Interpolation

Basics

Two continuous sets $X \in R^p$, $Y \in R^q$.

Data set: $(x_i, y_i)_{i \in [1,N]} \in X \times Y$

Take $\tilde{x} \in X \setminus \{x_i\}_{i \in [1,N]}$. What should be the matching $\tilde{y}$?

Discover implicit relation $y = f(x)$ (the model) then compute $\tilde{y} = f(\tilde{x})$.

f is chosen from a family $\mathcal{F}$ of functions parameterized by a parameter θ, the approximation family.

Interpolation vs. Regression

- *Interpolation*: f is chosen such that $\forall n, y_n = f(x_n)$
- *Regression*: f is chosen so as to minimize a fitness criterium such as
  - $\min_f \sum_n (y_n - f(x_n))^2$
  - or $\min_\theta \sum_n (y_n - f(x_n; \theta))^2 + \lambda ||\theta||^2$ with $\lambda > 0$
- Remarks:
  - some applied mathematicians tend to mix the two (interpolate=evaluate f outside of X)

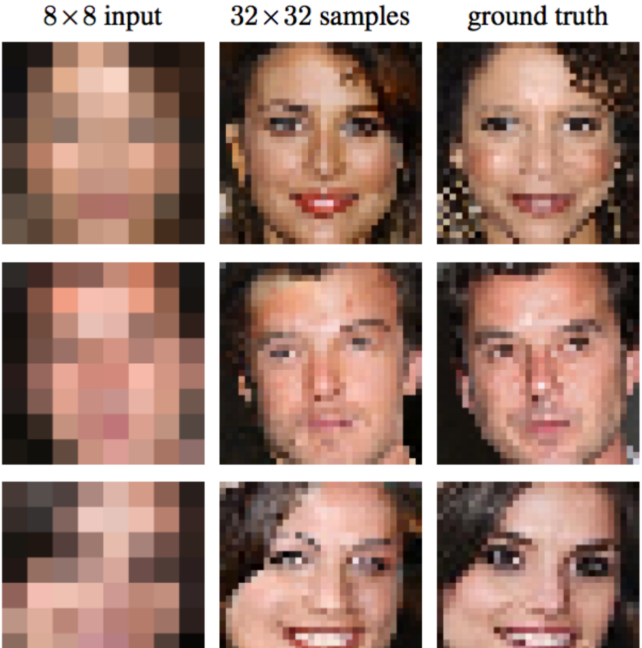
Examples (1): Linear Interpolation

1d Graph. Join the dots. Linear/Spline

2d Graph: Regression

Conclusion: interpolate only if f is known precisely on X

Example (2)



Example (2)

- X and Y: large databases of low and high resolutions images
- $\mathcal{F}$: neural network

Why do we need it?

In economics, we often solve a problem $\Phi(f) = 0$ where f is a function: $\forall s, \Phi(f)(s) = 0$.

If we approximate f by some element $f(;\theta) \in \mathcal{F}$ we just need to identify a finite set of parameters θ.

Several interpolation flavours

- local vs spectral:
  - local: functions in f have compact support
  - spectral: noncompact support
- linear vs nonlinear:
  - $\mathcal{F}$ is a vector space: $f(x) \approx \sum_{i=1}^{N} \theta_n b_n(x)$ where b_n is a base of $\mathcal{F}$
  - nonlinear: wavelets, neural networks, ...

Splines

Linear

- Take function f defined on an interval $[a, b]$. Suppose the value is known at $(a = x_1, ...x_N = b)$. Denote $y_i = f(x_i)$.
- Join the dots: define a piecewise linear function as

$$\forall x \in [x_i, x_{i+1}], \tilde{f}(x) = y_i + \underbrace{\frac{x - x_i}{x_{i+1} - x_i}}_{\text{barycentric coordinate}} (y_{i+1} - y_i)$$

Linear

- Alternate view:

$$\tilde{f}(x) = \sum_{i=1}^{N} y_i B_1^i(x)$$

where $b_1^i(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}.1_{x \in [x_{i-1}, x_i]} + (1 - \frac{x - x_i}{x_{i+1} - x_i}).1_{x \in [x_i, x_{i+1}]}$

