Continuous Geometry of Surfaces

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Surfaces

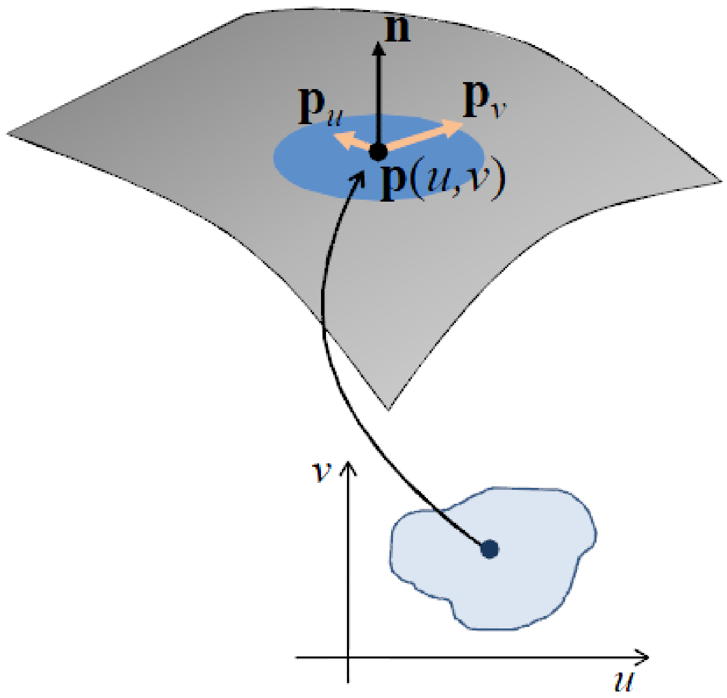
Parametric form

- Continuous surface

$$\mathbf{p}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \quad (u,v) \in \Re^2$$

- Tangent plane at point p(u,v) is spanned by

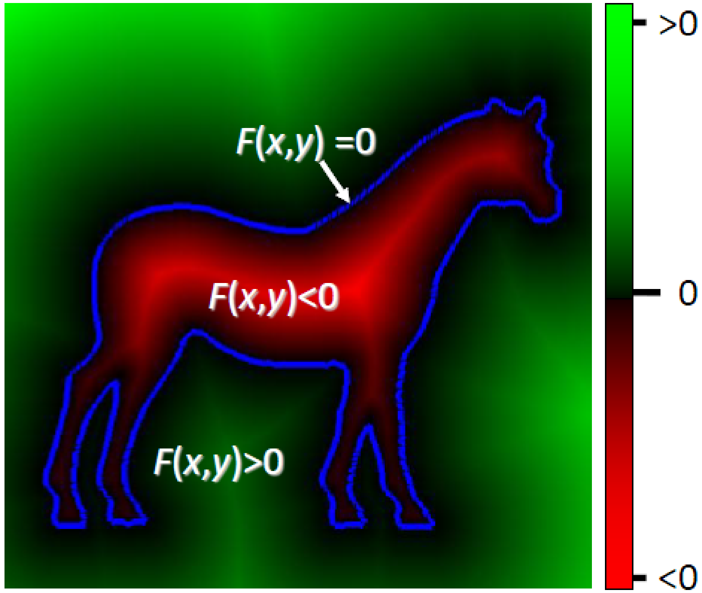
$$\mathbf{p}_u = \frac{\partial \mathbf{p}(u,v)}{\partial u}, \qquad \mathbf{p}_v = \frac{\partial \mathbf{p}(u,v)}{\partial v}$$



Shape Representations

- Parametric
  - Represent a surface as (continuous) injective function from a domain $\Omega \subset \boldsymbol{R}^2$ to $S \subset \boldsymbol{R}^3$
- Implicit
  - Represent a surface as the zero set of a scalar-valued function defined in R^3.

$$K = g^{-1}(0) = \{\mathbf{p} \in \mathrm{R}^3 : g(\mathbf{p}) = 0\}$$

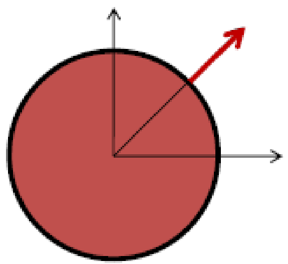


Implicit Surfaces

Gradient

- Represent a surface as the zero set of a (regular) function defined in R^3.
- The normal vector to the surface is given by the gradient of the (scalar) implicit function

