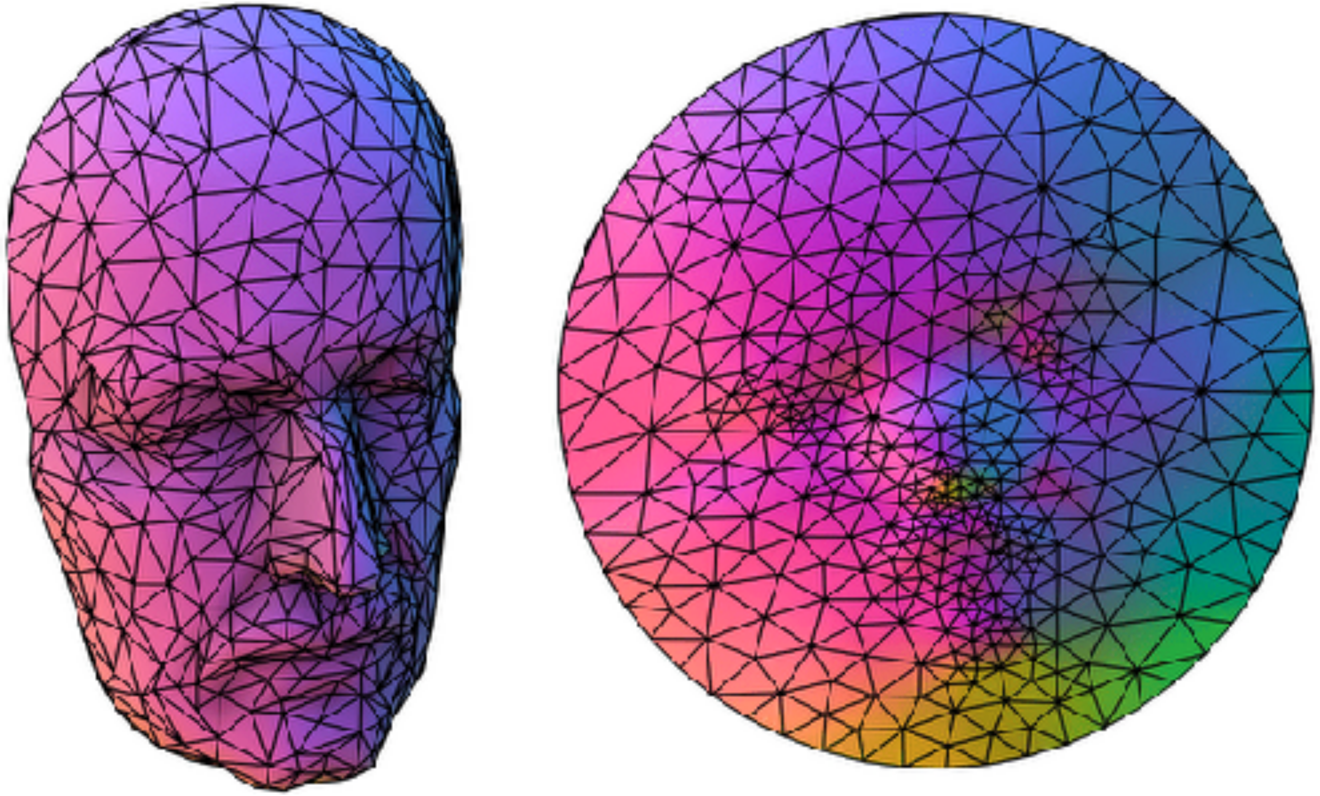
# Digital Geometry
# - Surface Parameterization



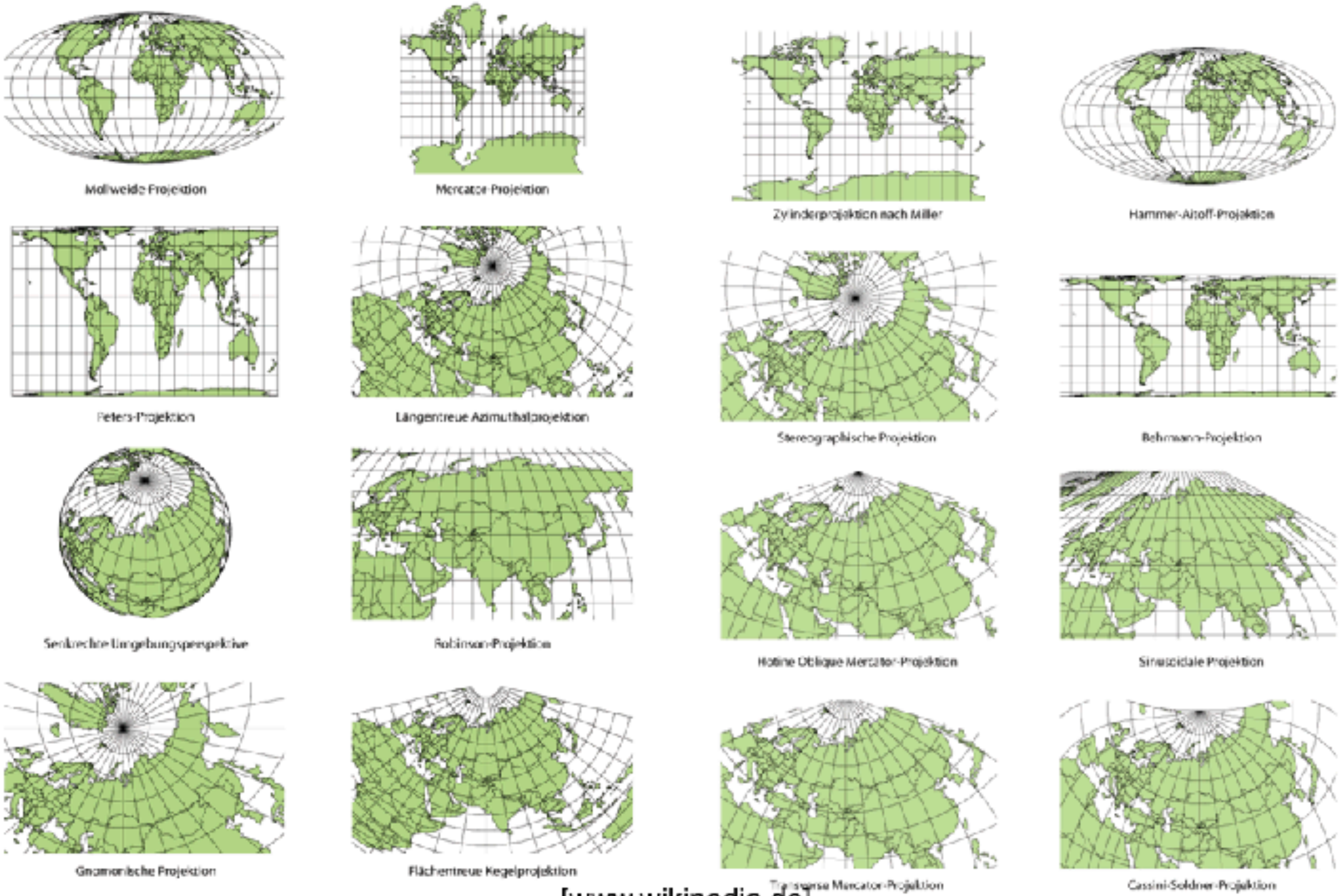
Junjie Cao @ DLUT

Spring 2019

# Outline

- Definition & Motivation
- Angle Preservation
  - Discrete Harmonic Maps
  - Discrete Conformal Maps
  - Angle Based Flattening
- Reducing Area Distortion
- Alternative Domains

# Surface Parameterization




[www.wikipedia.de]

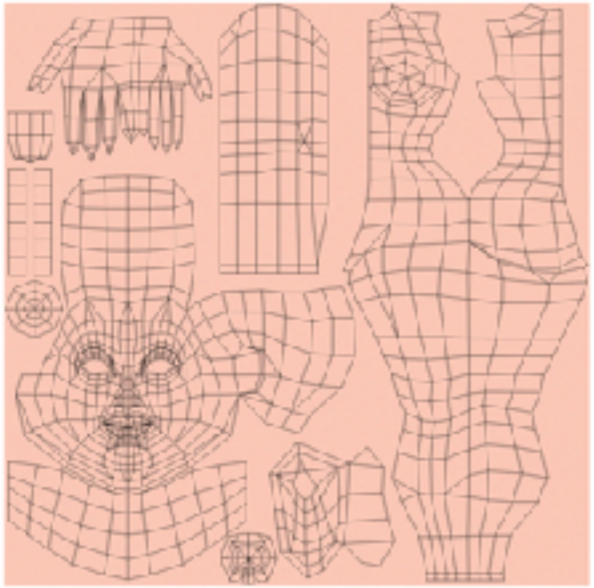
# Motivation

- Texture mapping

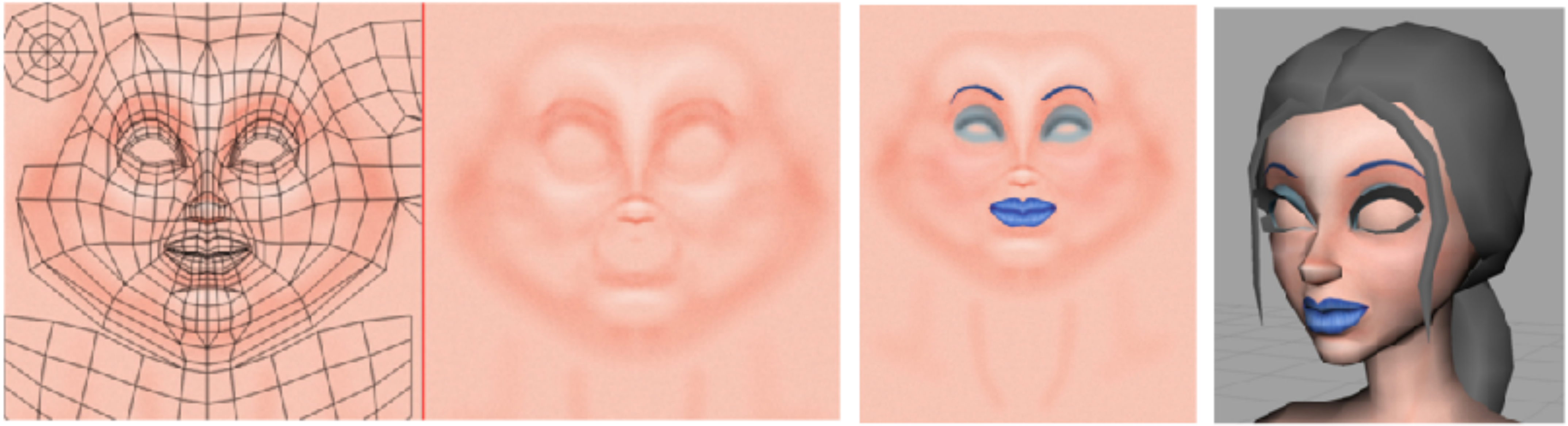
Lévy, Petitjean, Ray, and Maillot: Least squares conformal maps for automatic texture atlas generation, SIGGRAPH 2002

# Motivation

- Texture mapping



Base skin texture

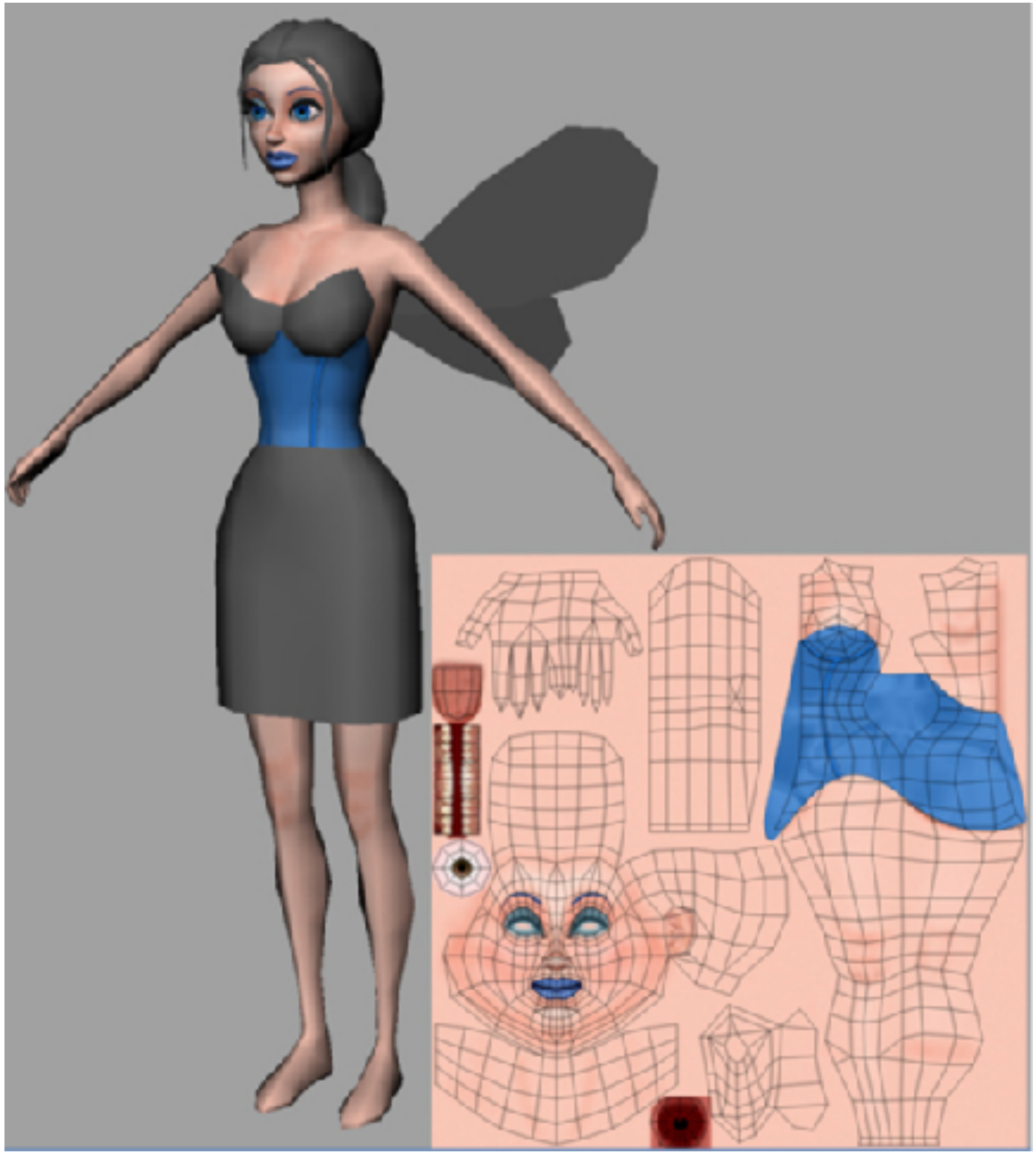


Artist paints in some highlights & make-up.

See details of texture creation: https://software.intel.com/en-us/articles/creating-textures-for-characters-in-autodesk-maya

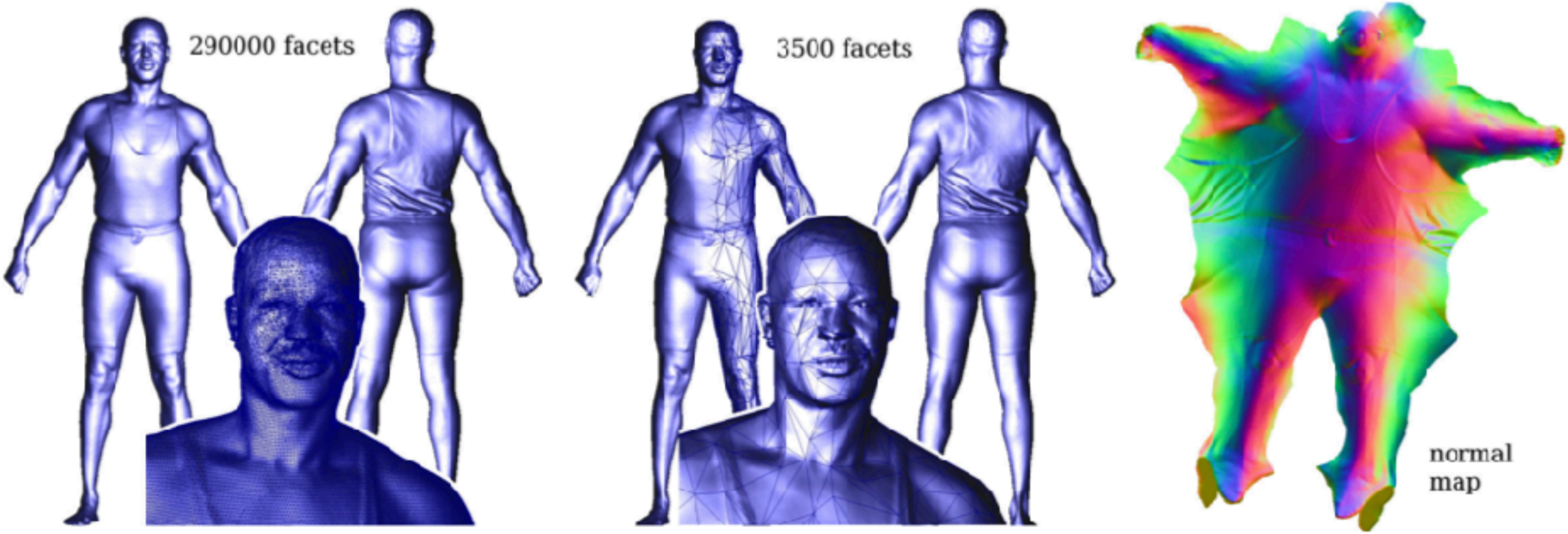
# Texture Mapping

- Unfold/flatten/unwrap/ parameterize a surface mesh in 3D to 2D plane.
- Pull back whatever structure you need: texture, vector field, grid pattern, etc.
- Here we pull back color for texture mapping application.



