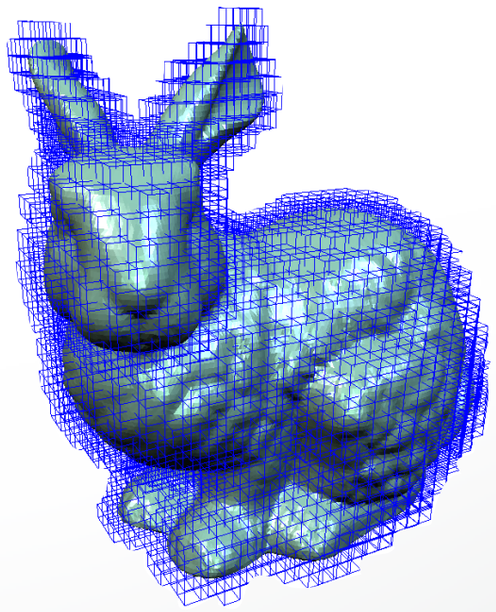
Spring 2017

CSCI 621: **Digital Geometry Processing**

3.1 Explicit & Implicit Surfaces



Hao Li

http://cs621.hao-li.com

Administrative

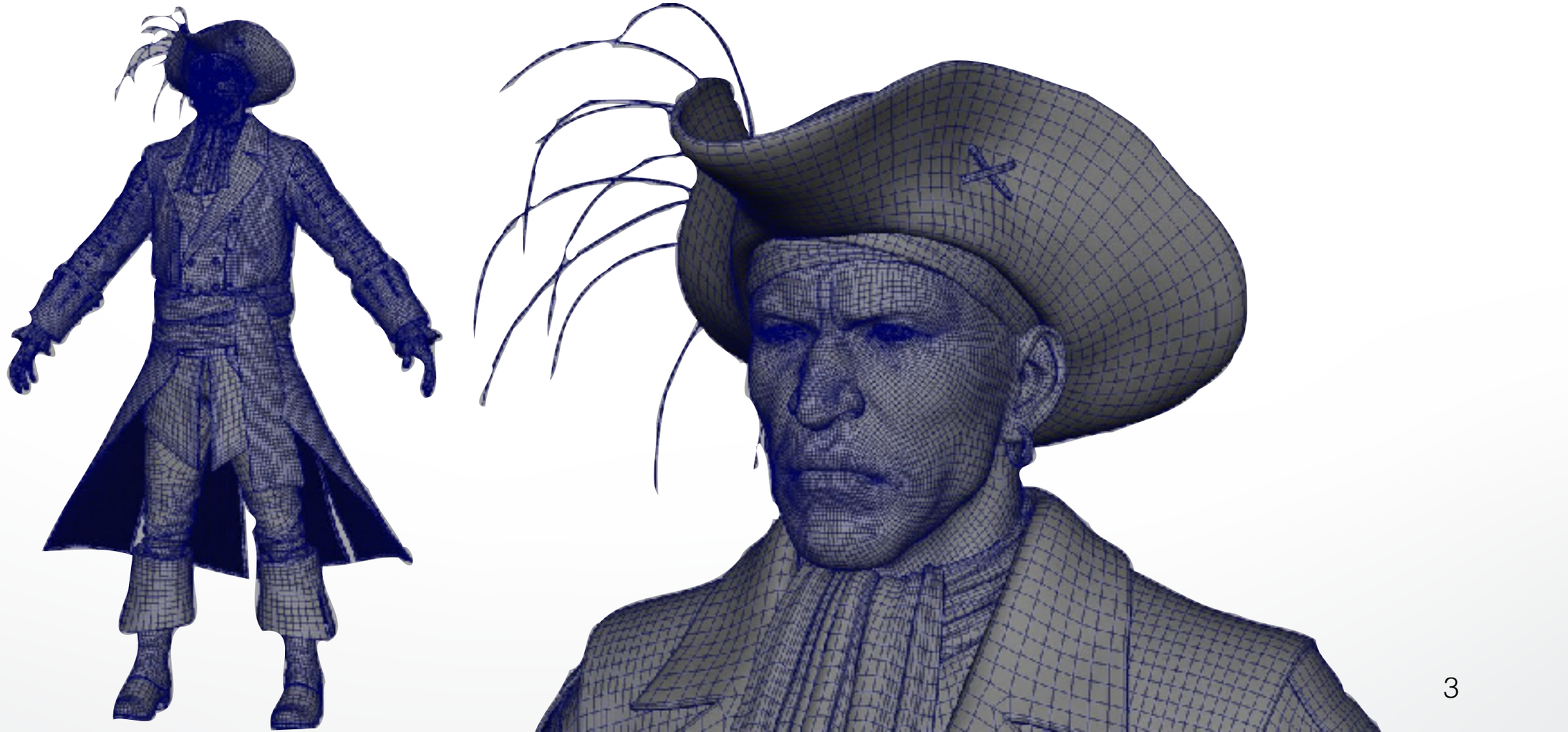
- **Exercise 1 discussion: Later**
- **Hao Li (Instructor)**
  - Office Hour: Tue 2:30 PM - 3:00 PM, SAL 244

