Digital Geometry
- Surface Registration

Junjie Cao @ DLUT

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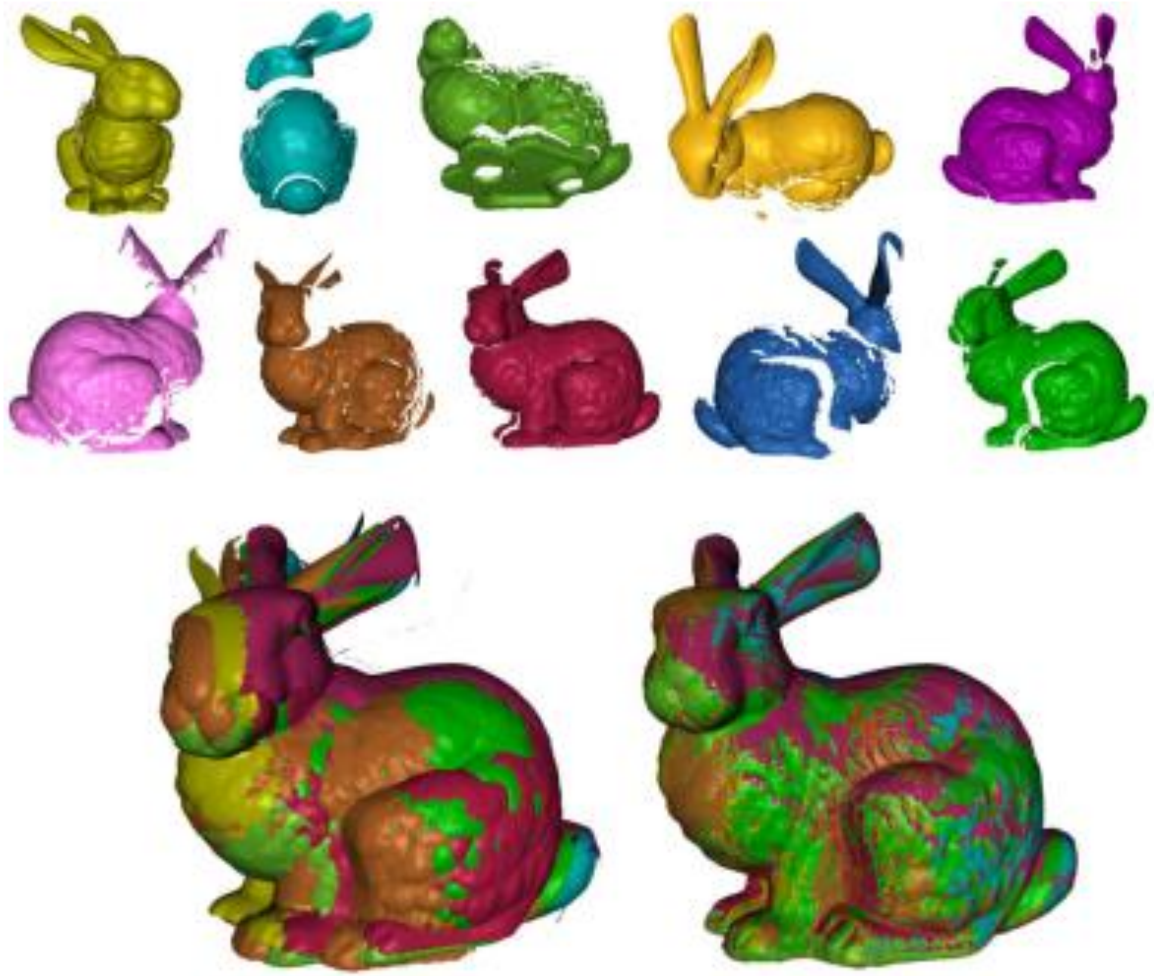
http://jjcao.github.io/DigitalGeometry/

The purpose of computing is insight, not numbers

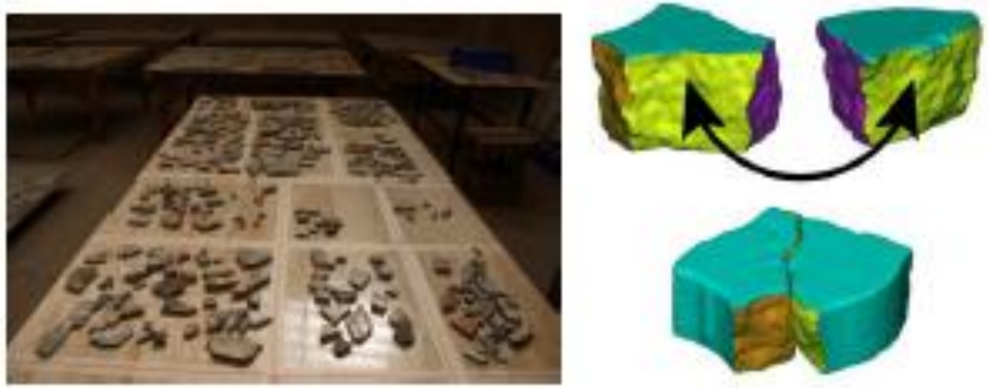
Definition

- Surface registration is the process of identifying and **matching corresponding regions**
- across **multiple scans** given in **arbitrary initial positions**,
- and estimating the corresponding **rigid transforms** that best align the scans to each other.

