Digital Geometry
-Continuous Geometry of Curves & Surfaces

Junjie Cao @ DLUT

Spring 2019

http://jjcao.github.io/DigitalGeometry/

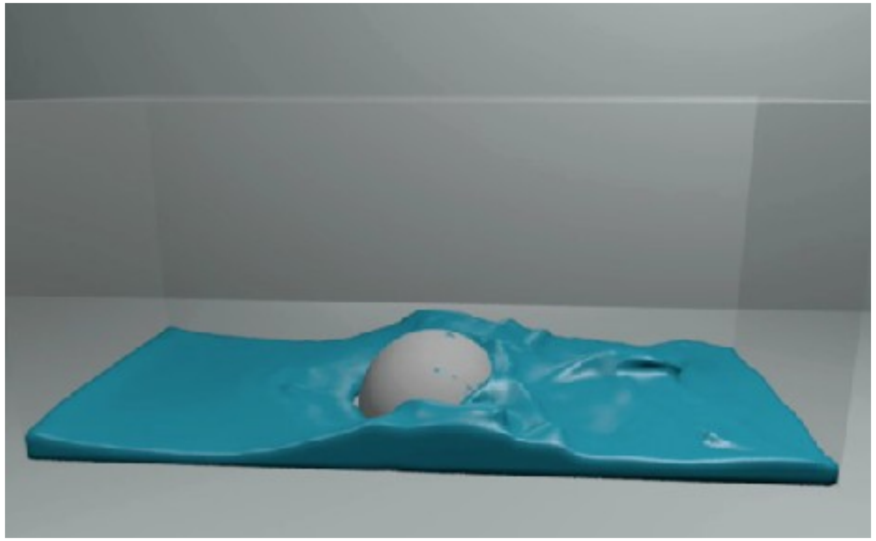
Pleasure may come from illusion, but happiness can come only of reality.

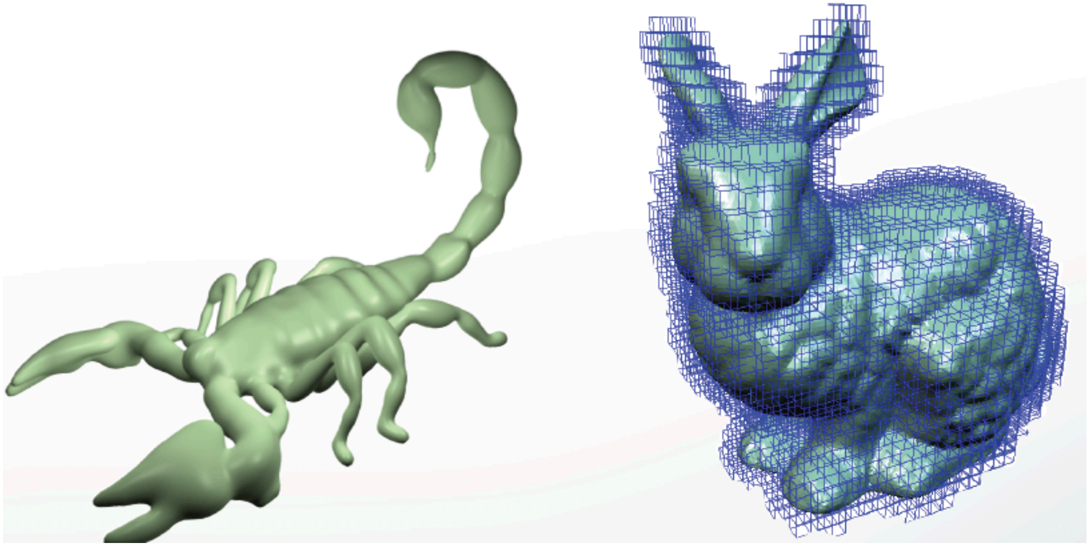
Last Time

- **Discrete Representations**
  - Explicit (parametric, polygonal meshes)
  - Implicit Surfaces (SDF, grid representation)
- Conversions
  - E→I: Closest Point, SDF, Fast Marching
  - I→E: Marching Cubes Algorithm

Geometry

Topology

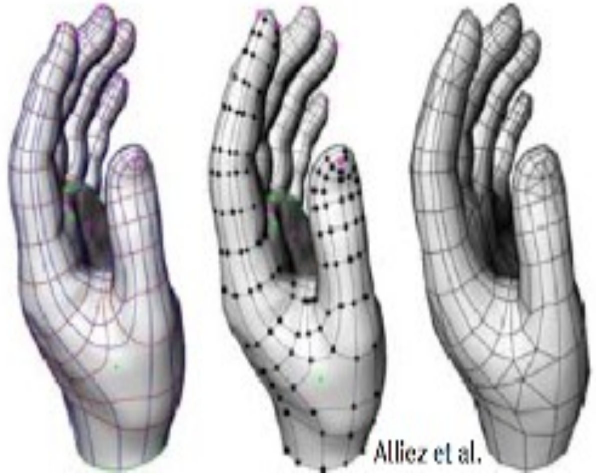


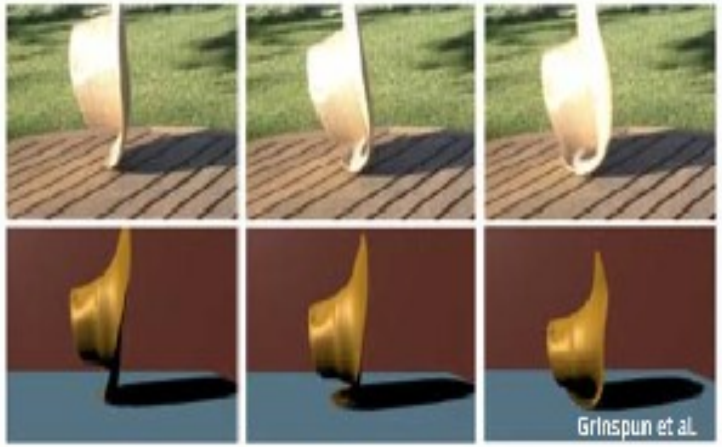


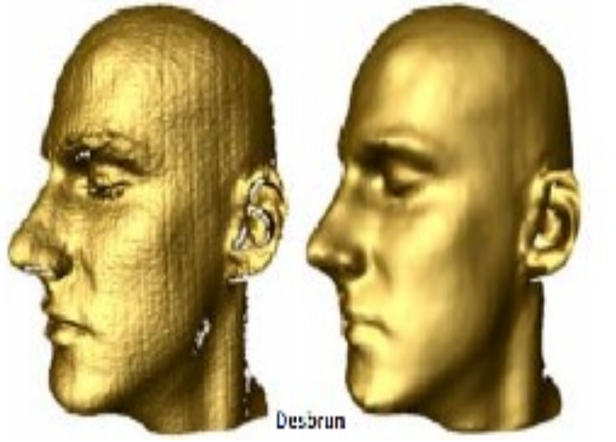
Differential Geometry

Why do we care?

- Geometry of surfaces
- Mothertongue of physical theories
- Computation: processing / simulation

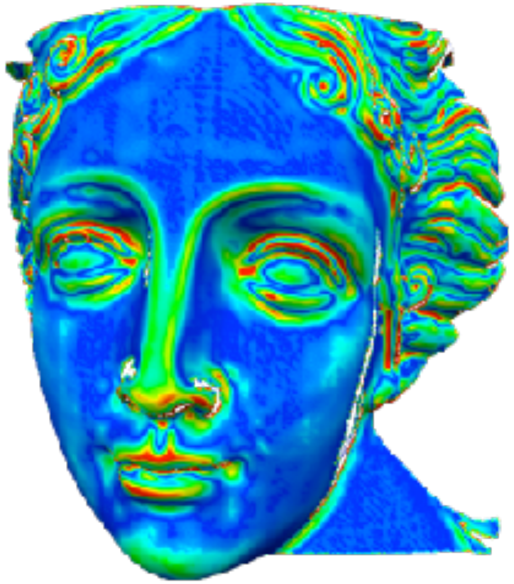


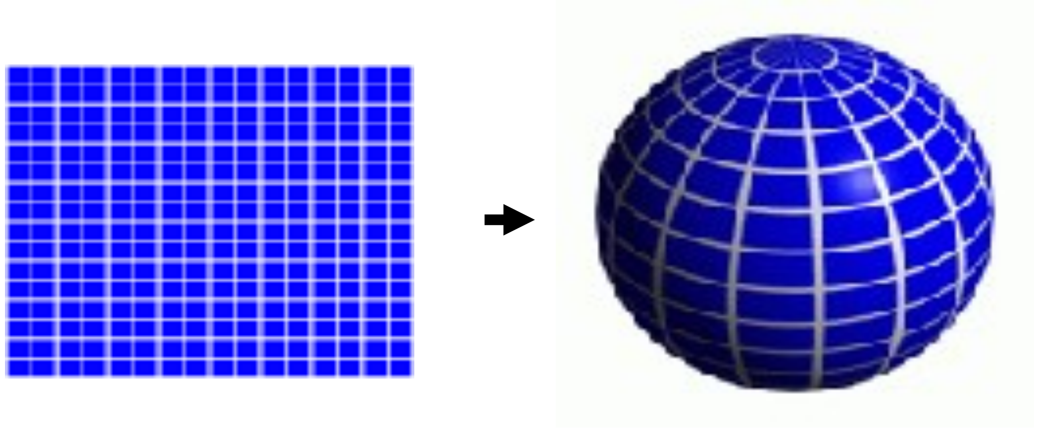


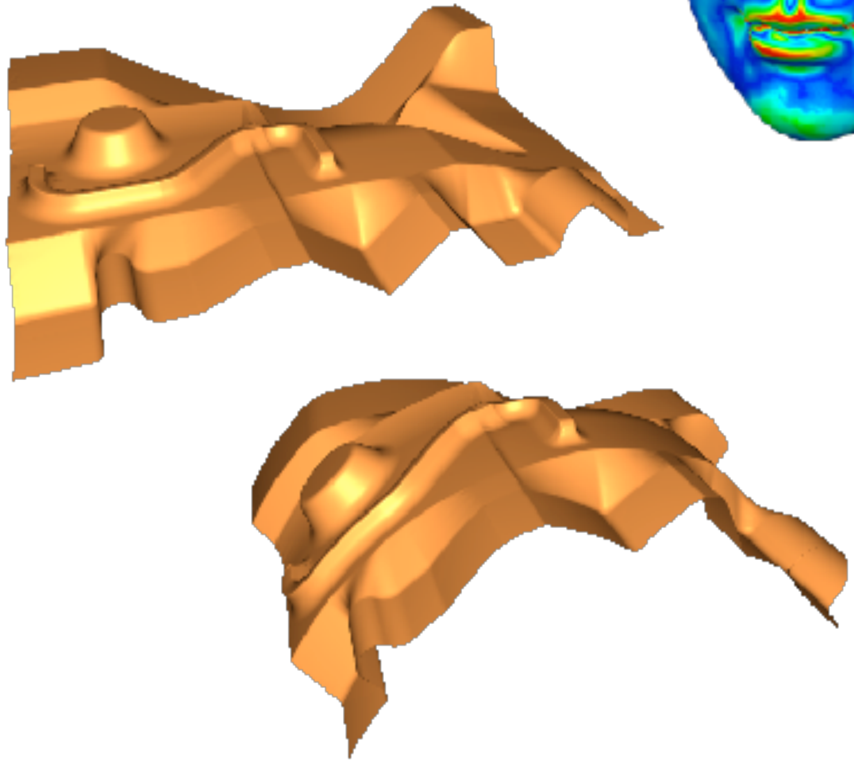


Motivation

- We need differential geometry to compute
  - surface curvature
  - parameterization distortion
  - deformation energies







Getting Started - How to apply DiffGeo ideas?

- surfaces as a collection of samples
  - and topology (connectivity)
- apply continuous ideas
  - BUT: setting is discrete
- what is the right way?
  - discrete vs. discretized

Let’s look at that first

Differential Geometry

- Parametric Curves
- Parametric Surfaces

Formalism & Intuition

What characterizes Surfaces/Shape?

- Intrinsic descriptor
  - quantities which do **not depend on a coordinate** frame / Euclidean motions
  - metric and curvatures

Metric on Surfaces

- Measure Stuff
  - angle, length, area
  - **requires** an inner product

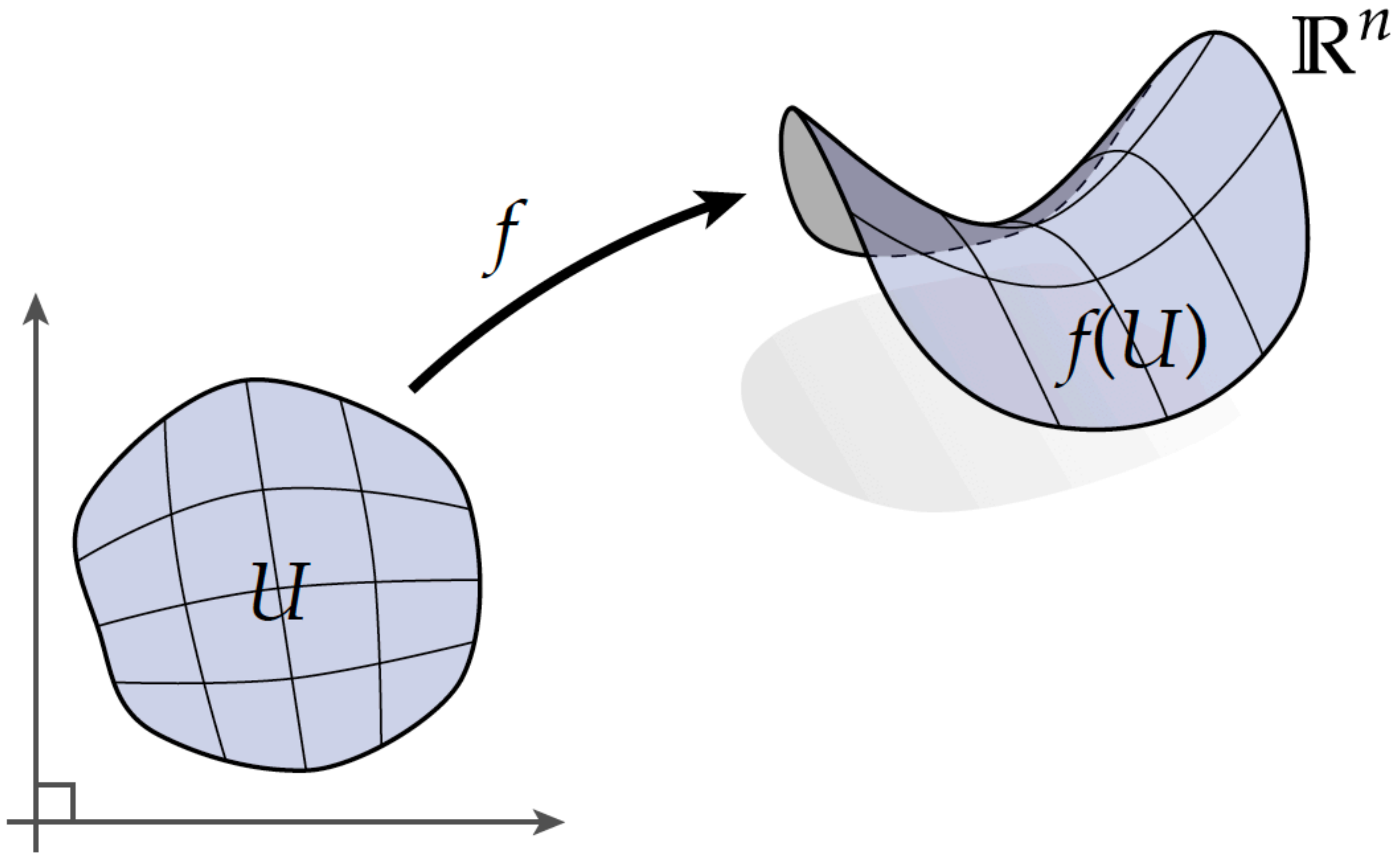
- we have:
  - Euclidean inner product in domain

- we want to turn this into:
  - **inner product on surface**

Parameterized Surface

A **parameterized surface** is a map from a two-dimensional region $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$:

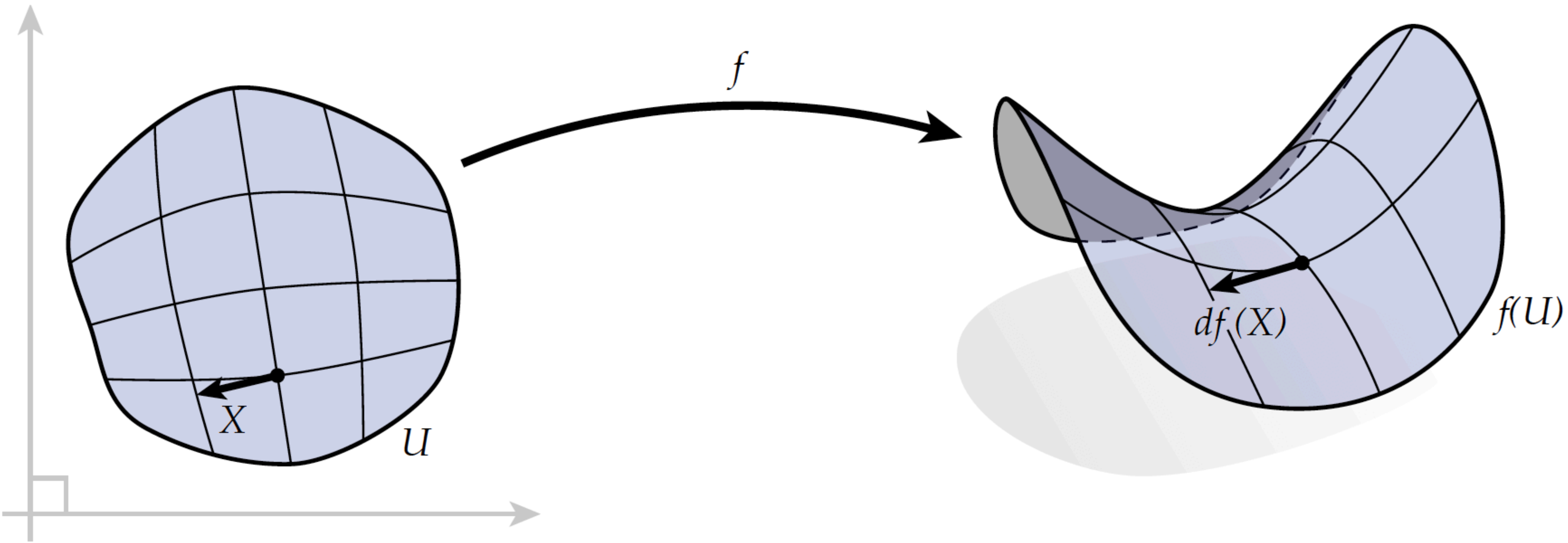
$$f : U \to \mathbb{R}^n$$



The set of points $f(U)$ is called the **image** of the parameterization.