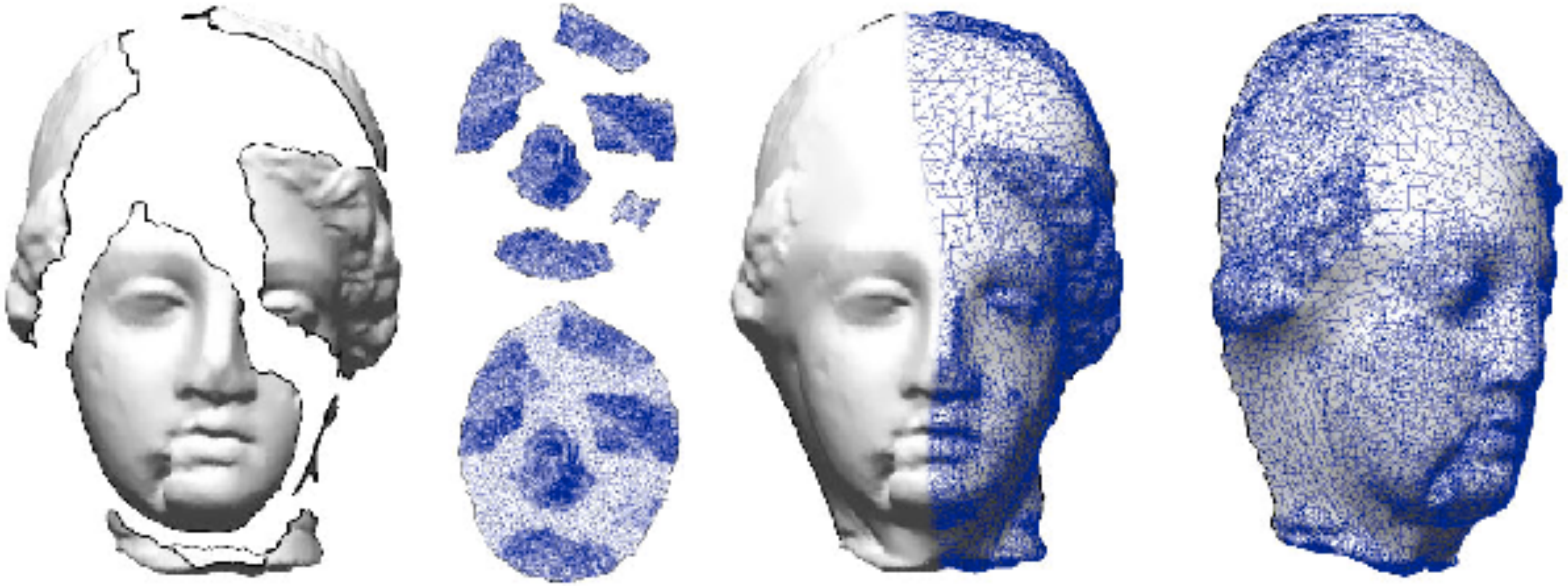
# Motivation

- Normal mapping

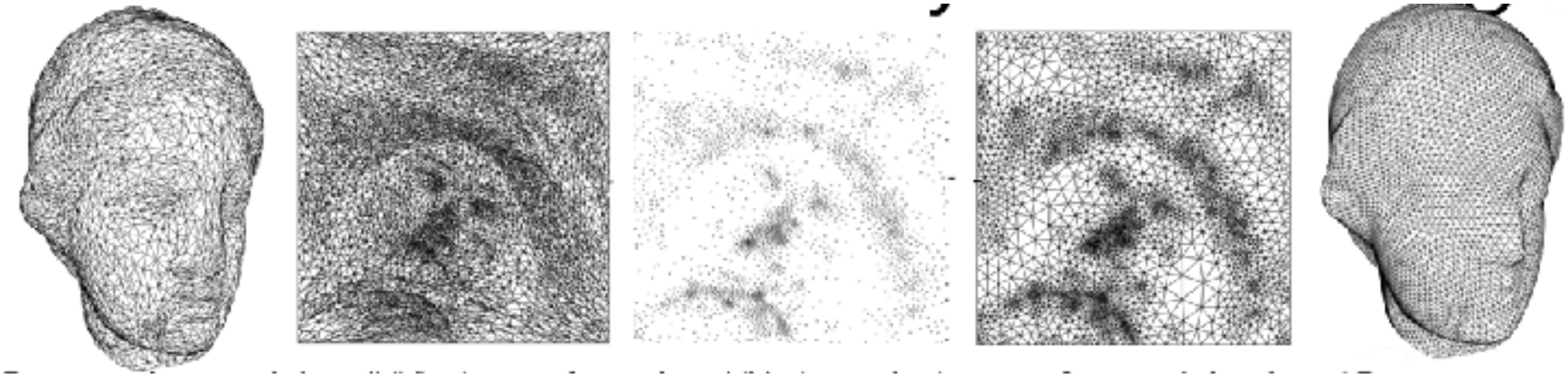


# Motivation

- Many operations are simpler on planar domain



Lévy: Dual Domain Exrapolation, SIGGRAPH 2003

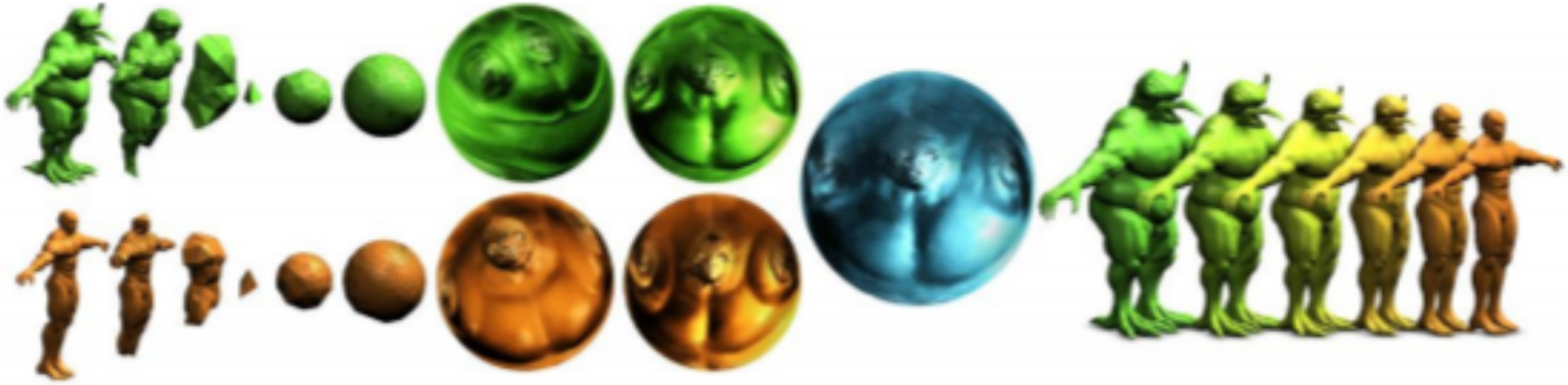


remeshing

# Apps for spherical parameterization

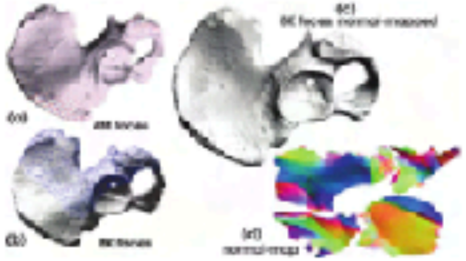
Correspondence:

Morphing:



# Motivation

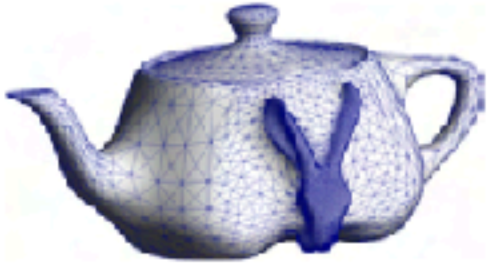

Texture Mapping


Normal Mapping


Detail Transfer

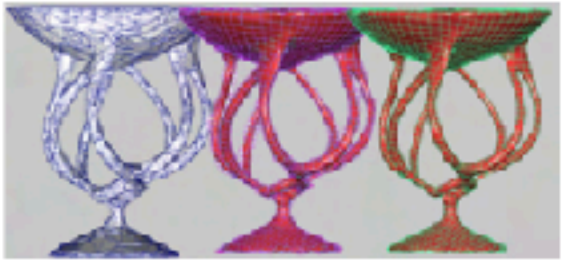

Morphing


Mesh Completion


Editing

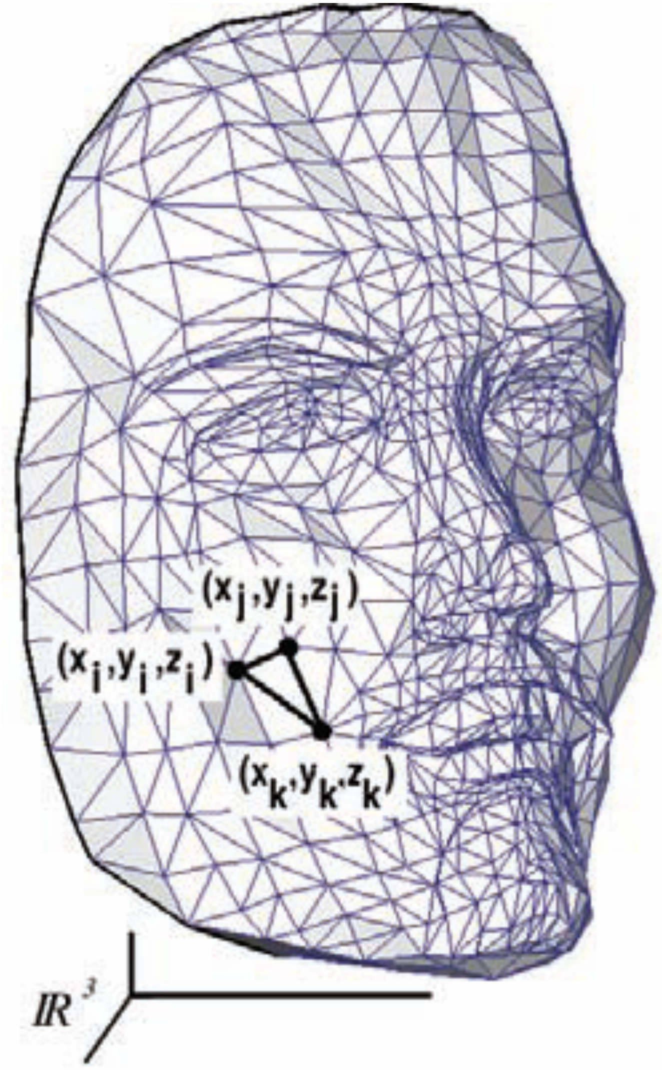

Databases

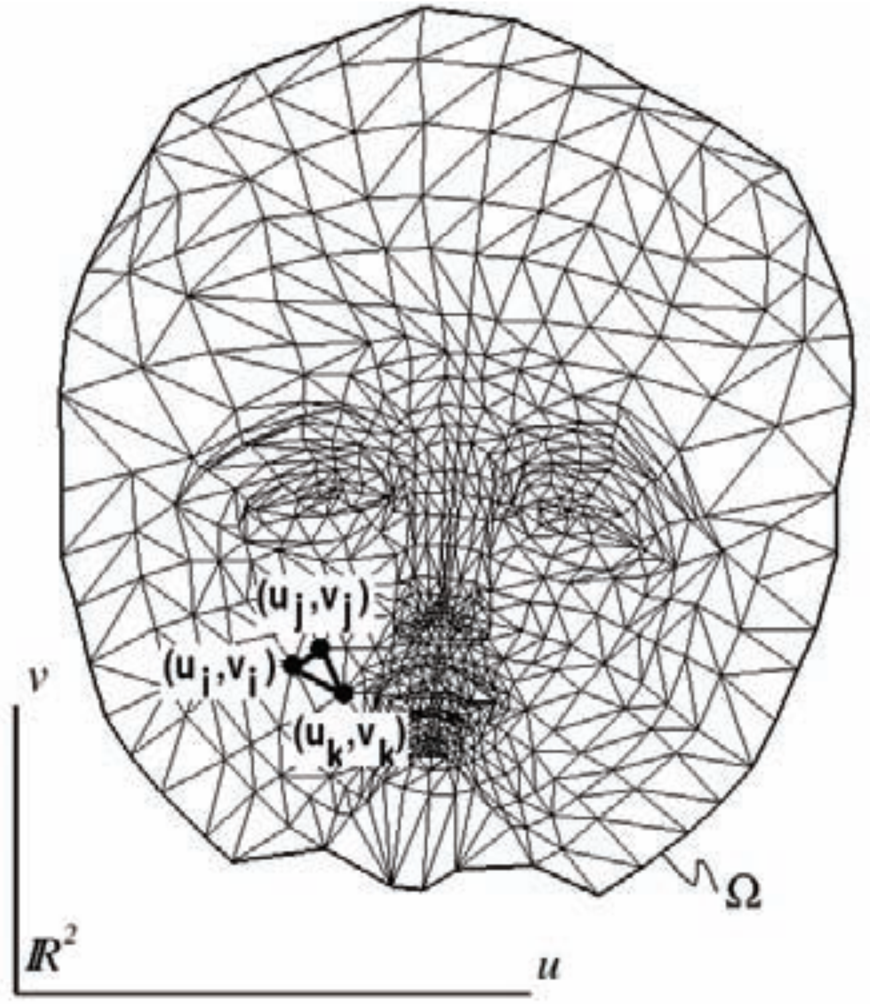

Remeshing


Surface Fitting

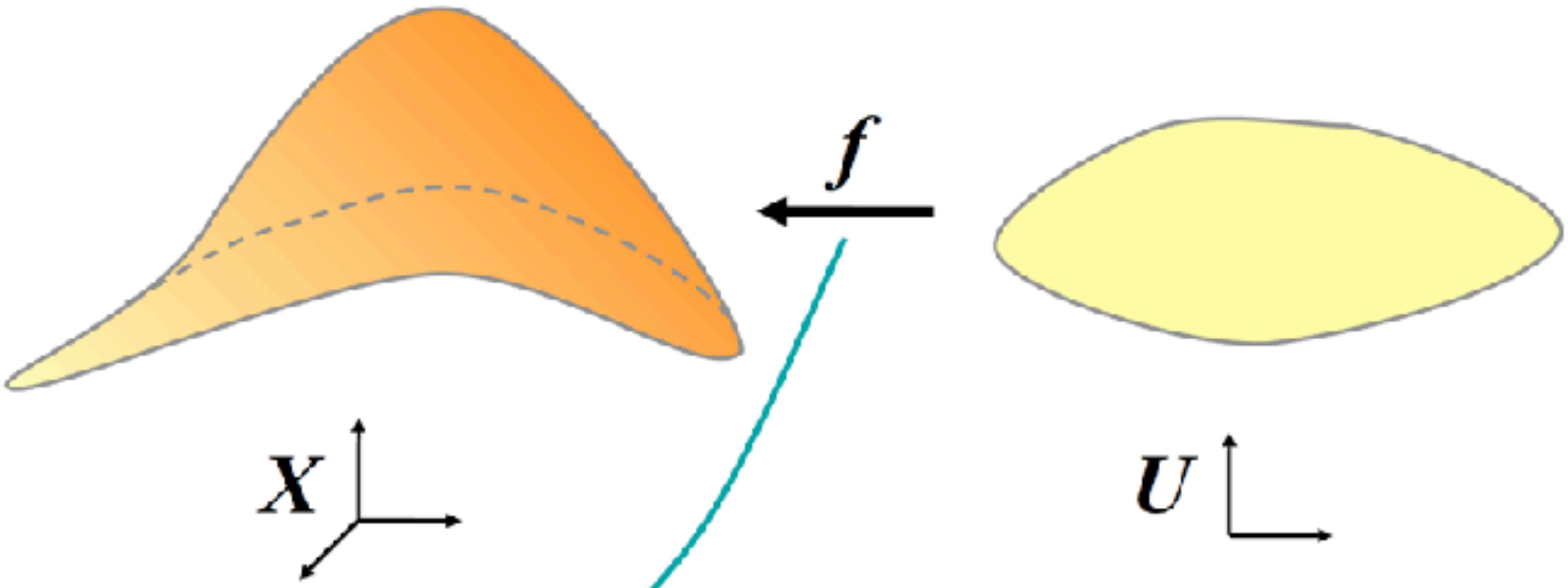
# Mesh Parameterization

- Find coordinates (ui,vi) associated to each vertex i.



