for statistical estimation and inference.

The financial statistician observes the returns r_t but not the state vector M_t. She therefore seeks to compute the conditional probability distribution $\Pi_t = \left(\Pi_t^1, \ldots, \Pi_t^d\right) \in \mathbb{R}_+^d$, where for every $j \in \{1, \ldots, d\}$,

$$\Pi_t^j \equiv \mathbb{P}\left(M_t = m^j | r_1, \ldots, r_t\right).$$

Conditional on the volatility state, the return has Gaussian density $\omega_j(r_t) = n[(r_t - \mu)/\sigma(m^j)]/\sigma(m^j)$, where $n(\cdot)$ denotes the density of a standard normal. Bayes' rule implies that

$$\Pi_t^j \propto \omega_j(r_t) \mathbb{P}\left(M_t = m^j | r_1, \ldots, r_{t-1}\right), \tag{3.4}$$

or $\Pi_t^j \propto \omega_j(r_t) \sum_{i=1}^d a_{i,j} \mathbb{P}\left(M_{t-1} = m^i | r_1, \ldots, r_{t-1}\right)$. The vector Π_t is therefore a function of its lagged value and the contemporaneous return r_t :

$$\Pi_t = \frac{\omega(r_t) \circ (\Pi_{t-1} A)}{[\omega(r_t) \circ (\Pi_{t-1} A)]\, \iota'}, \tag{3.5}$$

where $\omega(r_t) = [\omega_1(r_t), \ldots, \omega_d(r_t)]$, $\iota = (1, \ldots, 1) \in \mathbb{R}^d$, and $x \circ y$ denotes the Hadamard product $(x_1 y_1, \ldots, x_d y_d)$ for any $x, y \in \mathbb{R}^d$. The vector Π_t can therefore be computed recursively, as is familiar in regime-switching models (**[H]**). In empirical applications, the initial vector Π_0 is set equal to the ergodic distribution $\Pi_\infty = \iota/d$ of the Markov chain M_t.

Let $L(r_1, ..., r_T; \psi)$ denote the probability density function of the time series $r_1, ..., r_T$ under the MSM model with parameter vector ψ. We easily check that:

$$\log L\left(r_1, ..., r_T; \psi\right) = \sum_{t=1}^T \log[\omega(r_t) \cdot (\Pi_{t-1} A)]. \tag{3.6}$$

For a fixed $\bar{k}$, the maximum likelihood estimator (ML),

$$\hat{\psi} = \operatorname{argmax}_\psi \log L\left(r_1, ..., r_T; \psi\right),$$

is consistent and asymptotically normal: $\sqrt{T}(\hat{\psi} - \psi) \xrightarrow{d} \mathcal{N}(0, V)$. The ML estimator is asymptotically efficient, in the sense that no other estimator has a smaller asymptotic variance-covariance matrix V (see, e.g., **[C]**). In the case of MSM, $\hat{\psi}$ also performs well in finite samples (**[CF04]**).

---

[6]The transition probabilities of MSM are given by:

$$a_{ij} = \prod_{k=1}^{\bar{k}} \left[(1 - \gamma_k)\, 1_{\left\{m_k^i = m_k^j\right\}} + \gamma_k \mathbb{P}(M = m_k^j) \right],$$

where m_k^i denotes the mth component of vector m^i, and $1_{\{m_k^i = m_k^j\}}$ is the dummy variable equal to 1 if $m_k^i = m_k^j$, and 0 otherwise.

MSM specifications with different values of $\bar{k}$ are non-nested but are specified by the same number of parameters for every $\bar{k} \geq 2$. Comparing their likelihoods therefore provides meaningful information about the goodness-of-fit. The standard Vuong test ([**V**]), or alternatively a version adjusted for heteroskedasticity, is a simple and appropriate model selection criterion, as illustrated in [**CF04**] and [**CF**]. Filtering and parameter estimation are therefore remarkably convenient with MSM.

3.3. Empirical Estimation and Forecasting. We apply MSM to the Japanese yen / U.S. dollar exchange rate series illustrated in Figure 1. The daily logarithmic returns are calculated from exchange rates beginning in June 1973, extending to the end of our sample at the end of March, 2012. Overall, the series contains 9751 observations.

Table 1 reports the maximum likelihood estimates. For convenience, we set the drift parameter equal to zero: $\mu = 0$. In Panel A, following [**CF04**] and [**CFT**], we estimate the four remaining parameters m_0, $\bar{\sigma}$, $\gamma_{\bar{k}}$, and b, for a number of components $\bar{k}$ varying from 1 to 12. The first column corresponds to a standard Markov-switching model with only two volatility states. As $\bar{k}$ increases, the number of states increases geometrically as $2^{\bar{k}}$. There are over four thousand states when $\bar{k} = 12$.

The estimate of m_0 declines monotonically with $\bar{k}$: as more components are added, less variability is required in each $M_{k,t}$ to match the fluctuations in volatility exhibited by the data. The estimates of $\bar{\sigma}$ vary across $\bar{k}$ with no particular pattern; their standard errors increase with $\bar{k}$, consistent with the fact that long-run averages are difficult to identify in models permitting long volatility cycles. We next examine the frequency parameters $\gamma_{\bar{k}}$ and b. When $\bar{k} = 1$, the single multiplier has a duration $1/\gamma_1 = 1/0.192$ of about 5 business days, which corresponds to one calendar week. As $\bar{k}$ increases, the switching probability of the highest frequency multiplier increases until a switch occurs almost once a day for large $\bar{k}$. At the same time, the estimate of b decreases steadily with $\bar{k}$. The increasing number of frequencies permits durations to fan out to both very short and very long values, ranging from 1 day to decades, while the spacing of durations becomes tighter.