Differential of a Surface

Intuitively, the *differential* of a parameterized surface tells us how tangent vectors on the domain get mapped to vectors in space:



We say that df "*pushes forward*" vectors X into R^n, yielding vectors $df(X)$

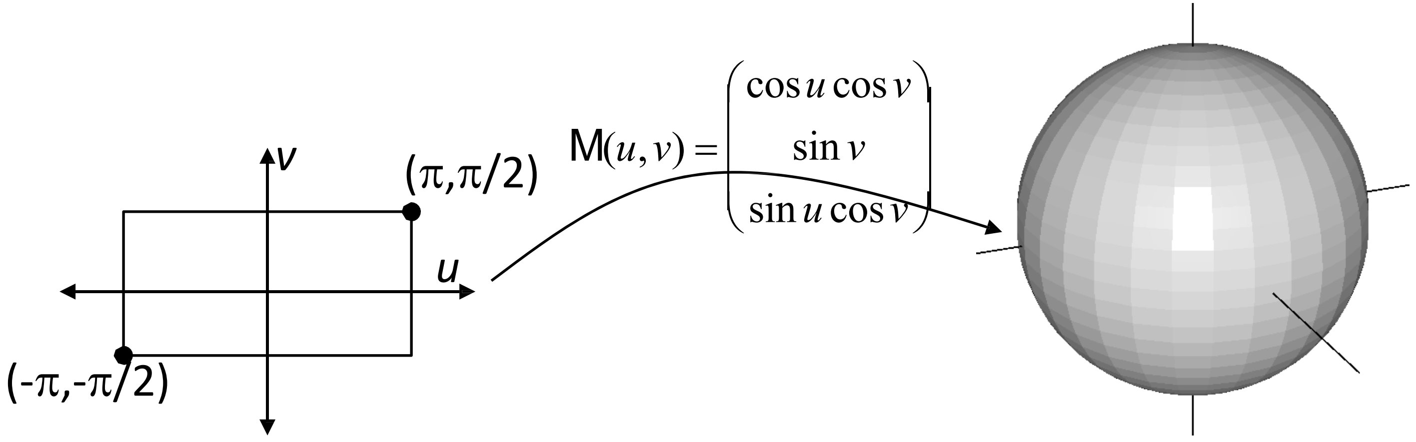
Differentiable Surfaces

Definition:

A *parameterized* ***differentiable*** *surface* is a differentiable map $\boldsymbol{M}$: $\Omega \rightarrow \boldsymbol{R}^3$ of an open domain $\Omega \subset \boldsymbol{R}^2$ into $\boldsymbol{R}^3$:

$$\boldsymbol{M(u,v)=(x(u,v),y(u,v),z(u,v))}$$

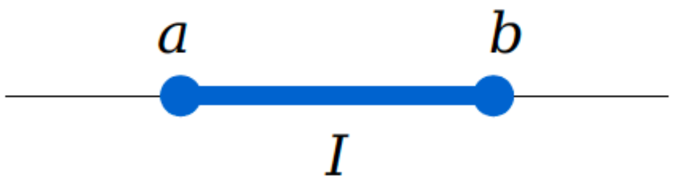
where $x(u,v)$, $y(u,v)$, and $z(u,v)$ are differentiable functions.



Curves and surfaces in 3D

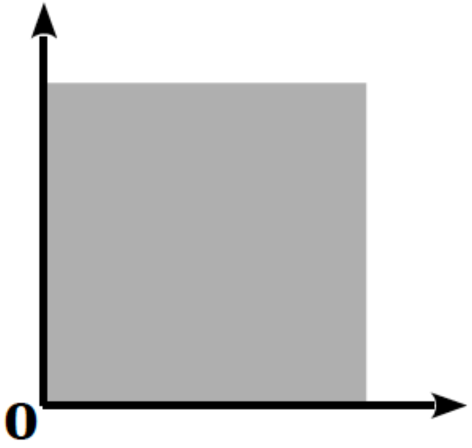
- For our purposes:
  - A **curve** is a map $\boldsymbol{\alpha} : \mathbb{R} \rightarrow \mathbb{R}^3$ (or from some subset I of $\mathbb{R}$)

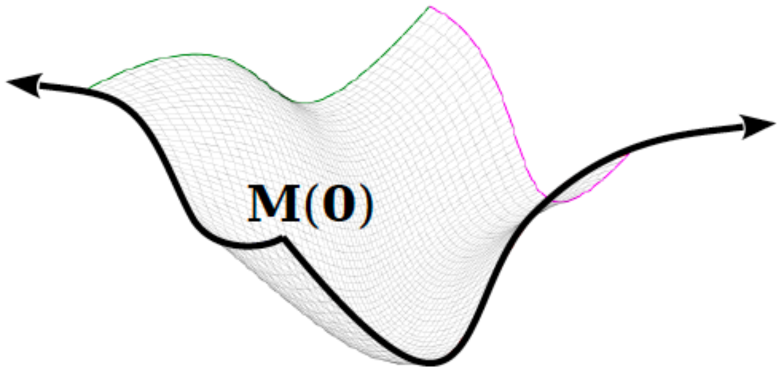
$\boldsymbol{\alpha}(t) = (x, y, z)$



  - A **surface** is a map $\mathbf{M} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ (or from some subset Ω of $\mathbb{R}^2$)

$\mathbf{M}(u, v) = (x, y, z)$



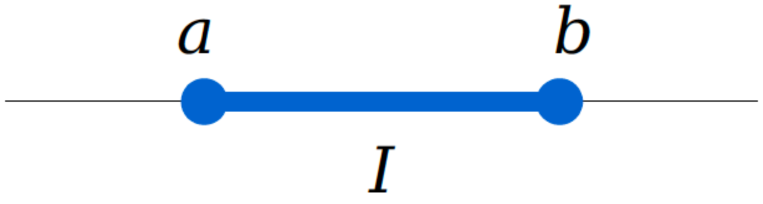


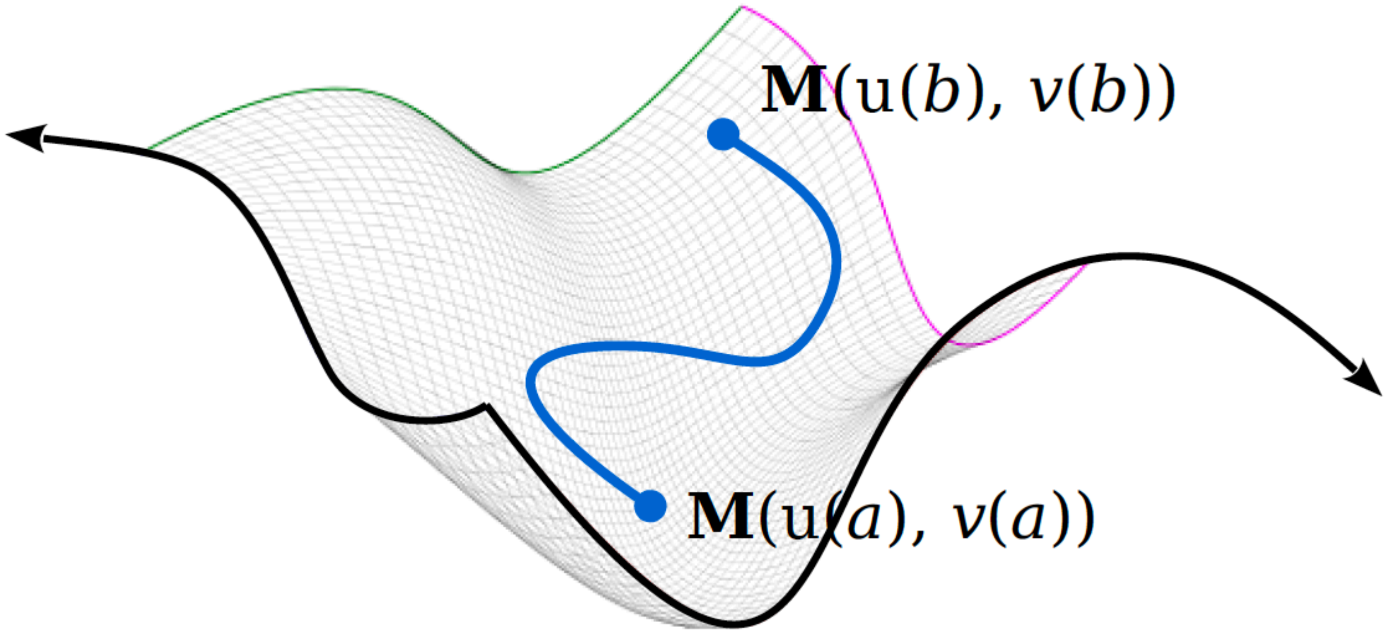
Curve on a surface

- A curve C on surface M is defined as a map
- **C(t) = M(c(t))**, c(t) = (u(t), v(t)) is preimage/inverse image of C(t)

$$= \mathbf{M}(\mathbf{u}(\mathbf{t}), \mathbf{v}(\mathbf{t})) = \begin{pmatrix} x(u(t), v(t)) \\ y(u(t), v(t)) \\ z(u(t), v(t)) \end{pmatrix}$$

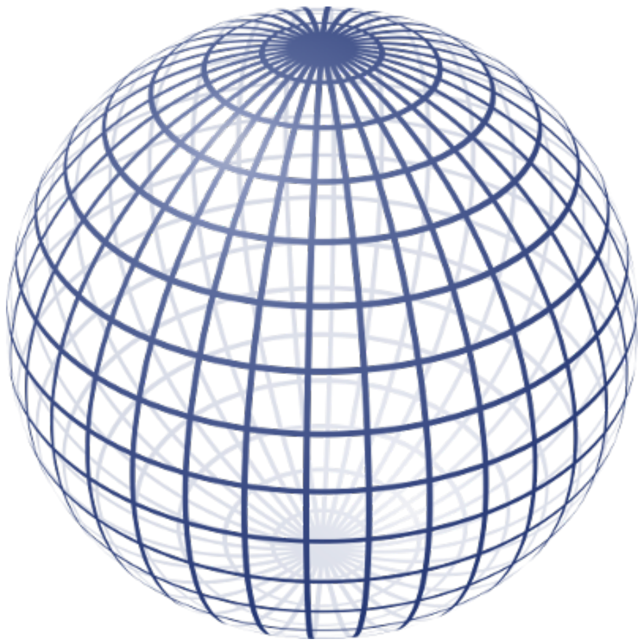
where u and v are smooth scalar functions





Special cases

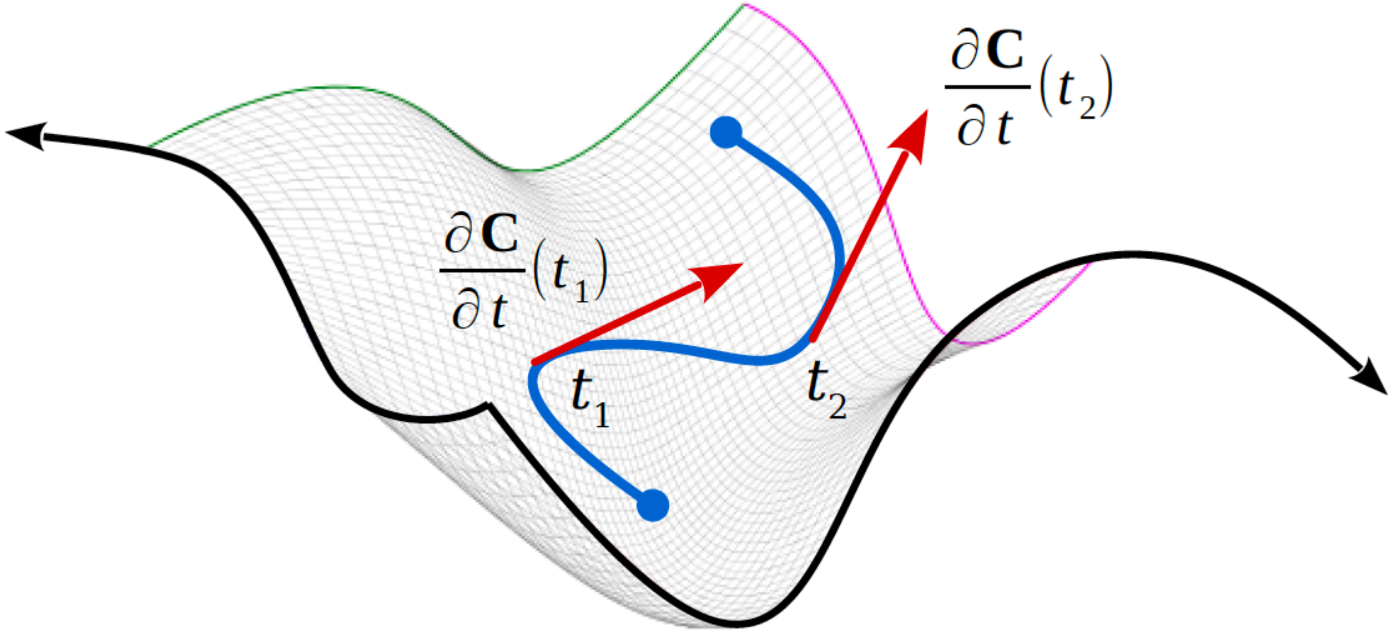
- The curve $C(t) = M(u_0, v(t))$ for constant u_0 is called a u-curve
- The curve $C(t) = M(u(t), v_0)$ for constant v_0 is called a v-curve
- These are collectively called coordinate curves



Tangent vector

- **C(t) = M(u(t), v(t))** = M(c(t)), c(t) = (u(t), v(t))
- The **tangent vector** to the surface curve C at t can be found by the chain rule

$$\frac{\partial \mathbf{C}}{\partial t} = \frac{\partial \mathbf{M}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{M}}{\partial v}\frac{dv}{dt}$$



We will use the following shorthand

$$\mathrm{M_u}(u,v) = \frac{\partial \mathrm{M}(u,v)}{\partial u} = \begin{pmatrix} \partial x / \partial u \\ \partial y / \partial u \\ \partial z / \partial u \end{pmatrix} \qquad \mathrm{M_v}(u,v) = \frac{\partial \mathrm{M}(u,v)}{\partial v} = \begin{pmatrix} \partial x / \partial v \\ \partial y / \partial v \\ \partial z / \partial v \end{pmatrix}$$

$$\mathbf{M}_u := \frac{\partial \mathbf{M}}{\partial u} \qquad \mathbf{M}_v := \frac{\partial \mathbf{M}}{\partial v}$$

$\mathbf{J=(M_u,M_v)}$

$$\dot{u} := \frac{du}{dt} \qquad \dot{v} := \frac{dv}{dt} \qquad \dot{\mathbf{C}} := \frac{\partial \mathbf{C}}{\partial t}$$

- Then the tangent vector is $\dot{\mathbf{C}} = \mathbf{M}_u \dot{u} + \mathbf{M}_v \dot{v}$

Tangent vector

- **C(t) = M(u(t), v(t))** = M(c(t)), c(t) = (u(t), v(t))
- The **tangent vector** to the surface curve C at t: $\dot{C} = M_u \dot{u} + M_v \dot{v} = J \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix}$

- **What is $(\dot{u}, \dot{v})$?**
  - c'(t) = (du/dt, dv/dt)= du/dt $\mathbf{e_1}$ + dv/dt $\mathbf{e_2}$
    - a tangent vector in parameter domain
    - with basis: e_1=(1,0), e_2=(0,1), and origin p=c(t)
  - J is a linear transformation
    - $\dot{c} -> \dot{C}$
    - transfers basis to basis, & coefficients are kept.

- **What is M_u and M_v?**

$J=(M_u,M_v)$, **taking** T_pR^2 **to** $T_{M(p)}R^3$

- **What is $\boldsymbol{M_u}$ and $\boldsymbol{M_v}$?** or what is the preimage of M_u, M_v?
  - J is a linear transformation
    - transfers basis to basis, & coefficients are kept.
    - **$M_u=Je_1$, $M_v=Je_2$**
      - $e_1=(1,0)'$, $e_2=(0,1)'$ are "pushed forward" to basis M_u, M_v
    - $\dot{C} = J\begin{bmatrix}\dot{u}\\ \dot{v}\end{bmatrix}$
      - Coefficients du/dt, dv/dt are kept

$J=(M_u,M_v)$, **taking T_pR^2 to $T_{M(p)}R^3$**

- **J:** T_pR^2 **->** $T_{M(p)}R^3$
  - Frame of T_pR^2 : $e_1=(1,0)$, $e_2=(0,1)$, p
  - Frame of $T_{M(p)}R^3$: $\boldsymbol{M_u}$, $\boldsymbol{M_v}$, M(p)

- ***J*** is the Jacobian matrix **taking directions/tangent vectors in Ω to tangent vectors** on the surface.

