Scalar field & its gradient, Laplacian operators

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Function on a mesh M={V,E,F}

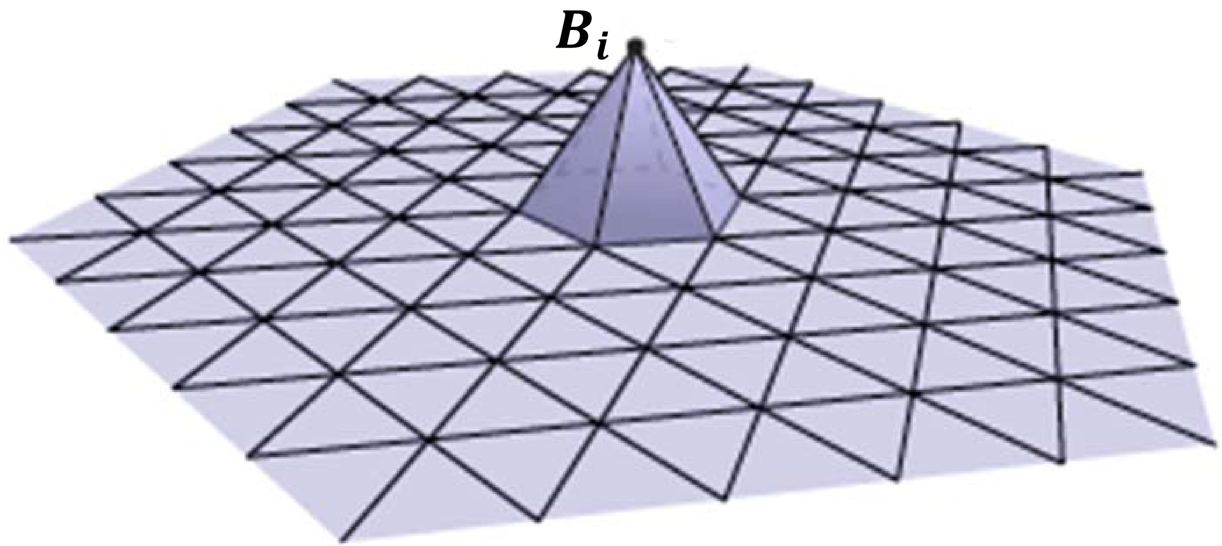
- A function is a discrete set of values or vectors defined at each vertex location. Then we have these two definitions,
- **Definition 2.1 (Scalar function on a mesh)**
- *A scalar function **f** on a mesh with n vertices is a discrete set of values defined at each vertex, and can be viewed equivalently as an n-vector, that is*

$$f : \begin{cases} \mathcal{V} & \longrightarrow & \mathbb{R} \\ x_i & \longmapsto & f(x_i) \end{cases} \iff f : \begin{cases} V & \longrightarrow & \mathbb{R} \\ i & \longmapsto & f_i \end{cases} \iff f = (f_i)_{i \in V} \in \mathbb{R}^n.$$

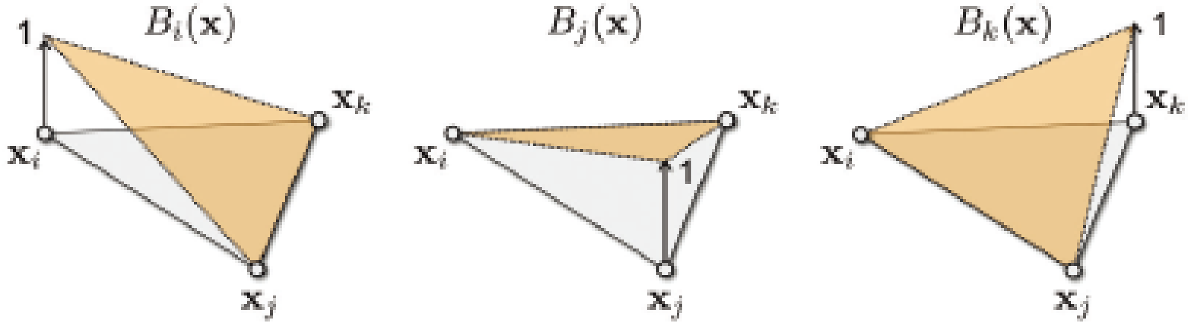
- **Definition 2.2 (Vector function on a mesh)**
- *A d-vector function on a mesh with n vertices is a discrete set of d-vectors defined at each vertex, and can be viewed equivalently as an n*d matrix, that is* $f = (f_i)_{i \in V} \in R^{n*d}$

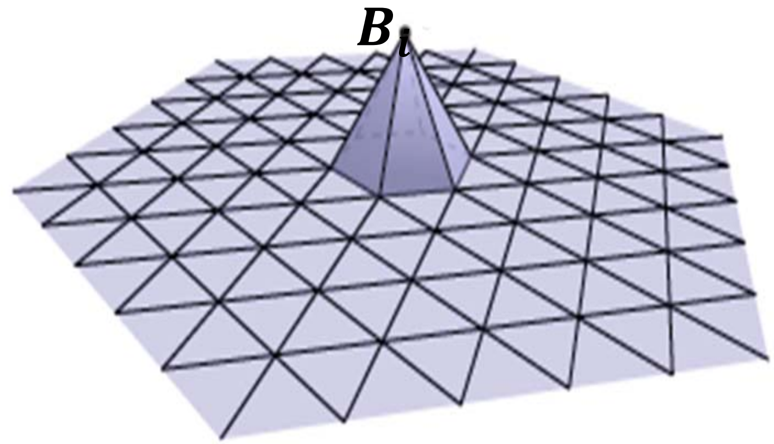
Discrete piecewise linear function

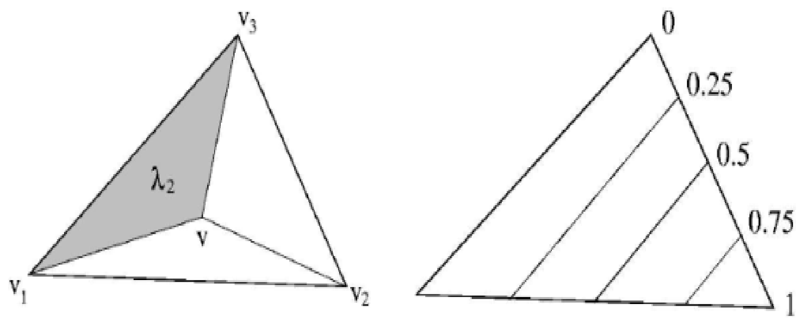
- $\mathrm{f}(\mathbf{x}) = \sum_{\mathrm{i}} \mathrm{B_i}(\mathbf{x})\mathrm{f_i}$, with basis function $\mathrm{B_i}$ being the piecewise-linear hat function valued 1 at vertex I and 0 at all other vertices.