$$\nabla g(x,y,z) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)^T$$



$$\nabla g(x,y,z) = (2,2,0)^T$$

  - Example

$$g(x,y,z) = x^2 + y^2 + z^2 - r^2$$

$$\nabla g(x,y,z) = (2x,2y,2z)^T$$

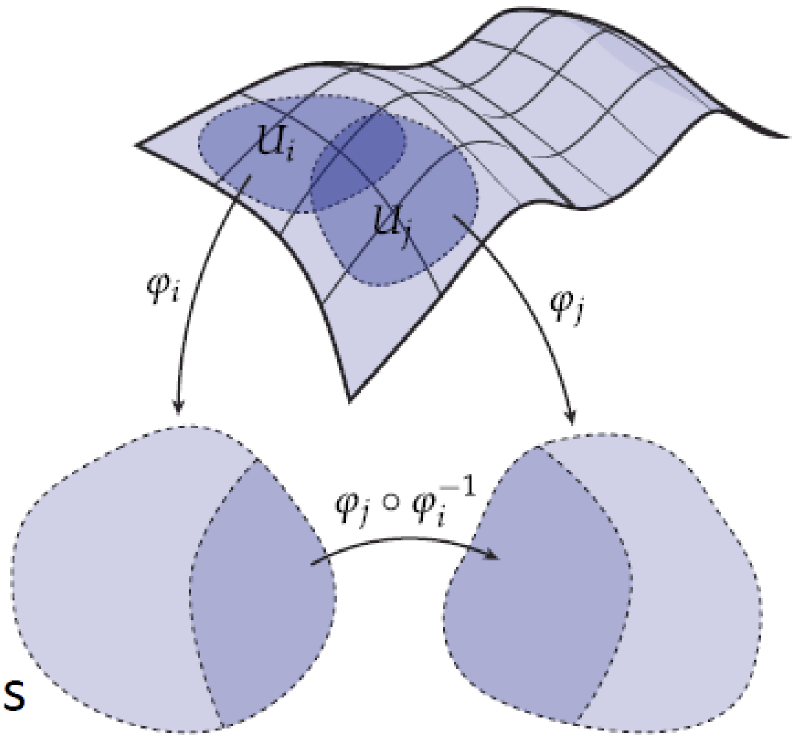
- Why is the condition that the function be regular (i.e. have non-vanishing derivative) necessary?
- How smooth is the surface?

Normal field is a gradient field of a scalar function g(x,,y,z)=r!

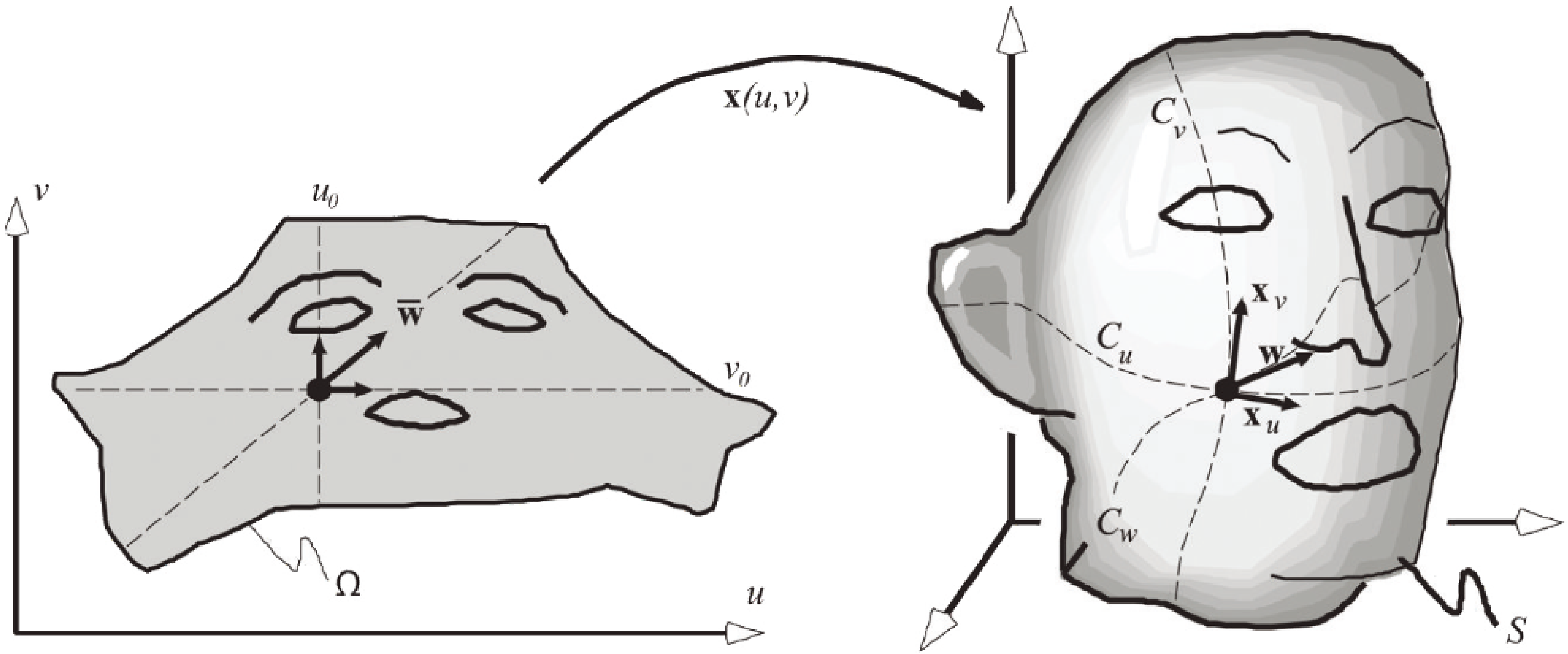
Mesh Representations

- Parametric
  - Represent a surface as (continuous) injective function from a domain $\Omega \subset \boldsymbol{R}^2$ to $S \subset \boldsymbol{R}^3$

