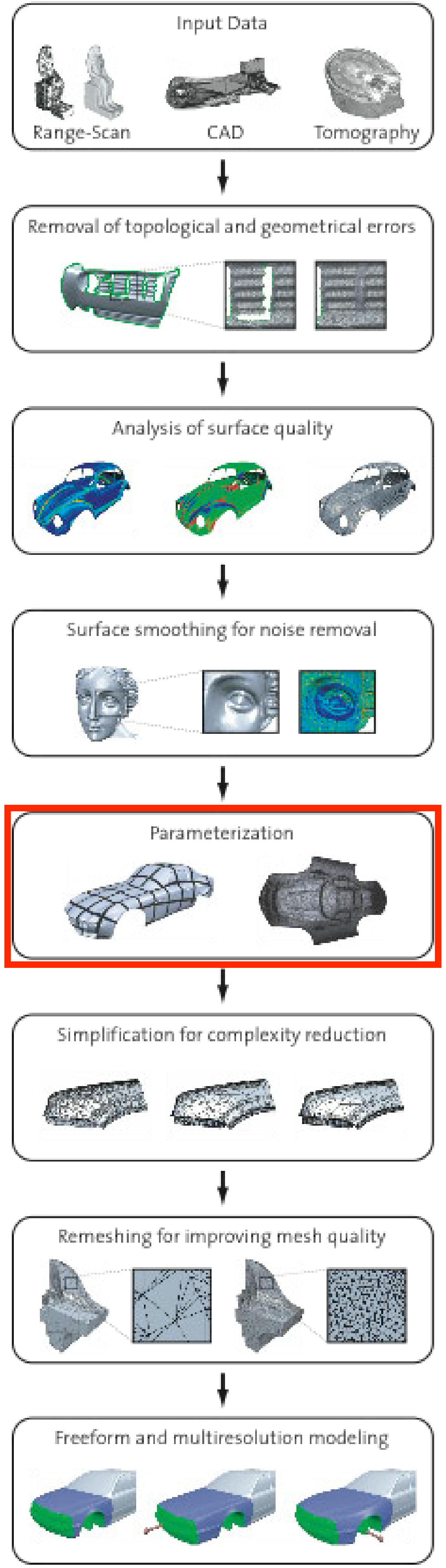


Surface Parameterization

Christian Rössl
INRIA Sophia-Antipolis

SIGGRAPH2006

Outline

- Motivation
- Objectives and Discrete Mappings
- Angle Preservation
  - Discrete Harmonic Maps
  - Discrete Conformal Maps
  - Angle Based Flattening
- Reducing Area Distortion
- Alternative Domains

Surface Parameterization

Mercator-Projektion

Mollweide-Projektion

[www.wikipedia.de]

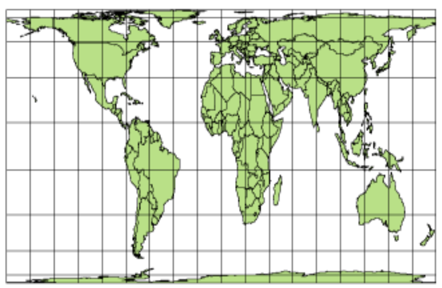
Surface Parameterization

Mollweide-Projektion

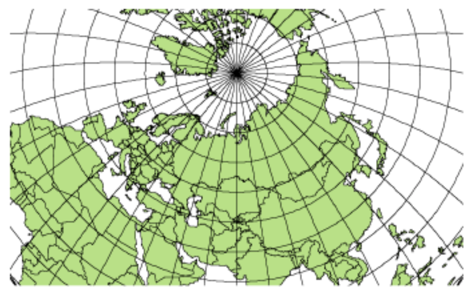
Mercator-Projektion

Zylinderprojektion nach Miller

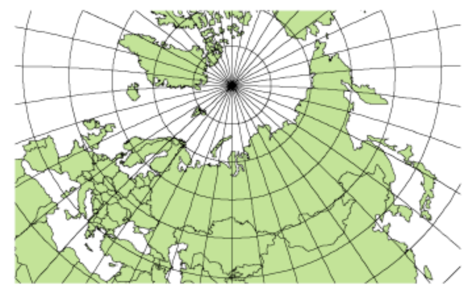
Hammer-Aitoff-Projektion



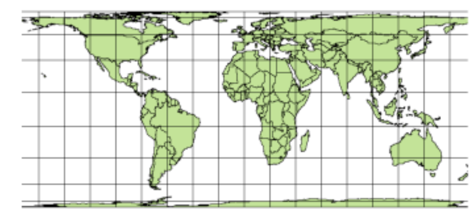
Peters-Projektion



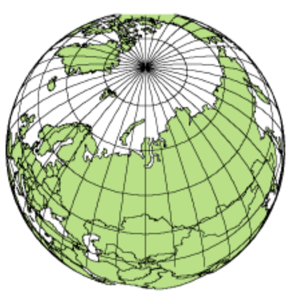
Längentreue Azimuthalprojektion



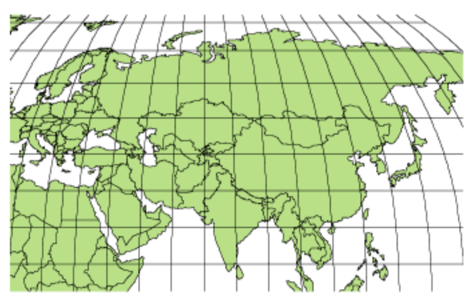
Stereographische Projektion



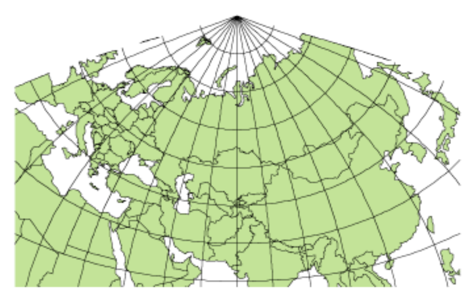
Behrmann-Projektion



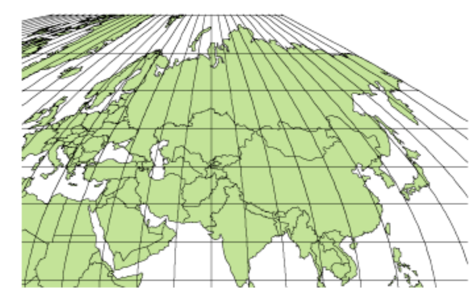
Senkrechte Umgebungsperspektive



Robinson-Projektion

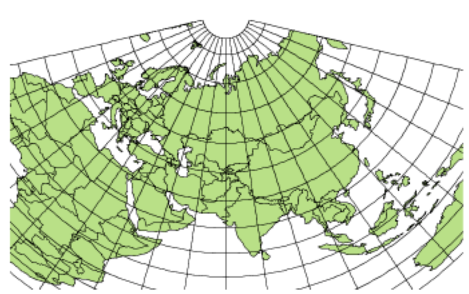


Hotine Oblique Mercator-Projektion



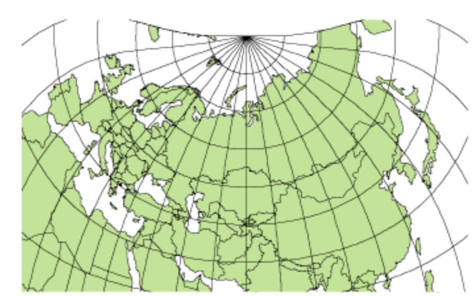
Sinusoidale Projektion

Gnomonische Projektion



Flächentreue Kegelprojektion

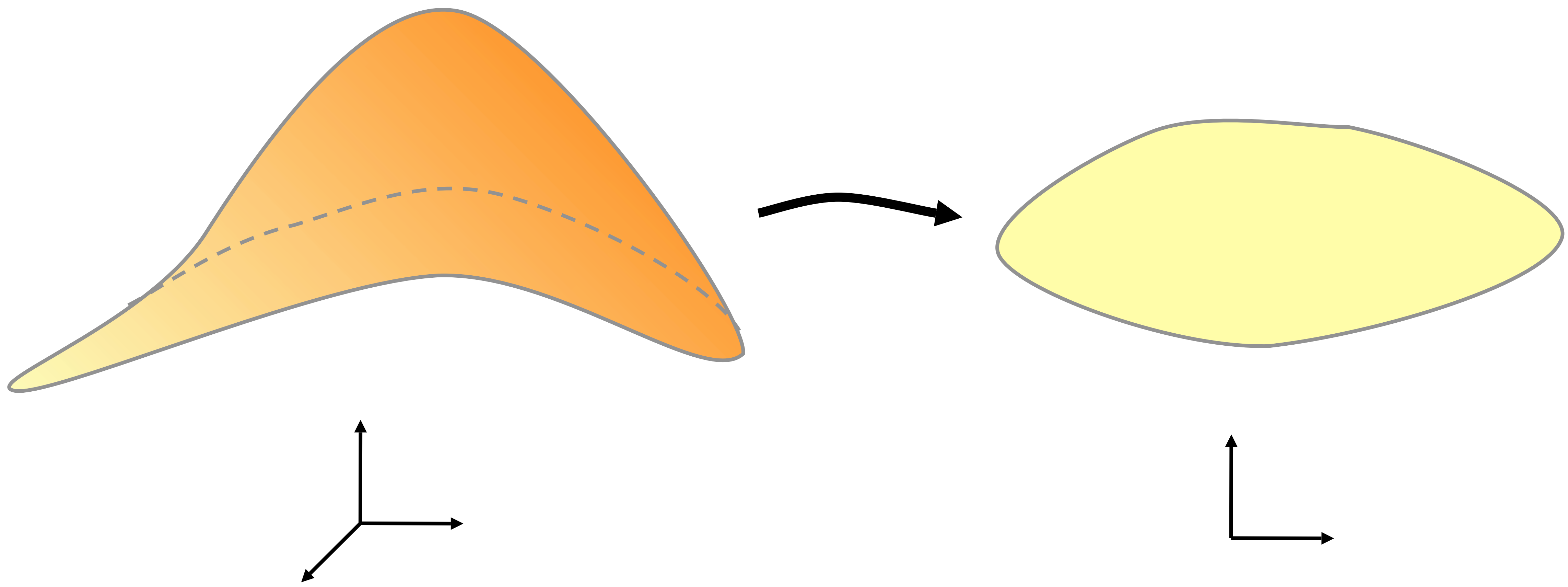
Transverse Mercator-Projektion



Cassini-Soldner-Projektion

[www.wikipedia.de]