The linear basis functions for barycentric interpolation on a triangle.

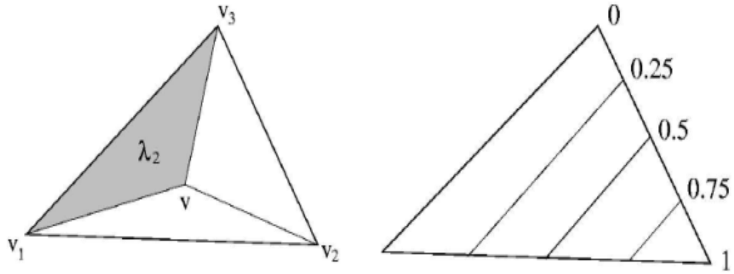






Triangle barycentric coordinates. Left: λ_2; Right: iso-curve of λ_2.

Triangle barycentric coordinates



- Barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3)$are such a coordinate system. Since three numbers are used when only two are needed, we **add one other condition**: $\sum_i \lambda_i = 1$.

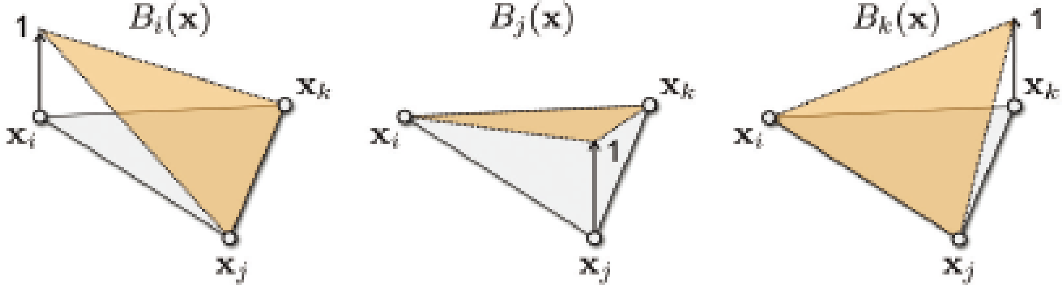
- All the followings follow from Ceva' s Theorem **[Ceva1678]** that for any point v inside a planar triangle $T[v_1, v_2, v_3]$:
- $V = \frac{\sum_i \omega_i v_i}{\sum_i \omega_i} = \sum_i \lambda_i v_i$
- where $\{\omega_i\}$ are called Homogeneous barycentric coordinate (HBC); $\{\lambda_1 = \frac{A(v,v_2,v_3)}{A(v_1,v_2,v_3)}, \lambda_2 = \frac{A(v_1,v,v_3)}{A(v_1,v_2,v_3)}, \lambda_3 = \frac{A(v_1,v_2,v)}{A(v_1,v_2,v_3)}\}$ are called **Normalized barycentric coordinates** (NBC).

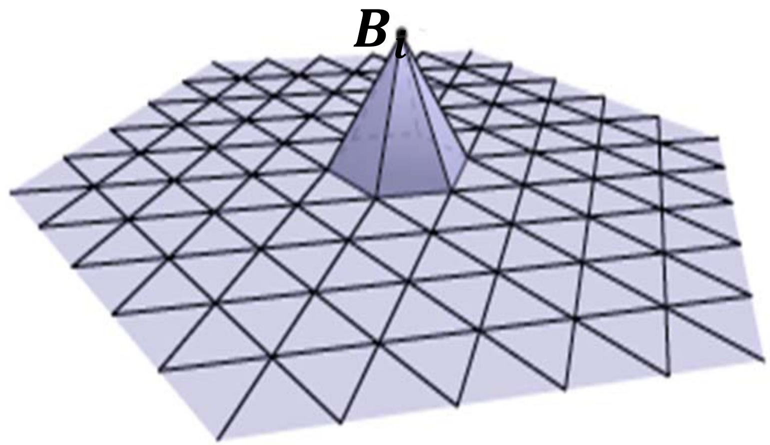
- $\omega_1(v) = A(v, v_2, v_3) = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix} = \frac{1}{2}[x(y_2 - y_3) + y(x_3 - x_2) + x_2 y_3 - x_3 y_2]$ is linear in v.
- **Mobius [Mobius1827]** was the first to study ω_i and he defined ω_i as the barycentric coordinates of v.

Function on mesh

$$f(v_i) = f(\mathbf{x}_i) = f(\mathbf{u}_i) = f_i$$

- $\mathrm{f}(\boldsymbol{u}) = \sum_{\mathrm{i}} \mathrm{B_i}(\boldsymbol{u})\mathrm{f_i}$,
- Where **u**=(u,v) is the parameter pair corresponding to the surface point x in a 2D conformal parameterization induced by the triangle.





Operators on a mesh

Operators on a mesh M={V,E,F}

- **Definition 2.3 (Linear operator A)**
- *A linear operator is defined as* $A = (a_{ij})_{i,j \in V} \in \mathbb{R}^{n \times n}$ *(matrix).*
- *And operate on a function as follow*

$$(Af)(x_i) = \sum_{j \in V} a_{ij} f(x_j) \Longleftrightarrow \quad (Af)_i = \sum_{j \in V} a_{ij} f_j .$$

- **Definition2. 4 (Local operator)**
- *A local operator* $W \in R^{n*n}$ *satisfies* $w_i = 0,\ if\ (i,j) \notin E$ *if*, *that is*

$$(\mathbf{W}f)_i = \sum_{(i,j) \in E} w_{ij} f_j$$

In most applications, we restrict our attention to local operators that can be conveniently stored as sparse matrices.