- In practice, it's not easy to find a single function that parameterizes the surface.
- So instead, we represent a surface as a collection of functions (charts) from (simple) 2D domains into 3D.



Given a set of charts, we say that the manifold S is "smooth" if for any two charts $\phi_1 : \Omega_1 \to S$ and $\phi_2 : \Omega_2 \to S$, the map $\phi_2^{-1} \circ \phi_1$ is smooth.



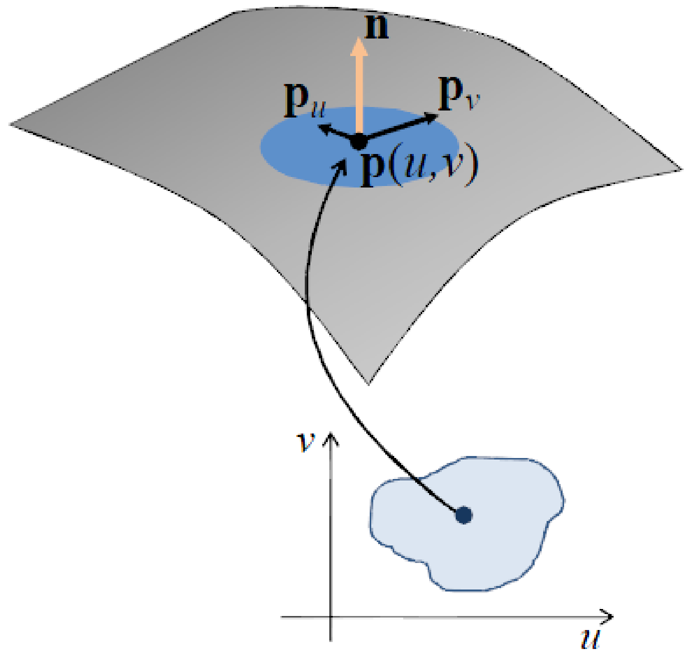
- 2 partial derivatives $\mathbf{x}_u(u_0, v_0) := \frac{\partial \mathbf{x}}{\partial u}(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0) := \frac{\partial \mathbf{x}}{\partial v}(u_0, v_0)$
- 2 iso-parameter curves $\mathbf{C}_\mathbf{u}(t) = \mathbf{x}(u_0 + t, v_0)$ and $\mathbf{C}_\mathbf{v}(t) = \mathbf{x}(u_0, v_0 + t)$

Surface normal

$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$

- Assuming regular parameterization, i.e.,

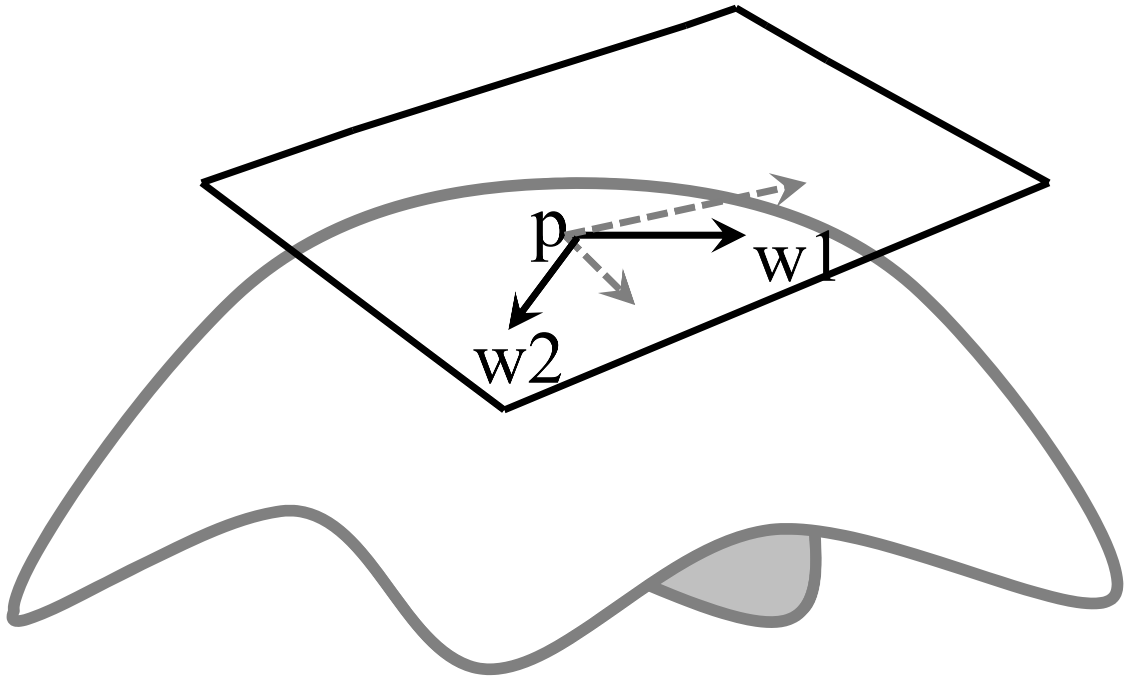
$$\mathbf{p}_u \times \mathbf{p}_v \neq 0$$