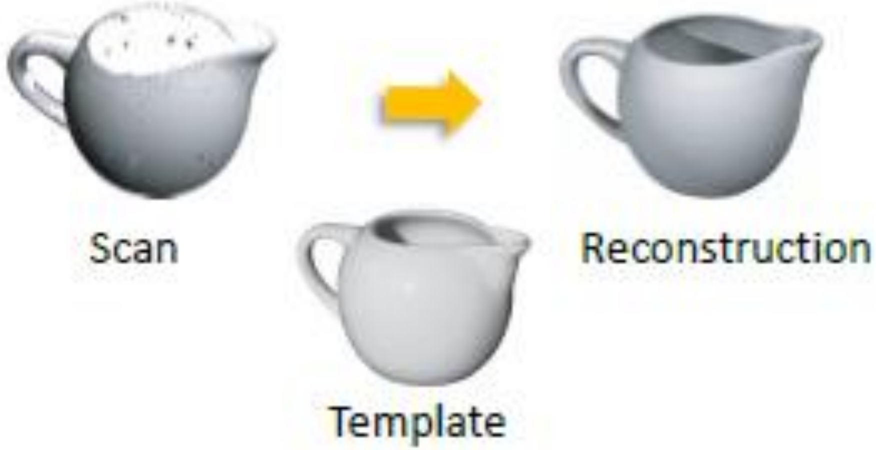


Applications

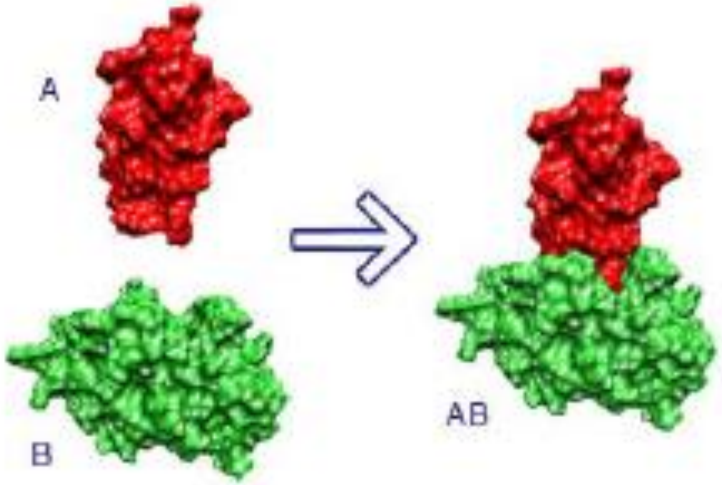
Surface reconstruction



Fragment assembly

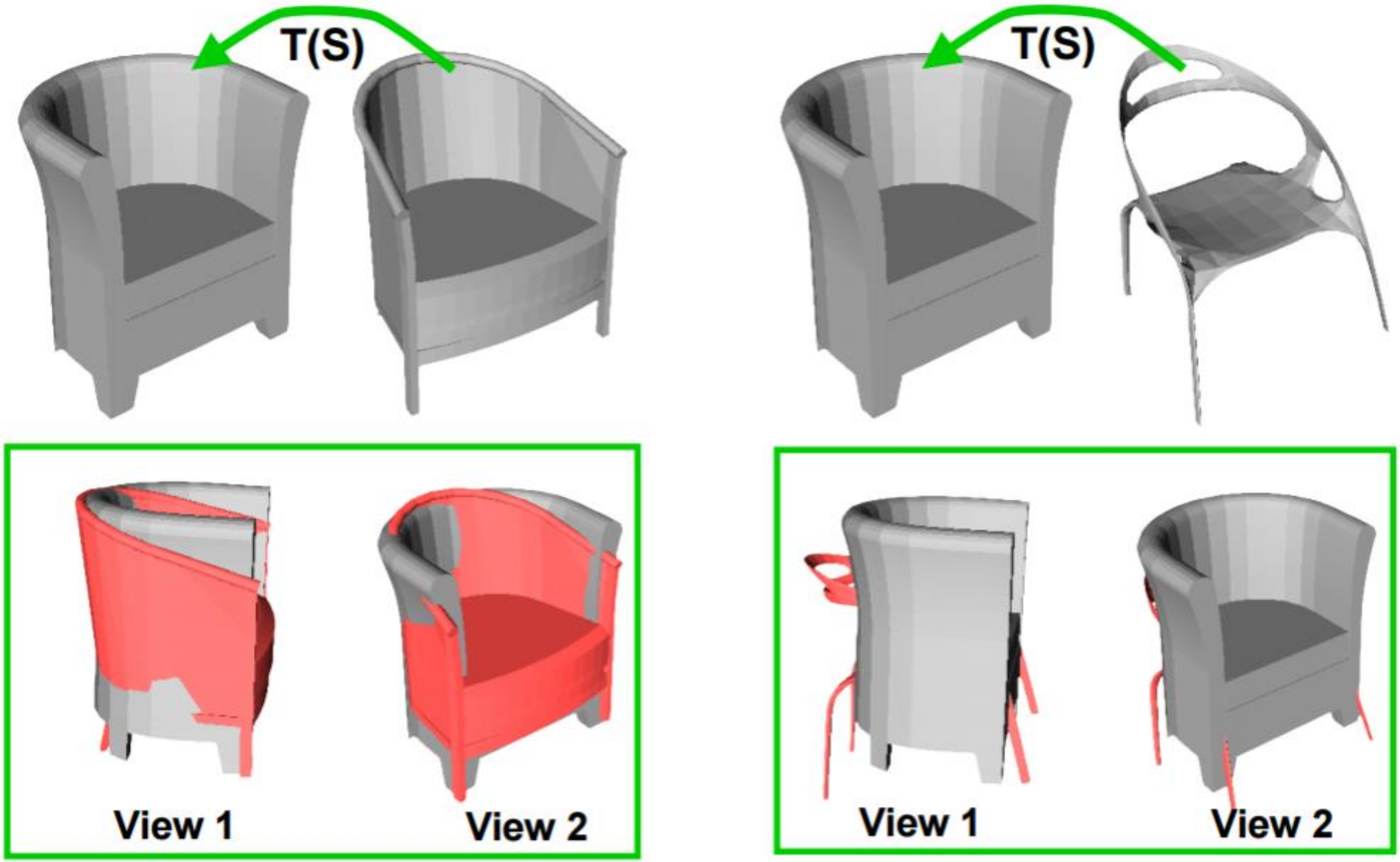


Object completion



Protein docking

Rigid Shape Matching – Search a transformation

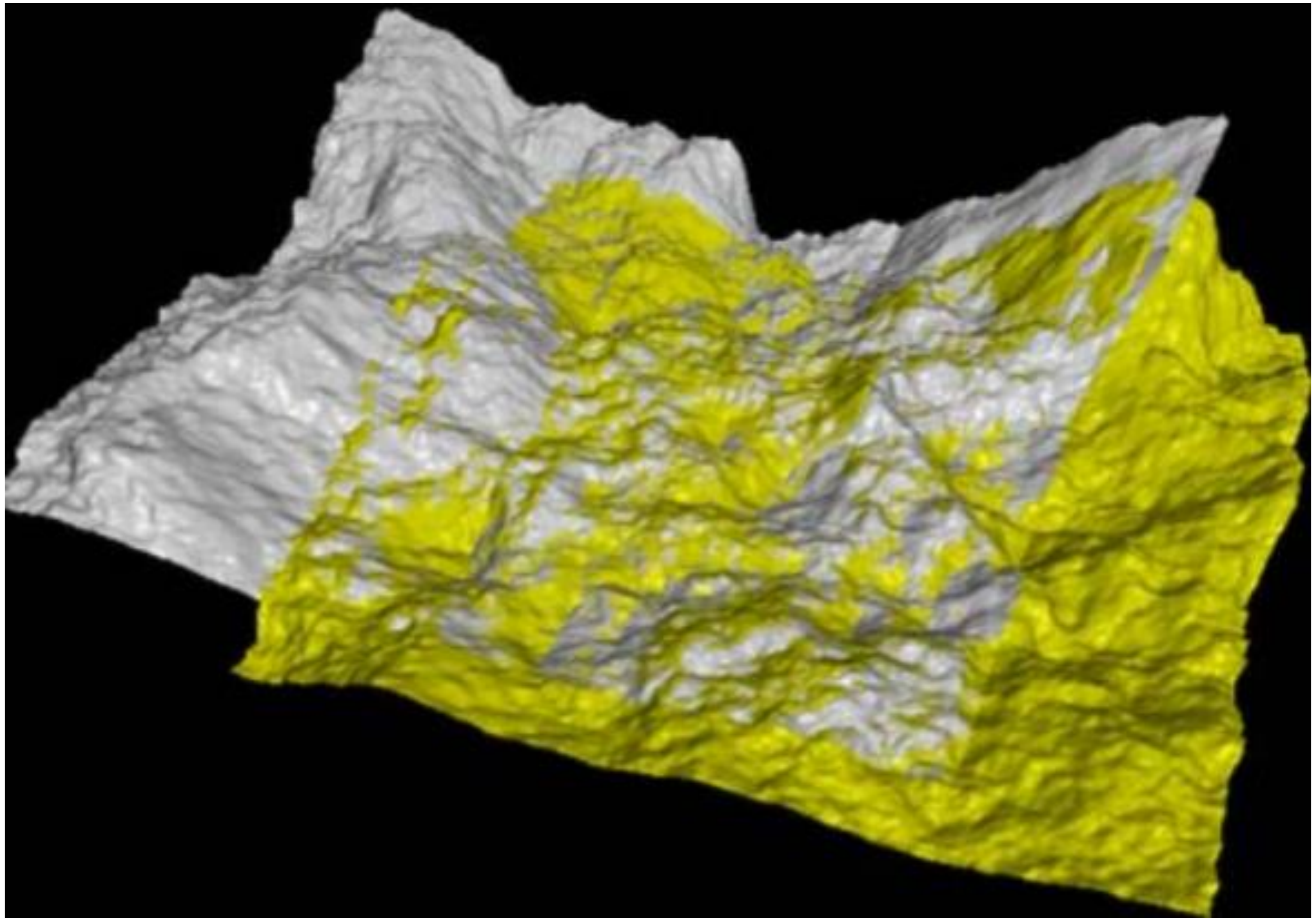


Correspondence Problem Classification

- How many meshes?
  - Two: Pairwise registration
  - More than two: multi-view registration
- Initial registration available?
  - Yes: Local optimization methods
  - No: Global methods
- Class of transformations?
  - Rotation and translation: Rigid-body
  - Non-rigid deformations

Pairwise Rigid Registration Goal

- Align two partially overlapping meshes given initial guess for relative transform



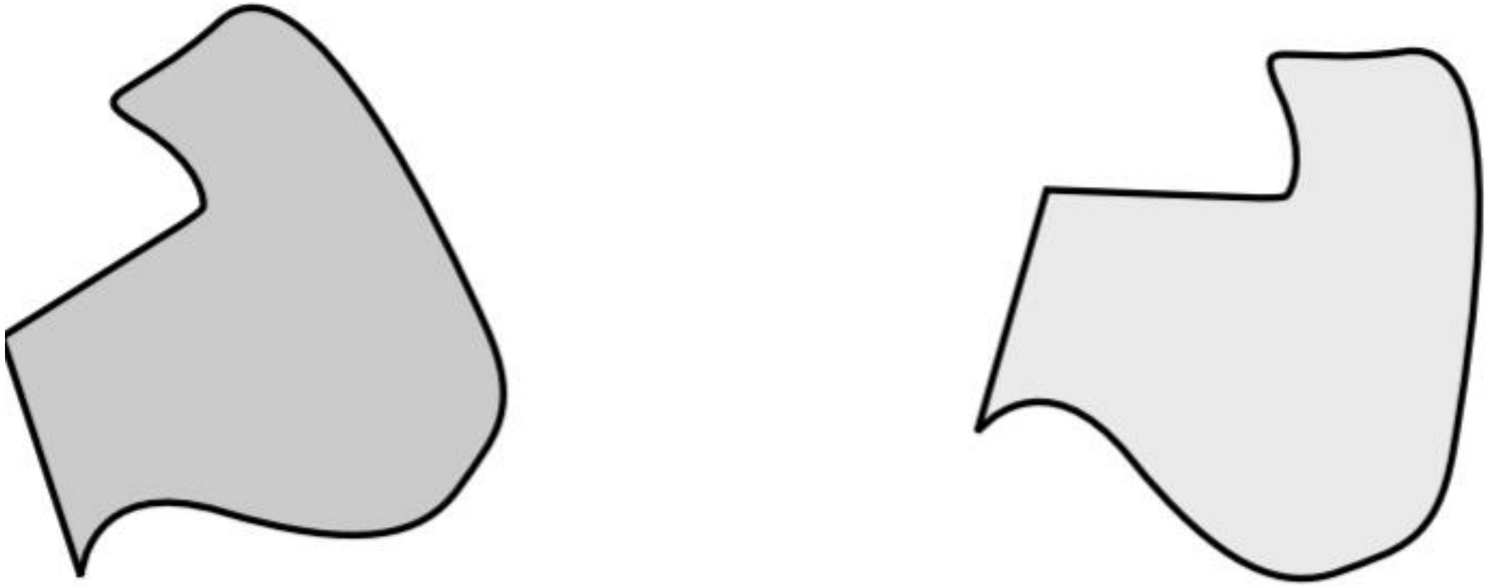
Outline

- Basic ICP: Iterative Closest Points
- Classification of ICP variants
  - Faster alignment
  - Better robustness
- Global Registration

ICP: Iterative Closest Points

Objective

M_1 M_2

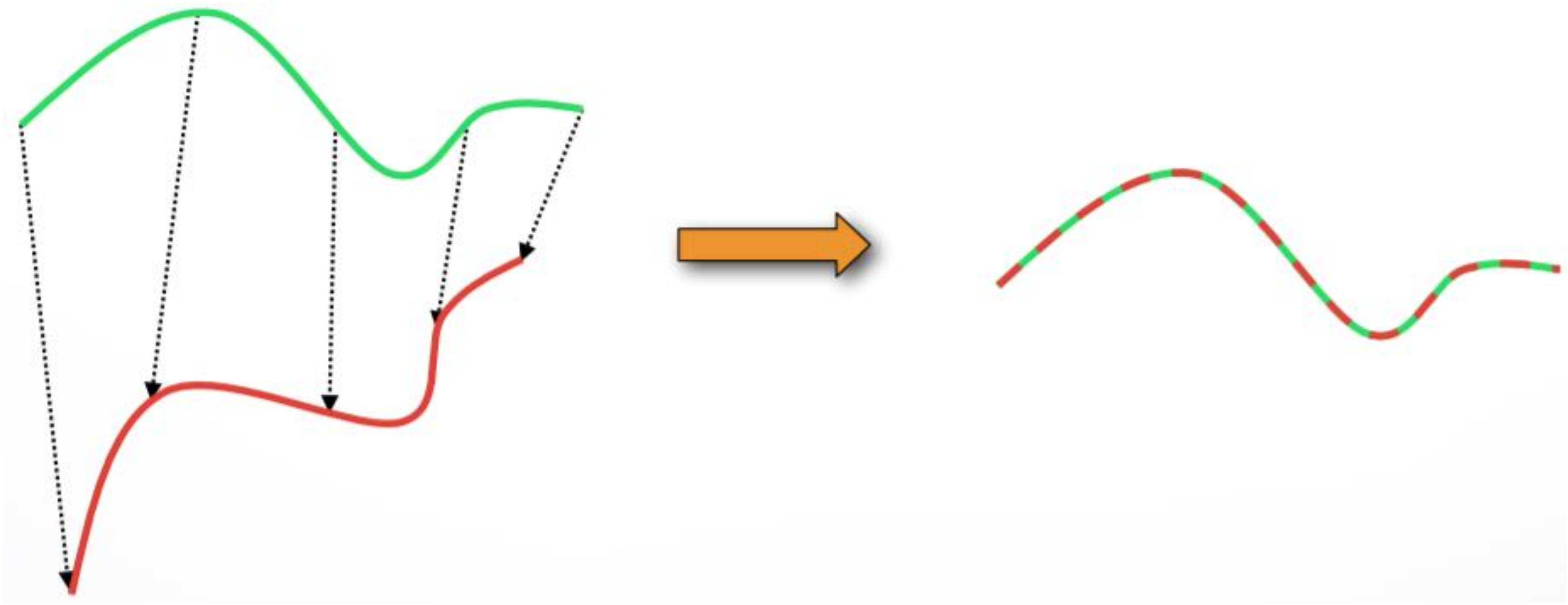


$$M_1 \approx T(M_2)$$

T : **translation + rotation**

Aligning 3D Data

- If correct **correspondences** are **known**, can find correct relative rotation/translation

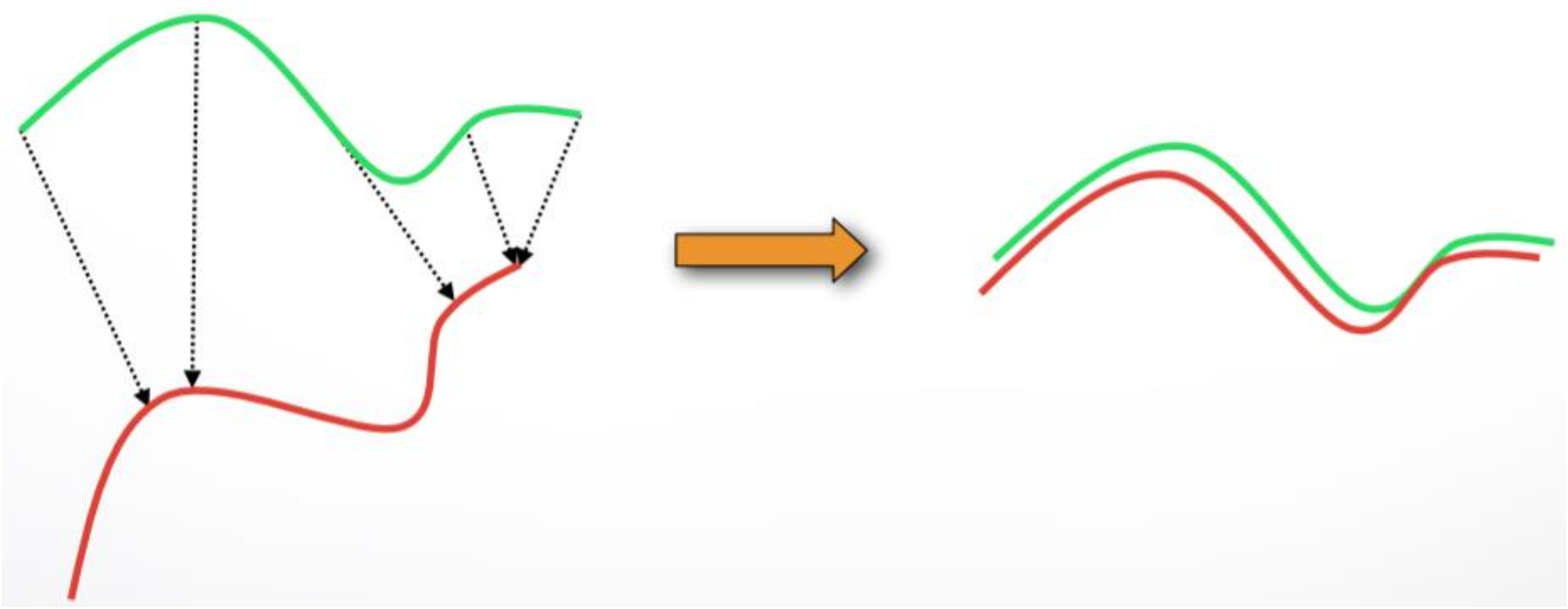


Aligning 3D Data

- How to find correspondences: User input? Feature detection? Signatures?

