

Spring 2015

CSCI 599: Digital Geometry Processing

8.1 Surface Parameterization

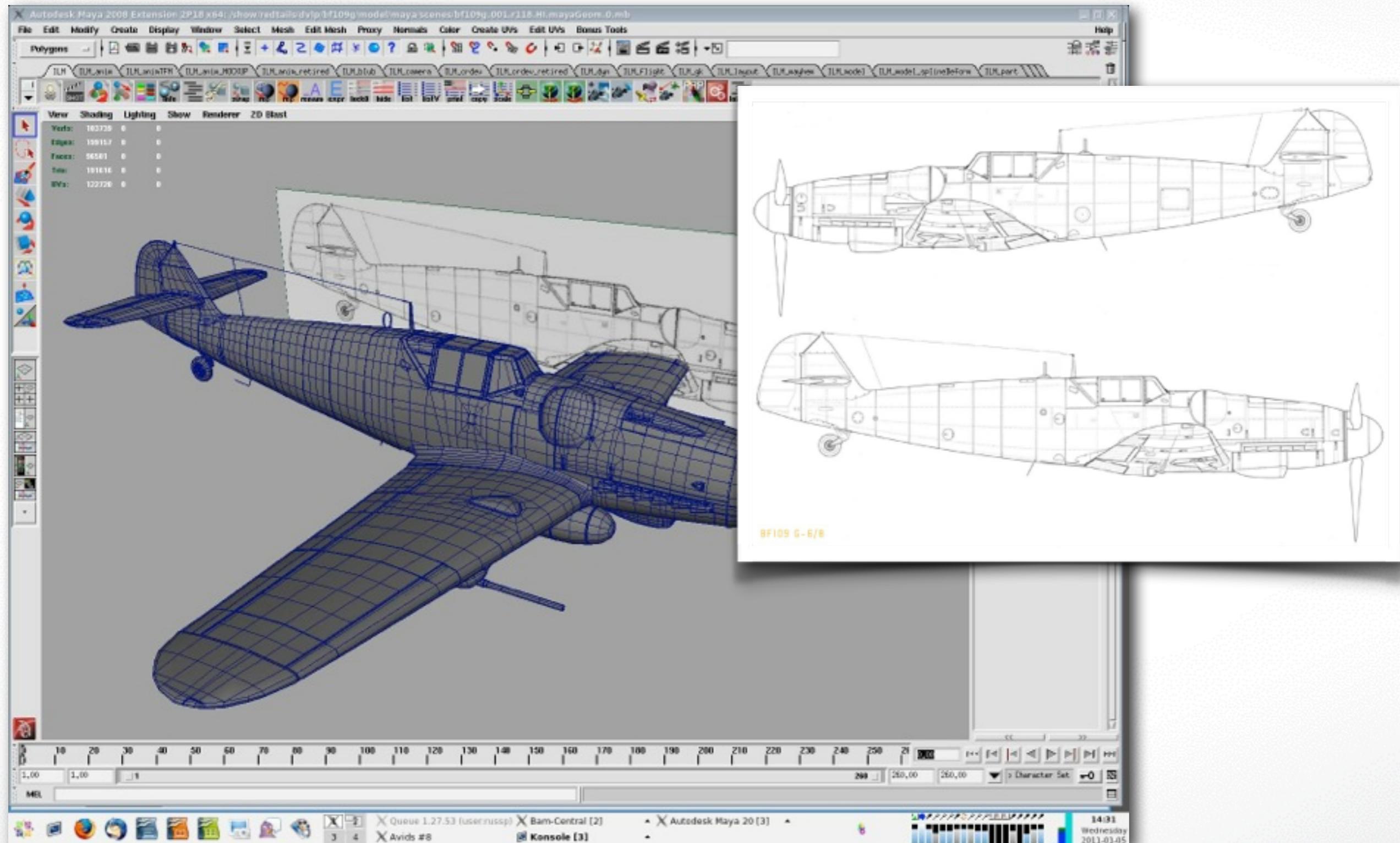


Hao Li
<http://cs599.hao-li.com>

Modeling



Modeling

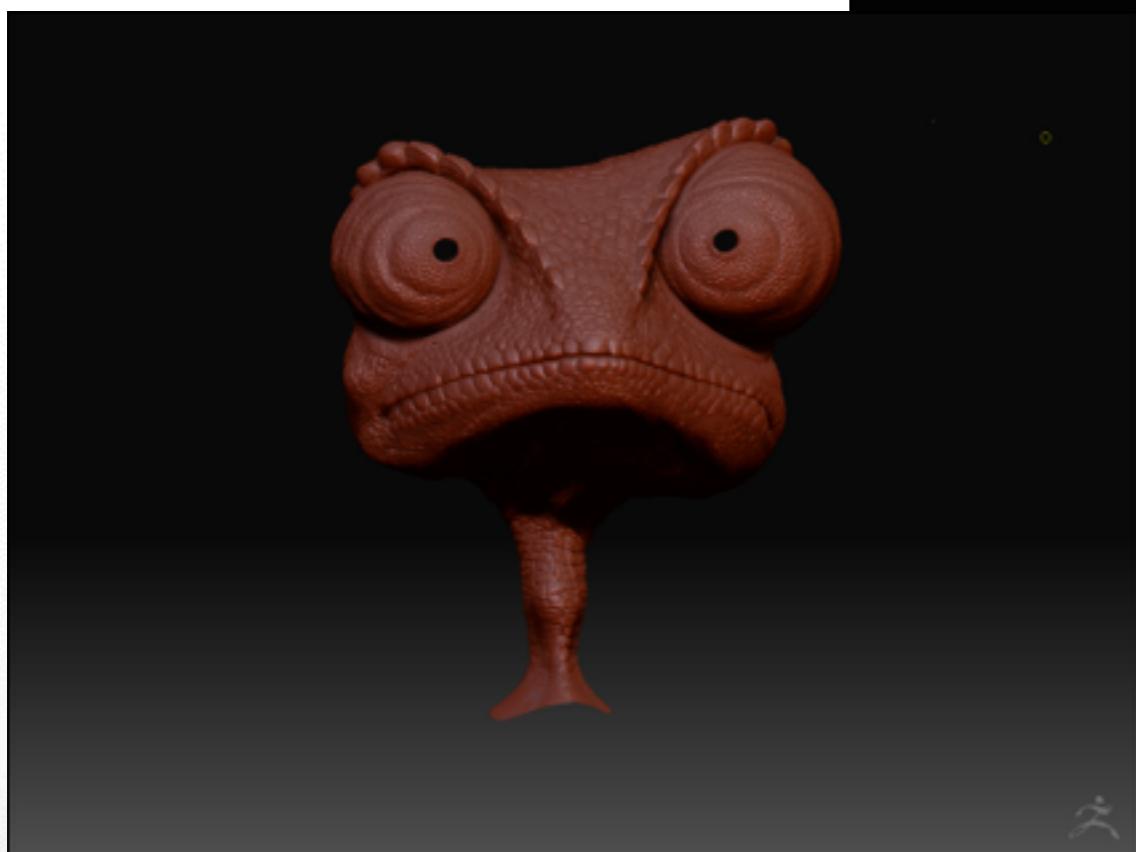


Viewpaint

The creation of a 3D assets surface, including that surface's color, texture, opacity, and reflectivity (or specularity).

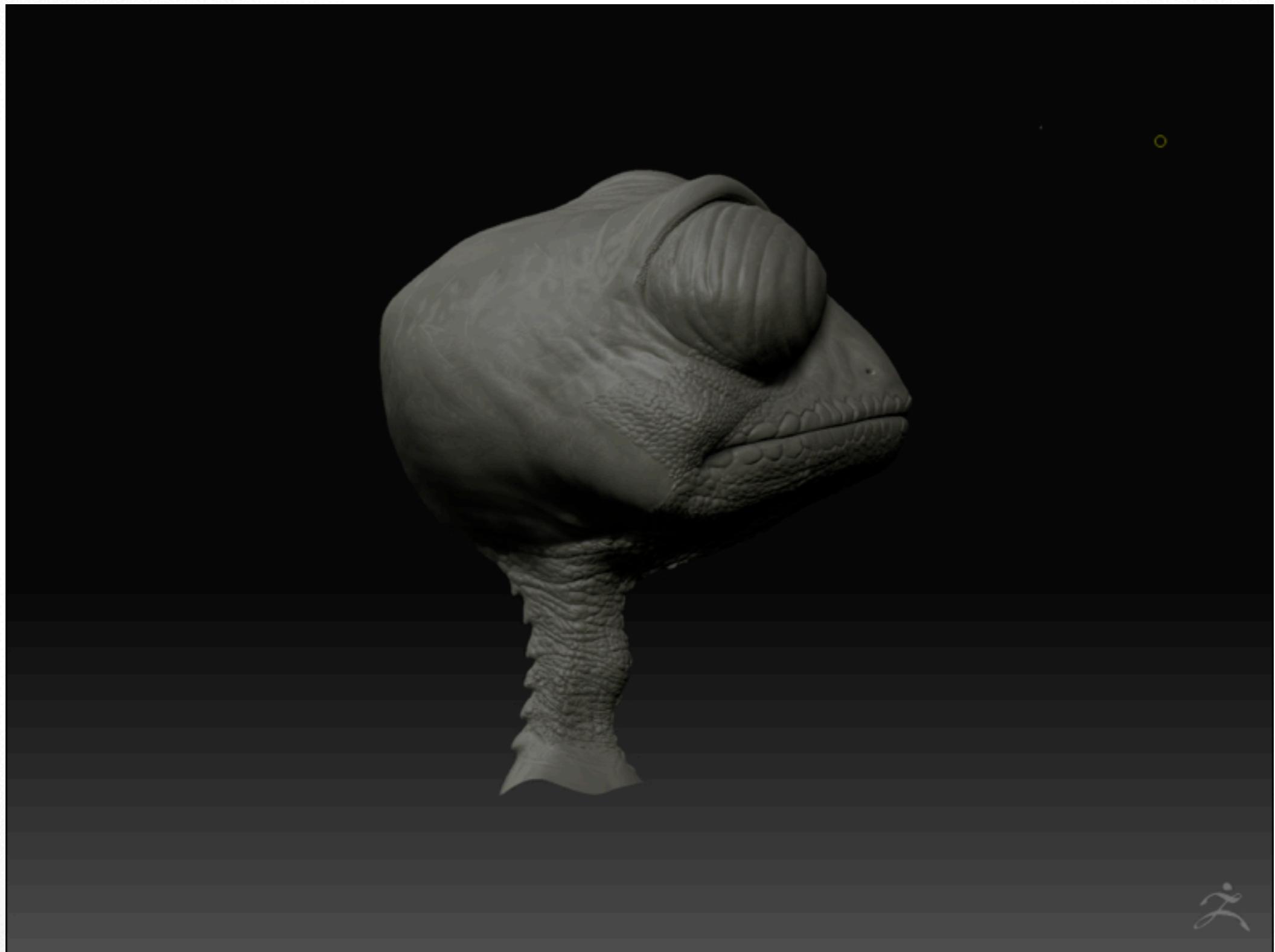
Viewpaint

Rango: Creating
creature scale textures
in ZBrush...

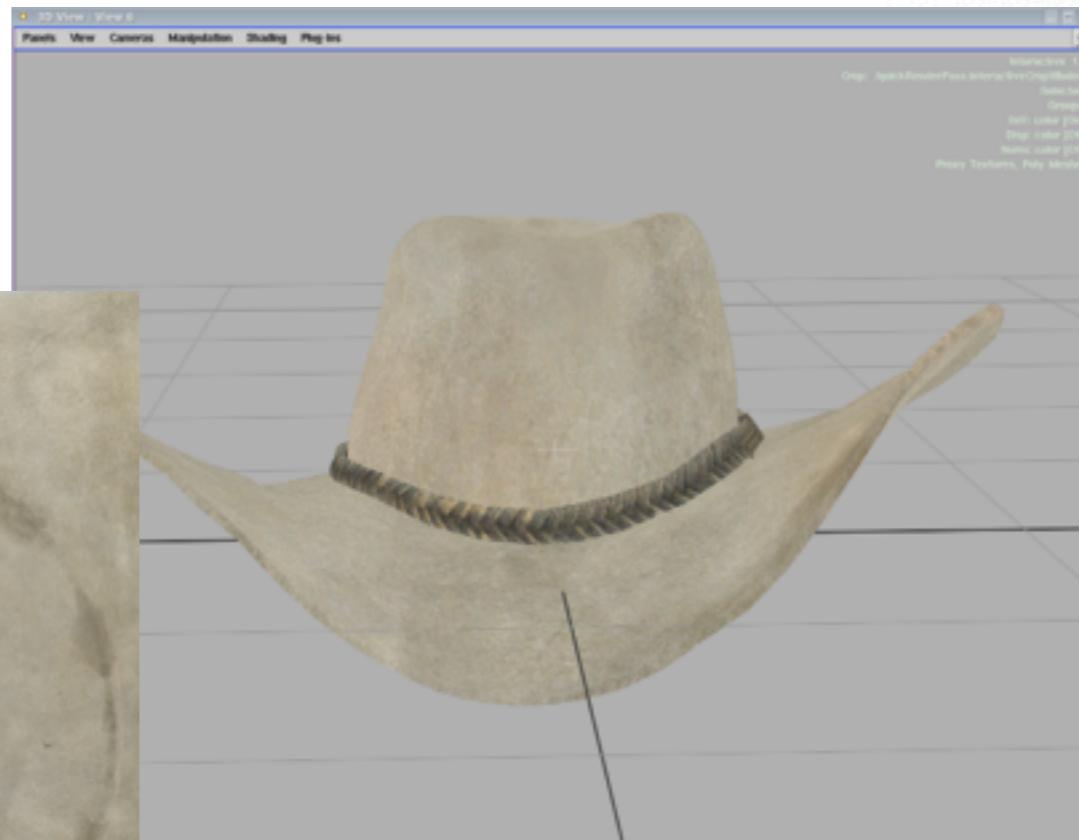


Viewpaint

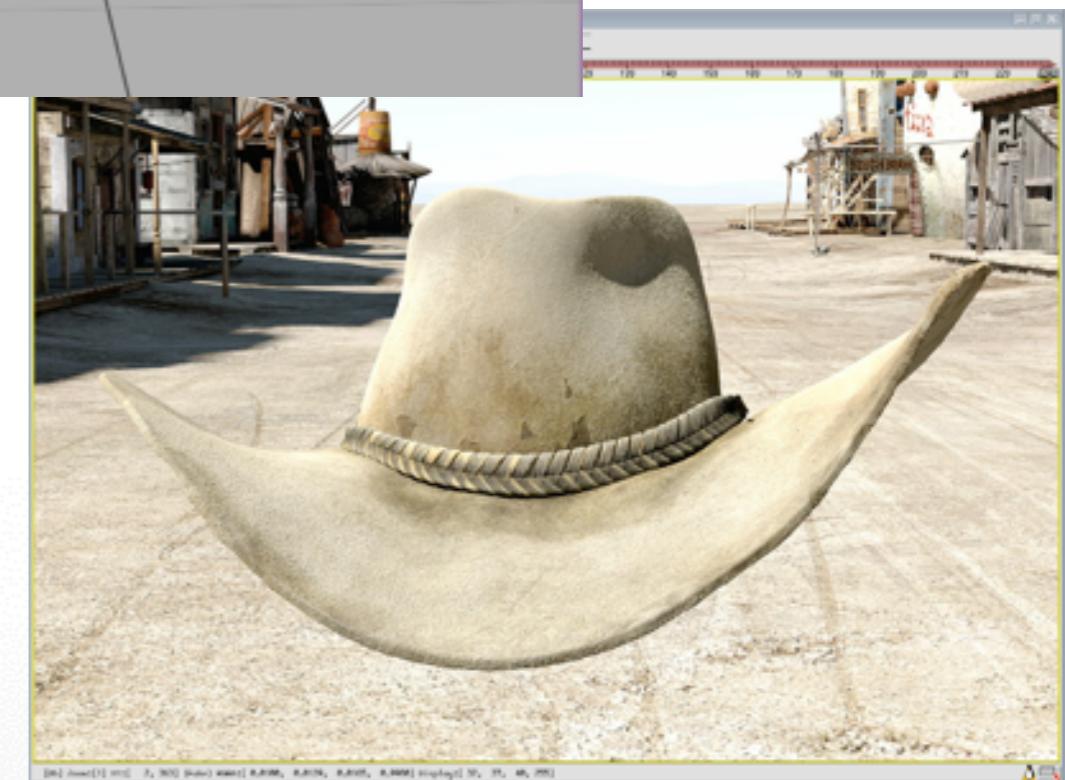
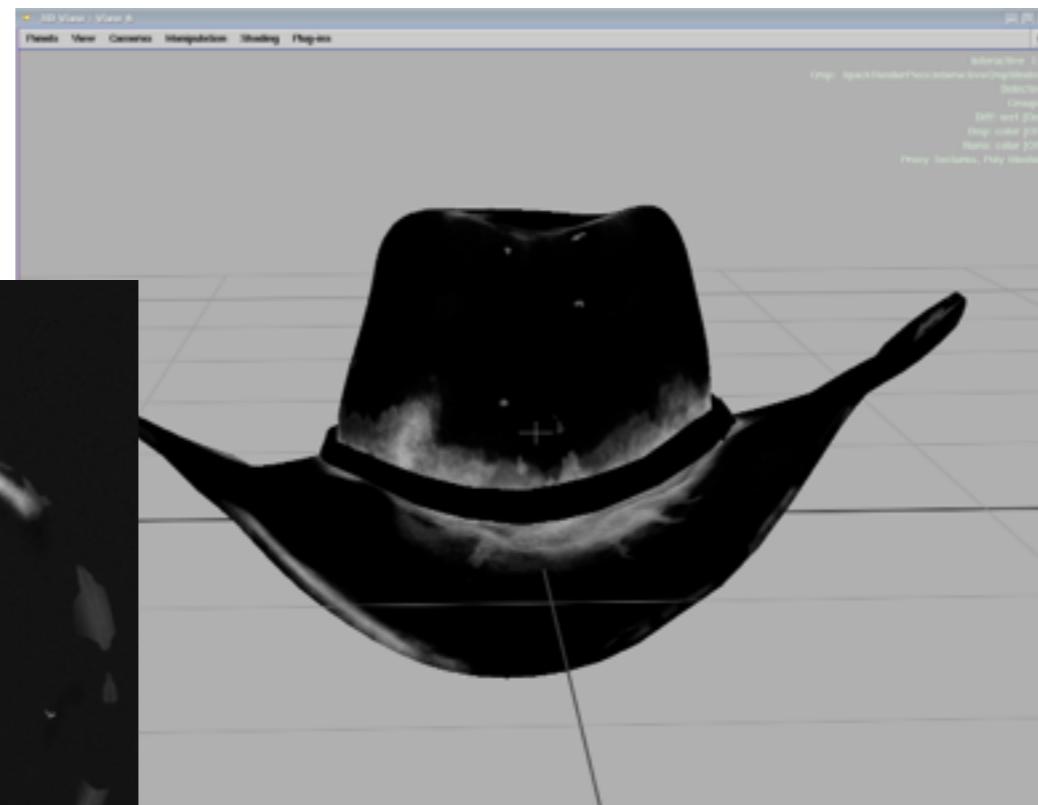
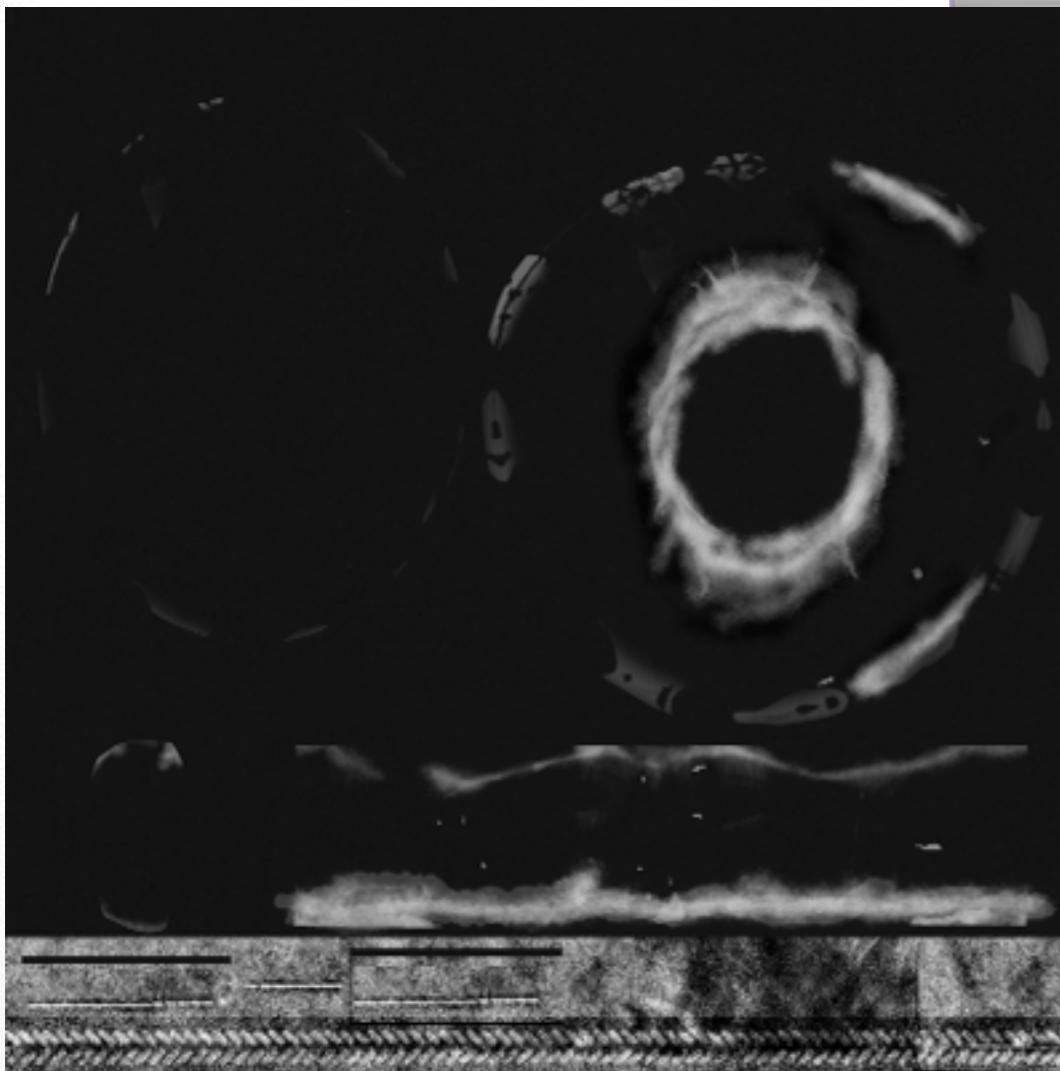
(Wrinkle Pass)