Normalization and some famous weights $(\mathbf{W}f)_i = \sum_{(i,j)\in E} w_{ij} f_j$

A particularly important class of local operators are local smoothings (also called filterings) that perform a local weighted sum around each vertex of the mesh. For this averaging to be consistent, we define a normalized operator W˜ whose set of weights sum to one

Definition 9 (Local averaging operator). *A local normalized averaging is* $\tilde{W} = (\tilde{w}_{ij})_{i,j\in V} \geqslant 0$ *where*

$$\forall (i,j) \in E, \quad \tilde{w}_{ij} = \frac{w_{ij}}{\sum_{(i,j)\in E} w_{ij}}.$$

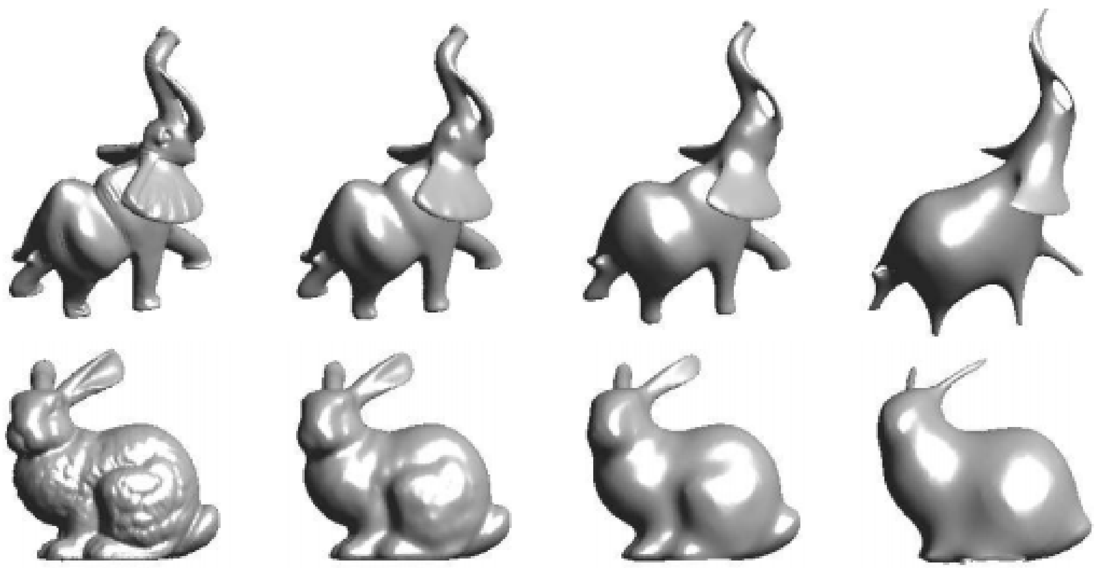
It can be equivalently expressed in matrix form as

$$\tilde{W} = D^{-1}W \quad \textit{with} \quad D = \mathrm{diag}_i(d_i) \quad \textit{where} \quad d_i = \sum_{(i,j)\in E} w_{ij}.$$

- Combinatorial/uniform weights $\forall (i,j) \in E, \quad w_{ij} = 1.$
- Distance weights $\forall (i,j) \in E, \quad w_{ij} = \frac{1}{\|x_j - x_i\|^2}.$

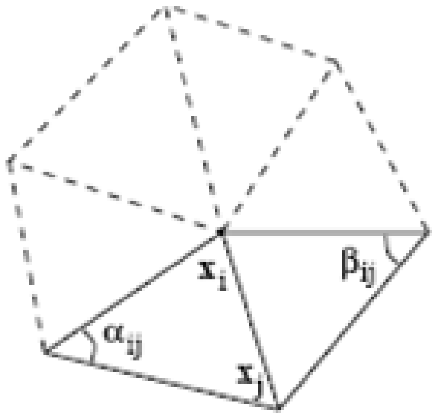
An **example** of such iterations applied to the three coordinates of mesh.

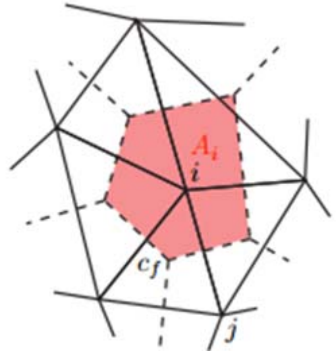
- One can use iteratively a smoothing in order to further filter a function on a mesh. The resulting vectors Wf, W^2 ,. .. , W^kf are increasingly smoothed version of f.
- The sharp features of the mesh tend to disappear during iterations.



Normalization and some famous weights

$$(\mathbf{W}f)_i = \sum_{(i,j)\in E} w_{ij} f_j$$





Cotangent weights

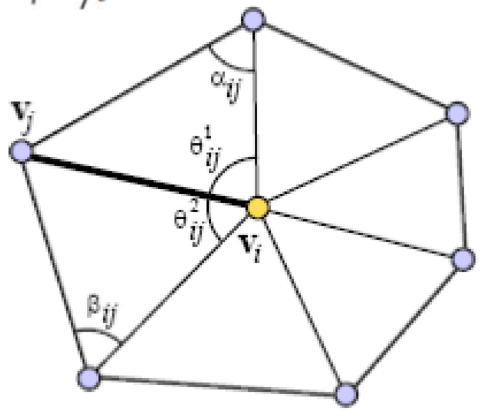
$$w_{ij} = \cot \alpha_{ij} + \cot \beta_{ij}\,.$$

Mean curvature weights

$$w_{ij} = \frac{1}{4A(v_i)}(\cot \alpha_{ij} + \cot \beta_{ij})\,.$$

Mean value weights

$$w_{ij} = \frac{\tan(\theta_{ij}^1/2) + \tan(\theta_{ij}^2/2)}{\left\| v_i - v_j \right\|}.$$