- **Yi Zhou & Shunsuke Saito (TA)**
  - Office Hour: Still TBD :-)

Last Time

Polygonal meshes are

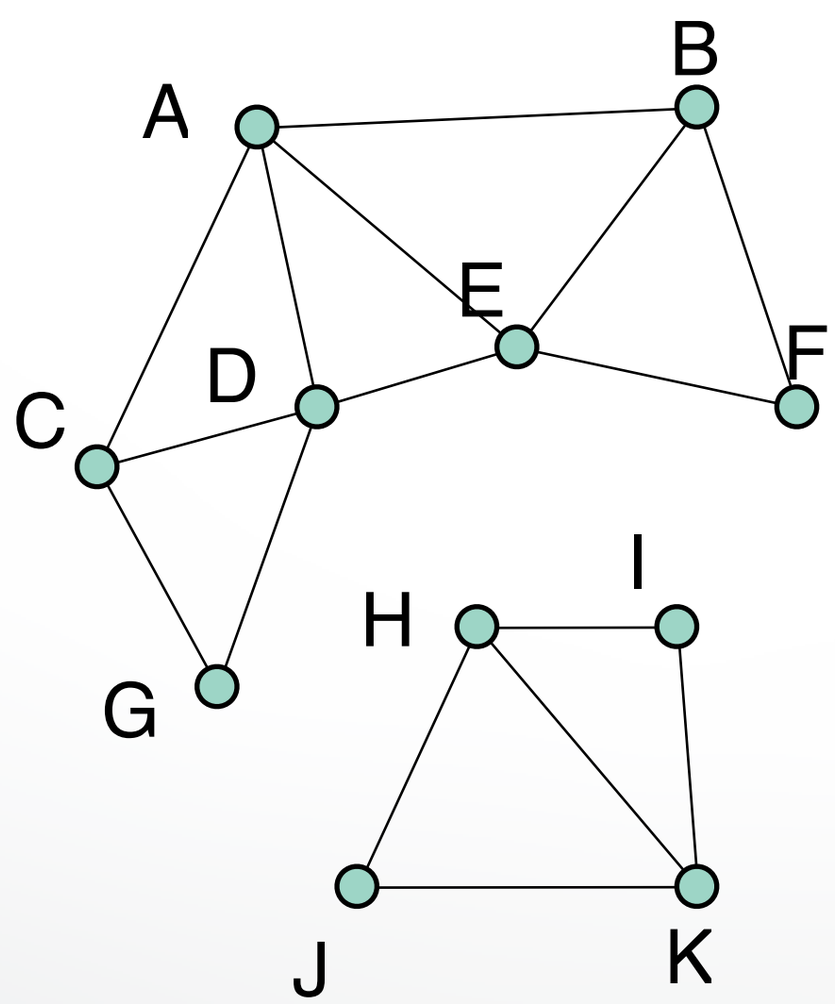
- Effective representations
- Flexible
- Efficient, simple, enables unified processing