Surface Parameterization



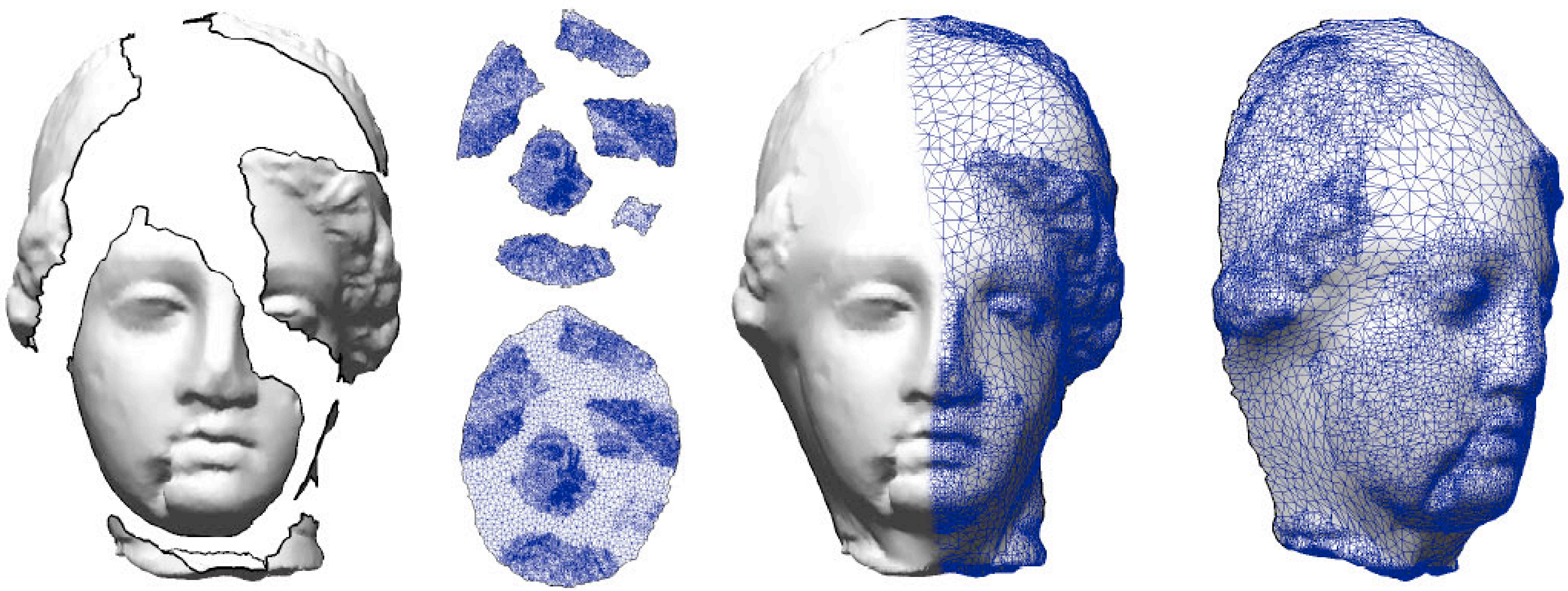
Motivation

- Texture mapping

Lévy, Petitjean, Ray, and Maillot: *Least squares conformal maps for automatic texture atlas generation*, SIGGRAPH 2002

Motivation

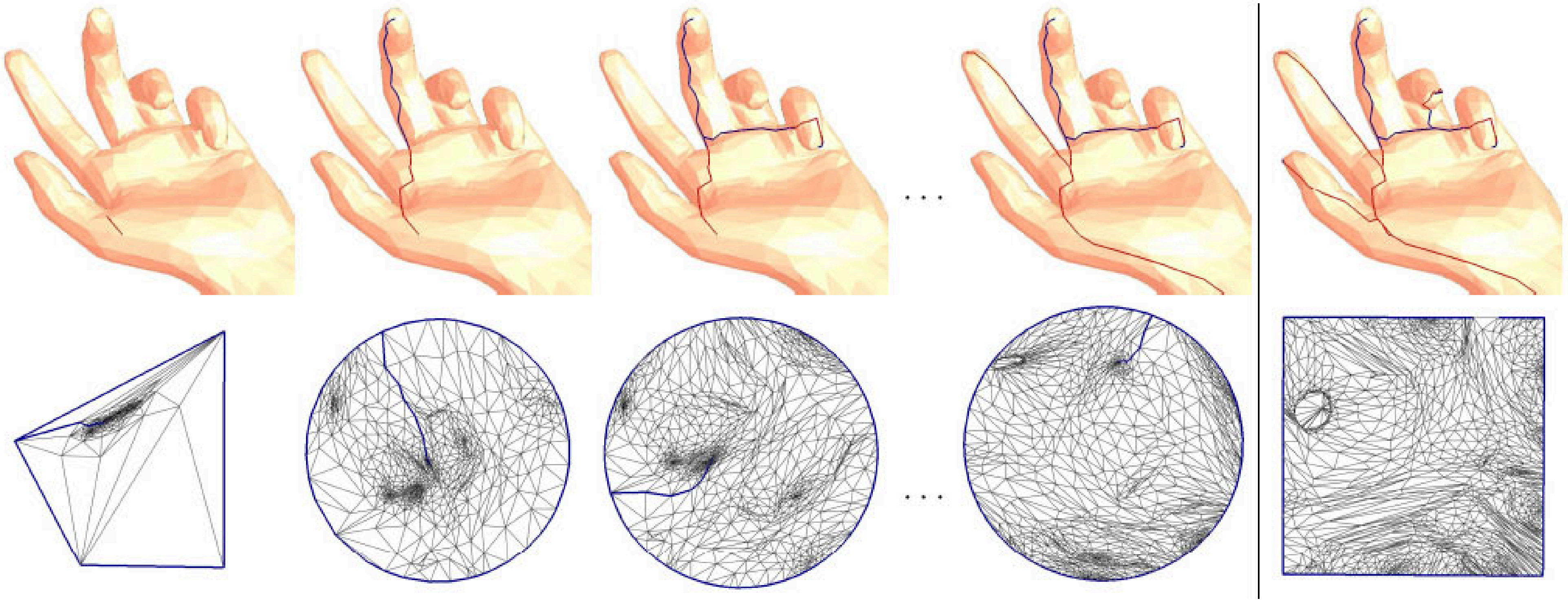
- Many operations are simpler on planar domain