Riemannian Metric & first fundamental form

first fundamental form

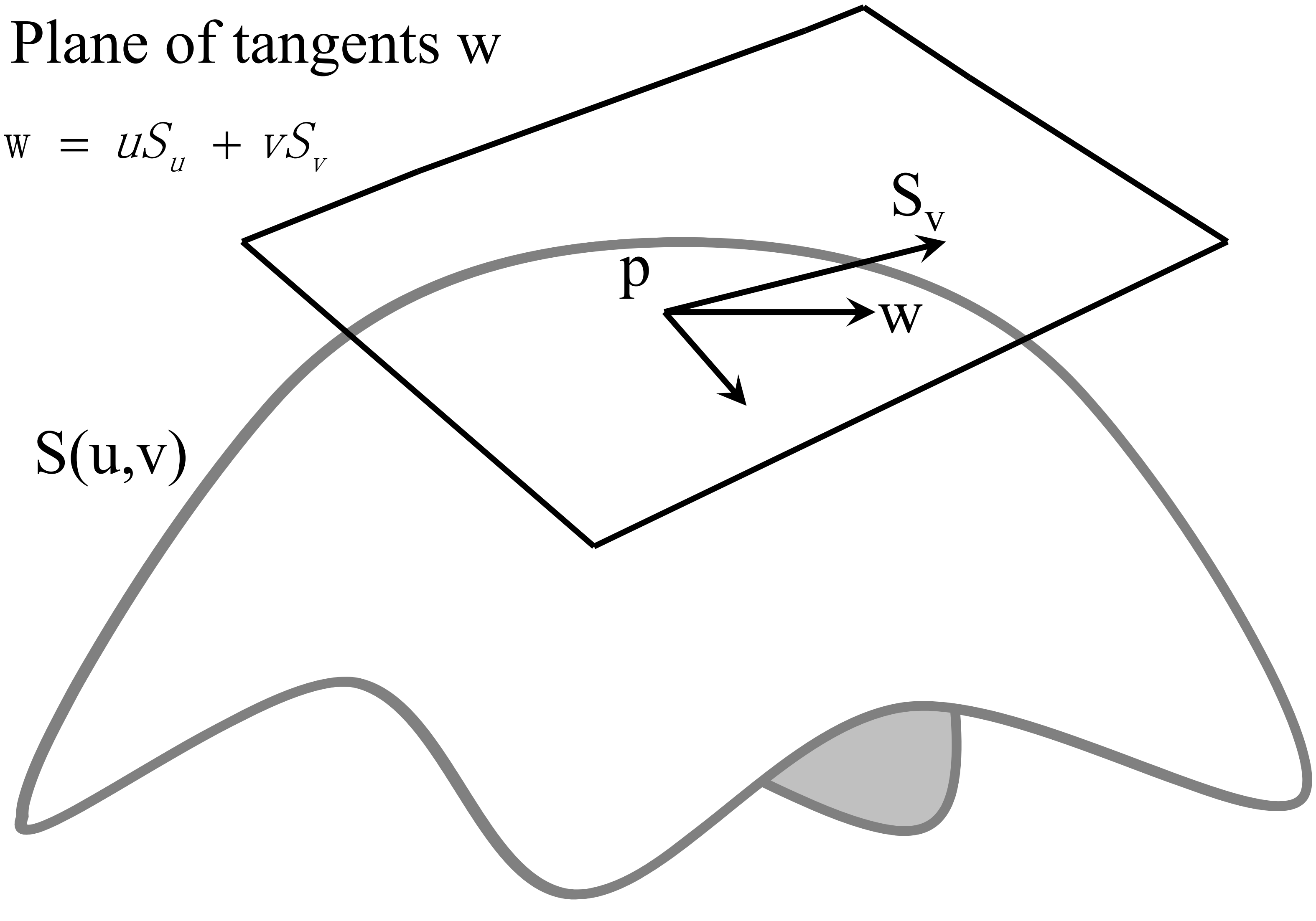
- It is the inner product on the tangent space of a surface in three-dimensional Euclidean space which is induced canonically from thedot product of $\mathbf{R}^3$