We finally examine the behavior of the log-likelihood function as the number of frequencies $\bar{k}$ increases from 1 to 12. The likelihood rises substantially as $\bar{k}$ increases from low to moderate values, and continues to rise at a decreasing rate as we add components. The likelihood function eventually flattens out when $\bar{k} \geq 10$. The monotonic relationship between the likelihood and $\bar{k}$ confirms one of the main premises of MSM: fluctuations in volatility occur with heterogeneous degrees of persistence, and explicitly incorporating a larger number of frequencies results in a better fit.

In Panel B, we restrict two of the MSM parameters. Consistent with the idea that the long-run mean of volatility is poorly identified, we set the unconditional volatility $\bar{\sigma}$ equal to the sample standard deviation of returns. Since the lowest-frequency volatility component is difficult to identify even in a long data sample, we set $\gamma_1 = 1/(4T)$, so that a switch in this component is expected to occur once in a sample containing four times as many observations as the available sample. With these restrictions, we only need to estimate the remaining parameters m_0 and b. Empirically, these restrictions reduce the likelihood substantially when $\bar{k}$ is small, but for large values of $\bar{k}$ the restricted likelihood is very close to the unrestricted likelihoods shown in Panel A. These results suggest that restricting the values of $\bar{\sigma}$

TABLE 1. – MAXIMUM LIKELIHOOD ESTIMATION

	$k = 1$	2	3	4	5	6	7	8	9	10	11	12
						Japanese Yen / US Dollar						
						A. Four Parameters						
m_0	1.732	1.730	1.663	1.625	1.563	1.541	1.493	1.488	1.435	1.405	1.400	1.376
	(0.012)	(0.014)	(0.010)	(0.011)	(0.009)	(0.012)	(0.009)	(0.010)	(0.010)	(0.009)	(0.009)	(0.009)
$\bar{\sigma}$	0.658	0.563	0.564	0.483	0.493	0.592	0.599	0.497	0.519	0.516	0.448	0.442
	(0.009)	(0.014)	(0.015)	(0.024)	(0.016)	(0.025)	(0.020)	(0.016)	(0.022)	(0.009)	(0.012)	(0.024)
$\gamma_{\bar{k}}$	0.192	0.352	0.309	0.719	0.856	0.932	0.992	0.989	1.000	1.000	1.000	1.000
	(0.022)	(0.041)	(0.059)	(0.080)	(0.053)	(0.050)	(0.015)	(0.013)	(0.000)	(0.000)	(0.000)	(0.000)
b	-	53.96	13.57	16.81	11.14	8.86	7.07	6.48	4.80	4.03	3.81	3.20
		(23.65)	(2.08)	(3.34)	(1.39)	(0.91)	(0.80)	(0.60)	(0.48)	(0.46)	(0.02)	(0.67)
$\ln L$	-8887.13	-8520.07	-8339.72	-8269.27	-8233.90	-8217.42	-8208.84	-8203.64	-8200.27	-8199.34	-8197.46	-8196.81
						B. Two Parameters						
m_0	-	1.716	1.710	1.640	1.612	1.552	1.506	1.471	1.453	1.427	1.403	1.382
		(0.008)	(0.012)	(0.010)	(0.008)	(0.009)	(0.008)	(0.008)	(0.010)	(0.009)	(0.008)	(0.008)
b	-	5460	86.84	42.32	15.40	10.12	7.73	6.53	4.43	3.94	3.61	3.33
		(522)	(7.97)	(3.88)	(0.67)	(0.42)	(0.38)	(0.31)	(0.17)	(0.13)	(0.11)	(0.11)
$\ln L$		-8573.37	-8420.14	-8308.86	-8236.95	-8222.57	-8213.22	-8214.04	-8208.38	-8202.43	-8200.51	-8199.23

Notes: This table reports the maximum likelihood estimation of binomial MSM on the yen/U.S. dollar dataset containing $T = 9751$ daily returns. Panel A shows unrestricted estimation of the four parameters m_0, $\bar{\sigma}$, $\gamma_{\bar{k}}$, and b. In Panel B, we set $\bar{\sigma}$ equal to the sample standard deviation and γ_1 equal to $1/(4T)$, which corresponds to a lowest frequency arrival occurring on average once in a period four times the sample size. Columns correspond to the number of frequencies $\bar{k}$ in the estimated model. Asymptotic standard errors are in parenthesis.

and γ_1 can be a pragmatic empirical approach that further simplifies the estimation of MSM.

It is natural to compare the MSM maximum likelihood results with estimates from a standard volatility process. Generalized Auto-Regressive Conditional Heteroskedasticity ("GARCH", **[E82]**, **[Bo87]**) assumes that returns are of the form $r_t = h_t^{1/2} e_t$, where h_t is the conditional variance of r_t at date $t-1$. The innovations $\{e_t\}_{t \geq 1}$ are independent and identically distributed as centered Student variables with a unit variance and ν degrees of freedom. In GARCH(1,1), the conditional variance satisfies the recursion $h_{t+1} = \omega + \alpha r_t^2 + \beta h_t$, and the return process is overall defined by the four parameters ν, ω, α, and β. We estimate GARCH(1,1) on the Yen / U.S. dollar exchange rate data, and find a likelihood of -8299.20, almost 100 points lower than MSM.