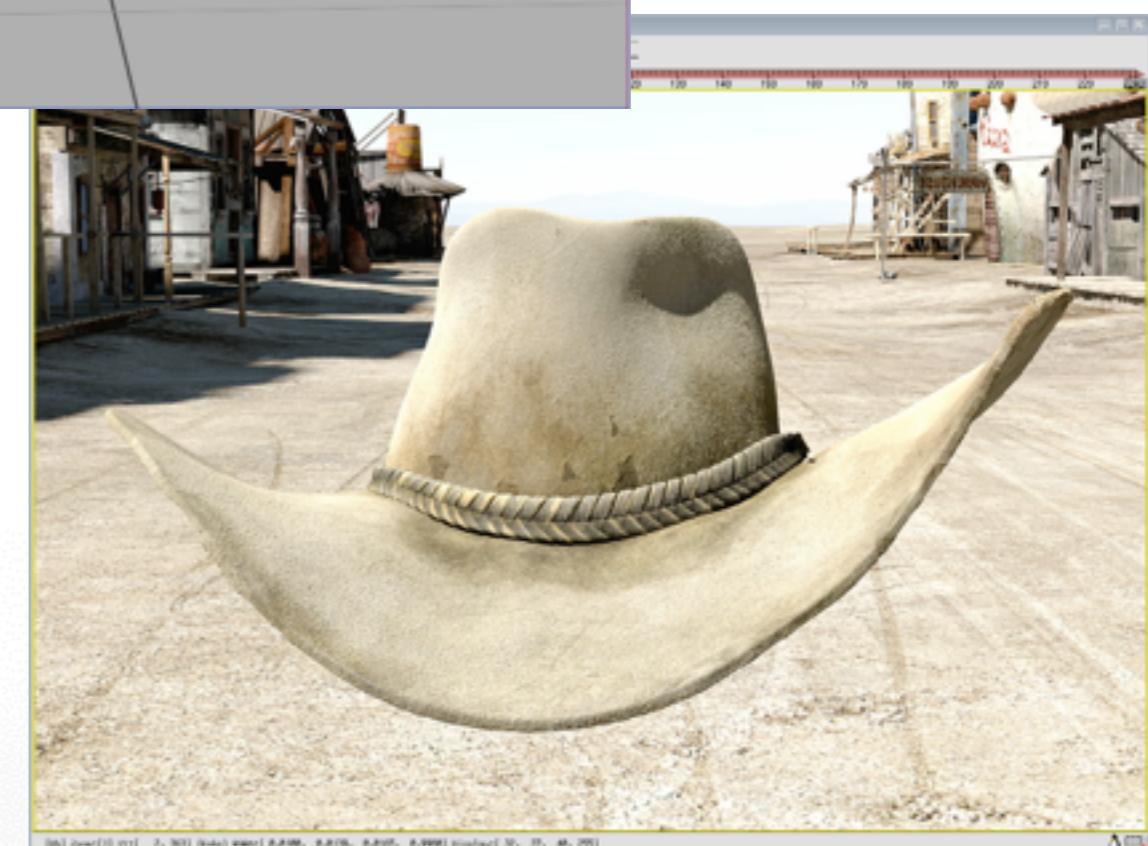
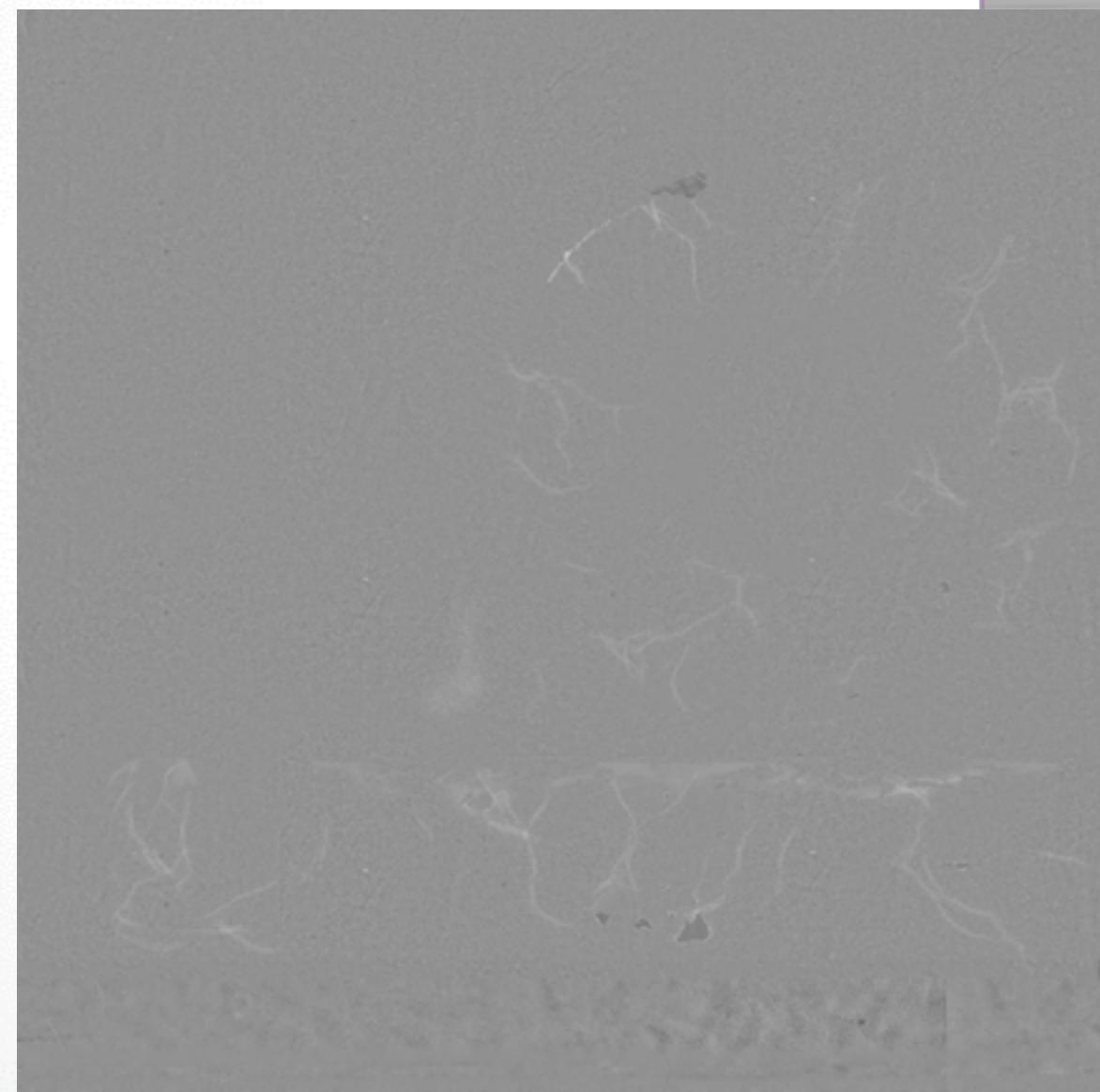
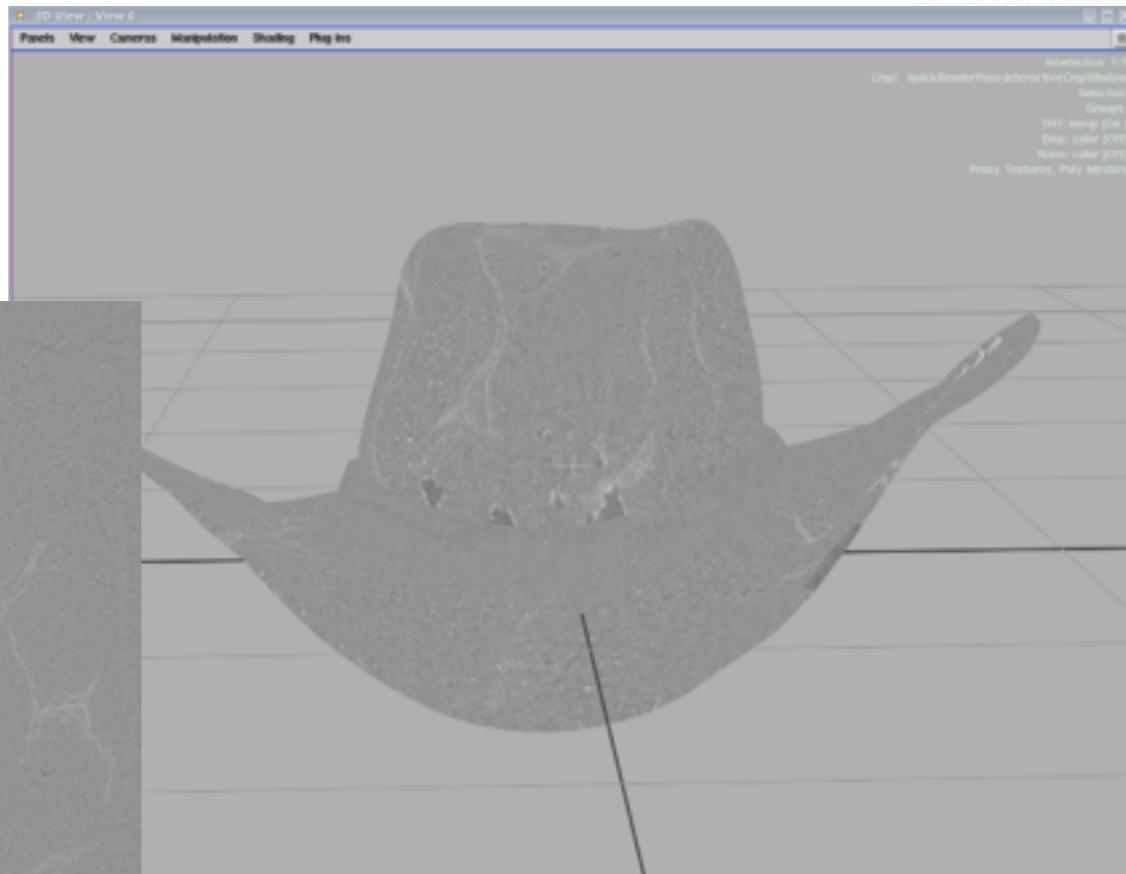
Color Maps



Wet Maps

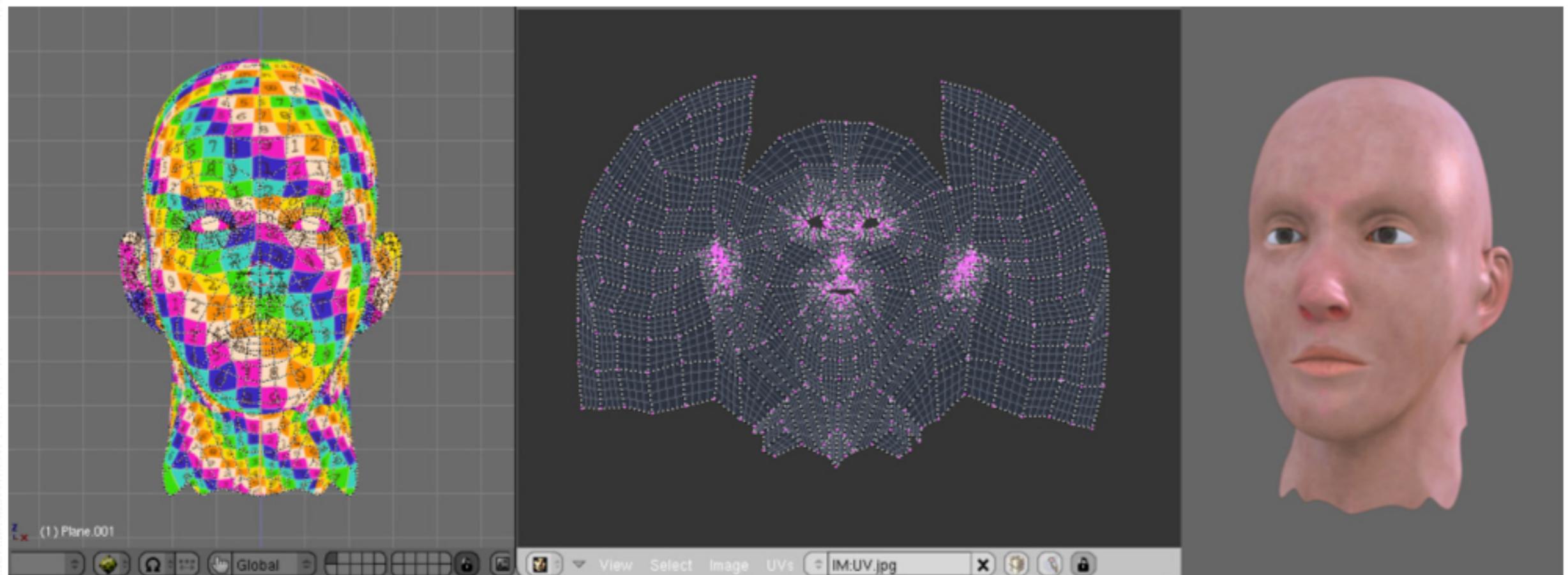


bump Maps



Motivation

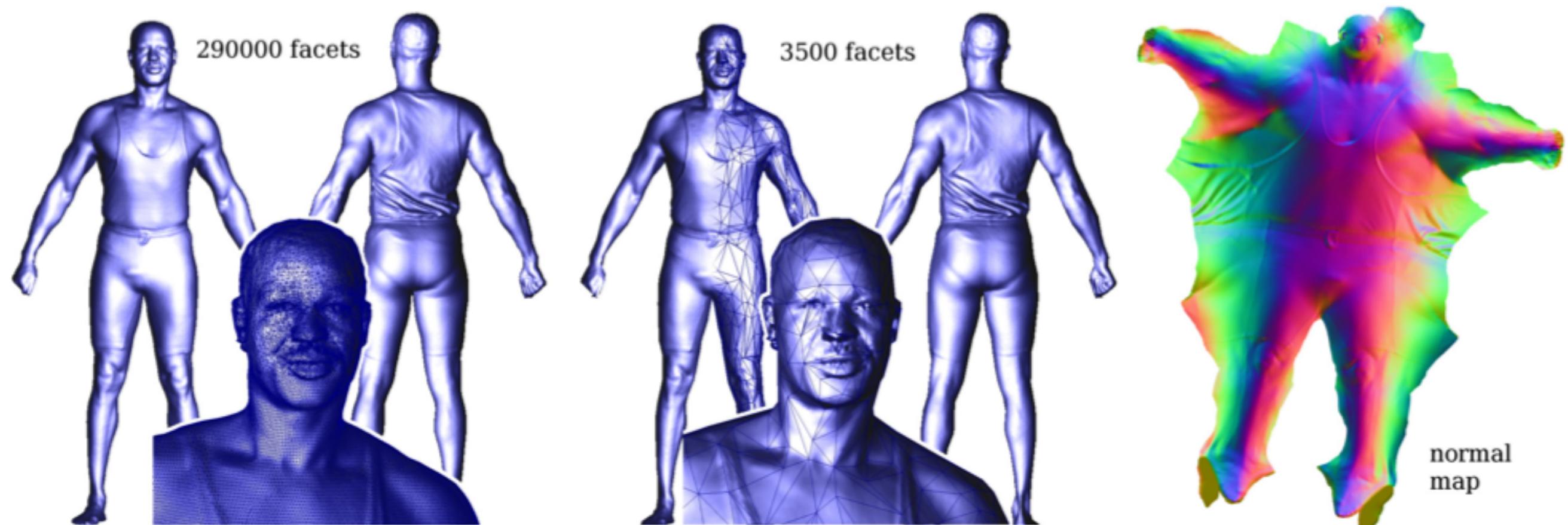
Texture Mapping



Levy et al.: *Least squares conformal maps for automatic texture atlas generation*, SIGGRAPH 2002.