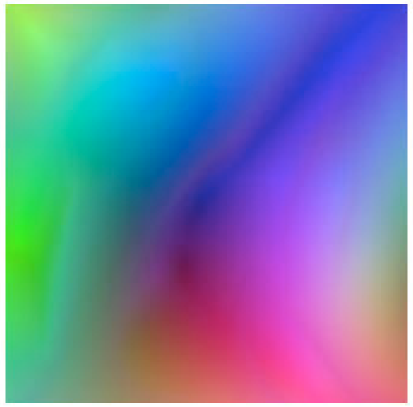


Lévy: *Dual Domain Exrapolation*, SIGGRAPH 2003

Motivation

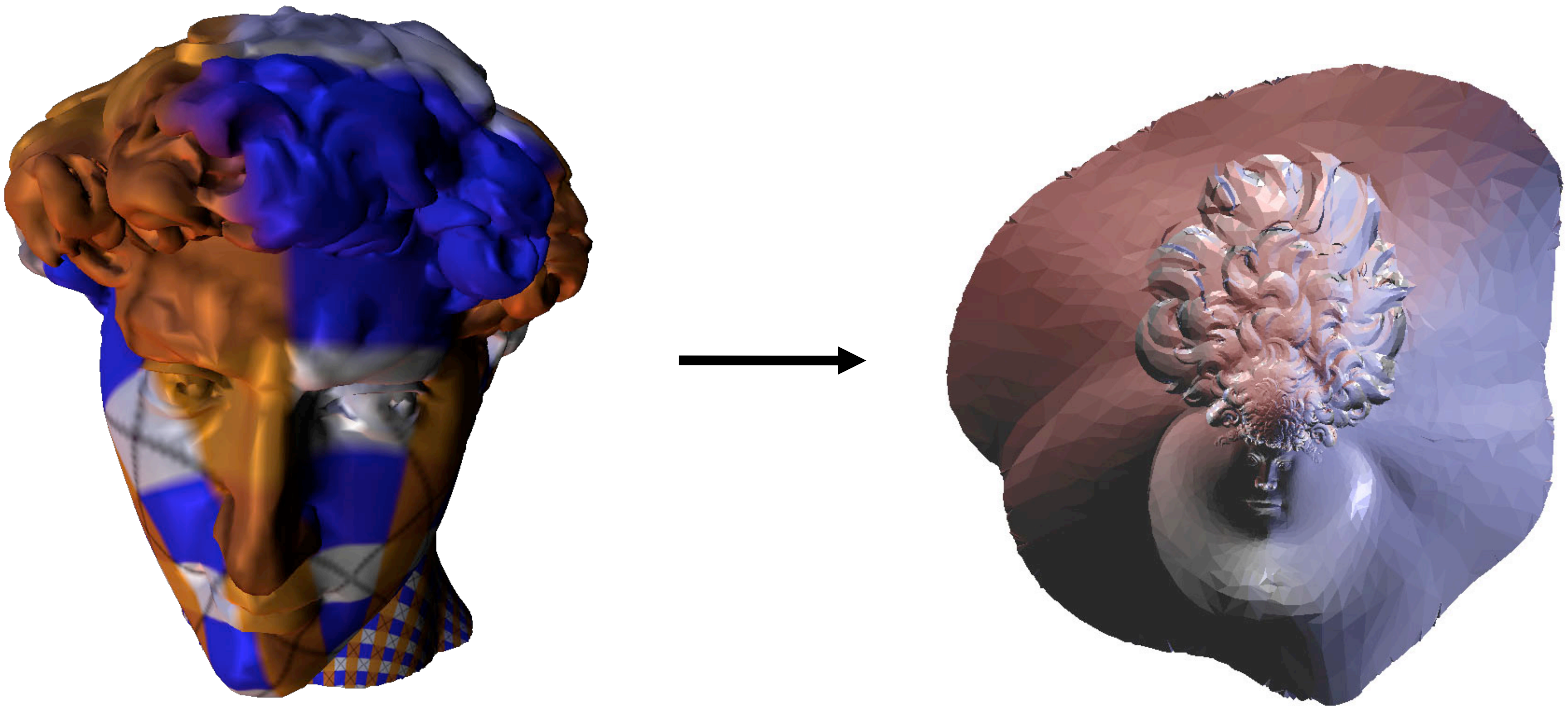
- Exploit regular structure in domain



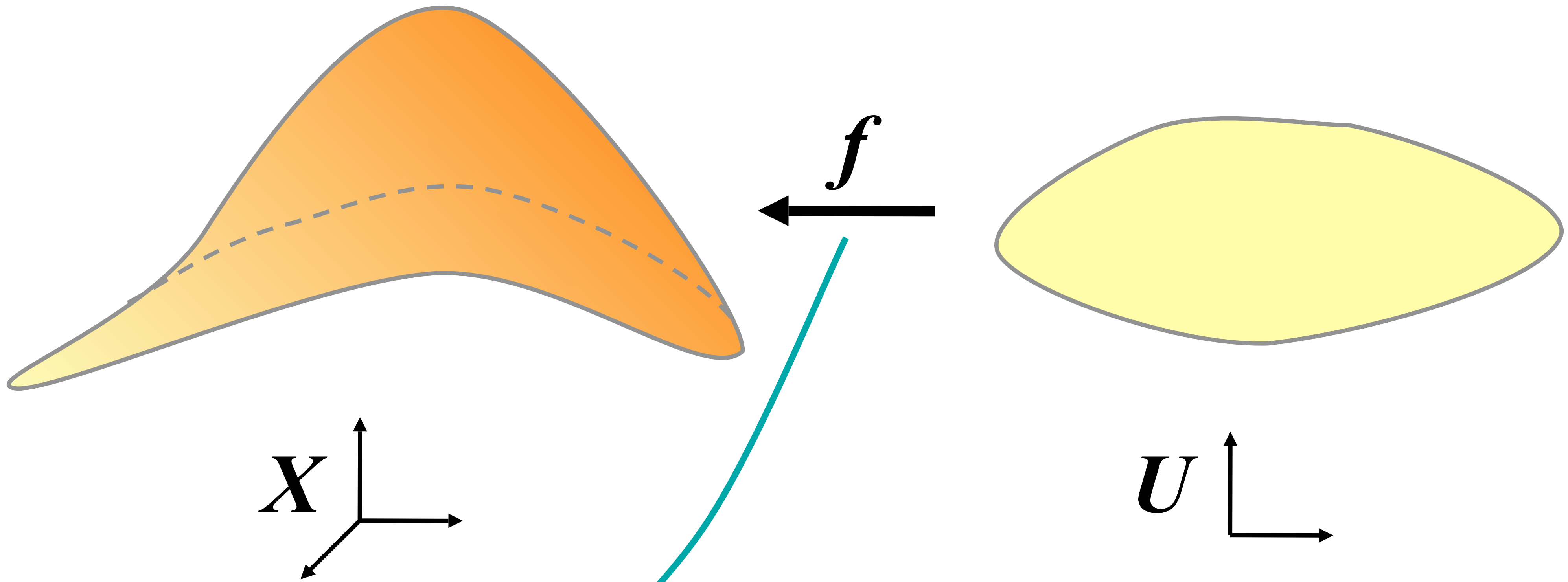


Gu, Gortler, Hoppe: *Geometry Images*, SIGGRAPH 2002

Surface Parameterization



Surface Parameterization

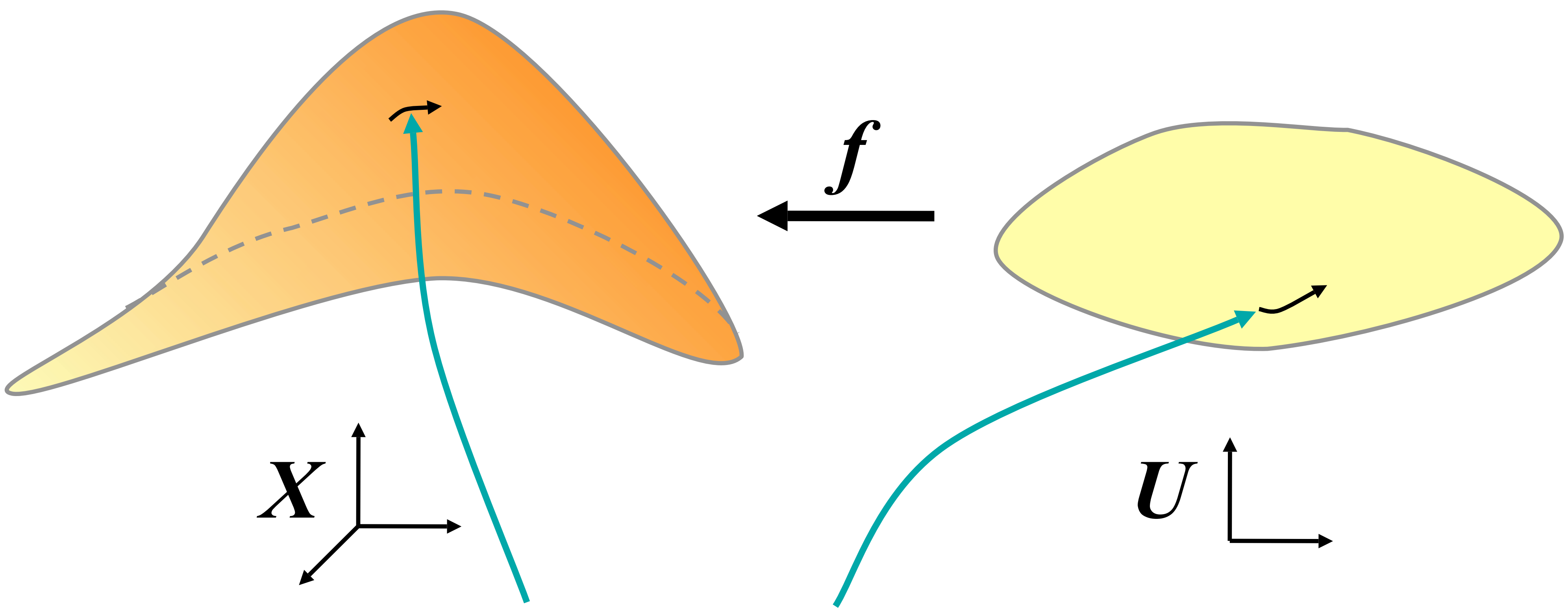


$$f(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

$$J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$

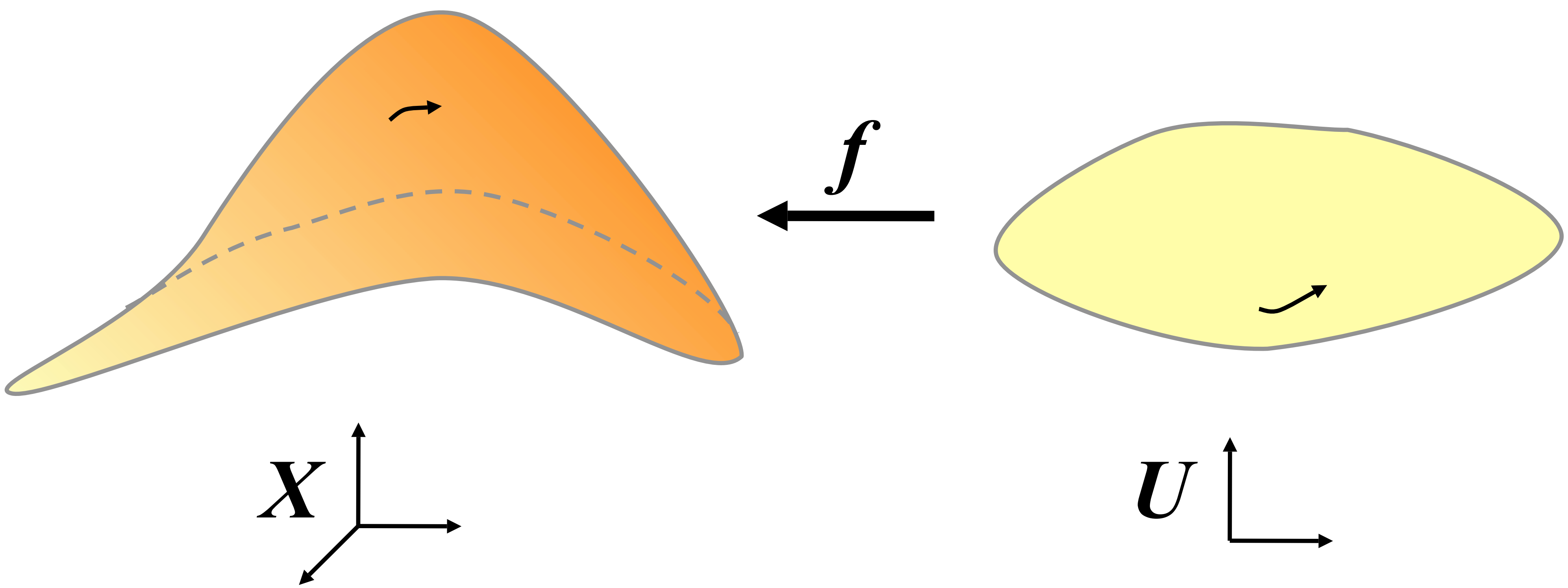
Jacobian

Surface Parameterization



$$d\boldsymbol{X} = \boldsymbol{J}\, d\boldsymbol{U}$$

Surface Parameterization

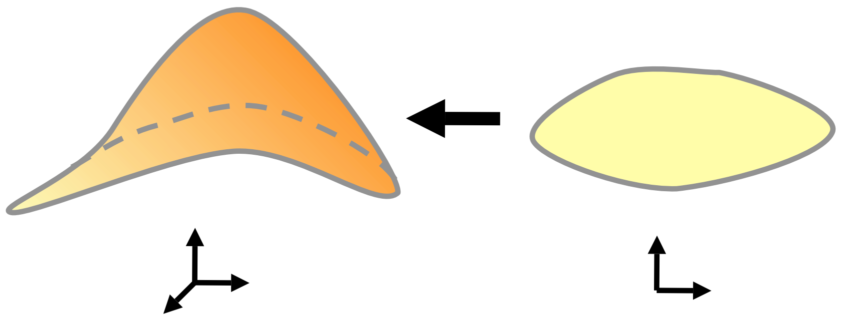


$$d\mathbf{X} = \mathbf{J}\, d\mathbf{U}$$

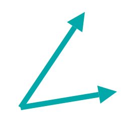
$$\|d\mathbf{X}\|^2 = d\mathbf{U}\, \underbrace{\mathbf{J}^T\mathbf{J}}\, d\mathbf{U}$$ First Fundamental Form

$$\mathbf{I} = \begin{pmatrix} x_u x_u & x_u x_v \\ x_u x_v & x_v x_v \end{pmatrix}$$

Characterization of Mappings



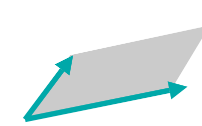
- By first fundamental form $\boldsymbol{I}$
  - Eigenvalues $\lambda_{1,2}$ of $\boldsymbol{I}$
  - Singular values $\sigma_{1,2}$ of $\boldsymbol{J}$ $(\sigma_i^2 = \lambda_i)$
- *Isometric*
  - $\boldsymbol{I} = \boldsymbol{Id}$, $\lambda_1 = \lambda_2 = 1$



- *Conformal*
  - $\boldsymbol{I} = \mu\, \boldsymbol{Id}$, $\lambda_1 / \lambda_2 = 1$

angle preserving

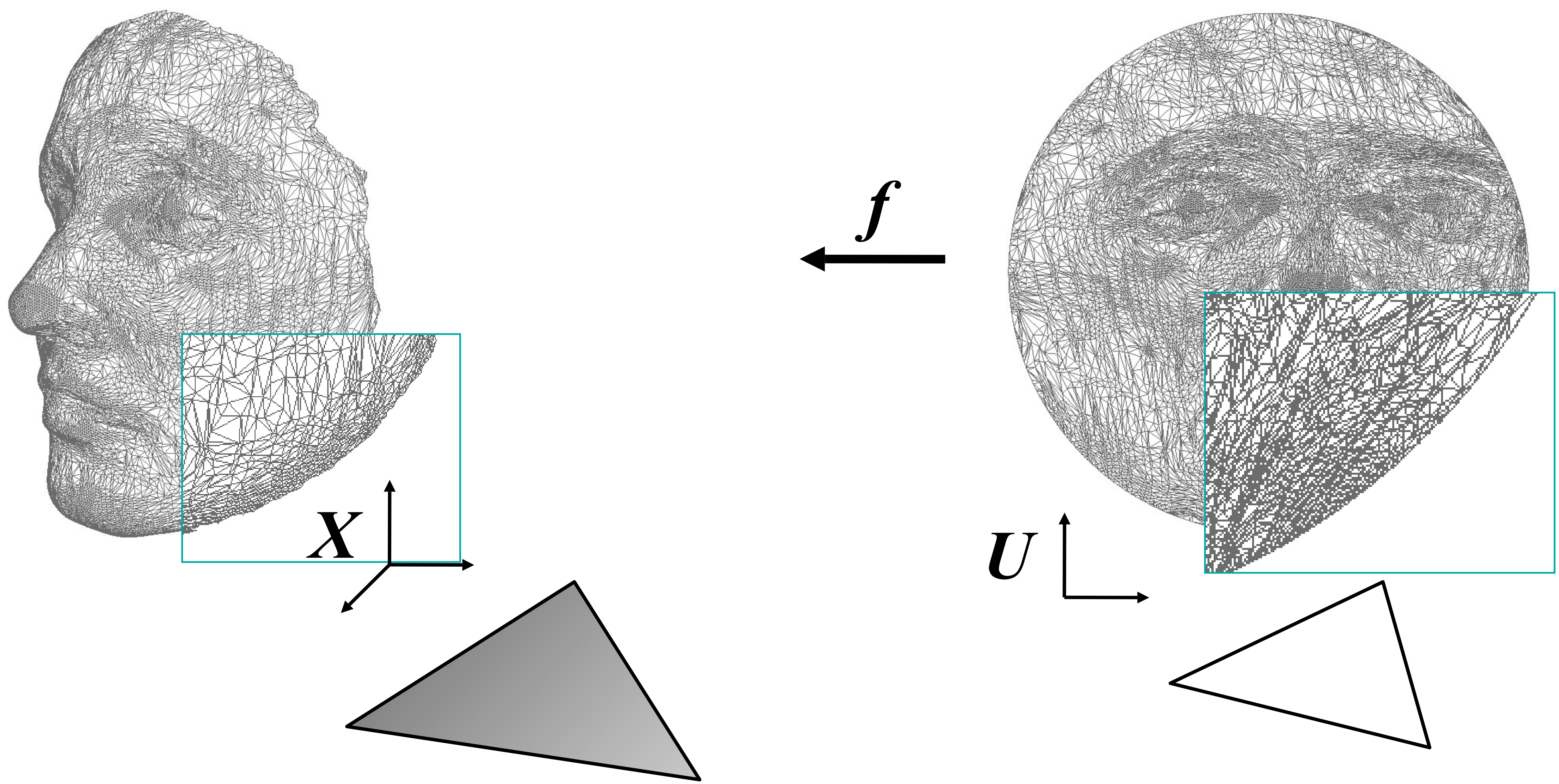
- *Equiareal*
  - $\det \boldsymbol{I} = 1$, $\lambda_1 \lambda_2 = 1$



area preserving

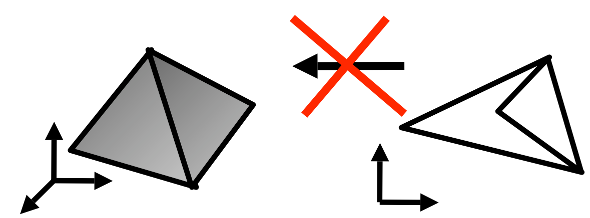
Piecewise Linear Maps

- Mapping = 2D mesh with same connectivity



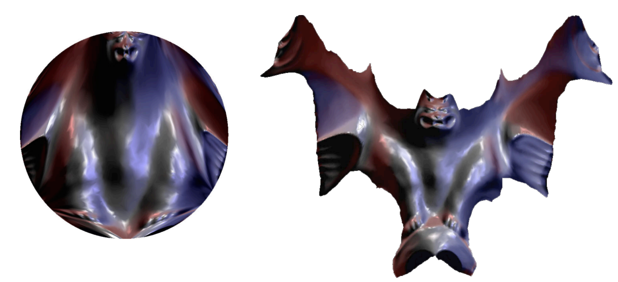
Objectives

- Isometric maps are rare
- Minimize distortion w.r.t. a certain measure
  - Validity (bijective map)