The MSM model produces accurate out-of-sample forecasts, as we now show. For both MSM and GARCH we estimate the models in-sample using returns from the beginning of the sample until the end of 1995. We then use returns from the beginning of 1996 to the end of the sample to evaluate out of sample performance. Each model is used to predict the realized volatility

$$RV_{t,n} = \sum_{s=t-n+1}^{t} r_s^2$$

for forecasting horizons n ranging from 1 to 100 days. Let the out-of-sample period begin on date T_0 and assume a forecasting horizon n. The $N = T - (n-1) - T_0$ realized volatility observations in the out-of-sample period have mean $\overline{RV}_n = N^{-1} \sum_{t=T_0}^{T-(n-1)} RV_{t,n}$. The out-of-sample forecasting R^2 is given by $R^2 = 1 - MSE/TSS$, where the total sum of squares (TSS) is the out-of-sample variance of realized volatility: $TSS = N^{-1} \sum_{t=T_0}^{T-(n-1)} (RV_{t,n} - \overline{RV}_n)^2$, the mean squared error (MSE) quantifies forecast errors: $MSE = N^{-1} \sum_{t=T_0}^{T-(n-1)} [RV_{t,n} - \mathbb{E}_{t-1} RV_{t,n}]^2$, and the conditional expectation is taken under the assumption that the model holds.

Table 2 reports summary forecasting results for horizons of 1, 5, 10, 20, 50, and 100 days. In addition to the yen / dollar series, we consider three additional currencies: the euro, the British pound and the Canadian dollar, all against the U.S. dollar.[7] MSM shows robust good performance at all horizons and for all currencies, with particular strength appearing over longer horizons of 50 and 100 days. **[CF04]**, **[CFT]**, **[L08]**, **[BKM]**, **[CDS]**, **[BSZ]**, and **[I]** confirm the excellent in- and out-of-sample performance of the multifractal approach applied to a variety of financial series. **[C09]** obtains similar results with a reduced-form version of a multifractal model.

3.4. Long Memory in Volatility and Moment-Scaling. MSM generates a hyperbolic decline in the autocorrelation $\rho_q(n)$ defined in (2.6) for a range of lags n. Consider two arbitrary numbers α_1 and α_2 in the open interval $(0, 1)$. The set of integers $I_{\bar{k}} = \{n : \alpha_1 \log_b(b^{\bar{k}}) \leq \log_b n \leq \alpha_2 \log_b(b^{\bar{k}})\}$ contains a large range of intermediate lags. We show in **[CF04]**:

[7]The Euro / U.S. dollar series is obtained by splicing the Deutschemark exchange rate with the Euro exchange rate, using the official Deutschemark / Euro exchange rate instituted at the end of 1998.

TABLE 2. – VOLATILITY FORECASTS

	Horizon (Days)					
	1	5	10	20	50	100
	Forecasting R^2					
	Euro/U.S. Dollar					
Binomial MSM	0.036	0.205	0.298	0.347	0.280	0.088
GARCH	0.045	0.197	0.285	0.328	0.174	-0.396
	Japanese Yen/U.S. Dollar					
Binomial MSM	0.052	0.120	0.166	0.206	0.166	0.103
GARCH	0.051	0.094	0.101	0.071	-0.172	-0.384
	British Pound/U.S. Dollar					
Binomial MSM	0.085	0.352	0.414	0.418	0.343	0.181
GARCH	0.117	0.410	0.485	0.489	0.452	0.118
	Canadian Dollar/U.S. Dollar					
Binomial MSM	0.100	0.270	0.324	0.316	0.257	0.219
GARCH	0.142	0.430	0.574	0.574	0.378	0.170

Notes: This table summarizes out-of-sample forecasting performance for MSM and GARCH across multiple forecasting horizons. We estimate the models in-sample using data from the beginning of the sample until the end of 1995. We then use the data from the beginning of 1996 to March 30, 2012 to evaluate out of sample performance. For each model we evaluate ability to forecast realized volatility $RV_{t,n} = \sum_{s=t-n+1}^{t} r_s^2$, for forecasting horizons n ranging from 1 to 100 days as described in the text. To obtain the Euro series, we splice the Deutsche Mark / U.S. Dollar series from the beginning of the sample to December 31, 1998, with the Euro / U.S. Dollar series from January 1, 1999 onwards, using the official Deutsche Mark / Euro conversion rate on January 1, 1999 to convert the Deutsche Mark series to Euros.

THEOREM 1 (Hyperbolic autocorrelation in volatility). *Consider a fixed vector ψ and let $q > 0$. The autocorrelation in levels satisfies*

$$\lim_{\bar{k}\to+\infty}\left(\sup_{n\in I_{\bar{k}}}\left|\frac{\log\rho_q(n)}{\log n^{-\delta(q)}}-1\right|\right)=0,$$

where $\delta(q) = \log_b \mathbb{E}(M^q) - 2\log_b \mathbb{E}(M^{q/2})$.

MSM mimics the hyperbolic autocorrelograms $\log\rho_q(n) \sim -\delta(q)\log n$ exhibited by many financial series (e.g., [**D**], [**DGE**], [**BBM**]).

MSM illustrates that a Markov-chain regime-switching model can theoretically exhibit one of the defining features of long memory, a hyperbolic decline of the autocorrelogram at long lags. Fractional Brownian motions ([**K**], [**M65**]) and their discrete-time equivalents ([**MV68**], [**GJ**], [**B**]) generate hyperbolic autocorrelograms by assuming that an innovation linearly affects future periods at a hyperbolically declining weight; as a result, fractional integration tends to produce