Motivation

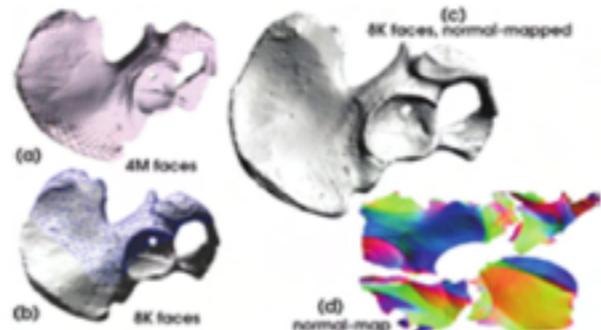
Normal Mapping



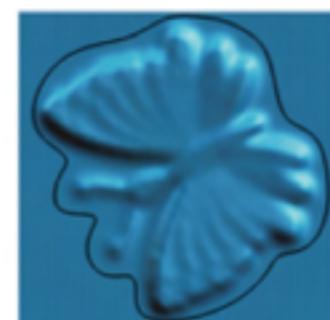
Motivation



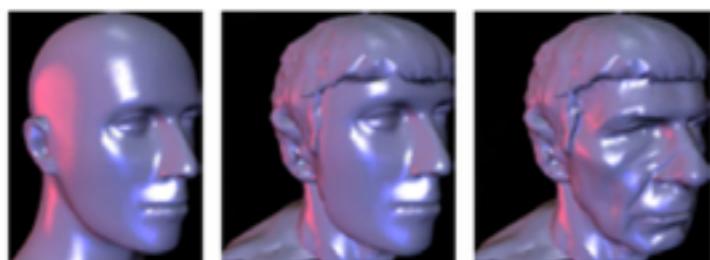
Texture Mapping



Normal Mapping



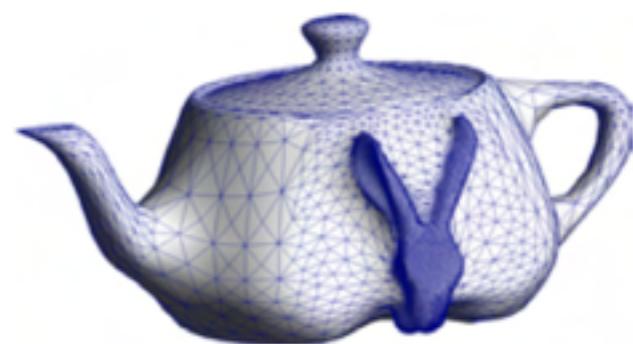
Detail Transfer



Morphing



Mesh Completion



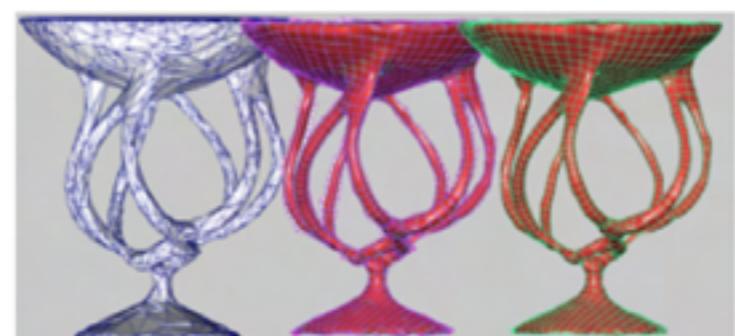
Editing



Databases

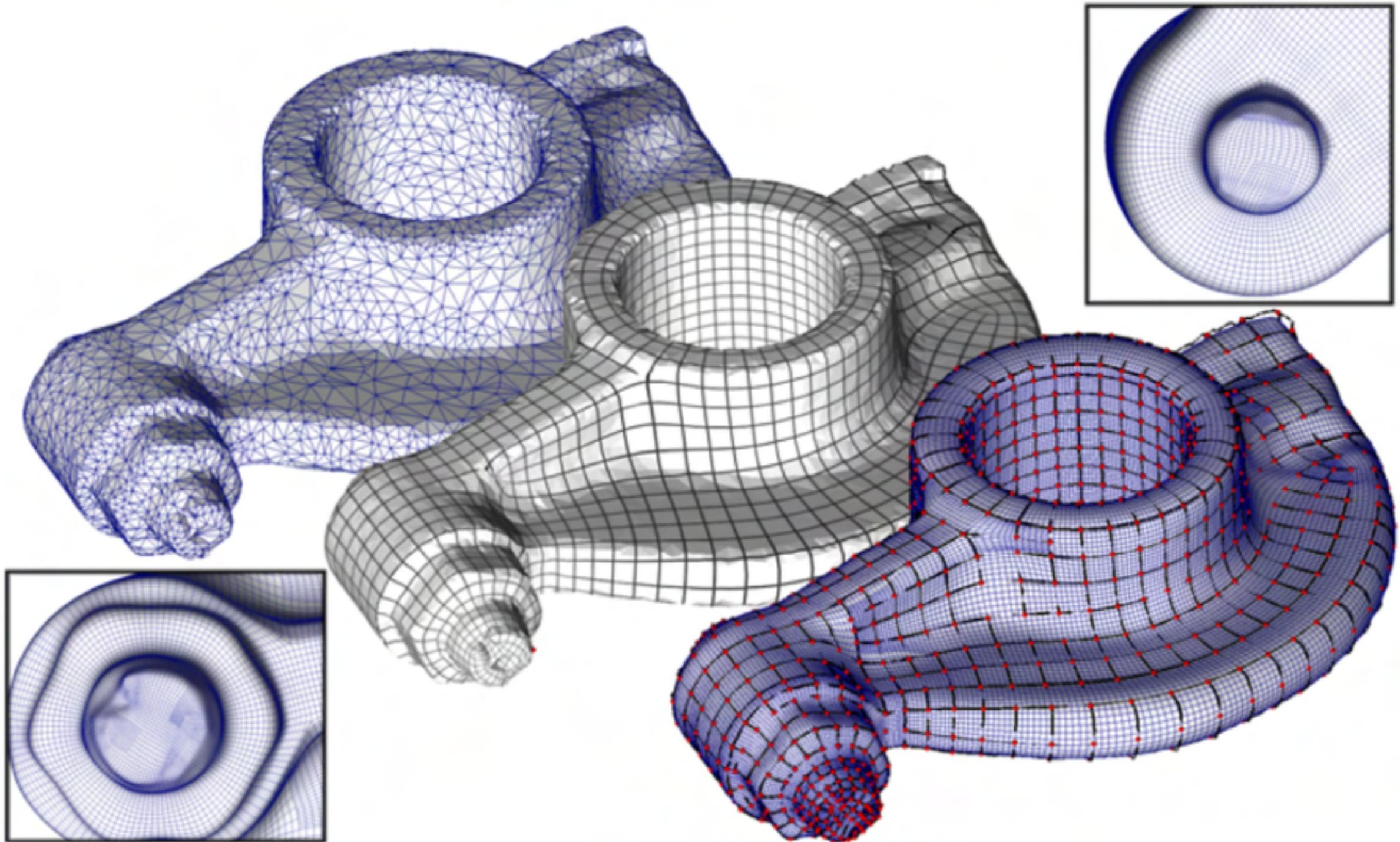


Remeshing

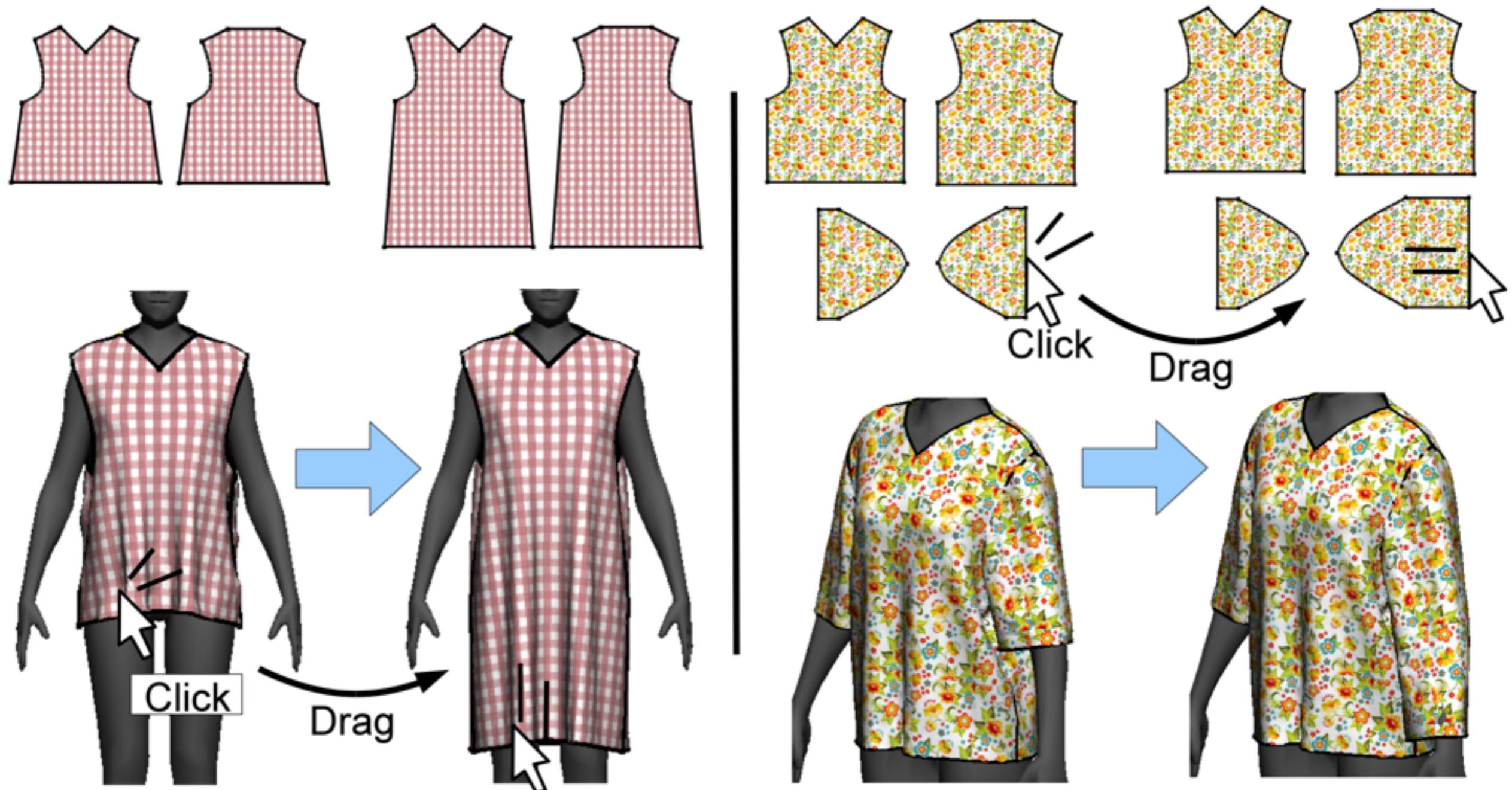


Surface Fitting

Motivation

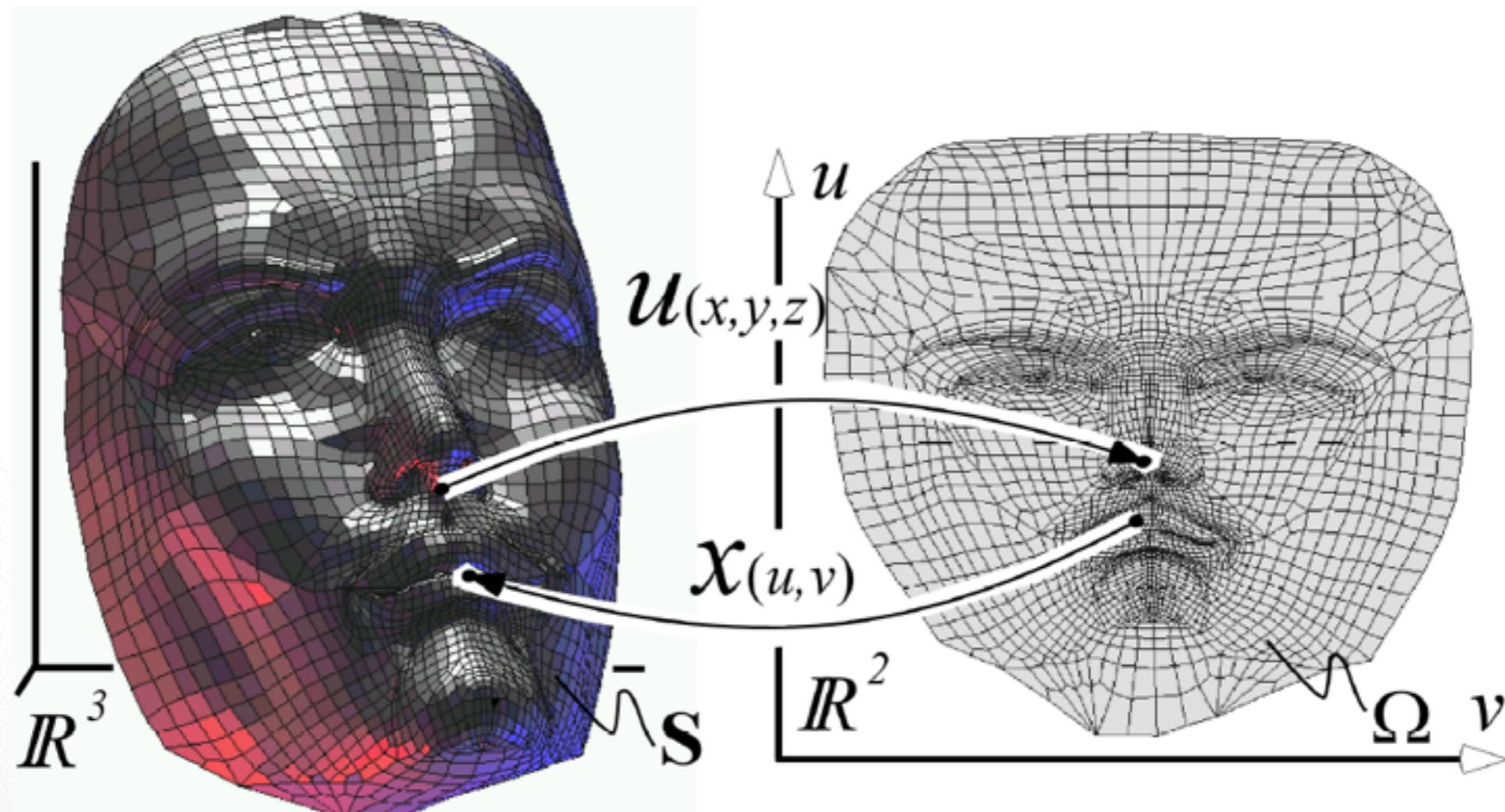


Motivation

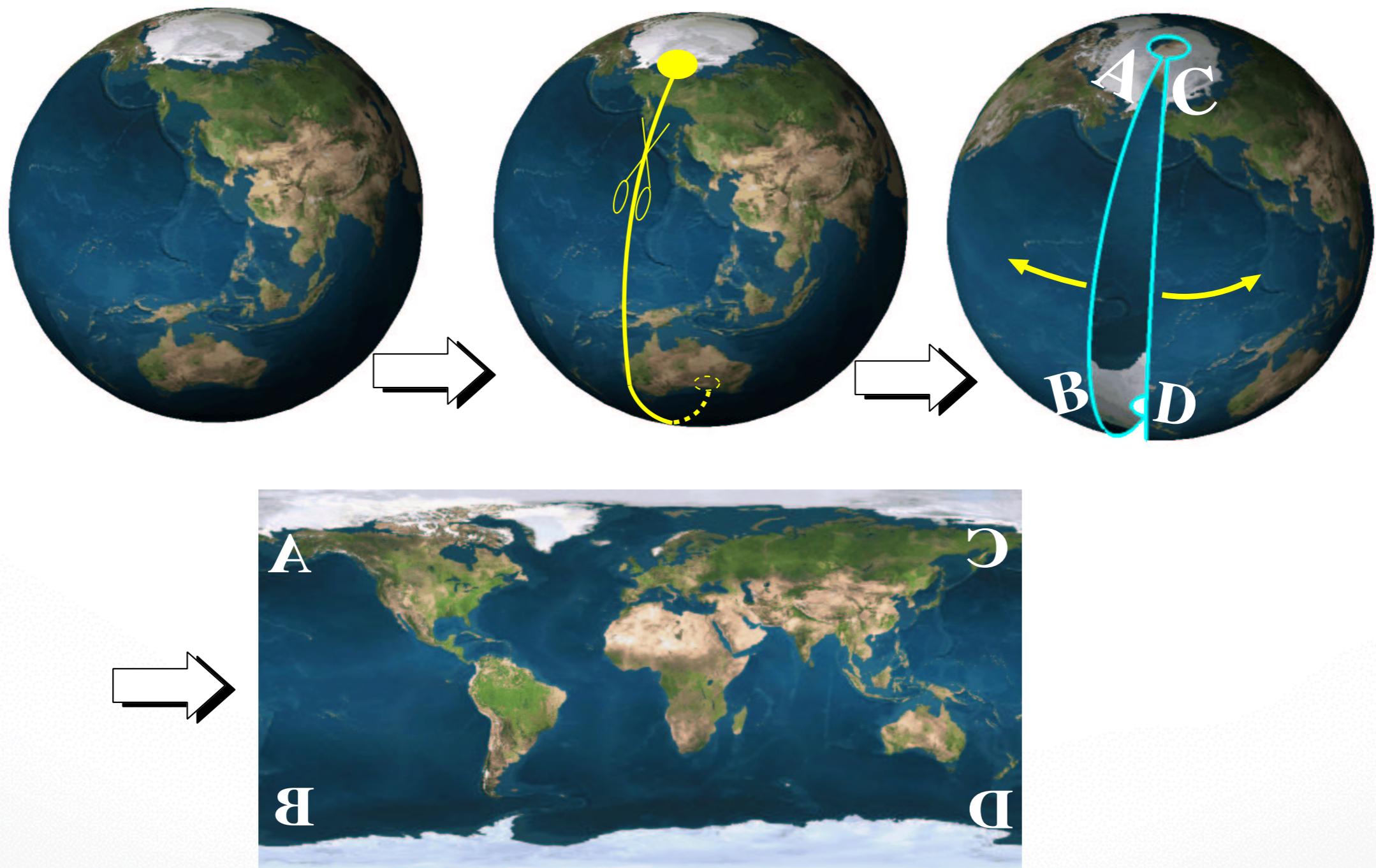


Mesh Parameterization

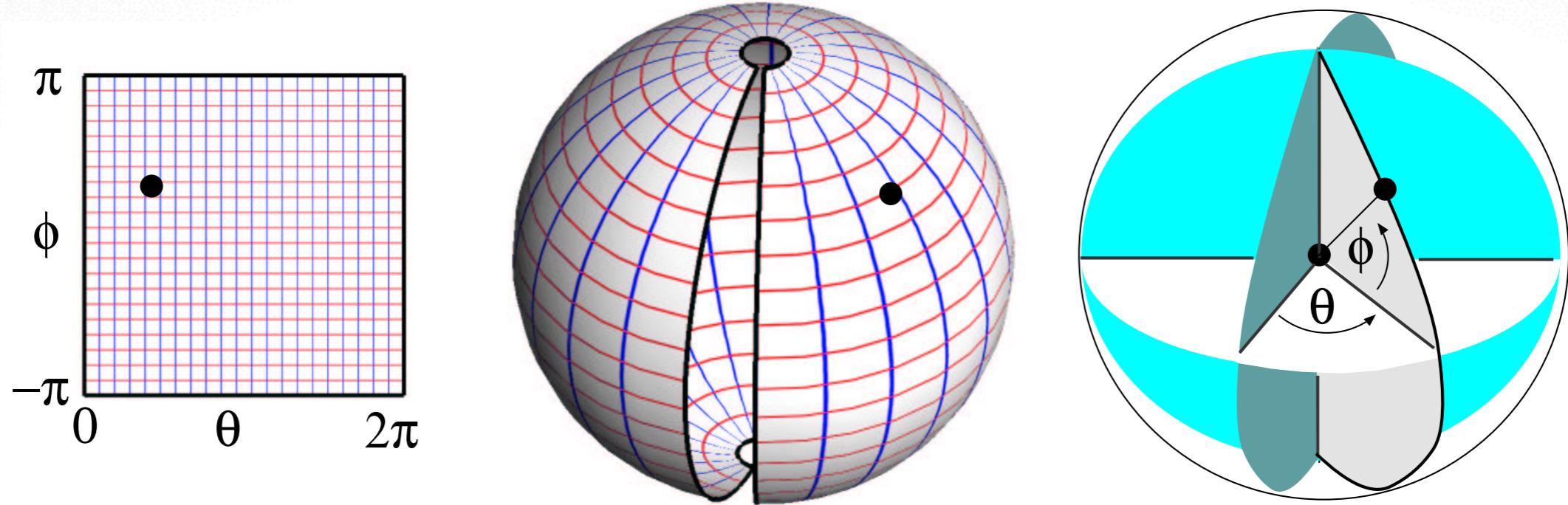
Find a 1-to-1 mapping between given surface mesh and 2D parameter domain



