Digital Geometry
-Continuous Geometry of Curves & Surfaces

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http://jjcao.github.io/DigitalGeometry/

Pleasure may come from illusion, but happiness can come only of reality.

Differential Geometry

- Curves
- Surfaces

Surfaces

- **What characterizes shape?**
  - shape does not depend on Euclidean motions
  - metric and curvatures

Metric on Surfaces

- **Measure Stuff**
  - angle, length, area
  - requires an inner product

- we have:
  - Euclidean inner product in domain

- we want to turn this into:
  - inner product on surface

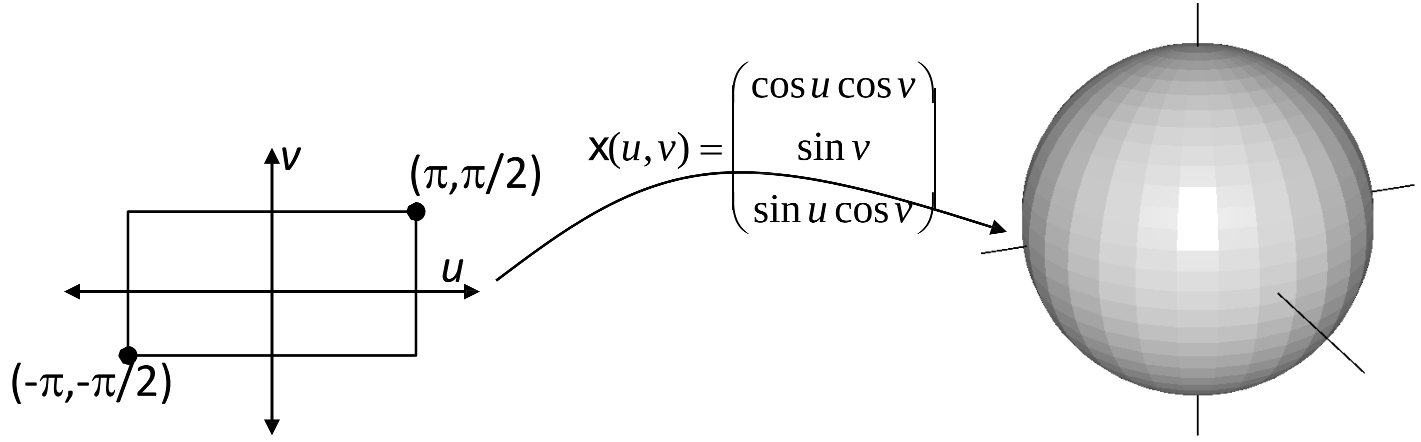
Differentiable Surfaces

Definition:

A *parameterized* ***differentiable*** *surface* is a differentiable map $\boldsymbol{x}:\Omega\to\boldsymbol{R}^3$ of an open domain $\Omega\subset\boldsymbol{R}^2$ into $\boldsymbol{R}^3$:

$$\boldsymbol{x}(u,v)=(x(u,v),y(u,v),z(u,v))$$

where $x(u,v)$, $y(u,v)$, and $z(u,v)$ are differentiable functions.

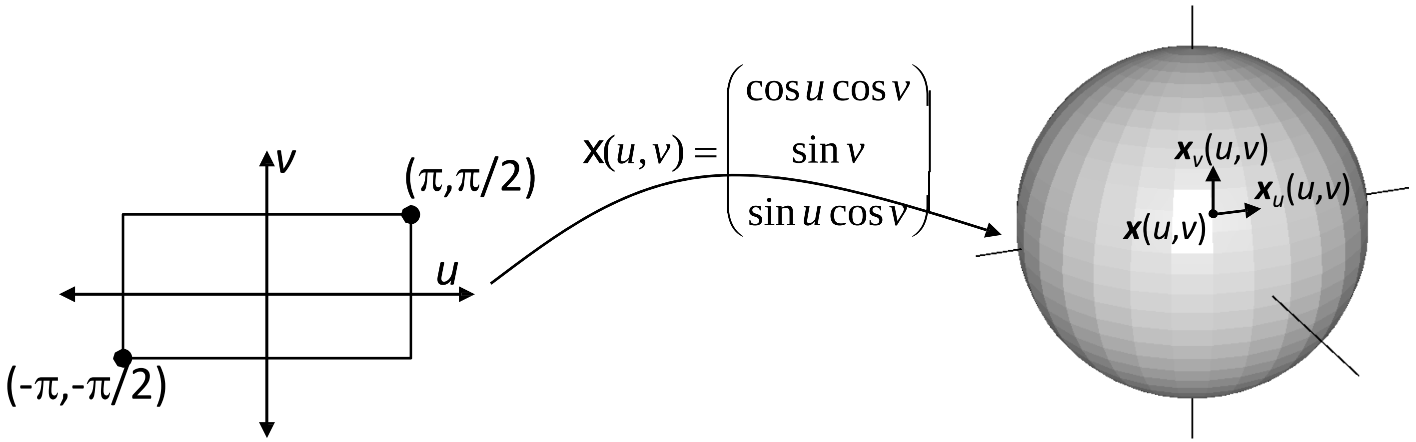


Differentiable Surfaces

Definition:

The ***derivatives*** of the surface at $\mathbf{x}(u,v)$ are the vectors:

$$\mathbf{x}_u(u,v) = \frac{\partial \mathbf{x}(u,v)}{\partial u} = \begin{pmatrix} \partial x / \partial u \\ \partial y / \partial u \\ \partial z / \partial u \end{pmatrix} \qquad \mathbf{x}_v(u,v) = \frac{\partial \mathbf{x}(u,v)}{\partial v} = \begin{pmatrix} \partial x / \partial v \\ \partial y / \partial v \\ \partial z / \partial v \end{pmatrix}$$



Differentiable Surfaces

$$\mathbf{x}_u(u,v) = \frac{\partial \mathbf{x}(u,v)}{\partial u} \qquad \mathbf{x}_v(u,v) = \frac{\partial \mathbf{x}(u,v)}{\partial v}$$

Definition:

The surface is said to be ***regular*** if at each point (u,v) the derivatives/tangents $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are linearly independent.

This is equivalent to the statement:

$$\mathbf{x}_u \times \mathbf{x}_v \neq 0$$

i.e. that a normal (line) can be defined everywhere.

Normal Vectors

- Continuous surface

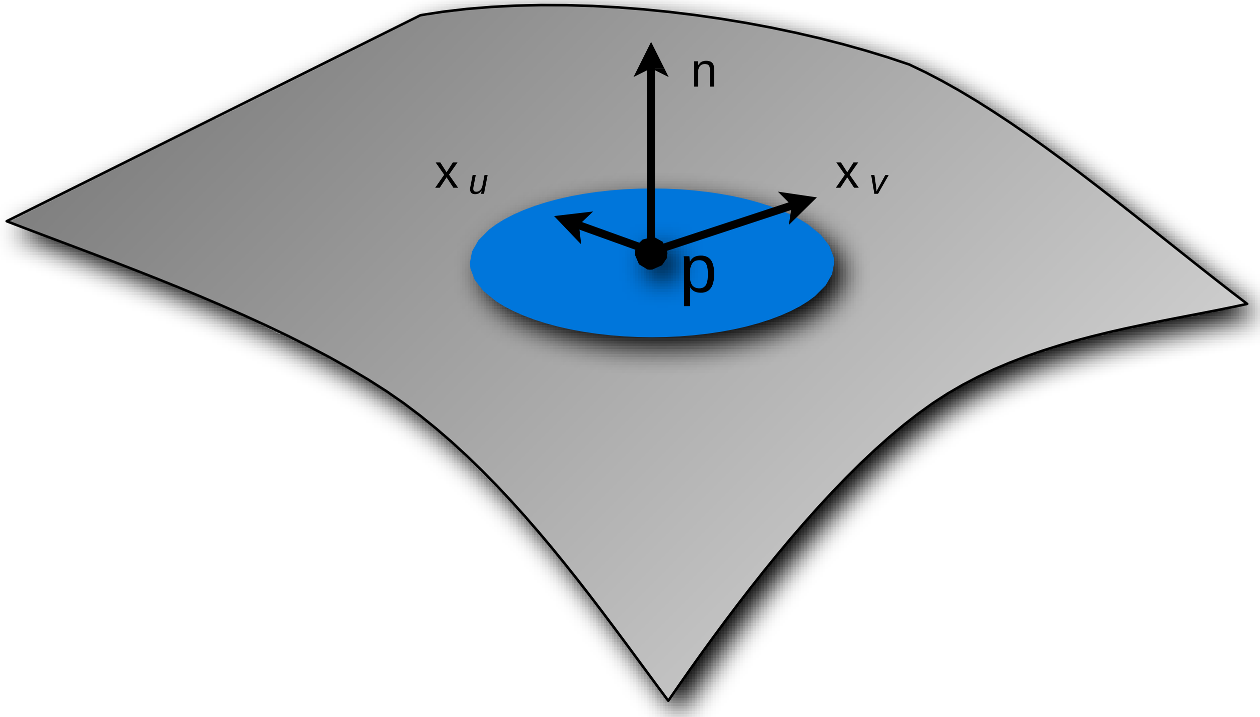
$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