Differential of a Function

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

Matrix:

$$Df = \left(\frac{\partial f_i}{\partial x_j}\right) \in \mathbb{R}^{m \times n}$$

Linear operator:

$$Df_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$$

Regular surface

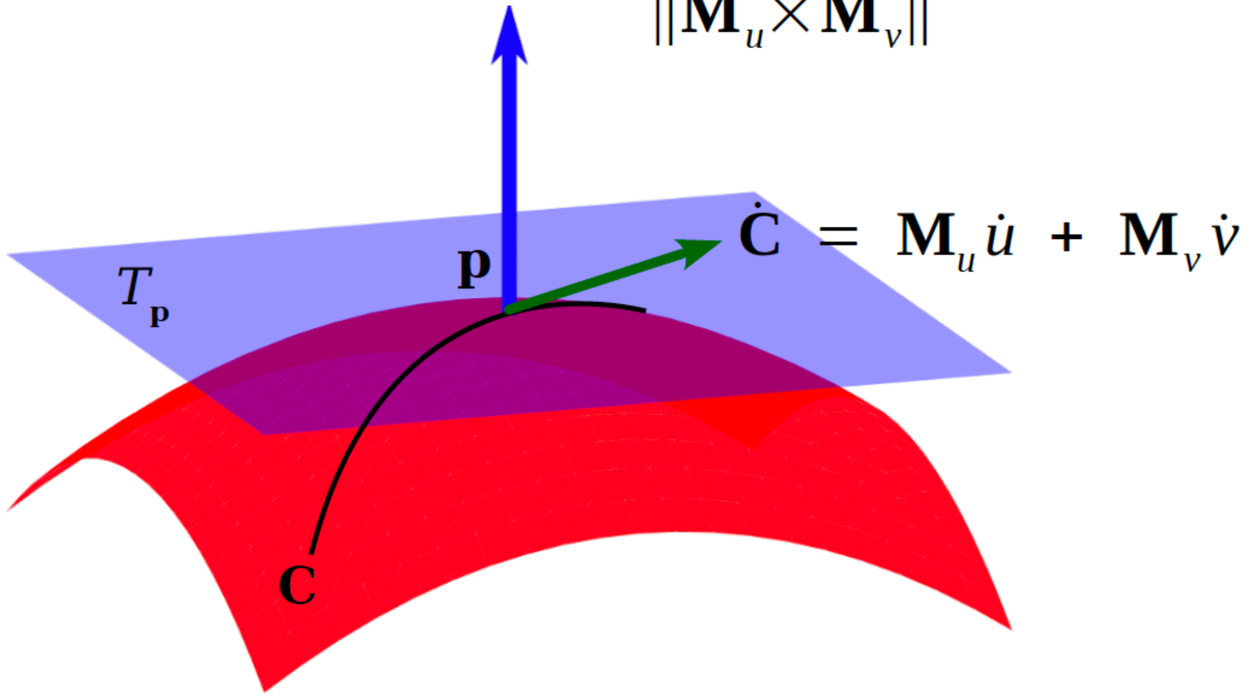
- A surface M is **regular** if $M_u x M_v \neq 0$ everywhere
  - (i.e. that a normal can be defined everywhere)

- A point where $M_u x M_v \neq 0$ is called a **regular point**
  - (else, it is a **singular point**)

Tangent space & Normal Vectors

- If the point is regular, the tangent vectors form a **2D space** called the **tangent space $\mathbf{T_p}$** at p
    - **Mu and Mv are basis vectors** for the tangent space
- The unit normal to the tangent space, also known as the normal to the surface at the point, is

$$\mathbf{N} = \frac{\mathbf{M}_u \times \mathbf{M}_v}{\|\mathbf{M}_u \times \mathbf{M}_v\|}$$

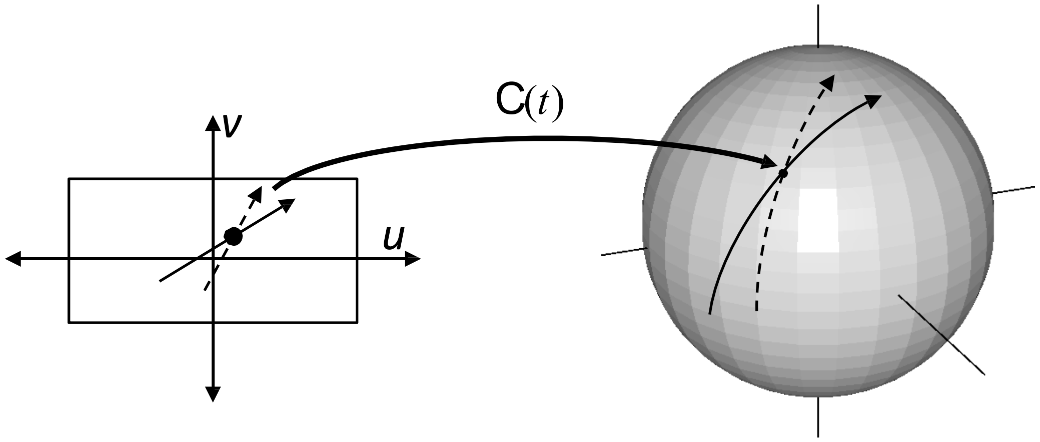


Curve in parameter domain => curve on surface

<u>Definition</u>:

Given a point $p_0=(u_0,v_0)\in\Omega$ and given a **direction $w=(w_u,w_v)$** in the parameter space, we can define the (3D) curve:

$$\mathrm{C}(t)=\mathrm{C}(p_0+tw),$$ (Special case: 2d line to 3d curve)



Directional derivatives

$$\frac{\partial \mathbf{C}}{\partial t} = \frac{\partial \mathbf{M}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{M}}{\partial v}\frac{dv}{dt}$$

<u>Definition</u>: $\mathrm{M}(t) = \mathrm{M}(p_0 + tw)$, $w=(w_u,w_v)$

Taking the derivative:

$$\mathrm{M}'(t) = w_u \mathrm{M}_u + w_v \mathrm{M}_v = Jw$$

J is the Jacobian matrix **taking directions in Ω to tangent vectors** on the surface:

$$\mathrm{J} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$ **i.e. $\mathbf{J=(M_u,M_v)}$,**

$$\mathrm{M}_u(u,v) = \frac{\partial \mathrm{M}(u,v)}{\partial u} \qquad \mathrm{M}_v(u,v) = \frac{\partial \mathrm{M}(u,v)}{\partial v}$$

Differential is a linear operator

$$\mathbf{x}'(t) = w_u \mathbf{x}_u + w_v \mathbf{x}_v = Jw$$

$$\frac{\partial \mathbf{C}}{\partial t} = \frac{\partial \mathbf{M}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{M}}{\partial v}\frac{dv}{dt}$$

- Basis of T_pR^2 : $e_1=(1,0)$, $e_2=(0,1)$
- Basis of $T_{f(p)}R^3$: x_u, x_v

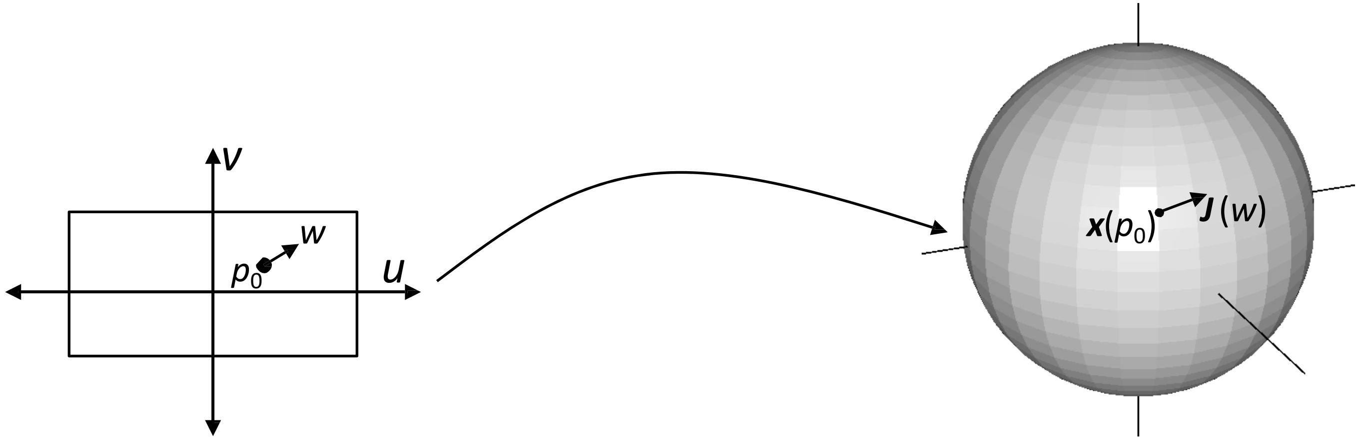
- Vector $w=(w_u,w_v)$ inT_pR^2 : $w= w_u e_1 + w_v e_2$
- To vector $x'(t)$ in $T_{f(p)}R^3$, coefficients are kept

Riemannian Metric & first fundamental form

Metric Properties - length

Thus, given a point $p_0=(u_0,v_0)\in\Omega$ and given a direction $w=(w_u,w_v)$, we can use the Jacobian to compute the length of the corresponding tangent vector over $\boldsymbol{x}(p_0)$:

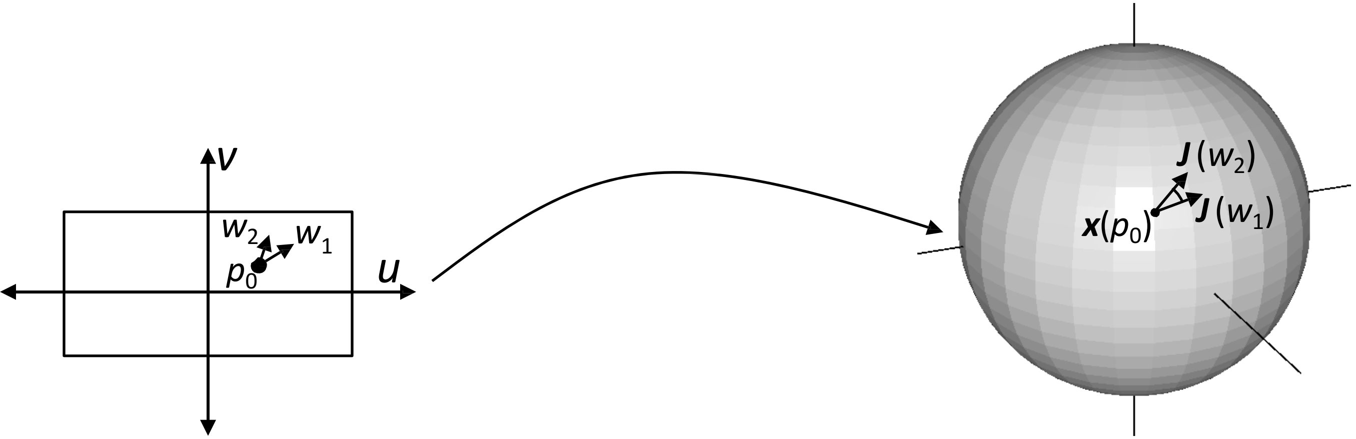
$$length^2 = \|\boldsymbol{J}w\|^2 = w^t \boldsymbol{J}^t \boldsymbol{J}w$$



Metric Properties - angle

- Similarly, given a point $p_0=(u_0,v_0)\in\Omega$ and given directions $w_1=(u_1,v_1)$ and $w_2=(u_2,v_2)$ we can use the Jacobian to compute the angle of the corresponding tangent vectors over $\boldsymbol{x}(p_0)$:

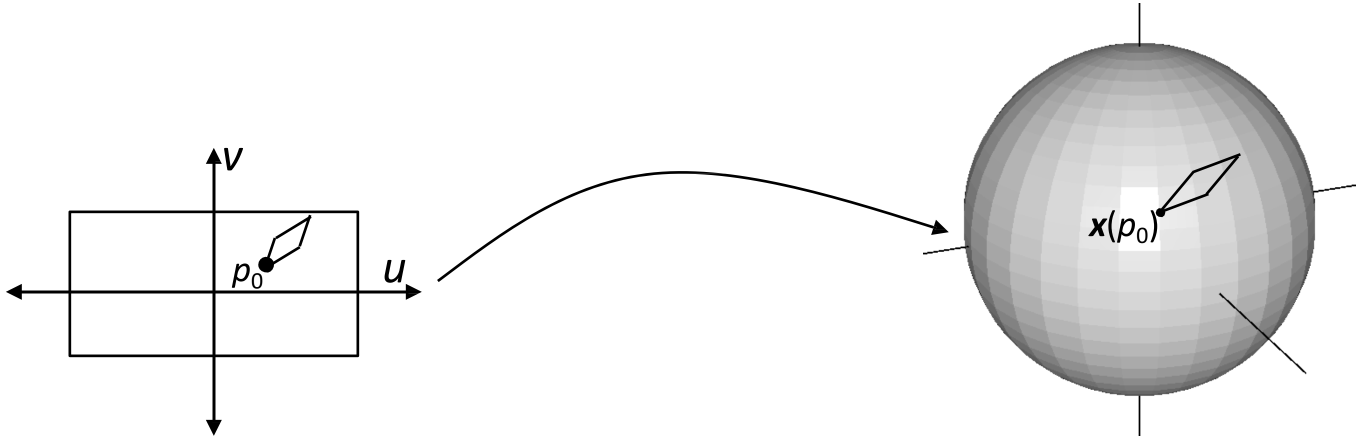
$$\cos(angle) = \frac{\langle \boldsymbol{J}w_1, \boldsymbol{J}w_2 \rangle}{\|\boldsymbol{J}w_1\|\|\boldsymbol{J}w_2\|} = \frac{w_1^t \boldsymbol{J}^t \boldsymbol{J}w_2}{\sqrt{w_1^t \boldsymbol{J}^t \boldsymbol{J}w_1}\sqrt{w_2^t \boldsymbol{J}^t \boldsymbol{J}w_2}}$$



Metric Properties - area

- Finally, given a point $p_0=(u_0,v_0)\in\Omega$ and given directions $w_1=(u_1,v_1)$ and $w_2=(u_2,v_2)$ we can use the Jacobian to compute the area of the corresponding parallelogram in the tangent space:

$$area = |w_1 \times w_2| = |w_1| \cdot |w_2| \cdot \sin(angle)$$

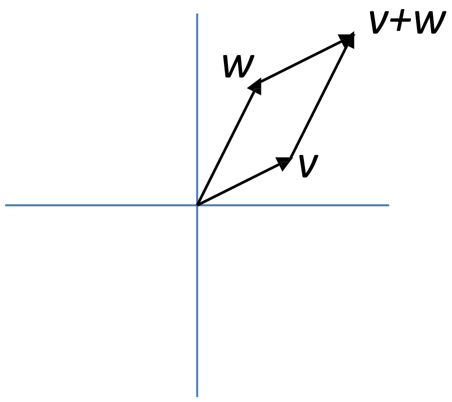


Metric Properties - area

<u>Note</u>:

Given vectors v and w in $\boldsymbol{R}^n$, the area of the parallelogram spanned by v and w is:

$$\begin{aligned} Area(v,w) &= |v|\cdot|w|\cdot\sin\big(Angle(v,w)\big) \\ &= |v|\cdot|w|\cdot\sqrt{1-\cos^2 Angle(v,w)} \\ &= |v|\cdot|w|\cdot\sqrt{1-\frac{\langle v,w\rangle^2}{|v|^2|w|^2}} \\ &= \sqrt{|v|^2|w|^2-\langle v,w\rangle^2} \end{aligned}$$

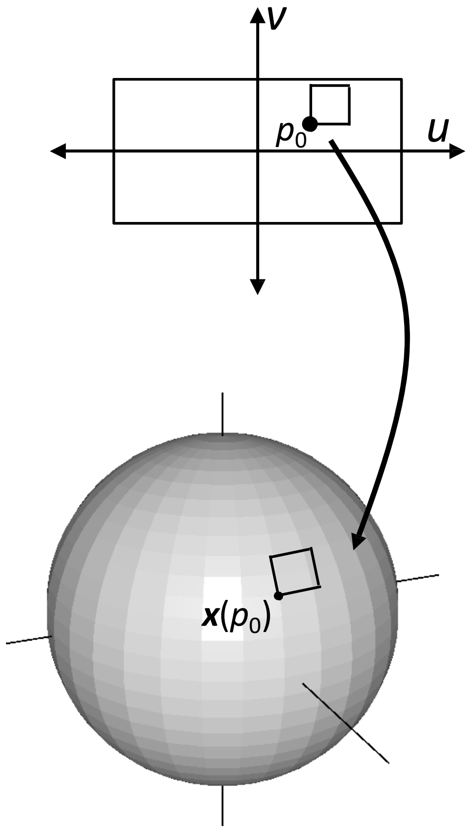


Metric Properties - area

- The area in tangent space is scaled by $\sqrt{\det(I)}$:

$$Area(Jw_1, Jw_2) = \sqrt{|Jw_1|^2|Jw_2|^2 - \langle Jw_1, Jw_2\rangle^2}$$
$$= \sqrt{\det(\boldsymbol{I})}Area(w_1, w_2),$$

where $I = J'J = \begin{bmatrix} \langle M_u, M_u\rangle & \langle M_u, M_v\rangle \\ \langle M_u, M_v\rangle & \langle M_v, M_v\rangle \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$

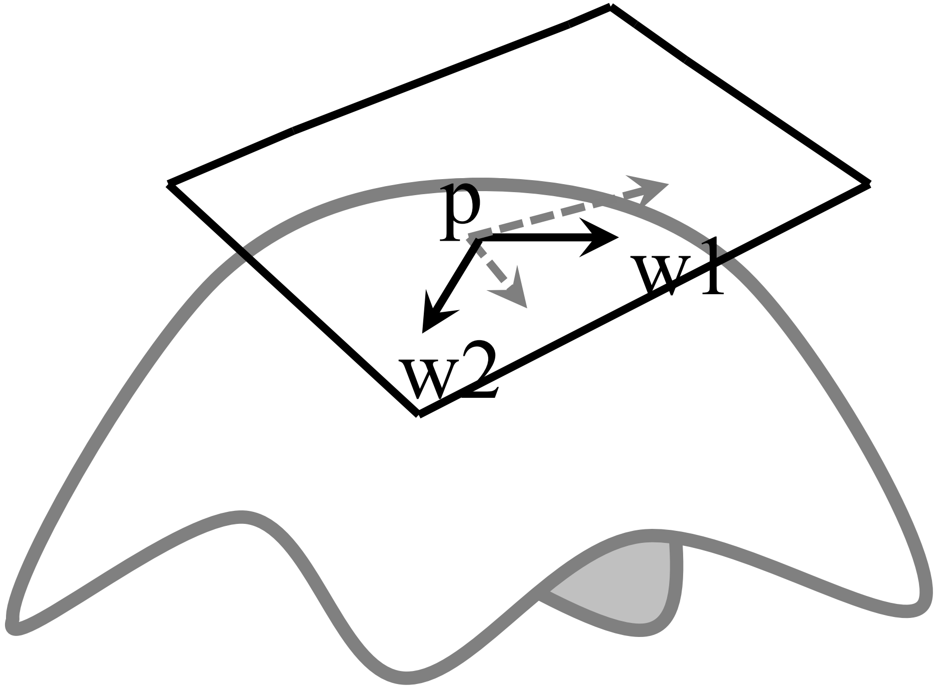


- When $w_1 = (du, 0), w_2 = (0, dv)$:

$$Area(w_1, w_2) = \text{dudv}$$
$$Area(Jw_1, Jw_2) = \sqrt{\det(I)}dudv$$

First Fundamental Form I_S

- **Riemannian metric**, Metric Tensor, Fundamental Tensor
- $I = J'J = \begin{bmatrix} \langle M_u, M_u \rangle & \langle M_u, M_v \rangle \\ \langle M_u, M_v \rangle & \langle M_v, M_v \rangle \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$
  - M(u,v)=(x(u,v), y(u,v),z(u,v))
  - Jacobian matrix $J = [M_u, M_v] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial \mathrm{v}} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$
  - $w = J\widehat{w} = [M_u, M_v] \begin{bmatrix} u \\ v \end{bmatrix}$
  - $< \widehat{w_1}, \widehat{w_2} >_S := \boldsymbol{I_S}(\widehat{\boldsymbol{w_1}}, \widehat{\boldsymbol{w_2}}) =< w_1, w_2 >= (J\widehat{w_1})^T (J\widehat{w_2}) = \widehat{w_1}^T (J^T J)\widehat{w_2}$



First Fundamental Form

First fundamental form $\mathbf{I}$ allows to measure

(w.r.t. surface metric)

Angles $\quad \mathbf{t}_1^\top \mathbf{t}_2 = \langle (\alpha_1, \beta_1), (\alpha_2, \beta_2) \rangle$