- (B^i) is an interpolation basis

Splines

- n-th order *spline*: piecewise polynomial function that is n times differentiable except on a finite set of break points (aka *knots*), where it is $(n-1)$ times differentiable.
- in practice the data points are the breakpoints
- example: order 2
  - suppose $\tilde{f}(x_i)$ and $\tilde{f}'(x_i)$ are known, choose the coefficients for the patch $p_{i+1}(x) = a_{i+1}x^2 + b_{i+1}x + c_{i+1}$
  - Already two constraints. Condition $p_{i+1}(x_{i+1}) = \tilde{f}(x_{i+1})$ supplies another one.
  - Do it for every patch. Not that it requires to set $f'(a)$ beforehand.

Basis Splines (much better)

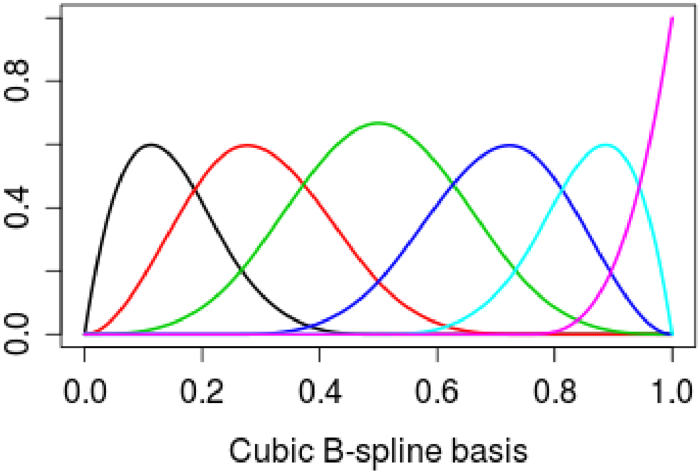
Define

$$B_{i,1}(x) = 1_{x \in [x_i, x_{i+1}]}$$

$$B_{i,k+1}(x) = \frac{x - x_i}{x_{i+k} - x_i} B_{i,k}(x) + \frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}} B_{i+1,k}(x)$$

- Theorem: Any spline of order k on the knots (x_i) can be expressed as a linear combination of the basis splines $(B_{i,k})$.
- All basis splines have compact support.
- If grid is regularly spaced there is B_k such that $B_{i,k}(x) = B_k(x - x_i)$

Basis splines



Basis splines are not interpolating

- Unfortunately basis splines are not "interpolating" in the sense that in general

$$f(x_i) \neq \sum_n f(x_n) B_{n,k}(x_i)$$

- One must choose other coefficients (c_n) which satisfy:

$$y_i = \sum_n c_n B_{n,k}(x_i)$$

  - there are more coefficients than data points: requires boundary conditions
    - f''=0: natural spline
  - going from y_n to c_n is called *prefiltering*

In practice: Interpolations

```
f(x) = log(x)
xs = 1:0.2:5
A = [f(x) for x in xs]

# linear interpolation
interp_linear = LinearInterpolation(xs, A)
interp_linear(1.3) # interpolate

# cubic spline interpolation
interp_cubic = CubicSplineInterpolation(xs, A)
interp_cubic(1.3) # interpolate
```

In practice: Interpolations (2)

Note that in $y_i = \sum_n c_n B_{n,k}(x_i)$, y_i and c_n could perfectly well be vectors. If we use a Vector type which implements all operations (zeros, *, ...) we can interpolate them with the same operations

```
using StaticArrays

f(x) = SVector(log(x), exp(x))
xs = 1:0.2:5
A = [f(x) for x in xs]

# linear interpolation
interp_linear = LinearInterpolation(xs, A)
interp_linear(1.3) # returns a 2d SVector
```

Polynomial approximation

Mental break: matrix conditioning

- Suppose you want to solve vector equation $Ax = y$. Will a small error in y affect a lot the value of x? (in particular round-off errors)
  - condition number: $\lim_{\epsilon \to 0} \sup_{\delta y \leq \epsilon} \frac{\delta x}{\delta y}$
  - or $\kappa(A) = ||A^{-1}||||A||$ where $||||$ is a subordonate norm.
  - if very-large: the matrix is ill conditioned
- What makes a matrix ill-conditioned?
  - some rows/columns are very small, others are gigantic
  - rows/columns are almost colinear

Fitting polynomials

Let's approximate: $f(;\theta) = \sum_{n=0}^{K} \theta_k x^k$.