Assume *regular* parameterization

$$\mathbf{x}_u \times \mathbf{x}_v \neq 0$$ normal exists



Riemannian Metric & first fundamental form

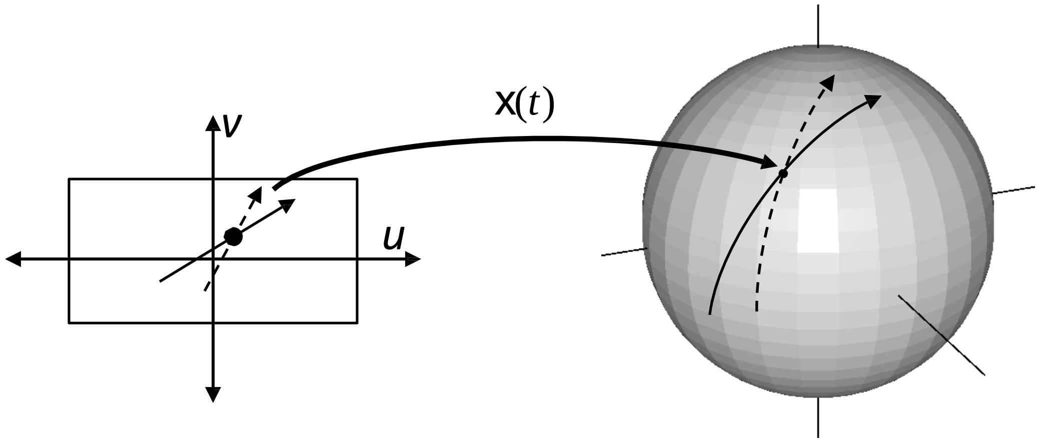
Curve in parameter domain => curve on surface

$$\mathbf{x}_u(u,v) = \frac{\partial \mathbf{x}(u,v)}{\partial u} \qquad \mathbf{x}_v(u,v) = \frac{\partial \mathbf{x}(u,v)}{\partial v}$$

Definition:

Given a point $p_0=(u_0,v_0)\in\Omega$ and given a **direction $w=(u_w,v_w)$** in the parameter space, we can define the (3D) curve:

$$\mathbf{x}(t) = \mathbf{x}(p_0 + tw)$$



Directional derivatives

$$\mathbf{x}_u(u,v) = \frac{\partial \mathbf{x}(u,v)}{\partial u} \qquad \mathbf{x}_v(u,v) = \frac{\partial \mathbf{x}(u,v)}{\partial v}$$

<u>Definition</u>:

$$\mathbf{x}(t) = \mathbf{x}(p_0 + tw)$$

Taking the derivative at t=0, we get:

$$\mathbf{x}'(0) = w_u \mathbf{x}_u + w_v \mathbf{x}_v = \mathrm{J}(w)$$

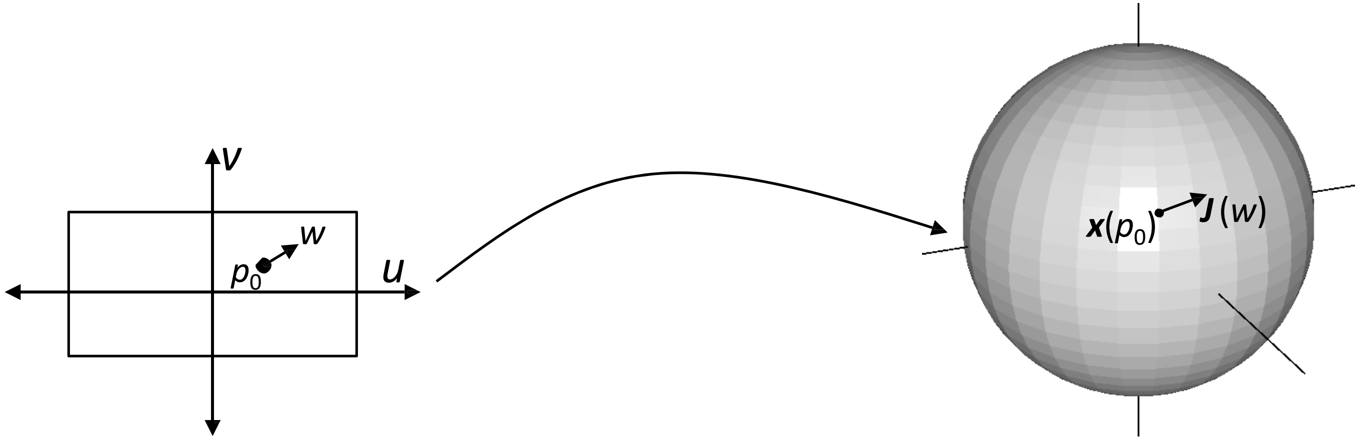
where ***J*** is the Jacobian matrix taking directions in Ω to tangent vectors on the surface:

$$\mathrm{J} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

Metric Properties - length

Thus, given a point $p_0=(u_0,v_0)\in\Omega$ and given a direction $w=(u_w,v_w)$, we can use the Jacobian to compute the length of the corresponding tangent vector over $\boldsymbol{x}(p_0)$:

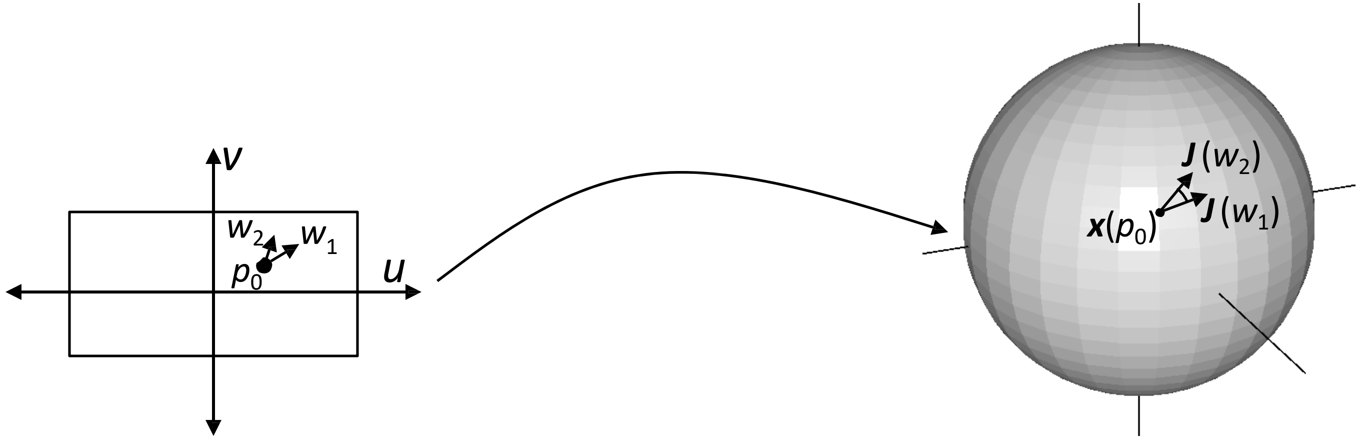
$$length^2 = \left\| \boldsymbol{J}w \right\|^2 = w^t \boldsymbol{J}^t \boldsymbol{J}w$$



Metric Properties - angle

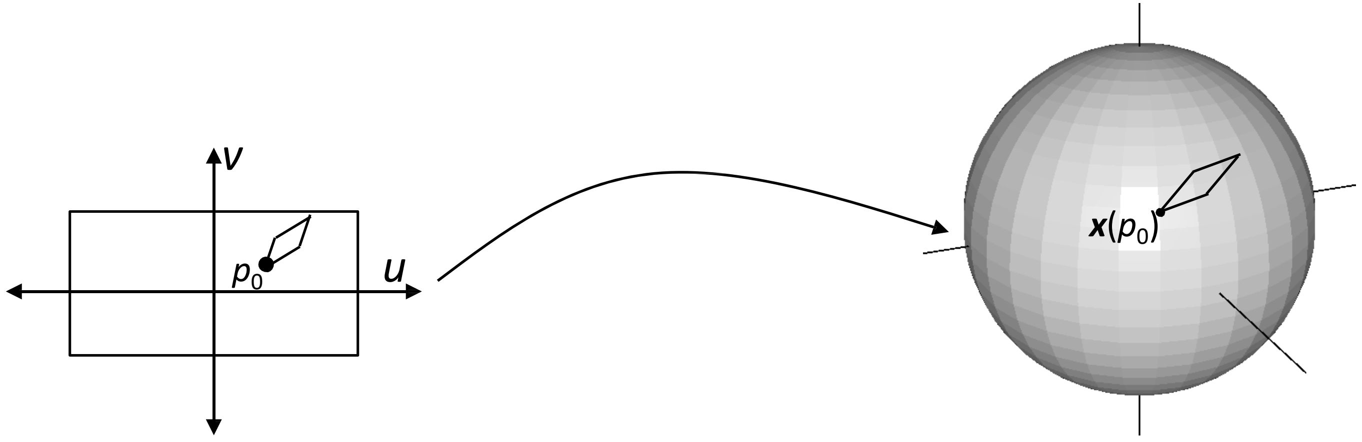
- Similarly, given a point $p_0=(u_0,v_0)\in\Omega$ and given directions $w_1=(u_1,v_1)$ and $w_2=(u_2,v_2)$ we can use the Jacobian to compute the angle of the corresponding tangent vectors over $\boldsymbol{x}(p_0)$:

$$\cos(angle)=\frac{\langle \boldsymbol{J}w_1,\boldsymbol{J}w_2\rangle}{\|\boldsymbol{J}w_1\|\|\boldsymbol{J}w_2\|}=\frac{w_1^t\boldsymbol{J}^t\boldsymbol{J}w_2}{\sqrt{w_1^t\boldsymbol{J}^t\boldsymbol{J}w_1}\sqrt{w_2^t\boldsymbol{J}^t\boldsymbol{J}w_2}}$$



Metric Properties - area

- Finally, given a point $p_0=(u_0,v_0)\in\Omega$ and given directions $w_1=(u_1,v_1)$ and $w_2=(u_2,v_2)$ we can use the Jacobian to compute the area of the corresponding parallelogram in the tangent space:
  - $area = length_1 \cdot length_2 \cdot \sin(angle)$

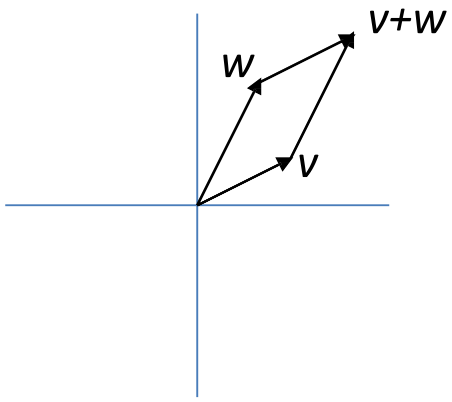


Metric Properties - area

<u>Note</u>:

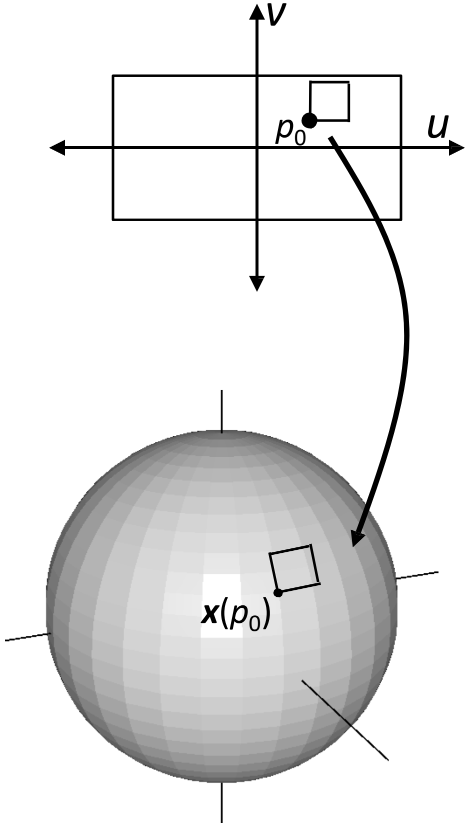
Given vectors *v* and *w* in $\mathbf{R}^n$, the area of the parallelogram spanned by *v* and *w* is:

$$
\begin{aligned}
Area(v,w) &= |v|\cdot|w|\cdot\sin\big(Angle(v,w)\big)\\
&= |v|\cdot|w|\cdot\sqrt{1-\cos^2 Angle(v,w)}\\
&= |v|\cdot|w|\cdot\sqrt{1-\frac{\langle v,w\rangle^2}{|v|^2|w|^2}}\\
&= \sqrt{|v|^2|w|^2-\langle v,w\rangle^2}
\end{aligned}
$$



Metric Properties - area

$$Area(v,w)=\sqrt{|v|^2|w|^2-\langle v,w\rangle}$$



<u>Note</u>:

Since the first fundamental form is defined as:

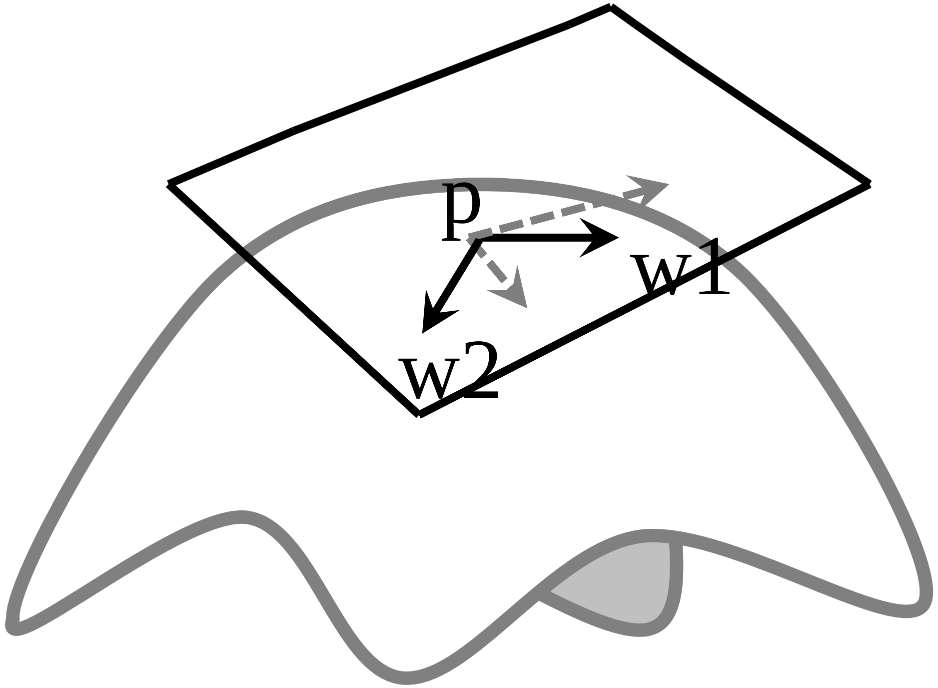
$$\mathrm{I}=\mathrm{J}\,\mathrm{J}^t=\begin{pmatrix}\langle \mathrm{x}_u,\mathrm{x}_u\rangle & \langle \mathrm{x}_u,\mathrm{x}_v\rangle\\ \langle \mathrm{x}_v,\mathrm{x}_u\rangle & \langle \mathrm{x}_v,\mathrm{x}_v\rangle\end{pmatrix}$$

in mapping from Ω to the surface, the area of a tiny patch of surface gets scaled by:

$$\sqrt{\|\mathrm{x}_u\|^2\|\mathrm{x}_v\|^2-\langle \mathrm{x}_u,\mathrm{x}_v\rangle^2}=\sqrt{\det \mathrm{I}}$$

First Fundamental Form I_S

- **Riemannian metric**, Metric Tensor, Fundamental Tensor
- S(u,v)=(x(u,v), y(u,v),z(u,v))
- Jacobian matrix $J = [S_u, S_v] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial u} \end{bmatrix}$
- $w = J\hat{w} = [S_u, S_v] \begin{bmatrix} u \\ v \end{bmatrix}$
- $< \widehat{w_1}, \widehat{w_2} >_S := \boldsymbol{I_S(\widehat{w_1}, \widehat{w_2})} =< w_1, w_2 >= (J\widehat{w_1})^T (J\widehat{w_2}) = \widehat{w_1}^T (J^T J)\widehat{w_2}$
- $I = J^T J = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$



First Fundamental Form

First fundamental form $\mathbf{I}$ allows to measure

(w.r.t. surface metric)

Angles $\quad \mathbf{t}_1^\top \mathbf{t}_2 = \langle(\alpha_1, \beta_1), (\alpha_2, \beta_2)\rangle$

Length
$$\begin{aligned} \mathrm{d}s^2 &= \langle(\mathrm{d}u, \mathrm{d}v), (\mathrm{d}u, \mathrm{d}v)\rangle \\ &= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 \end{aligned}$$

squared infinitesimal length

Area
$$\begin{aligned} \mathrm{d}A &= \|\mathbf{x}_u \times \mathbf{x}_v\| \,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{\mathbf{x}_u^T\mathbf{x}_u \cdot \mathbf{x}_v^T\mathbf{x}_v - (\mathbf{x}_u^T\mathbf{x}_v)^2}\,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v \end{aligned}$$

infinitesimal Area

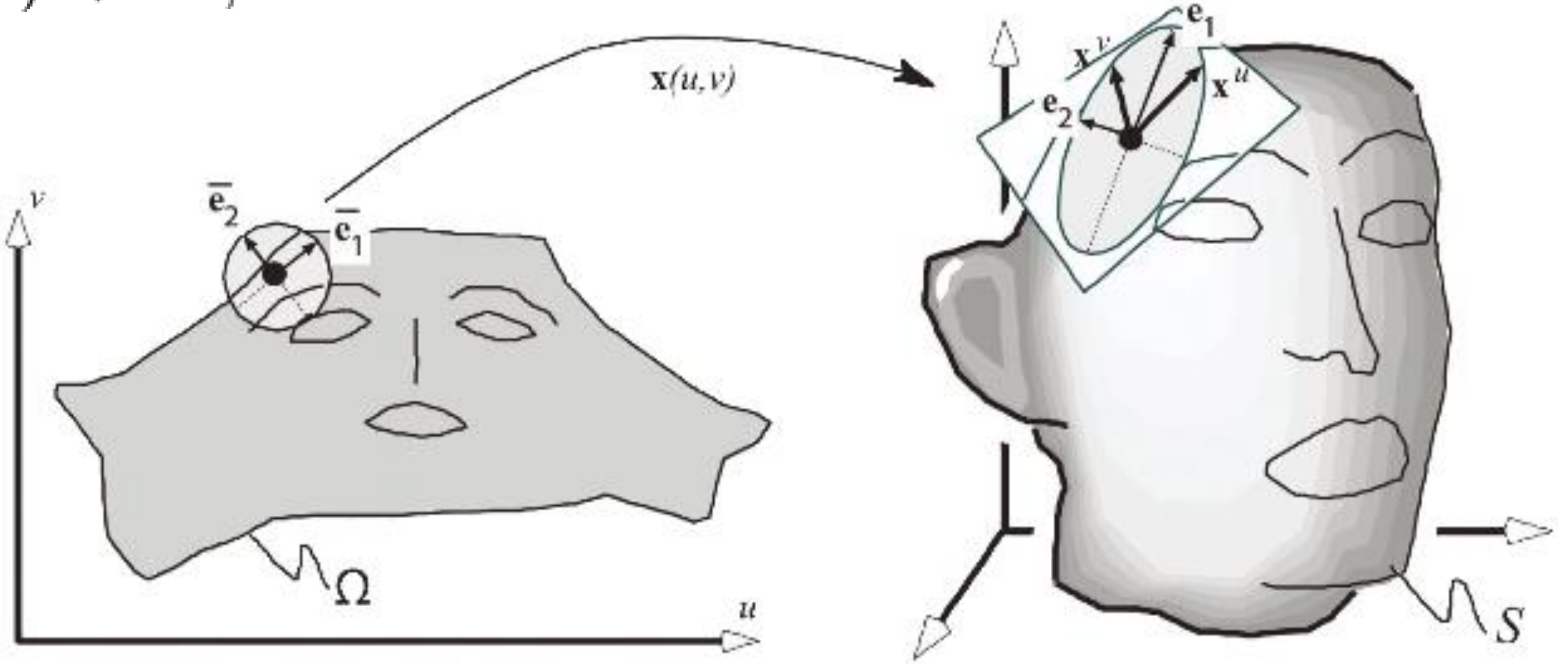
cross product → determinant with unit vectors → area