Unfolding Earth



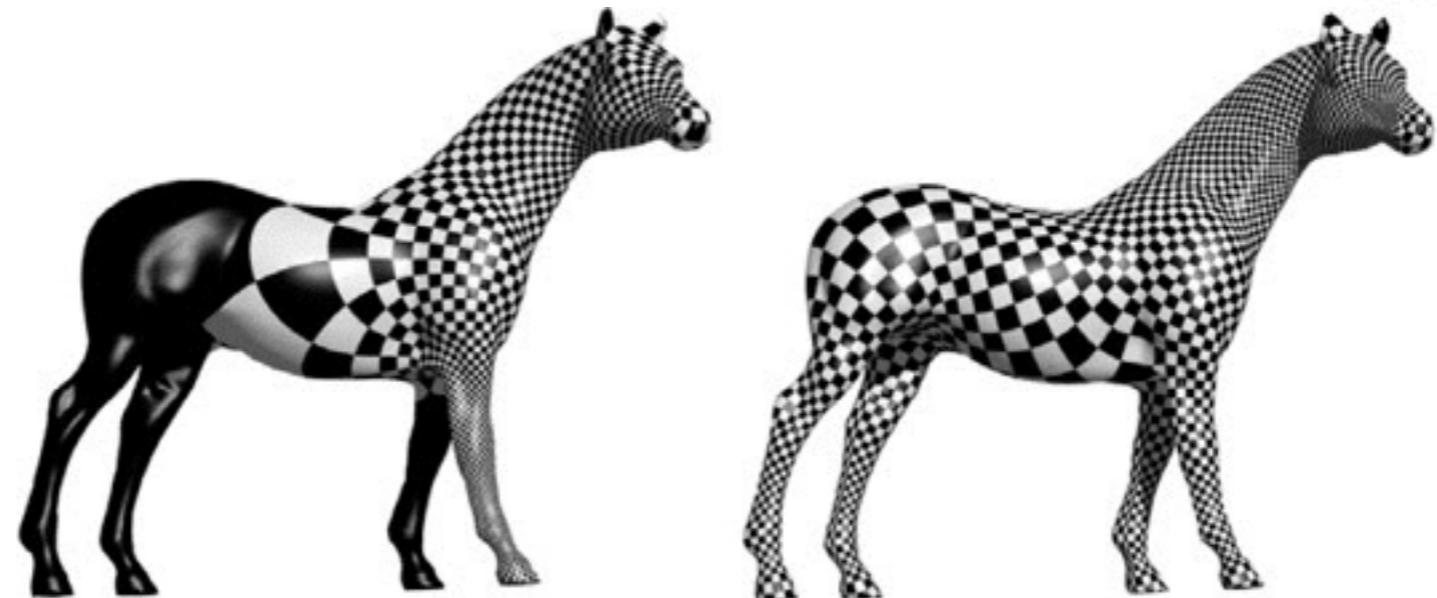
Spherical Coordinates



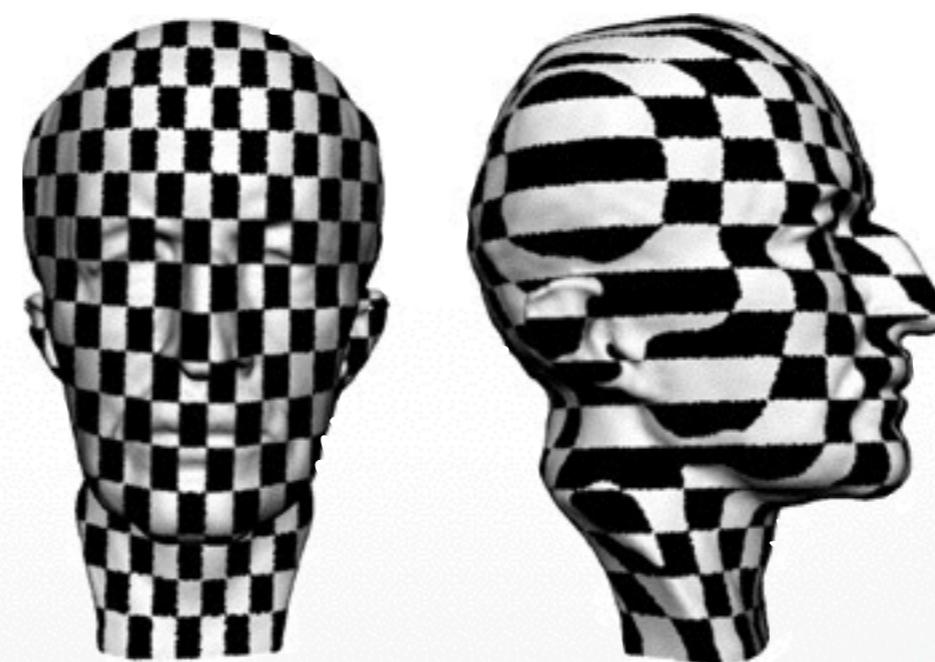
$$\begin{bmatrix} \theta \\ \phi \end{bmatrix} \mapsto \begin{bmatrix} \sin \theta \sin \phi \\ \cos \theta \sin \phi \\ \cos \phi \end{bmatrix}$$

Desirable Properties

Low distortion



Bijective mapping



Cartography



orthographic

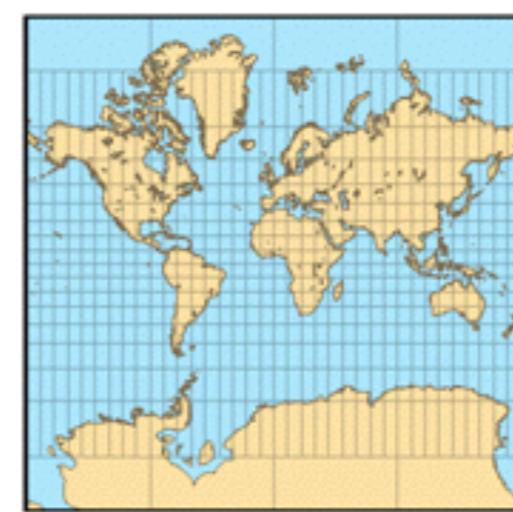


stereographic



preserves angles

= conformal



Mercator



Lambert



preserves area

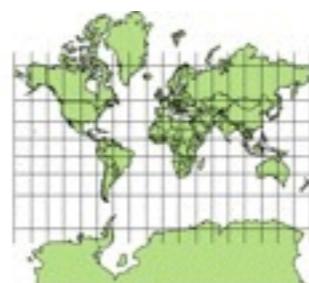
= equiareal

Floater, Hormann: *Surface Parameterization: A Tutorial and Survey*,
Advances in Multiresolution for Geometric Modeling, 2005

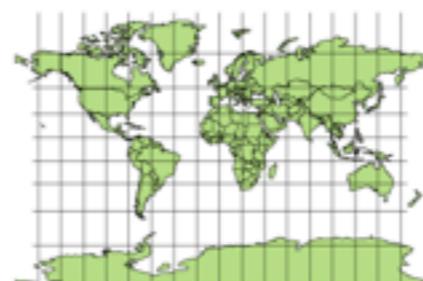
More Maps



Mollweide-Projektion



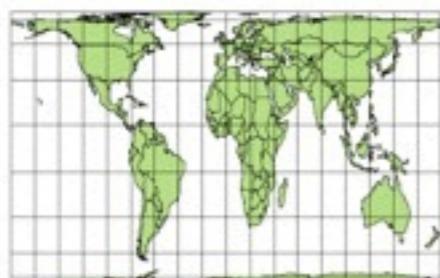
Mercator-Projektion



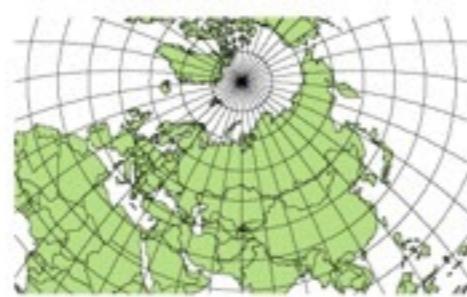
Zylinderprojektion nach Miller



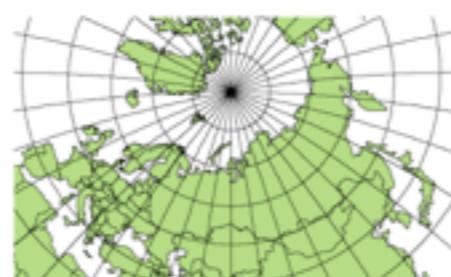
Hammer-Aitoff-Projektion



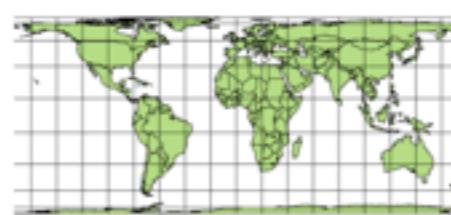
Peters-Projektion



Längentreue Azimuthalprojektion



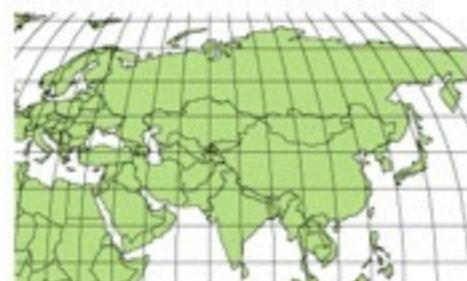
Stereographische Projektion



Behrmann-Projektion



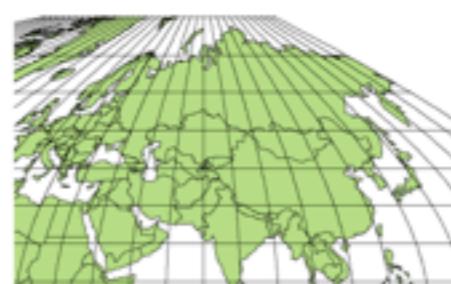
Senkrechte Umgebungs perspektive



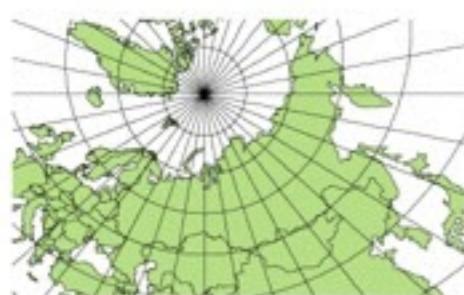
Robinson-Projektion



Hotine Oblique Mercator-Projektion



Sinusoidale Projektion



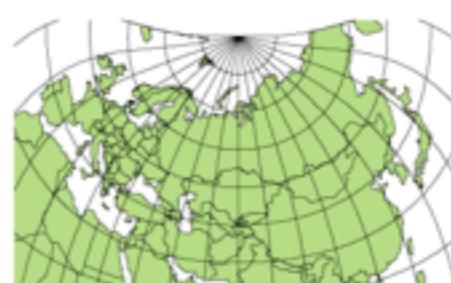
Gnomonische Projektion



Flächentreue Kegelprojektion

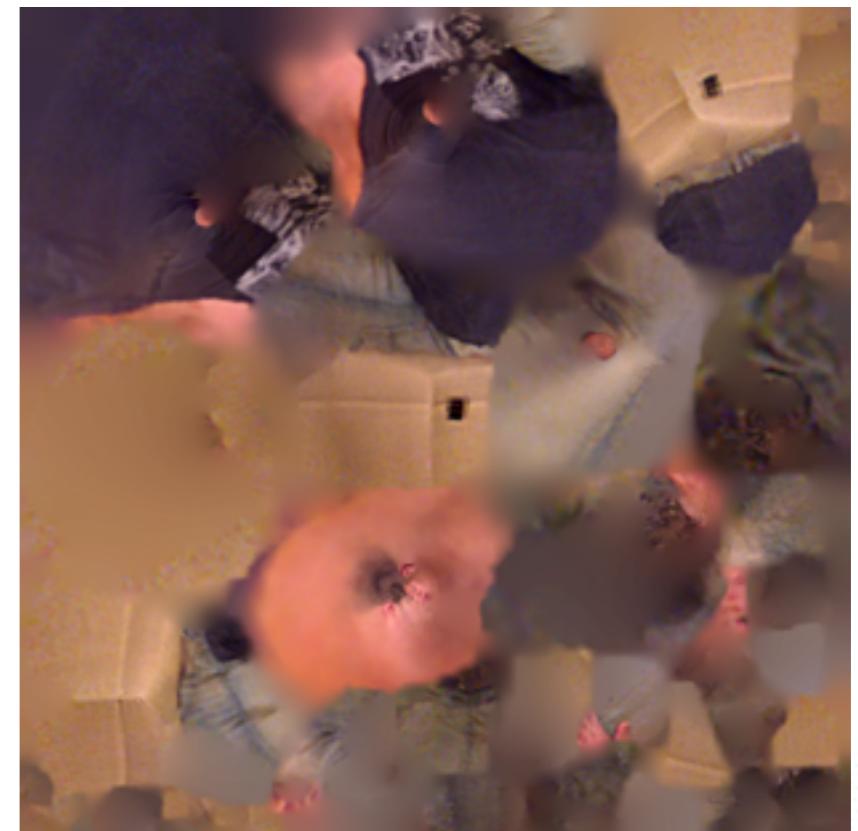
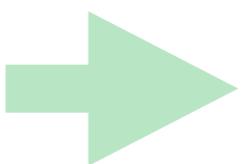


Transverse Mercator-Projektion



Cassini-Soldner-Projektion

Demo: Parameterization

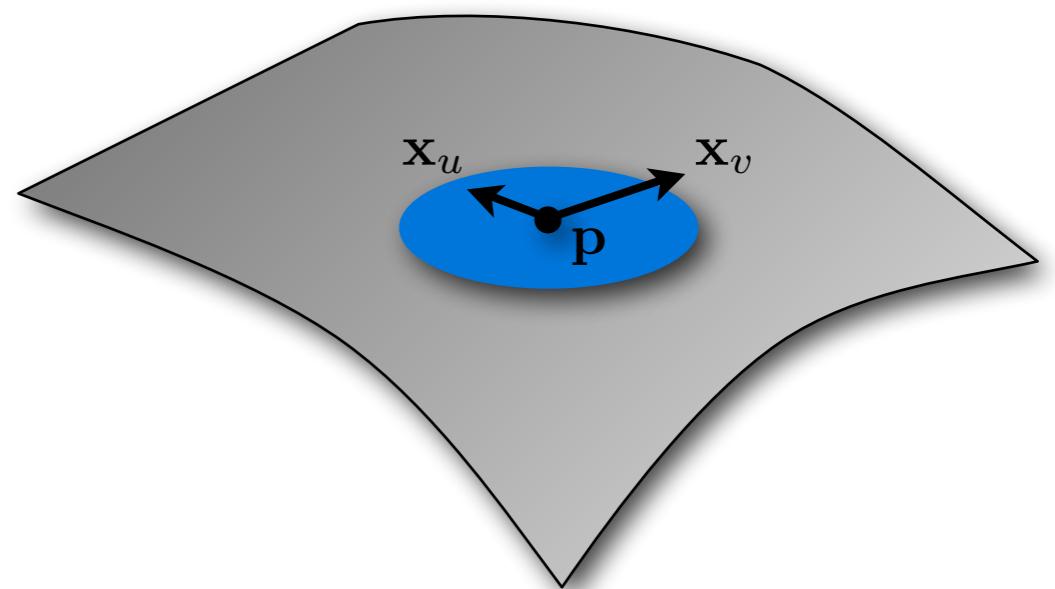


Recall: Differential Geometry

Parametric surface representation

$$\mathbf{x} : \Omega \subset \mathbb{R}^2 \rightarrow \mathcal{S} \subset \mathbb{R}^3$$

$$(u, v) \mapsto \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$



Regular if

- Coordinate functions x, y, z are smooth
- Tangents are linearly independent

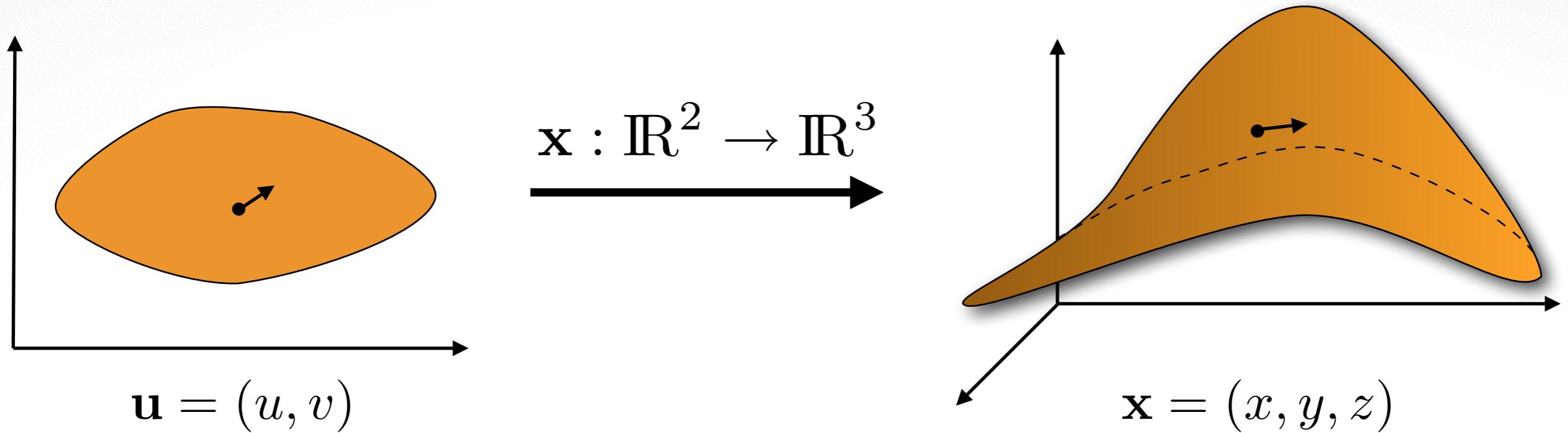
$$\mathbf{x}_u \times \mathbf{x}_v \neq 0$$

Definitions

A regular parameterization $\mathbf{x} : \Omega \rightarrow S$ is

- **Conformal** (angle preserving), if the angle of every pair of intersecting curves on S is the same as that of the corresponding pre-images in Ω .
- **Equiareal** (area preserving) if every part of Ω is mapped onto a part of S with the same area
- **Isometric** (length preserving), if the length of any arc on S is the same as that of its pre-image in Ω .