Differential Geometry of a Surface

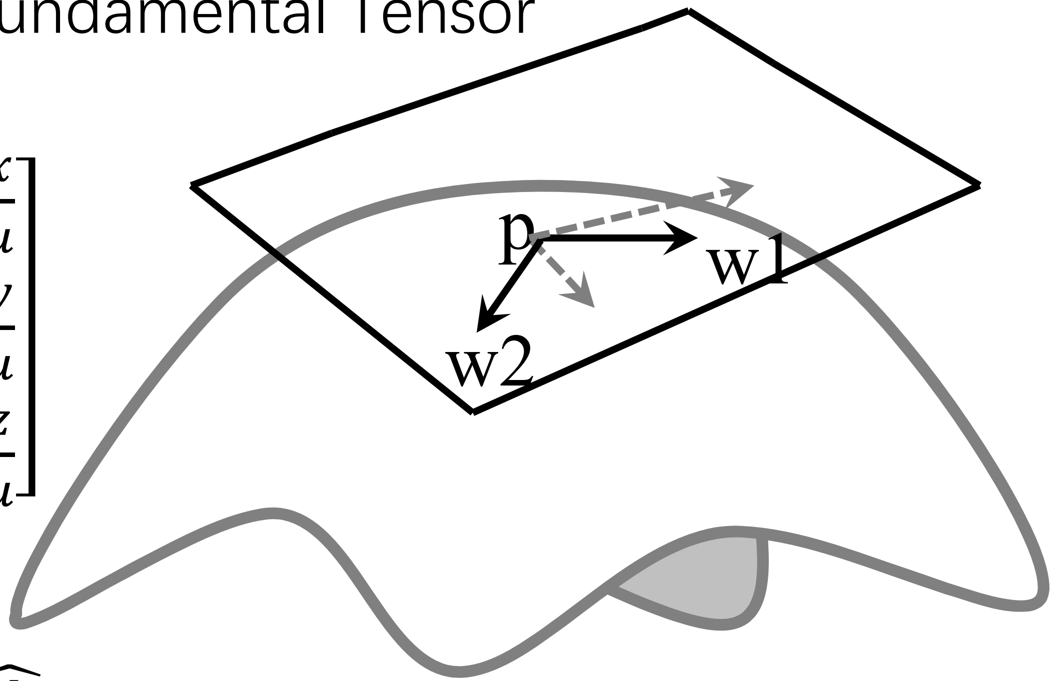
Plane of tangents w

$$w = uS_u + vS_v$$



First Fundamental Form I_S

- **Riemannian metric**, Metric Tensor, Fundamental Tensor
- S(u,v)=(x(u,v), y(u,v),z(u,v))
- Jacobian matrix $J = [S_u, S_v] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial u} \end{bmatrix}$
- $w = J\hat{w} = [S_u, S_v]\begin{bmatrix} u \\ v \end{bmatrix}$
- $< w_1, w_2 >= (J\widehat{w_1})^T(J\widehat{w_2})=\widehat{w_1}^T(J^T J)\widehat{w_2}$
- $I = J^T J = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$



- curve length $L = l(a,b) = \int_a^b \|\mathbf{x}'(u)\| \mathrm{d}u$

$$l(a,b) = \int_a^b \sqrt{(u_t, v_t)\mathbf{I}(u_t, v_t)^T} \mathrm{d}t$$
$$= \int_a^b \sqrt{Eu_t^2 + 2Fu_t v_t + Gv_t^2} \mathrm{d}t.$$