Length
$$\begin{aligned} \mathrm{d}s^2 &= \langle (\mathrm{d}u, \mathrm{d}v), (\mathrm{d}u, \mathrm{d}v) \rangle \\ &= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 \end{aligned}$$

squared infinitesimal length

Area
$$\begin{aligned} \mathrm{d}A &= \|\mathbf{x}_u \times \mathbf{x}_v\| \,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{\mathbf{x}_u^T\mathbf{x}_u \cdot \mathbf{x}_v^T\mathbf{x}_v - (\mathbf{x}_u^T\mathbf{x}_v)^2}\,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{EG - F^2}\mathrm{d}u\,\mathrm{d}v \end{aligned}$$

infinitesimal Area

cross product → determinant with unit vectors → area

- curve length

$$L = l(a,b) = \int_a^b \|\mathbf{x}'(u)\| \mathrm{d}u$$

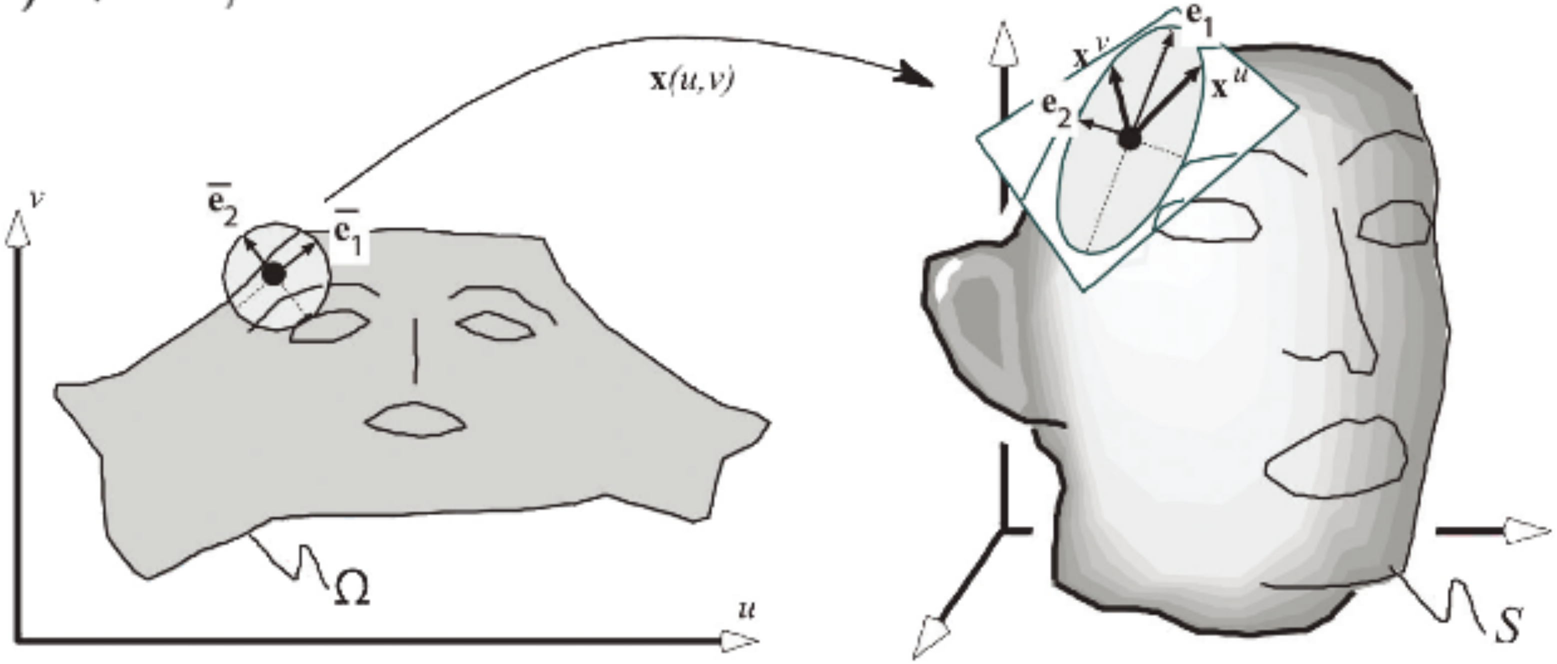
$$\begin{aligned} l(a,b) &= \int_a^b \sqrt{(u_t, v_t)\mathbf{I}(u_t, v_t)^T} \mathrm{d}t \\ &= \int_a^b \sqrt{Eu_t^2 + 2Fu_t v_t + Gv_t^2} \mathrm{d}t. \end{aligned}$$

- Surface area
- $\mathrm{A} = \mathrm{A}(\mathrm{X}) = \iint_{\mathrm{U}} |\mathrm{x_u} \times \mathrm{x_v}| \mathrm{dudv} = \iint_{\mathrm{U}} \sqrt{\mathrm{EG} - \mathrm{F}^2} \mathrm{dudv} = \iint_{\mathrm{U}} \sqrt{\det(\mathrm{I_X})} \mathrm{dudv}$

Anisotropy

- the axes of the anisotropy ellipse are $\mathbf{e}_1 = \mathbf{J}\bar{\mathbf{e}}_1$ and $\mathbf{e}_2 = \mathbf{J}\bar{\mathbf{e}}_2$;
- the lengths of the axes are $\sigma_1 = \sqrt{\lambda_1}$ and $\sigma_2 = \sqrt{\lambda_2}$.

$$\sigma_1 = \sqrt{1/2(E+G) + \sqrt{(E-G)^2 + 4F^2}},$$
$$\sigma_2 = \sqrt{1/2(E+G) - \sqrt{(E-G)^2 + 4F^2}},$$

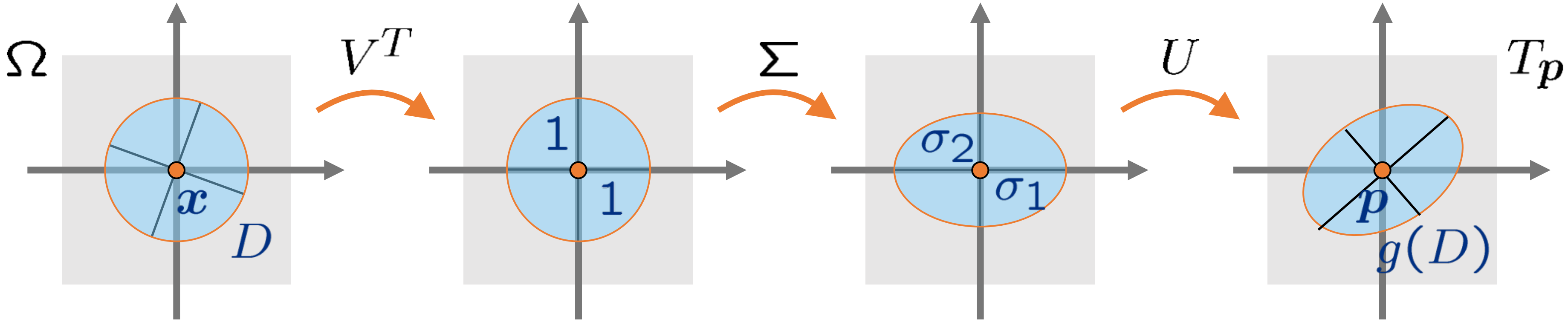


Linear Map Surgery

- **Singular Value Decomposition** (SVD) of J_f

$$J_f = U\,\Sigma\,V^T = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} V^T$$

with **rotations** $U \in \mathbb{R}^{3\times 3}$ and $V \in \mathbb{R}^{2\times 2}$
and **scale factors** (singular values) $\sigma_1 \geq \sigma_2 > 0$



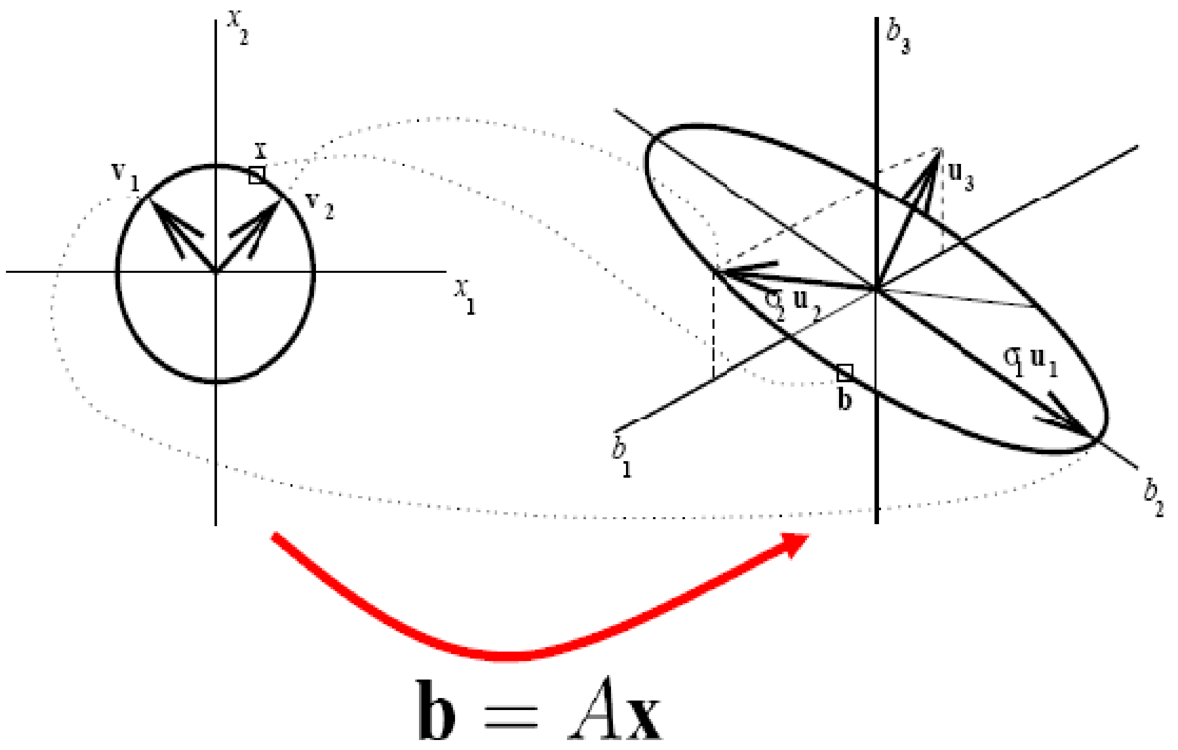
SVD

- Each matrix can be treated as a linear map or Jacobian Matrix of a map. Each owns a SVD decomposition, i.e. can be **described as an aligner followed by a stretch followed by a hanger.** (can be represented by a concatenation of rotation and scale.**)**

$J_f = (f_u \quad f_v)$ is a matrix of 3 by 2.

$J_f = U\Sigma V^T = (U_1 \quad U_2 \quad U_3)\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix}(V_1 \quad V_2)^T$, V_i are eigenvectors of ${J_f}^T J_f$, U_i are eigenvectors of $J_f {J_f}^T$.

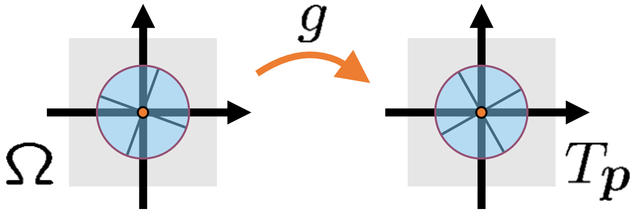
(Note: $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \lambda_1, \lambda_2$ are eigenvalues of ${J_f}^T J_f$, not $J_f {J_f}^T$)



Notion of Distortion

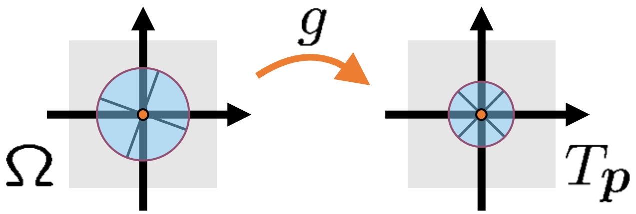
- **isometric** or **length**-preserving

$$\sigma_1 = \sigma_2 = 1$$



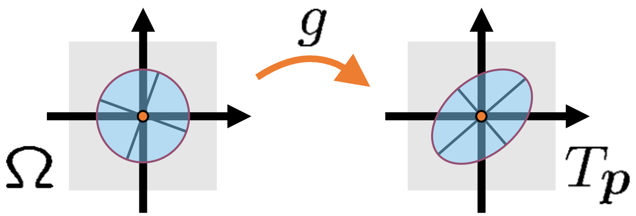
- **conformal** or **angle**-preserving

$$\sigma_1 = \sigma_2$$



- **equiareal** or **area**-preserving

$$\sigma_1 \cdot \sigma_2 = 1$$

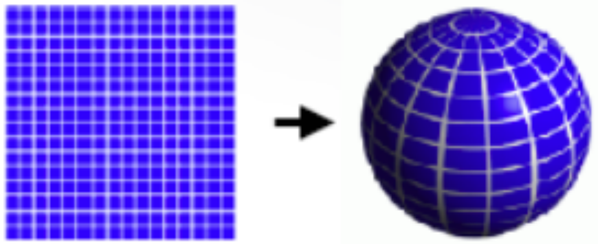


- everything defined **pointwise** on Ω

Theorem 4. *Every isometric mapping is conformal and equiareal, and every conformal and equiareal mapping is isometric, i.e.,*

$$\text{isometric} \Leftrightarrow \text{conformal} + \text{equiareal}.$$

Sphere Example



Spherical parameterization

$$\mathbf{x}(u,v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}, \quad (u,v) \in [0, 2\pi) \times [0, \pi)$$

Tangent vectors

$$\mathbf{x}_u(u,v) = \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix} \quad \mathbf{x}_v(u,v) = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{pmatrix}$$

First fundamental Form

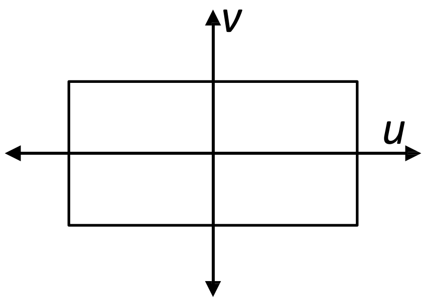
$$\mathbf{I} = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

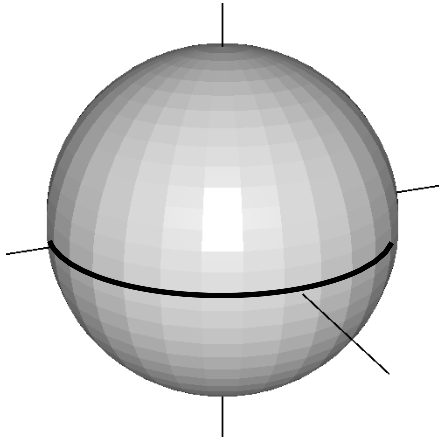
Metric Properties

$$\mathbf{x}(u,v) = \begin{pmatrix} \cos u \cos v & \sin v & \sin u \cos v \end{pmatrix} \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the length of the equator?





Metric Properties

$$\mathbf{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix} \qquad \mathrm{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

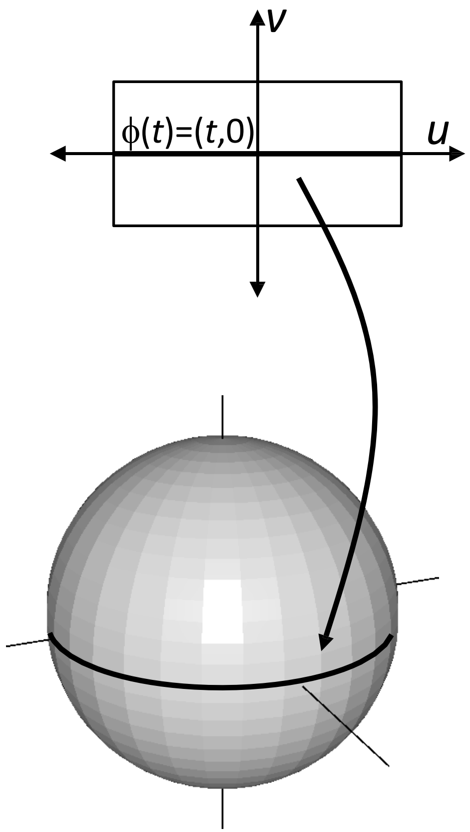
<u>Example (Sphere)</u>:

•What is the length of the equator?

The equator is the image of:

$\phi(t)=(t,0)$ with $t \in [-\pi, \pi]$

under the parameterization.



Metric Properties

$$\mathrm{x}(u,v) = (\cos u \cos v \quad \sin v \quad \sin u \cos v)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

Example (Sphere):

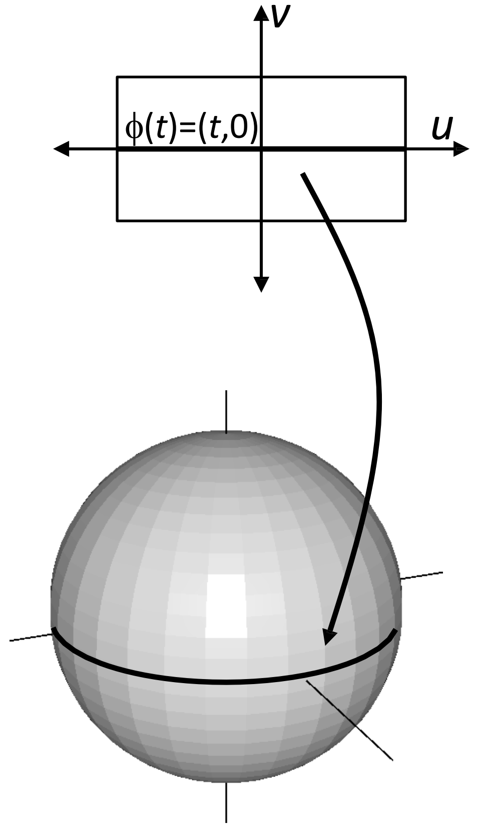
- What is the length of the equator?

$$length(\boldsymbol{x} \circ \phi) = \int_{-\pi}^{\pi} \sqrt{\phi'(t)^t \boldsymbol{I} \phi'(t)} dt$$

$$= \int_{-\pi}^{\pi} \sqrt{(1,0)^t \begin{pmatrix} \cos^2(0) & 0 \\ 0 & 1 \end{pmatrix} (1,0)} dt$$

$$= \int_{-\pi}^{\pi} dt$$

$$= 2\pi$$

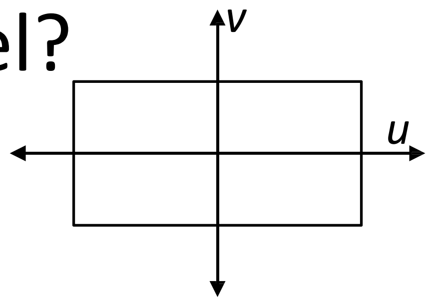


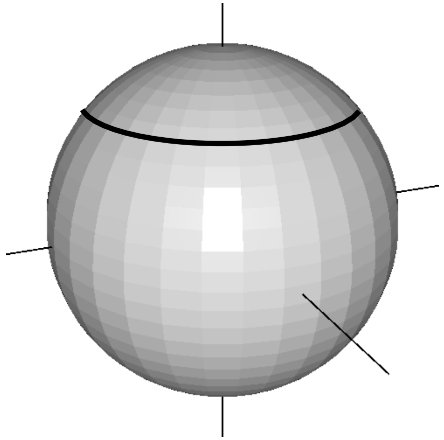
Metric Properties

$$\mathbf{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix} \qquad \mathrm{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

Example (Sphere):

- What is the length of the w^{th} parallel?