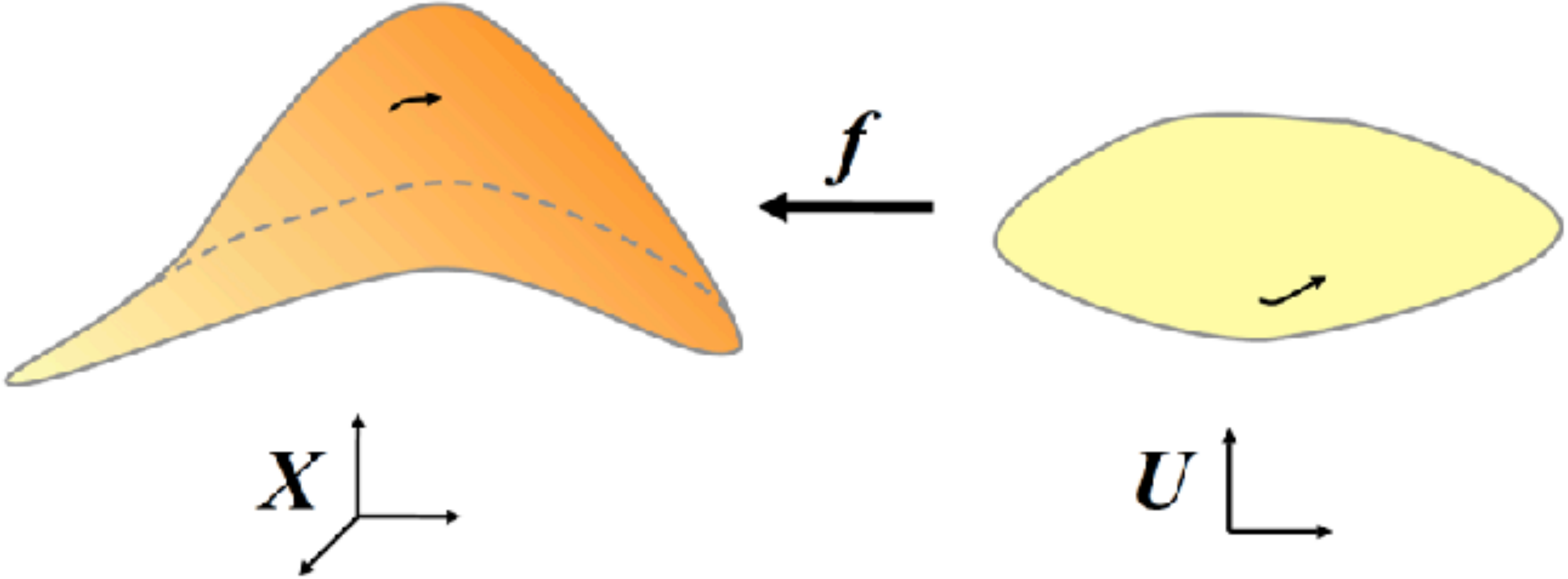
# Surface Parameterization




$$f(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} \qquad J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$

Jacobian

# Surface Parameterization

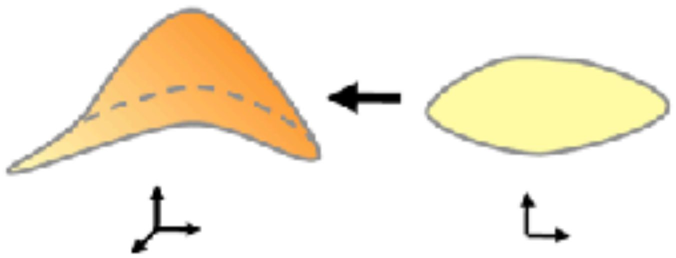



$$d\boldsymbol{X} = \boldsymbol{J}\, d\boldsymbol{U}$$

$$\|d\boldsymbol{X}\|^2 = d\boldsymbol{U}\, \underbrace{\boldsymbol{J}^{\mathsf{T}}\boldsymbol{J}}\, d\boldsymbol{U}$$ First Fundamental Form

$$\mathbf{I} = \begin{pmatrix} x_u x_u & x_u x_v \\ x_u x_v & x_v x_v \end{pmatrix}$$

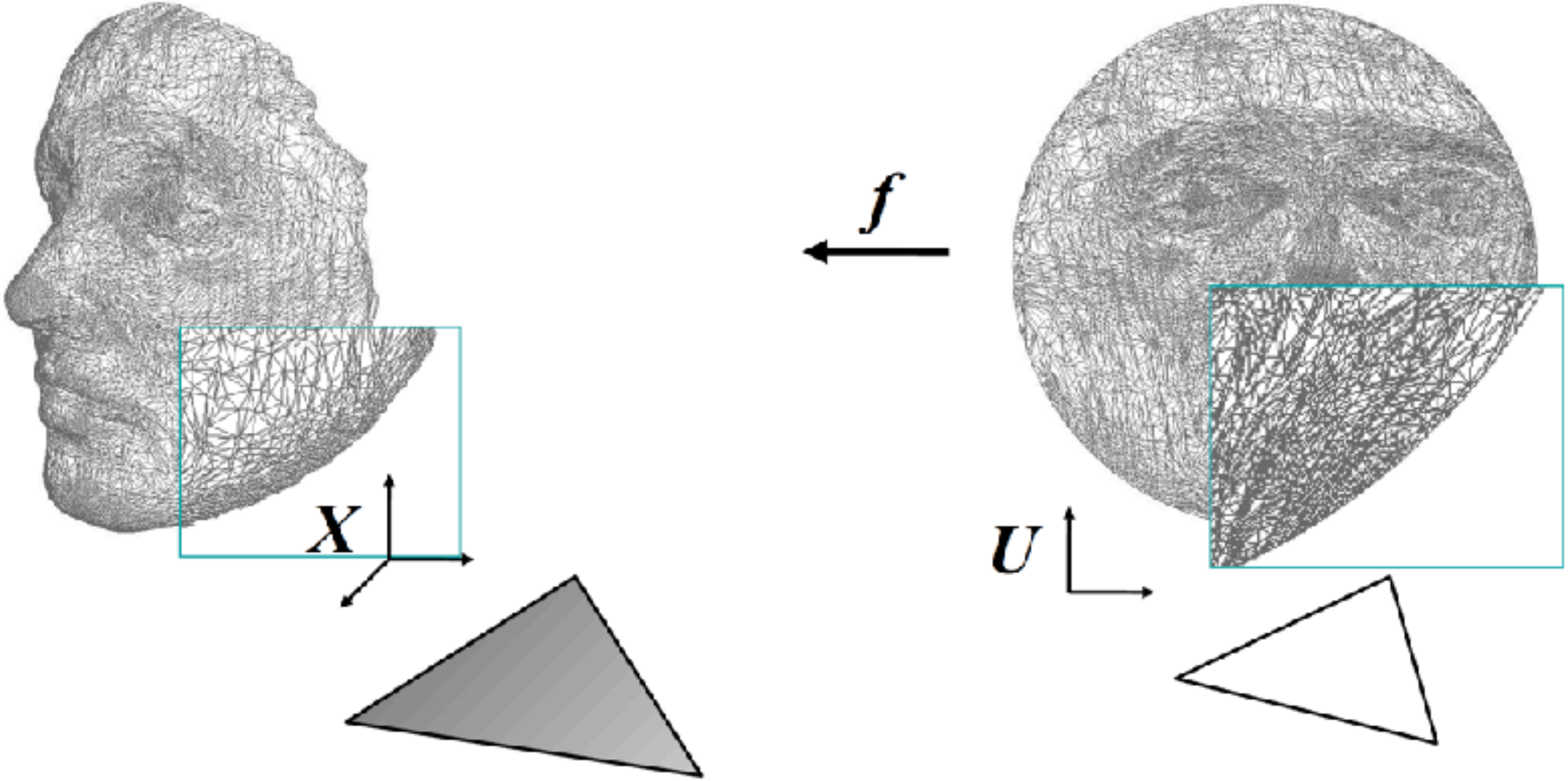
# Characterization of Mappings



- By first fundamental form $\boldsymbol{I}$
  - Eigenvalues $\lambda_{1,2}$ of $\boldsymbol{I}$
  - Singular values $\sigma_{1,2}$ of $\boldsymbol{J}$ ($\sigma_i^2 = \lambda_i$)
- *Isometric*
  - $\boldsymbol{I} = \boldsymbol{Id}$, $\quad \lambda_1 = \lambda_2 = 1$ 
- *Conformal*
  - $\boldsymbol{I} = \mu\, \boldsymbol{Id}$, $\quad \lambda_1 / \lambda_2 = 1$  angle preserving
- *Equiareal*
  - $\det \boldsymbol{I} = 1$, $\quad \lambda_1 \lambda_2 = 1$  area preserving

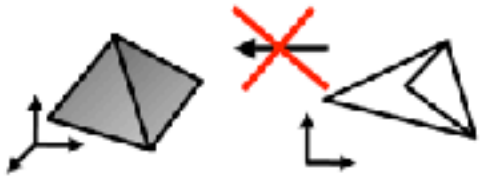
# Piecewise Linear Maps

- Mapping = 2D mesh with same connectivity



# Objectives

- Isometric maps are rare
- Minimize distortion w.r.t. a certain measure
  - Validity (bijective map)



triangle flip

  - Boundary

fixed / free?

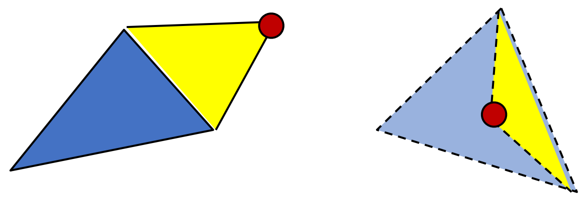
  - Domain

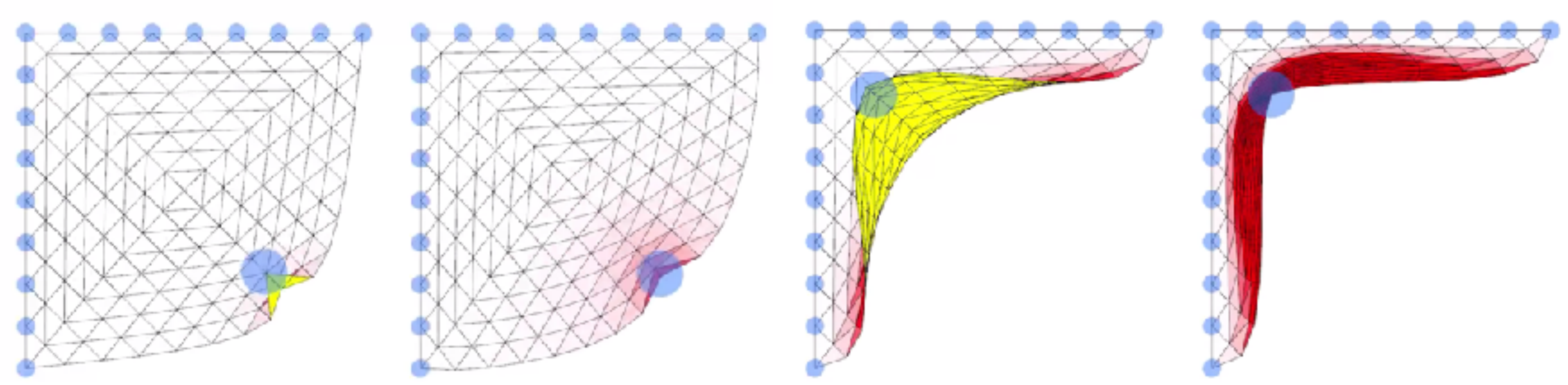
e.g.,spherical

  - Numerical solution

linear / non-linear?

# Parameterization Bijectivity





Local overlap.
(normal of the highlighted flipped triangle is inverted w.r.t. the other triangle normals).

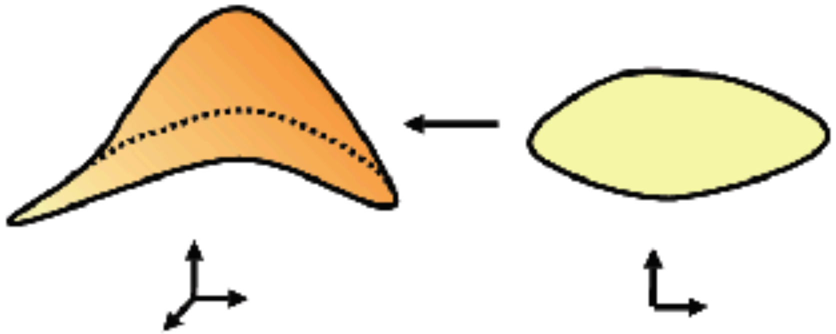
No local overlap.

Local overlap.

No local overlap.

# Discrete Harmonic Maps

- $f$ is *harmonic* if $\Delta f = 0$
- Solve Laplace equation

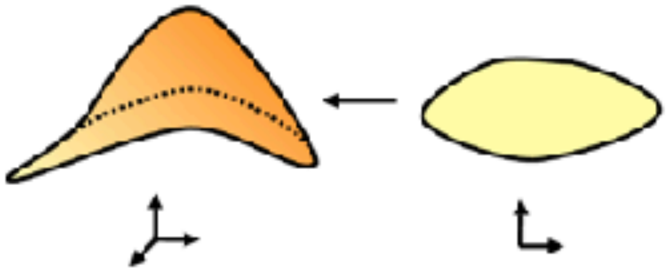


$$\begin{cases} \Delta u = 0 \\ \Delta v = 0 \\ (u,v)_{|\partial\Omega} = (u_0, v_0) \end{cases}$$

$u$ and $v$ are *harmonic*

Dirichlet boundary conditions

# Discrete Harmonic Maps



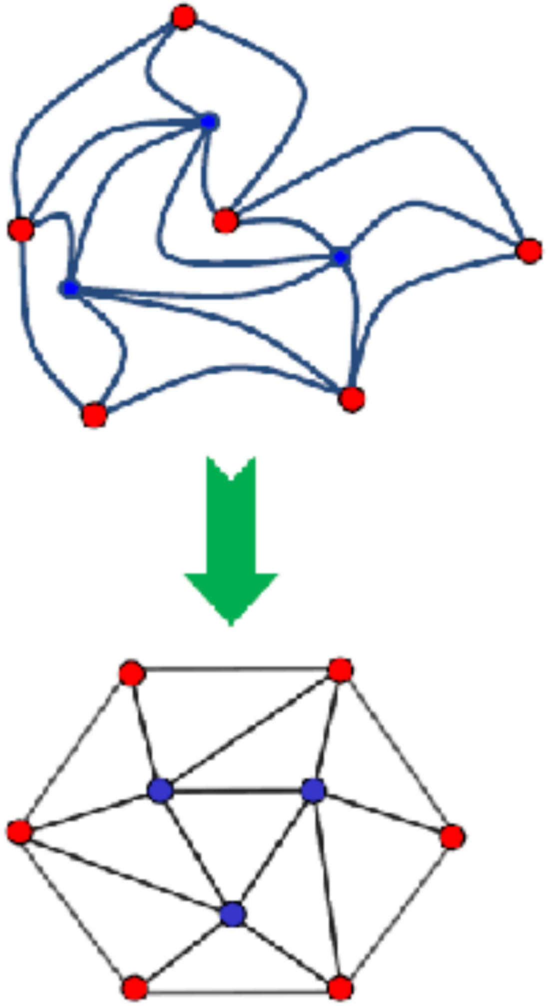
- $f$ is *harmonic* if $\Delta f = 0$
- Solve Laplace equation
- Yields linear system (again)

$$L(p_i) = \sum_{j \in N_i} w_{ij}(p_j - p_i) = 0 \qquad \text{vertices } 1 \le i \le n$$

- *Convex combination maps*
  - *Normalization* $\quad \sum_{j \in N_i} w_{ij} = 1$
  - *Positivity* $\quad w_{ij} > 0$

# Discrete Harmonic Maps

$B$ = Boundary vertices



Fix 2D boundary to **convex polygon**.