03 Discrete Differential-Geometry Operators for Triangulated 2-Manifolds

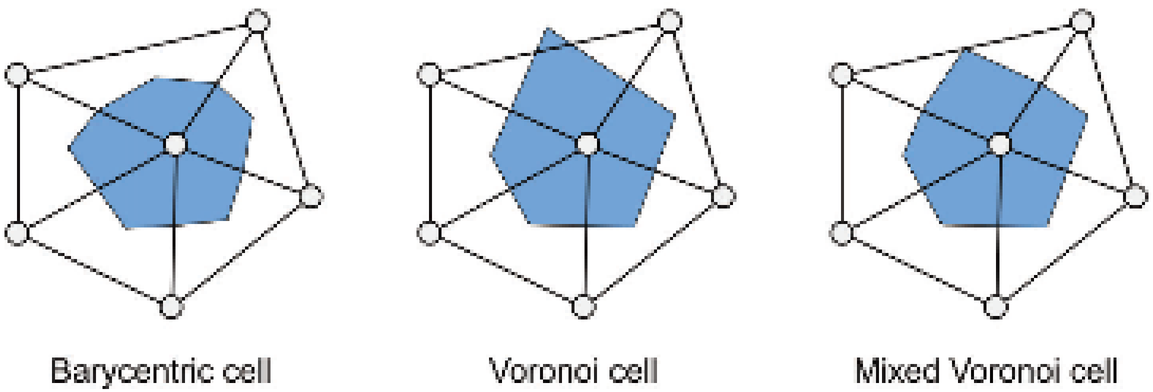
Approximating Integrals on a Mesh

- In the continuous domain, **filtering** is defined through **integration** of functions over the mesh.

- In order to descretize integrals, one needs to define a partition of the mesh into small cells centered around a vertex or an edge.

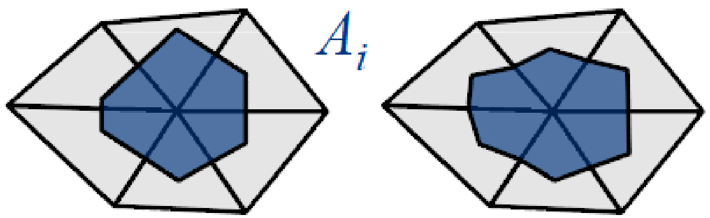
Local Averaging Region

- N-ring neighborhoods Nn(vi) of vertex vi, or
- Local geodesic balls



Mixed Regions

$\mathcal{A}_{\text{Mixed}} = 0$
For each triangle T from the 1-ring neighborhood of $\mathbf{x}$
    If T is non-obtuse,   `// Voronoi safe`
        `// Add Voronoi formula (see Section 3.3)`
        $\mathcal{A}_{\text{Mixed}} += $ Voronoi region of $\mathbf{x}$ in T
    Else   `// Voronoi inappropriate`
        `// Add either area(`T`)/4 or area(`T`)/2`
        If the angle of T at $\mathbf{x}$ is obtuse
            $\mathcal{A}_{\text{Mixed}} += \text{area}(T)/2$
        Else
            $\mathcal{A}_{\text{Mixed}} += \text{area}(T)/4$



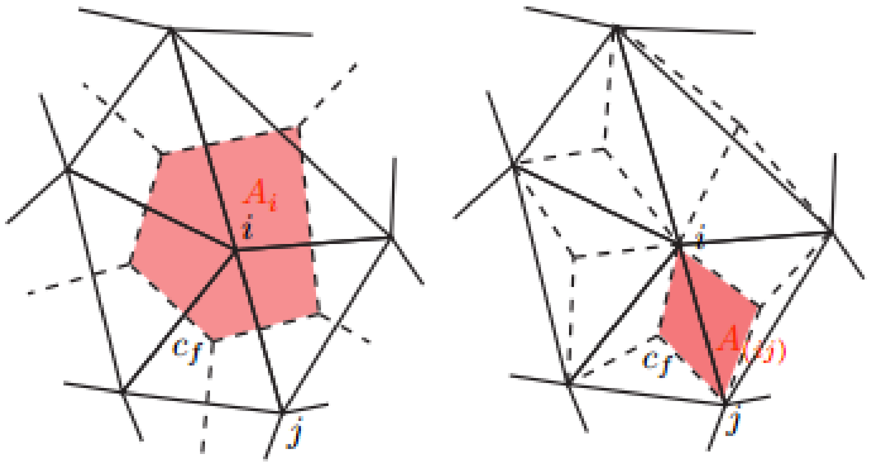
Approximating Integrals on a Mesh - Voronoi

Definition 10 (Vertices Voronoi). *The Voronoi diagram associated to the vertices is*

$$\forall i \in V, \quad E_i = \{x \in \mathcal{M} \setminus \forall j \neq i, \|x - x_i\| \leqslant \|x - x_j\|\}$$

Definition 11 (Edges Voronoi). *The Voronoi diagram associated to the edges is*

$$\forall e = (i, j) \in E, \quad E_e = \{x \in \mathcal{M} \setminus \forall e' \neq e, d(x, e) \leqslant d(x, e')\}$$



These Voronoi cells indeed form a partition of the mesh

$$\mathcal{M} = \bigcup_{i \in V} E_i = \bigcup_{e \in E} E_e.$$

Meyer, 2003, **Course;** ***Discrete Differential-Geometry Operators for Triangulated 2-Manifolds***

Approximating Integrals on a Mesh

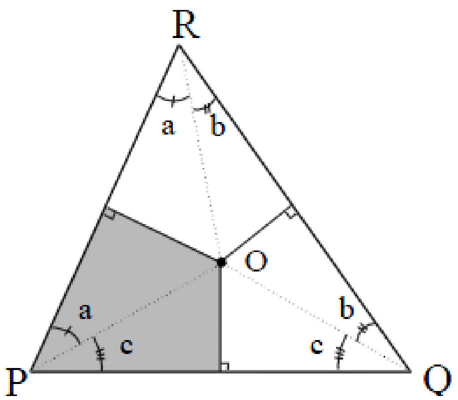
Theorem 1 (Voronoi area formulas). *For all $e = (i, j) \in E$, $\forall\, i \in V$, one has*