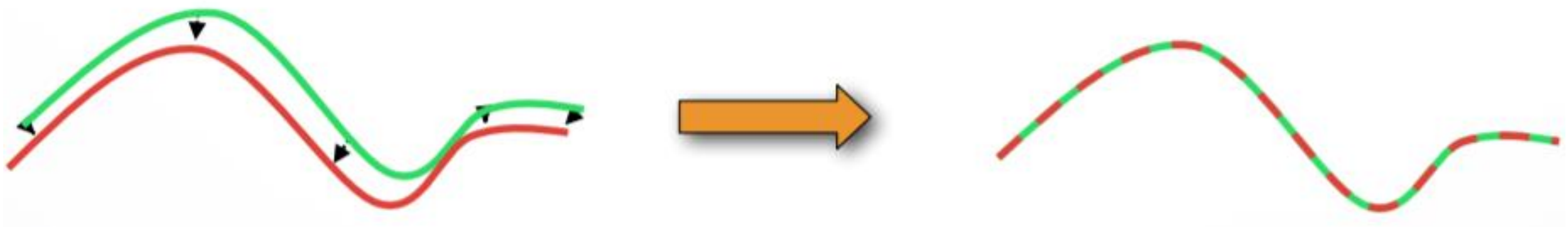
a seemingly-radical guess

- Alternatives: assume closest points correspond



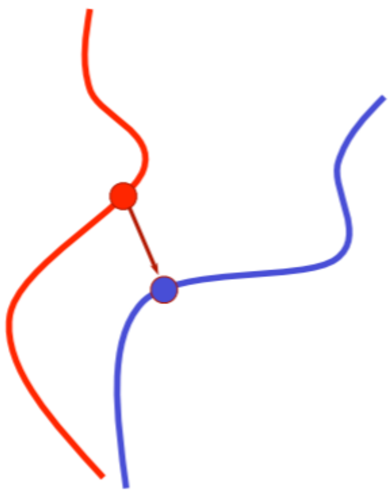
Aligning 3D Data

- … and iterate to find alignment
  - Iterative Closest Points (ICP) [Besl & Mckay]
- Converges if starting position “close enough”



Basic ICP

- **Select** e.g., 1000 random points



- **Match** each to closest point on other scan
- **Reject** pairs with distance > k times median

- Construct **error function**:

$$E = \sum \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|^2$$

- Minimize (closed form solution in [Horn 87]) (Also this note: Least-Squares Rigid Motion Using SVD by Olga Sorkine)

Shape Matching: Translation first. why?

Assume R is fixed and denote $F(\mathbf{t}) = \sum_{i=1}^{n} w_i \left\|(R\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\right\|^2$. We can find the optimal translation by taking the derivative of F w.r.t. $\mathbf{t}$ and searching for its roots:

$$\begin{aligned} 0 &= \frac{\partial F}{\partial \mathbf{t}} = \sum_{i=1}^{n} 2w_i \left(R\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\right) = \\ &= 2\mathbf{t}\left(\sum_{i=1}^{n} w_i\right) + 2R\left(\sum_{i=1}^{n} w_i \mathbf{p}_i\right) - 2\sum_{i=1}^{n} w_i \mathbf{q}_i. \end{aligned} \tag{2}$$

Denote

$$\bar{\mathbf{p}} = \frac{\sum_{i=1}^{n} w_i \mathbf{p}_i}{\sum_{i=1}^{n} w_i}, \quad \bar{\mathbf{q}} = \frac{\sum_{i=1}^{n} w_i \mathbf{q}_i}{\sum_{i=1}^{n} w_i}. \tag{3}$$

By rearranging the terms of (2) we get

$$\mathbf{t} = \bar{\mathbf{q}} - R\bar{\mathbf{p}}. \tag{4}$$

In other words, the optimal translation $\mathbf{t}$ maps the transformed weighted centroid of P to the weighted centroid of Q. Let us plug the optimal $\mathbf{t}$ into our objective function:

$$\mathbf{t} = \bar{\mathbf{q}} - R\bar{\mathbf{p}}. \tag{4}$$

In other words, the optimal translation $\mathbf{t}$ maps the transformed weighted centroid of P to the weighted centroid of Q. Let us plug the optimal $\mathbf{t}$ into our objective function:

$$\sum_{i=1}^{n} w_i \left\|(R\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\right\|^2 = \sum_{i=1}^{n} w_i \left\|R\mathbf{p}_i + \bar{\mathbf{q}} - R\bar{\mathbf{p}} - \mathbf{q}_i\right\|^2 = \tag{5}$$

$$= \sum_{i=1}^{n} w_i \left\|R(\mathbf{p}_i - \bar{\mathbf{p}}) - (\mathbf{q}_i - \bar{\mathbf{q}})\right\|^2. \tag{6}$$

We can thus concentrate on computing the rotation R by restating the problem such that the translation would be zero:

$$\mathbf{x}_i := \mathbf{p}_i - \bar{\mathbf{p}}, \quad \mathbf{y}_i := \mathbf{q}_i - \bar{\mathbf{q}}. \tag{7}$$

So we look for the optimal rotation R such that

$$R = \operatorname*{argmin}_R \sum_{i=1}^{n} w_i \left\|R\mathbf{x}_i - \mathbf{y}_i\right\|^2. \tag{8}$$

Shape Matching: Rotation

- Approximate nonlinear rotation by general matrix

$$\min_{\mathbf{R}} \sum_i \|\hat{\mathbf{p}}_i - \mathbf{R}\hat{\mathbf{q}}_i\|^2 \quad \rightarrow \quad \min_{\mathbf{A}} \sum_i \|\hat{\mathbf{p}}_i - \mathbf{A}\hat{\mathbf{q}}_i\|^2$$

- The least squares linear transformation is

$$\mathbf{A} = \left(\sum_{i=1}^{m} \hat{\mathbf{p}}_i \hat{\mathbf{q}}_i^T\right) \cdot \left(\sum_{i=1}^{m} \hat{\mathbf{q}}_i \hat{\mathbf{q}}_i^T\right)^{-1} \in \mathbb{R}^{3\times 3}$$

- SVD & Polar decomposition extracts rotation from $\mathbf{A}$

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \quad \rightarrow \quad \mathbf{R} = \mathbf{U}\mathbf{V}^T$$

Iterative Closest Points (ICP)-The classic version

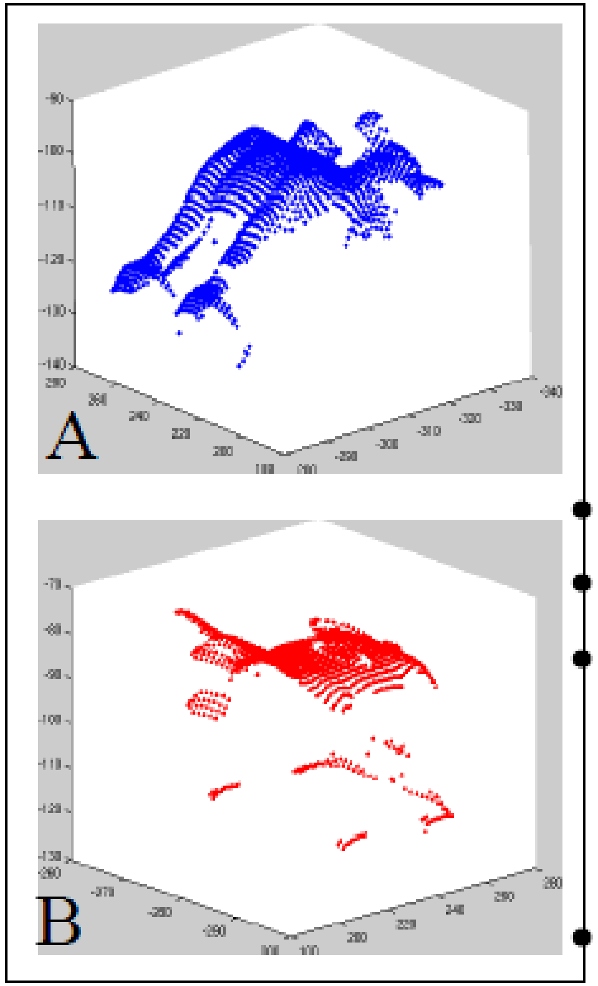
- Given two point sets P and Q, establish correspondence between each point at P and its closest point at Q.
- Iterate between two steps:
  1. Use the estimated correspondence to estimate the best rigid transformation and align the shapes.
  2. Derive new correspondence from the new alignment.
- Stop when there is no significant change.
- **The initial alignment is critical!**

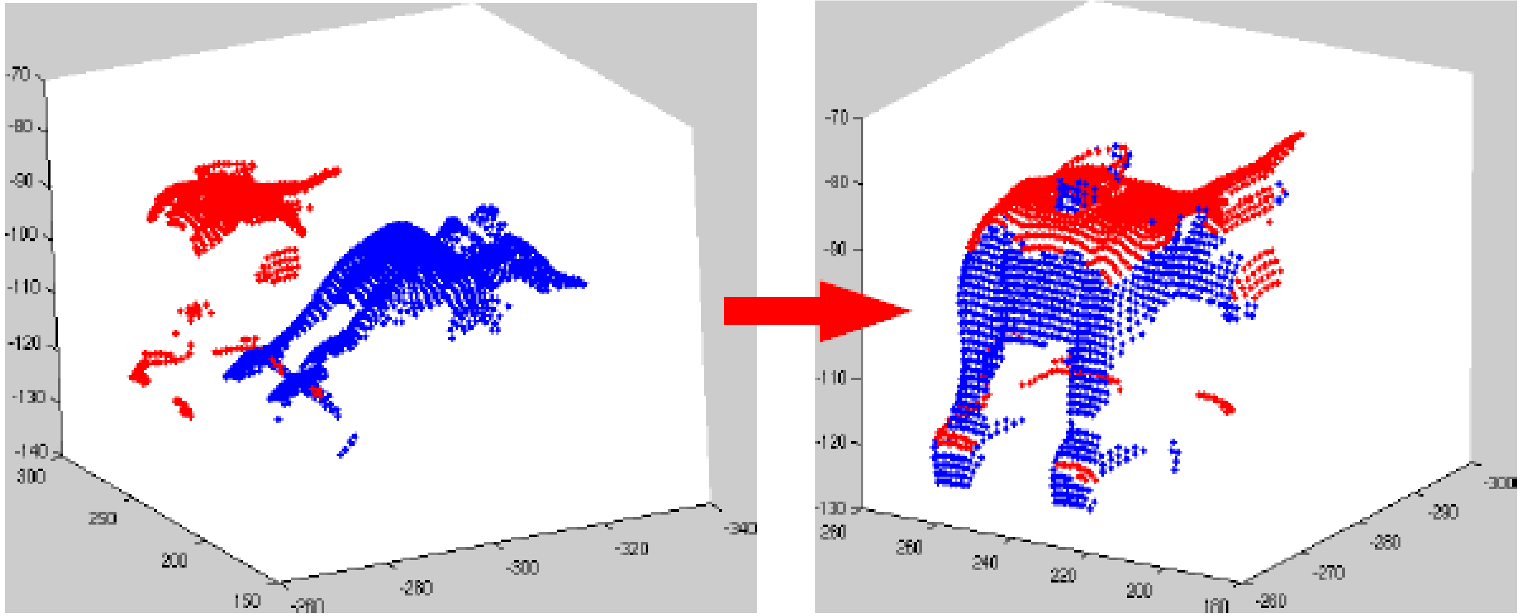
Suggestions:

1. Use a set of set of feature points/ user markers.
2. Use PCA to align the shapes.
3. Use the symmetry axes to align the shapes.

Registration – ICP algorithm

- Example : registration of 3D model parts (toy cow)





Input: point clouds acquired from non-aligned viewpoints (from 3D range scanner) & initial estimation of registration

Output: Transformation

ICP Variants

ICP Variants

Variants on the following stages of ICP have been proposed:

1. **Selecting source points (from one or both meshes)**
2. **Matching to points in the other mesh**
3. **Weighting the correspondences**
4. **Rejecting certain (outlier) point pairs**
5. **Assigning an error metric to the current transform**
6. **Minimizing the error metric w.r.t. transformation**

ICP Variants

- Can analyze various aspects of performance:
  - Speed
  - Stability
  - Tolerance of noise and/or outliers
  - Maximum initial misalignment
- Comparisons of many variants in
  - [Rusinkiewicz & Levoy, 3DIM 2001]: Efficient Variants of the ICP Algorithm

ICP Variants

1. Selecting source points (from one or both meshes)

2. Matching to points in the other mesh

3. Weighting the correspondences

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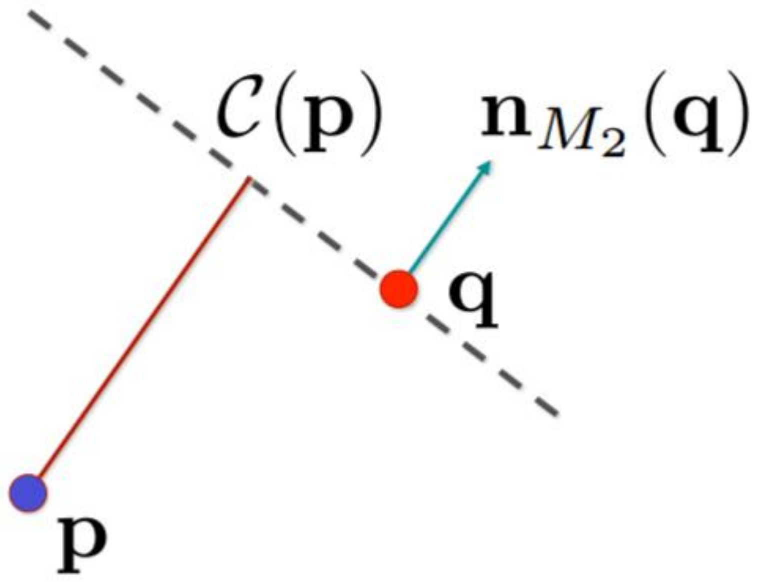
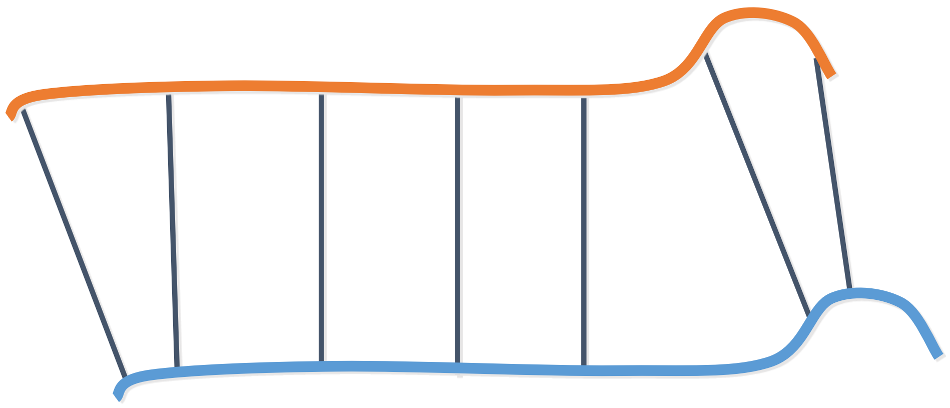
5. **Assigning an error metric to the current transform**