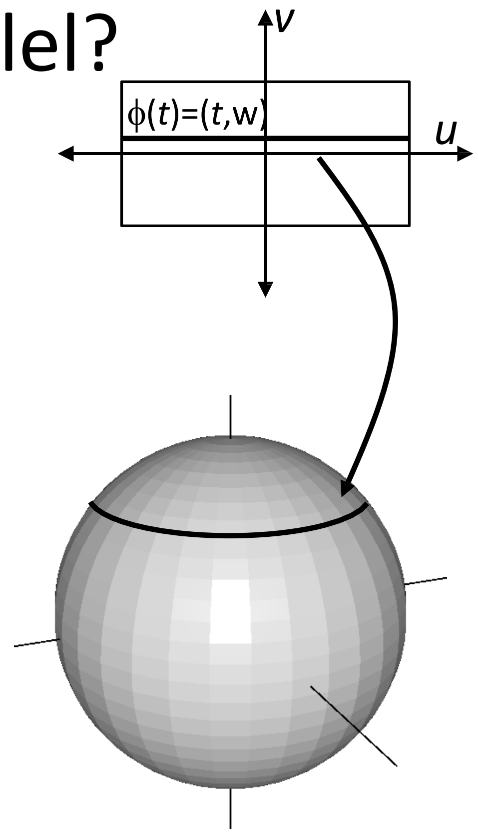


Metric Properties

$$\mathrm{x}(u,v) = (\cos u \cos v \quad \sin v \quad \sin u \cos v) \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the length of the w^{th} parallel?



The w^{th} parallel is the image of:

$\phi(t)=(t,w)$ with $t \in [-\pi, \pi]$

under the parameterization.

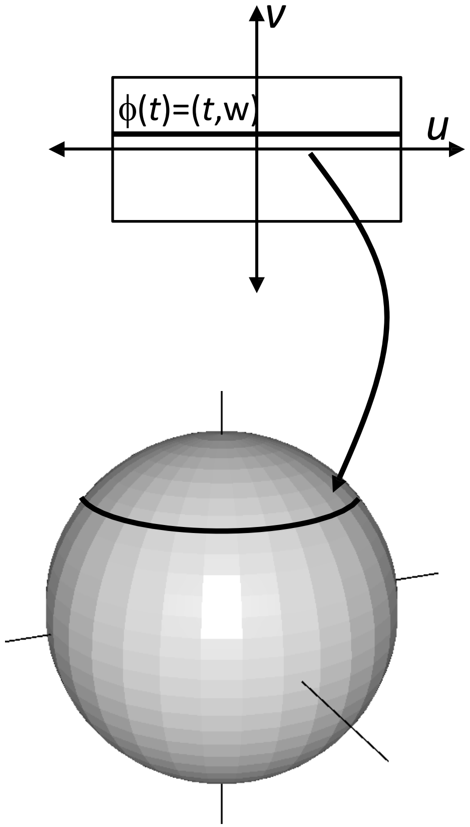
Metric Properties

$$\mathbf{x}(u,v) = (\cos u \cos v \quad \sin v \quad \sin u \cos v)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the length of the w^{th} parallel?

$$\begin{aligned} length(\boldsymbol{x} \circ \phi) &= \int_{-\pi}^{\pi} \sqrt{\phi'(t)^t \boldsymbol{I} \phi'(t)}\, dt \\ &= \int_{-\pi}^{\pi} \sqrt{(1,0)^t \begin{pmatrix} \cos^2 w & 0 \\ 0 & 1 \end{pmatrix} (1,0)}\, dt \\ &= \int_{-\pi}^{\pi} \cos w \, dt \\ &= 2\pi \cos w \end{aligned}$$

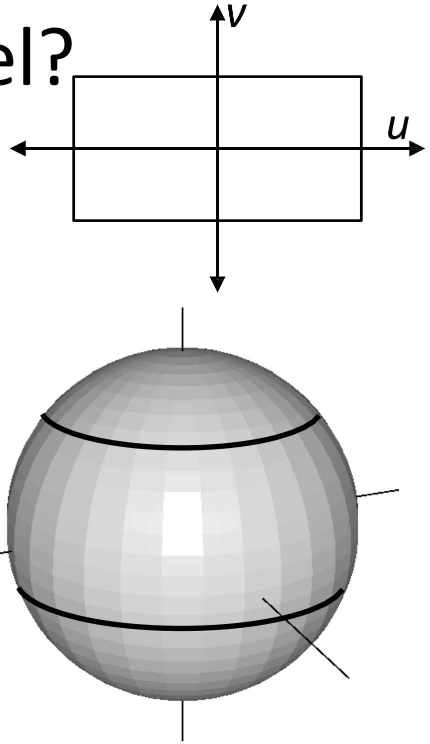


Metric Properties

$$\mathrm{x}(u,v) = \begin{pmatrix} \cos u \cos v & \sin v & \sin u \cos v \end{pmatrix} \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the area of the band between the w_1^{th} parallel and the w_2^{th} parallel?

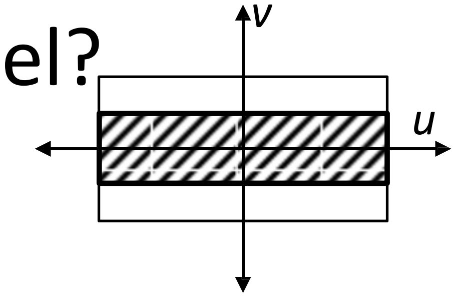


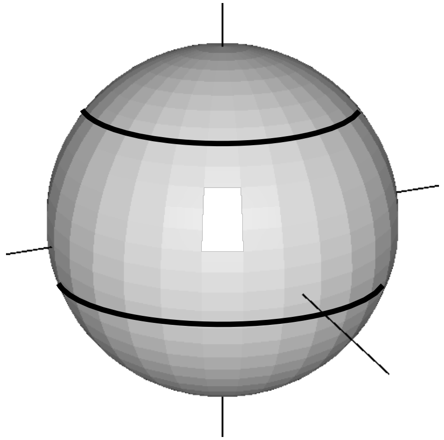
Metric Properties

$$\mathrm{x}(u,v) = (\cos u \cos v \quad \sin v \quad \sin u \cos v) \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the area of the band between the w_1^th^ parallel and the w_2^th^ parallel?





The band is the image of:

$\phi(s,t)=(s,t)$ with $s \in [-\pi,\pi]$, $t \in [w_1,w_2]$

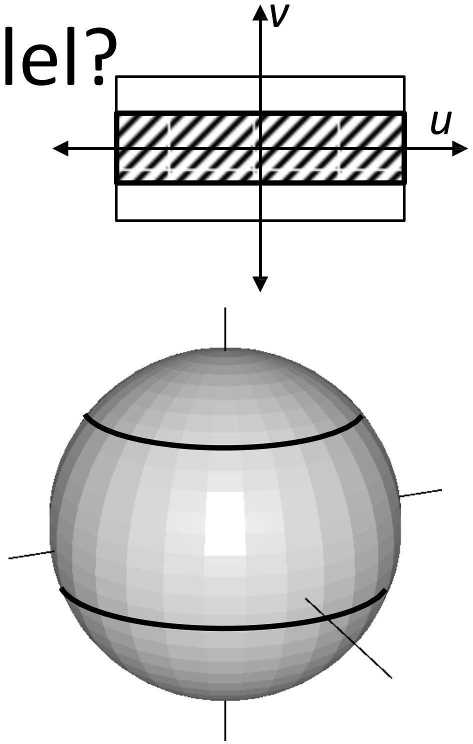
under the parameterization.

Metric Properties

$$\mathrm{X}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix} \qquad \mathrm{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the area of the band between the w_1th parallel and the w_2th parallel?



$$area(\mathrm{X}\circ\phi) = \int_{w_1}^{w_2}\int_{-\pi}^{\pi} \sqrt{\det \mathrm{I}}\, ds\, dt$$

$$= \int_{w_1}^{w_2}\int_{-\pi}^{\pi} \cos t\, ds\, dt$$

$$= 2\pi \int_{w_1}^{w_2} \cos t\, dt$$

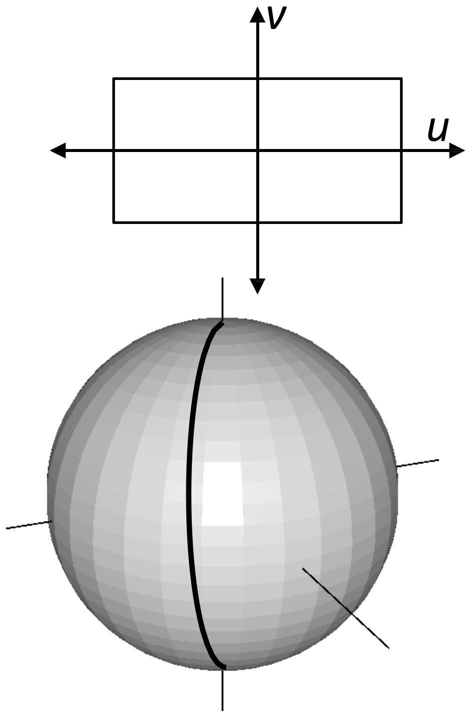
$$= 2\pi\left(\sin w_2 - \sin w_1\right)$$

Metric Properties

$$\mathrm{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix} \qquad \mathrm{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the area of the band between the w_1th and the w_2th meridians?



Metric Properties

$$\mathbf{x}(u,v) = \begin{pmatrix} \cos u \cos v & \sin v & \sin u \cos v \end{pmatrix} \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

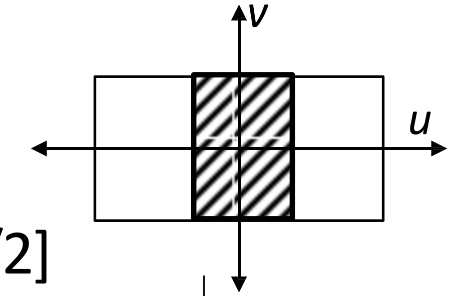
<u>Example (Sphere)</u>:

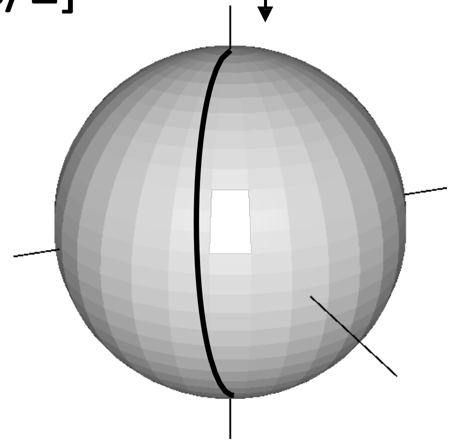
- What is the area of the band between the w_1th and the w_2th meridians?

The band is the image of:

$\phi(s,t)=(s,t)$ with $s \in [w_1, w_2]$, $t \in [-\pi/2, \pi/2]$

under the parameterization.





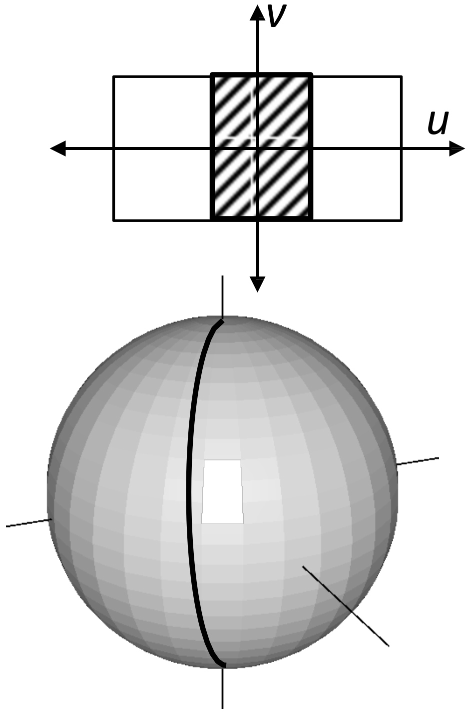
Metric Properties

$$\mathrm{x}(u,v) = (\cos u \cos v \quad \sin v \quad \sin u \cos v) \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

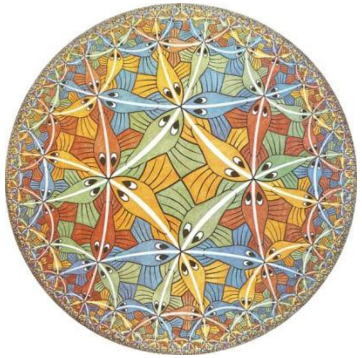
<u>Example (Sphere)</u>:

- What is the area of the band between the w_1th and the w_2th meridians?

$$\begin{aligned} area(\mathrm{x}\circ\phi) &= \int_{-\pi/2}^{\pi/2}\int_{w_1}^{w_2} \sqrt{\det \mathrm{I}}\, ds\, dt \\ &= \int_{-\pi/2}^{\pi/2}\int_{w_1}^{w_2} \cos t\, ds\, dt \\ &= (w_2 - w_1)\int_{-\pi/2}^{\pi/2} \cos t\ \ dt \\ &= (w_2 - w_1)\big(\sin(\pi/2) - \sin(-\pi/2)\big) \\ &= 2(w_2 - w_1) \end{aligned}$$



Metric Properties

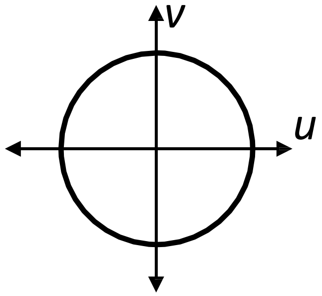


Example (Hyperbolic Plane):

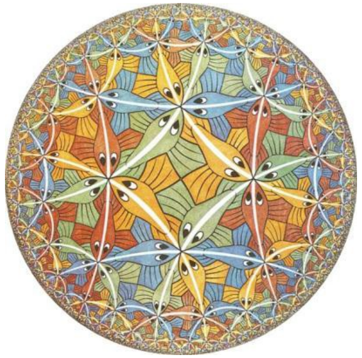
If we are given the first fundamental form, we can ignore the embedding of the surface in 3D, and integrate directly.

Consider the domain $\Omega=\{u,v \mid (u^2+v^2<1)\}$, with the first fundamental form:

$$\mathrm{I}(u,v)=\begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$



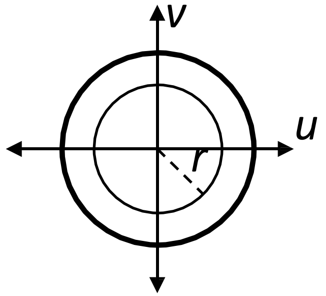
Metric Properties



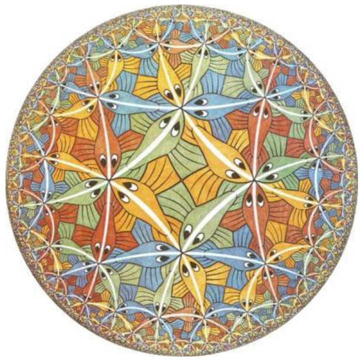
$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \quad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the length of the circle with radius *r*?



Metric Properties



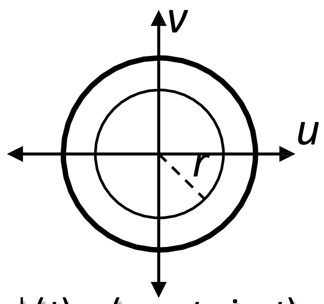
$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \qquad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

Example (Hyperbolic Plane):

•What is the length of the circle with radius *r*?

The circle is described by:

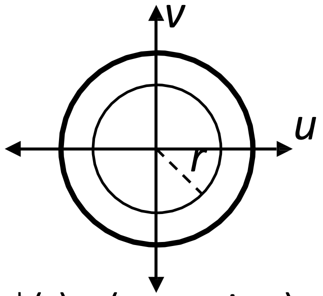
$\phi(s) = r(\cos s, \sin s)$ with $s \in [0, 2\pi]$.



$\phi(t) = r(\cos t, \sin t)$

Metric Properties

$$\Omega = \{(u,v) \mid u^2+v^2<1\} \qquad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$



$\phi(t)=r(\cos t, \sin t)$

<u>Example (Hyperbolic Plane)</u>:

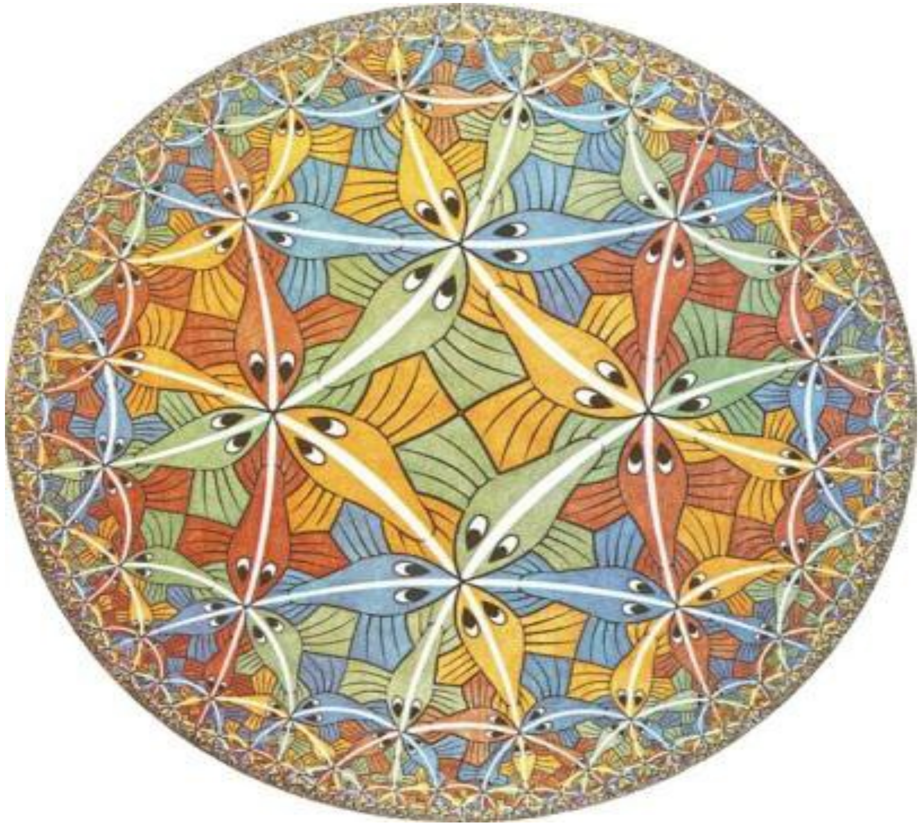
- What is the length of the circle with radius *r*?

$$length(\phi) = \int_0^{2\pi} \sqrt{\phi'(t)^t \, I \, \phi'(t)}\, \mathrm{dt}$$

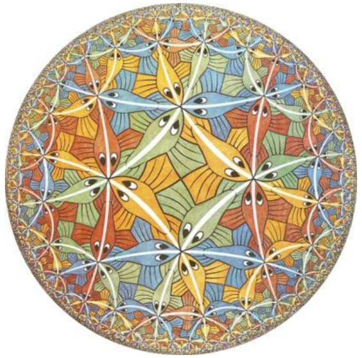
$$= \int_0^{2\pi} \sqrt{r(-\sin t, \cos t)^t \begin{pmatrix} \frac{1}{1-r^2} & 0 \\ 0 & \frac{1}{1-r^2} \end{pmatrix} r(-\sin t, \cos t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{\frac{r^2}{1-r^2}}\, dt$$

$$= 2\pi r \sqrt{\frac{1}{1-r^2}}$$



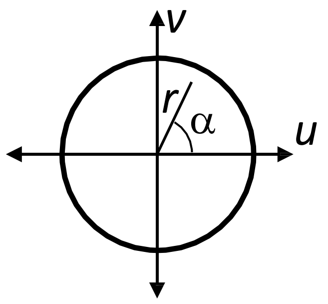
Metric Properties



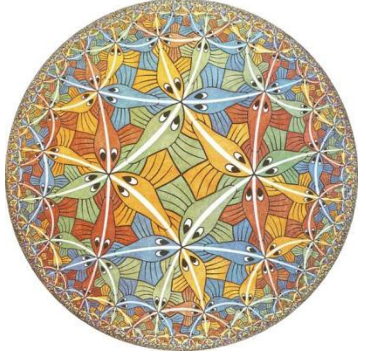
$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \qquad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the length of the segment with angle α and radius r?