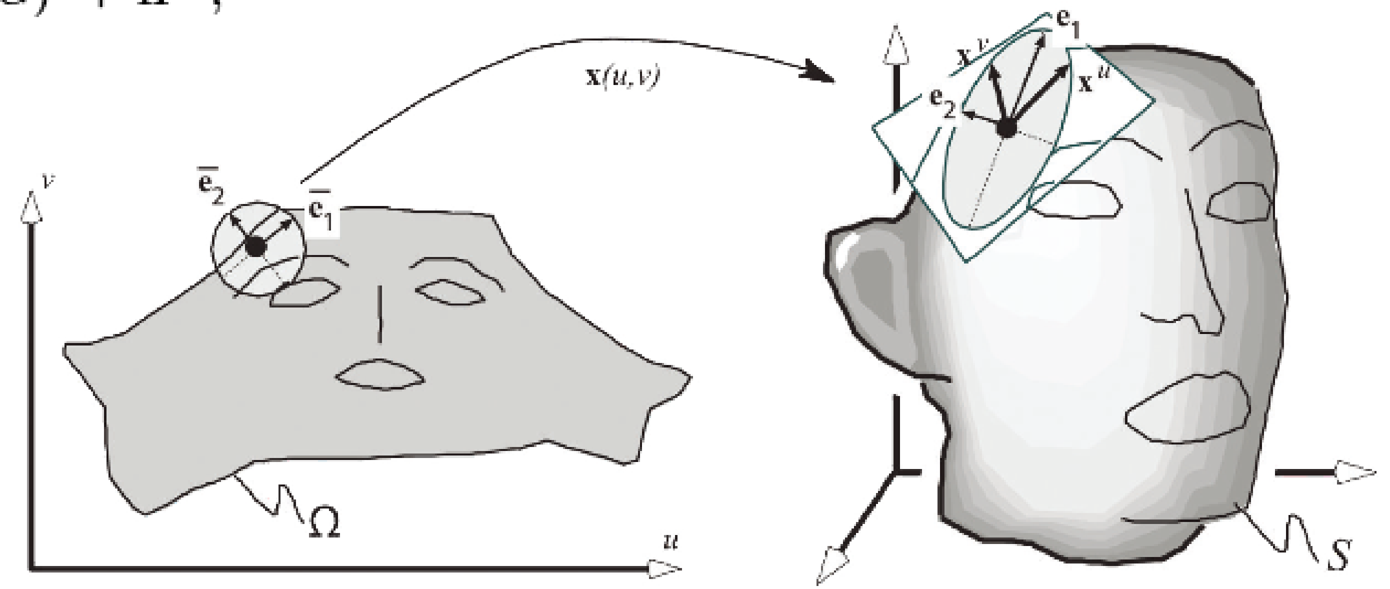
- Surface area
- $A = A(X) = \iint_U |x_u \times x_v| dudv = \iint_U \sqrt{EG - F^2} dudv = \iint_U \sqrt{\det(I_X)} dudv = \iint_U \det(J_X) dudv = \iint_U \text{Jacobian}(X) dudv$

Anisotropy

- the axes of the anisotropy ellipse are $\mathbf{e}_1 = \mathbf{J}\bar{\mathbf{e}}_1$ and $\mathbf{e}_2 = \mathbf{J}\bar{\mathbf{e}}_2$;
- the lengths of the axes are $\sigma_1 = \sqrt{\lambda_1}$ and $\sigma_2 = \sqrt{\lambda_2}$.

$$\sigma_1 = \sqrt{1/2(E+G) + \sqrt{(E-G)^2 + 4F^2}},$$

$$\sigma_2 = \sqrt{1/2(E+G) - \sqrt{(E-G)^2 + 4F^2}},$$

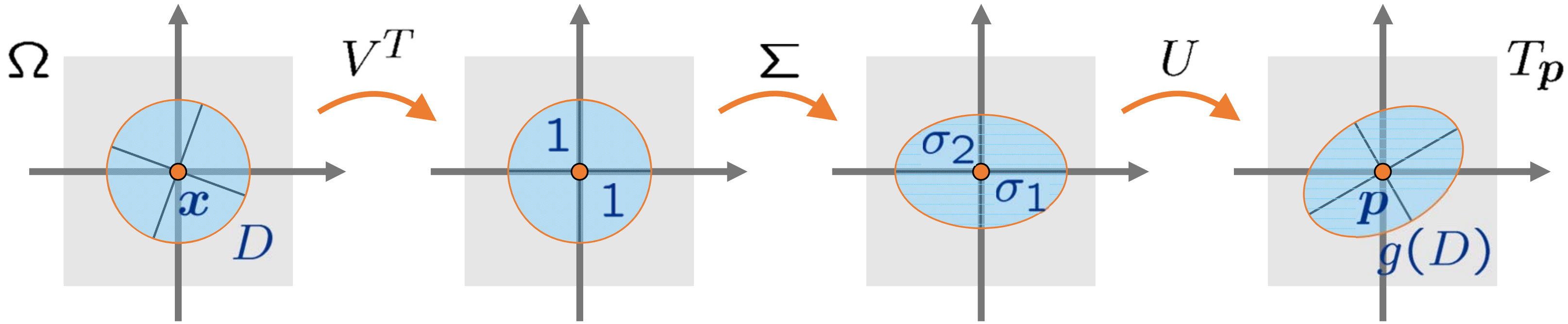


Linear Map Surgery

- **Singular Value Decomposition** (SVD) of J_f

$$J_f = U\,\Sigma\,V^T = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} V^T$$

with **rotations** $U \in \mathbb{R}^{3\times 3}$ and $V \in \mathbb{R}^{2\times 2}$
and **scale factors** (singular values) $\sigma_1 \geq \sigma_2 > 0$