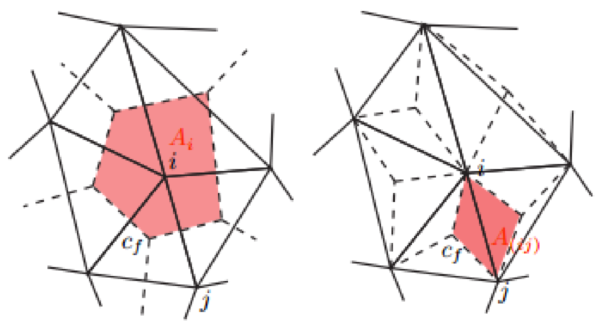
$$A_e = Area(E_e) = \frac{1}{2}\|x_i - x_j\|^2 \left(\cot(\alpha_{ij}) + \cot(\beta_{ij})\right)$$

$$A_i = Area(E_i) = \frac{1}{2}\sum_{j \in N_i} A_{(ij)}.$$

With these areas, one can approximate integrals on vertices and edges using

$$\int_{\mathcal{M}} f(x)\mathrm{d}x \approx \sum_{i \in V} A_i\, f(x_i) \approx \sum_{e=(i,j)\in E} A_e\, f([x_i, x_j]).$$





Dirichlet's energy of a function ($\mathbf{f}: \mathbf{M} \to \mathbf{R}, \mathbf{M} \sqsubseteq R^n$) on a manifold:

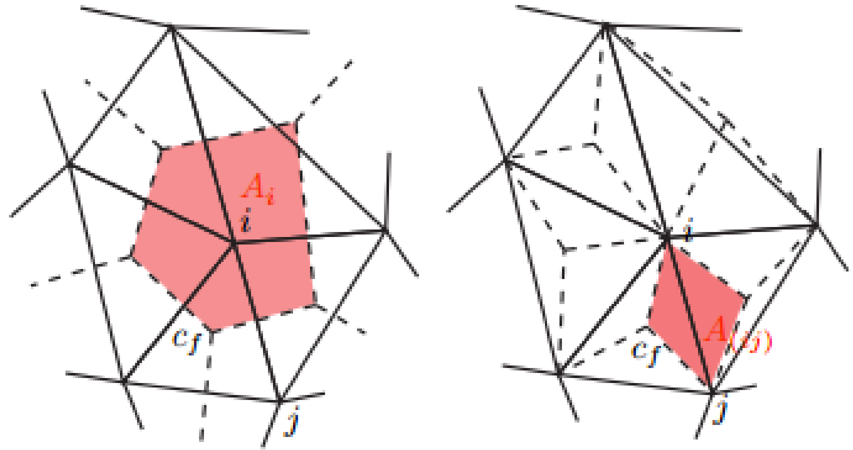
- **Dirichlet's energy** is a measure of how *variable* a function is.
- Solutions to $\Delta \mathbf{f} = \mathbf{0}$ are functions that make the Dirichlet energy functional stationary:

$$\mathrm{E_D(f)} = \frac{1}{2}\iint_{\mathrm{M}} |\nabla \mathrm{f}|^2 \mathrm{dudv}$$

- i.e. the Euler equation of the Dirichlet problem is a **Laplacian equation**: $\Delta \mathrm{f} = 0$

$$\int_{\mathcal{M}} f(x)\mathrm{d}x \approx \sum_{i\in V} A_i\, f(x_i) \approx \sum_{e=(i,j)\in E} A_e\, f([x_i, x_j]).$$

$$A_e = Area(E_e) = \frac{1}{2}\|x_i - x_j\|^2\,(\cot(\alpha_{ij}) + \cot(\beta_{ij}))$$



$$\int_{\mathcal{M}} \|\nabla_x f\|^2 \mathrm{d}x \approx \sum_{(i,j)\in E} A_{(i,j)} \frac{|f(x_j) - f(x_i)|^2}{\|x_j - x_i\|^2}$$

$$= \sum_{(i,j)\in E} w_{ij}|f(x_j) - f(x_i)|^2 \qquad \text{where} \qquad w_{ij} = \cot(\alpha_{ij}) + \cot(\beta_{ij}).$$

Approximate two vector fields

- M is a 2D manifold, and $\chi = (\mathrm{u}(\mathrm{x},\mathrm{y},\mathrm{z}), \mathrm{v}(\mathrm{x},\mathrm{y},\mathrm{z}))$ is an unknown map (vector function) defined on M. $\mathrm{G} = \begin{pmatrix} \mathrm{g}_1 \\ \mathrm{g}_2 \end{pmatrix}$ is a known 2*2 **tensor field** defined on M, formed by two vector fields $\mathbf{g_1}$ **and** $\mathbf{g_2}$.
- If we want find a χ, using $\nabla\chi$ to approximate the tensor field G, we can get the following formula:
- $\min\limits_{\chi} \int_{\mathrm{M}} \|\nabla\chi - \mathrm{G}\|^2 = \min\limits_{(\mathrm{u},\mathrm{v})} \int_{\mathrm{M}} \|\nabla \mathrm{u} - \mathrm{g}_1\|^2 + \|\nabla \mathrm{v} - \mathrm{g}_2\|^2 = \min\limits_{(\mathrm{u},\mathrm{v})} \int_{\mathrm{M}} \mathrm{F}(\mathrm{u}, \mathrm{v}, \nabla \mathrm{u}, \nabla \mathrm{v})$

Poisson Equation

- $\min\limits_{\chi} \int_{M} \|\nabla\chi - G\|^2 = \min\limits_{(u,v)} \int_{M} \|\nabla u - g_1\|^2 + \|\nabla v - g_2\|^2 = \min\limits_{(u,v)} \int_{M} F(u, v, \nabla u, \nabla v)$

- The Euler equation of the above problem is:
- $\frac{\partial E(u,v)}{\partial u} = -\mathrm{div}(\nabla u - g_1) = 0, \frac{\partial E(u,v)}{\partial v} = -\mathrm{div}(\nabla v - g_1) = 0$
- i.e. a **Poisson Equation**
- $\begin{cases} \Delta u = \mathrm{div}(g_1) \\ \Delta v = \mathrm{div}(g_2) \end{cases}$ or $\Delta\chi = \mathrm{div}(G)$