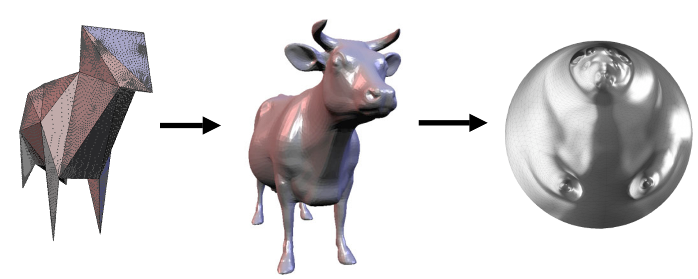
triangle flip

  - Boundary



fixed / free?

  - Domain



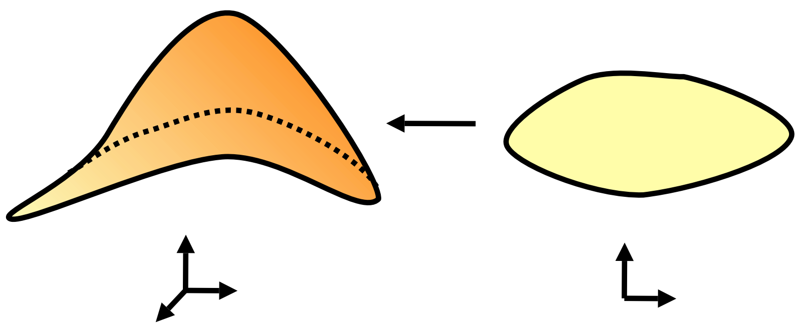
e.g.,spherical

  - Numerical solution

linear / non-linear?

Discrete Harmonic Maps

- f is *harmonic* if $\Delta f = 0$
- Solve Laplace equation



$$\begin{cases} \Delta u = 0 \\ \Delta v = 0 \\ (u,v)_{|\partial\Omega} = (u_0, v_0) \end{cases}$$

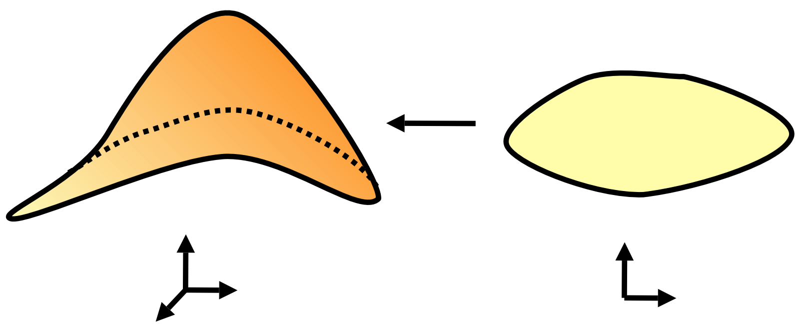
u and v are *harmonic*

Dirichlet boundary conditions

- In 3D: "fix planar boundary and smooth"

Discrete Harmonic Maps

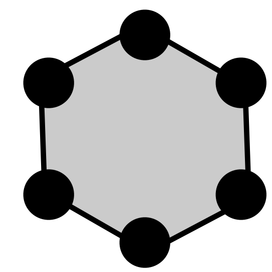
- *f is harmonic* if $\Delta f = 0$
- Solve Laplace equation
- Yields linear system (again)



$$L(p_i) = \sum_{j \in N_i} w_{ij} (p_j - p_i) = 0 \qquad \text{vertices } 1 \le i \le n$$

- *Convex combination maps*
  - *Normalization* $\quad \sum_{j \in N_i} w_{ij} = 1$
  - *Positivity* $\quad w_{ij} > 0$

Convex Combination Maps



- Every (interior) planar vertex is a *convex combination* of its neighbors
- Guarantees *validity* if boundary is mapped to a convex polygon (e.g., rectangle, circle)
- Weights
  - Uniform *(barycentric mapping)*
  - Shape preserving [Floater 1997]
  - Mean Value Coordinates [Floater 2003]
    - Use mean value property of harmonic functions

Reproduction of planar meshes

Conformal Maps

- Planar *conformal mappings* $f(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ satisfy the *Cauchy-Riemann conditions*

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \quad \text{and} \quad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}$$

Conformal Maps

- Planar *conformal mappings* $f(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ satisfy the *Cauchy-Riemann conditions*

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

- Differentiating once more by x and y yields

$$u_{xx} = v_{xy} \text{ and } u_{yy} = -v_{xy} \qquad \Rightarrow \qquad u_{xx} + u_{yy} = \Delta u = 0$$

and similar $\Delta v = 0$

- conformal $\Rightarrow$ harmonic

Discrete Conformal Maps

- Planar *conformal mappings* $f(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ satisfy the *Cauchy-Riemann conditions*

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

- *In general, there are no conformal mappings for piecewise linear functions!*

Discrete Conformal Maps

- Planar *conformal mappings* $f(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ satisfy the *Cauchy-Riemann conditions*

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

- Conformal energy *(per triangle T)*

$$E_T = (u_x - v_y)^2 + (u_y + v_x)^2$$

- Minimize

$$\sum_{T \in \mathrm{T}} E_T A_T \to \min$$

Discrete Conformal Maps

- Least-squares conformal maps [Lévy et al. 2002]

$$\sum_{T \in \mathrm{T}} E_T A_T \rightarrow \min \quad \text{where} \quad E_T = (u_x - v_y)^2 + (u_y + v_x)^2$$

- Satisfy Cauchy-Riemann conditions in *least-squares* sense
- Leads to solution of linear system

- *Alternative formulation leads to same solution…*

Discrete Conformal Maps

- Same solution is obtained for

$$\Delta_S u = 0$$

$$\Delta_S v = 0$$

cotangent weights

$$n \times \nabla u \,|_{\partial\Omega} = c$$

$$n \times \nabla v \,|_{\partial\Omega} = c$$

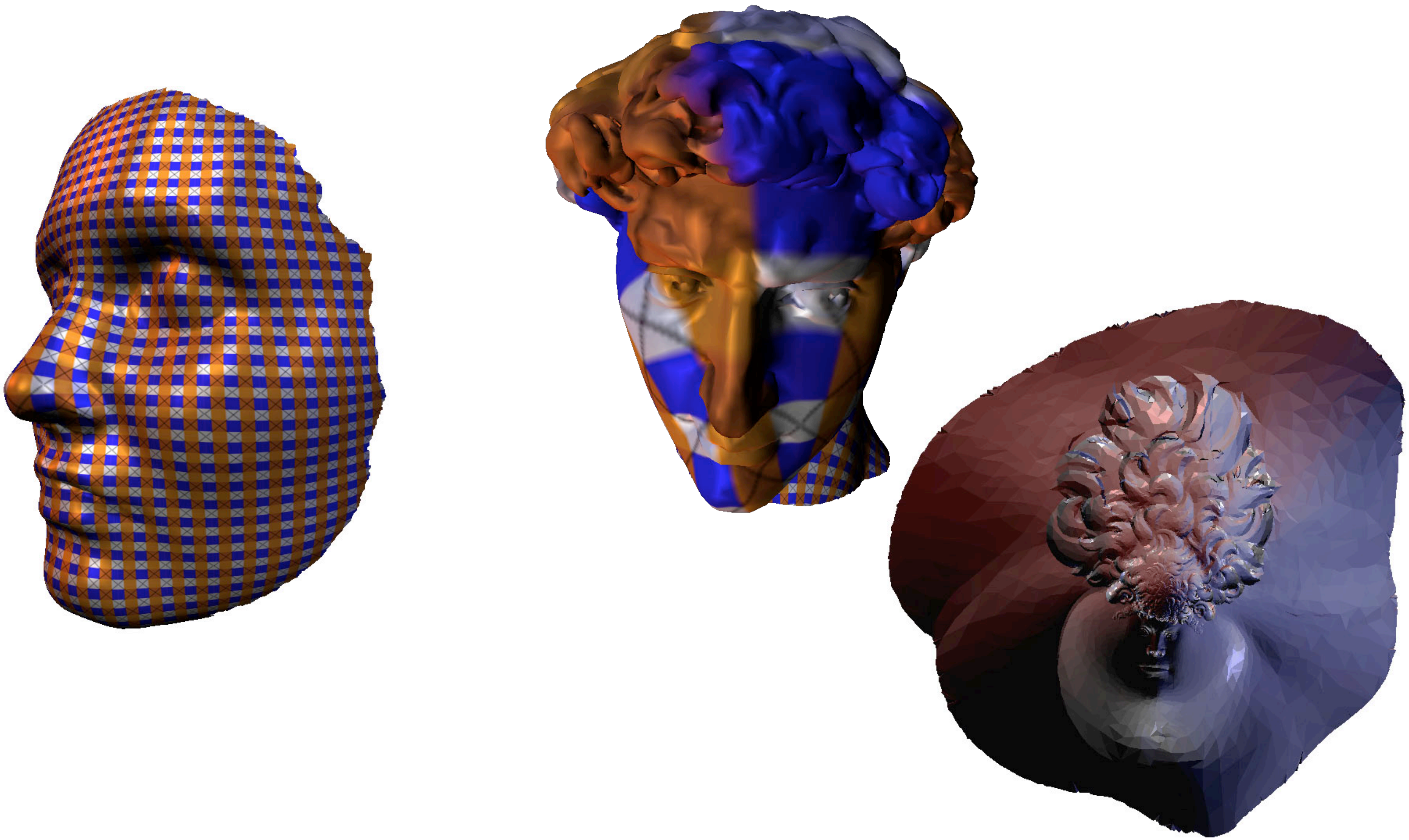
Neumann boundary conditions

$$(u, v)_{|\partial\Omega_0} = (u_0, v_0)$$

+ fixed vertices

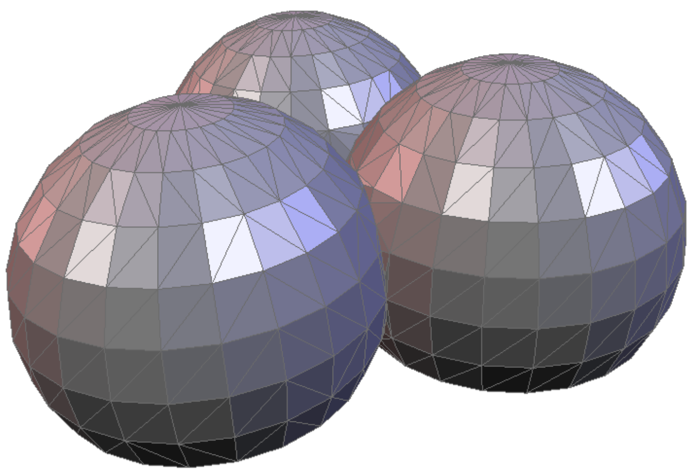
Discrete Conformal Maps [Desbrun et al. 2002]