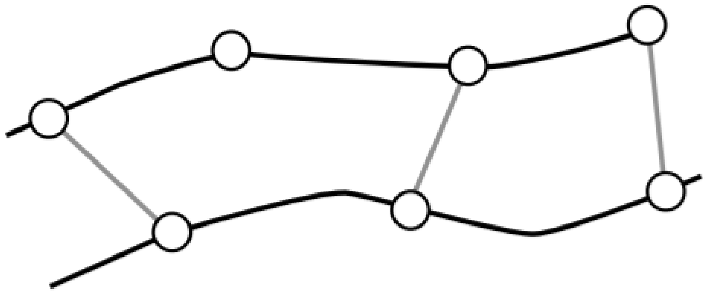
6. Minimizing the error metric w.r.t. transformation

Point-to-Plane Error Metric

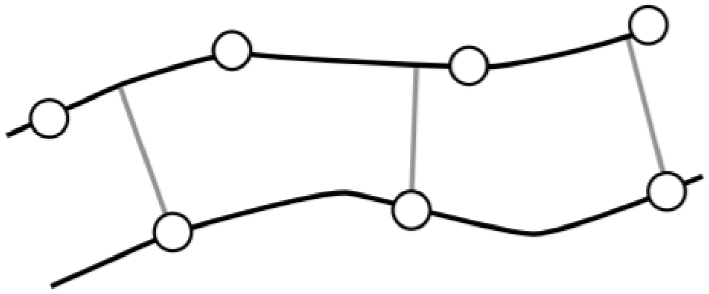
- Using point-to-plane distance instead of point-to-point
- allows flat regions slide along each other [Chen & Medioni 91]







point-to-point



point-to-plane

Point-to-Plane Error Metric

- Error function:

$$E = \sum \left((\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i)^\top \mathbf{n_i} \right)^2$$

where $\mathbf{R}$ is a rotation matrix, $\mathbf{t}$ is a translation vector

- Linearize (i.e. assume that $\sin\theta \approx \theta$, $\cos\theta \approx 1$):

$$E \approx \sum \left((\mathbf{p}_i - \mathbf{q}_i)^\top \mathbf{n}_i \right) \; + \mathbf{r}^\top (\mathbf{p}_i \times \mathbf{n}_i) + \mathbf{t}^\top \mathbf{n}_i)^2 \qquad \mathbf{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

- Result: overconstrained linear system

Point-to-Plane Error Metric

- Overconstrained linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} \leftarrow & \mathbf{p}_1 \times \mathbf{n}_1 & \rightarrow & \leftarrow & \mathbf{n}_1 & \rightarrow \\ \leftarrow & \mathbf{p}_2 \times \mathbf{n}_1 & \rightarrow & \leftarrow & \mathbf{n}_2 & \rightarrow \\ & \vdots & & & \vdots & \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} r_x \\ r_y \\ r_z \\ t_x \\ t_y \\ t_z \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -(\mathbf{p}_1 - \mathbf{q}_1)^\top \mathbf{n}_1 \\ -(\mathbf{p}_2 - \mathbf{q}_2)^\top \mathbf{n}_2 \\ \vdots \end{bmatrix}$$

- Solve using least squares

$$\mathbf{A}^\top \mathbf{A}\mathbf{x} = \mathbf{A}^\top \mathbf{b}$$

$$\mathbf{x} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b}$$

Rotation matrix from axis and angle

For some applications, it is helpful to be able to make a rotation with a given axis. Given a unit vector $\mathbf{u} = (u_x, u_y, u_z)$, where $u_x^2 + u_y^2 + u_z^2 = 1$, the matrix for a rotation by an angle of θ about an axis in the direction of $\mathbf{u}$ is[3]

$$R = \begin{bmatrix} \cos\theta + u_x^2\left(1-\cos\theta\right) & u_x u_y\left(1-\cos\theta\right) - u_z\sin\theta & u_x u_z\left(1-\cos\theta\right) + u_y\sin\theta \\ u_y u_x\left(1-\cos\theta\right) + u_z\sin\theta & \cos\theta + u_y^2\left(1-\cos\theta\right) & u_y u_z\left(1-\cos\theta\right) - u_x\sin\theta \\ u_z u_x\left(1-\cos\theta\right) - u_y\sin\theta & u_z u_y\left(1-\cos\theta\right) + u_x\sin\theta & \cos\theta + u_z^2\left(1-\cos\theta\right) \end{bmatrix}$$

A derivation of this matrix from first principles can be found in section 9.2. here.[4]

This can be written more concisely as

$$R = \cos\theta\mathbf{I} + \sin\theta[\mathbf{u}]_{\times} + (1-\cos\theta)\mathbf{u}\otimes\mathbf{u}\,,$$

where $[\mathbf{u}]_{\times}$ is the cross product matrix of **u**, $\otimes$ is the tensor product and *I* is the Identity matrix, or alternatively as $R_{jk} = \begin{cases} \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\left(2u_j^2 - 1\right), & \text{if } j = k \\ 2u_j u_k \sin^2\frac{\theta}{2} + \epsilon_{jkl} u_l \sin\theta, & \text{if } j \neq k \end{cases}$

where ϵ_{jkl} is the Levi-Civita symbol with $\epsilon_{123} = 1$. This is a matrix form of Rodrigues' rotation formula, (or the equivalent, differently parameterized Euler–Rodrigues formula) with[5]

$$\mathbf{u}\otimes\mathbf{u} = \mathbf{u}\mathbf{u}^T = \begin{bmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 \end{bmatrix}, \qquad [\mathbf{u}]_{\times} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}.$$

$$E = \sum \left((\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i)^\top \mathbf{n}_i \right)^2 \Longrightarrow E \approx \sum \left((\mathbf{p}_i - \mathbf{q}_i)^\top \mathbf{n}_i \right) + \mathbf{r}^\top (\mathbf{p}_i \times \mathbf{n}_i) + \mathbf{t}^\top \mathbf{n}_i)^2$$

$$R = cos\theta I + sin\theta [r]_\times + (1 - cos\theta I)\ r(r)^T$$

- Key: $(Rp_i)^T n_i = \left((I + [r]_\times)p_i\right)^T n_i$
- $([r]_\times p_i)^T n_i \ ?= \ (r)^T(p_i \times n_i)$

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- $(r \times p_i)^T n_i \ ?= \ (r)^T(p_i \times n_i)$
- $(r \times p_i)^T n_i = n_i \cdot (r \times p_i);\ \ (r)^T(p_i \times n_i) = r \cdot (p_i \times n_i)$
- $n_i \cdot (r \times p_i) ?= r \cdot (p_i \times n_i)$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

Improving ICP Stability

- Closest compatible point
- Stable sampling

ICP Variants

1. Selecting source points (from one or both meshes)
2. **Matching to points in the other mesh**
3. Weighting the correspondences
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Closest Compatible Point

- Closest point often a bad approximation to corresponding point
- Can improve matching effectiveness by restricting match to compatible points
  - Compatibility of colors [Godin et al. ’94]
  - Compatibility of normals [Pulli ’99]
  - Other possibilities: curvature, higher-order derivatives, and other local features (remember: data is noisy)

ICP Variants

1. **Selecting source points (from one or both meshes)**

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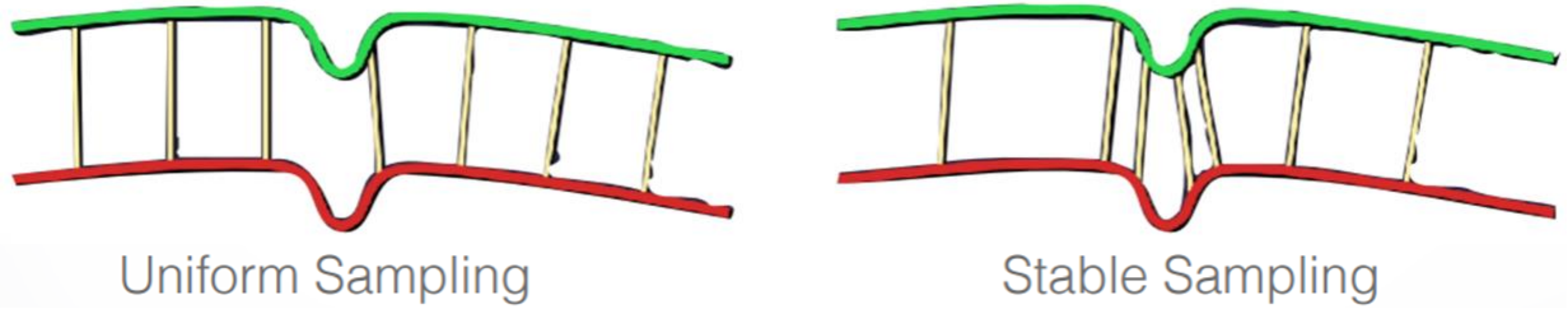
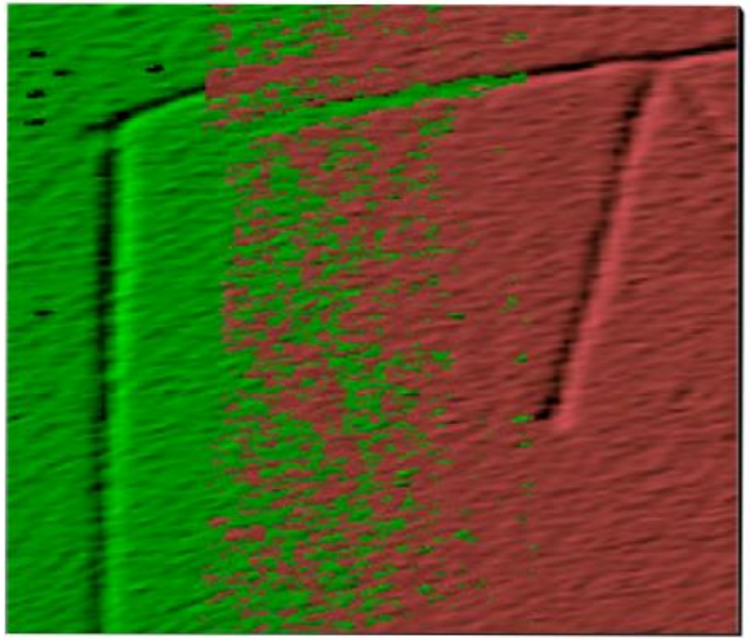
Selecting Source Points

- Use all points
- Uniform subsampling
- Random sampling
- Stable sampling [Gelfand et al. 2003]
  - Select samples that constrain all degrees of freedom of the rigid-body transformation

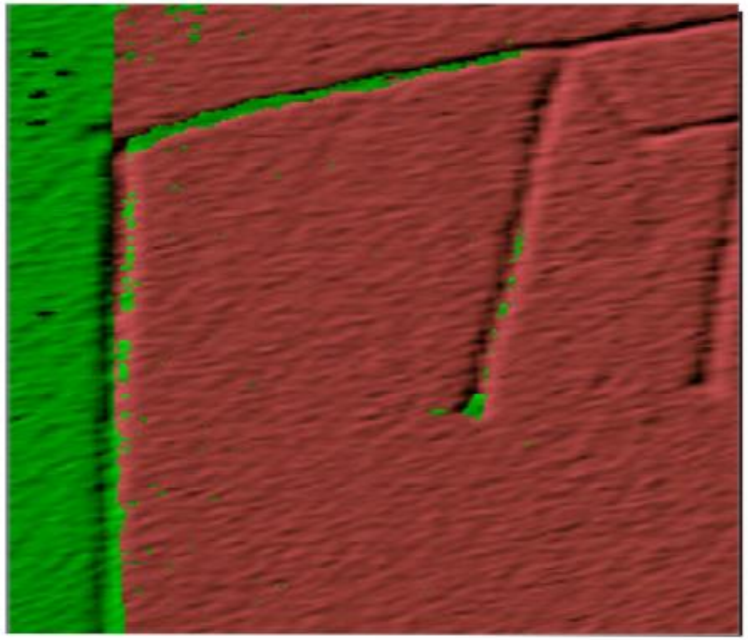


Sample Selection

- Simpler variant: normal-space sampling
  - select points with uniform distribution of normals
  - Pro: faster, does not require eigenanalysis
  - Con: only constrains translation
- Stability-based or normal-space sampling important for smooth areas with small features