Define drawing as a solution of

$$Wx = b_x \qquad Wy = b_y$$

$$w_{ij} = \begin{cases} \boxed{> 0} & (i,j) \in E \quad i \notin B \\ -\sum_{j \neq i} w_{ij} & (i,i), i \notin B \\ 1 & (i,i), i \in B \\ 0 & otherwise \end{cases}$$

$$W \text{ is } symmetric: \; w_{ij} = w_{ji}$$

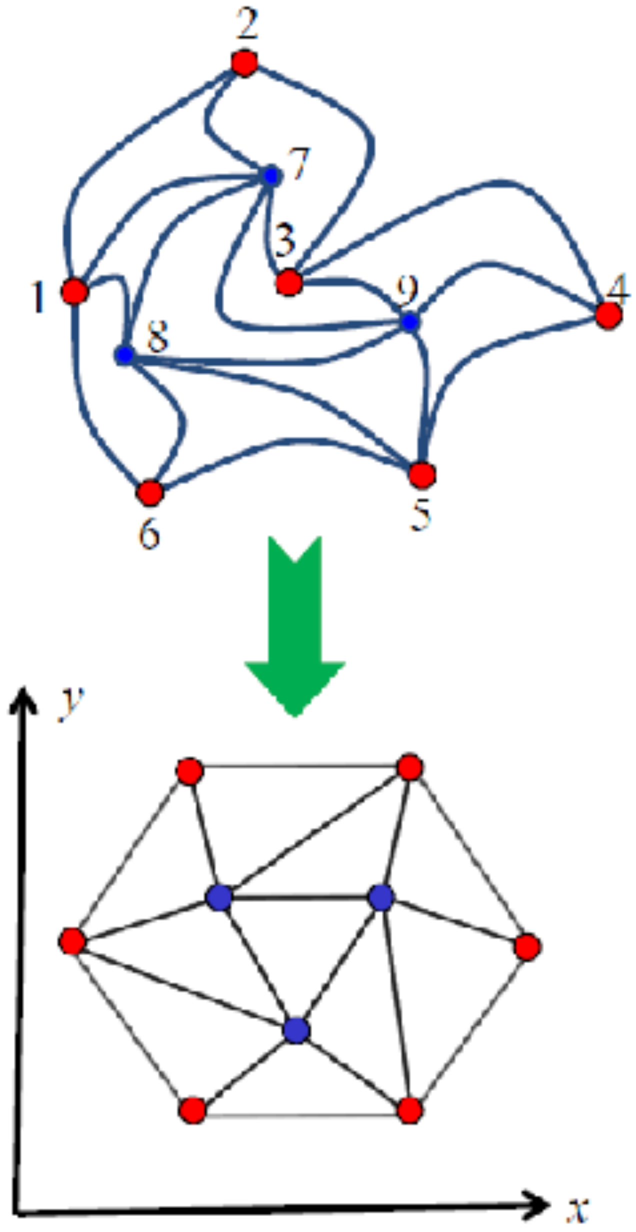
Weights $w_{ij}$ control triangle shapes

# Discrete Harmonic Maps

$$\boxed{w_{ij} = 1}$$

## Laplacian Matrix

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 5 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & -5 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & -5 \end{pmatrix} \quad b_x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad b_y = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



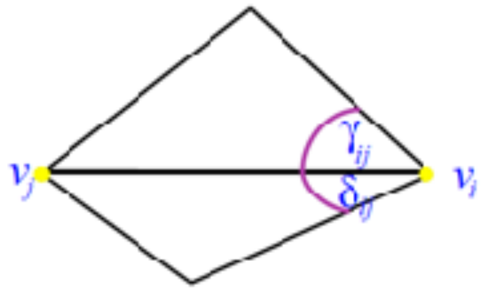
# Convex Combination Maps

- Every (interior) planar vertex is a convex combination of its neighbors
- Based on Tutte's barycentric mapping theorem from graph theory [Tutte60]

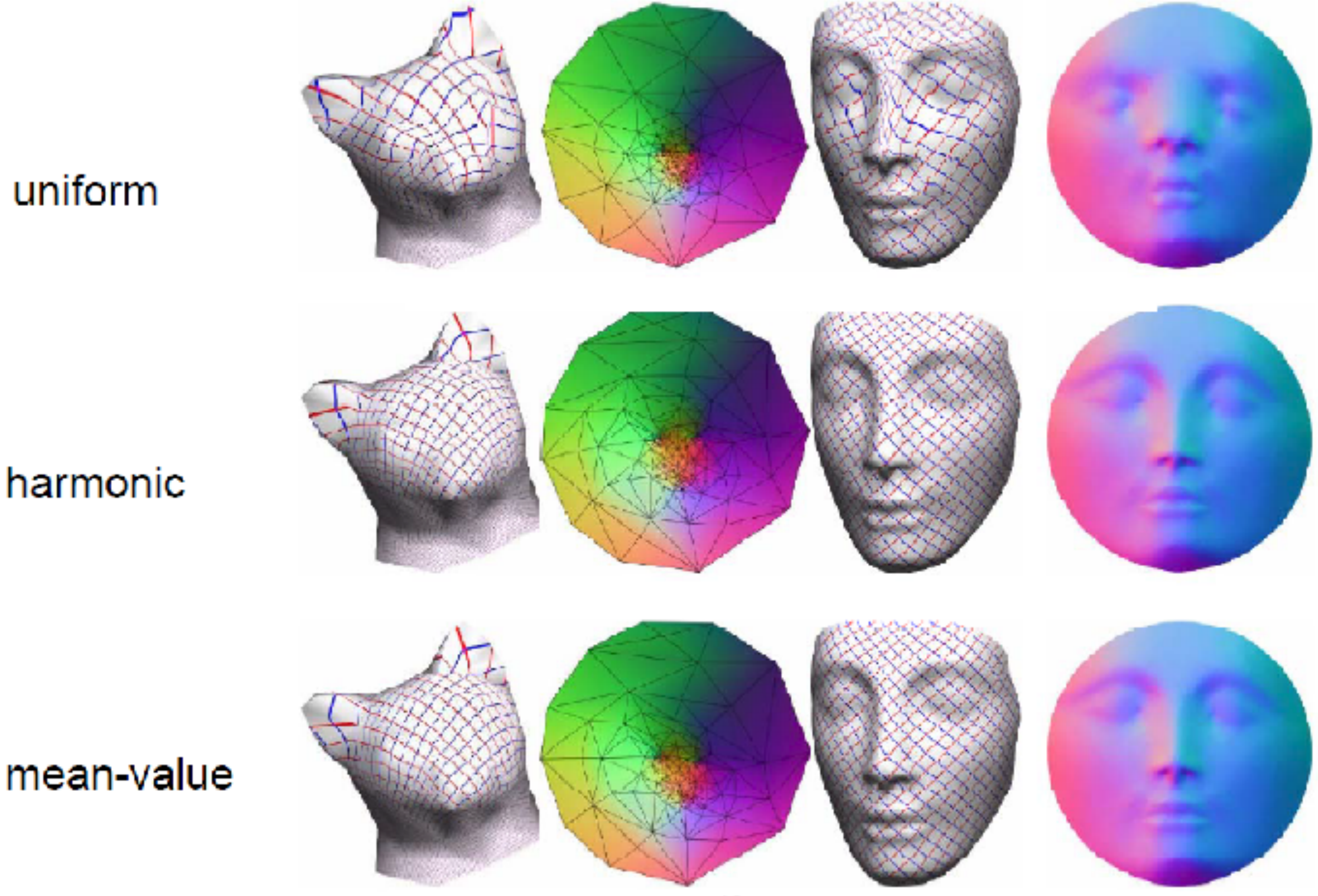
Given a triangulated surface homeomorphic to a disk, if the $(u, v)$ coordinates at the boundary vertices lie on a convex polygon, and if the coordinates of the internal vertices are a convex combination of their neighbors, then the $(u, v)$ coordinates form a valid parameterization (without self-intersections).

- Weights
  - Uniform (barycentric mapping)
  - Shape preserving [Floater 1997] (Reproduction of planar meshes)
  - Mean Value Coordinates [Floater 2003]
    - Use mean value property of harmonic functions

$$\boxed{w_{ij} = \frac{\tan(\gamma_{ij}/2) + \tan(\delta_{ij}/2)}{2\,\| v_i - v_j \|}}$$



# Methods compared



# Conformal Maps

- Planar *conformal mappings* $f(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ satisfy the *Cauchy-Riemann conditions*

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

- Differentiating once more by $x$ and $y$ yields

$$u_{xx} = v_{xy} \text{ and } u_{yy} = -v_{xy} \quad \Rightarrow \quad u_{xx} + u_{yy} = \Delta u = 0$$

$$\text{and similar} \quad \Delta v = 0$$

- conformal $\Rightarrow$ harmonic

# Discrete Conformal Maps

- Planar *conformal mappings* $f(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ satisfy the *Cauchy-Riemann conditions*

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

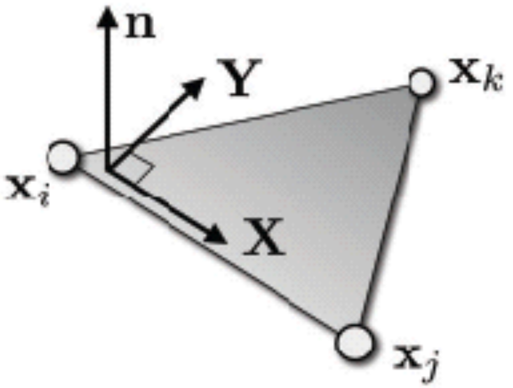
- Conformal energy *(per triangle T)*

$$E_T = (u_x - v_y)^2 + (u_y + v_x)^2$$

- Minimize

$$\sum_{T \in \Gamma} E_T A_T \to \min$$

# Gradient in a Triangle




$$
\begin{aligned}
\mathbf{X} &= \frac{\mathbf{x}_j - \mathbf{x}_i}{\|\mathbf{x}_j - \mathbf{x}_i\|} \\
\mathbf{n} &= \frac{\mathbf{X} \times (\mathbf{x}_k - \mathbf{x}_i)}{\|\mathbf{X} \times (\mathbf{x}_k - \mathbf{x}_i)\|} \\
\mathbf{Y} &= \mathbf{n} \times \mathbf{X}
\end{aligned}
$$

X, Y: orthonormal basis of the triangle

$x_i$, $x_j$, $x_k$: vertex coordinates in the XY basis

Study the inverse of parameterization: maps (X,Y) of the triangle to a point (u,v)

$$
\nabla u = \begin{bmatrix} \partial u / \partial X \\ \partial u / \partial Y \end{bmatrix} = \underbrace{\frac{1}{2\,A_T} \begin{bmatrix} Y_j - Y_k & Y_k - Y_i & Y_i - Y_j \\ X_k - X_j & X_i - X_k & X_j - X_i \end{bmatrix}}_{=\mathbf{M}_T} \begin{pmatrix} u_i \\ u_j \\ u_k \end{pmatrix}
$$

$\nabla u$ intersects the iso-u lines; $\nabla v$ intersects the iso-v line

Conformality condition
iso-u lines ⊥ iso-v lines

$\Updownarrow$

$\nabla u \perp \nabla v$

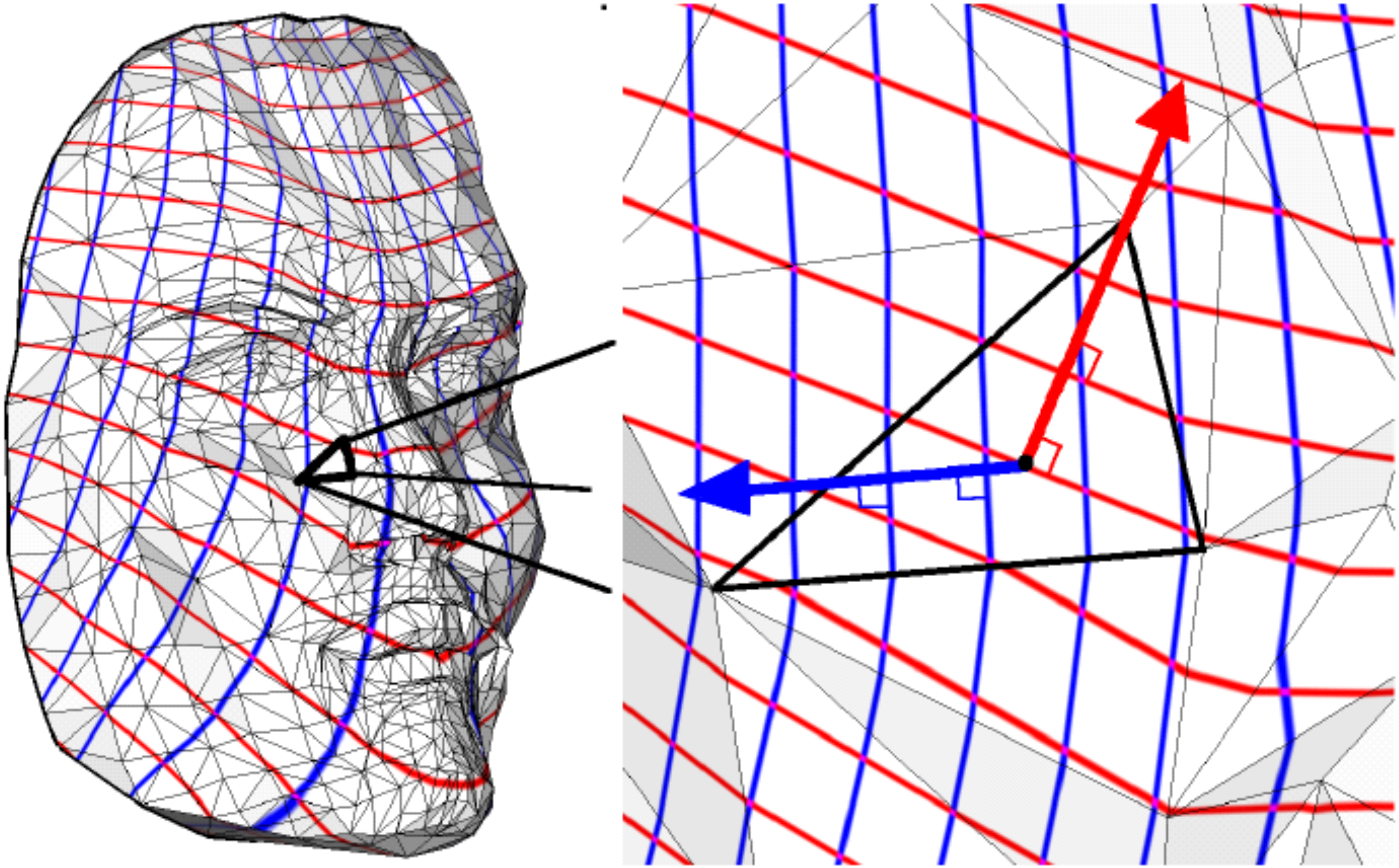


Figure 4.5: Iso-u,v curves and associated gradients.

# Least Square Conformal Map

$$\nabla v = \mathbf{n} \times \nabla u \quad \Rightarrow \quad \nabla v = (\nabla u)^{\perp} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla u$$

$$\mathbf{M}_T \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{M}_T \begin{pmatrix} u_i \\ u_j \\ u_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$E_{\mathrm{LSCM}} = \sum_{T=(i,j,k)} A_T \left\| \mathbf{M}_T \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{M}_T \begin{pmatrix} u_i \\ u_j \\ u_k \end{pmatrix} \right\|^2$$

Mimizing a quadratic form

- $E_{LSCM}$ is invariant to translation and rotation in the parametric space. To have a unique minimizer, it is required to fixed at least two vertices.
- From [Levy et al.2002], if the pinned vertices are chosen on the boundary, all the triangles are consistently oriented (no flips).