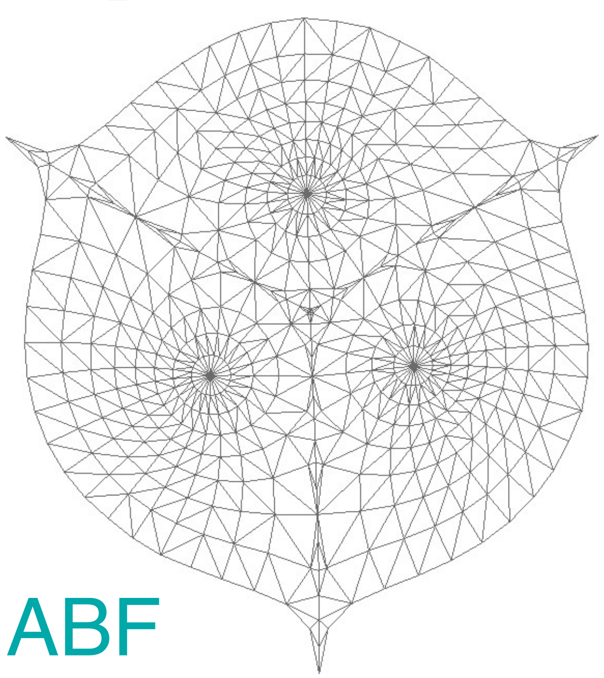
Discrete Conformal Maps



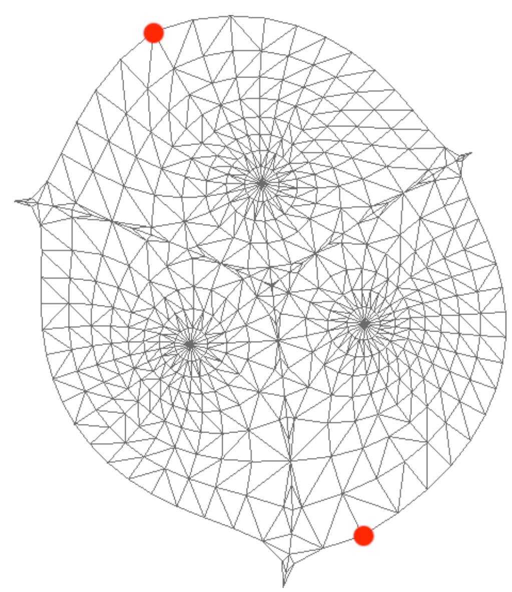
Discrete Conformal Maps

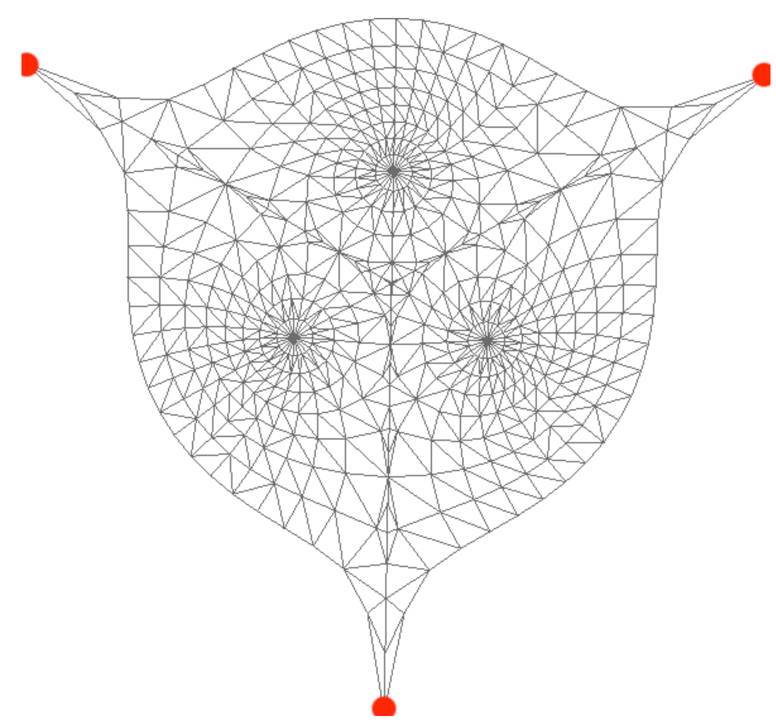
- Free boundary depends on choice of *fixed* vertices (>1)

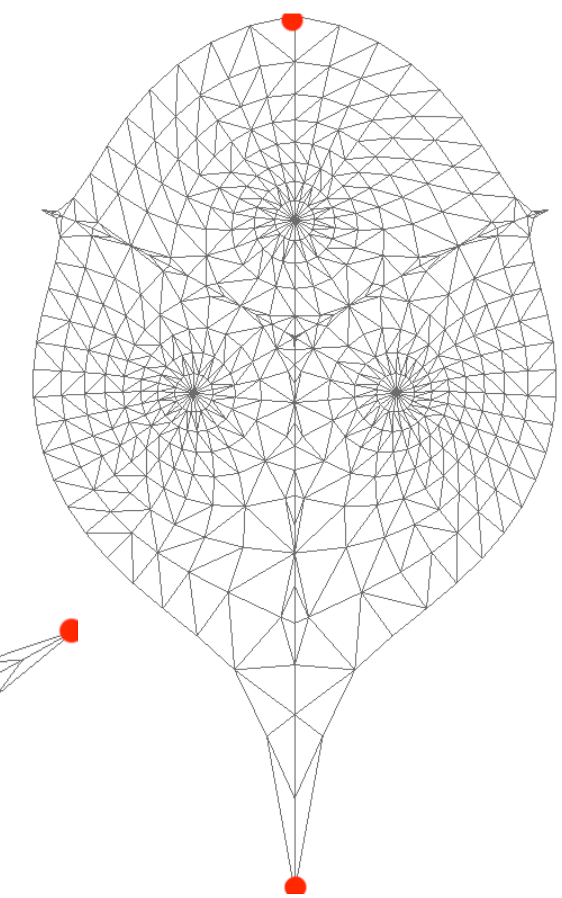




ABF





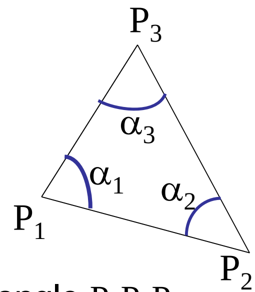


LSCM – Geometric Interpretation

- Algebraic Interpretation:
  - Minimize conformal energy

$$E_C = (\sigma_1 - \sigma_2)^2/2$$

- Geometric Interpretation:
  - Use triangle similarity
  - Given angles α_1, α_2, α_3 of a triangle $P_1P_2P_3$ in 2D we have



$$P_3 - P_1 = \frac{\sin\alpha_2}{\sin\alpha_3} R_{\alpha_1}(P_2 - P_1),$$

$$R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

University of British Columbia

LSCM

- In map from 3D to 2D might be impossible to keep angles exactly
  - Use least-squares

$$\min \sum_i \left(P_3^i - P^i{}_1 - \frac{\sin\alpha^i{}_2}{\sin\alpha^i{}_3} R_{\alpha^i{}_1}(P^i{}_2 - P_1^i)\right)^2$$

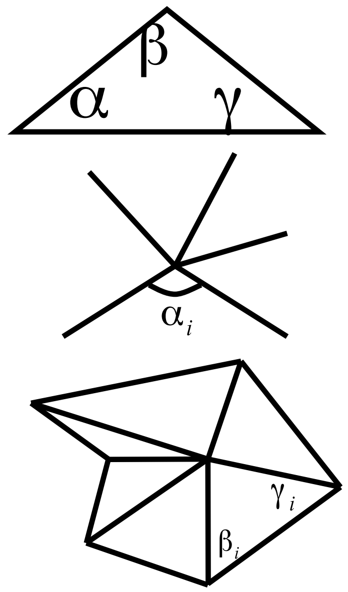
- To solve need to fix two vertices
  - Obtain linear system
  - Choice of vertices affects solution
- Can have flips

University of British Columbia

Angle Based Flattening [Sheffer&de Sturler 2000]

- Perserve angles ⇨ specify problem in angles
  - Constraints
    - triangle $\alpha + \beta + \gamma - \pi = 0$
    - Internal vertex $\sum_i \alpha_i - 2\pi = 0$
    - Wheel consistency $\prod_i \sin(\beta_i) - \prod_i \sin(\gamma_i) = 0$

ensure validity



  - Objective function

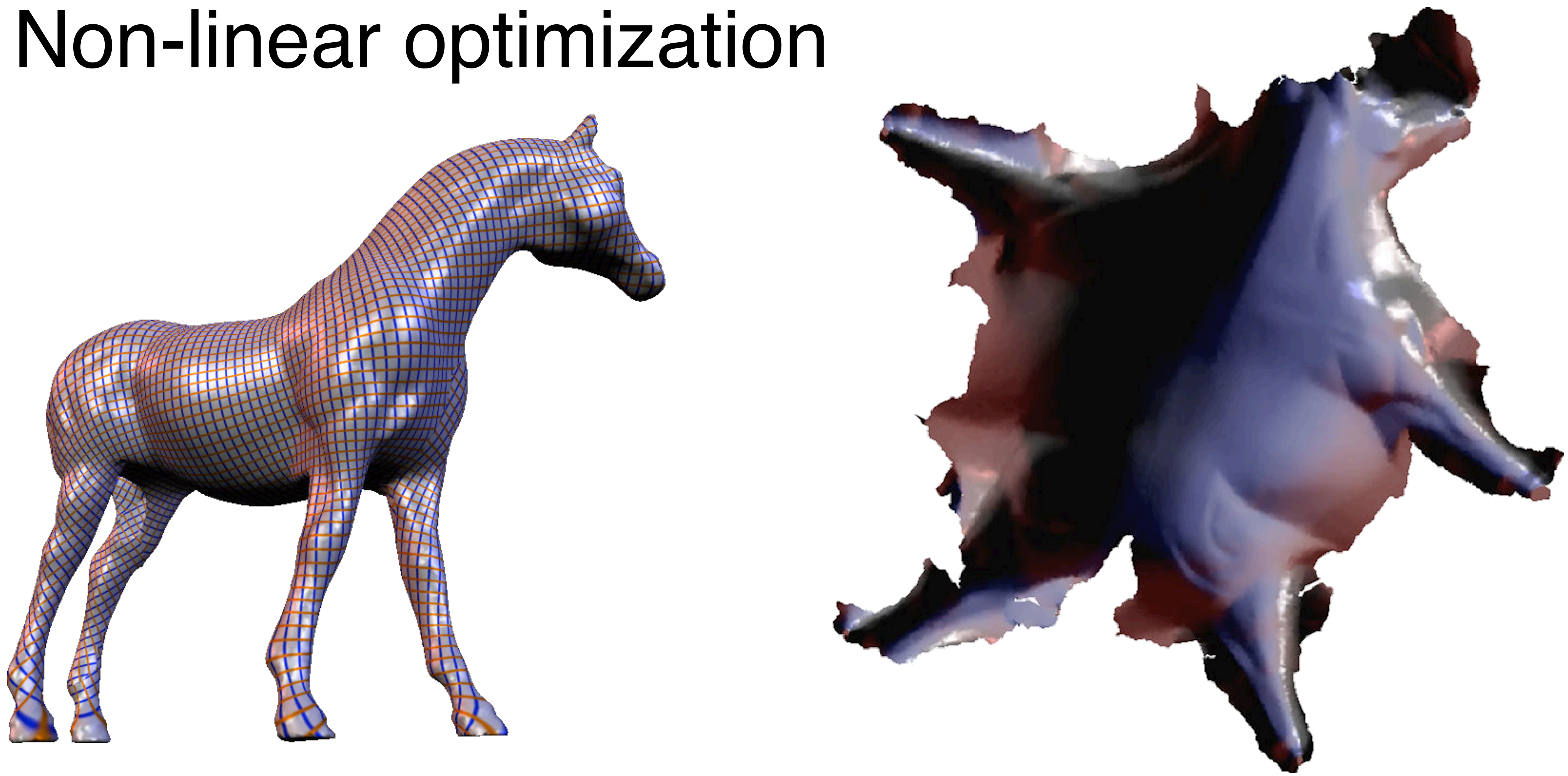
$$f(x) = \sum_{i=1}^{N} w_i (\alpha_i - \alpha_i^*)^2$$

preserve angles 2D ~3D

"optimal" angles (uniform scaling)