Random Sampling



Normal-space Sampling

Selection vs. Weighting

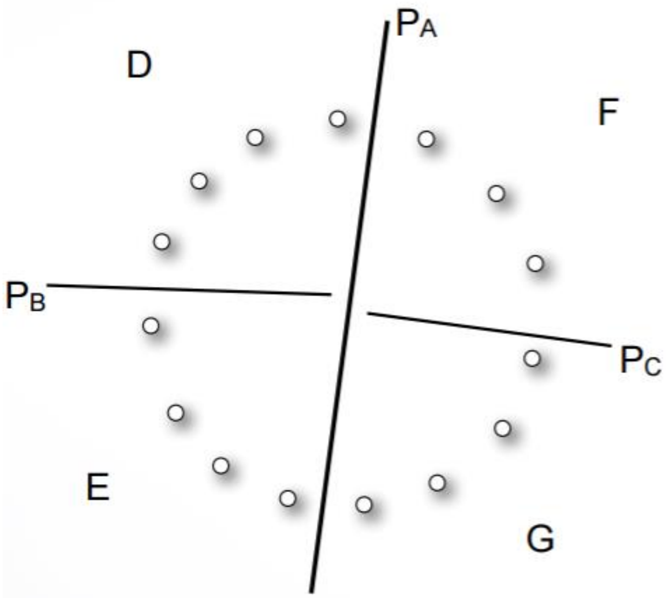
- Could achieve same effect with weighting
- Hard to ensure enough samples in features except at high sampling rates
- However, have to build special data structure
- Preprocessing / run-time cost tradeoff

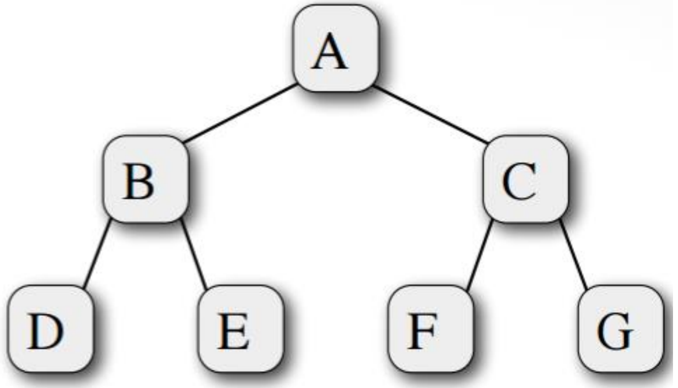
ICP Variants

- Can analyze various aspects of performance:
  - **Speed**
  - Stability
  - Tolerance of noise and/or outliers
  - Maximum initial misalignment
- Comparisons of many variants in
  - [Rusinkiewicz & Levoy, 3DIM 2001]

Closest Point Search

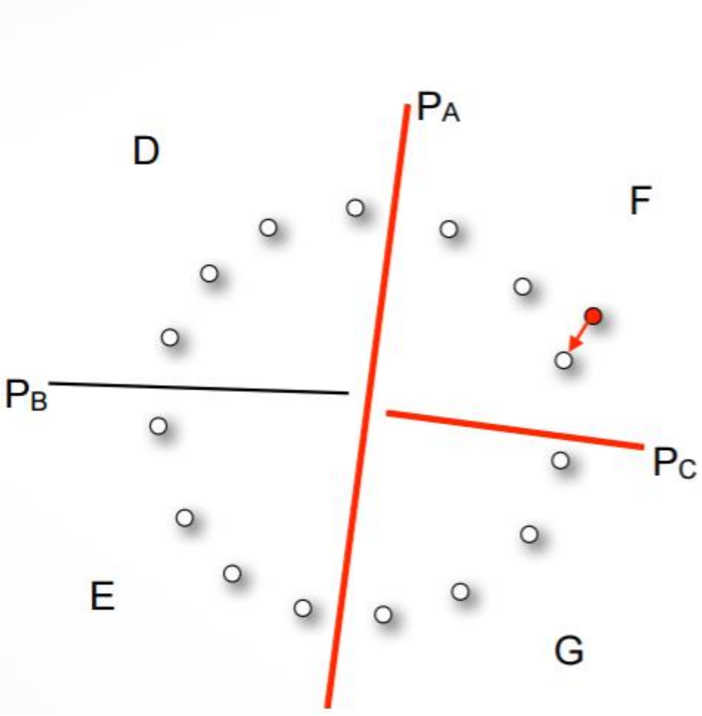
- most expensive stage of the ICP algorithm
  - Brute force search – O(n)

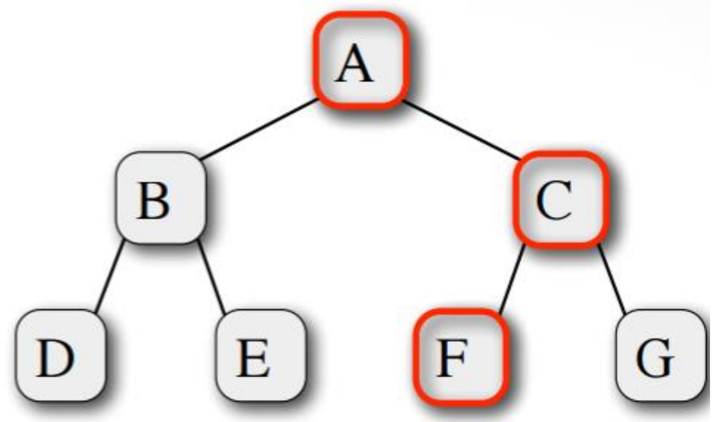




- Use Hierarchical BSP tree
  - Binary space partitioning tree (general kD-tree)
  - Recursively partition 3D space by planes
  - Tree should be balanced, put plane at median
  - log(n) tree levels, complexity O(nlog n)

BSP Closest Point Search

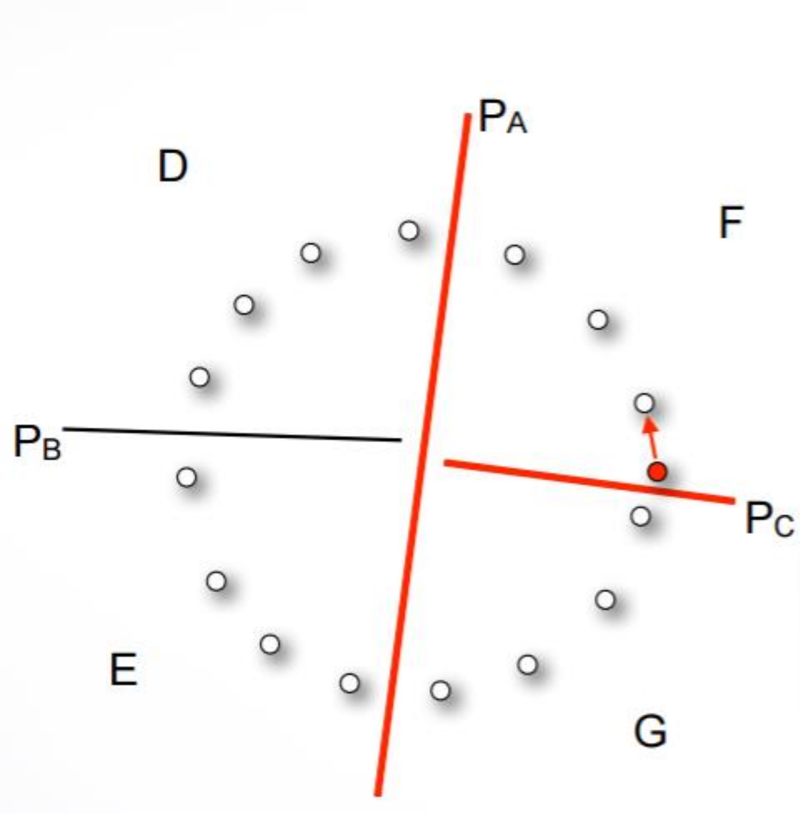


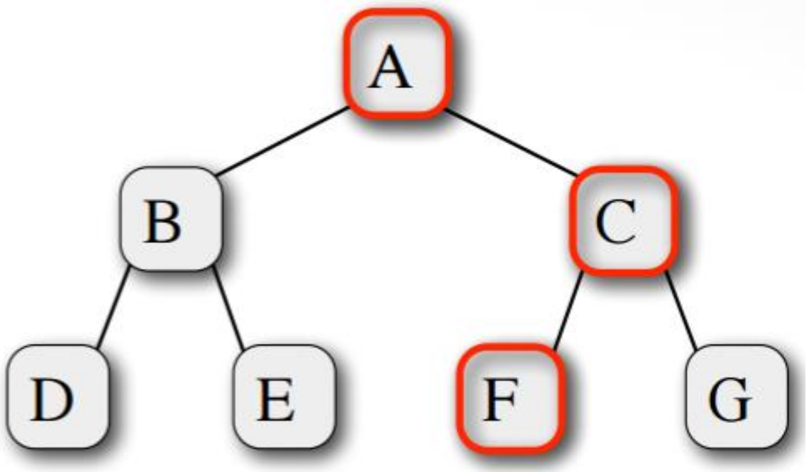


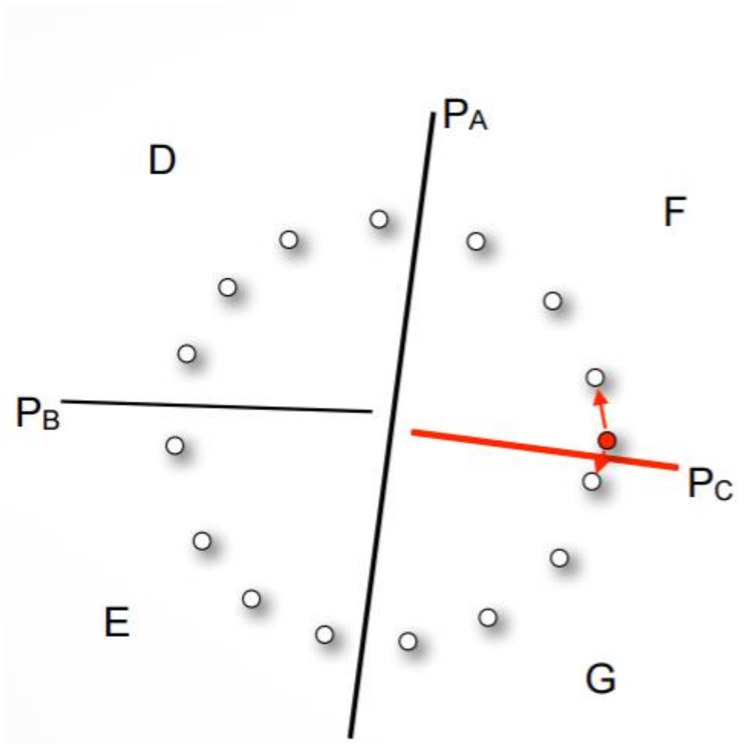
Implement BSPNode::dist() with the following info:

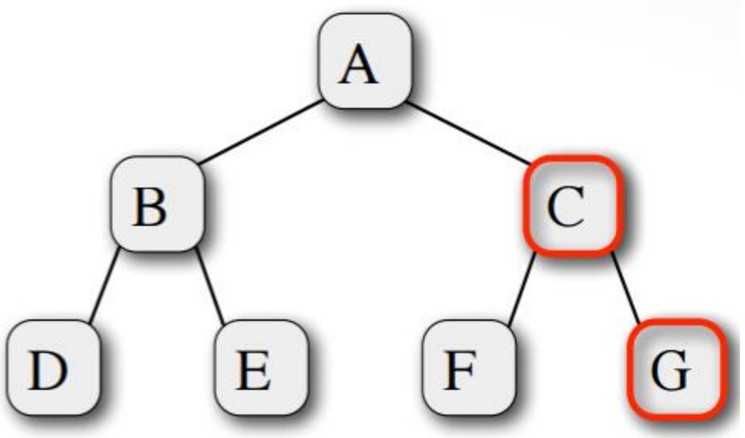
BSPNode{
BSPNode left_child, right_child;
vector<Point> p; // p[i] is the ith point
...
bool leaf_node();
void dist(Point x, Scalar& dmin): x: the query point, dmin: min distance between x and its closest point in the Tree.
};
float dist = dist_to_plane(x)

How to handle this?