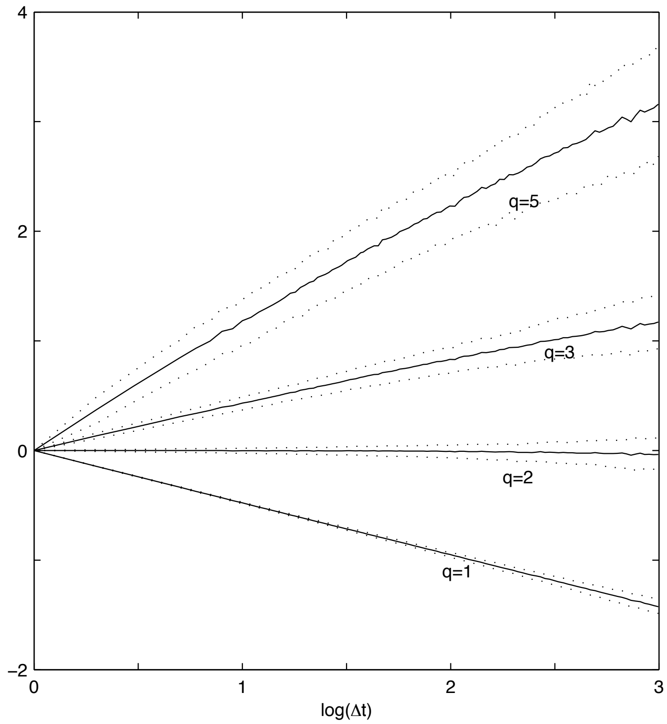


FIGURE 5. Moment Scaling of the Markov-Switching Multifractal. The figure illustrates moment scaling in binomial MSM. We simulate 500 independent samples of length $T = 20,000$ of the binomial MSM process with parameters $m_0 = 1.4$, $b = 2$, and $\gamma_1 = b/T$. For each simulated path, we calculate the partition function $S_q(T, \Delta t) \equiv \sum_{i=0}^{N-1} |p(i\Delta t + \Delta t) - X(i\Delta t)|^q$ for a set of interval lengths Δt. The solid lines plot averages across the random samples of the logarithm of the sample moments $S_q(T, \Delta t)$ against the logarithm of Δt, for moments $q = 1, 2, 3, 5$. For convenience, the lines are vertically displaced to begin at zero. Dotted lines show the 20th and 80th percentiles for each moment. The plots show approximate moment scaling, consistent with (2.8).

smooth processes. By contrast, MSM generates long cycles with a switching mechanism that also gives abrupt volatility changes. The combination of long-memory behavior with sudden volatility movements has a natural appeal for financial modeling.

MSM also captures the moment-scaling properties of financial series. Intuitively, MSM is a randomized version of the MMAR, and therefore inherits the moment-scaling properties of its precursor. Figure 5 shows moment scaling in binomial MSM. We simulate 500 random paths of length $T = 20,000$, and for each sample calculate an empirical estimate of $\mathbb{E}\left(|p(t + \Delta t) - p(t)|^q\right)$, as explained in the figure caption, for a variety of moments q. We take the averages across the random samples of the logarithm of the sample moments and plot these against the logarithm of the interval length Δt. The plots are approximately linear, consistent with the scaling relation (2.8). We refer the reader to **[CF]** for theoretical results on the asymptotic scaling of MSM and statistical tests of the ability of MSM to replicate scaling in empirical data.

3.5. Continuous-Time MSM. The MSM construction works just as well in continuous time. We now assume that time is defined on the interval $[0, +\infty)$. Given the Markov state vector

$$M_t = (M_{1,t}; M_{2,t}; \ldots; M_{\overline{k},t}) \in \mathbb{R}_+^{\bar{k}}$$

the dynamics over an infinitesimal interval are defined as follows. For each $k \in \{1, \ldots, \bar{k}\}$, a change in $M_{k,t}$ may be triggered by a Poisson arrival with intensity λ_k. The component $M_{k,t+dt}$ is drawn from a fixed distribution M if there is an arrival, and otherwise remains at its current value: $M_{k,t+dt} = M_{k,t}$. The construction can be summarized as:

$$\begin{array}{ll} M_{k,t+dt} \text{ drawn from the distribution of } M & \text{with probability } \lambda_k dt \\ M_{k,t+dt} = M_{k,t} & \text{with probability } 1 - \lambda_k dt. \end{array}$$

The Poisson arrivals and new draws from M are independent across k and t. The sample paths of a component $M_{k,t}$ are càdlàg, i.e. are right-continuous and have a limit point to the left of any instant.[8]

The arrival intensities are specified by

$$\lambda_k = \lambda_1 b^{k-1}, \qquad k \in \{1, \ldots, \bar{k}\}. \tag{3.7}$$

The parameter λ_1 determines the persistence of the lowest frequency component, and b the spacing between component frequencies.

Finally, we assume that the log price process $p(t)$ satisfies the stochastic differential equation diffusion

$$dp(t) = \mu dt + \sigma(M_t) dZ_t, \tag{3.8}$$

where Z_t is a standard Brownian motion and $\sigma(M_t)$ follows the maintained equation (3.3). The price

$$p(t) = p(0) + \mu t + \int_0^t \sigma(M_s) dZ_s \tag{3.9}$$

is a continuous Itô diffusion with constant drift μ and time-varying multifrequency volatility $\sigma(M_t)$. **[CF01]** and **[CF08]** investigate the tight link between the discrete and continuous time constructions of MSM, and show that the transition probabilities (3.1) are discretized versions of the geometric intensities (3.7). Multifrequency switches in the drift μ can also be useful for asset pricing, permitting the construction of multifrequency long-run risk models (**[BY04]**), as in **[CF07]**.

3.6. Limiting Process with Countably Many Frequencies. The MSM construction can accommodate an infinity of frequencies, as we now show. For given parameters $(\mu, \bar{\sigma}, m_0, \lambda_1, b)$, let $M_t = (M_{k,t})_{k=1}^{\infty} \in \mathbb{R}_+^{\infty}$ denote an MSM Markov state process with countably many components. Each component $M_{k,t}$ is characterized by the arrival intensity $\lambda_k = \lambda_1 b^{k-1}$. For any finite $\bar{k}$, stochastic volatility is defined as the product of the first $\bar{k}$ components of the state vector: $\sigma_{\bar{k}}(M_t) \equiv \bar{\sigma} \left(\prod_{k=1}^{\bar{k}} M_{k,t} \right)^{1/2}$.