Angle Based Flattening

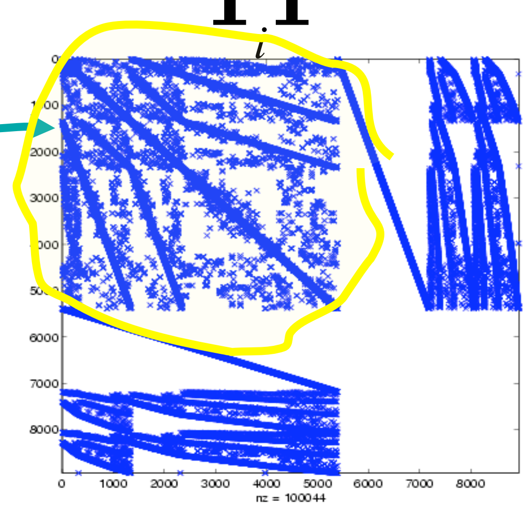
- Free boundary
- Validity: no local self-intersections
- Non-linear optimization



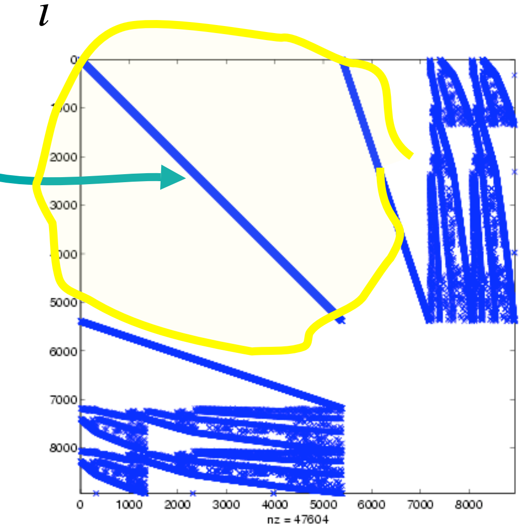
Angle Based Flattening

- Free boundary
- Non-linear optimization
  - Newton iteration
  - Solve linear system in every step

$$\prod_i \sin(\alpha_i) - \prod_i \sin(\beta_i) = 0$$

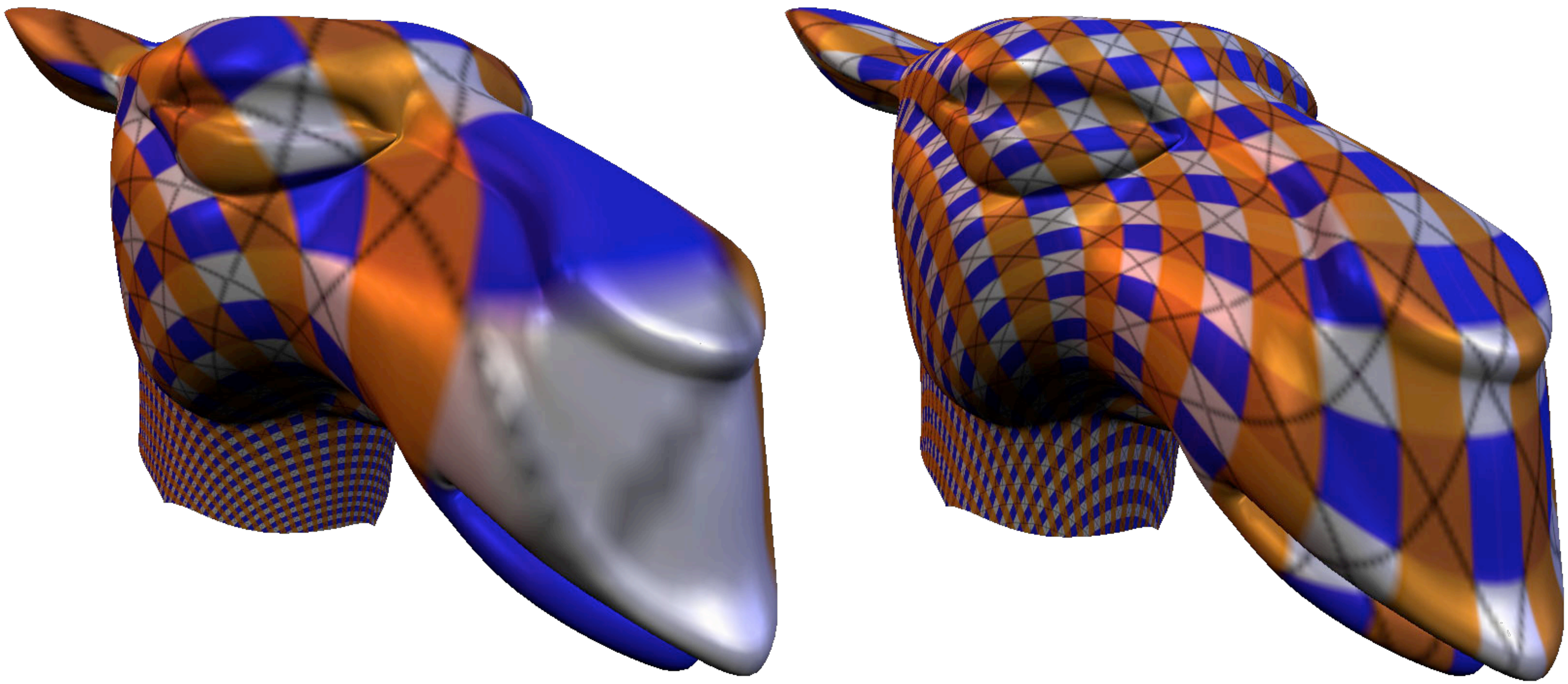


$$\prod_i \log \sin(\alpha_i) - \prod_i \log \sin(\beta_i) = 0$$

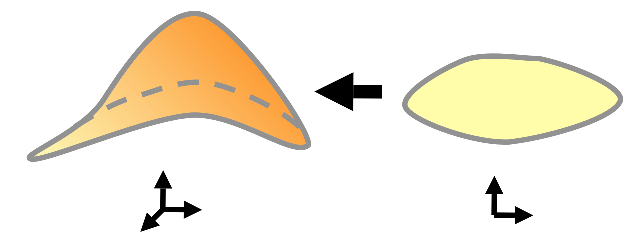


[Zayer et al. 2005]

And how about area distortion?



Reducing Area Distortion



- Energy minimization based on
  - MIPS [Hormann & Greiner 2000]

    $$\| J \|_F \| J^{-1} \|_F = \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}$$

  - *modification* [Degener et al. 2003]

    $$\det J + \frac{1}{\det J} = \sigma_1 \sigma_2 + \frac{1}{\sigma_1 \sigma_2}$$

  - "Stretch" [Sander et al. 2001]

    $$\| J \|_F = \sqrt{\sigma_1 + \sigma_2}$$ or $$\| J \| \infty = \sigma_1$$

  - *modification* [Sorkine et al. 2002]

    $$\max \{ \sigma_1, \frac{1}{\sigma_2} \} \,|$$

Non-Linear Methods

- Free boundary
- Direct control over distortion

- No convergence guarantees
- May get stuck in local minima
- May not be suitable for large problems
- May need feasible point as initial guess
- May require hierarchical optimization even for moderately sized data sets

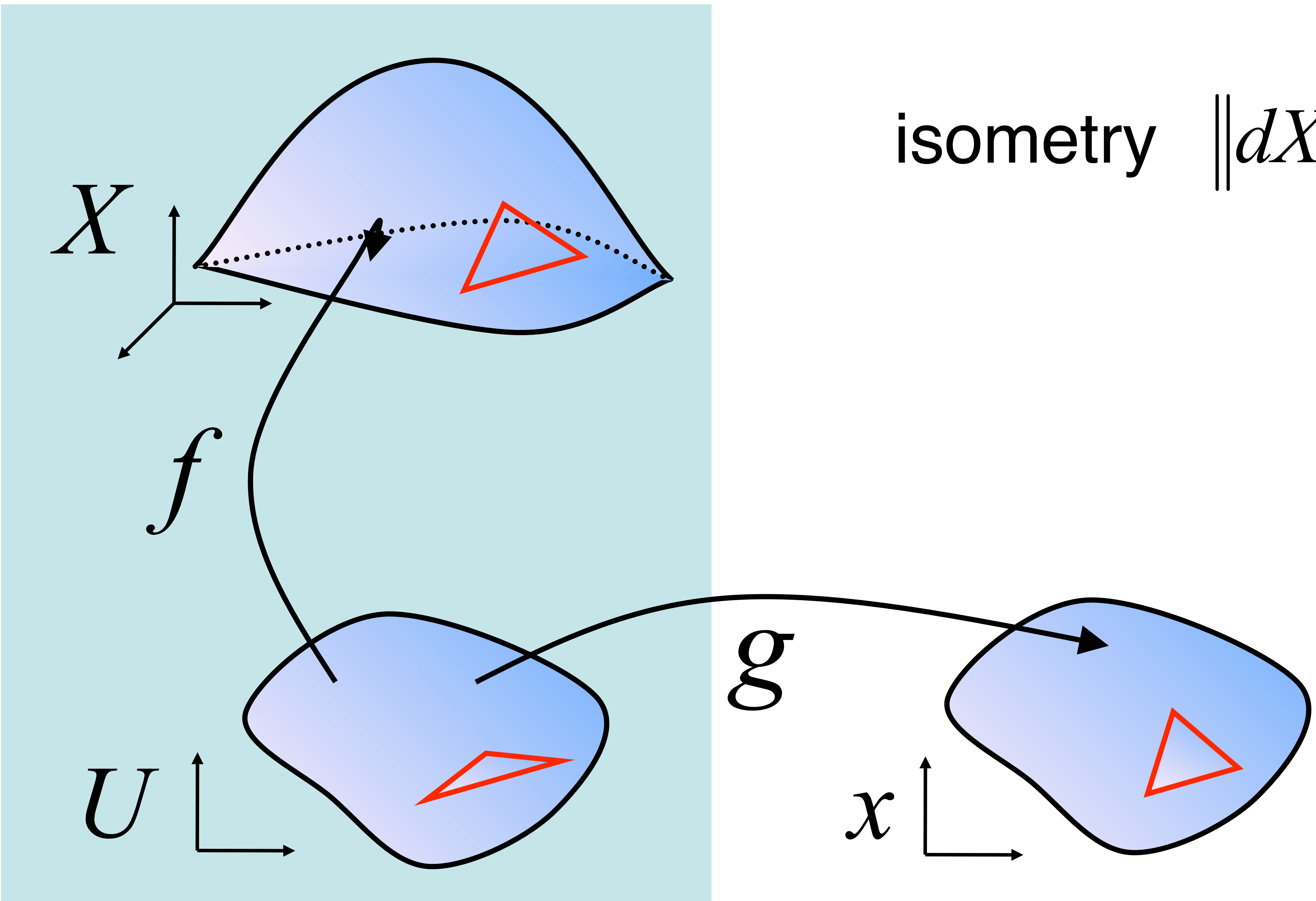
Linear Methods

- Efficient solution of a sparse linear system
- Guaranteed convergence

- Fixed convex boundary
- May suffer from area distortion for complex meshes

- *An alternative approach to reducing area distortion…*
  - *How accurately can we reproduce a surface on the plane?*
  - *How do we characterize the mapping?*