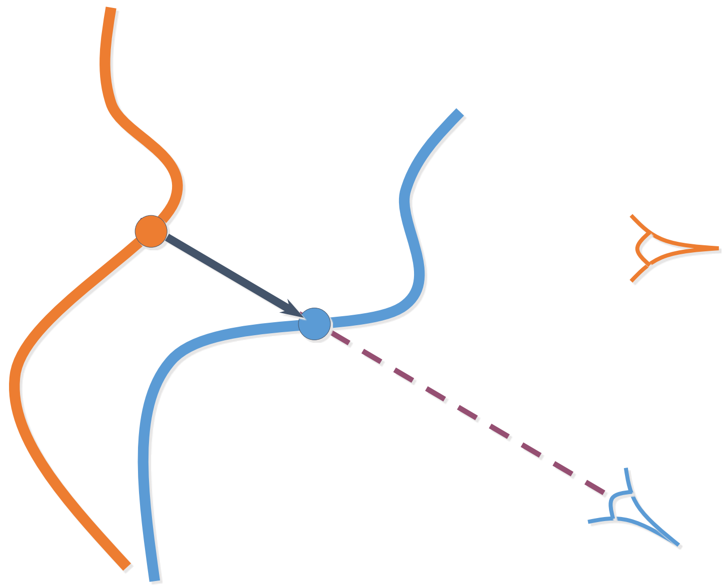
BSP Closest Point Search

```
BSPNode::dist(Point x, Scalar& dmin)
{
  if (leaf_node())
    for each sample point p[i]
      dmin = min(dmin, dist(x, p[i]));

  else
  {
    d = dist_to_plane(x);
    if (d < 0)
    {
      left_child->dist(x, dmin);
      if (|d| < dmin) right_child->dist(x, dmin);
    }
    else
    {
      right_child->dist(x, dmin);
      if (|d| < dmin) left_child->dist(x, dmin);
    }
  }
}
```

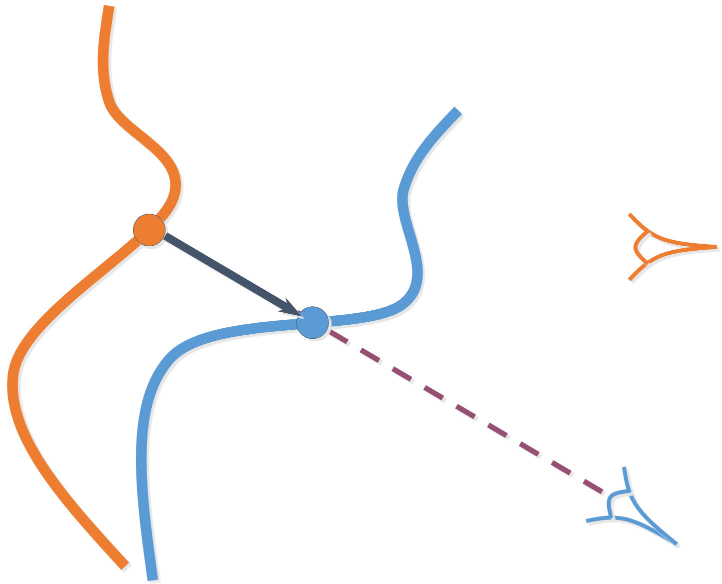
Closest Point Search

- most expensive stage of the ICP algorithm
  - Brute force search – O(n)
  - Binary space partitioning tree (general kD-tree) - O(nlog n)



Projection to Find Correspondence

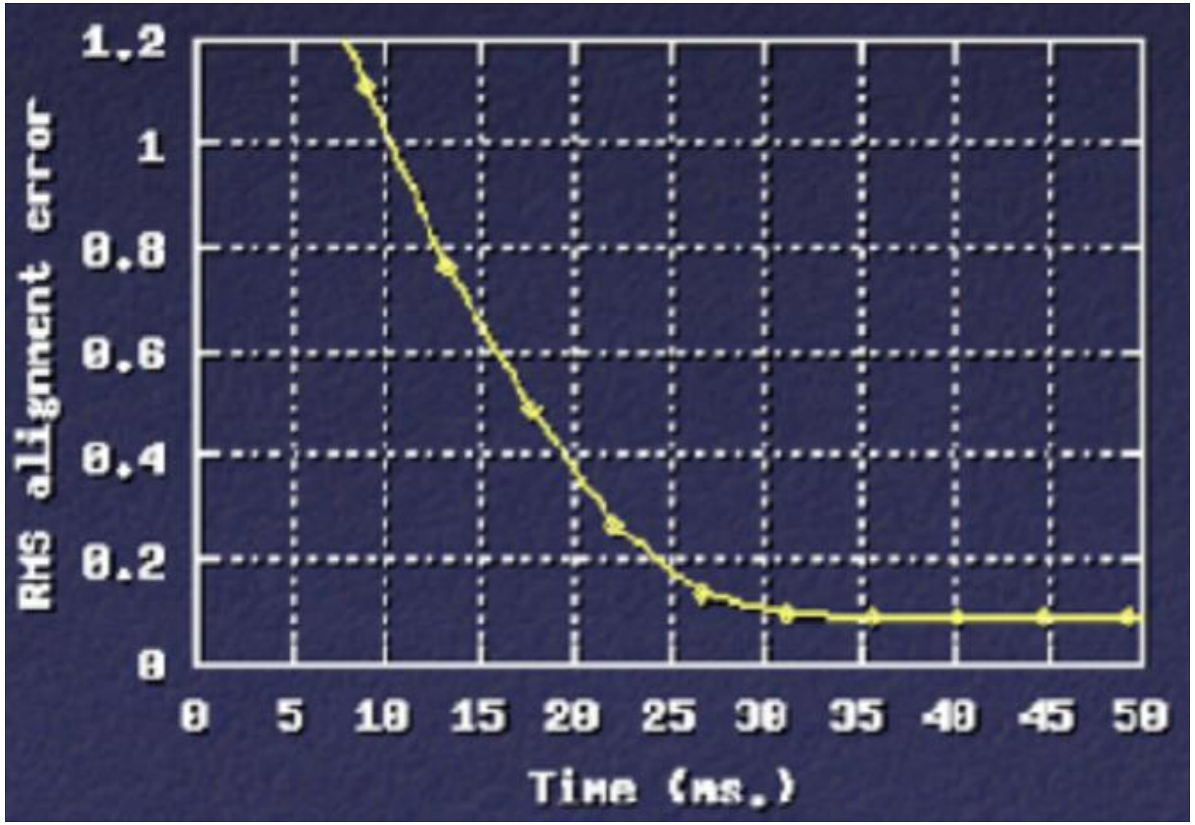
- Idea: use a simpler algorithm to find correspondences
- For range images, can simply project point by “reverse calibration” [Blais 95]
  - Constant-time
  - Does not require precomputing a spatial data structure



$$\tilde{\boldsymbol{x}}_s = \underset{\substack{\uparrow \\ \text{Calibration Matrix}}}{\boldsymbol{K}} \left[\, \boldsymbol{R} \,\middle|\, \boldsymbol{t} \,\right] \boldsymbol{p}_w = \underset{\substack{\uparrow \\ \text{Camera Matrix}}}{\boldsymbol{P}} \boldsymbol{p}_w$$

Projection-Based Matching

- Slightly worse performance per iteration
- Each iteration is one to two orders of magnitude faster than closest point
- Result: can align two range images in a few milliseconds, vs. a few seconds

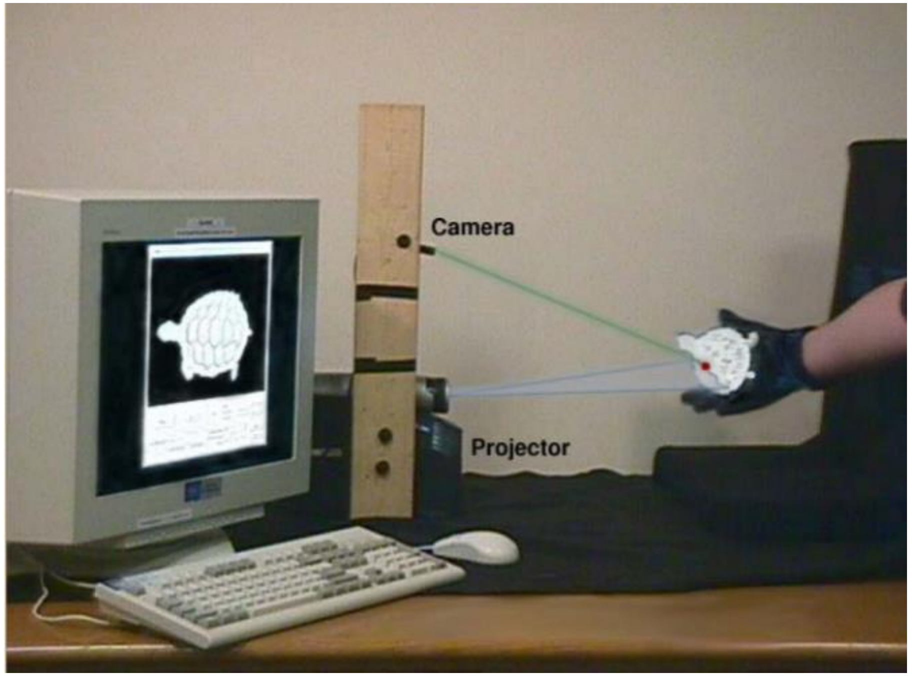


Applications

- Given:
  - A scanner that returns range images in real time
  - Fast ICP
  - Real-time merging and rendering

- Result: 3D model acquisition
  - Tight feedback loop with user
  - Can see and fill holes while scanning

Scanner Layout



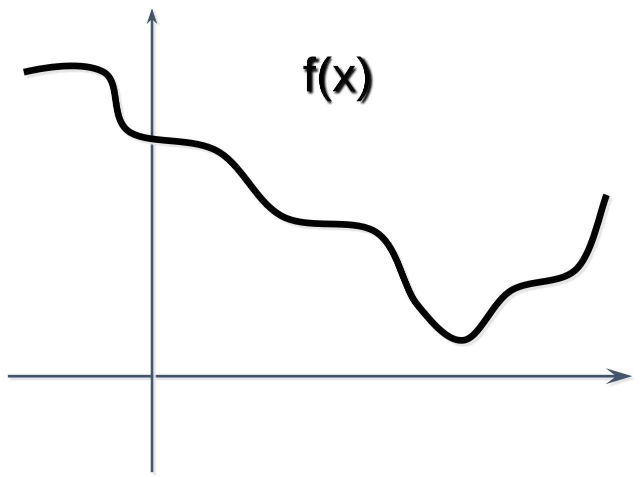
Artec Group

[Newcombe et al. '11]
KinectFusion

What Does ICP Do?

- Two ways of thinking about ICP:
  - Solving the correspondence problem
  - Minimizing point-to-surface squared distance

- ICP is like Newton's method on an approximation of the distance function



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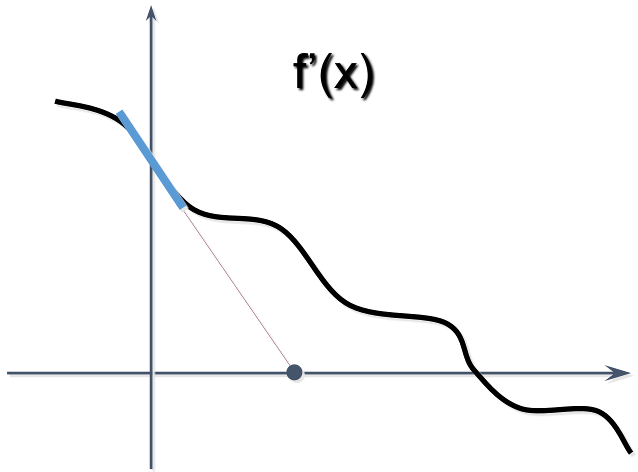
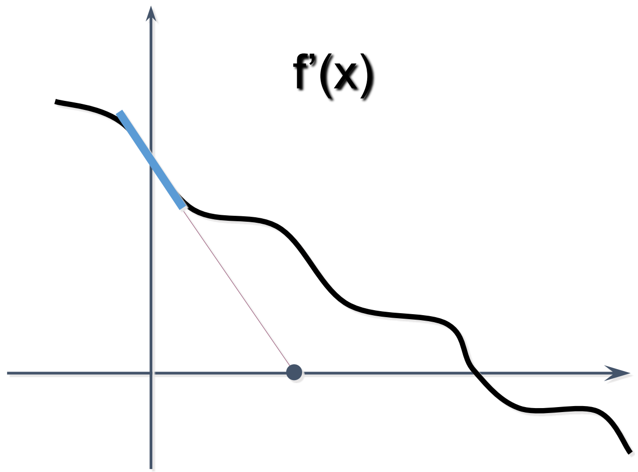


What Does ICP Do?