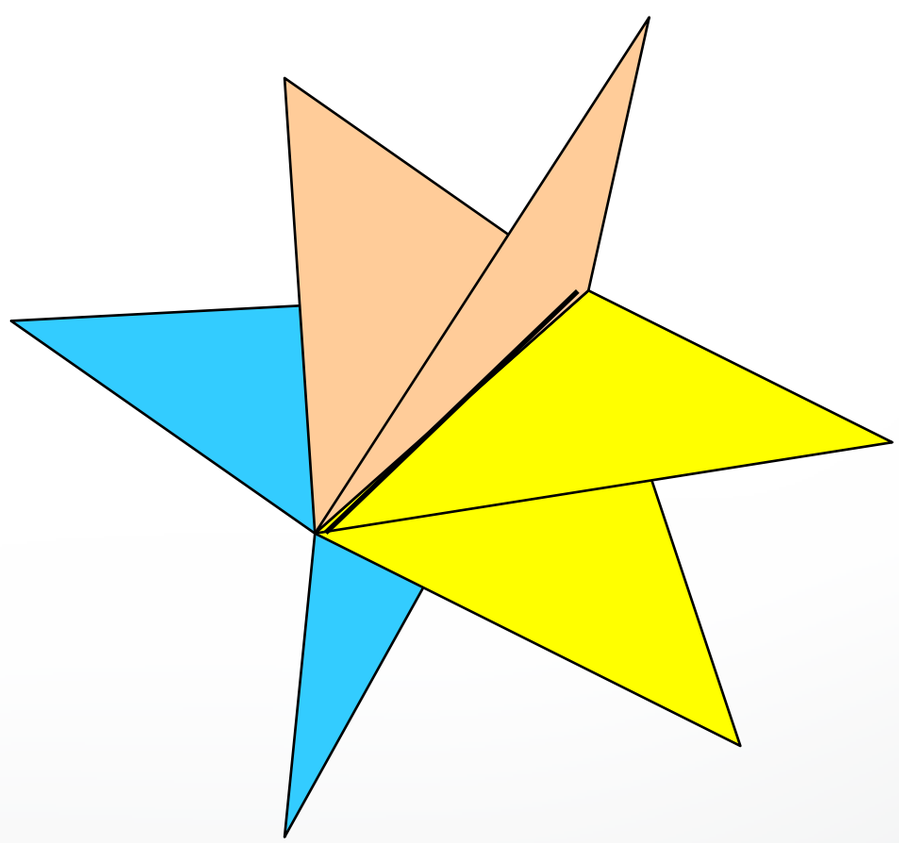


Last Time

Connection between Meshes and Graphs

- Formalism (valence, connections, subgraph, embedding…)
- Definitions (boundary, regular edge, singular edge, closed mesh)
- triangulation → triangle mesh

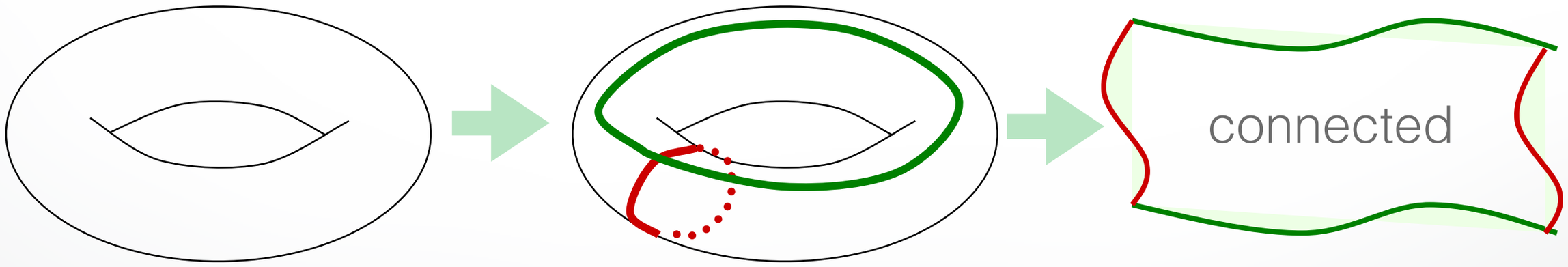




Last Time

Topology

- Genus, Euler characteristic
- Euler Poincaré formula $V - E + F = 2(1 - g)$
- Average valence of triangle mesh: 6
- Triangles: F = 2V, E = 3V
- Quads: F = V, E = 2V