Anisotropy

- the axes of the anisotropy ellipse are $\mathbf{e}_1 = \mathbf{J}\bar{\mathbf{e}}_1$ and $\mathbf{e}_2 = \mathbf{J}\bar{\mathbf{e}}_2$;
- the lengths of the axes are $\sigma_1 = \sqrt{\lambda_1}$ and $\sigma_2 = \sqrt{\lambda_2}$.

$$\sigma_1 = \sqrt{1/2(E+G) + \sqrt{(E-G)^2 + 4F^2}},$$

$$\sigma_2 = \sqrt{1/2(E+G) - \sqrt{(E-G)^2 + 4F^2}},$$

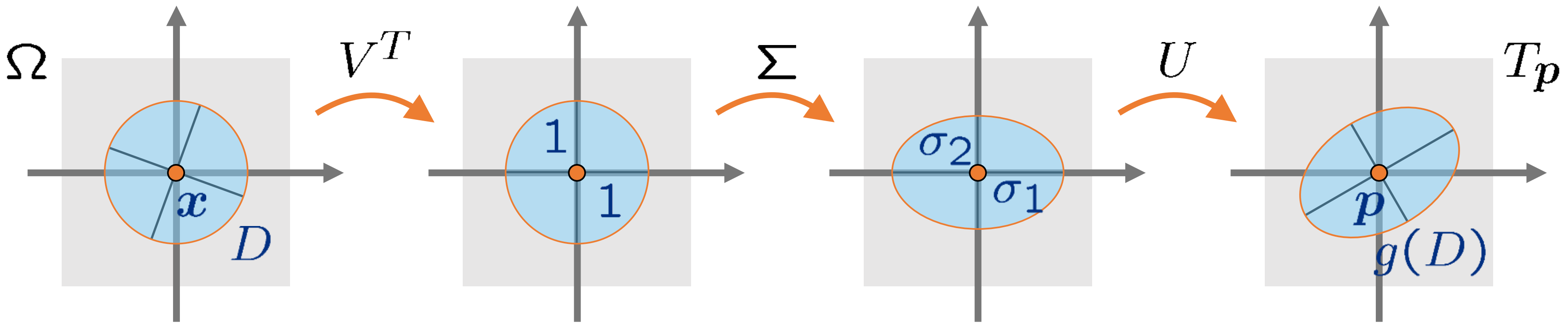


Linear Map Surgery

- **Singular Value Decomposition** (SVD) of J_f

$$J_f = U\,\Sigma\,V^T = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} V^T$$

with **rotations** $U \in \mathbb{R}^{3\times 3}$ and $V \in \mathbb{R}^{2\times 2}$
and **scale factors** (singular values) $\sigma_1 \geq \sigma_2 > 0$



Notion of Distortion

- **isometric** or **length**-preserving

$$\sigma_1 = \sigma_2 = 1$$

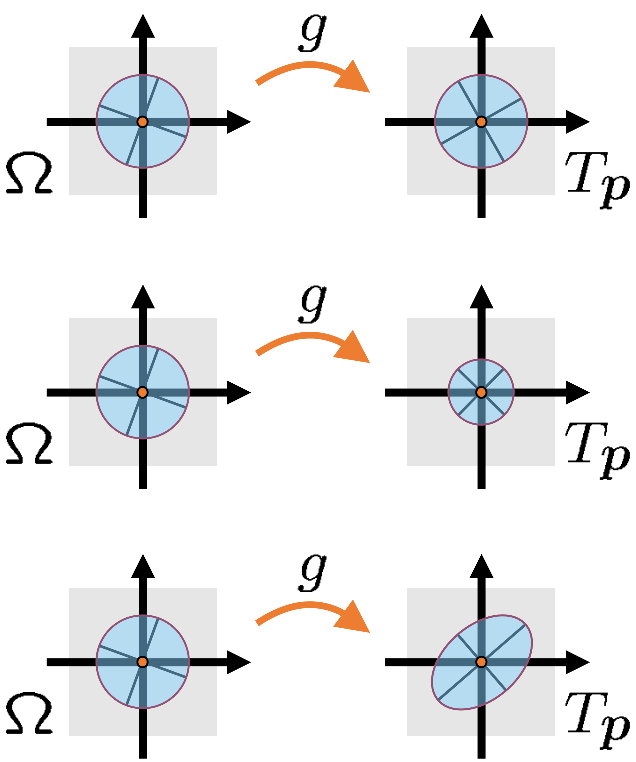
- **conformal** or **angle**-preserving

$$\sigma_1 = \sigma_2$$

- **equiareal** or **area**-preserving

$$\sigma_1 \cdot \sigma_2 = 1$$

- everything defined **pointwise** on Ω

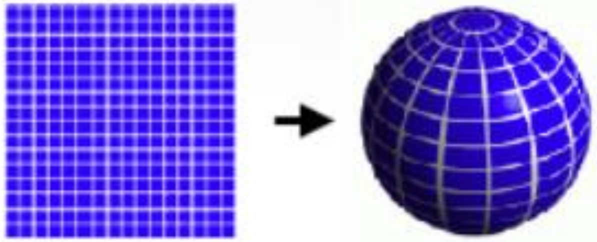


Theorem 4. *Every isometric mapping is conformal and equiareal, and every conformal and equiareal mapping is isometric, i.e.,*

$$\text{isometric} \Leftrightarrow \text{conformal} + \text{equiareal}.$$

Sphere Example

Spherical parameterization



$$\mathbf{x}(u,v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}, \quad (u,v) \in [0,2\pi) \times [0,\pi)$$

Tangent vectors

$$\mathbf{x}_u(u,v) = \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix} \quad \mathbf{x}_v(u,v) = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{pmatrix}$$

First fundamental Form

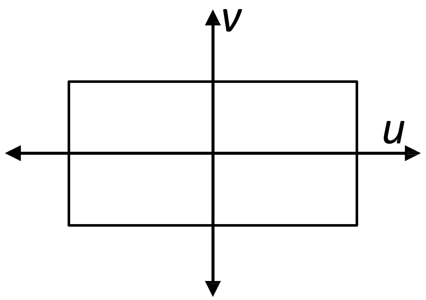
$$\mathbf{I} = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

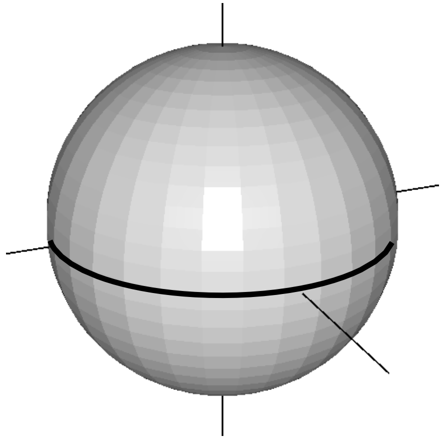
Metric Properties

$$\mathrm{x}(u,v) = \begin{pmatrix} \cos u \cos v & \sin v & \sin u \cos v \end{pmatrix}^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the length of the equator?





Metric Properties

$$\mathrm{x}(u,v) = \begin{pmatrix}\cos u\cos v & \sin v & \sin u\cos v\end{pmatrix}^t \qquad \mathrm{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

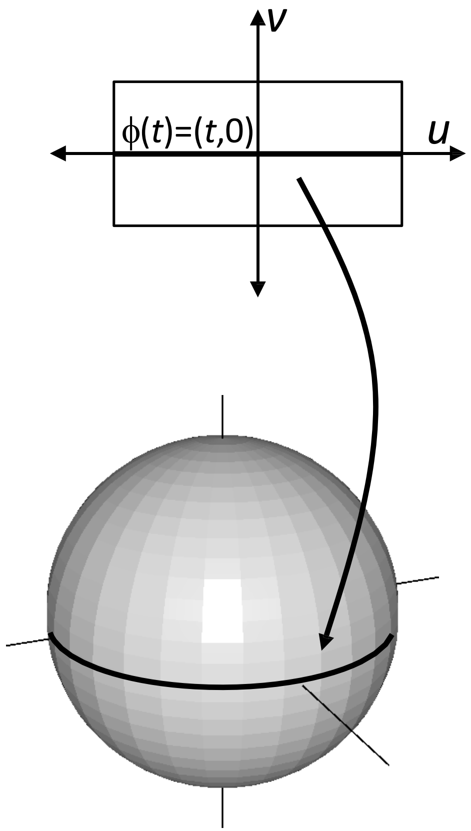
<u>Example (Sphere)</u>:

- What is the length of the equator?

The equator is the image of:

$$\phi(t)=(t,0) \quad \text{with } t\in[-\pi, \pi]$$

under the parameterization.