Reducing Area Distortion



isometry $\|dX\| = \|dx\|$

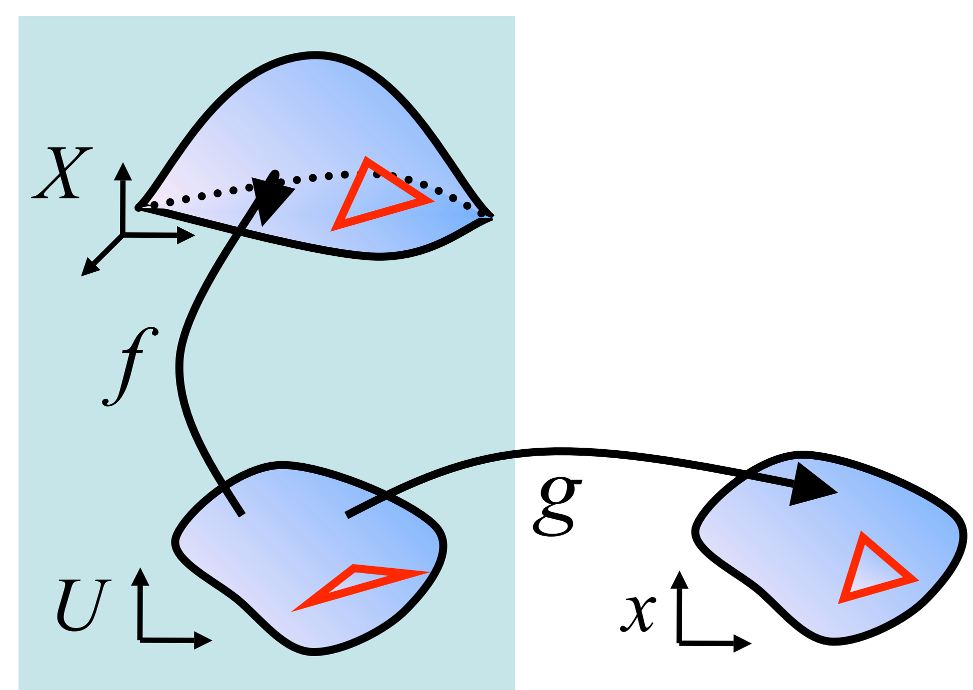
Reducing Area Distortion

- Quasi-harmonic maps [Zayer et al. 2005]

$$\int C \nabla g \times \nabla g \text{ ® } \min$$

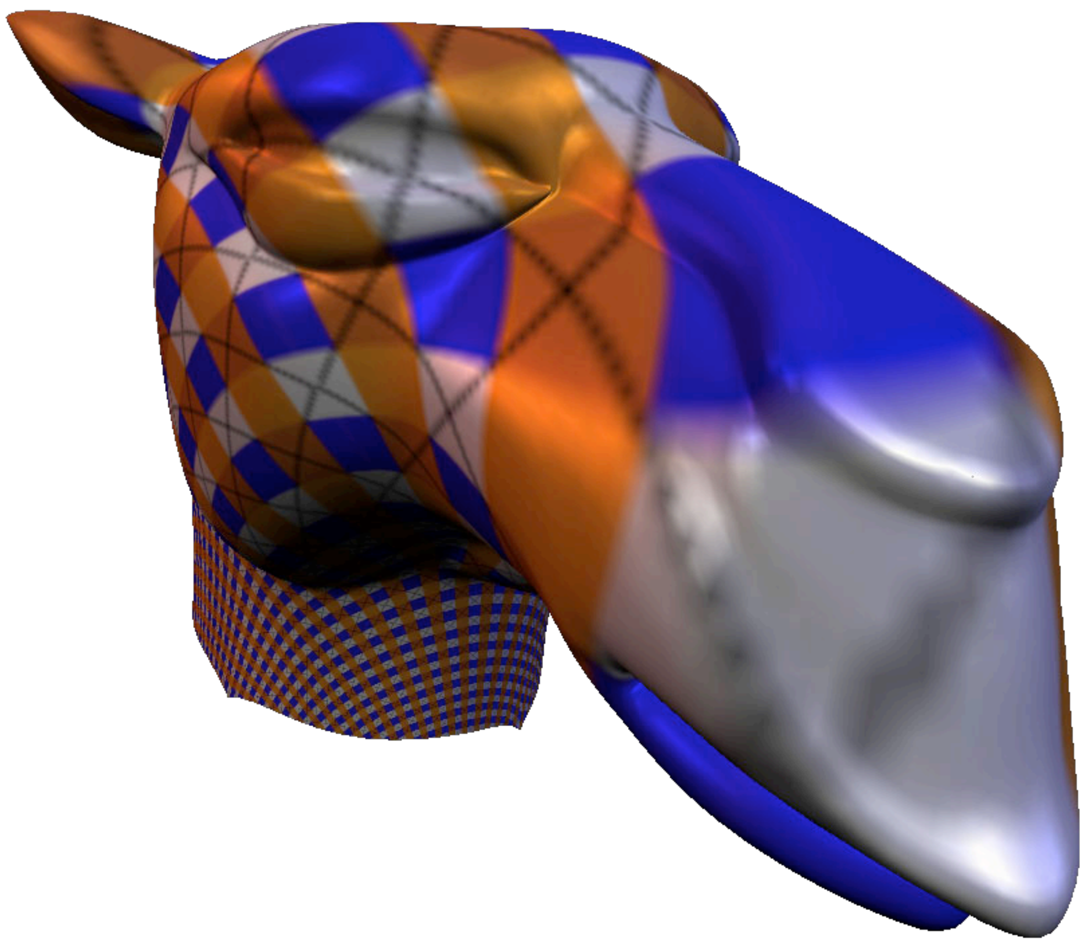
estimate from f

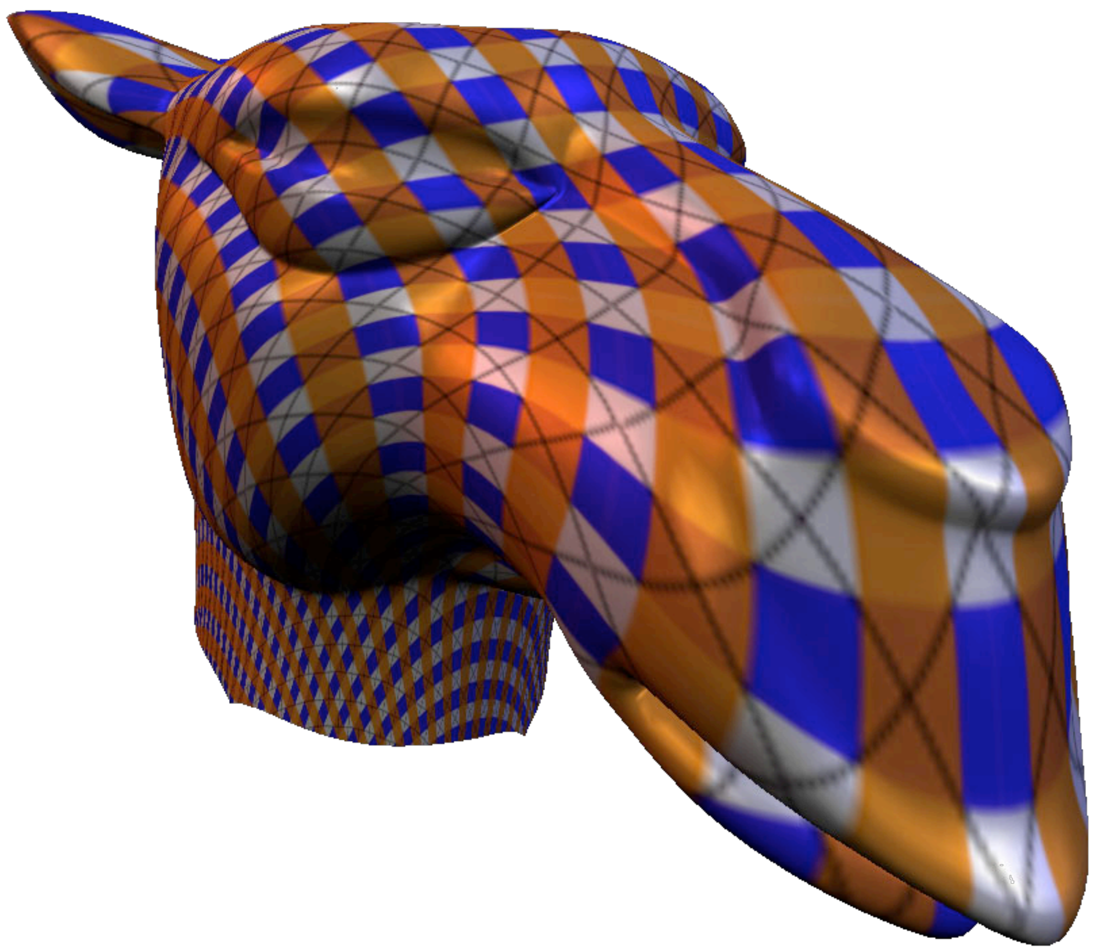
$$\mathrm{div}(C \nabla g) = 0$$



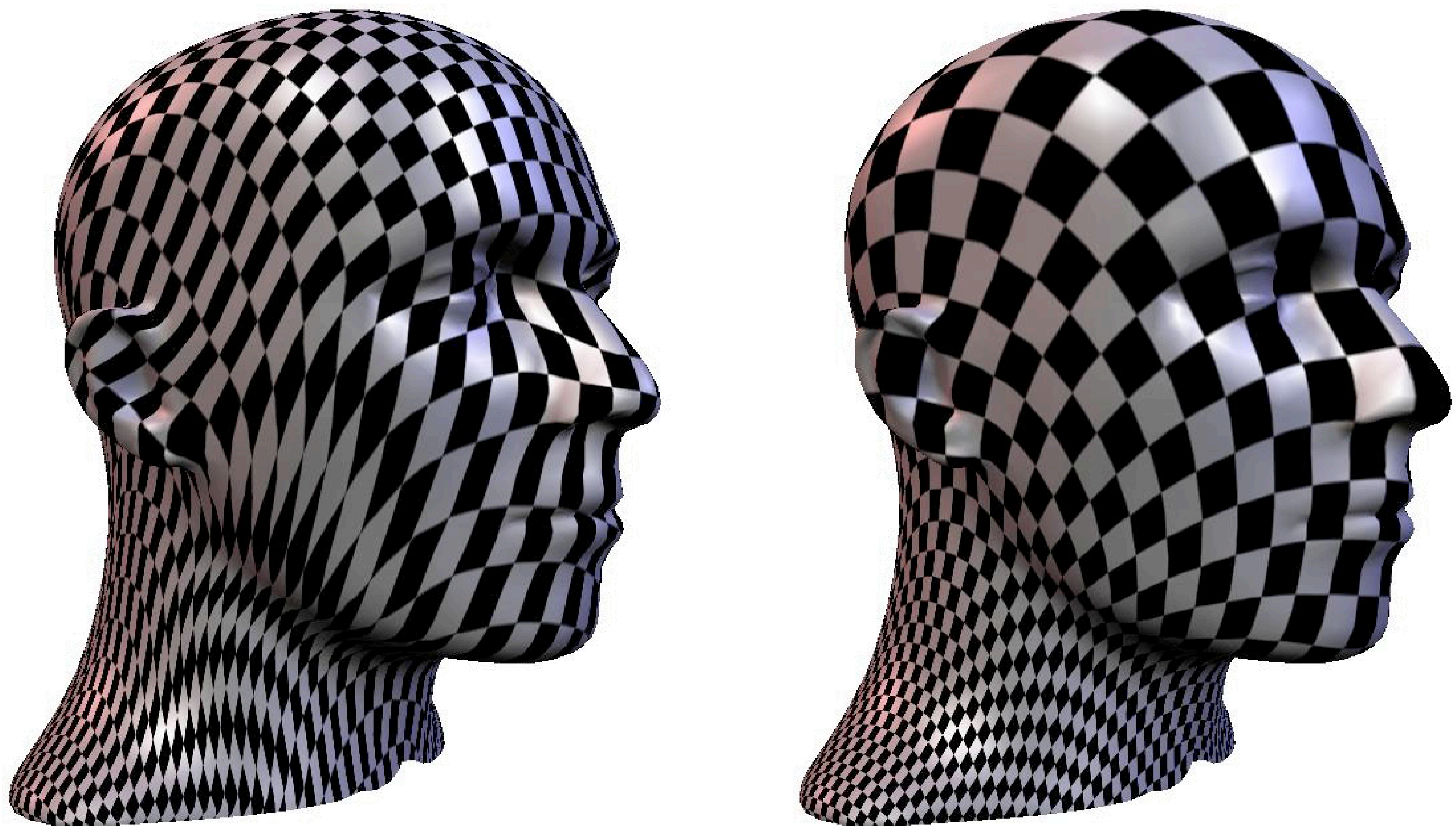
- Iterate *(few iterations)*
  - Determine tensor C from f
  - Solve for g