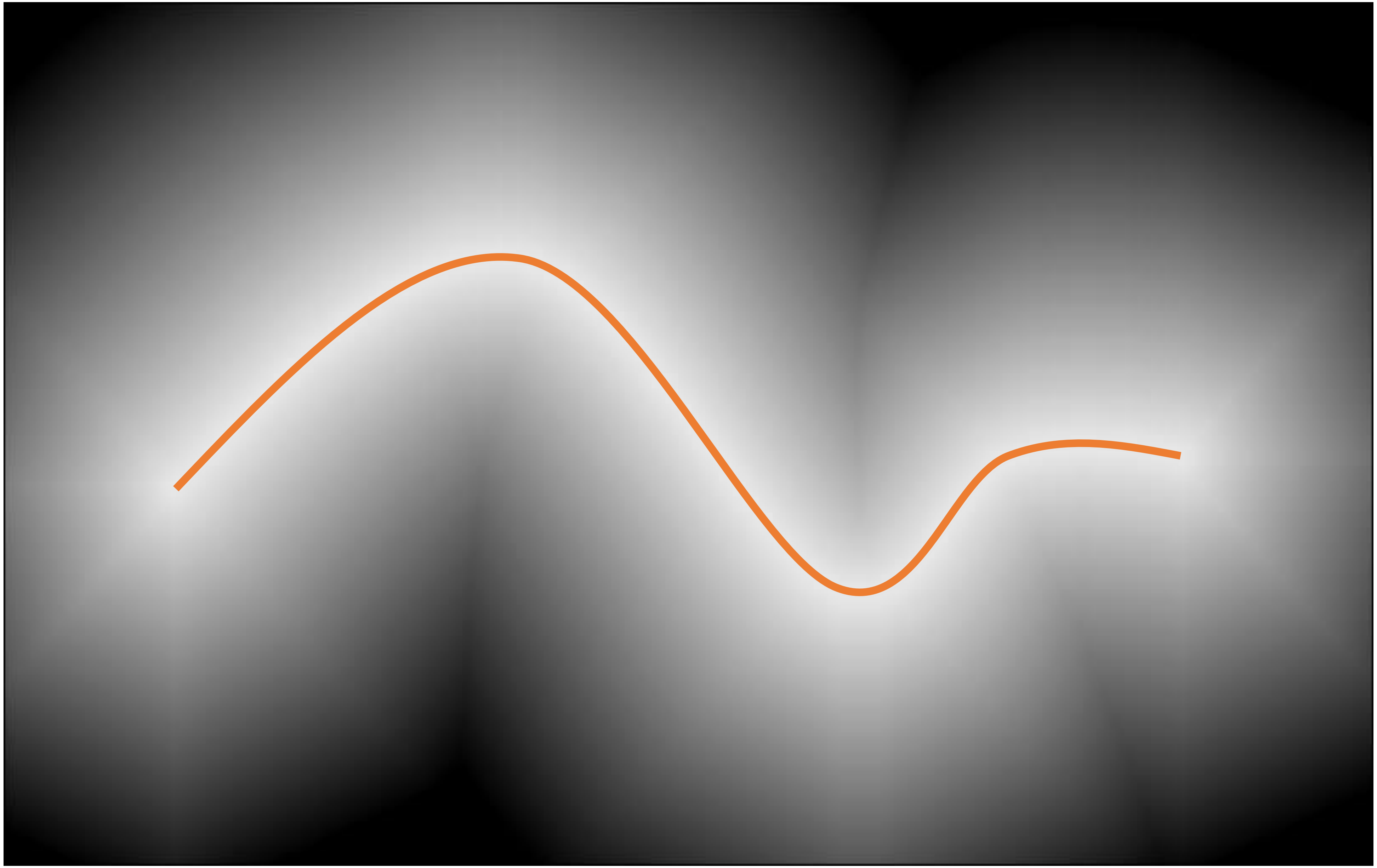
- Two ways of thinking about ICP:
  - Solving the correspondence problem
  - Minimizing point-to-surface squared distance

- ICP is like Newton's method on an approximation of the distance function
  - ICP variants affect shape of global error function