Notion of Distortion

- **isometric** or **length**-preserving

$$\sigma_1 = \sigma_2 = 1$$

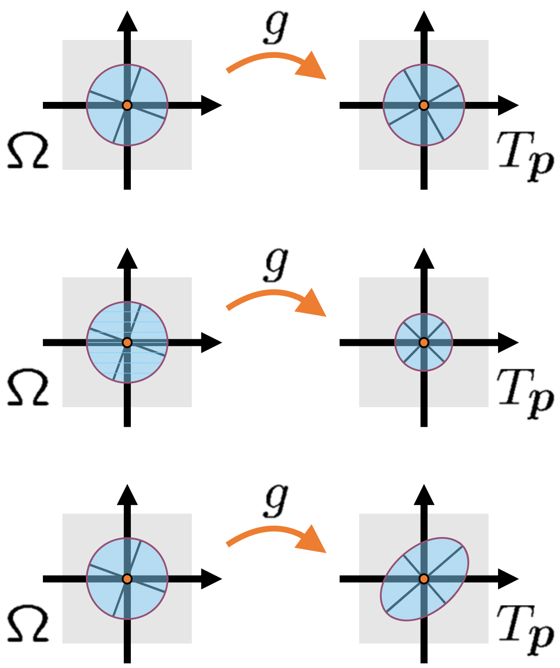
- **conformal** or **angle**-preserving

$$\sigma_1 = \sigma_2$$

- **equiareal** or **area**-preserving

$$\sigma_1 \cdot \sigma_2 = 1$$

- everything defined **pointwise** on Ω



Theorem 4. *Every isometric mapping is conformal and equiareal, and every conformal and equiareal mapping is isometric, i.e.,*

$$\text{isometric} \Leftrightarrow \text{conformal} + \text{equiareal}.$$

Curvature

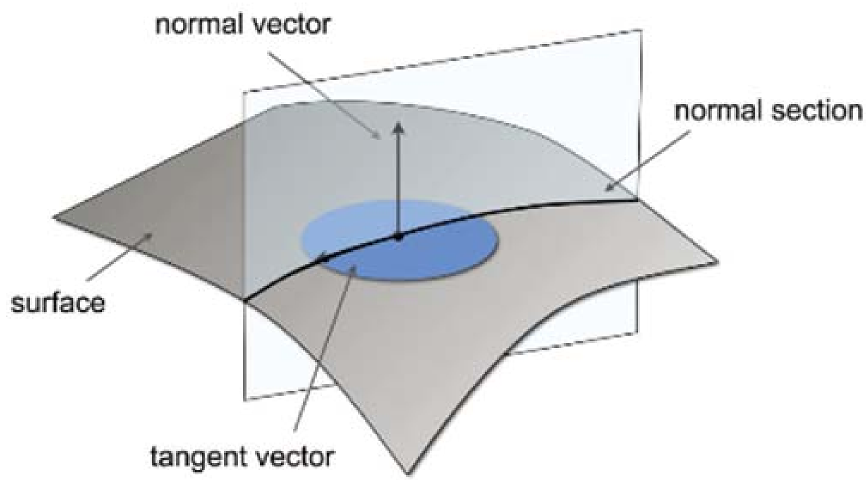
normal curvature $\kappa_n(\bar{\mathbf{t}})$ at $\mathbf{p}$

curvature of curves embedded in the surface. Let $\mathbf{t} = u_t\mathbf{x}_u + v_t\mathbf{x}_v$ be a tangent vector at a surface point $\mathbf{p} \in \mathcal{S}$ represented as $\bar{\mathbf{t}} = (u_t, v_t)^T$ in Parameter space

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{e u_t^2 + 2 f u_t v_t + g v_t^2}{E u_t^2 + 2 F u_t v_t + G v_t^2},$$

where $\mathbf{II}$ denotes the *second fundamental form* defined as

$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{bmatrix}.$$



Surfaces Curvatures

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{e u_t^2 + 2 f u_t v_t + g v_t^2}{E u_t^2 + 2 F u_t v_t + G v_t^2}$$

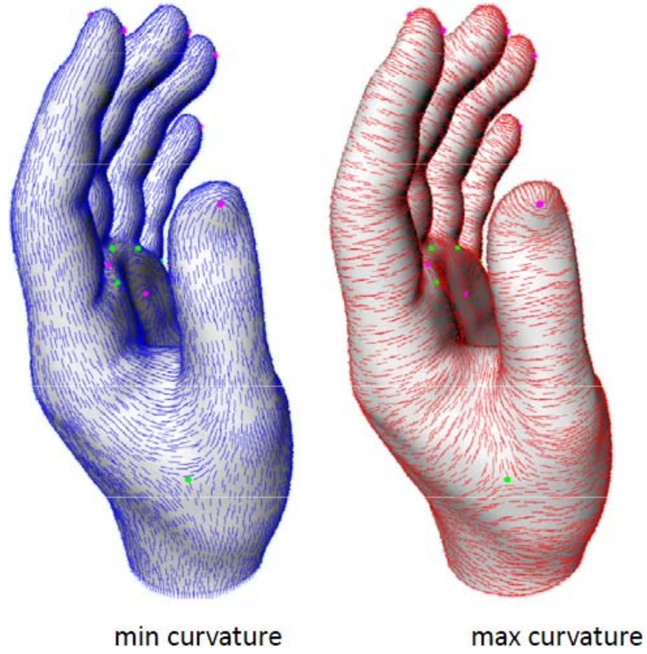
- Principal curvatures
  - Maximal curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
  - Minimal curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$
- Mean curvature: $k_H = \frac{k_1 + k_2}{2} = \frac{1}{2\pi}\int_0^{2\pi} k_n(\theta) d\,\theta = \lim_{diam(A)\to 0} \frac{\nabla A}{A}$
- Gaussian curvature: $k_G = k_1 \cdot k_2 = \lim_{diam(A)\to 0} \frac{A^G}{A}$
- Curvature tensor: $C = PDP^{-1}$, with P=[t1, t2, n] and D=diag(k1, k2, 0)