Examples





Examples



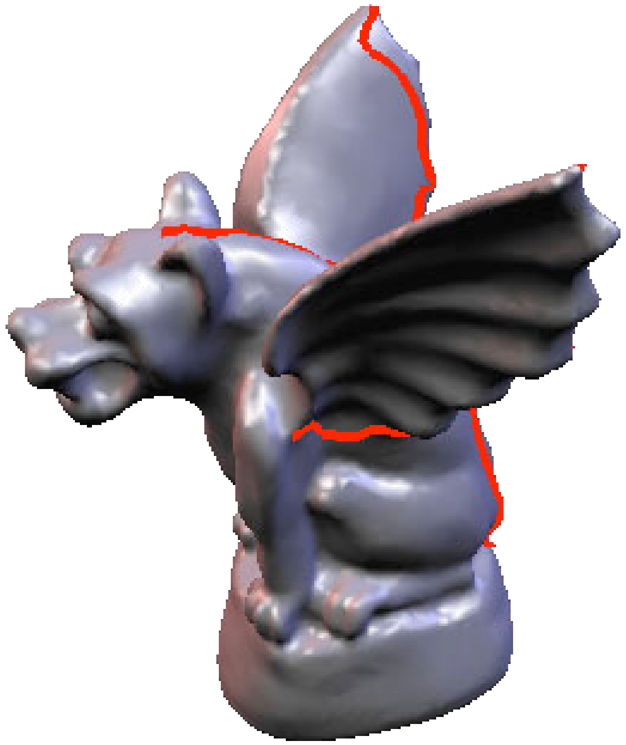
$\sqrt{\sigma_1 + \sigma_2} \rightarrow \min$

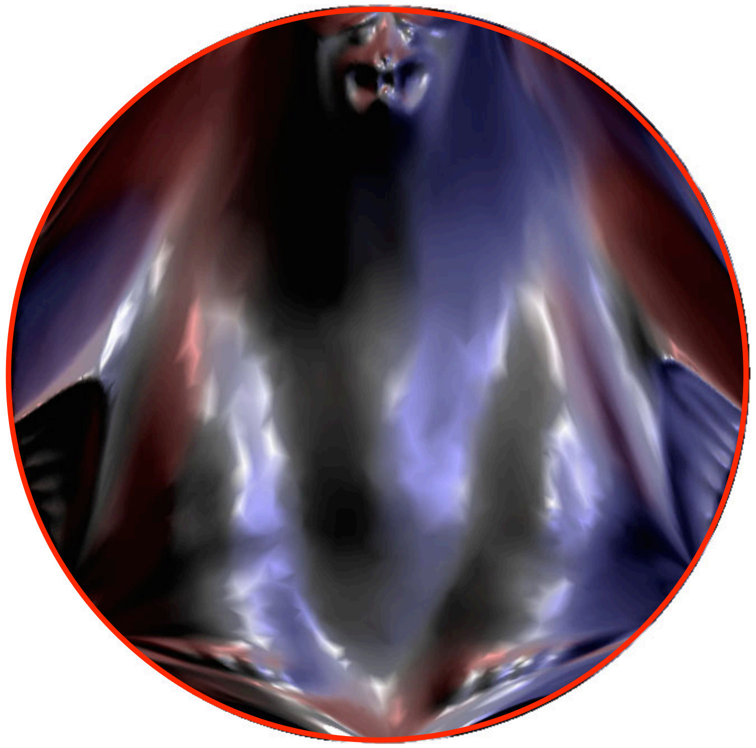
Stretch metric minimization

Using [Yoshizawa et. al 2004]

Reducing Area Distortion

- Introduce cuts ⇨ area distortion *vs.* continuity
- Often cuts are unavoidable (e.g., open sphere)

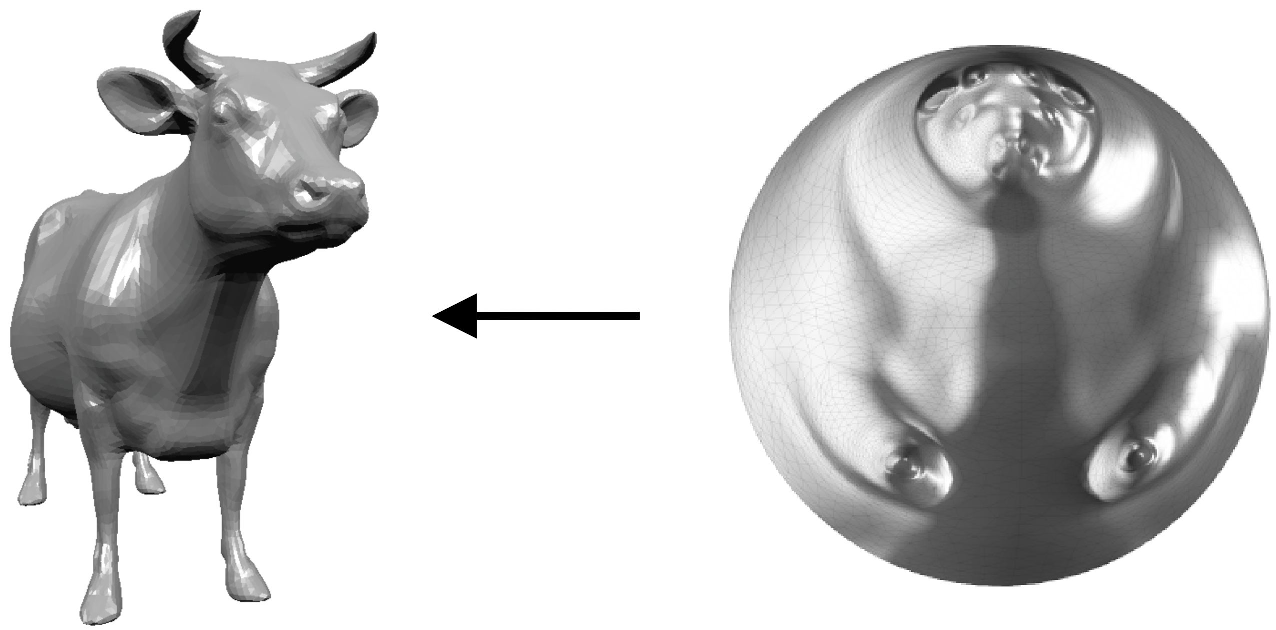




Treatment of boundary is important!

Spherical Parameterization

- Sphere is natural domain for genus-0 surfaces
- Additional constraint $\| U \|^2 = 1$



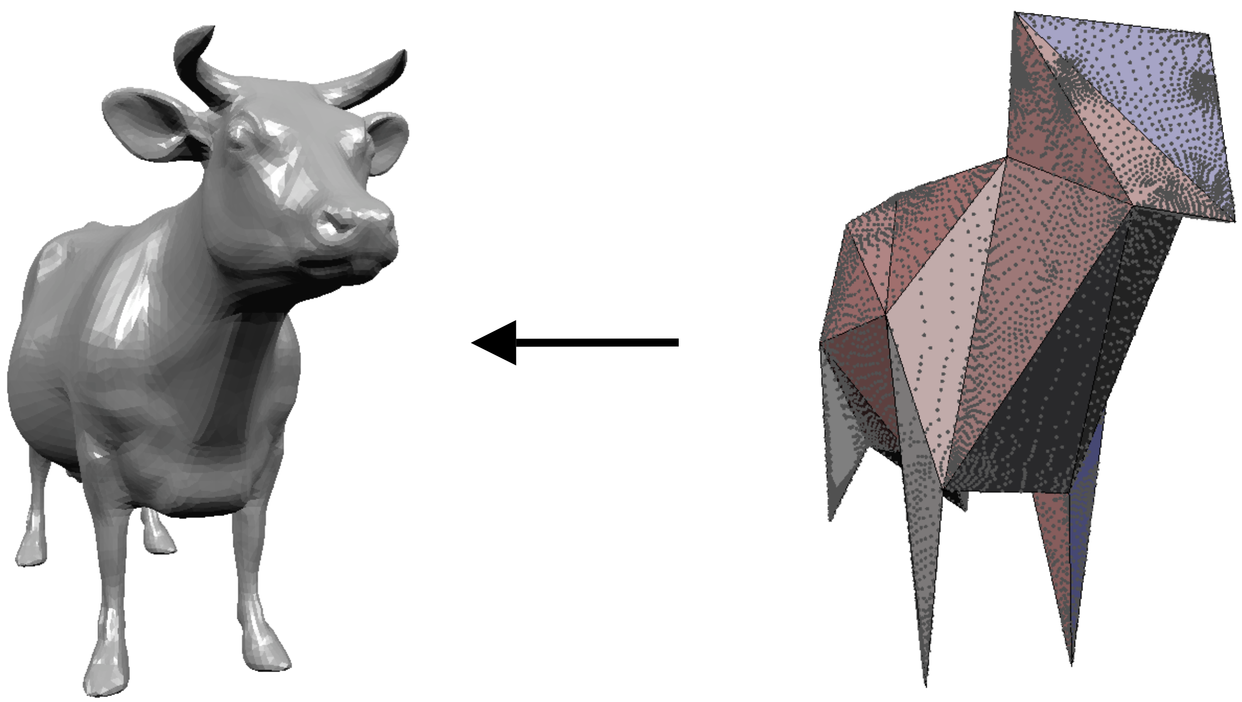
- Naïve approach
  - Laplacian smoothing and back-projection
  - Obtain minimum for degenerate configuration

Spherical Parameterization

- (Tangential) Laplacian Smoothing and back-projection
  - Minimum energy is obtained for *degenerate* solution
- Theoretical guarantees are expensive
  - [Gotsman et al. 2003]
- A compromise?!
  - Stereographic projection
  - Smoothing in curvilinear coordinates

Arbitrary Topology

- Piecewise linear domains
  - *Base mesh* obtained by *mesh decimation*
  - Piecewise maps
  - Smoothness



Literature

- Floater & Hormann: *Surface parameterization: a tutorial and survey,* Springer, 2005
- Lévy, Petitjean, Ray, and Maillot: *Least squares conformal maps for automatic texture atlas generation*, SIGGRAPH 2002
- Desbrun, Meyer, and Alliez: *Intrinsic parameterizations of surface meshes*, Eurographics 2002
- Sheffer & de Sturler: *Parameterization of faceted surfaces for meshing using angle based flattening*, Engineering with Computers, 2000.