# Least Square with Reduced D.O.F.

$x = [x_f | x_l]$. Free parameters $x_f = [x_1 \quad ? \quad x_{nf}]$

Lock parameters $x_l = [x_{nf+1} \quad ? \quad x_n]$

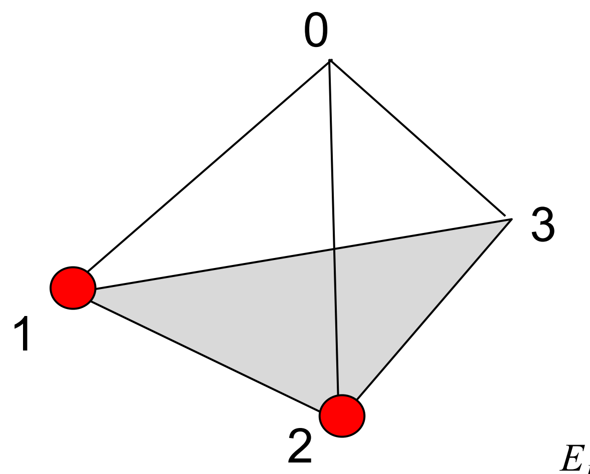
$$F(x_f) = \|Ax - b\|^2 = \left\| [A_f \mid A_l] \begin{bmatrix} x_f \\ \hline x_l \end{bmatrix} - b \right\|^2$$

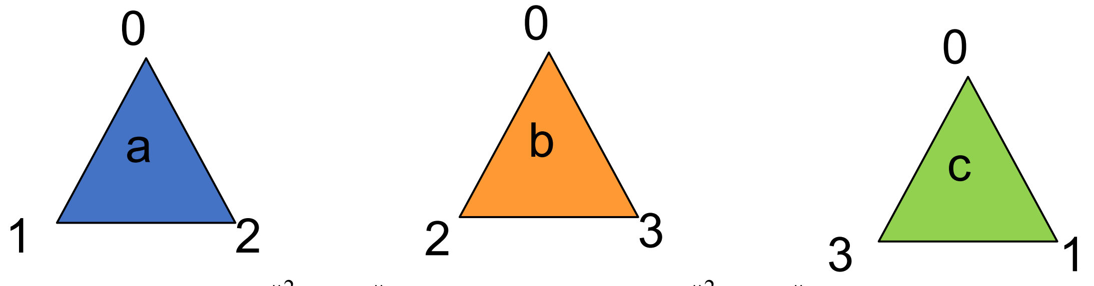
$$F(x_f) = \|A_f x_f + A_l x_l - b\|^2$$

$$b' = A_l x_l - b$$

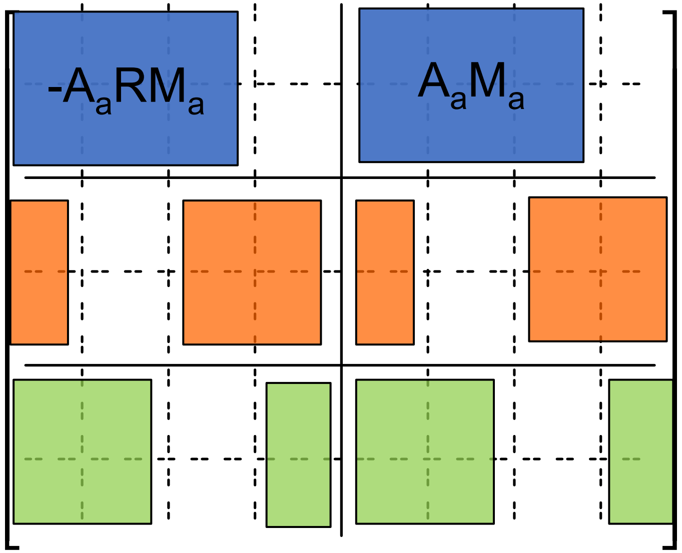
$$A_f^t A_f x_f = A_f^t b' \quad \text{or} \quad A_f^t A_f x_f = A_f^t b - A_f^t A_l x_l$$

$$E_{\mathrm{LSCM}} = \sum_{T=(i,j,k)} A_T \left\| \mathrm{M}_T \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathrm{M}_T \begin{pmatrix} u_i \\ u_j \\ u_k \end{pmatrix} \right\|^2$$



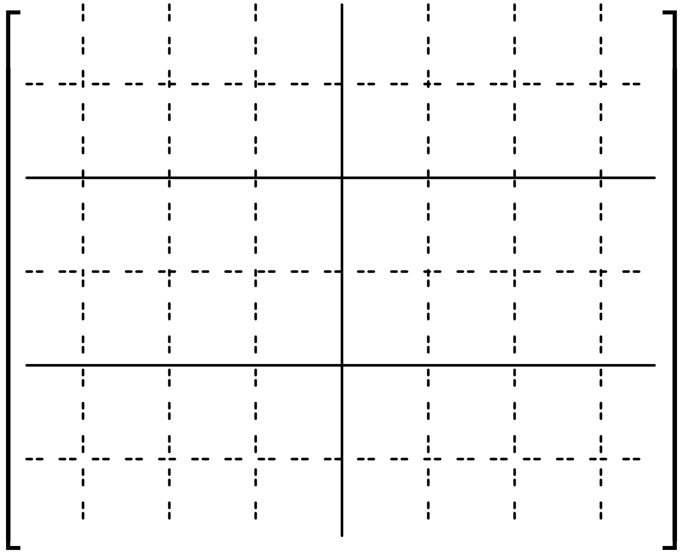

$$E_{LSCM} = A_a \left\| M_a \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} - RM_a \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} \right\|^2 + A_b \left\| M_b \begin{pmatrix} v_0 \\ v_2 \\ v_3 \end{pmatrix} - RM_b \begin{pmatrix} u_0 \\ u_2 \\ u_3 \end{pmatrix} \right\|^2 + A_c \left\| M_c \begin{pmatrix} v_0 \\ v_3 \\ v_1 \end{pmatrix} - RM_c \begin{pmatrix} u_0 \\ u_3 \\ u_1 \end{pmatrix} \right\|^2$$




$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \hline v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

Fixed vertices 1 & 2

$(u_1,v_1)$ & $(u_2,v_2)$ locked



$$\begin{bmatrix} u_0 \\ u_3 \\ v_0 \\ v_3 \\ \hline u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} = 0$$

Variable change→ columns swap

$$\begin{bmatrix} A_f & A_l \end{bmatrix} \begin{bmatrix} u_0 \\ u_3 \\ v_0 \\ v_3 \\ \hline u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} A_f \end{bmatrix} \begin{bmatrix} u_0 \\ u_3 \\ v_0 \\ v_3 \end{bmatrix} = -\begin{bmatrix} A_l \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix}$$

$E_{LSCM}$ is invariant to translation and rotation in the parametric space. To have a unique minimizer, it is required to fixed at least two vertices.

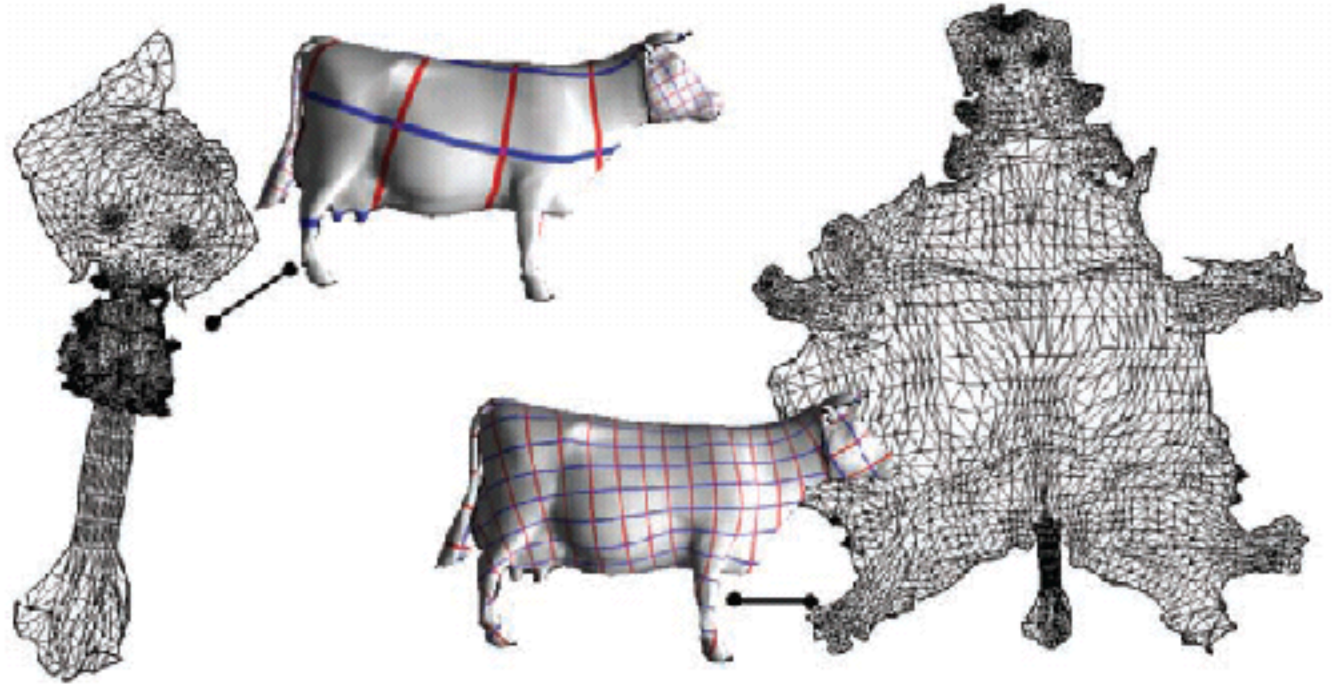
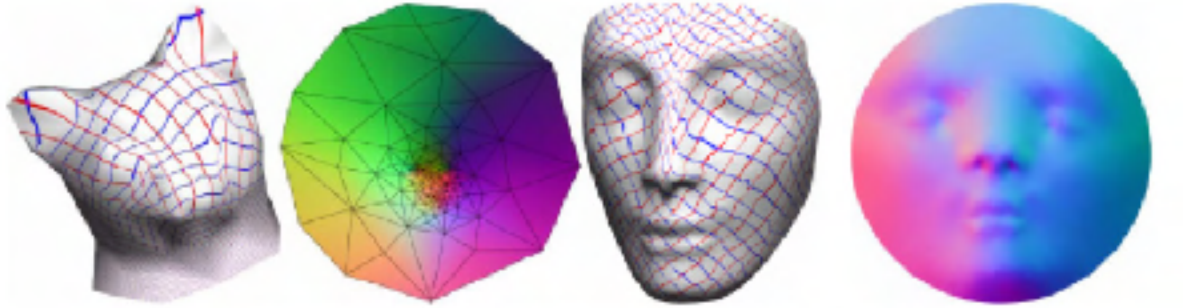
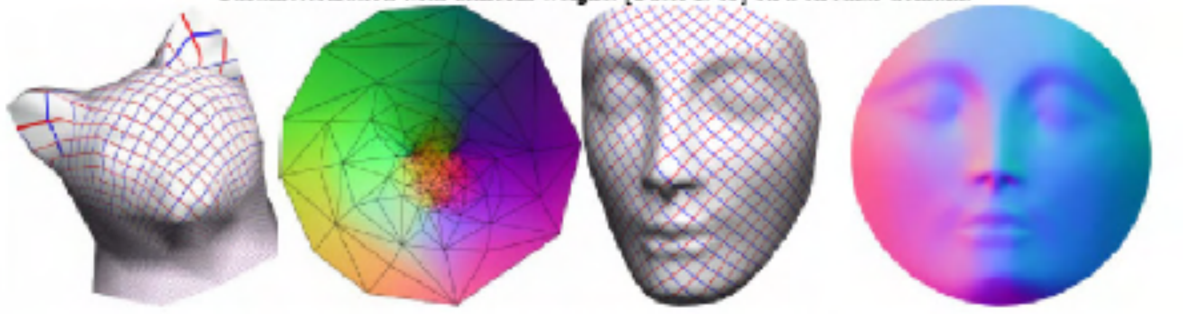


**Figure 5.10.** For surfaces that have a high Gaussian curvature, conformal methods may generate highly distorted results, different from what the user might expect (left). The ABF method and its variants better balance the distortions and give better results (right). (Image taken from [Hormann et al. 07]. ©2007 ACM, Inc. Included here by permission.)



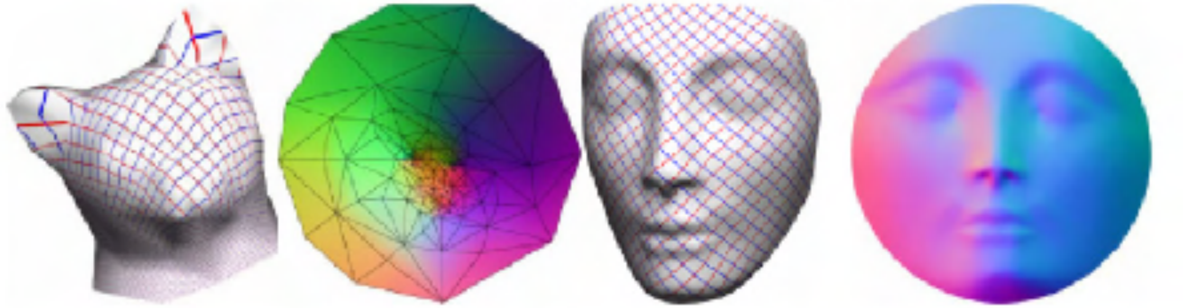
1963

Parameterization with uniform weights [Tutte 1963] on a circular domain.



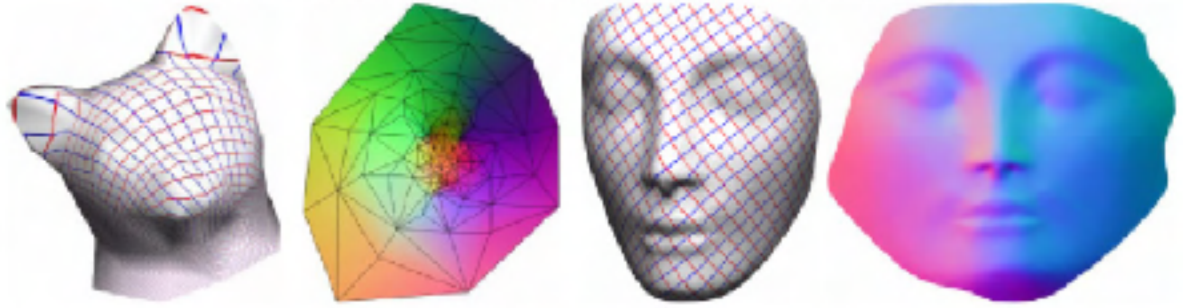
1995

Parameterization with harmonic weights [Eck et al. 1995] on a circular domain.



2003

Parameterization with mean value weights [Floater 2003] on a circular domain.



2002

Parameterization with LSCM [Lévy et al. 2002].

# Angle-Based Flattening (ABF) [Sheffer & de Sturler 2000]

unknown 2D angles $\alpha_k^T$

the "optimal" angles $\beta_k^T$ measured on the 3D mesh

$$E_{\mathrm{ABF}}(\alpha) = \sum_{T \in \mathcal{F}} \sum_{k=1}^{3} \left( \frac{\alpha_k^T - \beta_k^T}{\beta_k^T} \right)^2$$

Constraints:

$$\forall T \in \mathcal{F}: \quad \alpha_1^T + \alpha_2^T + \alpha_3^T = \pi$$

$$\forall v \in \mathcal{V}_{\mathrm{int}}: \quad \sum_{(T,k) \in v^*} \alpha_k^T = 2\pi$$

$$\forall v \in \mathcal{V}_{int}: \quad \prod_{(T,k) \in v^*} \sin \alpha_{k \oplus 1}^T = \prod_{(T,k) \in v^*} \sin \alpha_{k \ominus 1}^T$$

Constrained quadratic optimization with equality constraints

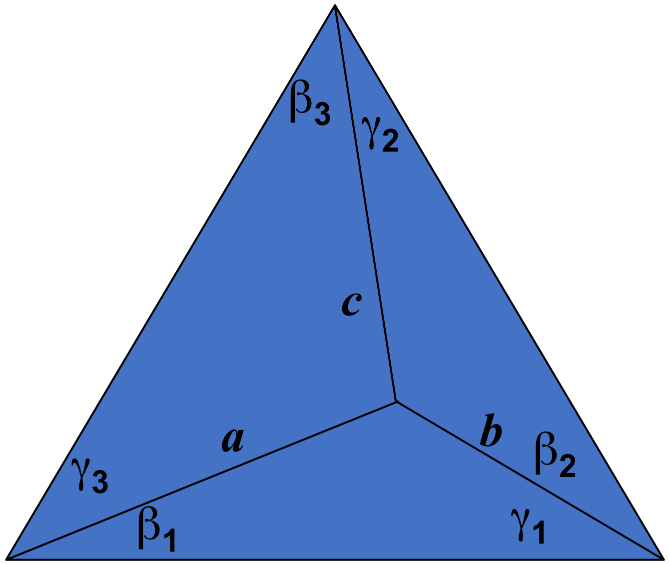
**Nonlinear optimization**

(wheel consistency)

Finding (ui,vi) coordinates, in terms of angles, $\alpha$

Stable (ui,vi) to $\alpha$i conversion

# Wheel Consistency




$$\sin\beta_1 \sin\beta_2 \sin\beta_3 - \sin\gamma_1 \sin\gamma_2 \sin\gamma_3 = 0$$

$$\sin\beta_1 \sin\beta_2 \sin\beta_3 = \sin\gamma_1 \sin\gamma_2 \sin\gamma_3$$

$$\frac{\sin\beta_1}{\sin\gamma_1} \cdot \frac{\sin\beta_2}{\sin\gamma_2} \cdot \frac{\sin\beta_3}{\sin\gamma_3} = 1$$