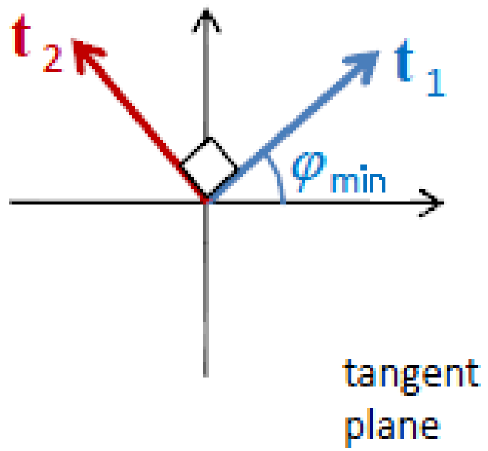
Discrete Differential-Geometry Operators for triangulated 2-manifolds_02

Euler theorem $\kappa_n(\bar{\mathrm{t}}) = \kappa_1 \cos^2\psi + \kappa_2 \sin^2\psi,$

- ψ is the angle between $\bar{t}$ and t1, t1 is the
- Principal directions: tangent vectors corresponding to φ_{max} & φ_{min}

- any normal curvature is a convex combination of the minimum and maximum curvature
- principal directions are orthogonal to each other



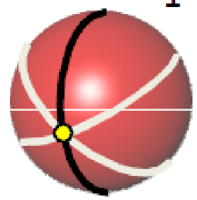


Classification

A point p on the surface is called

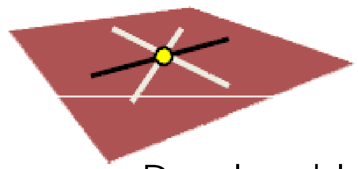
Isotropic: all directions are principle directions

$K > 0, \kappa_1 = \kappa_2$



spherical (umbilical)

$K = 0$

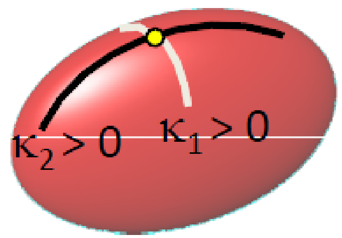


Developable surface ⇔ K=0

planar

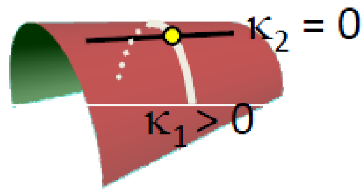
Anisotropic: 2 distinct principle directions

$K > 0$



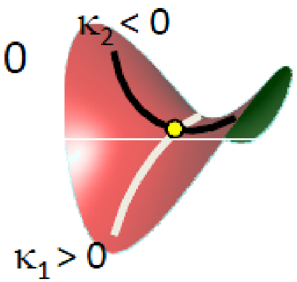
elliptic

$K = 0$



parabolic

$K < 0$



hyperbolic

Laplace & Laplace-Beltrami Operator

Laplace Operator: $\mathbf{divF} = \nabla \cdot F$

- $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
- $f = f(x, y, z)$, $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$
- $F = (U(x, y, z), V(x, y, z), W(x, y, z))$
- $\mathrm{divF} = \nabla \cdot F = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}$

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \mathrm{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

function in Euclidean space

divergence operator

Cartesian coordinates

Laplace-Beltrami Operator: $\Delta_S \mathbf{f} = \mathbf{div}_S \nabla_S f$

gradient operator

Laplace-Beltrami

mean curvature

$$\Delta_{\mathcal{S}}\mathbf{x} = \operatorname{div}_{\mathcal{S}} \nabla_{\mathcal{S}}\mathbf{x} = 2H\mathbf{n}$$

coordinate function

divergence operator

surface normal

For researchers in CG (for differential coordinates), $\Delta_s = -2Hn$

For mathematician, $\Delta_s = 2Hn$

The only difference is the sign.

References

2010_Polygon Mesh Processing