[8] Càdlàg is a French acronym for *continue à droite, limite à gauche.*

Since instantaneous volatility $\sigma_{\bar{k}}(M_t)$ depends on an increasing number of components, the differential representation (3.8) becomes unwieldy as $\bar{k} \to \infty$. In fact, the instantaneous volatility $\sigma_{\bar{k}}(M_t)$ converges almost surely to zero as $\bar{k} \to \infty$. Since volatility is unbounded, however, the Lebesgue dominated convergence does not apply. We consider instead the time deformation

$$\theta_{\bar{k}}(t) \equiv \int_0^t \sigma_{\bar{k}}^2(M_s)ds. \tag{3.10}$$

At any given instant t, the sequence $\{\theta_{\bar{k}}(t)\}_{\bar{k}=1}^{\infty}$ is a positive martingale with bounded expectation. By the martingale convergence theorem, the random variable $\theta_{\bar{k}}(t)$ converges to a limit distribution when $\bar{k} \to \infty$. A similar argument applies to any vector sequence $\{\theta_{\bar{k}}(t_1); \ldots; \theta_{\bar{k}}(t_d)\}$, guaranteeing that the stochastic process $\theta_{\bar{k}}$ has at most one limit point. As shown in **[CF01]**, the sequence $\{\theta_{\bar{k}}\}_{\bar{k}}$ is tight[9] under the following sufficient condition.

CONDITION 1 (Tightness). $\mathbb{E}(M^2) < b$.

Intuitively, tightness prevents the time deformation $\theta_{\bar{k}}$ from oscillating too wildly as $\bar{k} \to \infty$. Correspondingly, Condition 1 imposes that the volatility shocks are sufficiently small or that their durations λ_k^{-1} decrease sufficiently fast to guarantee convergence.[10] Let $D[0, \infty)$ denote the space of càdlàg functions defined on $[0, \infty)$, and let d_∞° denote the Skohorod distance.

THEOREM 2 (Time deformation with countably many frequencies). *Under Condition 1, the sequence $(\theta_{\bar{k}})_{\bar{k}}$ weakly converges as $\bar{k} \to \infty$ to a measure θ_∞ defined on the metric space $(D[0,\infty), d_\infty^\circ)$. Furthermore, the sample paths of θ_∞ are continuous almost surely.*

The limiting time deformation θ_∞ is driven by the state vector $M_t = (M_{k,t})_{k=1}^{\infty}$ and therefore has a Markov structure analogous to MSM with a finite $\bar{k}$.

The limiting price process

$$p_\infty(t) \stackrel{d}{=} p(0) + \mu t + B[\theta_\infty(t)] \tag{3.11}$$

has sample paths that are continuous but can be more irregular than a Brownian motion at some instants. Specifically, the local variability of a sample path at a given date t is characterized by the local Hölder exponent

$$\alpha(t) = \sup\{\beta \geq 0 : |p_\infty(t + \Delta t) - p_\infty(t)| = O(|\Delta t|^\beta) \text{ as } \Delta t \to 0\}.$$

Heuristically, we can express the infinitesimal variations of the price process as being of order $(dt)^{\alpha(t)}$ around instant t. Lower values of $\alpha(t)$ correspond to more abrupt variations. Traditional jump-diffusions impose that $\alpha(t)$ be equal to 0 at points of discontinuity, and to 1/2 otherwise.[11] In a multifractal diffusion such as p_∞, however, the exponent $\alpha(t)$ takes a continuum of values in any time interval.

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[9] We refer the reader to **[Bi]** for a detailed exposition of tightness and weak convergence in function spaces.

[10] Because volatility exhibits increasingly extreme behavior as $\bar{k}$ goes up, the time deformation θ_∞ cannot be computed by taking the pointwise limit of the integrand $\sigma_{\bar{k}}^2(M_t)$ in equation (3.10). Specifically, $\sigma_{\bar{k}}^2(M_s)$ converges almost surely to zero as $\bar{k} \to \infty$ (by the Law of Large Numbers), suggesting that $\theta_\infty \equiv 0$. This conclusion would of course be misleading. For every fixed t, Condition 1 implies that $\sup_k \mathbb{E}\left[\theta_{\bar{k}}^2(t)\right] < \infty$ (**[CF]**), and the sequence $\{\theta_{\bar{k}}(t)\}_{\bar{k}}$ is therefore uniformly integrable. Hence $\mathbb{E}\theta_\infty(t) = \mathbb{E}\theta_{\bar{k}}(t) = \bar{\sigma}^2\ t > 0$.

[11] See **[Ka]** for further discussion.

3.7. Extensions. MSM has been extended along several directions. [**CFT**] considers a multivariate version of MSM that captures both the correlation in levels and the correlation in volatility of the returns on several financial assets. The likelihood function and the Bayesian filter of multivariate MSM are available analytically, as in the univariate case. Multivariate MSM captures well the joint dynamics of asset returns and provides accurate forecasts of the value at risk of a portfolio of assets.

[**I**] develops an extension of bivariate MSM that incorporates dynamic correlation in the Gaussian innovations. The new model, which the author coins MSMDCC, combines the multifrequency structure of bivariate MSM with the flexible correlation of Engle's Dynamic Conditional Correlation model [**E02**]. The likelihood and Bayesian filter of MSMDCC are available analytically. MSMDCC outperforms its two building blocks – MSM and DCC – both in and out of sample.