Metric Properties

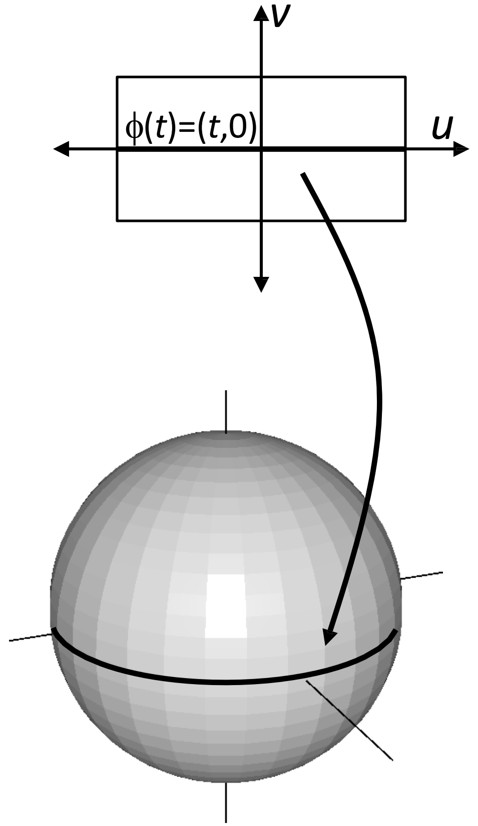
$\mathsf{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t$

$\mathsf{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$

Example (Sphere):

- What is the length of the equator?

$$\begin{aligned}
length(\boldsymbol{x} \circ \phi) &= \int_{-\pi}^{\pi} \sqrt{\phi'(t)^t \boldsymbol{I} \phi'(t)}\,dt \\
&= \int_{-\pi}^{\pi} \sqrt{(1,0)^t \begin{pmatrix}\cos^2(0) & 0 \\ 0 & 1\end{pmatrix} (1,0)}\,dt \\
&= \int_{-\pi}^{\pi} dt \\
&= 2\pi
\end{aligned}$$

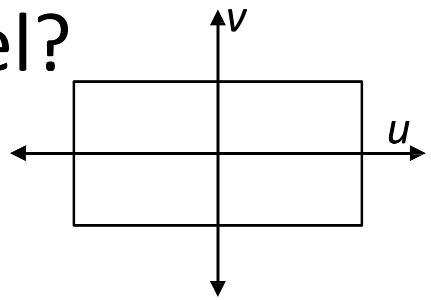


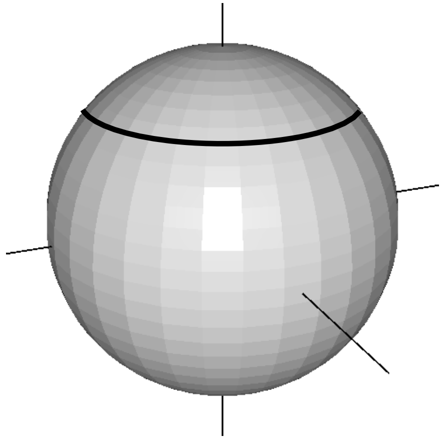
Metric Properties

$$\mathrm{x}(u,v) = (\cos u \cos v \quad \sin v \quad \sin u \cos v)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the length of the w^{th} parallel?





Metric Properties

$$\mathrm{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t \qquad \mathrm{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

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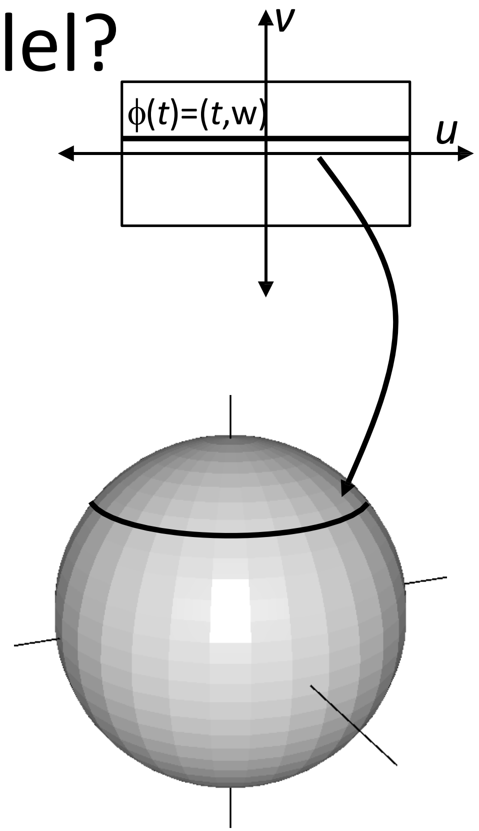
- What is the length of the w^{th} parallel?

The w^{th} parallel is the image of:

$$\phi(t)=(t,w) \quad \text{with } t\in[-\pi, \pi]$$

under the parameterization.



Metric Properties

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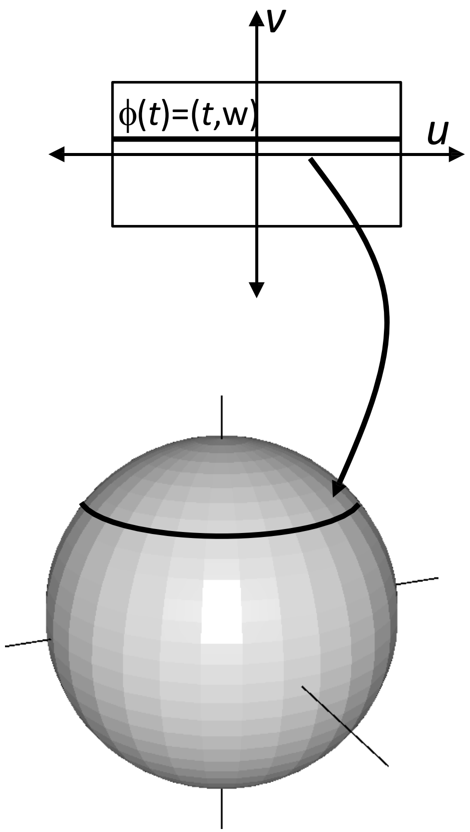
Example (Sphere):

- What is the length of the w^{th} parallel?

$$\begin{aligned} length(\boldsymbol{x} \circ \phi) &= \int_{-\pi}^{\pi} \sqrt{\phi'(t)^t \boldsymbol{I} \phi'(t)}\, dt \\ &= \int_{-\pi}^{\pi} \sqrt{(1,0)^t \begin{pmatrix} \cos^2 w & 0 \\ 0 & 1 \end{pmatrix} (1,0)}\, dt \\ &= \int_{-\pi}^{\pi} \cos w \, dt \\ &= 2\pi \cos w \end{aligned}$$

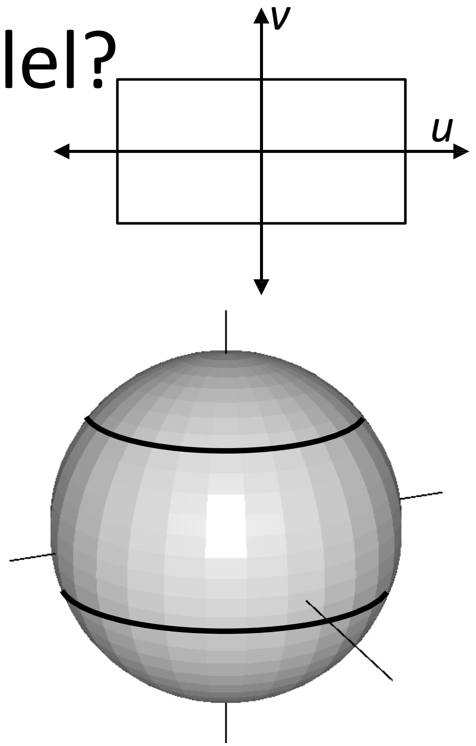


Metric Properties

$$\mathrm{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t \qquad \mathrm{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the area of the band between the w_1^{th} parallel and the w_2^{th} parallel?



Metric Properties

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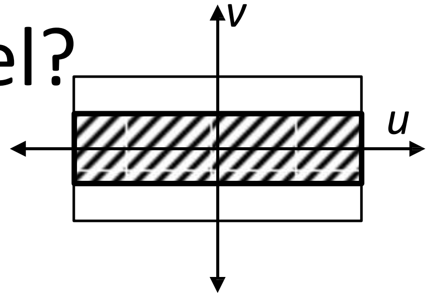
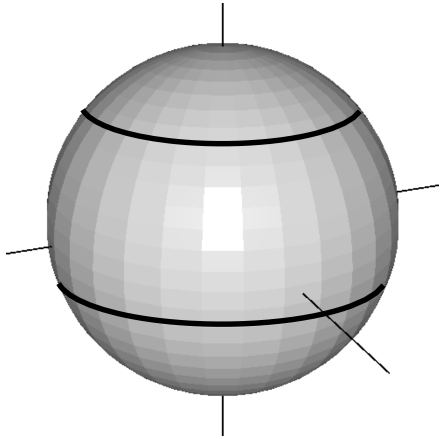
- What is the area of the band between the w_1th parallel and the w_2th parallel?



The band is the image of:

$$\phi(s,t)=(s,t) \quad \text{with } s\in[-\pi,\pi],\ t\in[w_1,w_2]$$

under the parameterization.



Metric Properties

$$\mathrm{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t \qquad \mathrm{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

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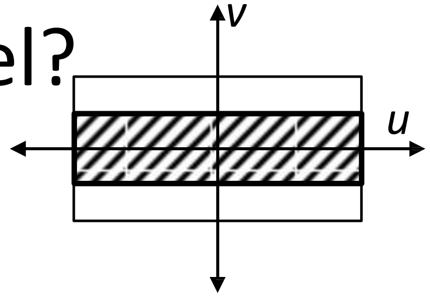
- What is the area of the band between the w_1th parallel and the w_2th parallel?