Metric Properties



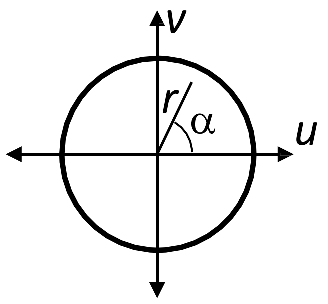
$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \qquad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the length of the segment with angle α and radius r?

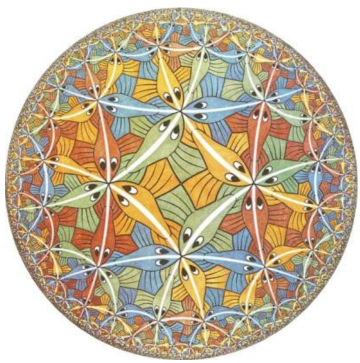
The segment is described by:

$\phi(s)=s(\cos\alpha, \sin\alpha)$ with $s \in [0,r]$.



$\phi(s)=s(\cos\alpha,\sin\alpha)$

Metric Properties



$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \quad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

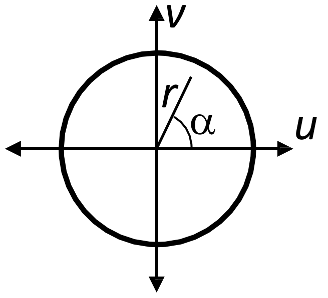
<u>Example (Hyperbolic Plane)</u>:

- What is the length of the segment with angle α and radius *r*?

$$length(\phi) = \int_0^r \sqrt{\phi'(s)^t \, \mathrm{I} \, \phi'(s)}\, ds$$

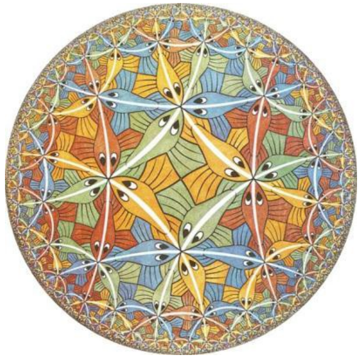
$$= \int_0^r \sqrt{(\cos\alpha, \sin\alpha)^t \begin{pmatrix} \frac{1}{1-s^2} & 0 \\ 0 & \frac{1}{1-s^2} \end{pmatrix} (\cos\alpha, \sin\alpha)}\, ds$$

$$= \int_0^r \frac{1}{1-s^2}\, ds = \frac{1}{2}\log\frac{1+r}{1-r}$$



$\phi(s) = s(\cos\alpha, \sin\alpha)$

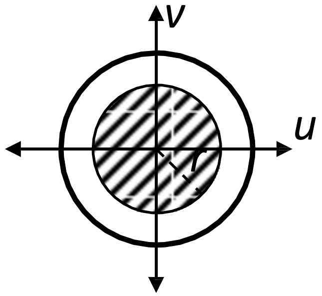
Metric Properties



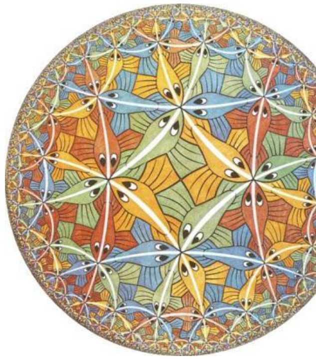
$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \quad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the area of the region with radius less than *r*?



Metric Properties



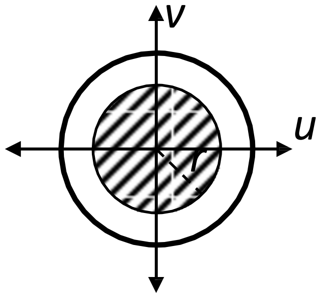
$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \qquad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the area of the region with radius less than *r*?

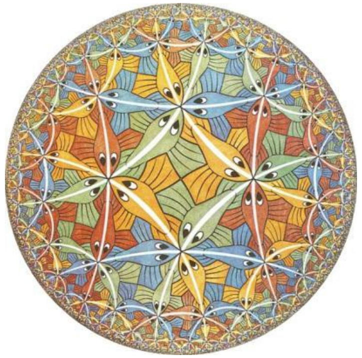
The region is the image of:

$\phi(s,t) = s(\cos t, \sin t)$ with $s \in [0,r]$, $t \in [-\pi,\pi]$.



$\phi(s,t)=s(\cos t,\sin t)$

Metric Properties



$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \quad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

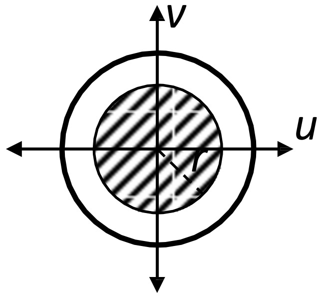
- What is the area of the region with radius less than *r*?

$$area(\phi) = \int_{-\pi}^{\pi}\int_{0}^{r} \sqrt{\det \mathrm{I} s}\, ds\, dt$$

$$= \int_{-\pi}^{\pi}\int_{0}^{r} \frac{s}{1-s^2}\, ds\, dt$$

$$= 2\pi \int_{0}^{r} \frac{s}{1-s^2}\, ds$$

$$= -\pi \ln\left(1-r^2\right)$$



$\phi(s,t)=s(\cos t,\sin t)$

Surfaces Curvatures

Quantify how a surface bends.

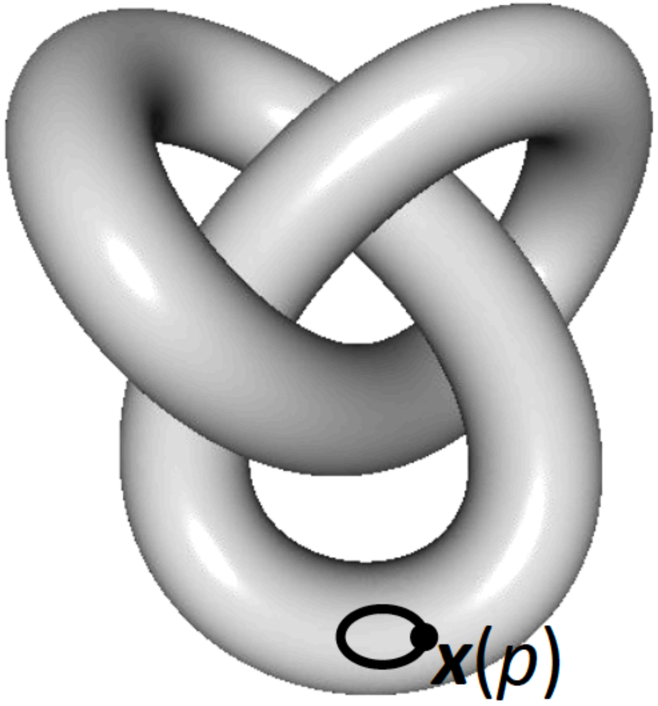
Curvatures of curves

$$\kappa(u) = -\frac{\langle \boldsymbol{n}'(u), \boldsymbol{t}(u) \rangle}{|\boldsymbol{x}'(u)|} = \frac{\langle \boldsymbol{n}(u), \boldsymbol{t}'(u) \rangle}{|\boldsymbol{x}'(u)|} = \ldots = \frac{\langle \boldsymbol{n}(u), \boldsymbol{x}''(u) \rangle}{\langle \boldsymbol{x}'(u), \boldsymbol{x}'(u) \rangle}$$

Curvature

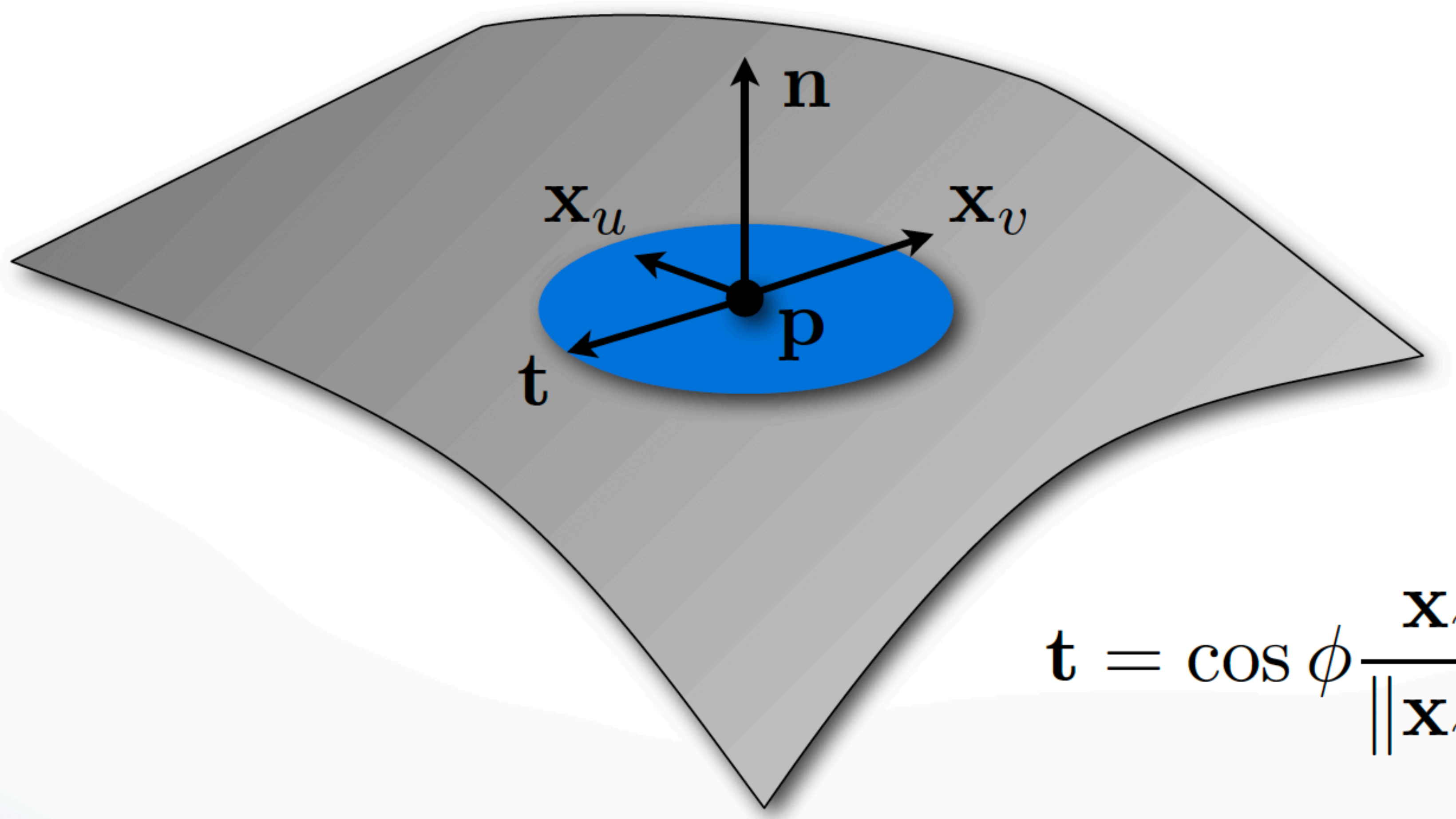
- We extend the notion to the curvature of a surface at the point $\boldsymbol{x}(p)$ by looking at the curvature of curves on the surface.

- Using arbitrary curves, we don't get a sense of the curvature as we go "around" the surface, e.g. we can get the curvature to be arbitrarily small.



Curvature

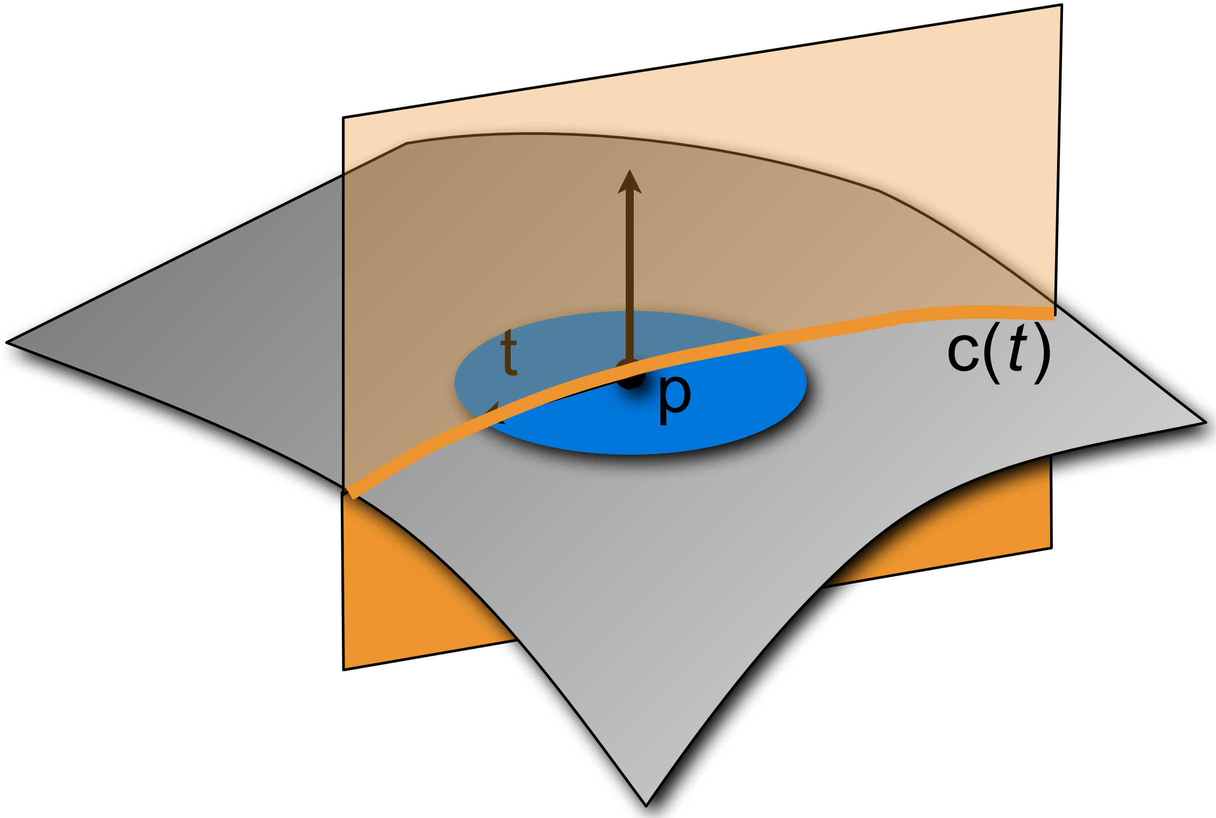
Tangent vector t …



$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Normal Curvature

Instead, we look at the curvature of *normal curves* c(t) – curves through $\boldsymbol{x}(p)$ obtained by intersecting the surface with a plane containing the normal at $\boldsymbol{x}(p)$.



$$\mathbf{t} = \cos\varphi \frac{x_u}{\|x_u\|} + \sin\varphi \frac{x_v}{\|x_v\|}$$

Regular Surfaces

$$\kappa(u)=\frac{\langle \boldsymbol{n}(u),\boldsymbol{x}''(u)\rangle}{\langle \boldsymbol{x}'(u),\boldsymbol{x}'(u)\rangle}$$

Computing the curvature of the curve $\boldsymbol{x}(\phi(t))$ at $\boldsymbol{x}(\phi(0))=\boldsymbol{x}(p)$ gives:

$$\kappa(0)=\frac{\langle \boldsymbol{n},(\boldsymbol{x}\circ\phi)''(0)\rangle}{\langle (\boldsymbol{x}\circ\phi)'(0),(\boldsymbol{x}\circ\phi)'(0)\rangle}$$

$$=\frac{\langle \boldsymbol{n},\left(\left(d^2\boldsymbol{x}\circ\phi\right)\cdot\phi'(0)\right)\cdot\phi'(0)+\left(\left(d\boldsymbol{x}\circ\phi\right)\cdot\phi''\right)(0)\rangle}{\langle \boldsymbol{J}w,\boldsymbol{J}w\rangle}$$

$$=\frac{w^t\begin{pmatrix}\langle \boldsymbol{n},\boldsymbol{x}_{uu}(p)\rangle & \langle \boldsymbol{n},\boldsymbol{x}_{vu}(p)\rangle\\ \langle \boldsymbol{n},\boldsymbol{x}_{uv}(p)\rangle & \langle \boldsymbol{n},\boldsymbol{x}_{vv}(p)\rangle\end{pmatrix}w}{w^t\boldsymbol{I}(p)w}$$