[**CDS**] and [**BSZ**] introduce Markov-switching multifractal models of inter-trade duration, that is the time interval between two consecutive trades on a given security. Inter-trade durations play an important role in the financial econometrics and microstructure literatures (e.g., [**ER**]) and can help design algorithmic trading strategies. The MSM duration models capture the key features of financial market inter-trade durations: long-memory dynamics and highly dispersed distributions. They also outperform their short-memory competitors in and out of sample.

4. Pricing Multifractal Risk

The integration of multifractal risk into asset pricing is now at the forefront of current research. We begin with an illustrative example drawn from [**CF08**].

4.1. An Equilibrium Model of Stock Prices. We consider an infinitely-lived asset, such as the stock of a corporation, that pays off a random cash flow D_t every period. Since the profitability of the company is impacted by multiple shocks that each have their own degrees of persistence, the cash flow process has multifractal characteristics and is therefore a source of multifractal risk. We know from financial theory that in the absence of arbitrage, the stock price at a given date t is the present value of expected future dividends, where the discount rate takes into account the risk aversion of investors ([**M**],[**DD**]). In the following example, we assume that the discount rates are obtained from the classic Lucas valuation model ([**Lu**]), as we now explain.

The model is formally defined as follows on the continuous time interval $[0, \infty)$. Let $Z(t) \in \mathbb{R}$ denote a standard Brownian motion, let $\bar{k} \in \mathbb{N}^*$, and let $M_t \in \mathbb{R}_+^{\bar{k}}$ denote an MSM state vector with $\bar{k}$ components. The processes Z and M are mutually independent. The stock pays off the continuous stream of cash flows D_t, which includes dividends and the proceeds from stock repurchases. For simplicity, we will simply refer to D_t as the dividend process.

Condition 2 (Dividends). The dividend process satisfies

$$\log(D_t) \equiv \log(D_0) + \int_0^t \left[\bar{g}_D - \frac{\sigma_D^2(M_s)}{2}\right] ds + \int_0^t \sigma_D(M_s) dZ_D(s)$$

at every instant $t \in [0, \infty)$, where $\bar{g}_D$ and $\bar{\sigma}_D$ are strictly positive elements of the real line and $\sigma_D(M_t) = \bar{\sigma}_D(\prod_{k=1}^{\bar{k}} M_{k,t})^{1/2}$.

The stock is priced by a collection of identical risk-averse agents, who observe the realization of the processes Z and M. Risk aversion is defined as follows. An agent ranks the desirability of a random consumption stream $\{C_t\}_{t\geq 0}$ according to the utility index

$$U(\{C_t\}) = \mathbb{E}\left[\int_0^{+\infty} e^{-\delta s} u(C_{t+s}) ds \,\middle|\, I_t\right],$$

where the discount rate δ is a strictly positive constant and the Bernoulli utility is:

$$u(C) \equiv \begin{cases} C^{1-\alpha}/(1-\alpha) & \text{if } \alpha \neq 1, \\ \log(C) & \text{if } \alpha = 1. \end{cases}$$

The agent strictly prefers the consumption stream $\{C_t\}$ to the consumption stream $\{C'_t\}$ if and only if $U(\{C_t\}) > U(\{C'_t\})$. We let $\rho = \delta - (1-\alpha)\bar{g}_D$, which we assume to be strictly positive. We use lower cases for the logarithms of all variables.

THEOREM 3 (Equilibrium stock price). *The stock price is in logs the sum of the continuous dividend process and the price:dividend ratio:*

$$p_t = d_t + q(M_t),$$

where

$$q(M_t) = \log \mathbb{E}\left(\int_0^{+\infty} e^{-\rho s - \frac{\alpha(1-\alpha)}{2}\int_0^s \sigma_D^2(M_{t+h})dh} ds \,\middle|\, M_t\right). \tag{4.1}$$

The price process therefore follows a jump-diffusion. A price jump occurs when there is a discontinuous change in the Markov state M_t driving the continuous dividend process.

The price jumps are endogenous implications of market pricing, and the discontinuities of the price $p(t)$ contrast with the continuous behavior of the dividend process $d(t)$. Over an infinitesimal time interval, the stock price changes by

$$d(p_t) = d(d_t) + \Delta(q_t),$$

where $\Delta(q_t) \equiv q(M_t) - q(M_{t^-})$ denotes the finite variation of the price:dividend ratio triggered by a Markov switch. If $\alpha < 1$, a switch that increases the volatility of current and future dividends induces a negative realization of $\Delta(q_t)$. Market pricing thus generates an endogenous negative correlation between volatility changes and price jumps.

The size of a jump $\Delta(q_t) = q(M_t) - q(M_{t^-})$ depends on the persistence of the component that changes. Low-frequency multipliers deliver persistent and discrete switches, which by (4.1) have a large price impact. By contrast, higher frequency components have no noticeable effect on prices, but give additional outliers in returns through their direct effect on the tails of the dividend process. The price process is therefore characterized by a large number of small jumps (high frequency $M_{k,t}$), a moderate number of moderate jumps (intermediate frequency $M_{k,t}$), and a small number of very large jumps. Earlier empirical research suggests that this is a good characterization of the dynamics of stock returns. The multifractal model avoids the difficult choice of a unique frequency and size for rare events, which is a common issue with traditional jump-diffusions.[12]

[12] In the simplest exogenously specified jump-diffusions, it is often possible that discontinuities of heterogeneous but fixed sizes and different frequencies can be aggregated into a single collective jump process with an intensity equal to the sum of all the individual jumps, and a random distribution of sizes. A comparable analogy can be made for the state vector M_t in our model,

Figure 6 illustrates the dynamics of the pricing model. The top two panels present a simulated dividend process, in growth rates and in logarithms of the level respectively. The middle two panels display the corresponding stock returns and log prices. The price series exhibits much larger movements than dividends, due to the presence of endogenous jumps in the price-dividend ratio, $e^{q(M_t)}$. To see this clearly, the bottom two panels show consecutively: 1) the "feedback" effects, defined as the difference between log stock returns and log dividend growth, and 2) the price:dividend ratio. Consistent with Theorem 3, we observe a few infrequent but large jumps in prices, with smaller but more numerous small discontinuities. The simulation demonstrates that the difference between stock returns and dividend growth can be large even when the price-dividend ratio varies in a plausible and relatively modest range. Overall, the equilibrium pricing model captures endogenous multifrequency price jumps, multifrequency stochastic volatility, and endogenous correlation between volatility and returns.