Distortion Analysis



Jacobian transforms infinitesimal vectors

$$d\mathbf{x} = \mathbf{J} d\mathbf{u}$$

$$\mathbf{J} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$

$$\|d\mathbf{x}\|^2 = (d\mathbf{u})^T \mathbf{J}^T \mathbf{J} d\mathbf{u} = (d\mathbf{u})^T \mathbf{I} d\mathbf{u}$$

First Fundamental Form

Characterizes the surface locally

$$\mathbf{I} = \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_v^T \mathbf{x}_u & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

Allows to measure on the surface

- Angles $\cos \theta = (\mathbf{d}\mathbf{u}_1^T \mathbf{I} \mathbf{d}\mathbf{u}_2) / (\|\mathbf{d}\mathbf{u}_1\| \cdot \|\mathbf{d}\mathbf{u}_2\|)$
- Length $ds^2 = \mathbf{d}\mathbf{u}^T \mathbf{I} \mathbf{d}\mathbf{u}$
- Area $dA = \det(\mathbf{I}) du dv$

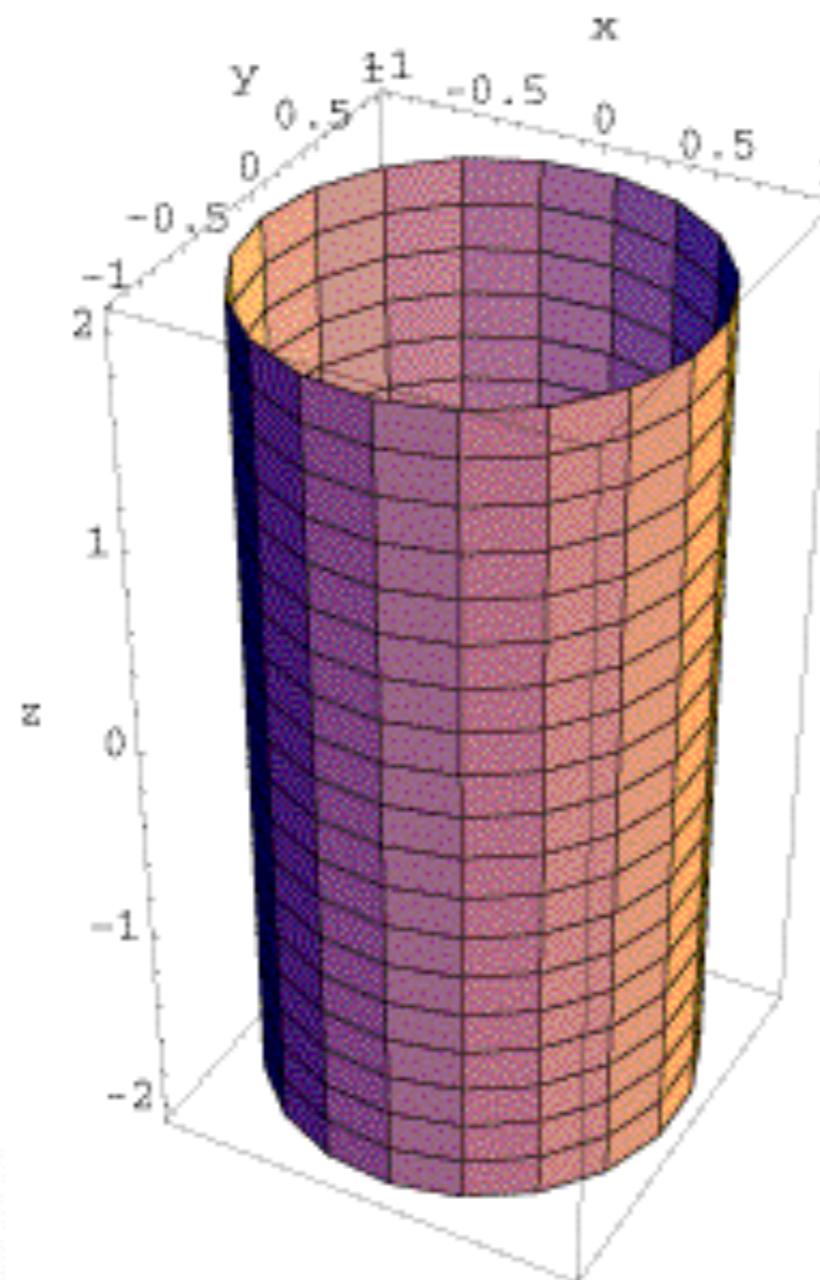
Isometric Maps

A regular parameterization $\mathbf{x}(u, v)$ is isometric, iff its first fundamental form is the identity:

$$\mathbf{I}(u, v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A surface has an isometric parameterization iff it has zero Gaussian curvature

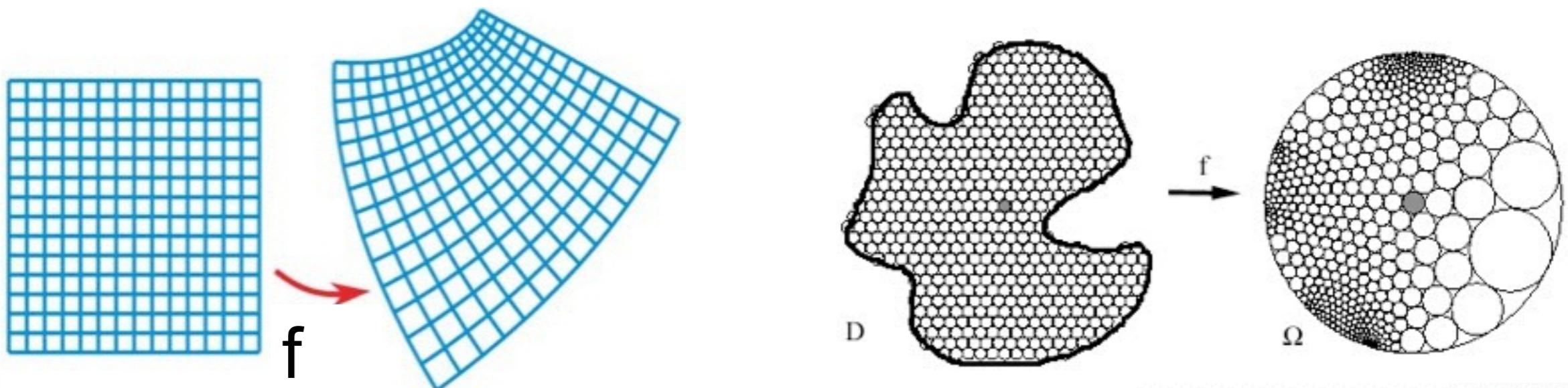
Cylinder



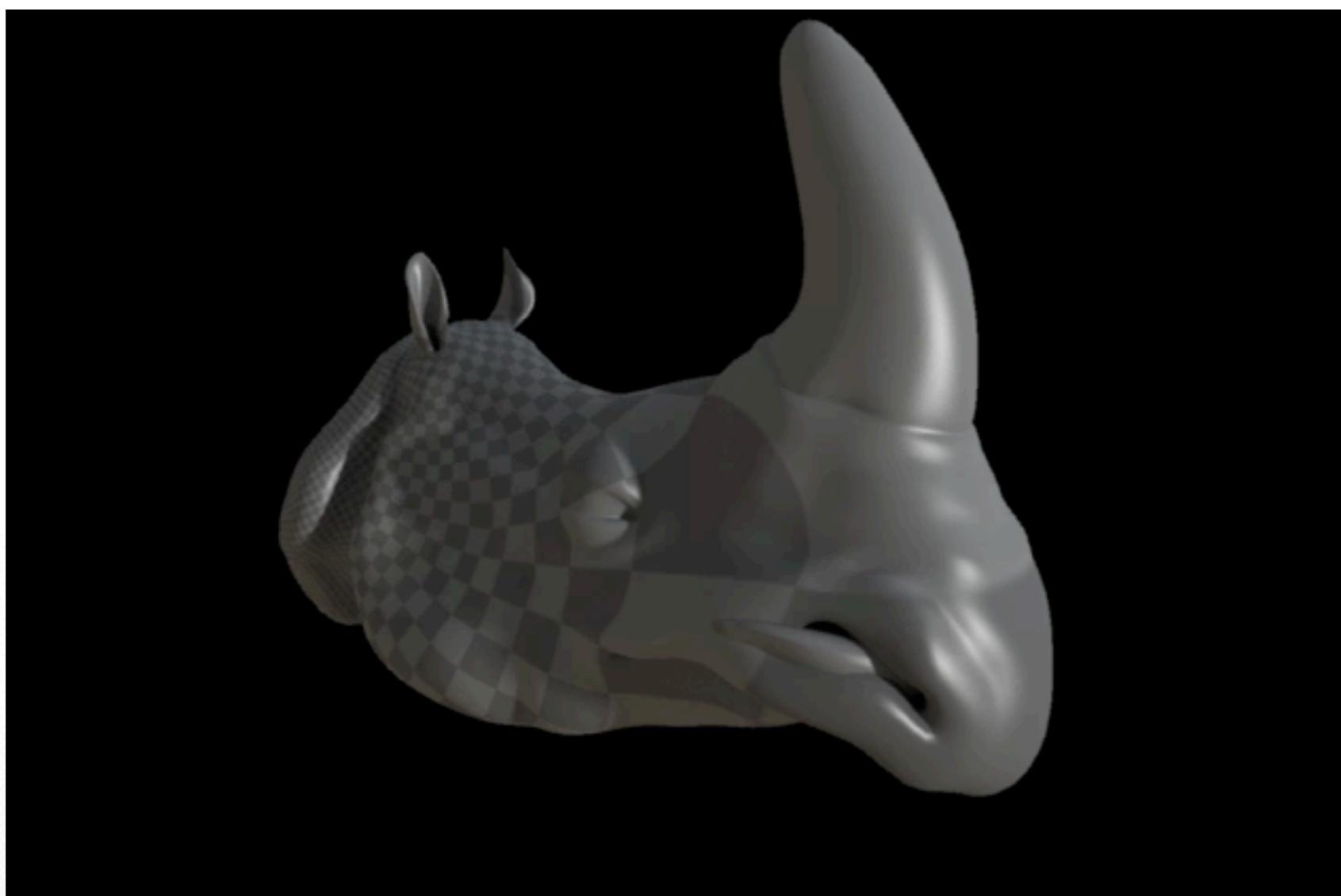
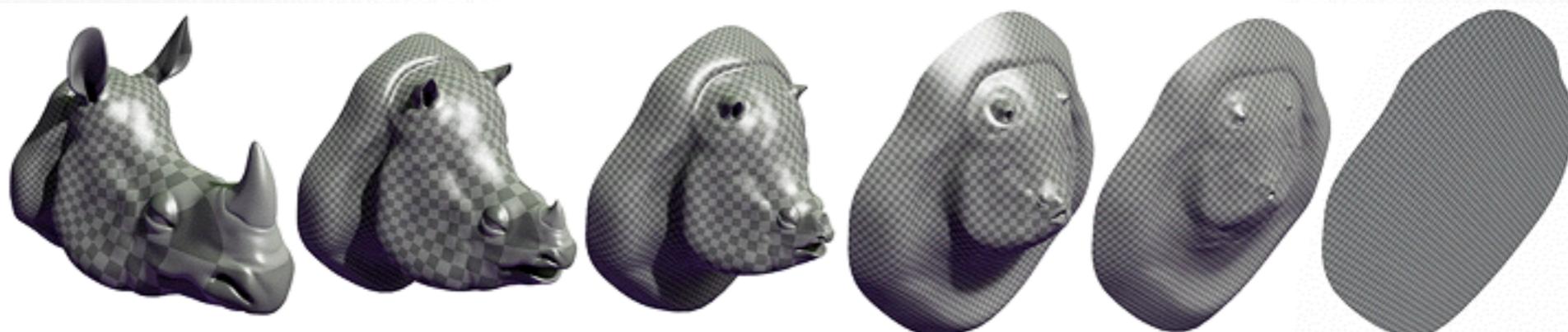
Conformal Maps (A-Similar-AP)

A regular parameterization $\mathbf{x}(u, v)$ is conformal, iff its first fundamental form is a scalar multiple of the identity:

$$\mathbf{I}(u, v) = s(u, v) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



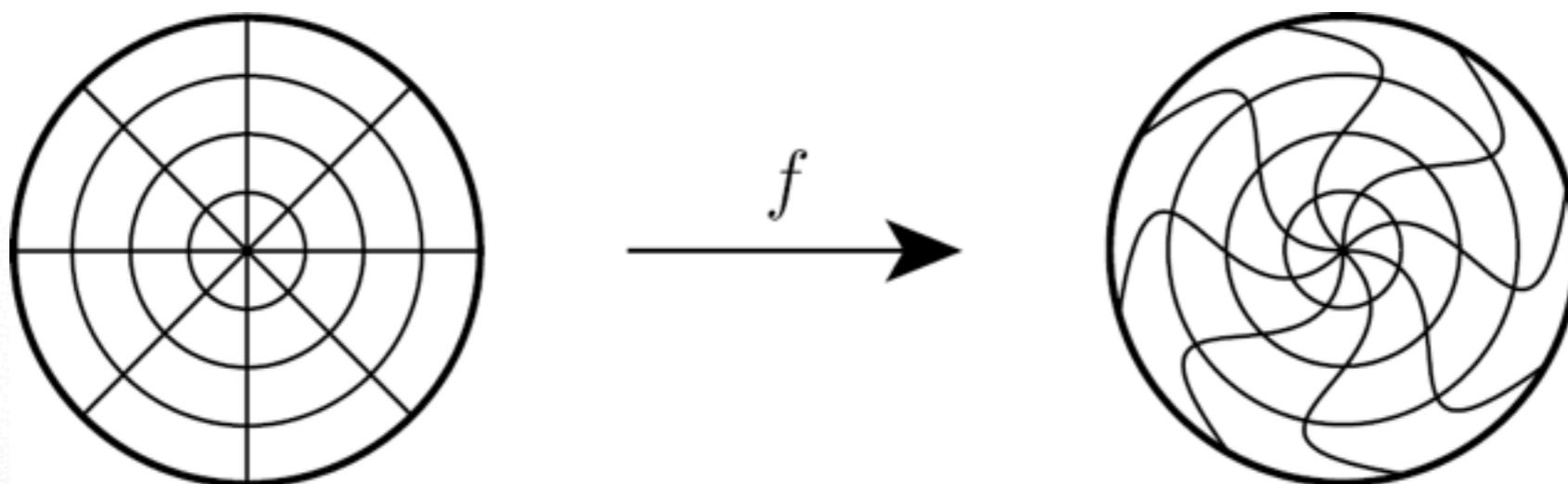
Conformal Flow



Equiareal Maps

A regular parameterization $\mathbf{x}(u, v)$ is equiareal, iff the determinant of its first fundamental form is 1:

$$\det(\mathbf{I}(u, v)) = 1$$



Relationships

An isometric parameterization is conformal and equiareal, and vice versa:

isometric \Leftrightarrow conformal + equiareal

Isometric is ideal, but rare. In practice, people try to compute:

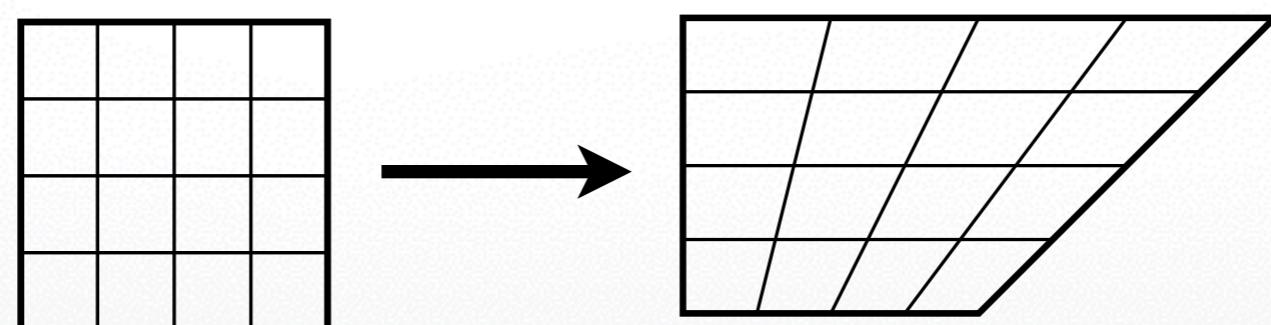
- Conformal
- Equiareal
- Some balance between the two

Harmonic Maps

- A regular parameterization $\mathbf{x}(u, v)$ is harmonic, iff it satisfies

$$\Delta \mathbf{x}(u, v) = 0$$

- isometric \Rightarrow conformal \Rightarrow harmonic
- Easier to compute than conformal, but does not preserve angles



Harmonic Maps

- A harmonic map minimizes the Dirichlet energy

$$\int_{\Omega} \|\nabla \mathbf{x}\|^2 = \int_{\Omega} \|\mathbf{x}_u\|^2 + \|\mathbf{x}_v\|^2 \, du \, dv$$

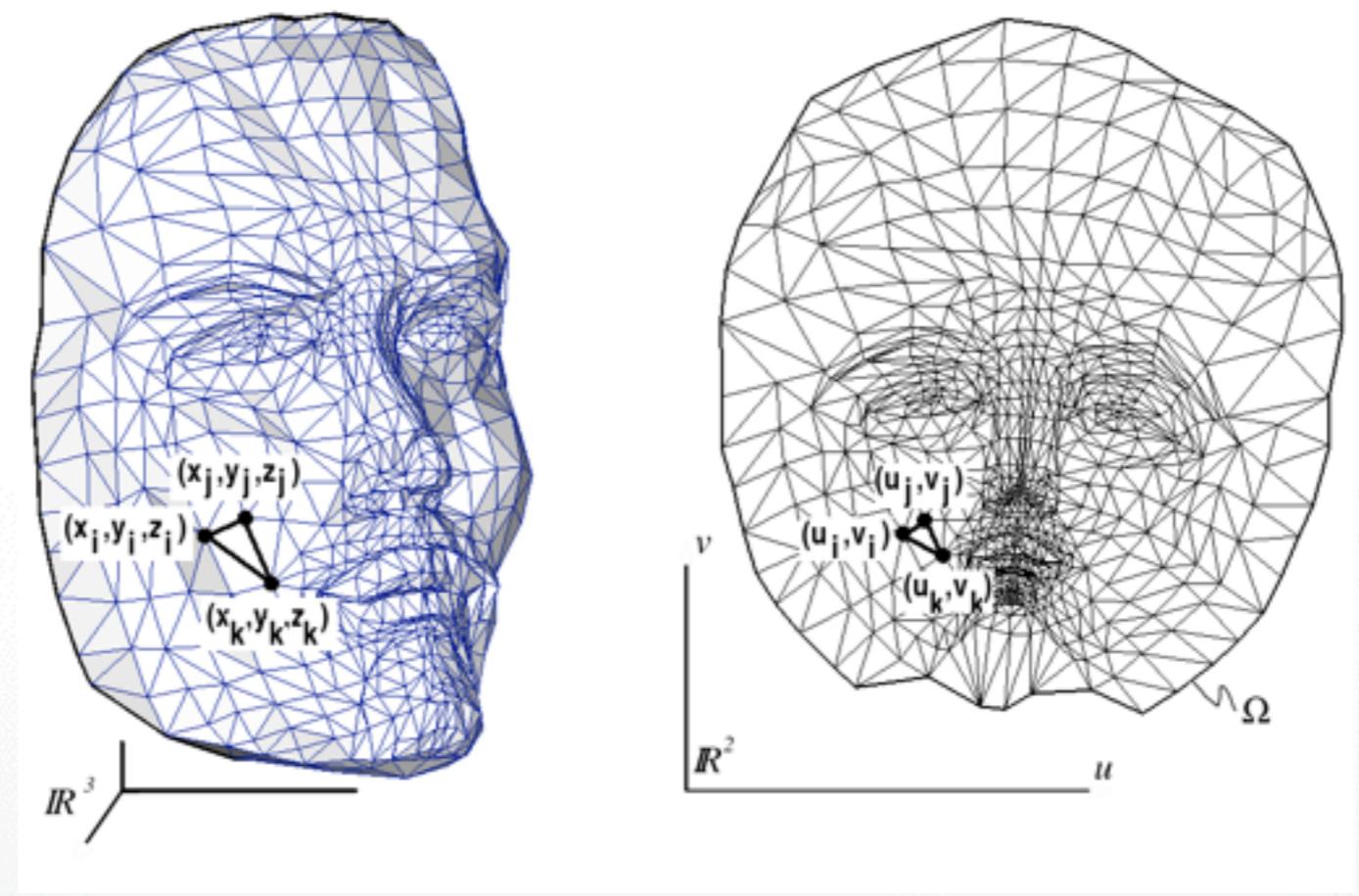
- Variational calculus then tells us that