Dirichlet energy

- When G is zero everywhere, we get the **Dirichlet energy**:
- $\mathrm{E_D}(\chi) = \frac{1}{2}\iint_{\mathrm{M}} \|\nabla\chi\|^2 \mathrm{ds}$
- The Euler equation of the Dirichlet problem is a **Laplacian equation**:
- $\Delta\chi = \mathrm{div}(\mathrm{G}) = 0$
- The **heat equation** is: $\mathrm{k}\Delta\chi = \frac{\partial\chi}{\partial \mathrm{t}}$

Variational: Euler equation

单元单标量函数	$E(u) = \int_0^{`} F(x, u, u', u'')\, dx$
	$\frac{\partial \mathbf{F}}{\partial \mathbf{u}} - \frac{\mathbf{d}}{\mathbf{dx}}\left(\frac{\partial \mathbf{F}}{\partial \mathbf{u}'}\right) + \frac{\mathbf{d}^2}{\mathbf{dx}^2}\left(\frac{\partial \mathbf{F}}{\partial \mathbf{u}''}\right) = 0$
多元单标量函数	$E(u) = \iint_\Omega F(x, y, u, u_x, u_y, u_{xx}, u_{yy})\, dxdy$
	$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) - \frac{d}{dy}\left(\frac{\partial F}{\partial u_y}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u_{xx}}\right) + \frac{d^2}{dy^2}\left(\frac{\partial F}{\partial u_{yy}}\right) = 0$ $= \mathbf{F_u} - \mathbf{div}\left(\mathbf{F_{u_x}}, \mathbf{F_{u_y}}\right) + \Delta\left(\mathbf{F_{u_{xx}}}, \mathbf{F_{u_{yy}}}\right) = 0$
多元多标量函数 Multi multivariable (scalar) function	$E[u, v] = \int_\Omega F\left(x, y, u, u_x, u_y, v, v_x, v_y\right) dxdy$
	$\begin{cases} \mathbf{F_u} - \mathbf{div}\left(\mathbf{F_{u_x}}, \mathbf{F_{u_y}}\right) = 0 \\ \mathbf{F_v} - \mathbf{div}\left(\mathbf{F_{v_x}}, \mathbf{F_{v_y}}\right) = 0 \end{cases}$

Gradient operator

Gradient operator – for edges

$$\forall\, (i,j) \in E,\; i < j, \quad (Gf)_{(i,j)} \stackrel{\text{def.}}{=} \sqrt{w_{ij}}(f_j - f_i) \in \mathbb{R}.$$

$$w_{ij} = \|x_i - x_j\|^{-2}, \quad (Gf)_{(i,j)} = \frac{f(x_j) - f(x_i)}{\|x_i - x_j\|}$$

which is exactly the finite difference discretization of a directional derivative.

Gradient operator – for faces

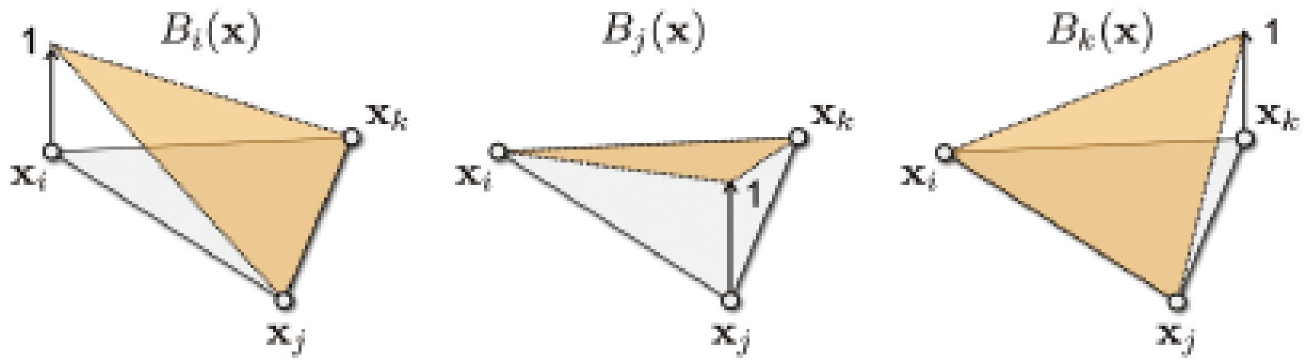
- $\mathrm{f}(\boldsymbol{u}) = \sum_{\mathrm{i}} \mathrm{B_i}(\boldsymbol{u})\mathrm{f_i}$

$$\nabla f(\mathbf{u}) = f_i \nabla B_i(\mathbf{u}) + f_j \nabla B_j(\mathbf{u}) + f_k \nabla B_k(\mathbf{u})$$

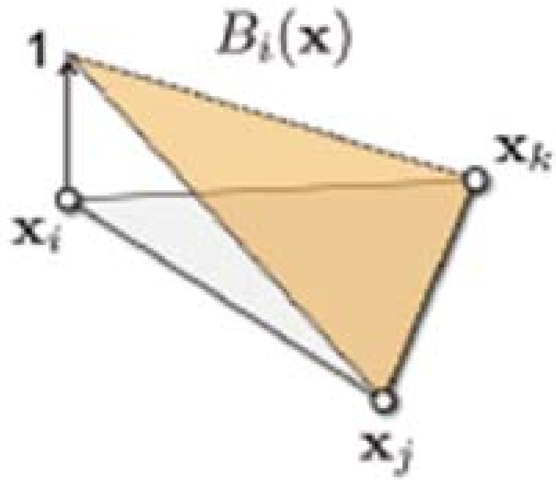
$$B_i(\mathbf{u}) + B_j(\mathbf{u}) + B_k(\mathbf{u}) = 1 \quad \Rightarrow \quad \nabla B_i(\mathbf{u}) + \nabla B_j(\mathbf{u}) + \nabla B_k(\mathbf{u}) = 0.$$

$$\nabla f(\mathbf{u}) = (f_j - f_i)\nabla B_j(\mathbf{u}) + (f_k - f_i)\nabla B_k(\mathbf{u})$$