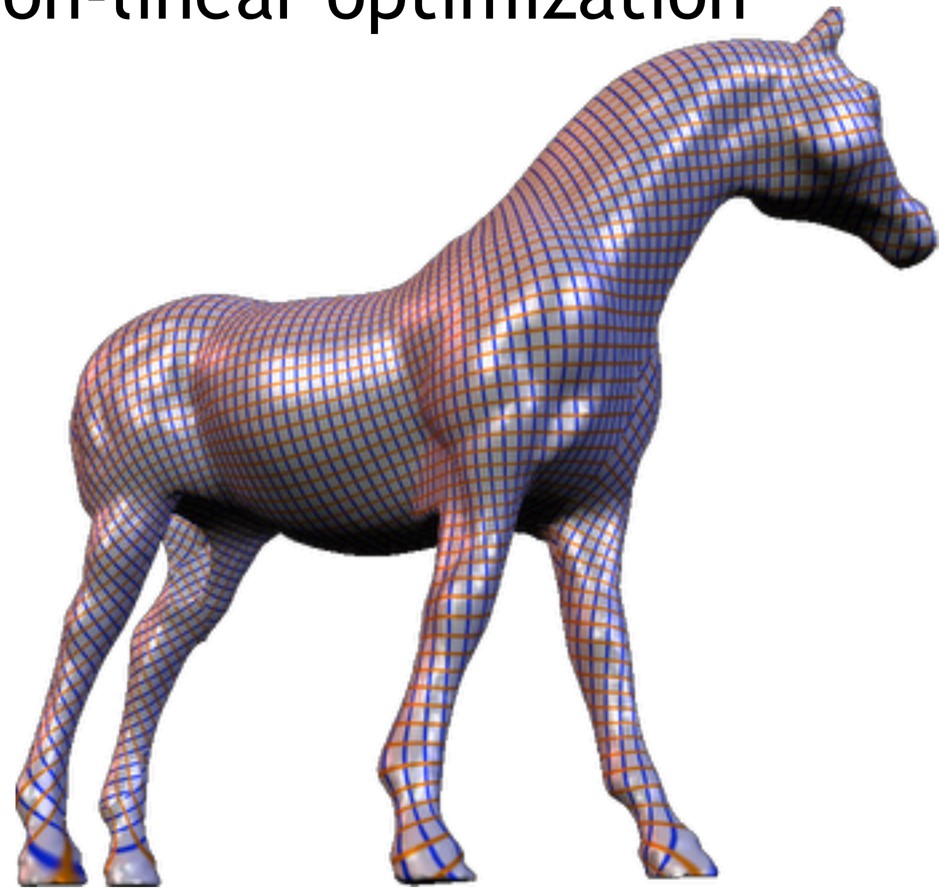
$$\text{Sine law}: \frac{a}{\sin\gamma_1} = \frac{b}{\sin\beta_1}$$

$$\frac{\sin\beta_1}{\sin\gamma_1} = \frac{b}{a}$$

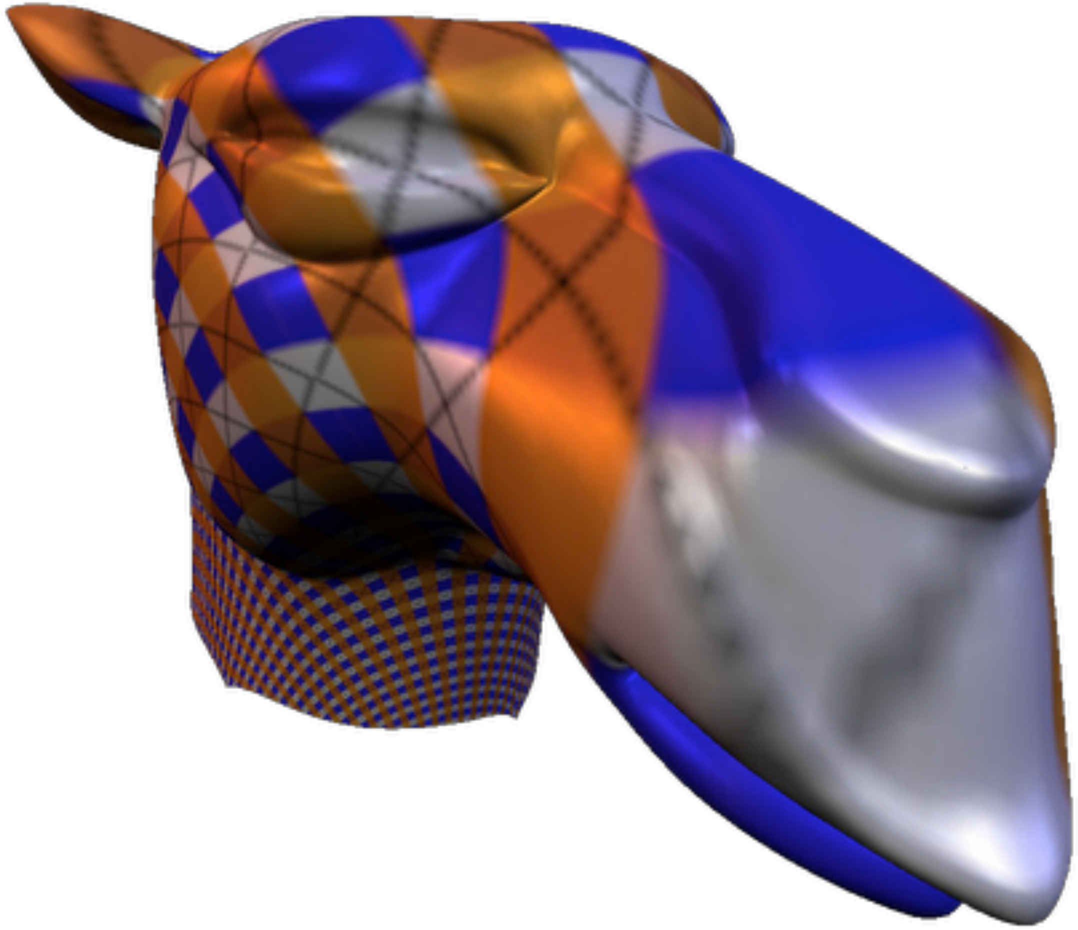
$$\frac{\sin\beta_1}{\sin\gamma_1} \cdot \frac{\sin\beta_2}{\sin\gamma_2} \cdot \frac{\sin\beta_3}{\sin\gamma_3} = \frac{b}{a} \cdot \frac{c}{b} \cdot \frac{a}{c} = 1$$

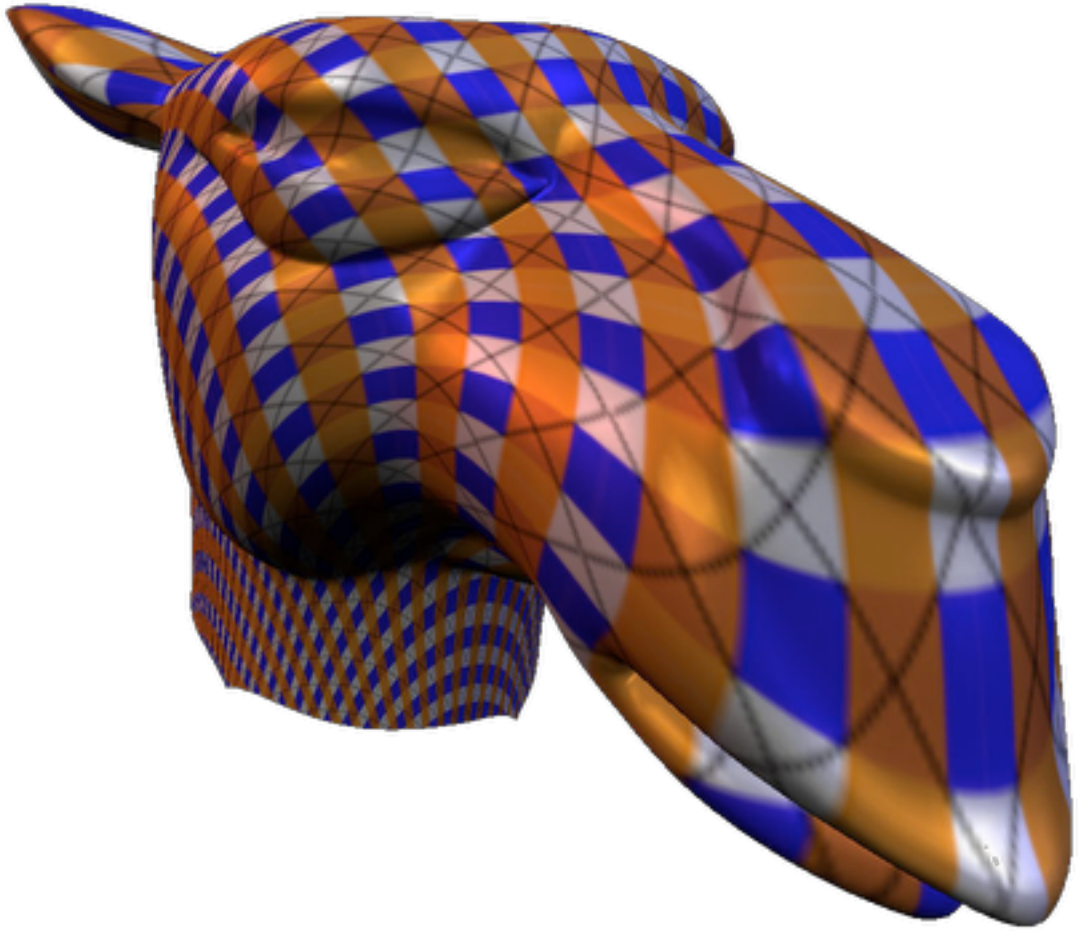
# Angle Based Flattening

- Free boundary
- Validity: no local self-intersections
- Non-linear optimization

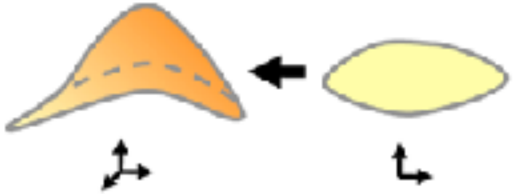


# And how about area distortion?





# Reducing Area Distortion



- Energy minimization based on
  - MIPS [Hormann & Greiner 2000]

$$\| J \|_F \| J^{-1} \|_F = \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}$$

  - modification [Degener et al. 2003]

$$\det J + \frac{1}{\det J} = \sigma_1 \sigma_2 + \frac{1}{\sigma_1 \sigma_2}$$

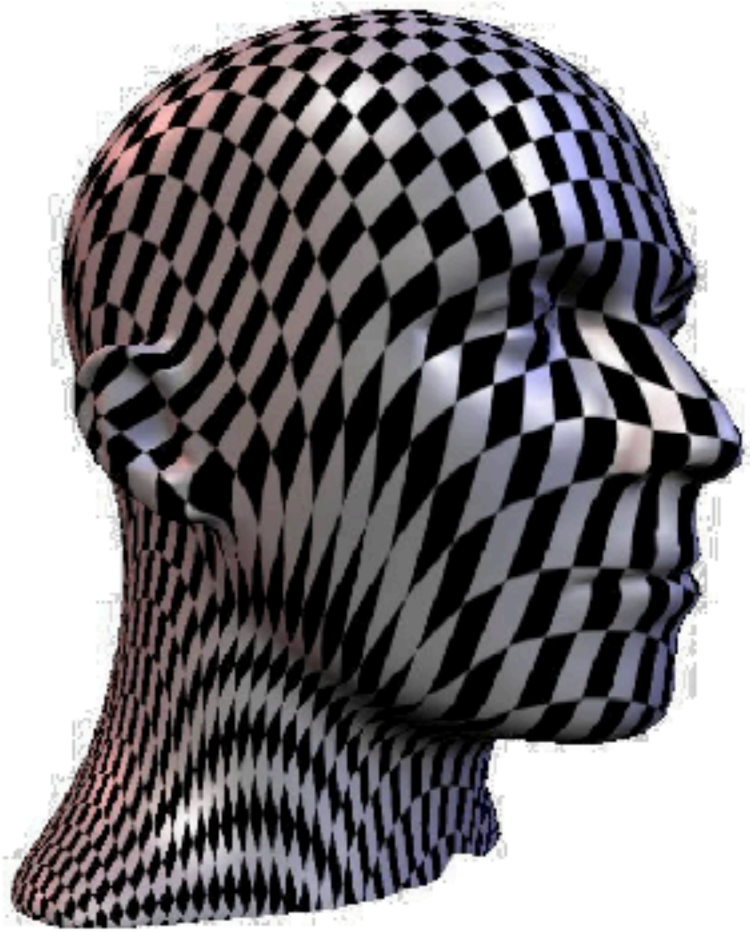
  - "Stretch" [Sander et al. 2001]

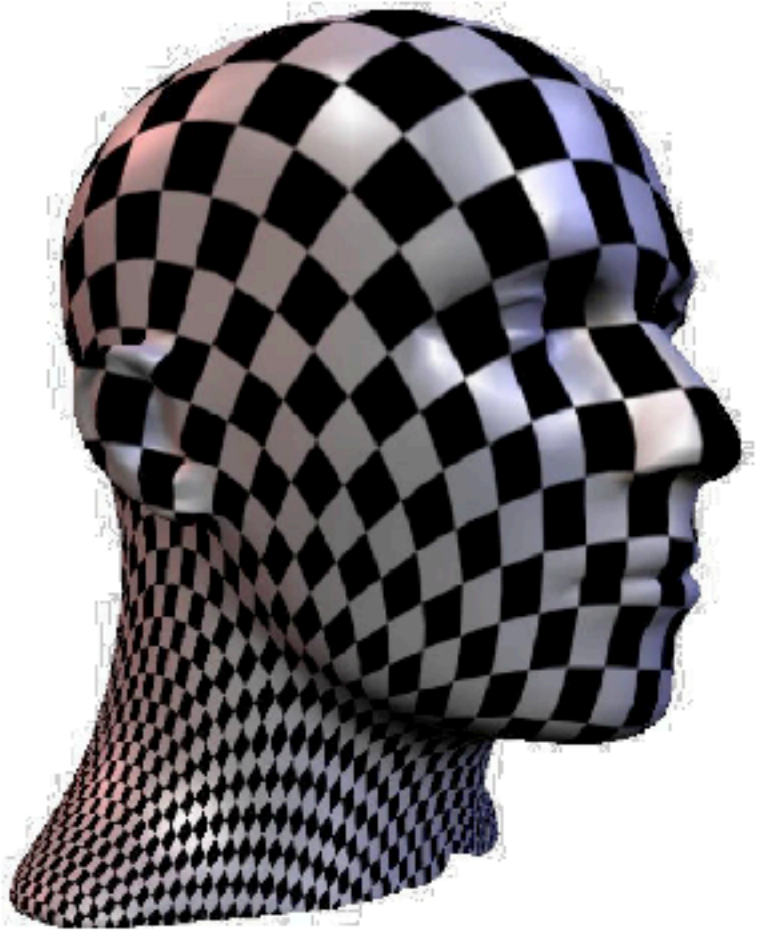
$$\| J \|_F = \sqrt{\sigma_1 + \sigma_2} \quad \text{or} \quad \| J \|_\infty = \sigma_1$$

  - modification [Sorkine et al. 2002]

$$\max\{\sigma_1, \frac{1}{\sigma_2}\}$$

# Examples





$\sqrt{\sigma_1 + \sigma_2} \to \min$

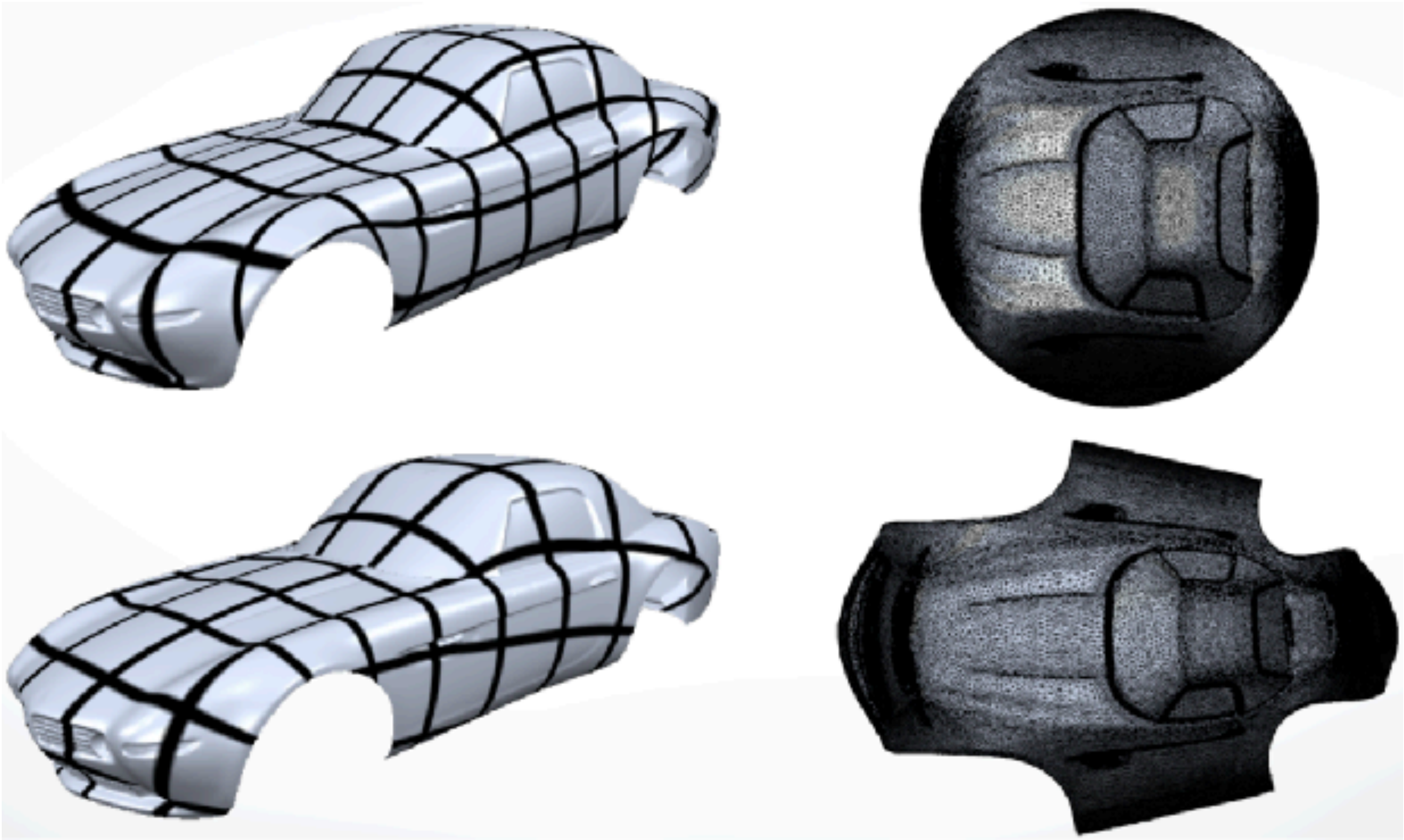
Stretch metric minimization

Using [Yoshizawa et. al 2004]