$$\Delta \mathbf{x}(u, v) = 0$$

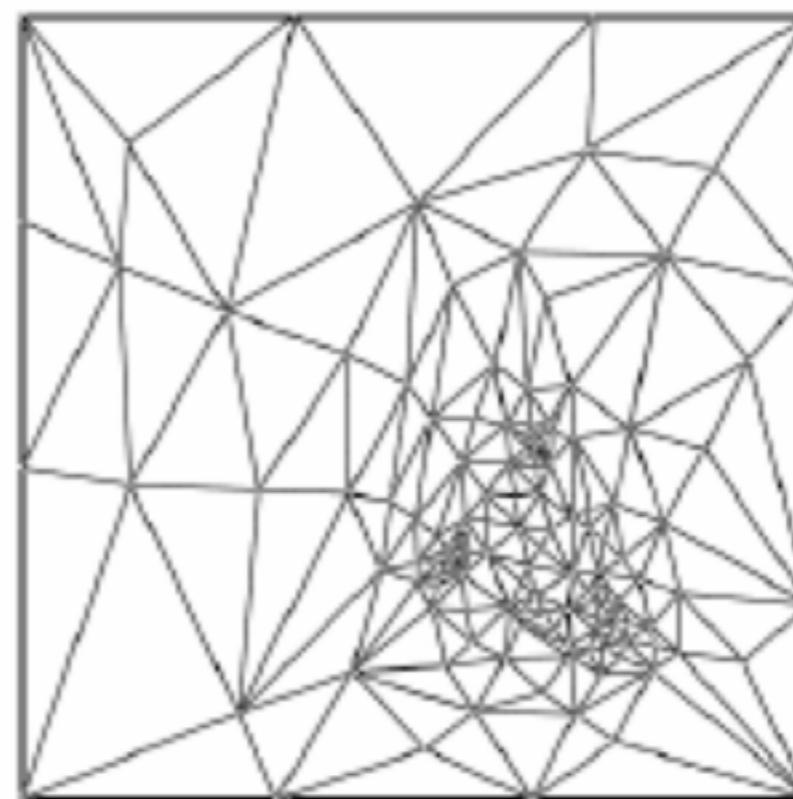
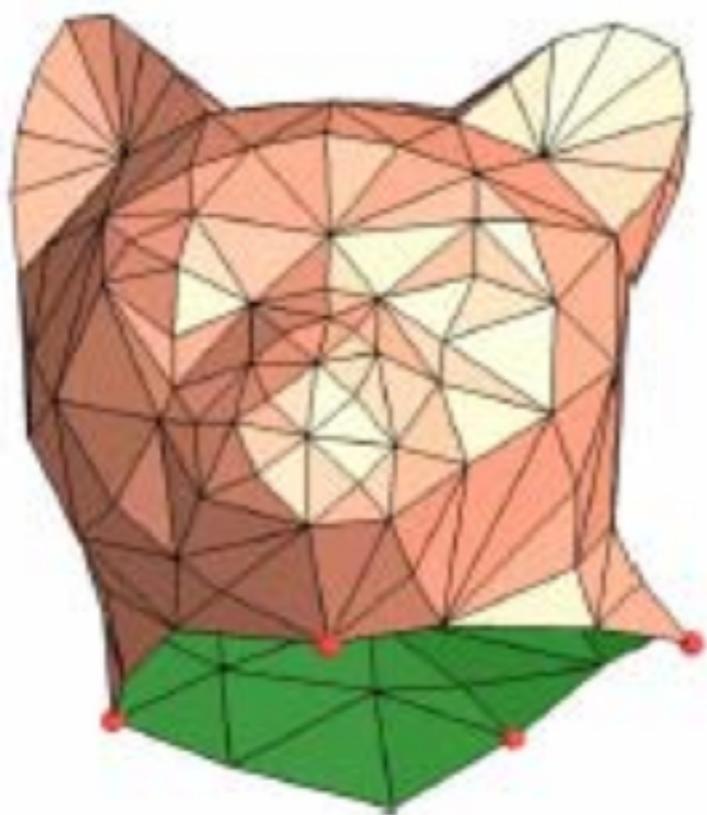
- If $\mathbf{x} : \Omega \rightarrow S$ is harmonic and maps the boundary $\partial\Omega$ of a convex region $\Omega \subset \mathbb{R}^2$ homeomorphically onto the boundary ∂S , then \mathbf{x} is one-to-one.

Parameterization Goal

- Piecewise linear mapping of a discrete 3D triangle mesh onto a planar 2D polygon
- Slightly different situation: Given a 3D mesh, compute the inverse parameterization



Floater's Parameterization



Floater's Parameterization

- For Quadrilateral Patch
- Fix the parameters of the boundary vertices on a unit square
- Derive the bijection \mathbf{u} for each of the interior vertices \mathbf{v}_i by solving

$$u(v_i) = \sum_{k \in v(i)} \lambda_{i,k} u(v_k)$$

where $\lambda_{i,k}$ satisfies shape preserving criteria

and $\sum_{k \in v(i)} \lambda_{i,k} = 1, \quad i = 1, 2, \dots, n$

Floater's Algorithm

- Compute for each i the $\lambda_{i,k}$, $k \in v(i)$
 - Compute a local parameterization for $v(i)$ that preserves the aspect ratio of the angle and length
 - Compute $\lambda_{i,k}$, $k \in v(i)$ that satisfies
 - Shape preserving criteria
 - and $\sum_{k \in v(i)} \lambda_{i,k} = 1$, $i = 1, 2, \dots, n$
- Solve the sparse equation for $u(v_i)$, $i = 1 \dots n$

$$u(v_i) = \sum_{k \in v(i)} \lambda_{i,k} u(v_k)$$

Discrete Harmonic Maps

- Map the boundary ∂S homeomorphically to some (convex) polygon $\partial\Omega$ in the parameter plane
- Minimize the Dirichlet energy of \mathbf{u} by solving the corresponding Euler-Lagrange PDE

$$\Delta_S \mathbf{u} = 0$$

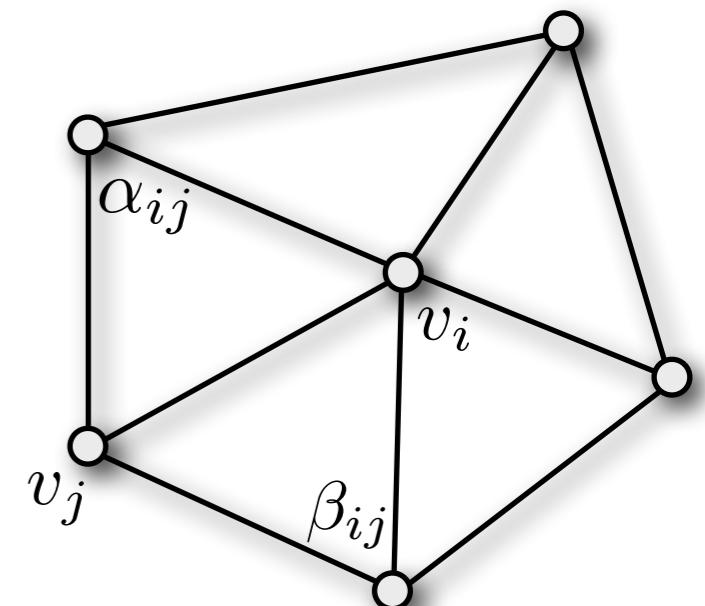
- Requires discretization of Laplace-Beltrami
- Compare to surface fairing

Discrete Harmonic Maps

- System of linear equations

$$\forall v_i \in \mathcal{S} : \sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} (\mathbf{u}(v_j) - \mathbf{u}(v_i))$$

$$w_{ij} = \cot\alpha_{ij} + \cot\beta_{ij}$$



- Properties of system matrix:
 - Symmetric + positive definite \rightarrow unique solution
 - Sparse \rightarrow efficient solvers

Discrete Harmonic Maps

- But...
 - Does the same theory hold for discrete harmonic maps as for harmonic maps?
 - In other words, is it possible for triangles to flip or become degenerate?

Convex Combination Maps

- If the linear equations are satisfied

$$\sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} (\mathbf{u}(v_j) - \mathbf{u}(v_i))$$

and if the weights satisfy

$$w_{ij} > 0 \quad \wedge \quad \sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} = 1$$

then we get a convex combination mapping.

Convex Combination Maps

- Each $\mathbf{u}(v_i)$ is a convex combination of $\mathbf{u}(v_j)$

$$\mathbf{u}(v_i) = \sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} \mathbf{u}(v_j)$$

- If $\mathbf{u} : S \rightarrow \Omega$ is a convex combination map that maps the boundary ∂S homeomorphically to the boundary $\partial\Omega$ of a convex region $\Omega \subset \mathbb{R}^2$, then \mathbf{u} is one-to-one.

Convex Combination Maps

- Uniform barycentric weights

$$w_{ij} = 1/\text{valence}(v_i)$$

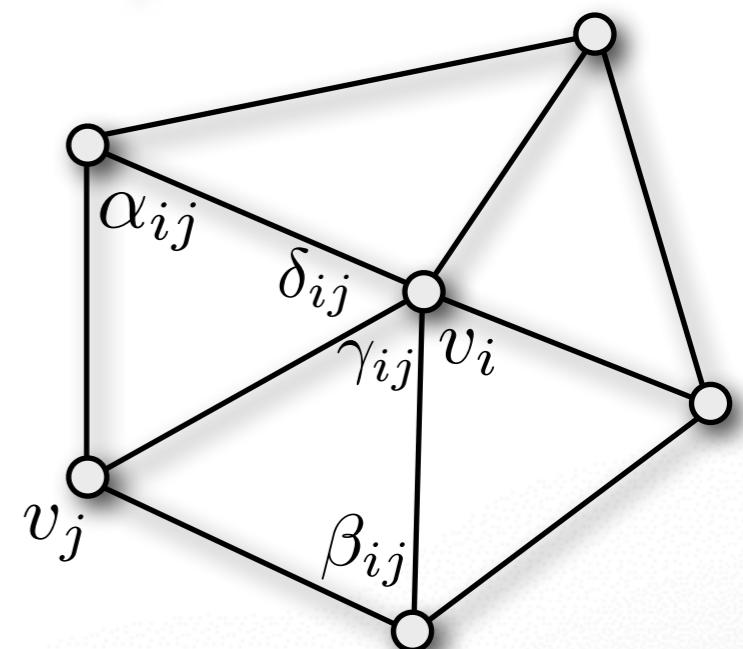
- Cotangent weights (> 0 if $\alpha_{ij} + \beta_{ij} < \pi$)

$$w_{ij} = \cot(\alpha_{ij}) + \cot(\beta_{ij})$$

- Mean value weights

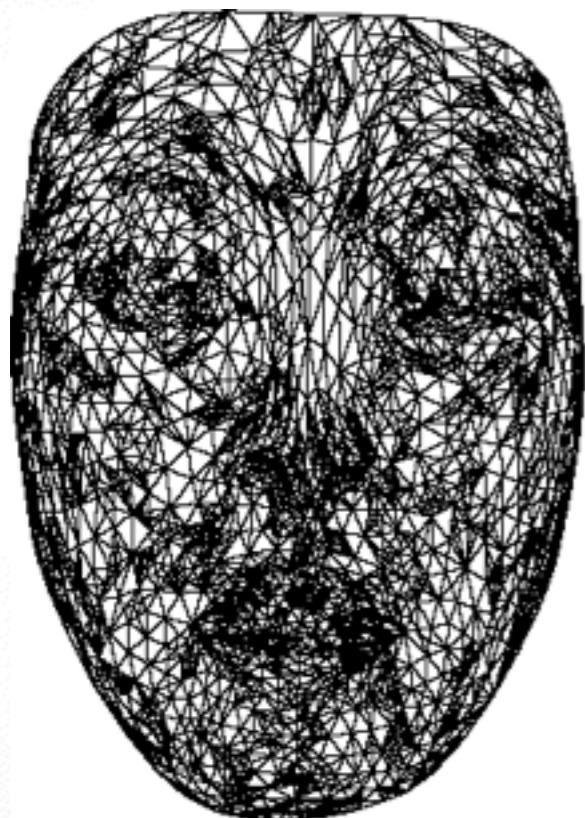
$$w_{ij} = \frac{\tan(\delta_{ij}/2) + \tan(\gamma_{ij}/2)}{\|\mathbf{p}_j - \mathbf{p}_i\|}$$

(no negative weights, even for obtuse angles)

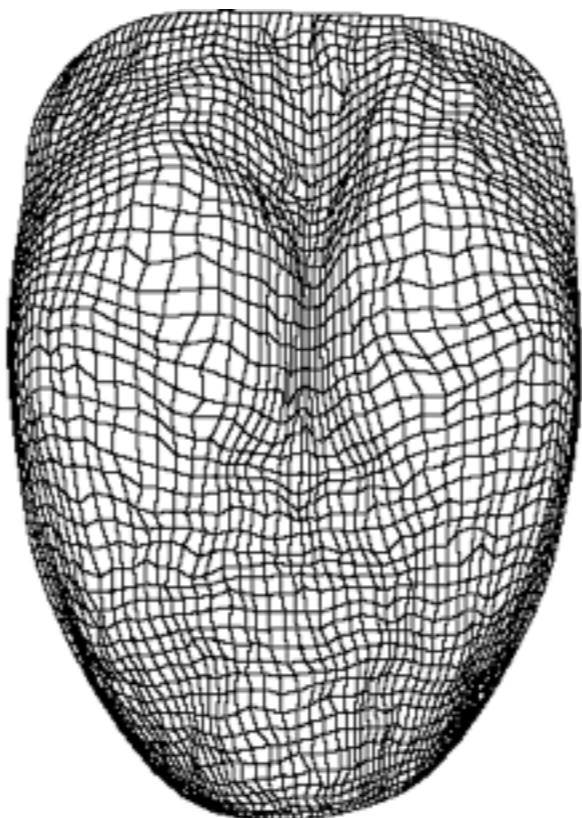


Convex Combination Maps

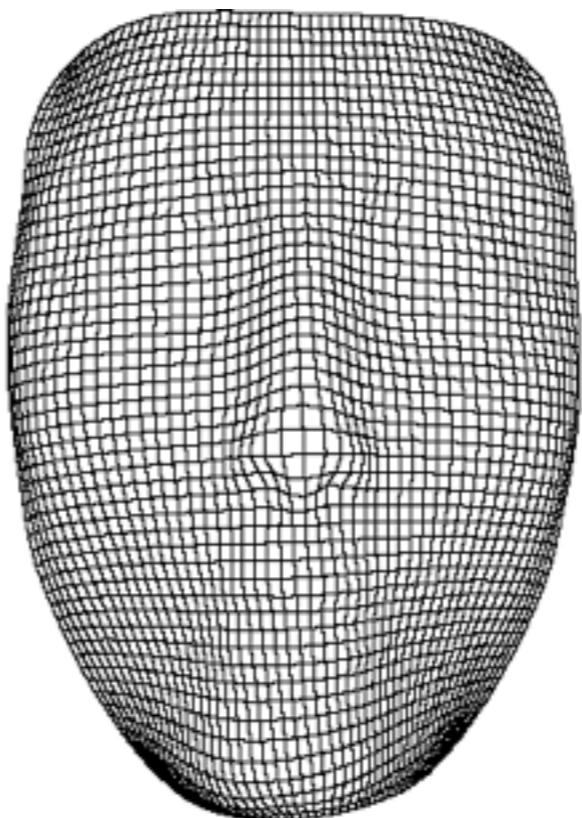
- Comparison



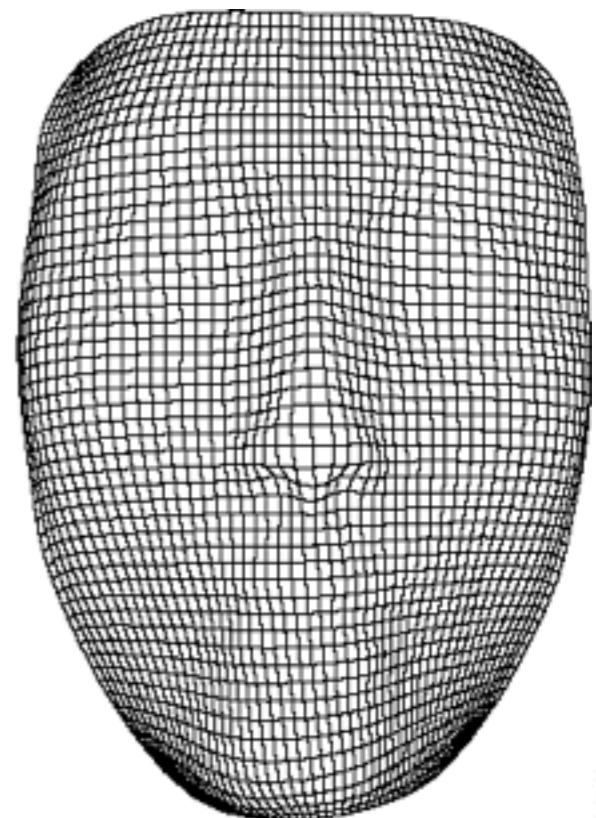
original
mesh



uniform
weights



cotan
weights
(shape preserving)