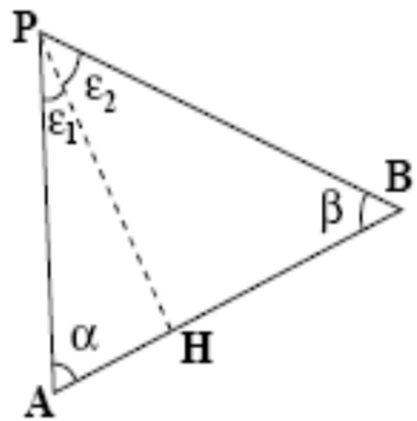
$$\boxed{\nabla B_i(\mathbf{u}) = \frac{(\mathbf{x}_k - \mathbf{x}_j)^{\perp}}{2A_T},}$$



$$\nabla B_i(u) = \frac{(x_k - x_j)^{\perp}}{2A_T}$$

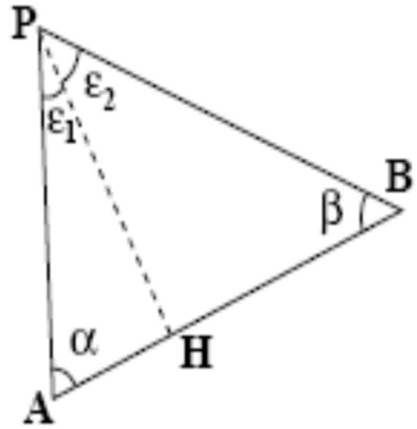


- $B_i(u) = \frac{A(x, x_j, x_k)}{A(x_i, x_j, x_k)}$
- $\nabla B_i(u) = \frac{\nabla A(x, x_j, x_k)}{A(x_i, x_j, x_k)}$
- $\nabla A(x, x_j, x_k) = \frac{(x_k - x_j)^{\perp}}{2}$
- $\text{Area}\Delta PAB = F(P) = \frac{1}{2}|AB||PH|$
- $\nabla F = \frac{1}{2}|AB|\nabla|PH| = \frac{1}{2}|AB|\frac{HP}{|PH|} = \frac{1}{2}AB^{90}$



$\nabla|PH| = \frac{HP}{|PH|}$: Unit vector in HP direction

$$\nabla|\mathrm{PH}| = \frac{\mathrm{HP}}{|\mathrm{PH}|}$$



- Proof:
- P(x,y), $\mathrm{v}_1 = \frac{\mathrm{AB}}{|\mathrm{AB}|} = \binom{\mathrm{x}_1}{\mathrm{y}_1}$, $\mathrm{v}_2 = {\mathrm{v}_1}^{\perp} = \binom{-\mathrm{y}_1}{\mathrm{x}_1}$,
- $\mathrm{H} = (\mathrm{v}_1 \quad \mathrm{v}_2)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}(\mathrm{v}_1 \quad \mathrm{v}_2)^{\mathrm{T}}\binom{\mathrm{x}}{\mathrm{y}} = \begin{pmatrix}{\mathrm{x}_1}^2 & \mathrm{x}_1\mathrm{y}_1\\ \mathrm{x}_1\mathrm{y}_1 & {\mathrm{y}_1}^2\end{pmatrix}\binom{\mathrm{x}}{\mathrm{y}}$
- $\mathrm{PH} = \mathrm{H} - \mathrm{P} = \begin{pmatrix}{\mathrm{x}_1}^2 - 1 & \mathrm{x}_1\mathrm{y}_1\\ \mathrm{x}_1\mathrm{y}_1 & {\mathrm{y}_1}^2 - 1\end{pmatrix}\binom{\mathrm{x}}{\mathrm{y}} = \begin{pmatrix}-{\mathrm{y}_1}^2 & \mathrm{x}_1\mathrm{y}_1\\ \mathrm{x}_1\mathrm{y}_1 & -{\mathrm{x}_1}^2\end{pmatrix}\binom{\mathrm{x}}{\mathrm{y}} = \begin{pmatrix}\mathrm{x}_1\mathrm{y}_1\mathrm{y} - {\mathrm{y}_1}^2\mathrm{x}\\ \mathrm{x}_1\mathrm{y}_1\mathrm{x} - {\mathrm{x}_1}^2\mathrm{y}\end{pmatrix}$
- $|\mathrm{PH}| = \sqrt{(\mathrm{x}_1\mathrm{y}_1\mathrm{y} - {\mathrm{y}_1}^2\mathrm{x})^2 + (\mathrm{x}_1\mathrm{y}_1\mathrm{x} - {\mathrm{x}_1}^2\mathrm{y})^2}$
- $\frac{\partial|\mathrm{PH}|}{\partial\mathrm{x}} = \frac{-(\mathrm{x}_1\mathrm{y}_1\mathrm{y} - {\mathrm{y}_1}^2\mathrm{x}){\mathrm{y}_1}^2 + (\mathrm{x}_1\mathrm{y}_1\mathrm{x} - {\mathrm{x}_1}^2\mathrm{y})\mathrm{x}_1\mathrm{y}_1}{\sqrt{(\mathrm{x}_1\mathrm{y}_1\mathrm{y} - {\mathrm{y}_1}^2\mathrm{x})^2 + (\mathrm{x}_1\mathrm{y}_1\mathrm{x} - {\mathrm{x}_1}^2\mathrm{y})^2}} = \frac{[(\mathrm{x}_1\mathrm{y}_1)^2 + (\mathrm{y}_1\mathrm{y}_1)^2]\mathrm{x} - [\mathrm{x}_1\mathrm{y}_1(\mathrm{y}_1)^2 + \mathrm{x}_1\mathrm{y}_1(\mathrm{x}_1)^2]\mathrm{y}}{|\mathrm{PH}|}$
- $= \frac{{\mathrm{y}_1}^2\mathrm{x} - \mathrm{x}_1\mathrm{y}_1\mathrm{y}}{|\mathrm{PH}|} = \frac{\mathrm{HP}_\mathrm{x}}{|\mathrm{PH}|}$
- $\nabla|\mathrm{PH}| = \frac{\mathrm{HP}}{|\mathrm{PH}|}$
- END.