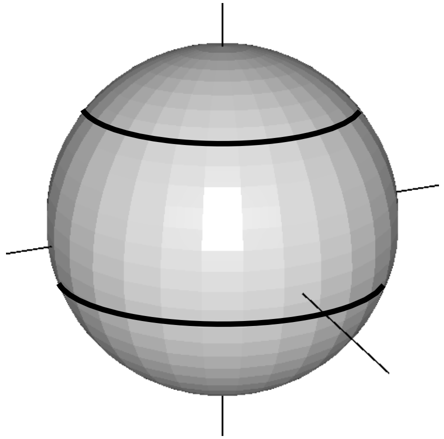


$$area(\mathrm{x} \circ \phi) = \int_{w_1}^{w_2}\int_{-\pi}^{\pi} \sqrt{\det \mathrm{I}}\, ds\, dt$$

$$= \int_{w_1}^{w_2}\int_{-\pi}^{\pi} \cos t\, ds\, dt$$

$$= 2\pi \int_{w_1}^{w_2} \cos t\, dt$$

$$= 2\pi(\sin w_2 - \sin w_1)$$

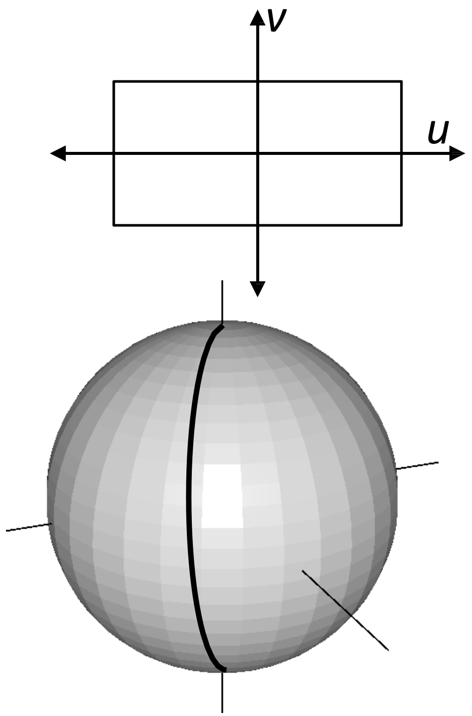


Metric Properties

$$\mathrm{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t \qquad \mathrm{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the area of the band between the w_1^{th} and the w_2^{th} meridians?



Metric Properties

$$\mathrm{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix} \qquad \mathrm{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

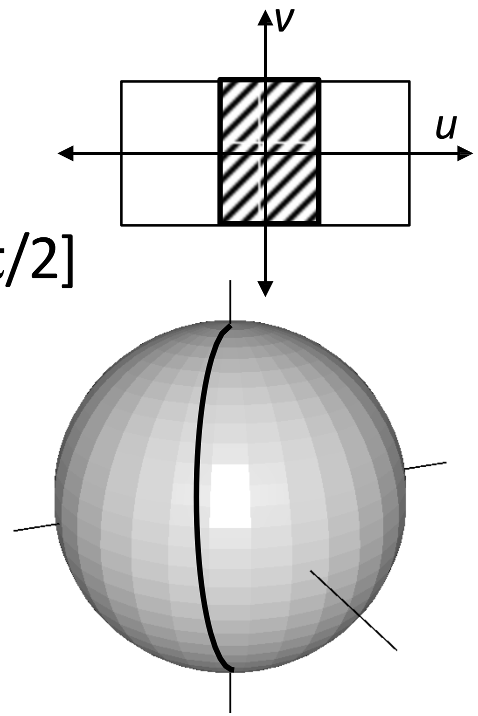
Example (Sphere):

- What is the area of the band between the w_1th and the w_2th meridians?

The band is the image of:

$\phi(s,t)=(s,t)$ with $s \in [w_1, w_2]$, $t \in [-\pi/2, \pi/2]$

under the parameterization.



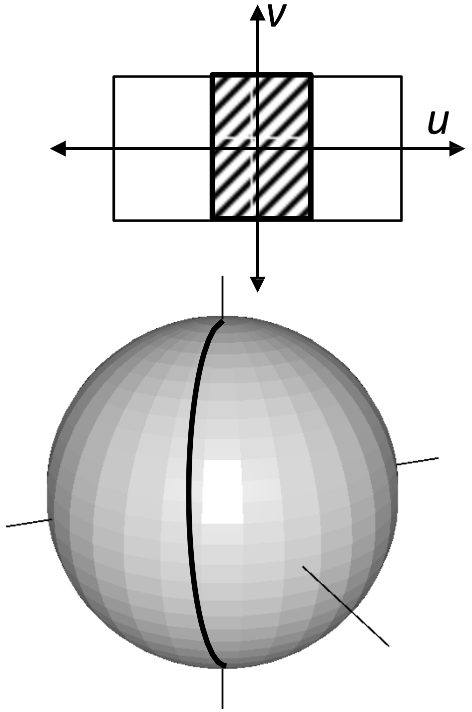
Metric Properties

$$\mathrm{X}(u,v) = \begin{pmatrix}\cos u\cos v & \sin v & \sin u\cos v\end{pmatrix}^t \qquad \mathrm{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

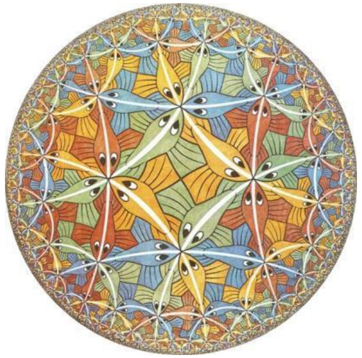
<u>Example (Sphere)</u>:

- What is the area of the band between the w_1^th and the w_2^th meridians?

$$\begin{aligned}
area(\mathrm{X}\ \phi) &= \int_{-\pi/2}^{\pi/2}\int_{w_1}^{w_2}\sqrt{\det \mathrm{I}}\,ds\,dt \\
&= \int_{-\pi/2}^{\pi/2}\int_{w_1}^{w_2}\cos t\,ds\,dt \\
&= (w_2 - w_1)\int_{-\pi/2}^{\pi/2}\cos t\ \ dt \\
&= (w_2 - w_1)\big(\sin(\pi/2) - \sin(-\pi/2)\big) \\
&= 2(w_2 - w_1)
\end{aligned}$$



Metric Properties

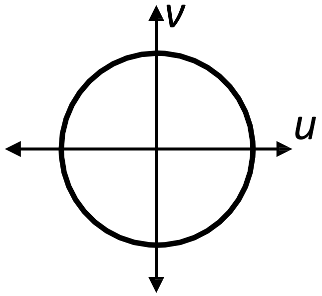


Example (Hyperbolic Plane):

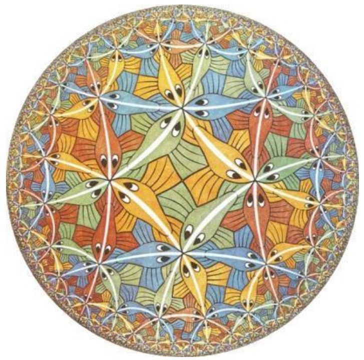
If we are given the first fundamental form, we can ignore the embedding of the surface in 3D, and integrate directly.

Consider the domain $\Omega=\{u,v \mid (u^2+v^2<1)\}$, with the first fundamental form:

$$\mathrm{I}(u,v)=\begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$



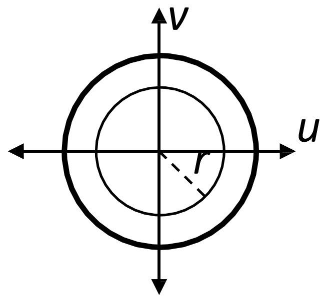
Metric Properties



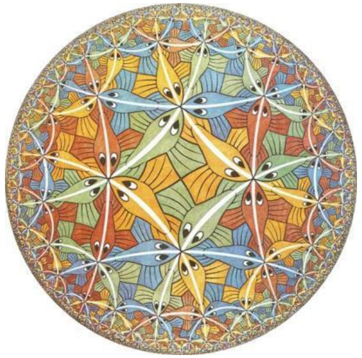
$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \quad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

Example (Hyperbolic Plane):

- What is the length of the circle with radius *r*?



Metric Properties



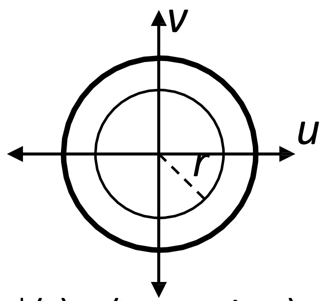
$$\Omega = \{(u,v) \mid u^2+v^2<1\} \qquad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

•What is the length of the circle with radius *r*?

The circle is described by:

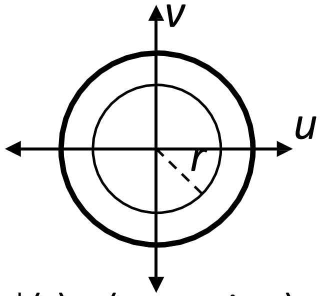
$\phi(s)=r(\cos s, \sin s)$ with $s\in[0,2\pi]$.



$\phi(t)=r(\cos t,\sin t)$

Metric Properties

$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \qquad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$



$\phi(t)=r(\cos t, \sin t)$

<u>Example (Hyperbolic Plane)</u>:

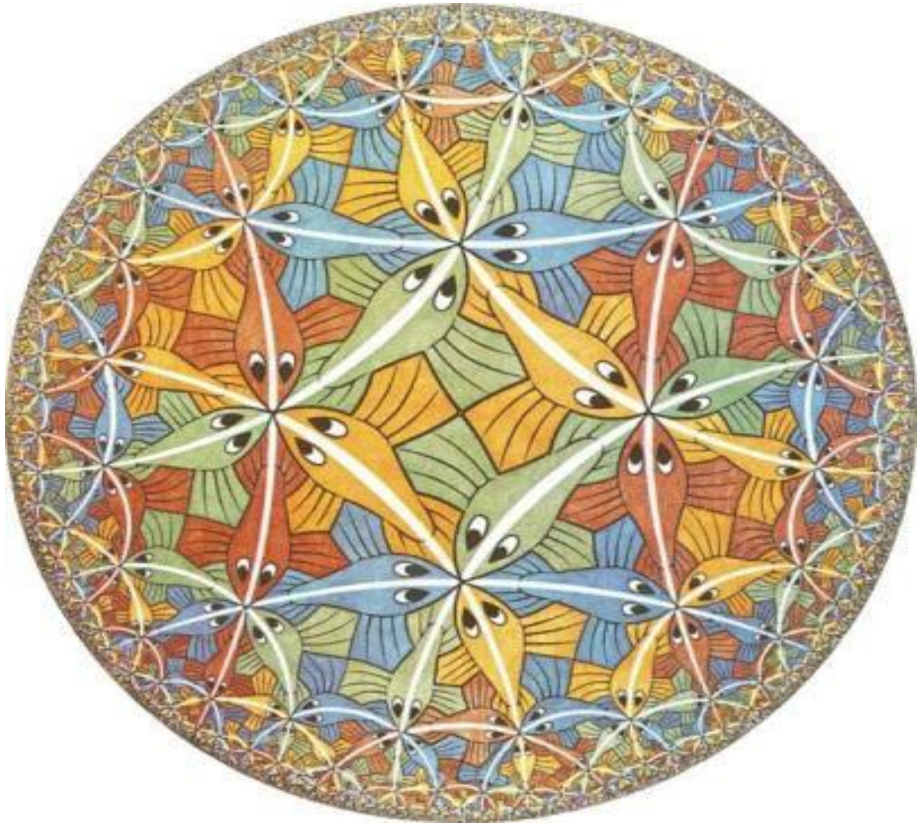
- What is the length of the circle with radius *r*?