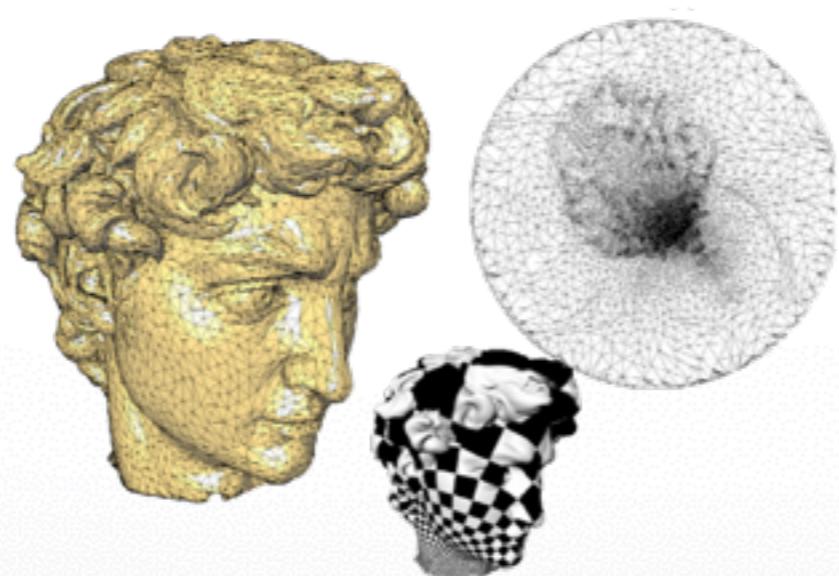


mean
value

Fixing the Boundary

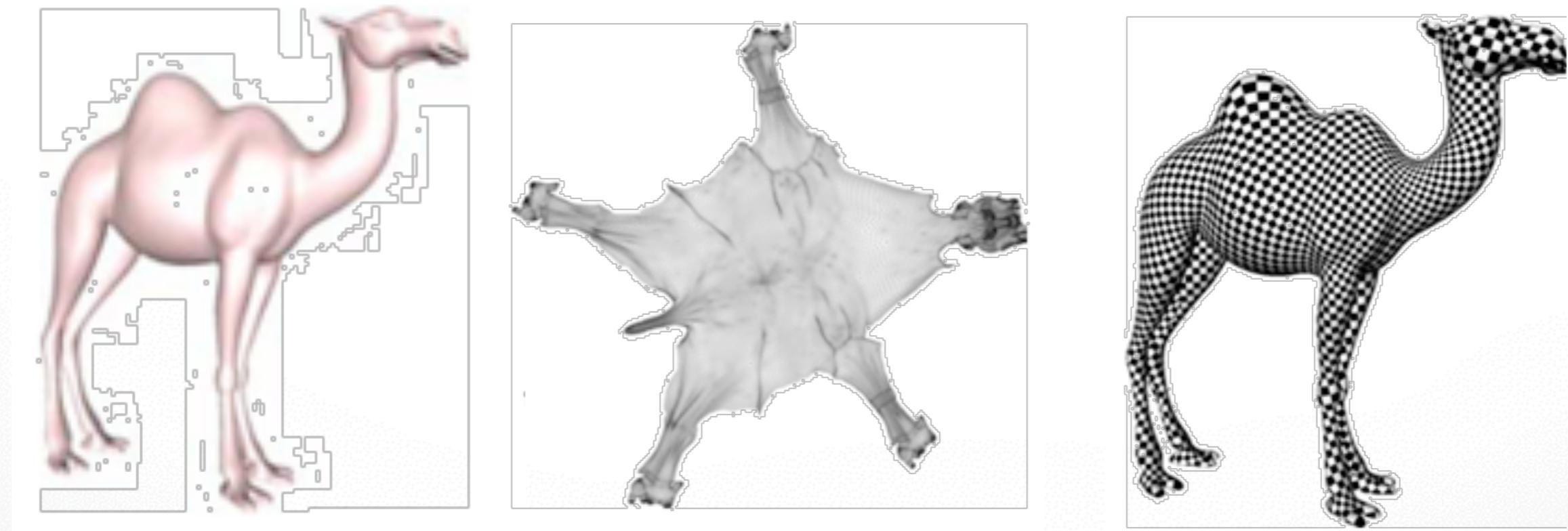
- Choose a simple convex shape
 - Triangle, square, circle
- Distribute points on boundary
 - Use chord length parameterization

Fixed boundary can create high distortion

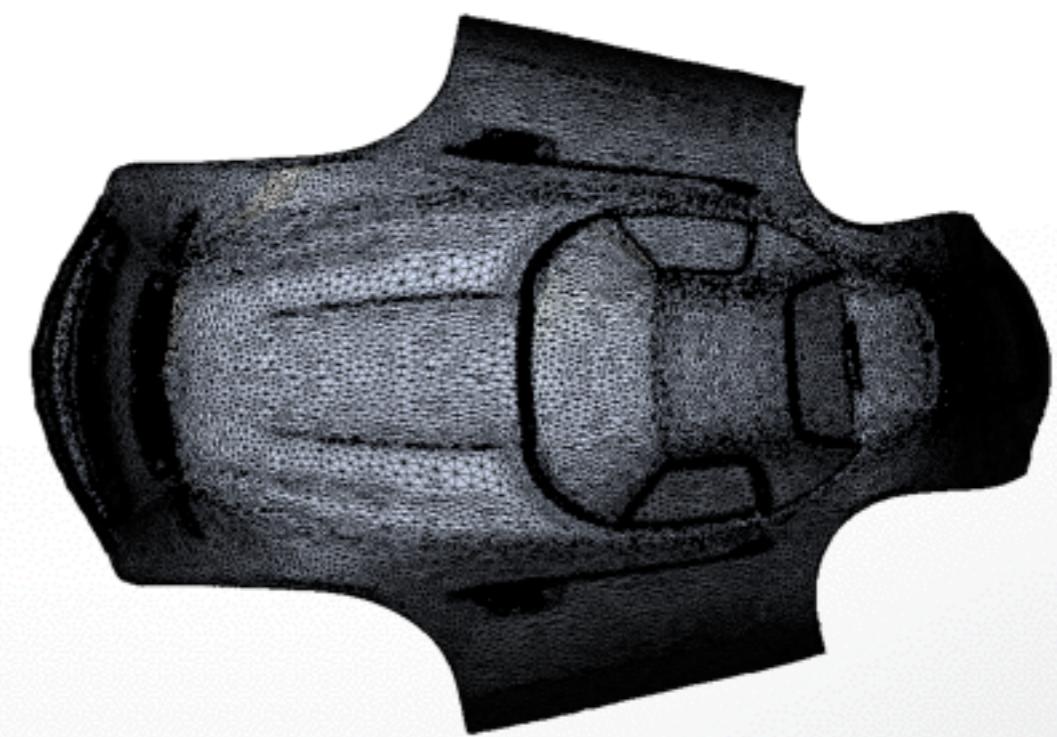
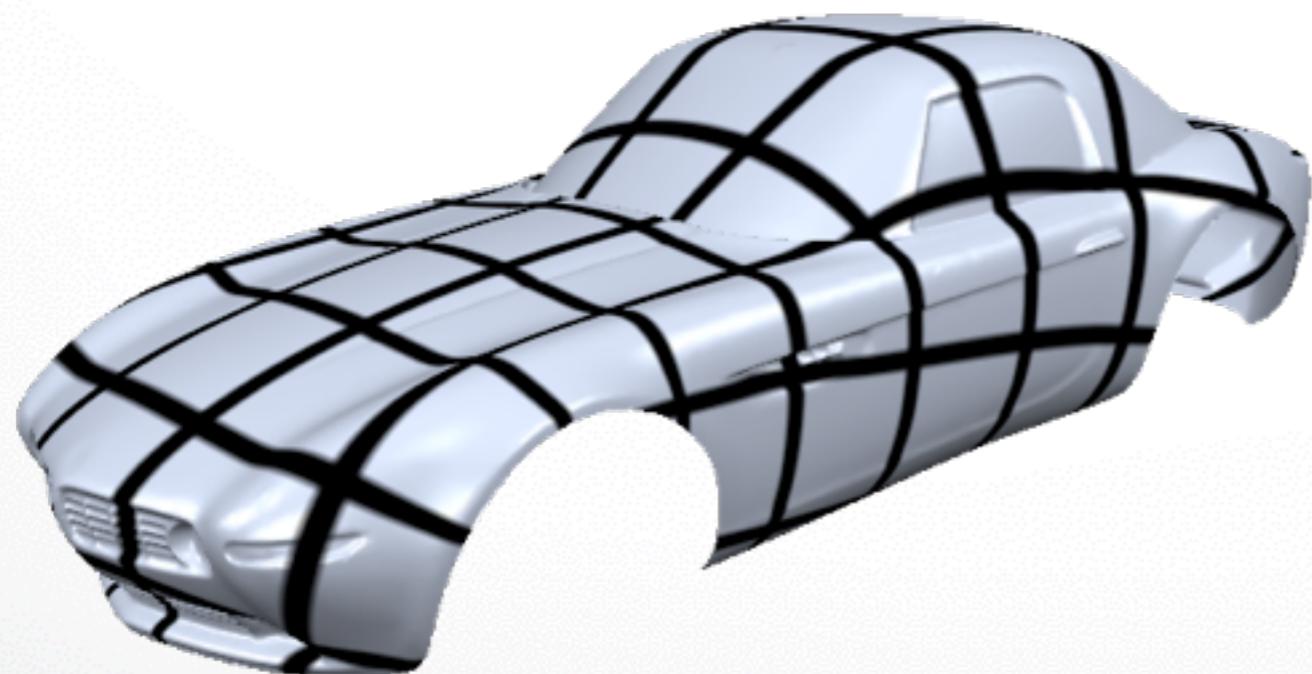
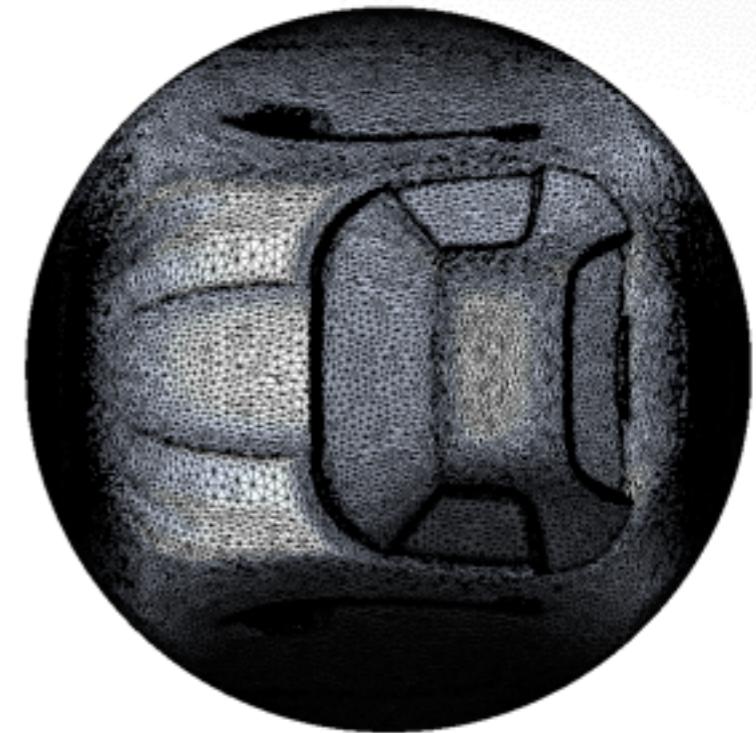
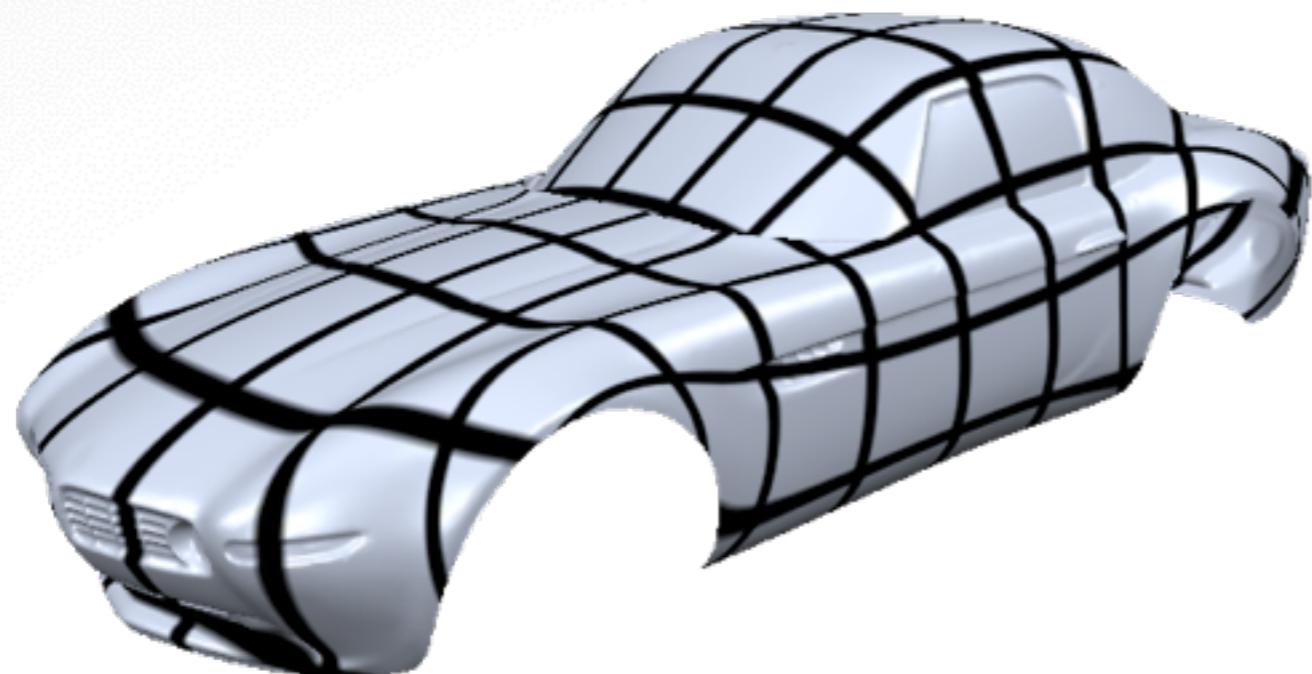


Open Boundary Mappings

- Include boundary vertices in the optimization
- Produces mappings with lower distortion