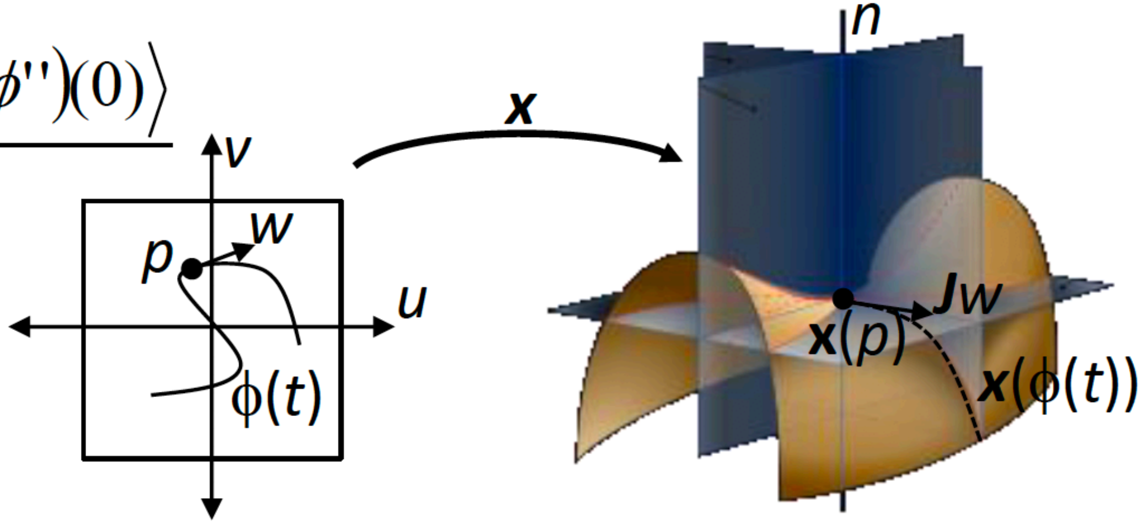
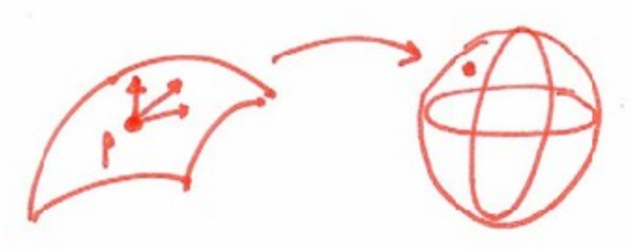


Image courtesy of Wikipedia

Geometry of the Normal

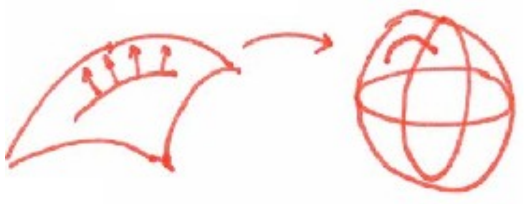
Gauss map

- normal at point



$$N(p) = \frac{S_{,u} \times S_{,v}}{|S_{,u} \times S_{,v}|}(p) \qquad N : S \to \mathbb{S}^2$$

- consider curve in surface again
  - study its curvature at p
  - normal “tilts” along curve



normal curvature $\kappa_n(\bar{\mathbf{t}})$ at $\mathbf{p}$

Let $t = u_t X_u + v_t X_v$ be a tangent vector at a surface point $p \in S$ represented as $\bar{t} = (u_t, v_t)^T$ in parameter domain

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{e u_t^2 + 2 f u_t v_t + g v_t^2}{E u_t^2 + 2 F u_t v_t + G v_t^2},$$

where $\mathbf{II}$ denotes the *second fundamental form* defined as

$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{bmatrix}.$$

Principal Curvatures

- Normal curvatures

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}}$$

- Principal curvatures
  - We can find the principal curvature values (and directions) by setting the derivative of normal curvature to 0:

$$\nabla \kappa_p(w) = 0 \quad \Rightarrow \quad \frac{\left(w^t I\!I w\right)}{\left(w^t I w\right)} I w = I\!I w$$

  - Thus, the principal curvature values (and directions) can be obtained by solving:

$$I^{-1} I\!I w = \lambda w$$

  - it has two distinct eigen values

$$I^{-1} I\!I w_1 = \kappa_1 w_1 \qquad I^{-1} I\!I w_2 = \kappa_2 w_2$$

  - We denote with k1 the minimum curvature and with k2 the maximum curvature.

$$I^{-1}IIw_1 = \kappa_1 w_1 \qquad I^{-1}IIw_2 = \kappa_2 w_2$$

- $I^{-1}II$ is also called the shape operator S
- This implies that mean and Gaussian curvatures are the trace and determinant of this matrix:
  - mean curvature $H = \mathrm{Tr(S)}$=k1+k2
  - Gaussian curvature $K = \mathrm{Det(S)}$ = k1*k2

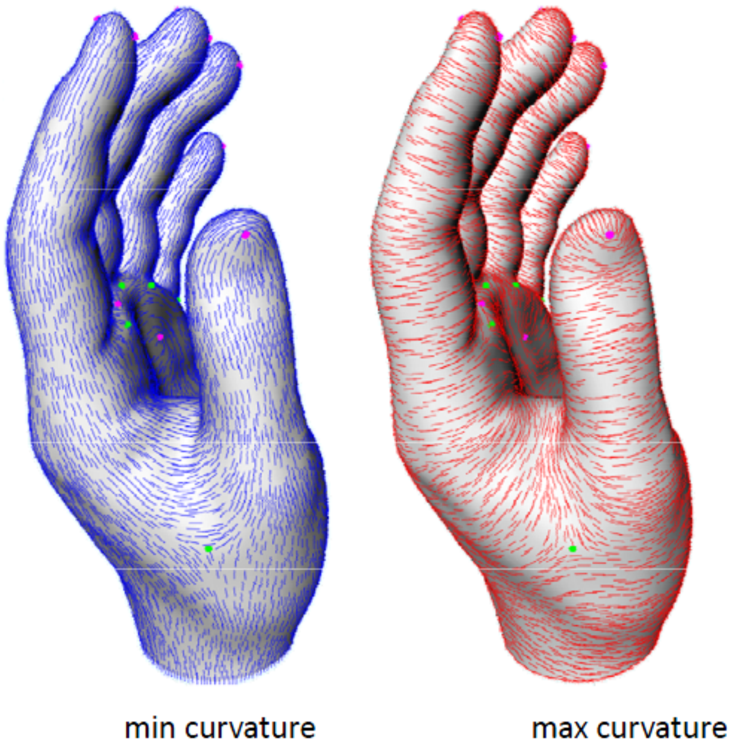
Principal Curvatures

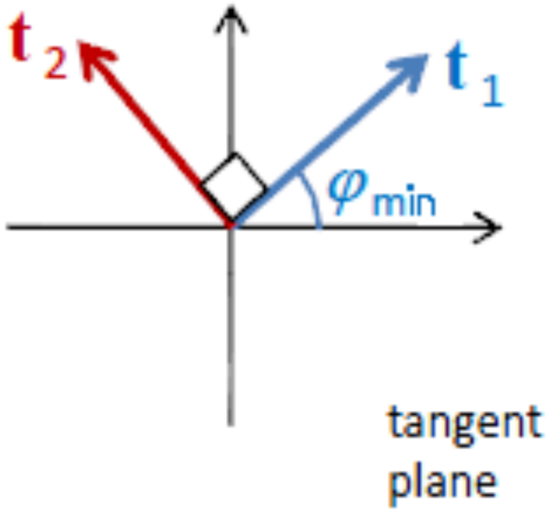
$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}}$$

- Euler theorem

$$\kappa_n(\bar{\mathbf{t}}) = \kappa_1 \cos^2 \psi + \kappa_2 \sin^2 \psi,$$

- ψ is the angle between t and t1
- t1 & t2 are principal directions: tangent vectors corresponding to φ_{min} & φ_{max}
- **any normal curvature is a convex combination of the minimum and maximum curvature**
- **principal directions are orthogonal to each other**





Curvature tensor

$$\kappa_p(w) = \kappa_1(p)\cos^2\alpha + \kappa_2(p)\sin^2\alpha$$

To prove it, we define curvature tensor

Given the **unit principal curvatures directions** ***J****w*1 and ***J****w*2, and the principal curvature k1 and k2, the *curvature tensor* is a 3x3 symmetric matrix associated to each point on the surface, defined by:

$$C(X(p)) = k_1 Jw_1 {Jw_1}^t + k_2 Jw_2 {Jw_2}^t$$

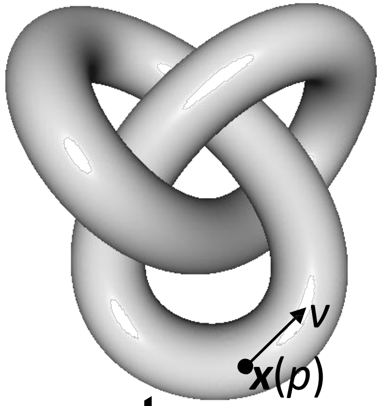
Curvature tensor & Euler theorem

$$C(X(p)) = k_1 Jw_1 Jw_1{}^t + k_2 Jw_2 Jw_2{}^t$$

Note:

Given a (non-tangent) vector *v* at the point ***x***(*p*), we can express *v* as:

$$v = cos\psi\, Jw_1 + sin\psi\, Jw_2 + \gamma\, n(p)$$



Applying the curvature tensor to *v*, gives:

$$v^t C(X(p)) v = k_1 cos\psi^2 + k_2 sin\psi^2$$

So the curvature tensor gives the curvature in the tangent component (scaled by square length).

$$\kappa_n(\bar{t}) = \kappa_1 \cos^2\psi + \kappa_2 \sin^2\psi,$$

Surfaces Curvatures

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{e u_t^2 + 2 f u_t v_t + g v_t^2}{E u_t^2 + 2 F u_t v_t + G v_t^2}$$

- Principal curvatures
  - Maximal curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
  - Minimal curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$
- Mean curvature: $k_H = \frac{k_1 + k_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} k_n(\theta) d\,\theta = \lim_{diam(A) \to 0} \frac{\nabla A}{A}$
- Gaussian curvature: $k_G = k_1 \cdot k_2 = \lim_{diam(A) \to 0} \frac{A^G}{A}$
- Curvature tensor: $C = PDP^{-1}$, with P=[t1, t2, n] and D=diag(k1, k2, 0)

Surfaces Curvatures

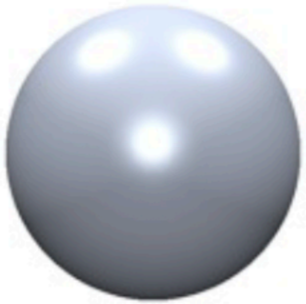
$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{e u_t^2 + 2 f u_t v_t + g v_t^2}{E u_t^2 + 2 F u_t v_t + G v_t^2}$$

- Principal curvatures: eigenvalues of the shape operator S: $I^{-1}II$
  - Maximal curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
  - Minimal curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$
- Mean curvature: $k_H = \frac{k_1+k_2}{2} = \frac{1}{2\pi}\int_0^{2\pi} k_n(\theta)d\,\theta = \lim_{diam(A)\to 0} \frac{\nabla A}{A}$
- Gaussian curvature: $k_G = k_1 \cdot k_2 = \lim_{diam(A)\to 0} \frac{A^G}{A}$

Gauss-Bonnet Theorem

For any closed manifold surface with Euler characteristic $\chi = 2 - 2g$

$$\int K = 2\pi\chi$$

$\int K($  $) = \int K($  $) = \int K($  $) = 4\pi$

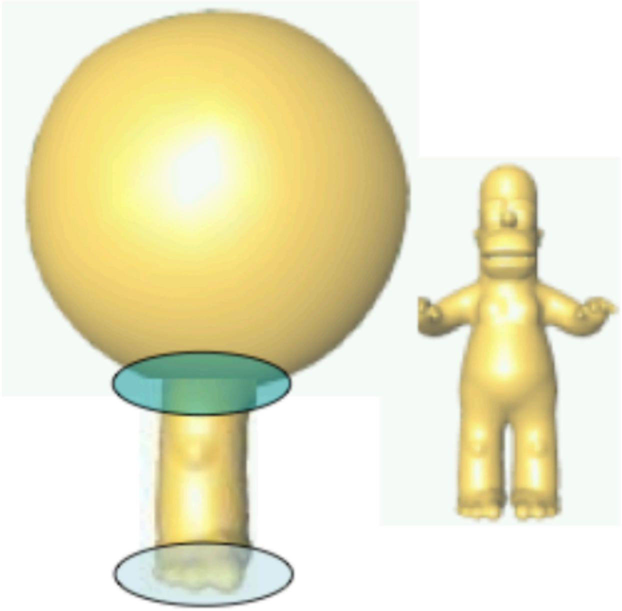
Gauss-Bonnet Theorem

Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

$$K = \kappa_1 \kappa_2 = 1/r^2$$

$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$



when sphere is deformed, new
positive and negative curvature cancel out

高斯曲率 反应了曲面的弯曲程度。在给出高斯曲率的几何解释之前，首先引入高斯映射的定义，设 A 是曲面上包含 p 点的一小片曲面（其面积仍用 A 表示），把 A 上的每点的单位法向量 n 平移到原点 O 处，那么 n 的终点轨迹是以 O为中心的单位球面 $S2$ 上的一块区域 A^*。这个对应称为高斯映射。则 p 点的高斯曲率可以表示为：

$$\kappa_G(p)=\lim_{A\to 0}\frac{A^*}{A}$$

其中高斯曲率 κ_G 和平均曲率 κ_H 都反映局部曲面的几何特征。

Lagrange注意到 $\kappa_H = 0$ 是极小曲面的Lagrange方程，于是就给出了一个极小曲面与平均曲率的直接关系：

$$2\kappa_H n = \lim_{diam(A)\to 0} \frac{\nabla A}{A}$$

其中，A 是点 p 处无穷小区域的面积，$diam(A)$ 是它的直径，∇ 是关于点 $p(x,y,z)$ 坐标的梯度，因此，定义算子 $K(p) = 2\kappa_H(p)n(p)$，这就是著名的Laplace-Beltrami算子。

Analogies with curves

Curves:

First derivative → arc length

Second derivative → curvature

Surfaces:

First fundamental form → distances

Second fundamental form → (extrinsic) curvatures

Intrinsic and Extrinsic Properties

- Properties of the surface related to the first fundamental form are called **intrinsic** properties
  - Determined only by measuring distances on the surface
- Properties of the surface related to the second fundamental form are called **extrinsic** properties
  - Determined by looking at the full embedding of the surface in $\mathbb{R}^3$

Gaussian Curvature

- The Gaussian curvature at a surface point is an intrinsic property

$$K = \frac{L\,N - M^2}{E\,G - F^2}$$

- But this involves L, M, N from the second fundamental form, how is this intrinsic?

Theorem Egregium of Gauss

- The Gaussian curvature can be expressed solely as a function of the coefficients of the first fundamental form and their derivatives

$$K = \frac{\det\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \det\begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG - F^2)^2}$$

Bonnet’s Theorem

- A surface in 3-space is uniquely determined upto rigid motion by its first and second fundamental forms

- Compare to the Fundamental Theorem of Space Curves:
  - curvature and torsion uniquely define a curve upto rigid motion.

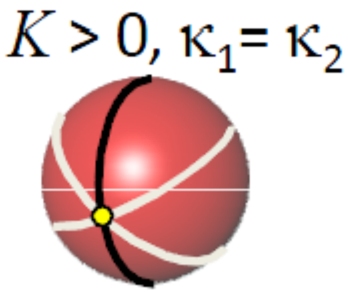
Who cares?

Curvature
completely determines
local surface geometry.

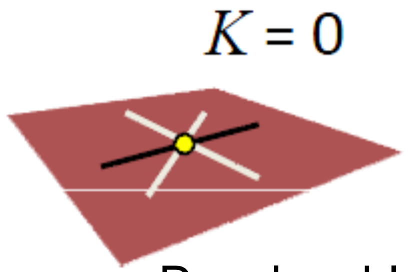
Classification

A point p on the surface is called

Isotropic: all directions are principle directions



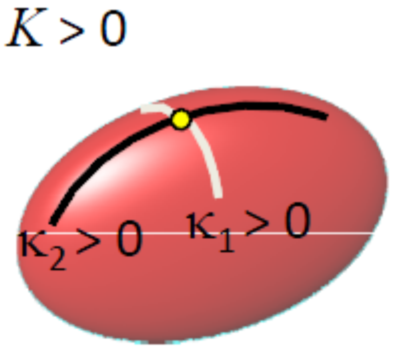
spherical (umbilical)



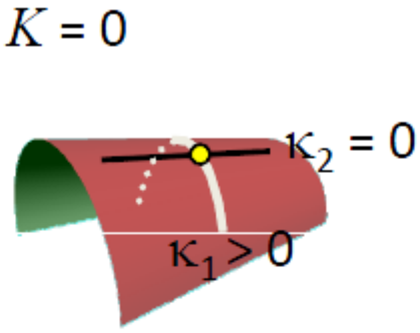
Developable surface ⇔ K=0

planar

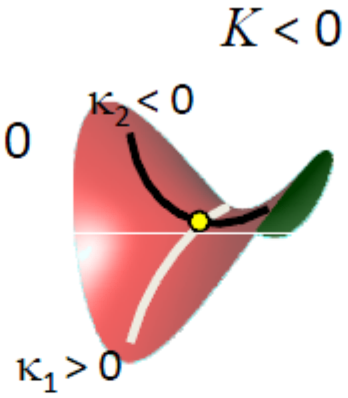
Anisotropic: 2 distinct principle directions