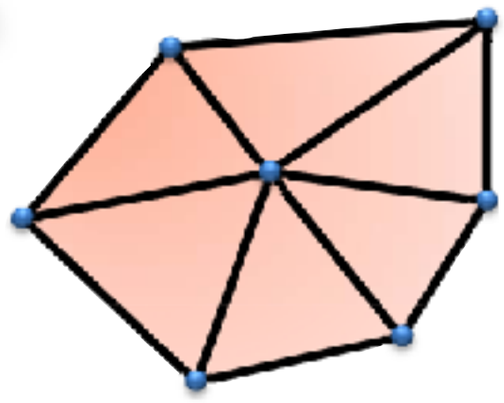


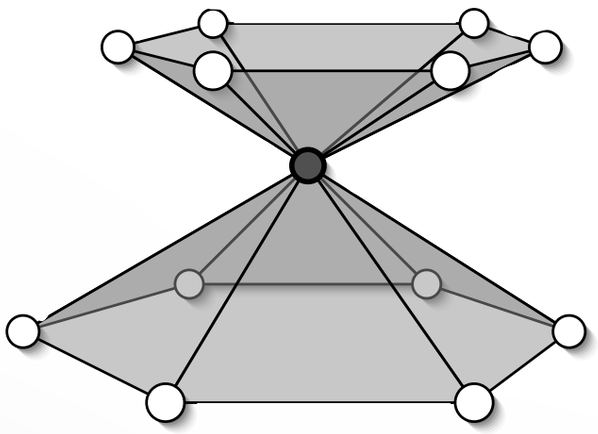
k=1 handle ≤2k edge loops

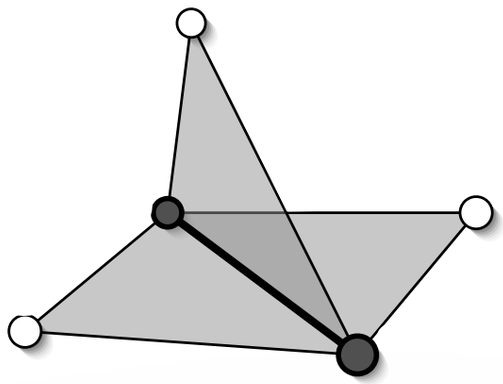
Last Time

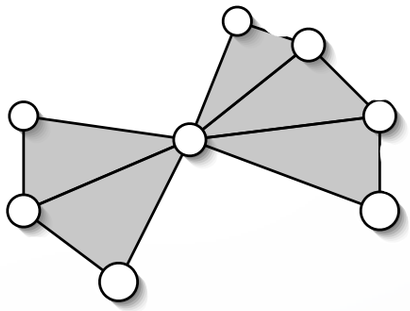
2-Manifold Surface

- Local Neighborhood is disk-shaped $\mathbf{f}(D_\epsilon[u,v]) = D_\delta[\mathbf{f}(u,v)]$
- Guarantees meaningful neighbor enumeration
- Non-manifold