$$length(\phi) = \int_0^{2\pi} \sqrt{\phi'(t)^t\, I\, \phi'(t)}\ \mathrm{dt}$$

$$= \int_0^{2\pi} \sqrt{r(-\sin t, \cos t)^t \begin{pmatrix} \frac{1}{1-r^2} & 0 \\ 0 & \frac{1}{1-r^2} \end{pmatrix} r(-\sin t, \cos t) dt}$$

$$= \int_0^{2\pi} \sqrt{\frac{r^2}{1-r^2}} dt$$

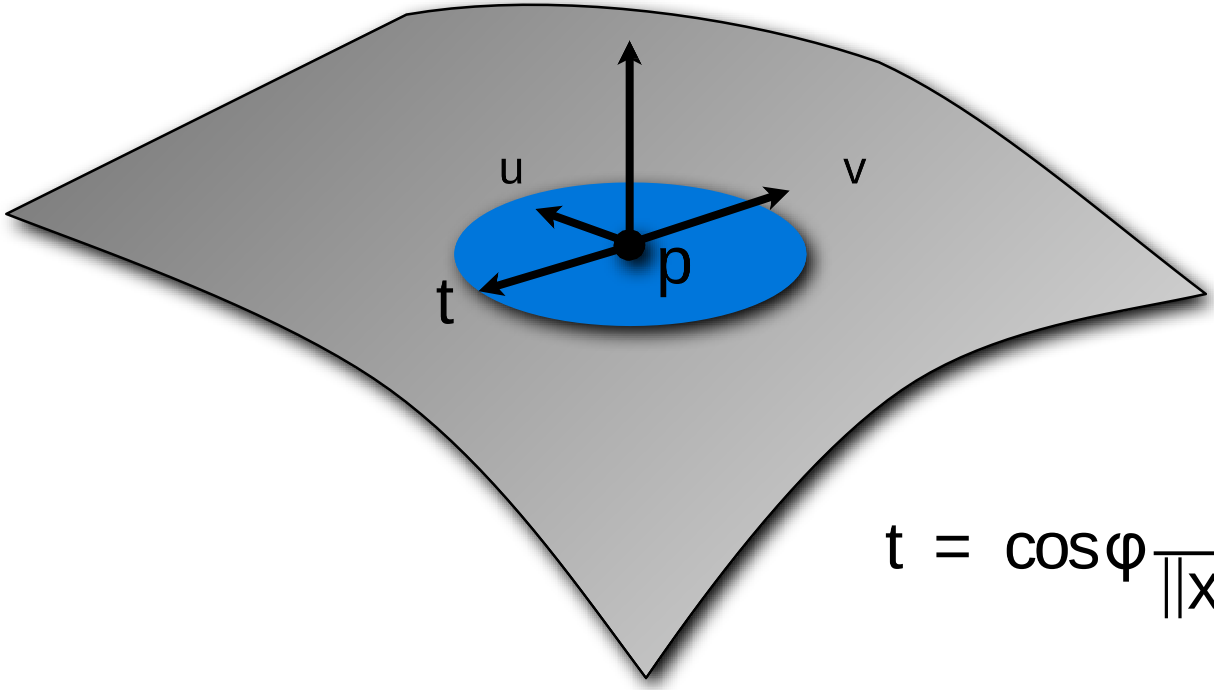
$$= 2\pi r \sqrt{\frac{1}{1-r^2}}$$



Metric on Surfaces

Normal Curvature

Tangent vector **t** …



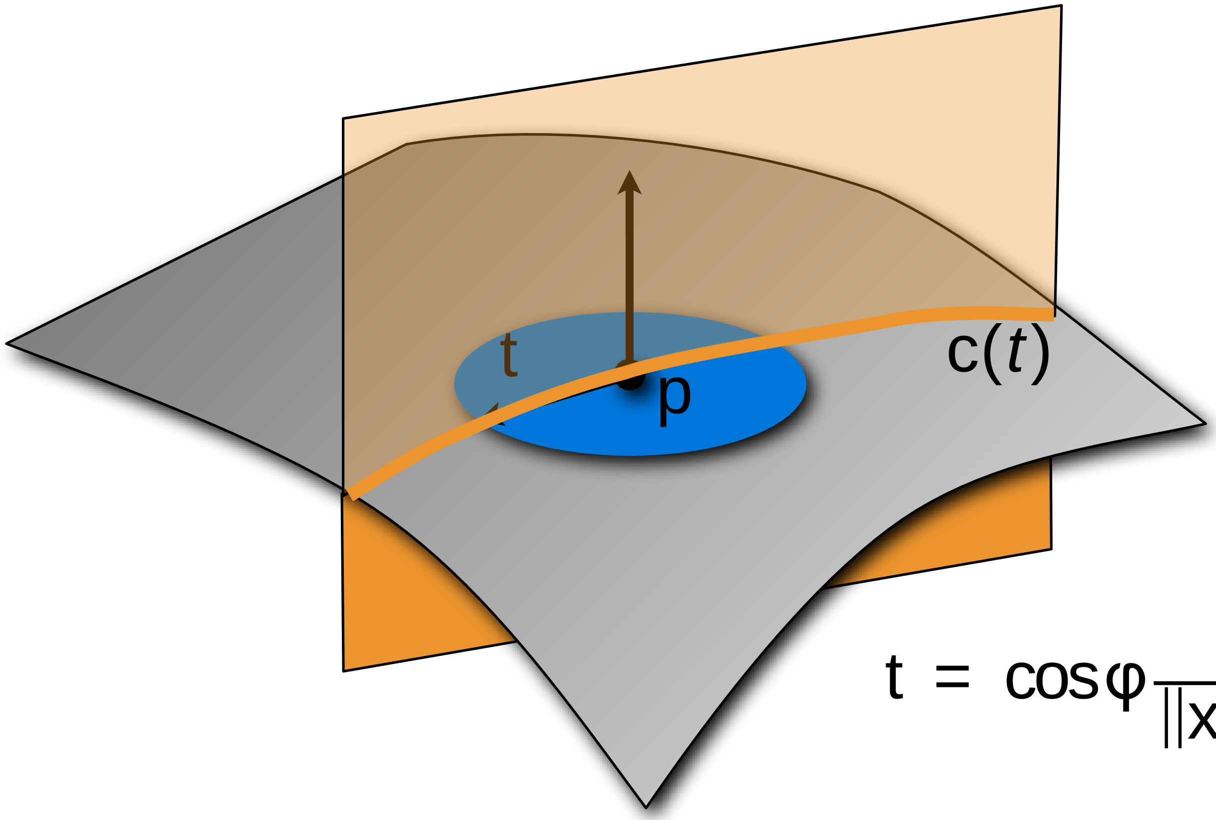
$$t = \cos\varphi \frac{u}{\|x_u\|} + \sin\varphi \frac{v}{\|x_v\|}$$

unit vector

Normal Curvature

… defines intersection plane, yielding curve c(t)

normal curve

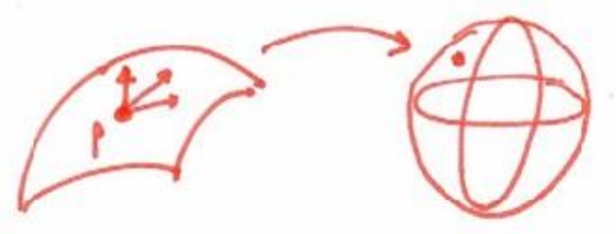


$$t = \cos\varphi \frac{u}{\|x_u\|} + \sin\varphi \frac{v}{\|x_v\|}$$

Geometry of the Normal

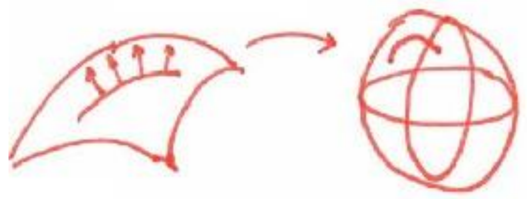
Gauss map

- normal at point



$$N(p) = \frac{S_{,u} \times S_{,v}}{|S_{,u} \times S_{,v}|}(p) \qquad N : S \to \mathbb{S}^2$$

- consider curve in surface again
  - study its curvature at p
  - normal “tilts” along curve