# Other Issues

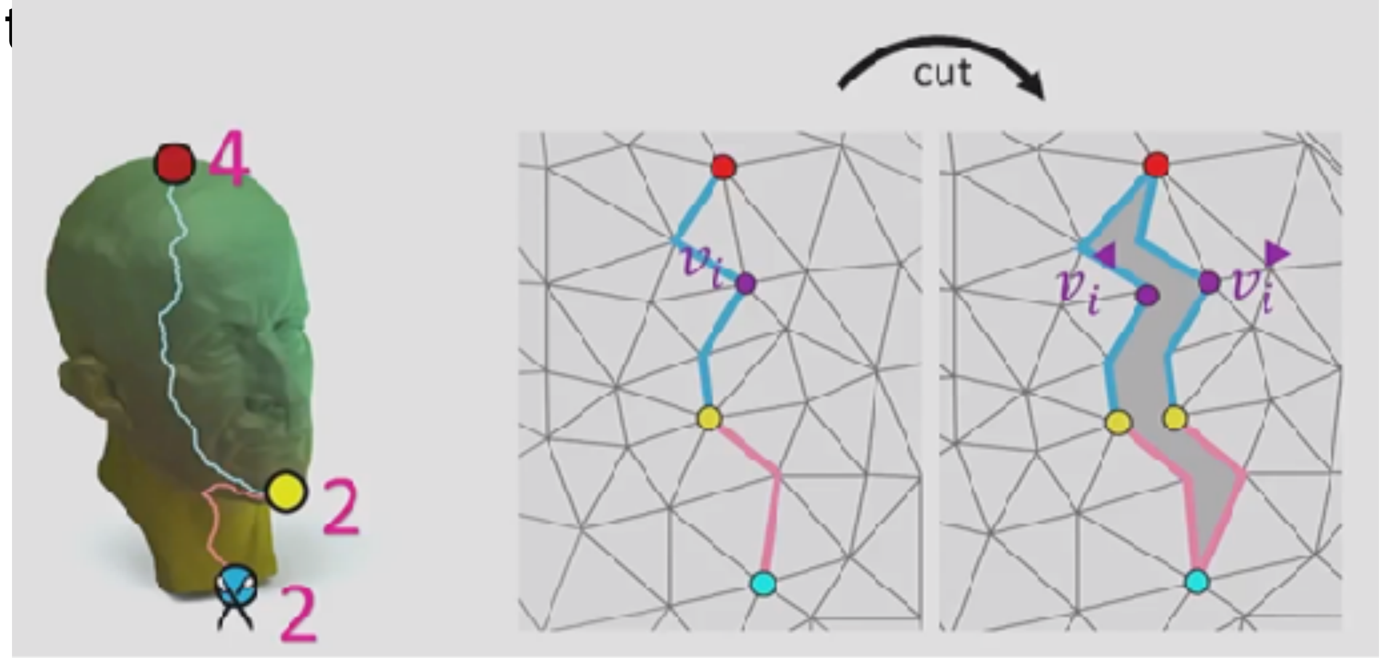
# Fixing the Boundary vs free

- Choose a simple convex shape
  - Triangle, square, circle
- Distribute points on boundary
  - Use chord length parameterization
- Fixed boundary can create high distortion



# Other Issues: Atlas -- Model Segmentation

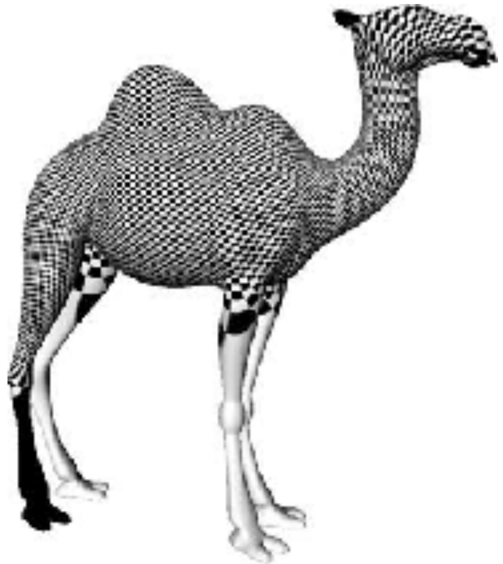
- Planar parameterization is only applicable to surfaces with **disk topology**
- **Closed** surfaces and surfaces with genus greater than zero have to be cut prior to planar parameterization
- **Cut** to **reduce complexity** (to reduce **distortion**)
  - Duplicate each vertex on the cut.
- Cut may lead to cross-discontinuit



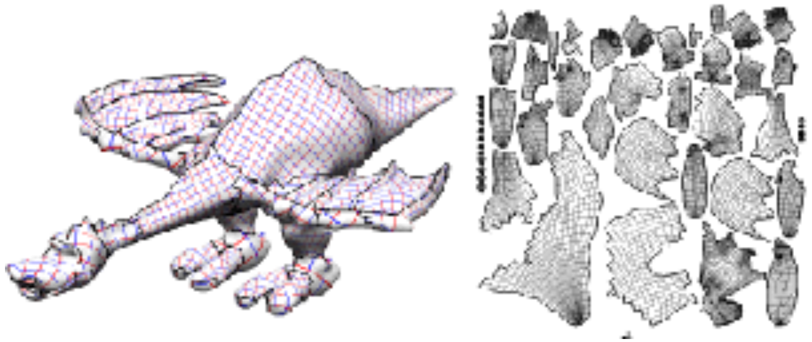

Lévy, Petitjean, Ray, and Maillot: Least squares conformal maps for automatic texture atlas generation, SIGGRAPH 2002

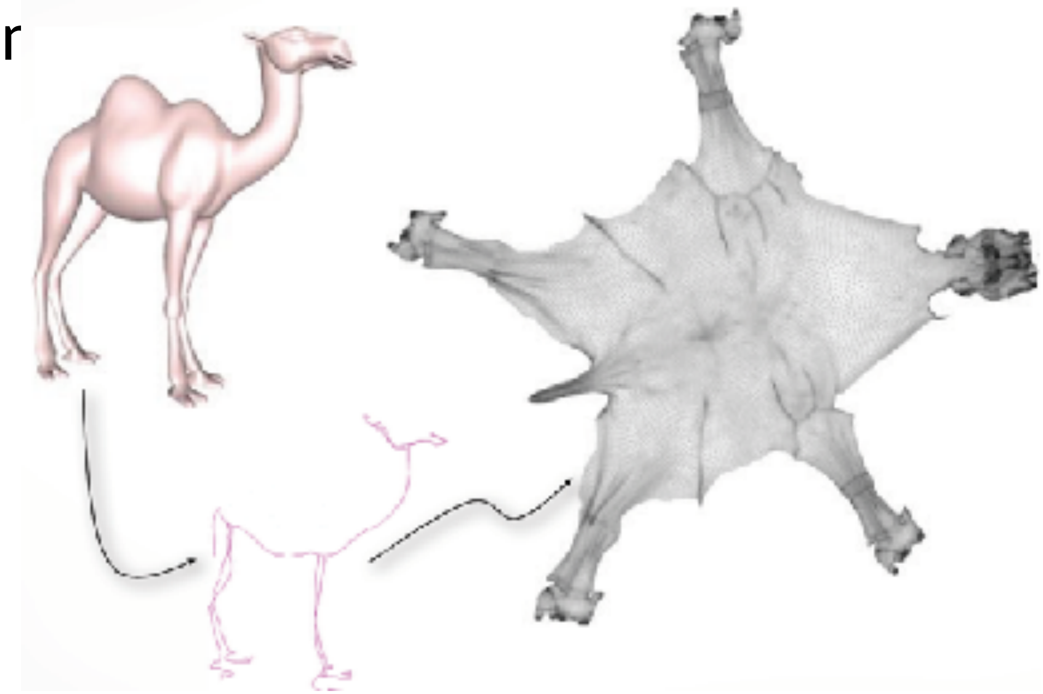
# Other Issues: Model Segmentation

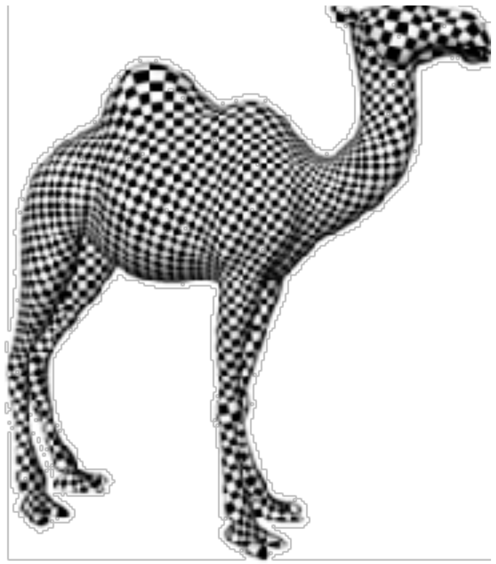
- **Cut** to **reduce complexity** (to reduce **distortion**)
  - Naïve Vs smart cut



- **Two approaches**
  - **Seam** (introduce cuts into the surface but keep it as a single chart)
  - **Segmentation** (partitior multiple charts: Atlas)

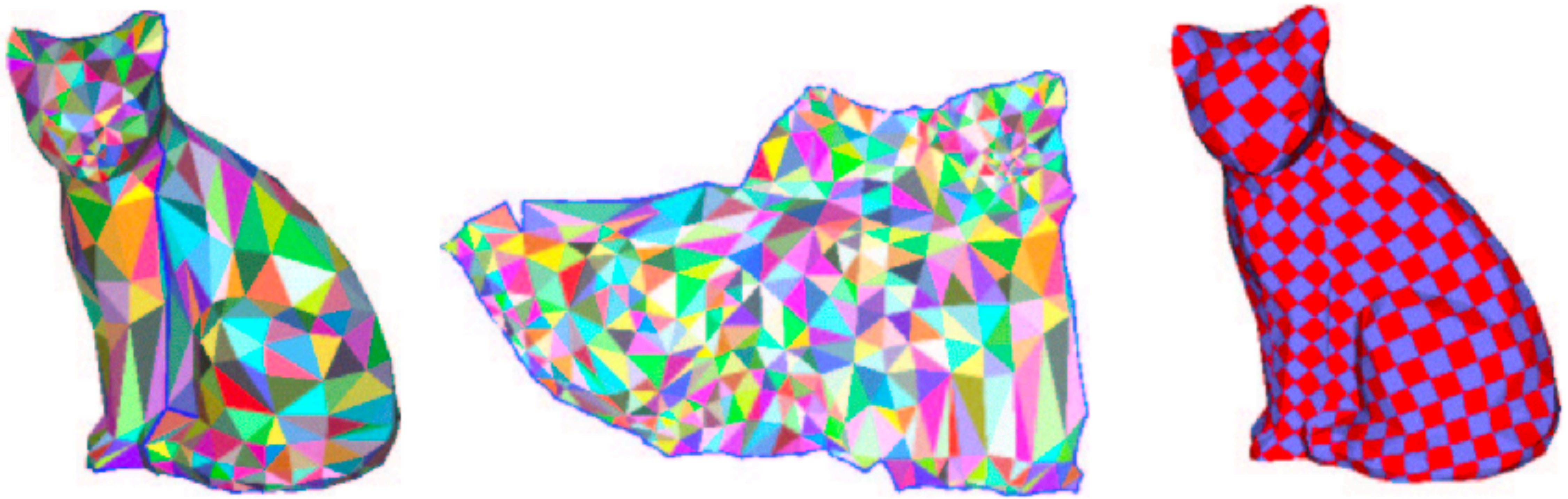






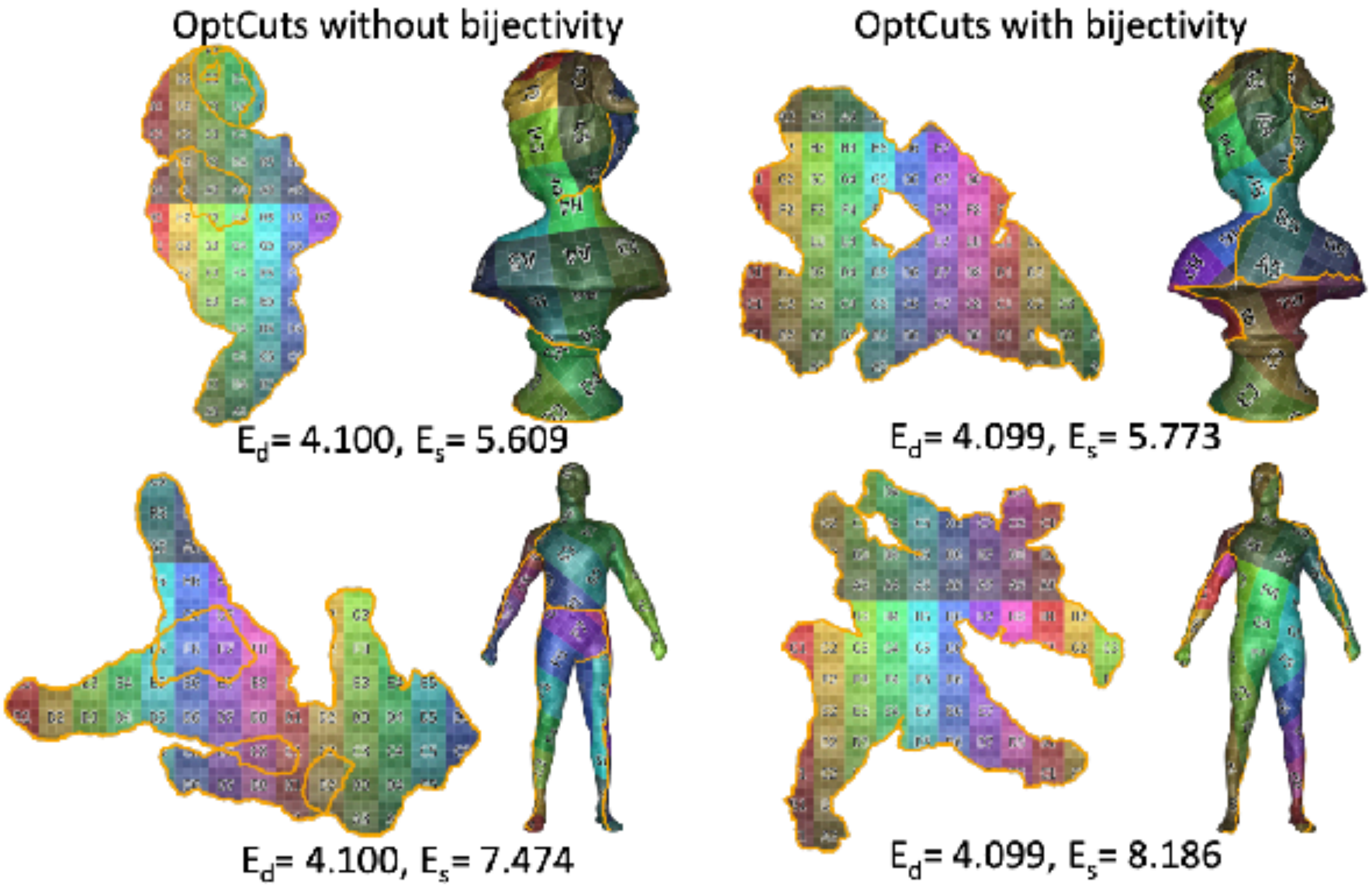
# A basic method for cut/seam computation

- Minimum Spanning Tree computation on high-curvature parts.
  - From the base to the cat's head, adding nodes on the ear tips as the extreme curvature points.
  - Blue is a connected cut w/o loops (tree), which turns a closed surf into one disk topology surface that can be mapped to a single disk.