We need $(K+1)$ points to fit a polynomial of order K. Let's take grid points $(x_0, ...x_K)$ and denote $y_k = f(x_k)$

We need to solve in $(\theta_k)_{k=[0,K]}$:

$$\forall n \in [0, K], \underbrace{\sum_k \theta_k (x_n)^k}_{M\theta} = y_k$$

Vandermonde Matrix

$$M = \begin{bmatrix} 1 & x_0 & x_0^2 \cdots & x_0^K \\ 1 & x_1 & x_1^2 \cdots & x_1^K \\ 1 & x_2 & x_2^2 \cdots & x_2^K \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_K & x_K^2 \cdots & x_K^K \end{bmatrix}$$

- Vandermonde matrix is ill-conditioned if points are too close or if K is high.

Orthogonal polynomials

- Define a scalar product over functions on the domain $[a, b]$ by choosing a positive weight function $w(x)$.

$$< P, Q >= \int_a^b w(x)P(x)Q(x)dx$$

- Construct an orthogonal base $(T_n)_{n=[1,K]}$.
- Approximate

$$f(x) \approx f(x;\theta) = \sum_{n=0}^{K} \theta_n T_n(x) = \sum_{n=0}^{K} < f|T_n > T_n(x)$$

  - this is optimal for the norm associated to $<>$ (projection on the orthogonal base)

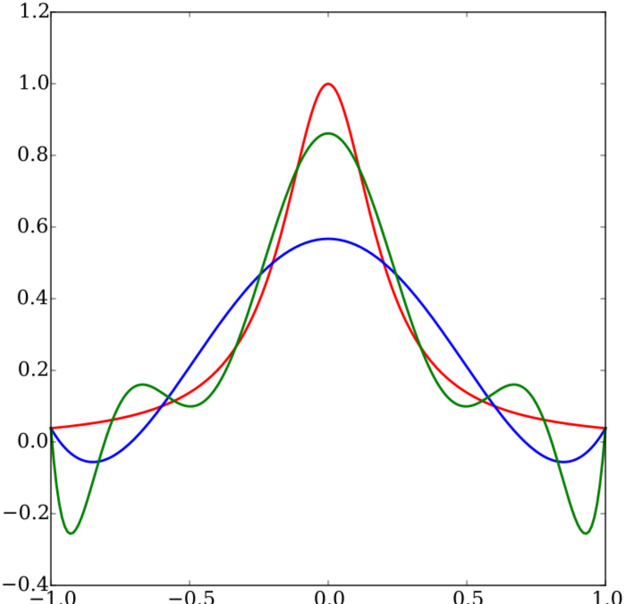
Vandermonde matrix

Coefficients can still be identified by inverting:

$$\forall n \in [0, K] \underbrace{\sum_k \theta_k T_k(x_n)}_{M\theta} = y_n$$

$$M = \begin{bmatrix} T_0(x_0) & T_1(x_0) & \cdots & T_K(x_0) \\ T_0(x_1) & T_1(x_1) & \cdots & T_K(x_1) \\ T_0(x_2) & T_1(x_2) & \cdots & T_K(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_K) & T_1(x_K) & \cdots & T_K(x_K) \end{bmatrix}$$

Problem: Runge error



Problem: Runge error (2)