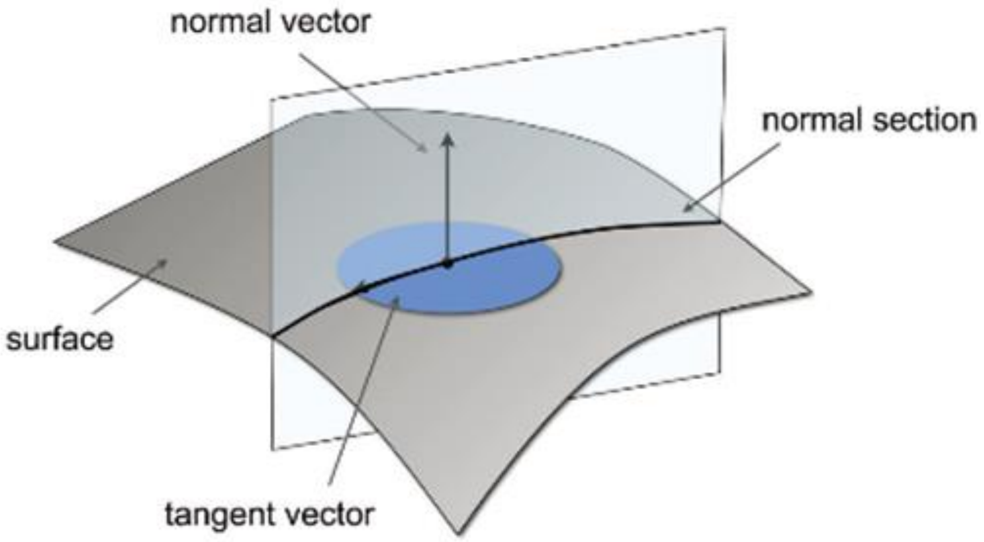
normal curvature $\kappa_n(\bar{\mathbf{t}})$ at $\mathbf{p}$

curvature of curves embedded in the surface. Let $\mathbf{t} = u_t \mathbf{x}_u + v_t \mathbf{x}_v$ be a tangent vector at a surface point $\mathbf{p} \in \mathcal{S}$ represented as $\bar{\mathbf{t}} = (u_t, v_t)^T$ in Parameter space

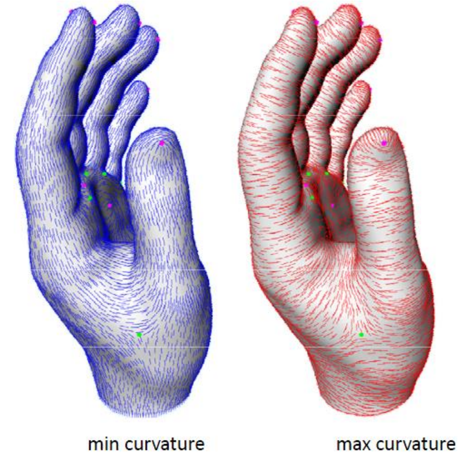
$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{e u_t^2 + 2 f u_t v_t + g v_t^2}{E u_t^2 + 2 F u_t v_t + G v_t^2},$$

where $\mathbf{II}$ denotes the *second fundamental form* defined as

$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{bmatrix}$$



Principal Curvatures



- Normal curvatures

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}}$$

- Principal curvatures
  - We can find the principal curvature values (and directions) by setting the derivative of normal curvature to 0:

$$\nabla \kappa_p(w) = 0 \quad \Rightarrow \quad \frac{\left(w^t IIw\right)}{\left(w^t Iw\right)} Iw = IIw$$

  - Thus, the principal curvature values (and directions) can be obtained by solving:

$$I^{-1} IIw = \lambda w$$

$$I^{-1} IIw_1 = \kappa_1 w_1 \qquad I^{-1} IIw_2 = \kappa_2 w_2$$

- Maximal curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
- Minimal curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$

$$I^{-1}IIw_1 = \kappa_1 w_1 \qquad I^{-1}IIw_2 = \kappa_2 w_2$$

- $I^{-1}II$ is also called the shape operator S
- $I^{-1}II$ = dNp (Np is the Gauss map)

$$\text{mean curvature } H = \mathrm{Tr}(S) = k1 + k2$$

$$\text{Gaussian curvature } K = \mathrm{Det}(S) = k1 * k2$$

Principal Curvatures

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}}$$

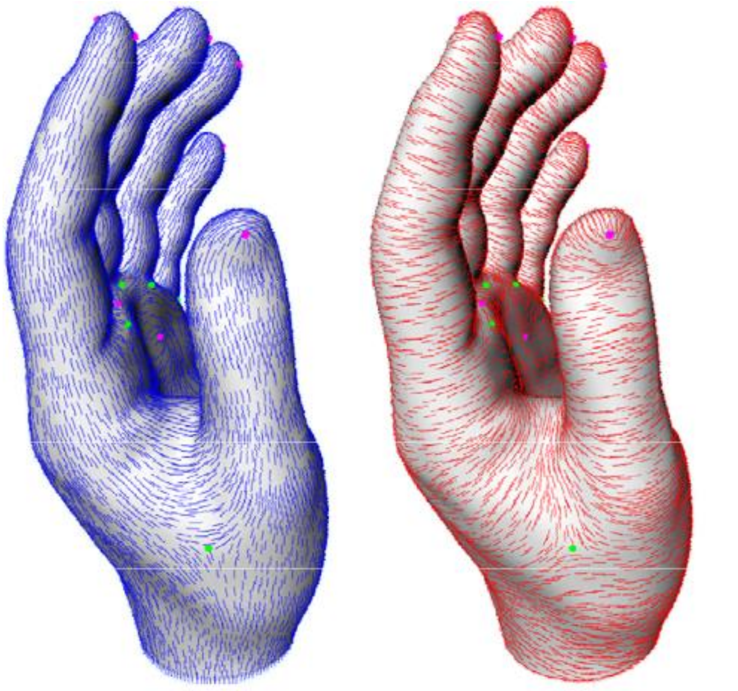
- Principal curvatures
  - Maximal curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
  - Minimal curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$
- Euler theorem

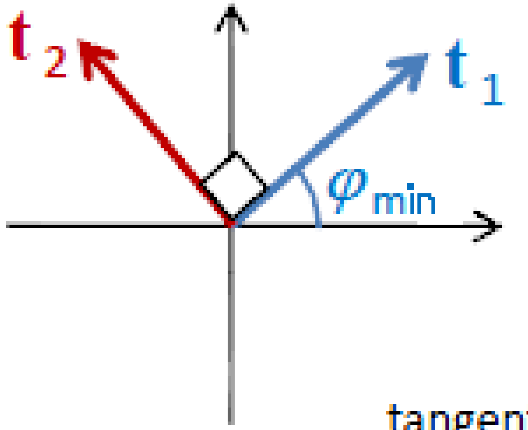
$$\kappa_n(\bar{\mathbf{t}}) = \kappa_1 \cos^2 \psi + \kappa_2 \sin^2 \psi,$$

- ψ is the angle between $\bar{t}$ and t1, t1 is the

Principal directions: tangent vectors corresponding to φ_{max} & φ_{min}

- any normal curvature is a convex combination of the minimum and maximum curvature
- **principal directions are orthogonal to each other**





Surfaces Curvatures

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{eu_t^2 + 2fu_tv_t + gv_t^2}{Eu_t^2 + 2Fu_tv_t + Gv_t^2}$$

- Principal curvatures
  - Maximal curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
  - Minimal curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$

- Mean curvature: $k_H = \frac{k_1+k_2}{2} = \frac{1}{2\pi}\int_0^{2\pi} k_n(\theta)d\,\theta = \lim_{diam(A)\to 0} \frac{\nabla A}{A}$

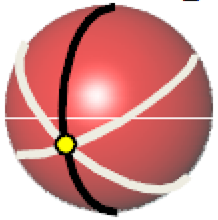
- Gaussian curvature: $k_G = k_1 \cdot k_2 = \lim_{diam(A)\to 0} \frac{A^G}{A}$

Classification

A point p on the surface is called

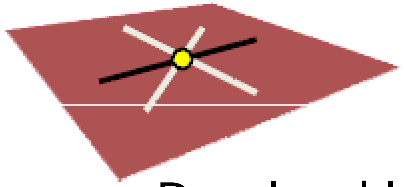
Isotropic: all directions are principle directions

$K > 0, \kappa_1 = \kappa_2$



spherical (umbilical)

$K = 0$

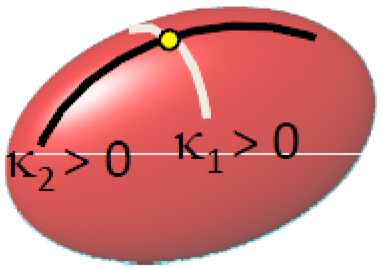


Developable surface ⇔ K=0

planar

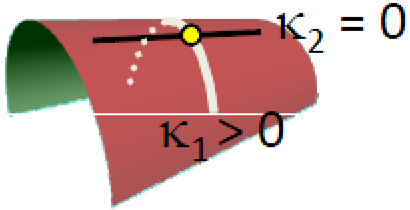
Anisotropic: 2 distinct principle directions

$K > 0$



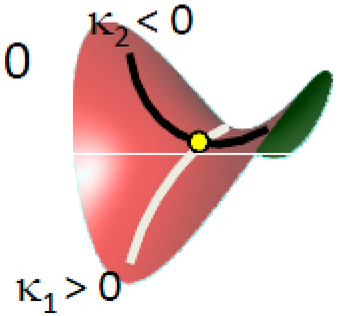
elliptic

$K = 0$



parabolic

$K < 0$



hyperbolic

Laplace & Laplace-Beltrami Operator

Laplace Operator: $\mathrm{divF} = \nabla \cdot F$

- $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
- $f = f(x, y, z)$, $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$
- $F = (U(x, y, z), V(x, y, z), W(x, y, z))$
- $\mathrm{divF} = \nabla \cdot F = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}$

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \mathrm{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

function in Euclidean space

divergence operator

Cartesian coordinates

Laplace-Beltrami Operator: $\Delta_S \mathrm{f} = \mathrm{div}_S \nabla_S f$

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{S}} \mathbf{x} = \mathrm{div}_{\mathcal{S}} \ \nabla_{\mathcal{S}} \mathbf{x} = \quad 2H\mathbf{n}$$

coordinate function

divergence operator

surface normal

For researchers in CG (for differential coordinates), $\Delta_s = -2Hn$
For mathematician, $\Delta_s = 2Hn$
The only difference is the sign.

Thanks