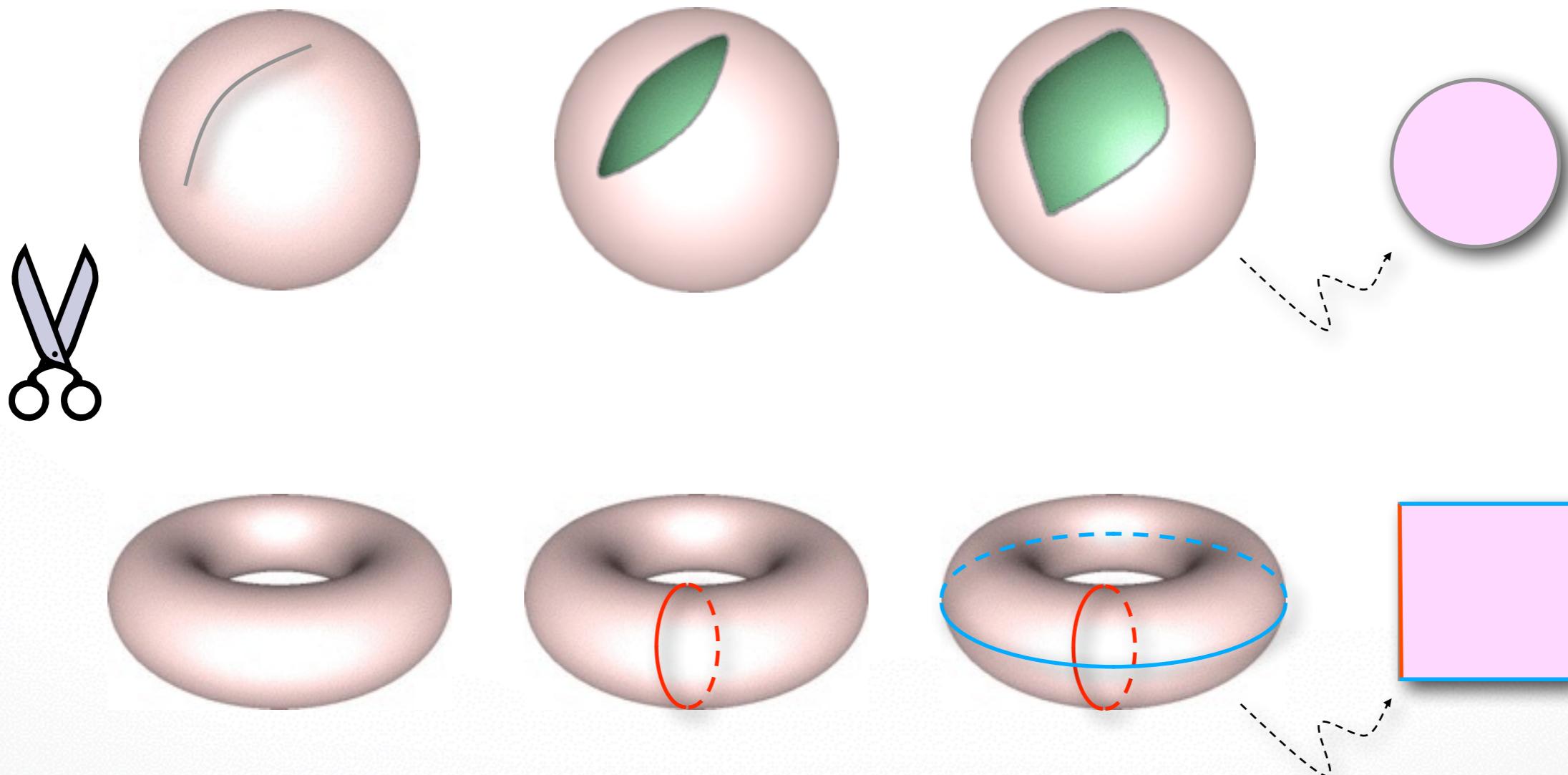


Open Boundary Mappings

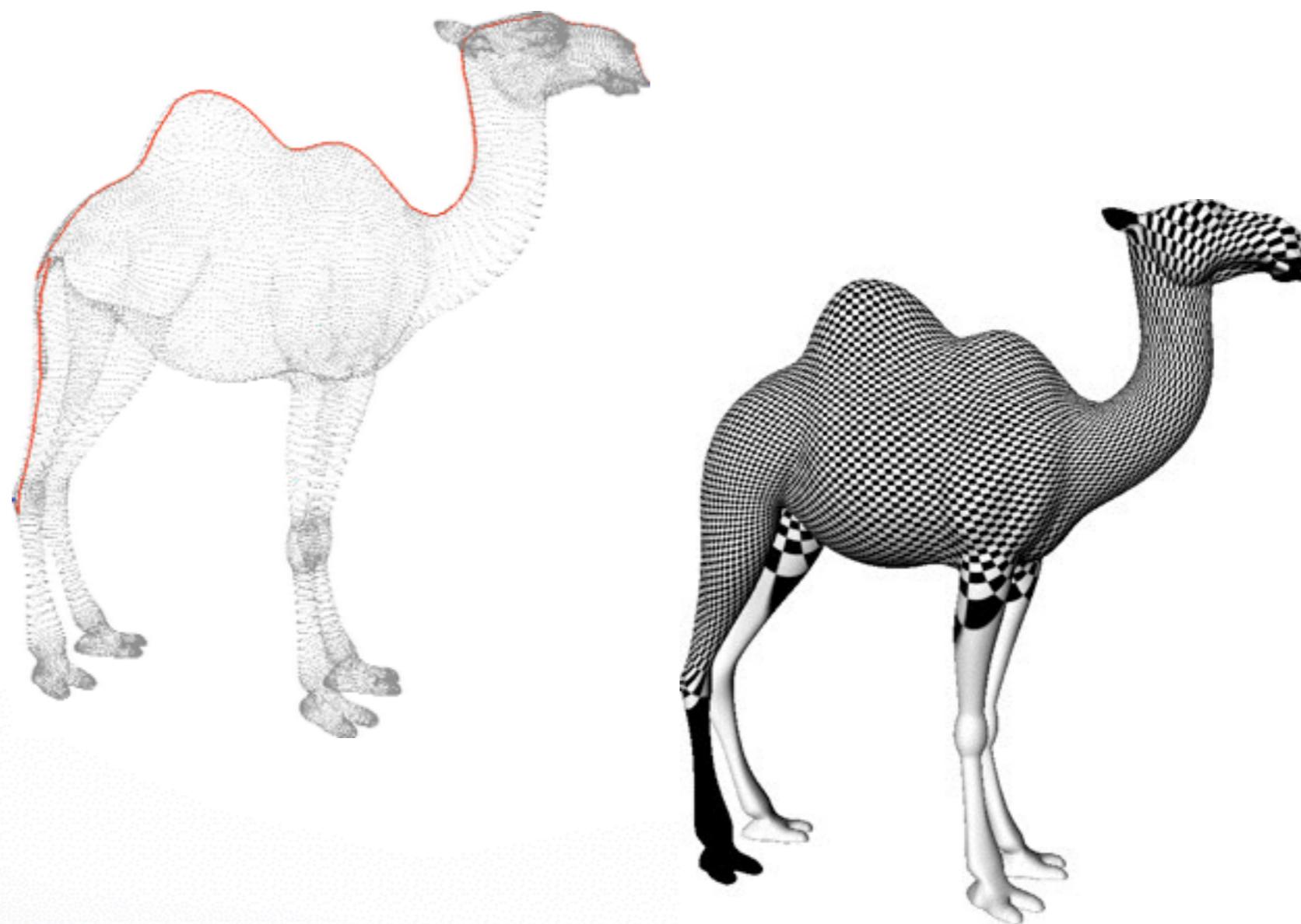


Need disk-like topology

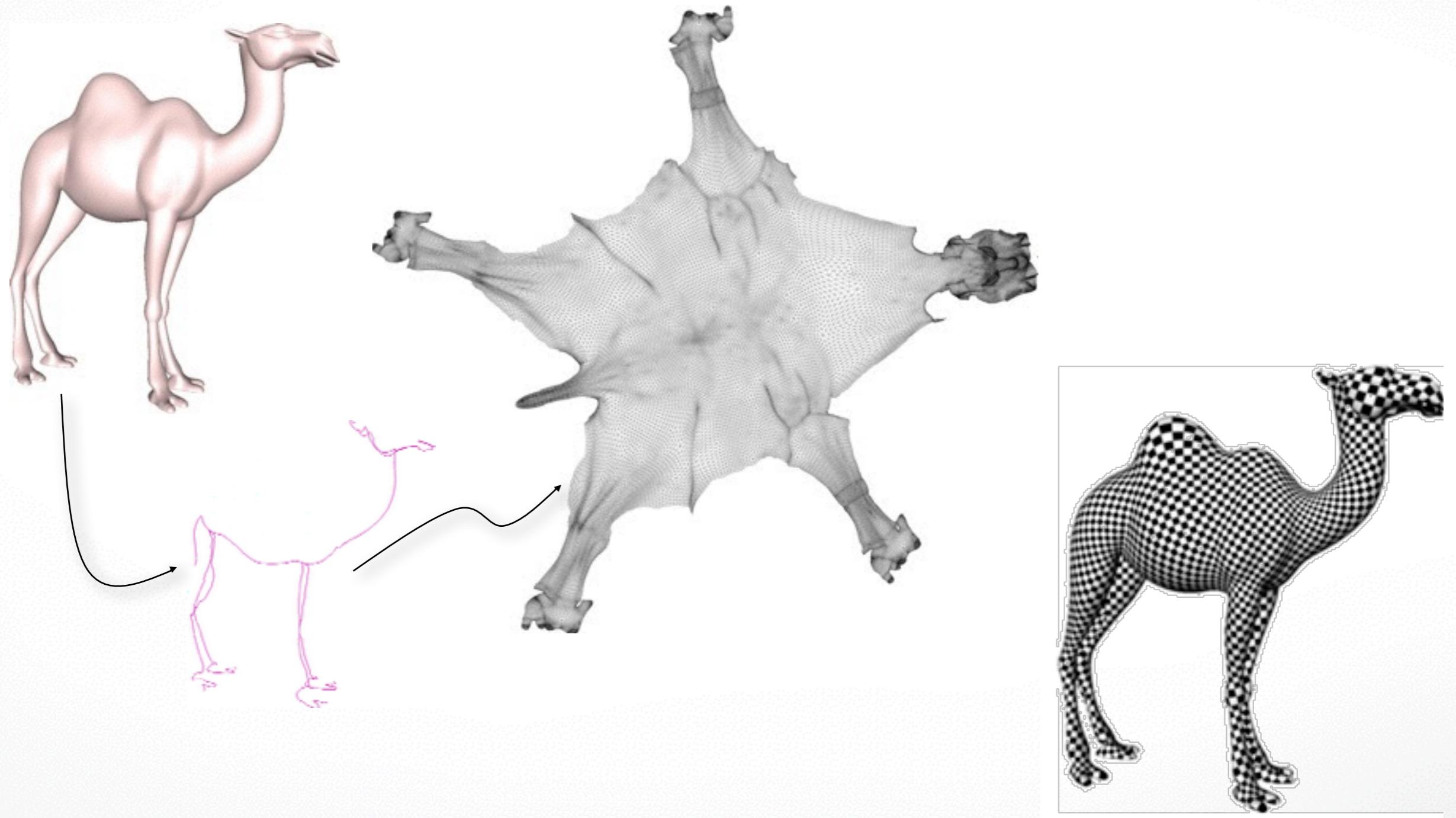
- Introduce cuts on the mesh



Naive Cut, Numerical Problems

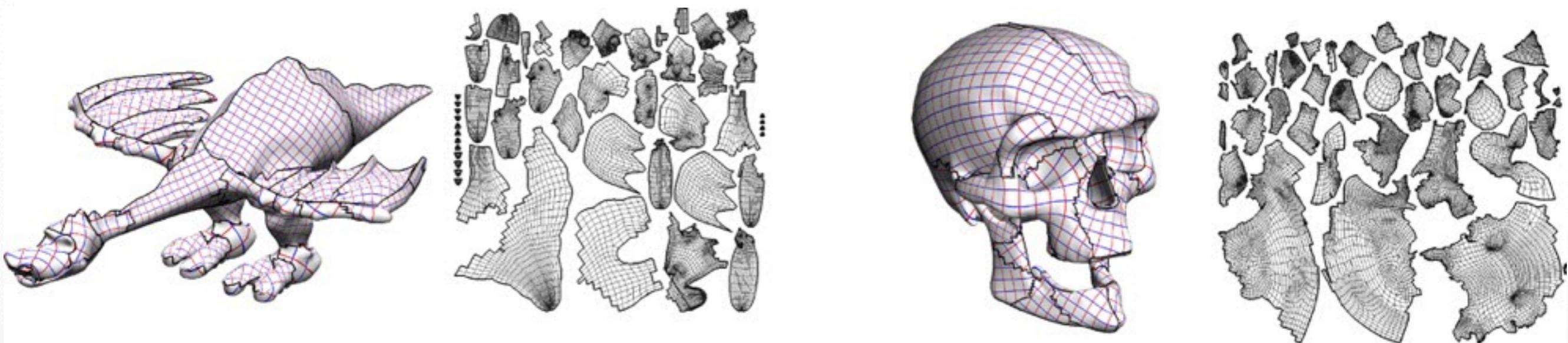


Smart Cut, Free Boundary



Texture Atlas Generation

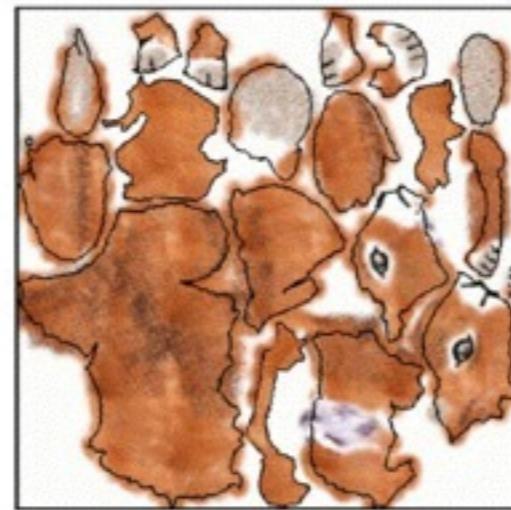
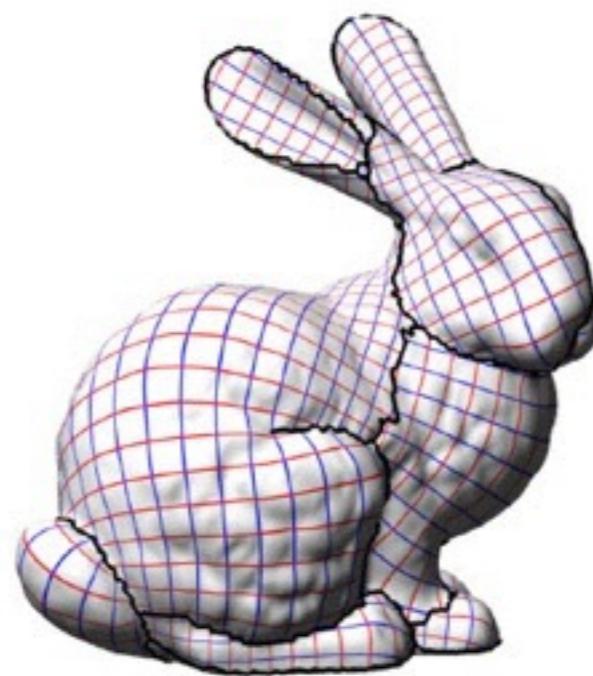
- Split model into number of patches (atlas)
 - because higher genus models cannot be mapped onto plane and/or
 - because distortion, the number of patches will be too high eventually



Levy, Petitjean, Ray, Maillot: *Least Squares Conformal Maps for Automatic Texture Atlas Generation*, SIGGRAPH, 2002

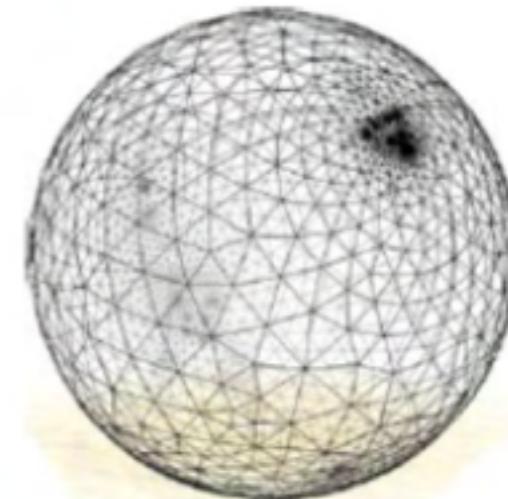
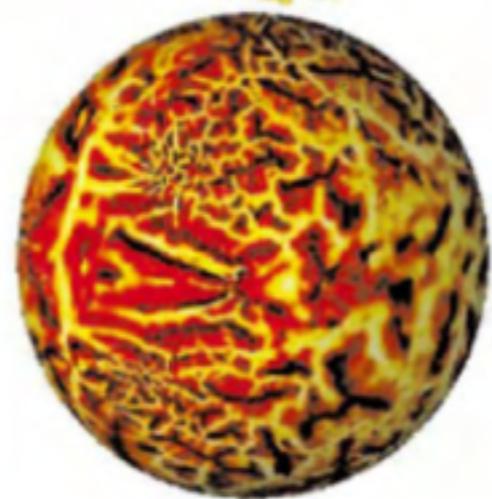
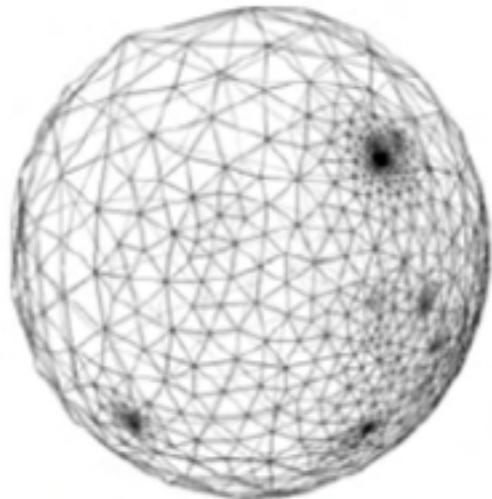
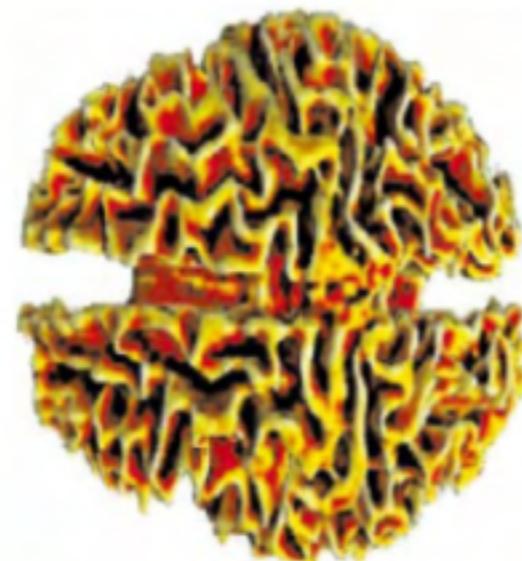
Texture Atlas Generation

- Split model into number of patches (atlas)
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Levy, Petitjean, Ray, Maillot: *Least Squares Conformal Maps for Automatic Texture Atlas Generation*, SIGGRAPH, 2002

Non-Planar Domains



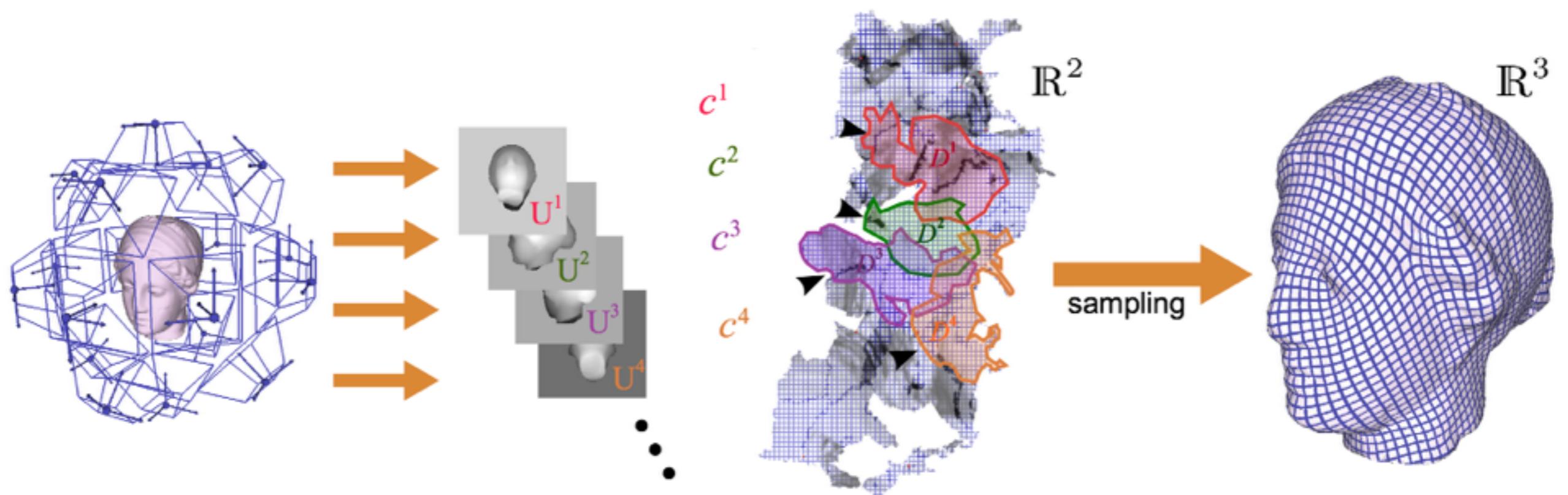
(a) [Alexa 2000]

(b) [Haker et al., 2000]

(c) [Isenburg et al., 2001]

seamless, continuous parameterization of genus-0 surfaces

Global Parameterization – Range Images



Constrained Parameterizations

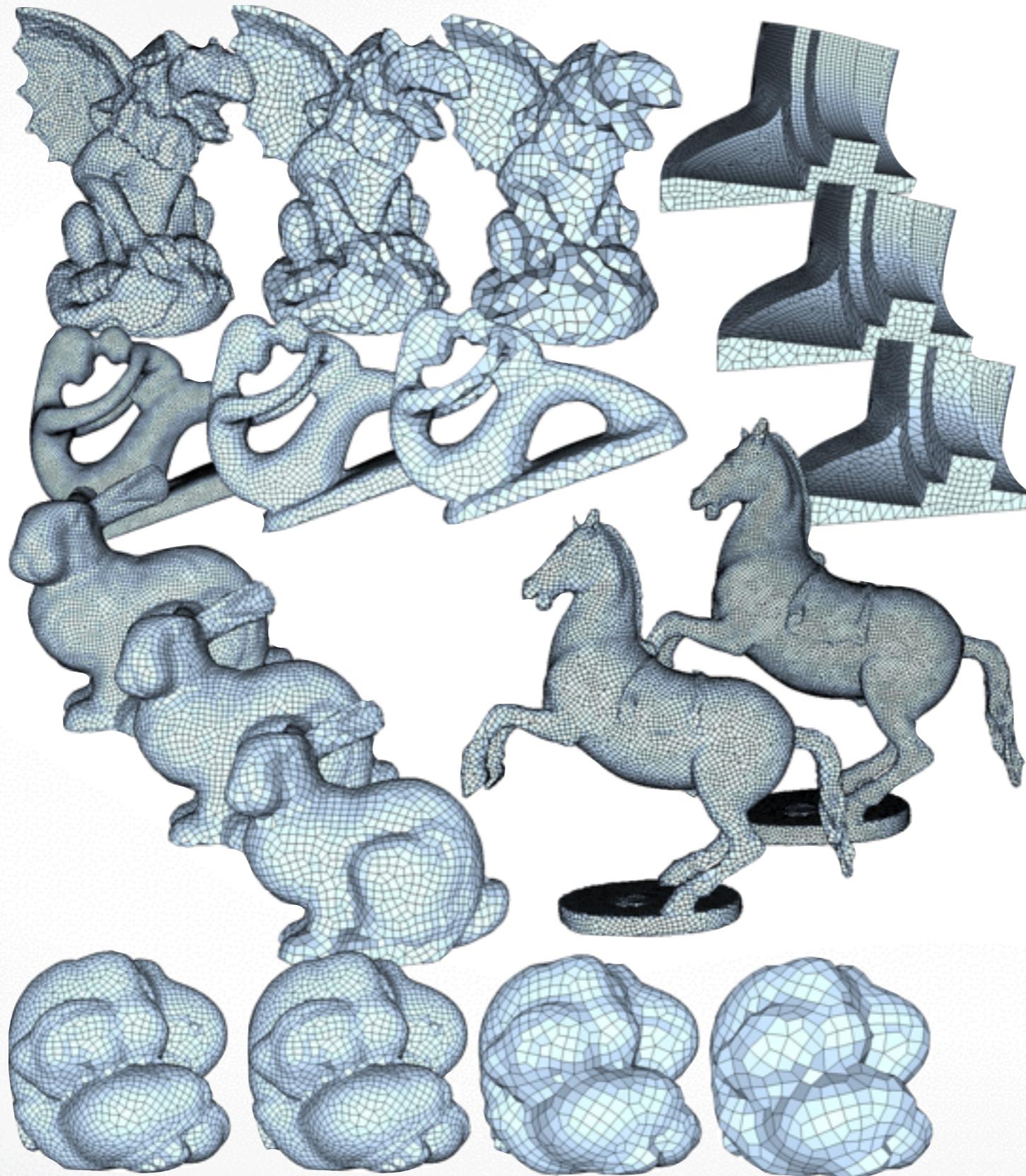


Levy: *Constraint Texture Mapping*, SIGGRAPH 2001.

Literature

- Book, Chapter 5
- Hormann et al.: Mesh Parameterization, Theory and Practice, Siggraph 2007 Course Notes
- Floater and Hormann: Surface Parameterization: a tutorial and survey, advances in multiresolution for geometric modeling, Springer 2005
- Hormann, Polthier, and Sheffer, Mesh Parameterization: Theory and Practice, SIGGRAPH Asia 2008 Course Notes

Next Time



Decimation

<http://cs599.hao-li.com>

Thanks!