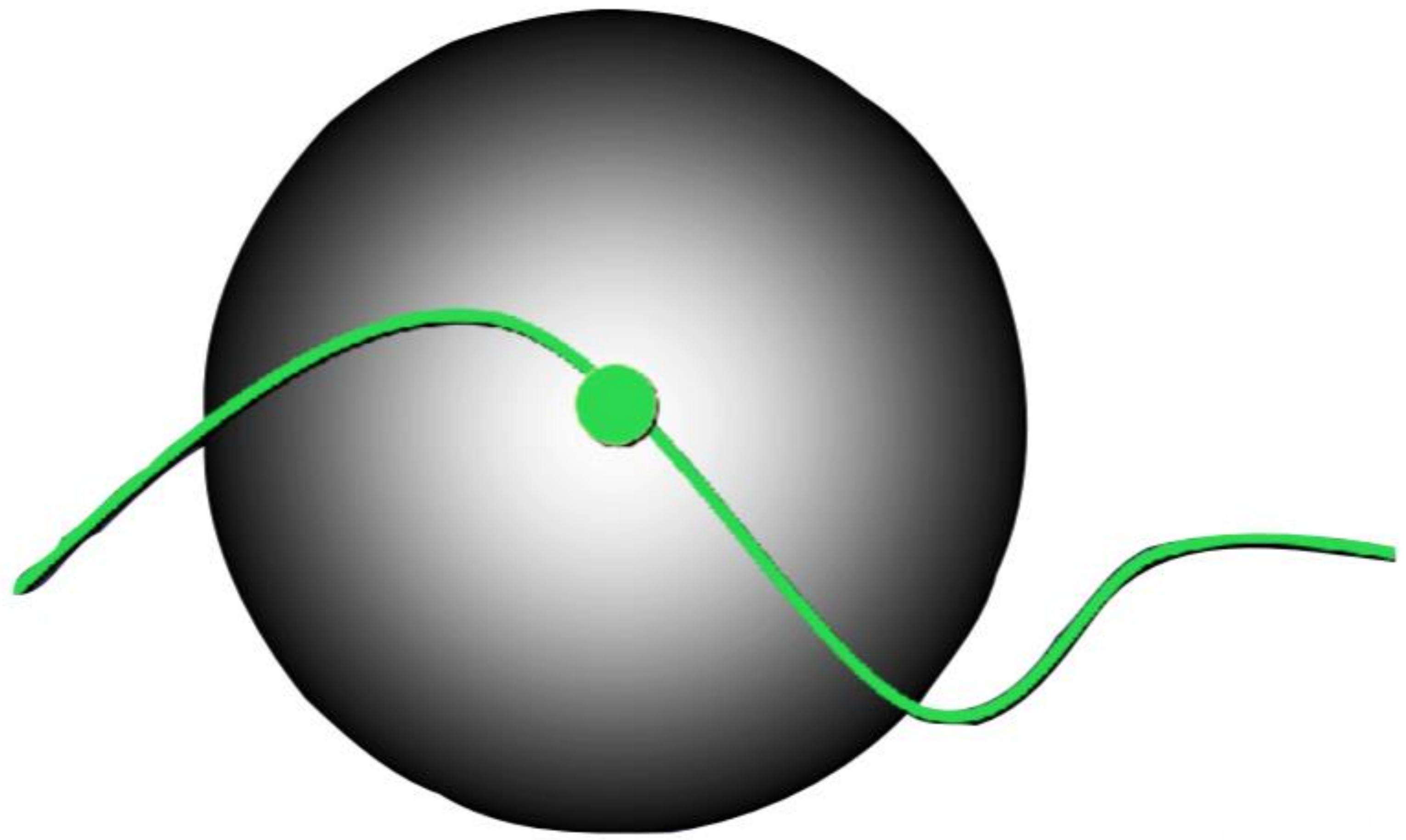
  or local approximation



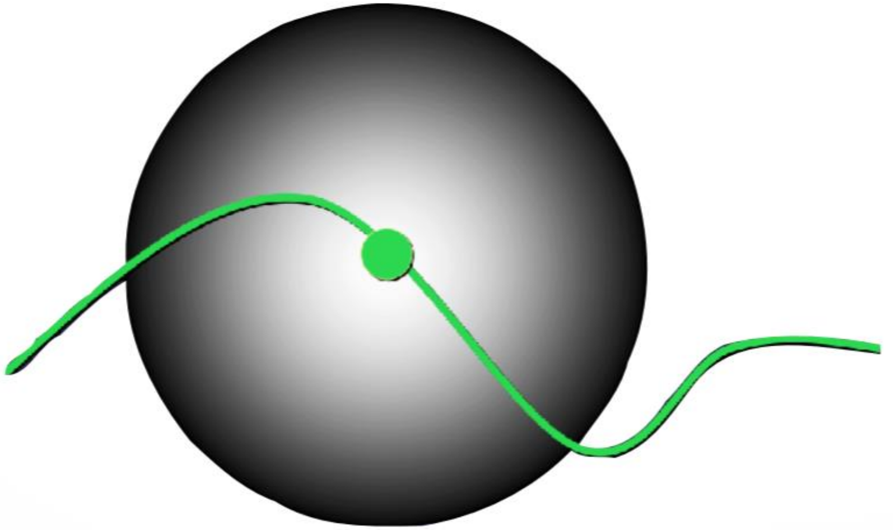
Point-to-Surface Distance

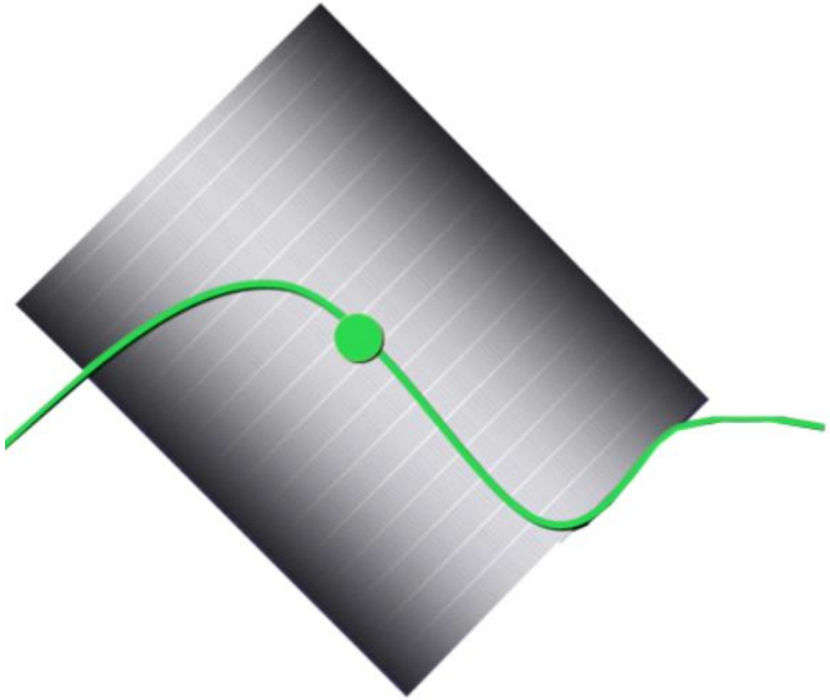


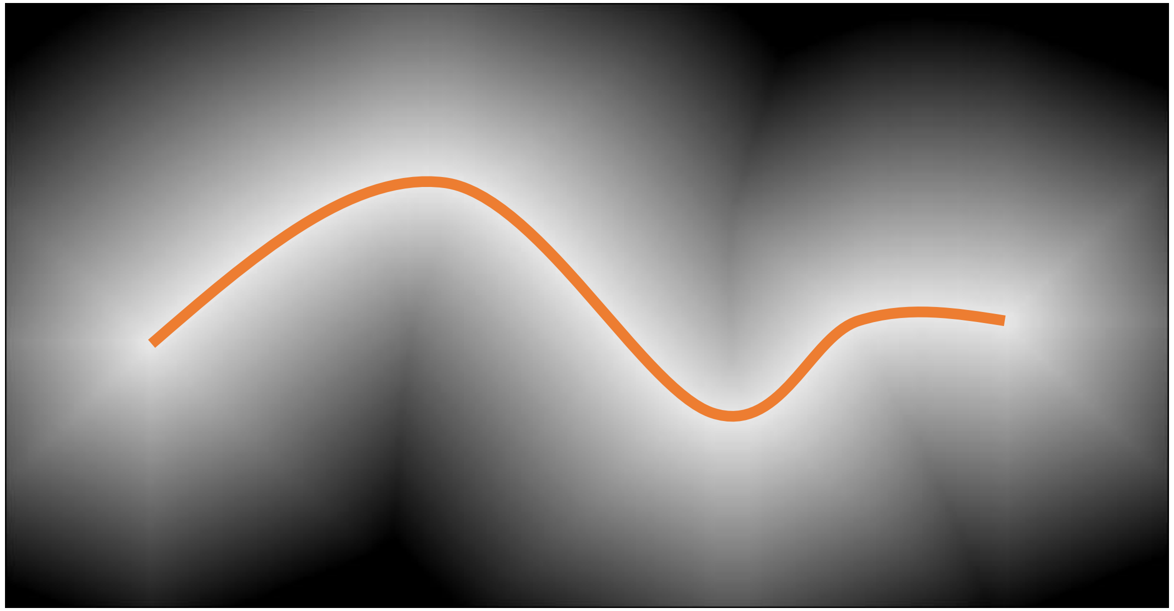
Point-to-Point Distance



Point-to-Plane Distance

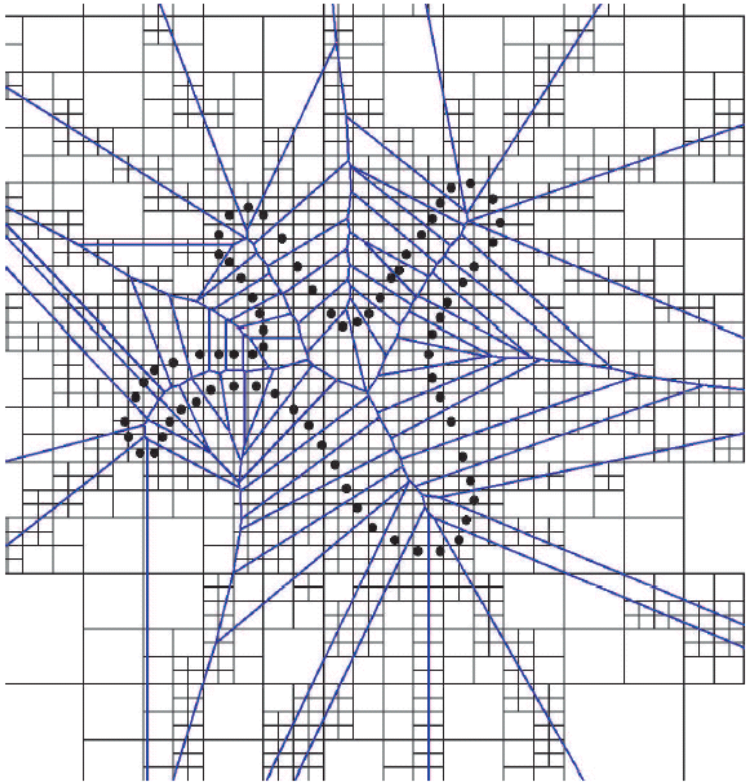




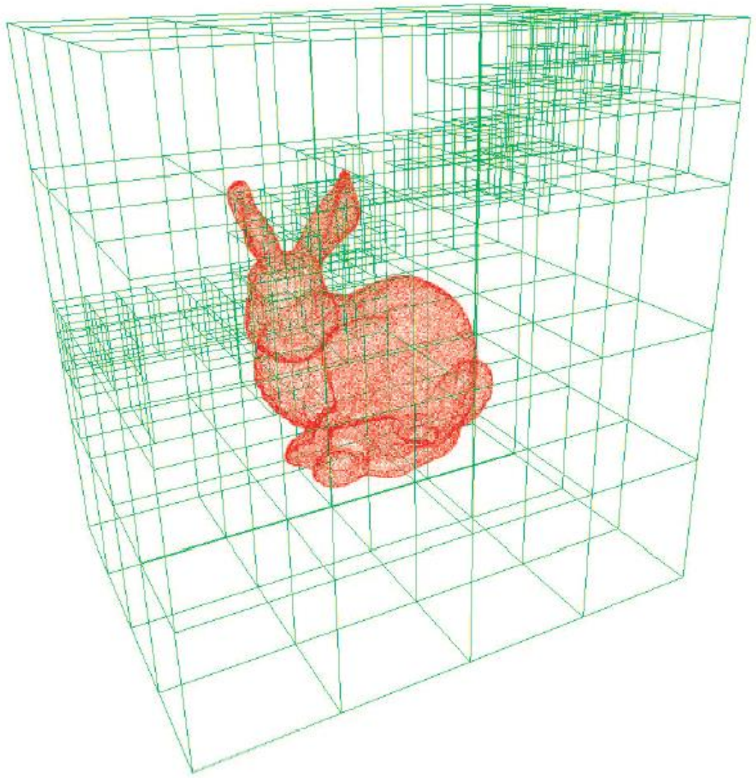


Mitra et al.'s Optimization

- Precompute piecewise-quadratic approximation to distance field throughout space
- Store in “d2tree” data structure



2D



3D

[Mitra et al. 2004]

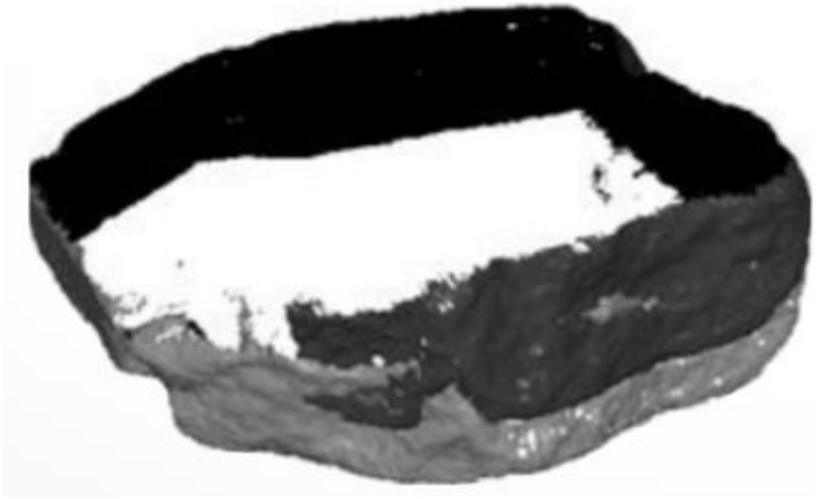
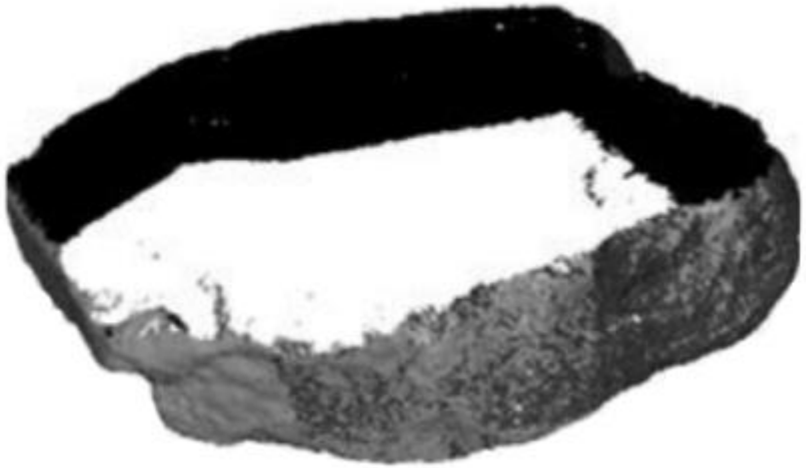
Mitra et al.’s Optimization

- Precompute piecewise-quadratic approximation to distance field throughout space
- Store in “d2tree” data structure

- At run time, look up quadratic approximants and optimize using Newton’s method
  - More robust, wider basin of convergence
  - Often fewer iterations, but more precomputation

Global Registration Goal

- Given: n scans around an object
- Goal: align them all
- First attempt: apply ICP to each scan to one other
- Want method for distributing accumulated error among all scans

Approach #1: Avoid the Problem

- In some cases, have 1 (possibly low-resolution) scan that covers most surface

- Align all other scans to this “anchor” [Turk 94]

- Disadvantage: not always practical to obtain anchor scan

Approach #2: The Greedy Solution

- Align each new scan to the union of all previous scans [Masuda ’96]
- Disadvantages:
  - Order dependent
  - Doesn’t spread out error
    - This sometimes avoids catastrophic accumulation of error, but really isn’t guaranteed to do anything.

Approach #3: The Brute-Force Solution

- While not converged:
  - For each scan:
    - For each point:
      - For every other scan
        - Find closest point
  - Minimize error w.r.t. transforms of all scans

- Disadvantage:
  - Solve (6n)x(6n) matrix equation, where n is number of scans

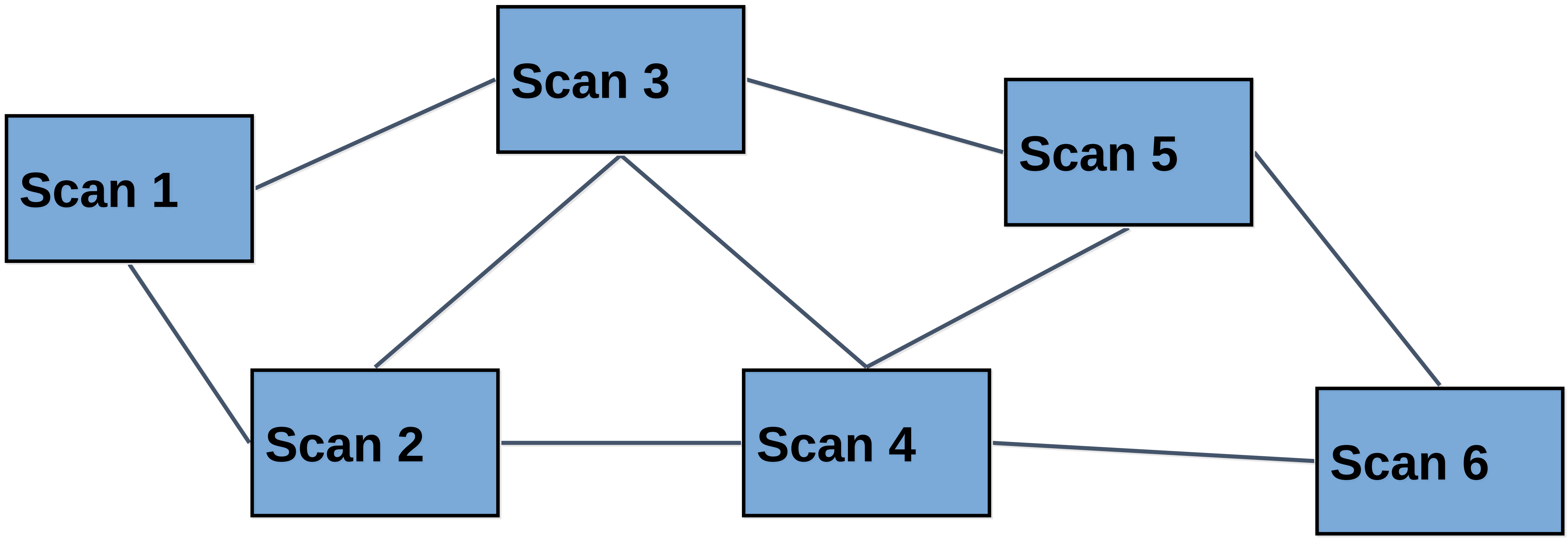
Approach #3a: Slightly Less Brute-Force Solution

- While not converged:
    - For each scan:
        - For each point:
            - For every other scan
                - Find closest point
    - Minimize error w.r.t. transforms of this scans

- Faster than previous method (matrices are 6x6) [Bergevin ’96, Benjemaa ‘97]
- Previous: Solve (6n)x(6n) matrix equation, where n is number of scans

Graph Methods

- Many globalreg algorithms create a graph of pairwise alignments between scans



Pulli’s Algorithm

- **Perform pairwise ICPs, record sample (e.g. 200) of corresponding points**
- For each scan, starting w. most connected
  - Align scan to existing set
  - While (change in error) > threshold
    - Align each scan to others
- **All alignments during globalreg phase use precomputed corresponding points**

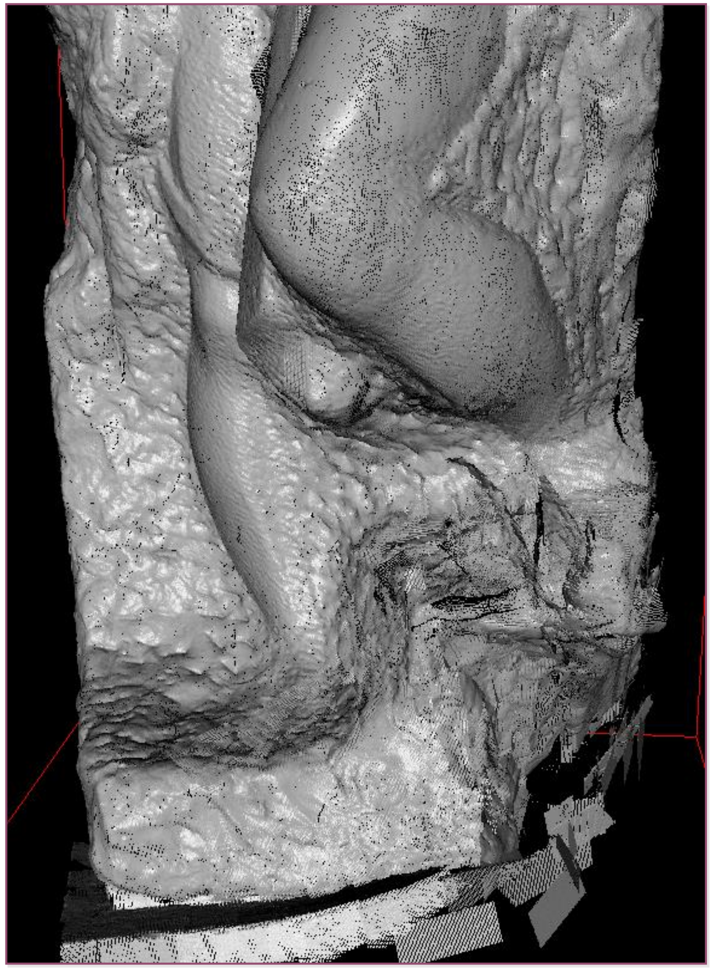
Sharp et al. Algorithm

- Perform pairwise ICPs, record only optimal rotation/translation for each
- Decompose alignment graph into cycles
- While (change in error) > tolerance
  - For each cycle:
    - Spread out error equally among all scans in the cycle
  - For each scan belonging to more than 1 cycle:
    - Assign average transform to scan

Bad ICP in Globalreg

One bad ICP can throw off the entire model!

Correct Globalreg



Globalreg Including Bad ICP

Acknowledgement

- Data-Driven Shape Anlaysis - CS468 @Stanford.edu, Spring 2014
  CSCI 621: Digital Geometry Processing SS 2017 @ USC
- 3D Scan Matching and Registration, Part II, ICCV 2005 Short Course

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Thanks