elliptic

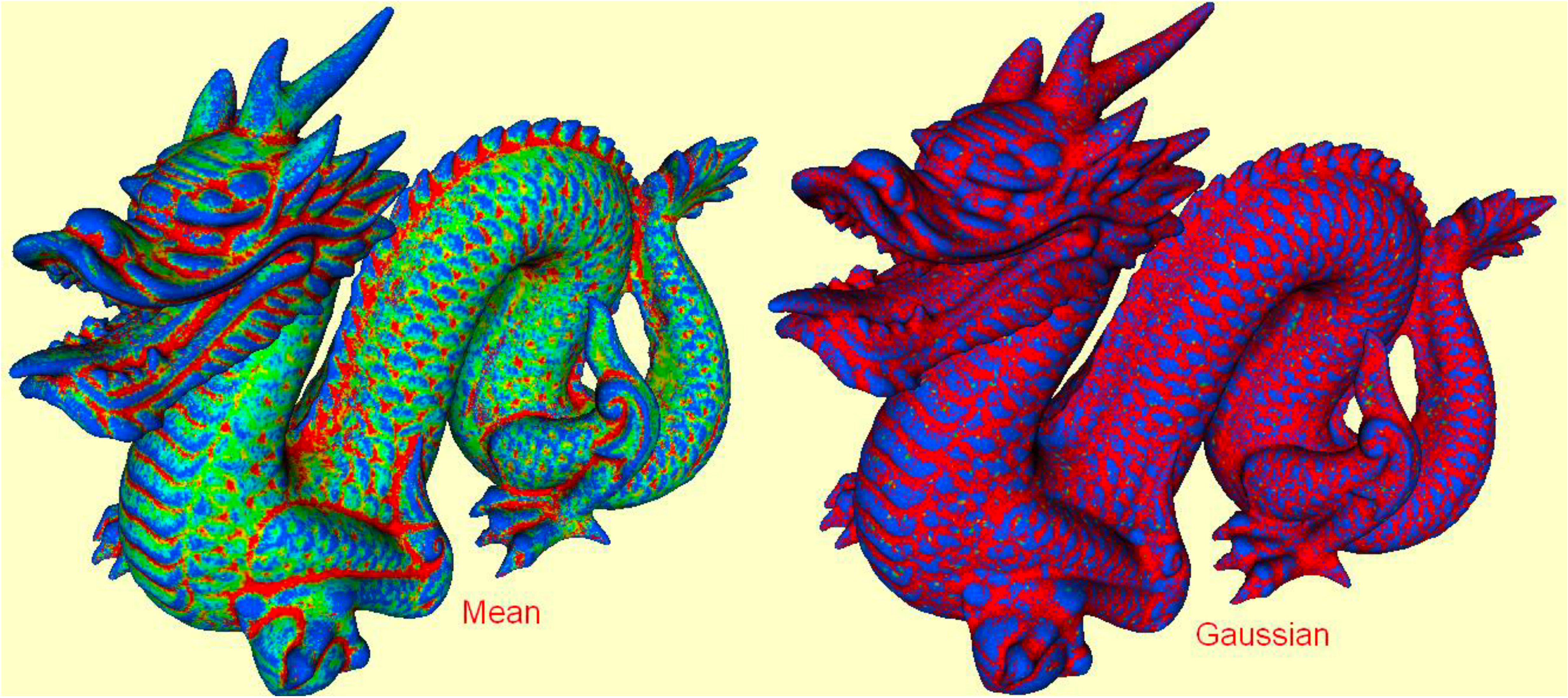


parabolic

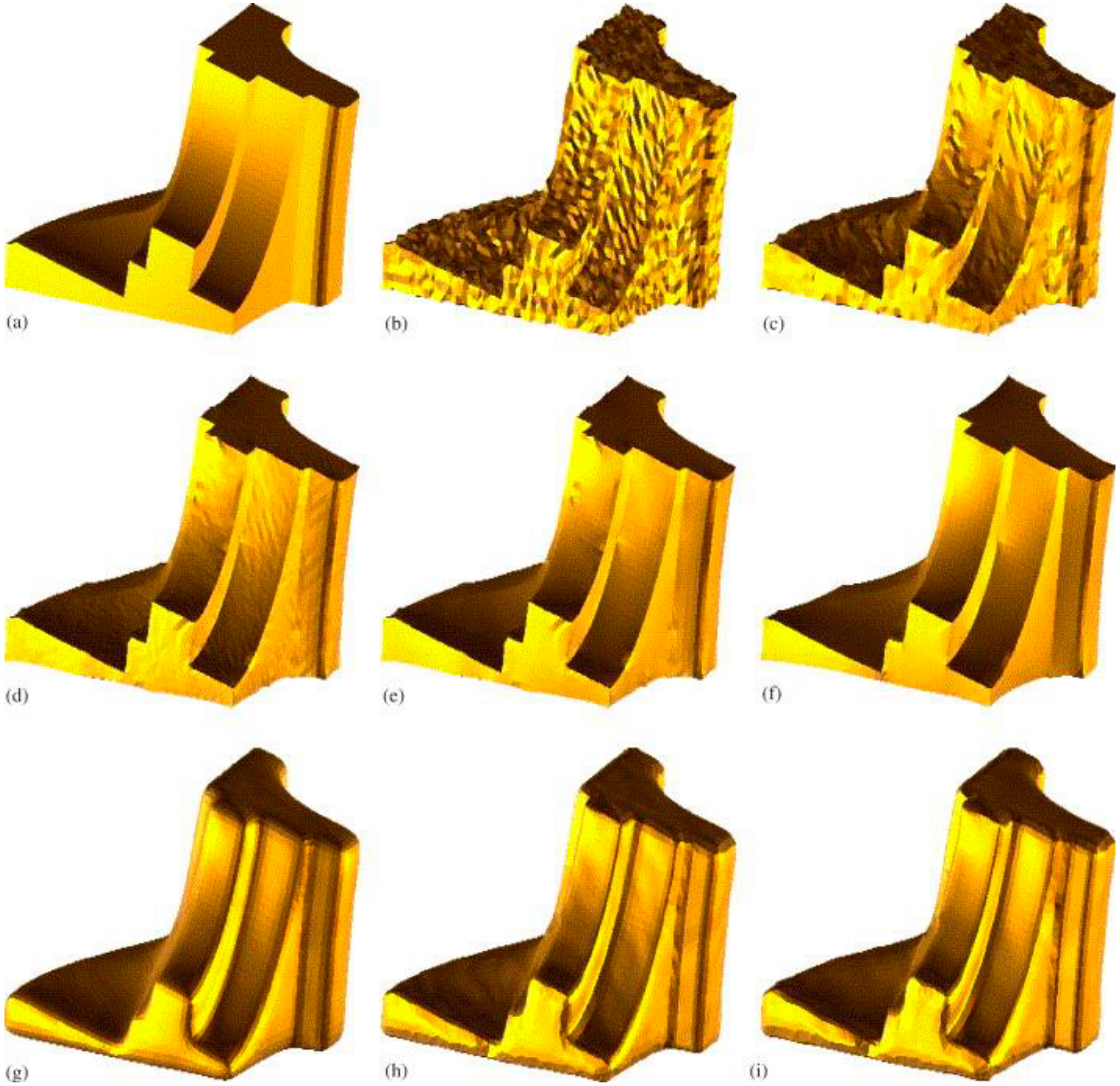


hyperbolic

Use as a descriptor



Fairness measure



Triangular Surface Mesh Fairing via Gaussian Curvature Flow

Zhao, Xu

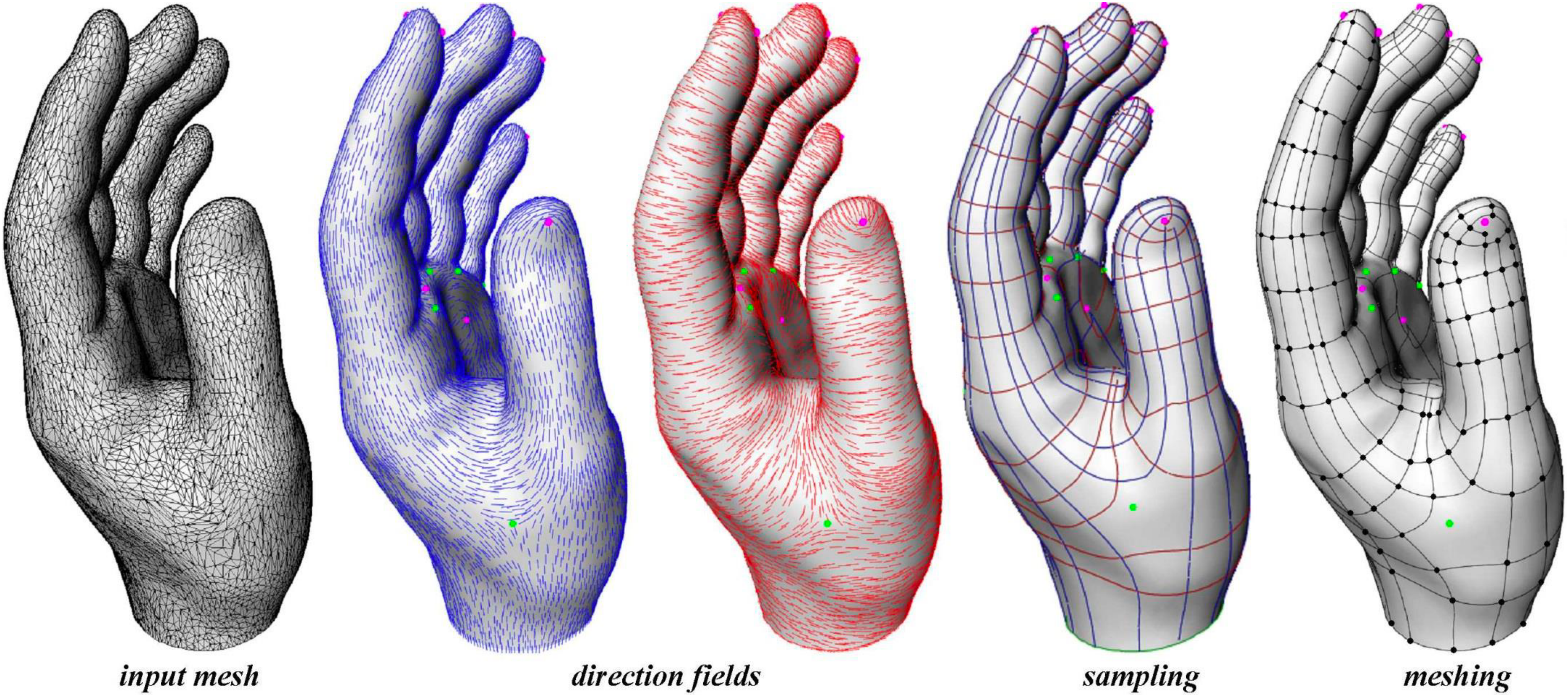
Journal of Computational and Applied Mathematics

(2006)

Guiding rendering

Highlight Lines for Conveying Shape
DeCarlo, Rusinkiewicz
NPAR(2007)

Guiding meshing

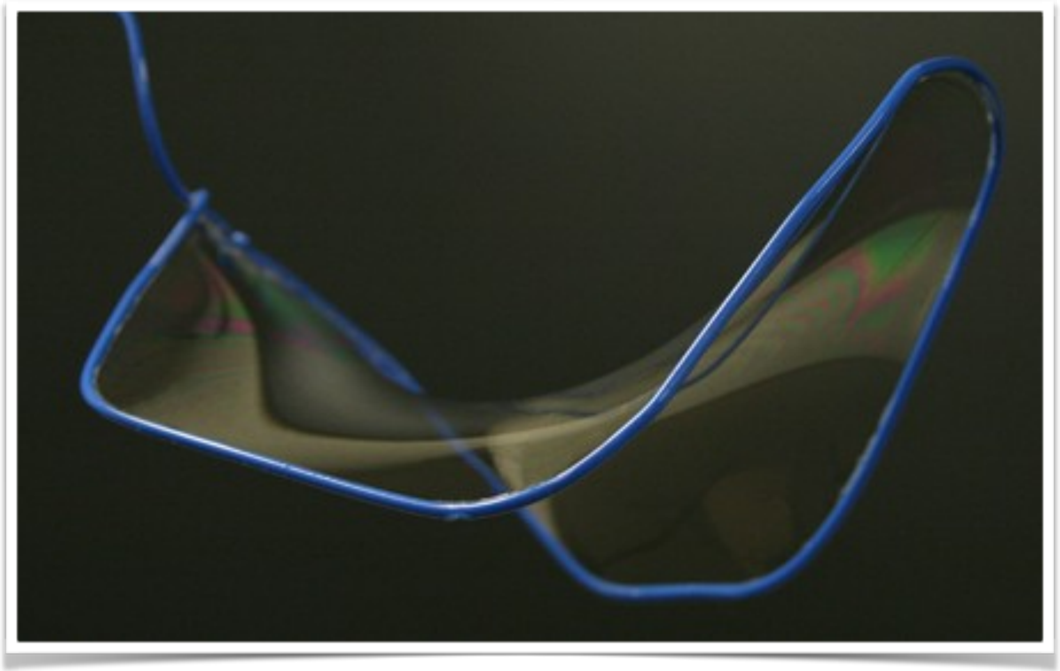


Anisotropic Polygonal Remeshing
Alliezet al.
SIGGRAPH(2003)

Curvature of Surfaces

Mean curvature $H = \dfrac{d_1 + d_2}{2}$

- $H = 0$ everywhere   minimal surface



soap film

Curvature of Surfaces

Mean curvature $H = \frac{d_1 + d_2}{2}$

- $H = 0$ everywhere   minimal surface

Green Void, Sydney
Architects: Lava

Curvature of Surfaces

Gaussian curvature $K = \kappa_1 \cdot \kappa_2$

- $K = 0$ everywhere developable surface

surface that can be flattened to a plane without distortion (stretching or compression)

Disney, Concert Hall, L.A.
Architects: Gehry
Partners

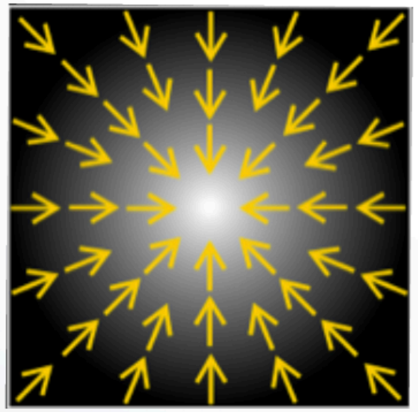
Timber Fabric
IBOIS, EPFL

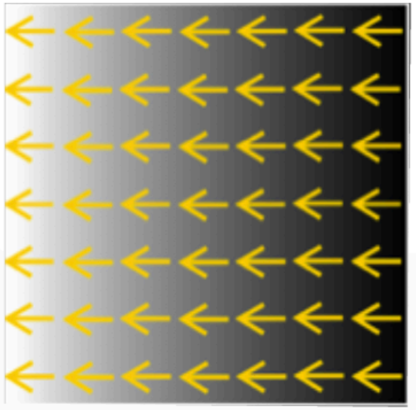
Differential Operators

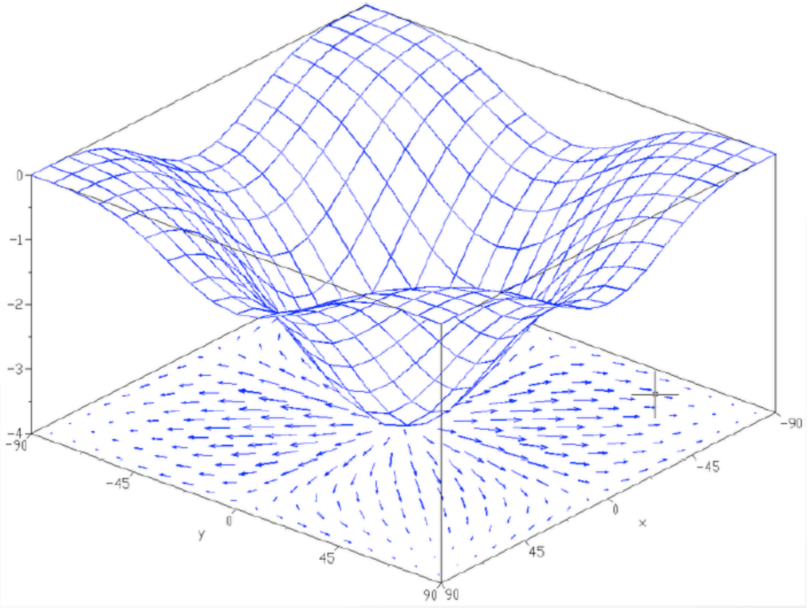
Gradient

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

- points in the direction of the steepest ascend





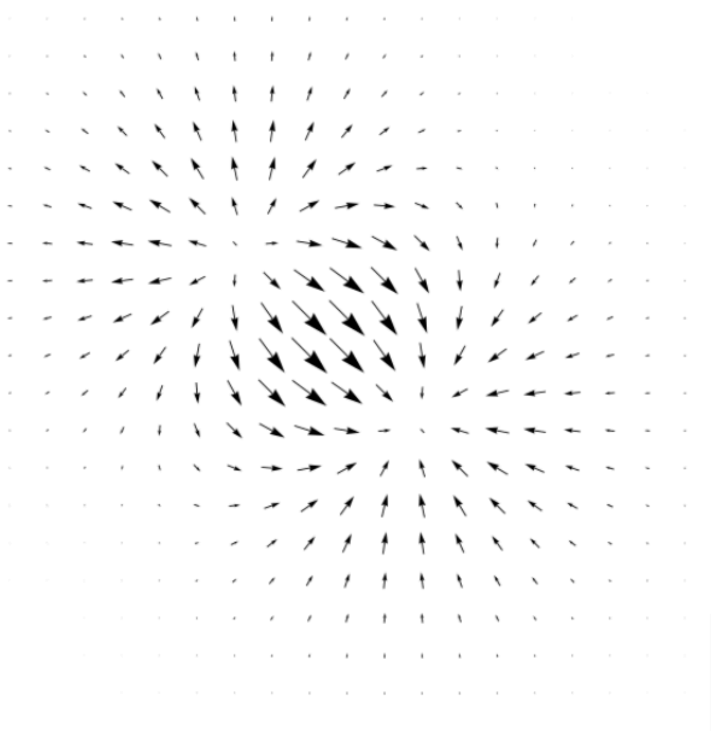


Differential Operators

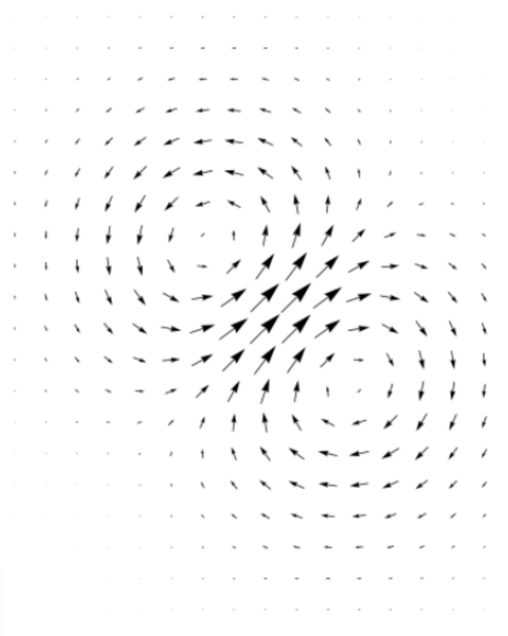
Divergence

$$\mathrm{div}F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$

- volume density of outw
- magnitude of source or
- Example: incompressib
  - velocity field is diver



high divergence



low divergence

Laplace Operator: $\text{divF} = \nabla \cdot F$

- $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
- $f = f(x, y, z)$, $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$
- $F = (U(x, y, z), V(x, y, z), W(x, y, z))$
- $\text{divF} = \nabla \cdot F = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}$

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \operatorname{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

function in Euclidean space

divergence operator

Cartesian coordinates

Laplace-Beltrami Operator

- Extension of Laplace of functions on manifolds

Laplace-Beltrami

gradient operator

...of the surface

$$\Delta_{\mathcal{S}} f = \mathrm{div}_{\mathcal{S}} \nabla_{\mathcal{S}} f$$

function on manifold S

divergence operator

Laplace on the surface

Laplace-Beltrami Operator

- Extension of Laplace of functions on manifolds

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{S}} \mathbf{x} = \mathrm{div}_{\mathcal{S}}\, \nabla_{\mathcal{S}} \mathbf{x} = -2H\mathbf{n}$$

function on manifold S

divergence operator

surface normal

Literature

- M. Do Carmo: Differential Geometry of Curves and Surfaces, Prentice Hall, 1976
- A. Pressley: Elementary Differential Geometry, Springer, 2010
- G. Farin: Curves and Surfaces for CAGD, Morgan Kaufmann, 2001
- W. Boehm, H. Prautzsch: Geometric Concepts for Geometric Design, AK Peters 1994
- H. Prautzsch, W. Boehm, M. Paluszny: Bézier and B-SplineTechniques, Springer 2002
- ddg.cs.columbia.edu
- http://graphics.stanford.edu/courses/cs468-13-spring/schedule.html