- Red: Runge function $f(x) = \frac{1}{1+25x^2}$
- When interpolation order increases, over regularly spaced-points,oscillation increase.
-

$$\boxed{\text{Does it contradict Stone-Weierstrass theorem ? No.}}$$

Chebychev Nodes

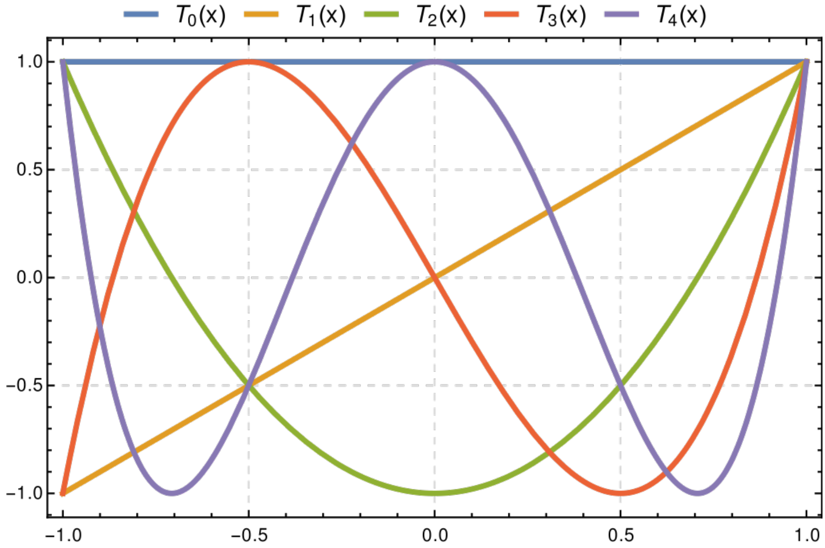
- There is an optimal way to choose the interpolation points:
  - the roots of $cos(\frac{2k-1}{2n}\pi)$ for [-1,1]
  - rescale for a finite interval [a,b]
- for the interpolating polynomial:

$$|f(x) - P_n(x)| \leq \frac{1}{2^n(n+1)!} \max_{\xi \in [-1,1]} |f^n(\xi)|$$

Chebychev polynomials

- Chebychev polynomials (of the first kind) have their zeros on the nodes.
- Definitions:
  - $T_n(x) = \cos(n \arccos(x))$ (in [0,1])
  - recursive: $T_0(x) = 1$, $T_1(x) = x$,
    $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$
- Very good choice:
  - matrix M is well conditioned: $\sqrt{2}$

Chebychev Polynomial



Additional topics

Multidimensional interpolation

Consider a function f defined on a space $X_1 \times X_d$ Take d grids $\mathcal{G}_1 \subset X_1, ..., \mathcal{G}_d \subset X_d$ with approximation bases $\mathcal{B}_1 = (b_1^1, ...b_1^{N_1}), ..., \mathcal{B}_d = (b_d^1, ...b_d^{N_d})$.

Then f can be approximated by

$$f(x_1, ...x_d; \theta) = \sum_{i_1=1}^{N_1} ... \sum_{i_d=1}^{N_d} \theta_{i_1,...i_d} \underbrace{b_{i_1}^1(x_1)...b_{i_d}^d(x_d)}_{\text{Product Base}}$$

Multidimensional interpolation (comments)

- Coefficients are still the solution of a linear system:

$$M\theta = y$$

  - but M has a special structure (tensor product)
- Number of coefficients to determine increases exponentially with number of dimensions:
  - "Curse of Dimensionality"
- Remedies: sparse grids, adaptive approximation, etc...

In Practice:

```
using BasisMatrices
```

Collocation

Back to original functional equation: $\Phi(f)(x)$. We want an approximation of f.

- choose the domain: $x \in [a, b]$
- choose an approximation grid $(x_1, ...x_N)$, and matching approximation basis $(b_1, ...b_N)$ so that $f(x; \theta) = \sum_n \theta_n b_n(x)$
- for any $\theta = (\theta_1, ...\theta_N)$ we can compute $\Phi(f(; \theta))(x)$
- we need N conditions to pin down $(\theta_1, ...\theta_N)$
  - *collocation*: set $\Phi(f(; \theta))(x_1) = ... = \Phi(f(; \theta))(x_N) = 0$ to pin down $\theta_1, ...\theta_N$
  - does not guarantee behaviour between the points, but they are well chosen