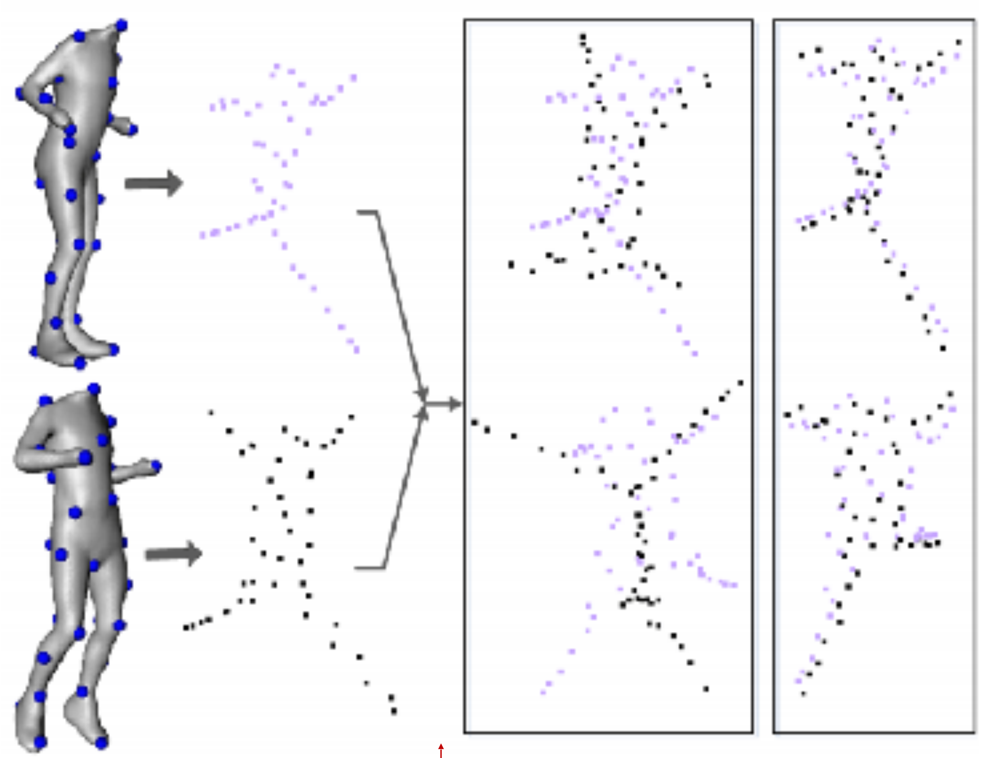
# Advance cut/seam computation

- OptCuts: Joint Optimization of Surface Cuts and Parameterization
- Autocuts: Simultaneous Distortion and Cut Optimization for UV Mapping



# MDS-based Parameterization & Atlas

- Embedding of the mesh into $R^k$, k=2 for planar parameterization
- pairwise dissimilarity values b/w points, e.g., geodesic distances are approximated by the Euclidean distances in the embed space.
- Here are 2 different MDS transformations from $R^3$ to $R^3$.



✓Useful for non-rigid shape correspondence problem.

✓Robust 3D Shape Correspondence in the Spectral Domain

✓Minimum-Distortion Isometric Shape Correspondence Using EM Algorithm.

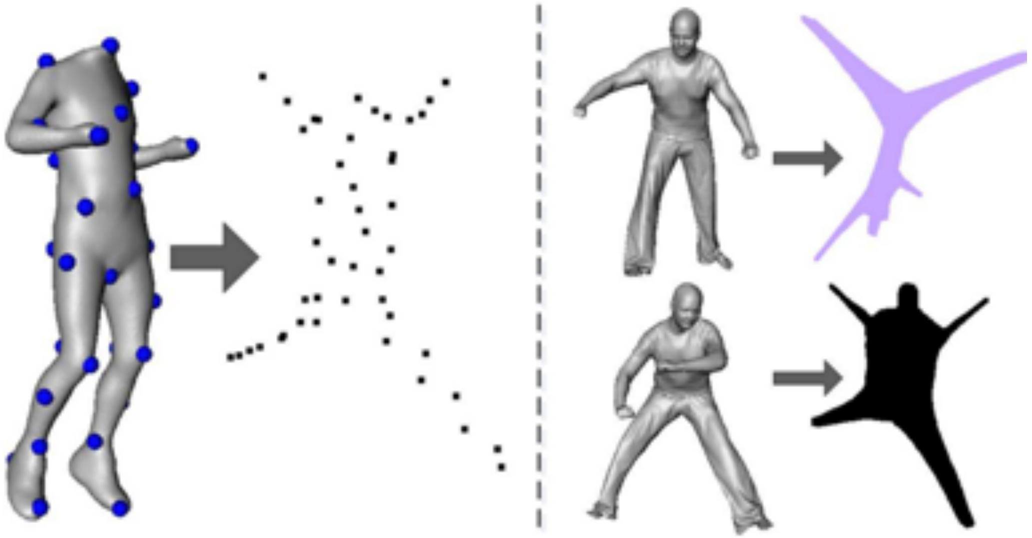
# MDS-based Parameterization & Atlas

- Classical method
  - k leading eigenvectors of A define MDS coords in $R^k$.
  - affinity matrix A saves pairwise dissimilarities
  - used in Texture Mapping Using Surface Flattening via Multidimensional Scaling to get parameterizations that are OK for non-complex surfaces such as:

- Least-squares method
  - minimize $$E(\mathbf{v}) = \sum_{i<j} (\|\mathbf{v}_i - \mathbf{v}_j\| - g(i, j))^2$$
  - using mass-spring system as described in Detail-Preserving Mesh Unfolding for Nonrigid Shape Retrieval.

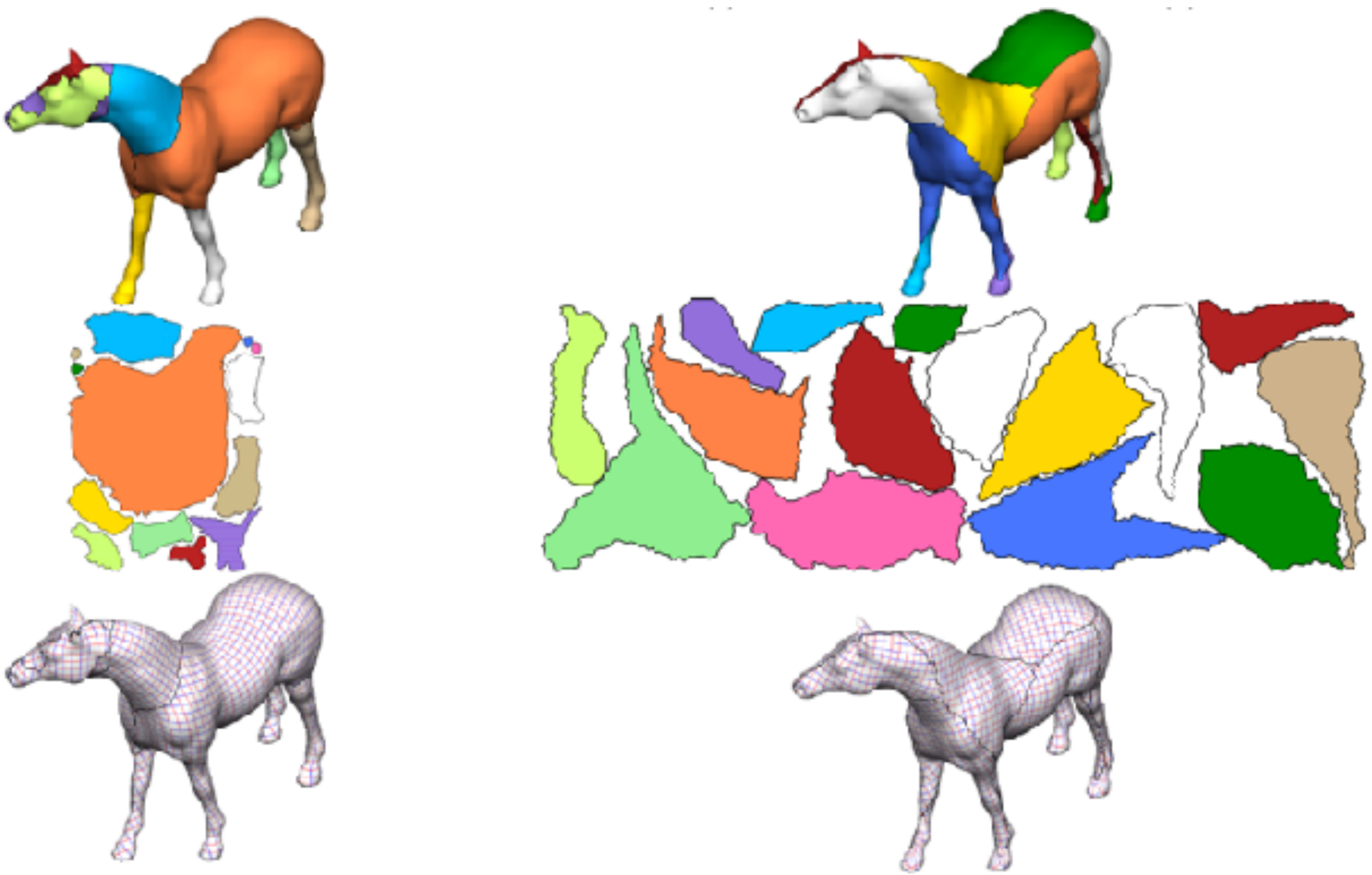
# MDS-based Parameterization & Atlas

- Classical method
  - k leading eigenvectors of A define MDS coords in $R^k$.
  - affinity matrix A saves pairwise dissimilarities
  - can be accelerated: Sparse Multidimensional Scaling Using Landmark Points.
    - Compute embedded landmark points first
    - Then computes embedding coordinates for the remaining data points based on their distances from the landmark points.

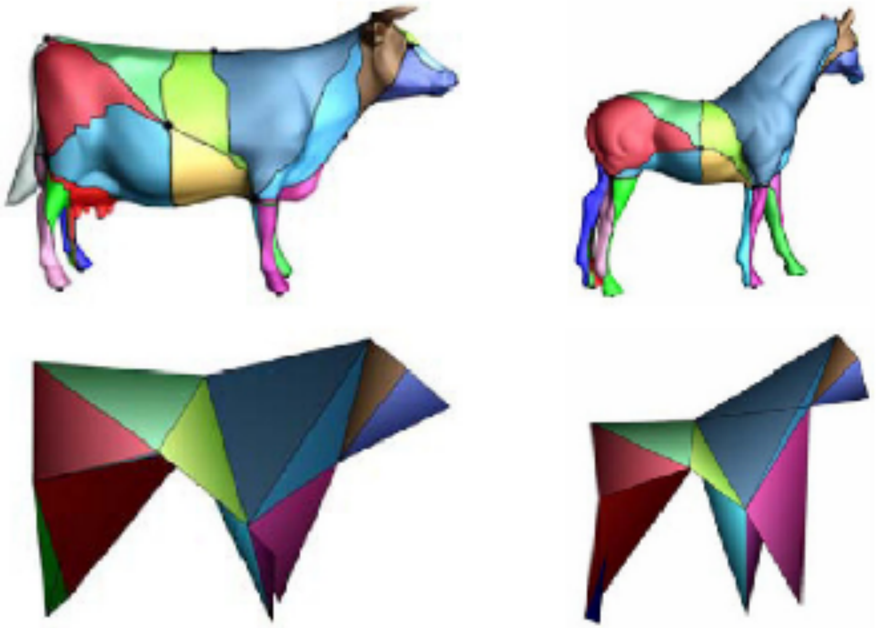


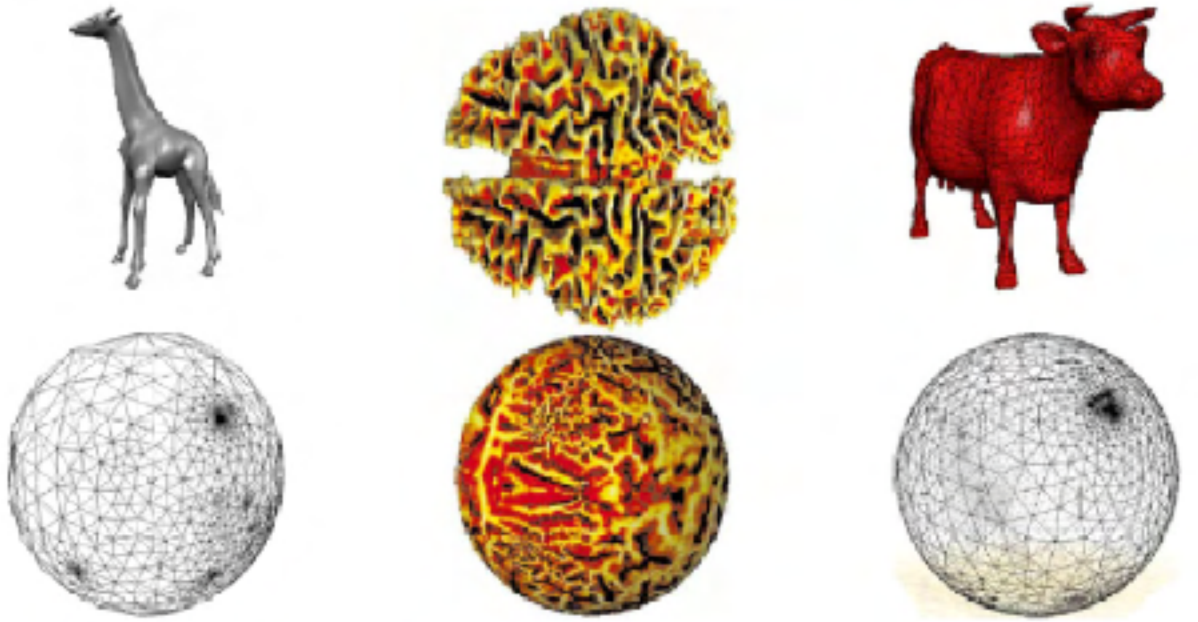
# MDS-based Parameterization & Atlas

- Segment into charts, parameterize each chart with MDS



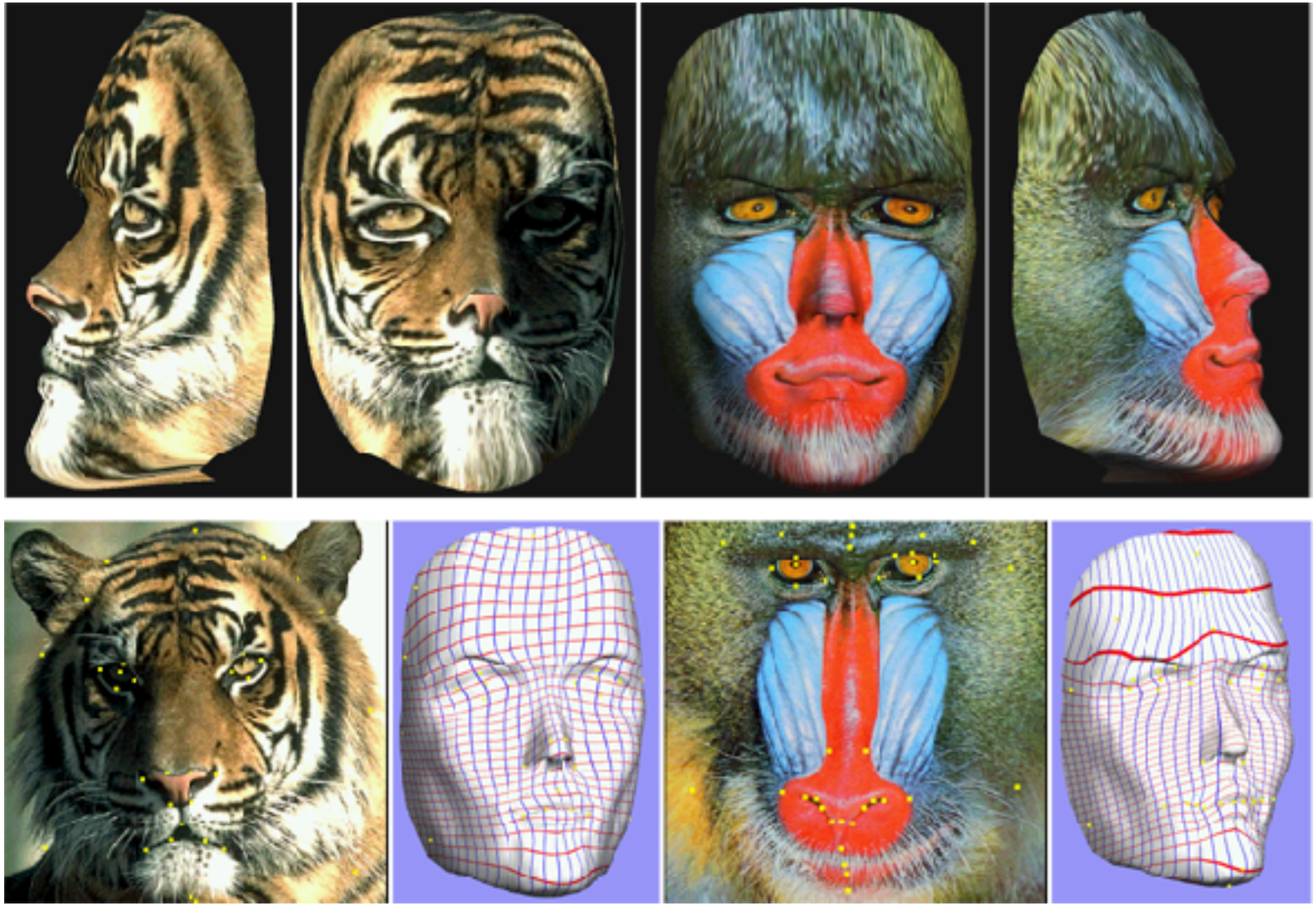
# Non-Planar Domains





(a) [Alexa 2000] (b) [Haker et al., 2000] (c) [Isenburg et al., 2001]

# Constrained Parameterizations



Levy: Constraint Texture Mapping, SIGGRAPH 2001.

# Literature

- Floater and Hormann: Surface Parameterization: a tutorial and survey, advances in multiresolution for geometric modeling, Springer 2005
- Hormann, Polthier, and Sheffer, Mesh Parameterization: Theory and Practice, SIGGRAPH Asia 2008 Course Notes

# Thanks