4.2. Convergence to a Multifractal Jump-Diffusion. We now investigate how the price diffusion evolves as $\bar{k} \to \infty$, i.e. as components of increasingly high frequency are added into the state vector. By Condition 2 and Theorem 2, the dividend process $d_{\bar{k}}(t)$ converges in distribution to

$$d_\infty(t) = d_0 + \bar{g}_D t - \theta_\infty(t)/2 + B[\theta_\infty(t)]$$

as $\bar{k} \to \infty$. By (4.1), the process $q_{\bar{k}}(t)$ is a positive submartingale, which also converges to a limit as $\bar{k} \to \infty$.

Theorem 4 (Jump-diffusion with countably many frequencies). *We assume that $\alpha < 1$ and that the maintained conditions 1 and 2 hold. When the number of frequencies goes to infinity, the log-price process weakly converges to*

$$p_\infty(t) \equiv d_\infty(t) + q_\infty(t),$$

where

$$q_\infty(t) = \log \mathbb{E}\left[\int_0^{+\infty} e^{-\rho s - \frac{\alpha(1-\alpha)}{2}[\theta_\infty(t+s)-\theta_\infty(t)]} ds \,\middle|\, (M_{k,t})_{k=1}^{\infty} \right]$$

is a pure jump process. The limiting price is thus a jump diffusion with countably many frequencies.

The limiting log-price process $p_\infty(t)$ is the sum of: (i) the continuous multifractal diffusion $d_\infty(t)$; and (ii) the pure jump process $q_\infty(t)$. We correspondingly call $p_\infty(t)$ a *multifractal jump-diffusion.*

When $\bar{k} = \infty$, the state space is a continuum and the multifractal jump-diffusion is tightly specified by the seven parameters $(\bar{g}_D, \bar{\sigma}_D, m_0, \gamma_1, b, \alpha, \rho)$. The limiting process $q_\infty(t)$ exhibits rich dynamic properties. Within any bounded time interval, there exists almost surely a multiplier $M_{k,t}$ that switches and triggers a jump in the stock price. Hence a jump in price occurs almost surely in the neighborhood of any instant. Furthermore, the number of switches is countable almost surely within any bounded time interval, implying that the process $q_\infty(t)$ has infinite activity and is continuous almost everywhere.

but due to the equilibrium linkages between jump size and the duration of volatility shocks, and the state dependence of price jumps, no such reduction to a single aggregated frequency is possible for the equilibrium stock price.

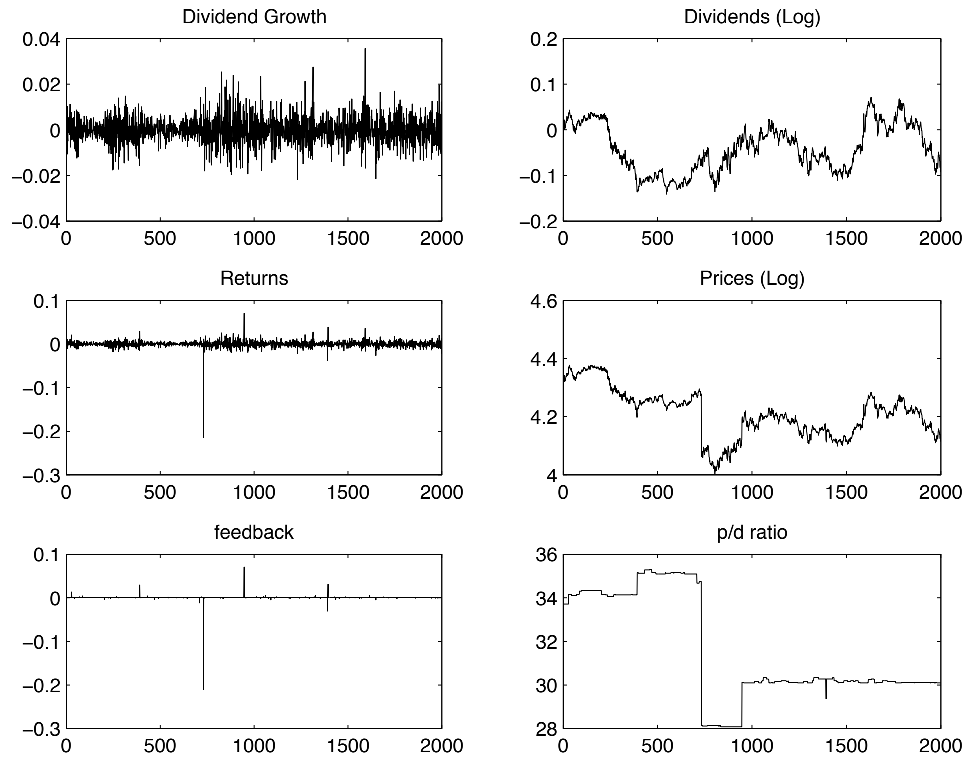

FIGURE 6. The Multifractal Jump-Diffusion. The figure illustrates the dynamics of the pricing model. The top two panels show dividend growth and dividend levels. The middle panel shows the equilibrium returns and price process, which are considerably more variable. The bottom panels isolate the endogenous portion of returns and prices. The bottom left panel displays the price jumps $\Delta(q_t) = q(M_t) - q(M_{t-})$. The bottom right panel shows the price:dividend ratio $\exp[q(M_t)]$.