Gradient operator – for vertices

The gradient defined on a vertex as

$$\nabla_M^{(A)} f(p_i) = \frac{1}{\mathcal{A}(p_i)} \sum_{j \in N_1(i)} A_j \nabla_{T_j} f,$$

Where

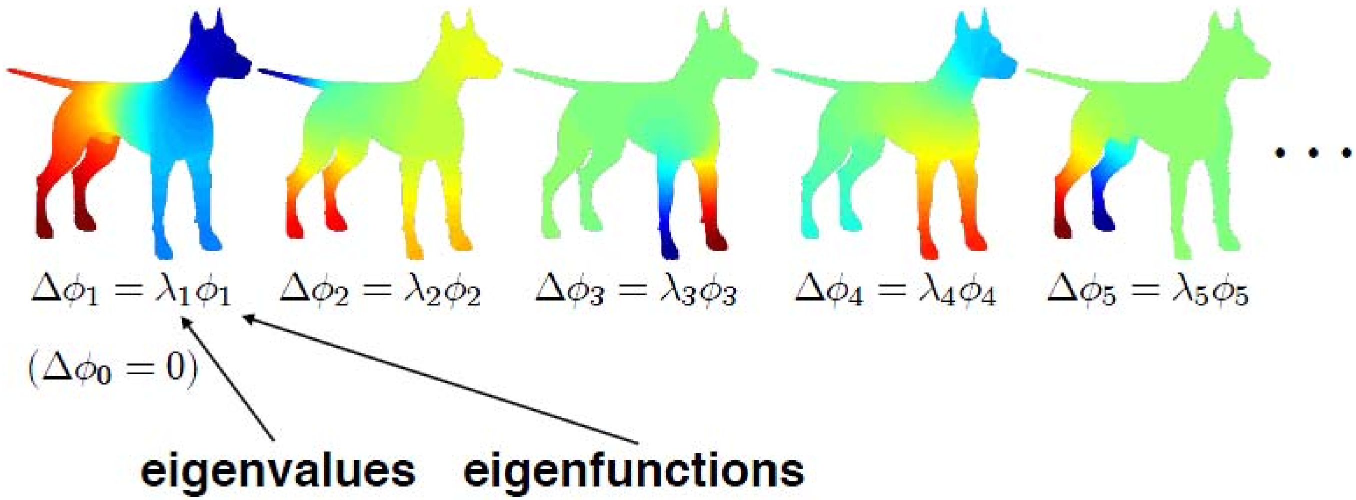
$$\mathcal{A}(p_i) = \sum_{j \in N_1(i)} A_j .$$

Laplacian operator

Laplace-Beltrami Operator

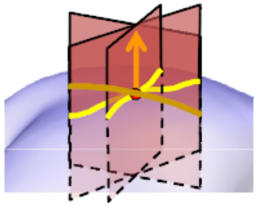
Functional basis on a surface

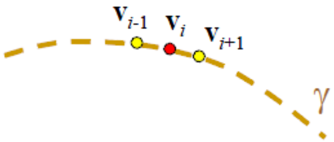
- Invariant to isometric deformations
- Low-frequency to high-frequency
- Has physics interpretation

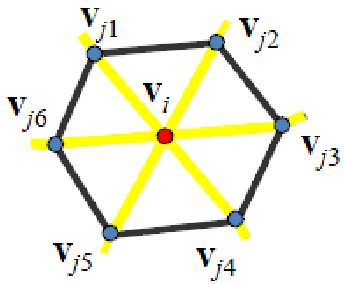


Discrete Laplace-Beltrami

- Intuition for uniform discretization







$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa(\varphi)\,\mathrm{d}\varphi$$

$$\kappa \mathbf{n} = \gamma''$$

$$\gamma'' \approx \frac{1}{t}\left(\left(\mathbf{v}_i - \mathbf{v}_{i-1}\right) - \left(\mathbf{v}_{i+1} - \mathbf{v}_i\right)\right) = -\frac{1}{t}\left(\mathbf{v}_{i-1} + \mathbf{v}_{i+1} - 2\mathbf{v}_i\right)$$

$$\begin{aligned} &\mathbf{v}_{j1} + \mathbf{v}_{j4} - 2\mathbf{v}_i \quad + \\ &\mathbf{v}_{j2} + \mathbf{v}_{j5} - 2\mathbf{v}_i \quad + \\ &\mathbf{v}_{j3} + \mathbf{v}_{j6} - 2\mathbf{v}_i \quad = \end{aligned}$$

$$6L(\mathbf{v}_i) = \sum_{k=1}^{6} \mathbf{v}_{jk} - 6\mathbf{v}_i \approx -6 \cdot 2H\mathbf{n}$$

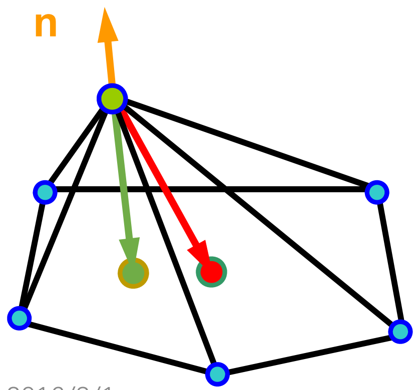
Discrete Laplacians

$$\boldsymbol{\delta}_i = \mathbf{x}_i - \frac{1}{\sum_{(i,j)\in E} w_{ij}} \sum_{(i,j)\in E} w_{ij}\mathbf{x}_j$$

$\delta_{\mathbf{uniform}}$: $w_{ij} = 1$

$\delta_{\mathbf{cotangent}}$: $w_{ij} = \cot\alpha_{ij} + \cot\beta_{ij}$

$\Delta_{\mathbf{mean\ curvature}}$: $w_{ij} = \dfrac{\cot\alpha_{ij} + \cot\beta_{ij}}{2A(v_i)}$



$$\boxed{\mathbf{L}}\ \boxed{\mathbf{v}} = \boxed{\delta}$$