The convergence results provide useful guidance on the choice of the number of frequencies in theoretical and empirical applications. On the one hand, the convergence of the price process implies that when $\bar{k}$ is large, the marginal contribution of additional components is likely to be small in applications concerned with fitting the price or return series. It is then convenient to consider a number of frequencies $\bar{k}$ that is sufficiently large to capture the heteroskedasticity of financial series, but sufficiently small to remain tractable. On the other hand, countably many frequencies might prove useful in more theoretical contexts, in which the local behavior of the price process needs to be carefully understood. Examples could include the construction of learning models or the design of dynamic hedging strategies.

4.3. Other Work. Several other papers derive the pricing implications of multifractal risk. [**CF07**] develops a discrete-time model of stock returns in which the volatility of dividend news follows an MSM. The resulting variance of stock returns is substantially higher than the variance of dividends, as is the case with the data. The MSM dividend specification improves on the classic [**CH**] model, which generates more modest amplification effects with a GARCH dividend process. [**CF07**] also investigates the dynamics of returns when the agent is not fully informed about the state M_t but must sequentially learn about it from dividends and other signals; the implied return process exhibits substantial negative skewness, which is again consistent with the data. [**Ki**] builds on [**CF07**] to explain a range of empirical findings.

Multifractal volatility has direct implications for option pricing. [**CFFL**] introduces an extension of MSM that can account for the variation in skewness and term-structure of option data. Jumps to the return process are triggered by changes in lower-frequency volatility components, and the "leverage effect" is generated by a negative correlation of high-frequency innovations to returns and volatility. Using S&P 500 index returns and a panel of options with multiple maturities and strikes, the latent volatility components enable the model to dynamically fit a wide range of option surfaces both in and out of sample.

Parsimonious models with multiple components have a natural use in interest rate modeling. [**CFW**] develops a class of dynamic term structure models that accommodates arbitrarily many interest-rate factors with a fixed number of parameters. The approach builds on a short-rate cascade, a parsimonious recursive construction that ranks the state variables by their rates of mean reversion, each revolving around the preceding lower-frequency factor. The cascade accommodates a wide range of volatility and risk premium specifications, and the forward curves implied by absence of arbitrage are smooth, dynamically consistent, and available in closed form. [**CFW**] provides conditions under which, as the number of factors goes to infinity, the construction converges to a well-defined, infinite-dimensional dynamic term structure. The cascade overcomes the curse of dimensionality associated with general affine models. Using a panel of 15 LIBOR and swap rates, [**CFW**] estimates specifications with a number of factors ranging from one to 15, all specified by only five parameters. In sample, the implied yield curve fits the data almost perfectly. Out of sample, interest rate forecasts significantly outperform prior benchmarks.

Overall, the results presented in this section show that multifractal risk has rich pricing implications that have already allowed researchers to overcome key shortcomings of standard financial models based on smaller state spaces. These early successes suggest that multifractals are promising powerful tools for asset pricing.

5. Conclusion

Fifty years ago, Benoît Mandelbrot discovered that financial returns exhibit strong departures from Gaussianity and advocated the use of self-similar Lévy-stable processes for modeling market fluctuations. These two insights sparked the introduction of fractal methods in finance. Since then, fat tails, fractional integration, and multifractal scaling have become familiar tools to financial practitioners, econometricians, statisticians and econophysicists. Fractal methods are now routinely combined with more traditional approaches, and have given rise to popular hybrid models such as fractionally integrated GARCH ([**BBM**]) or long-memory stochastic volatility ([**HMS**]). These advances are testimony to the successful integration of fractal methods into mainstream finance.

In the past fifteen years, fractal research in finance has centered on the development of multifractal models of returns, which can jointly capture fat tails, long-memory volatility persistence, multifractal moment scaling, and nonlinear changes in the distribution of returns observed over various horizons. Multifractal models capture these empirical regularities with a remarkably small number of parameters and are strong performers both in- and out-of-sample, as the empirical section of this article illustrates.

These developments in financial research have led to advances in multifractal methodology itself. Multifractal measures can now be constructed dynamically through time ([**CF01**], [**BDM**]), and several classes of multifractal diffusions are now available ([**CFM**], [**CF01**], [**BDM**]). These innovations provide new intuitions about the emergence of multifractal behavior in economic and natural phenomena. For instance, MSM shows that multifractality can be generated by a Markov process with multiple components, each of which has its own degree of persistence. MSM permits the application of efficient statistical methods, such as likelihood estimation and Bayesian filtering, to a multifractal process. These developments are new to the multifractal literature and are now spreading outside the field of finance (e.g., [**RR**]). Furthermore, incorporating multifractal risk into a pricing model generates multifractal jump-diffusions, an entirely new mathematical object that deserves further investigation.

Despite these successes, multifractal finance remains a young field and many challenges remain. The statistical methodology can be improved to incorporate finer features of financial returns, for instance along the lines of [**CFFL**]. Improvements in statistical inference are undoubtedly possible, for instance by using different distributions M, by exploring different transition probability specifications or by simplifying the estimation method. Last but not least, the integration of fractal risk into asset pricing offers considerable potential for financial economics, as illustrated by recent work on options and the term structure of interest rates.

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