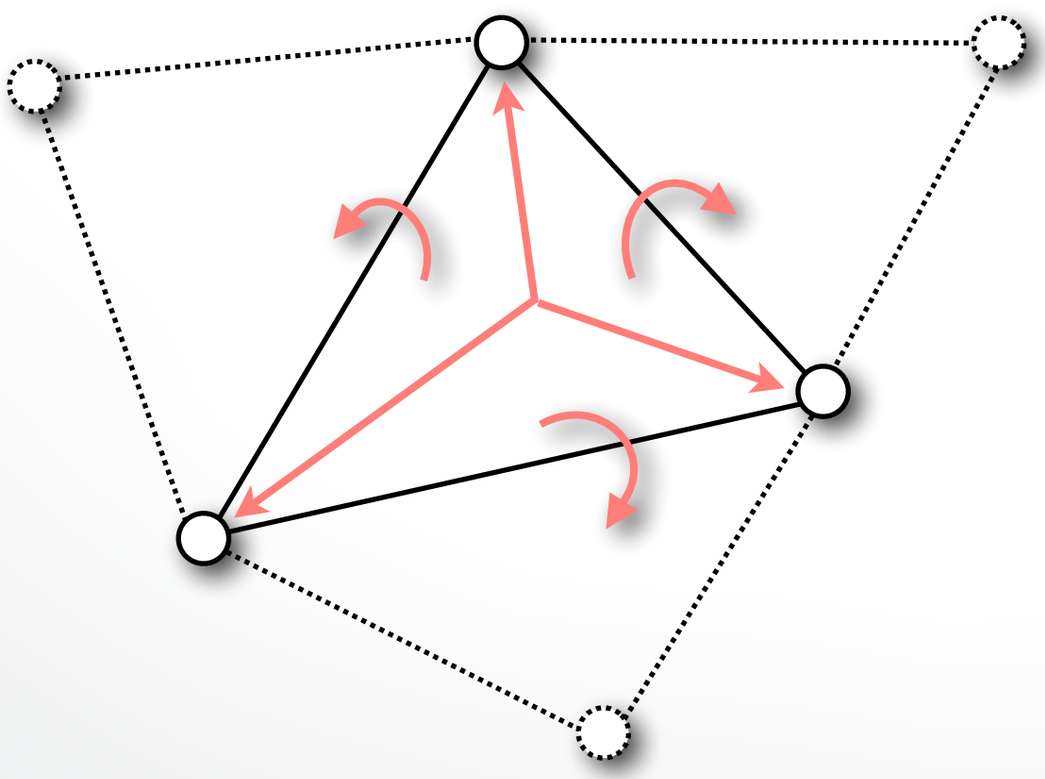


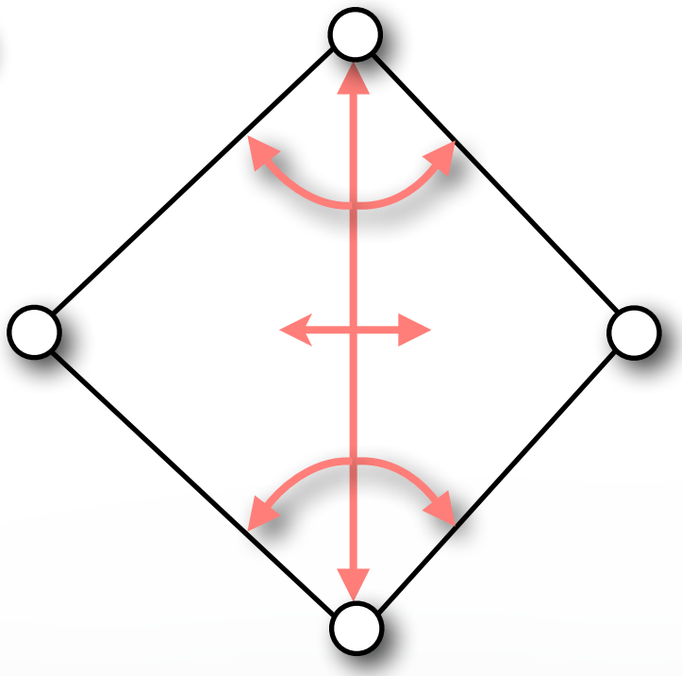


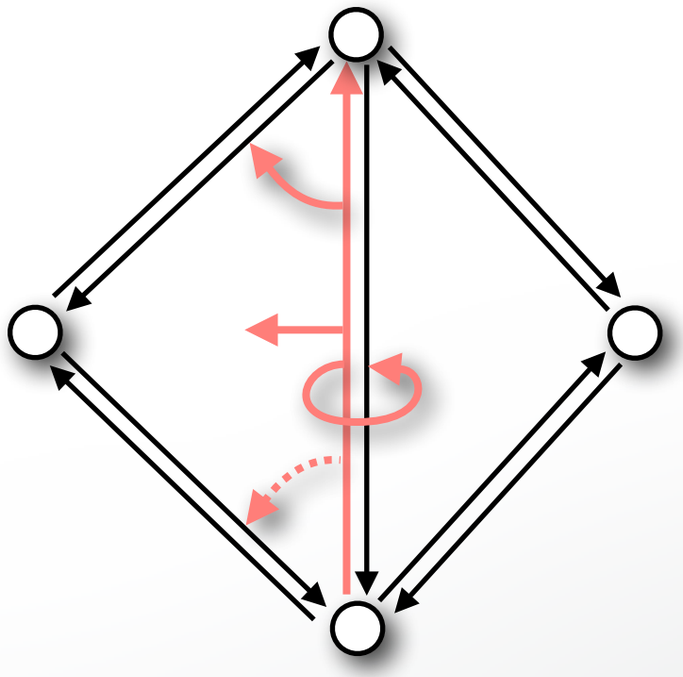
Last Time

Data Structures

- Face-Based
- Edge-Based, edges always have two faces
- Halfedge-Based

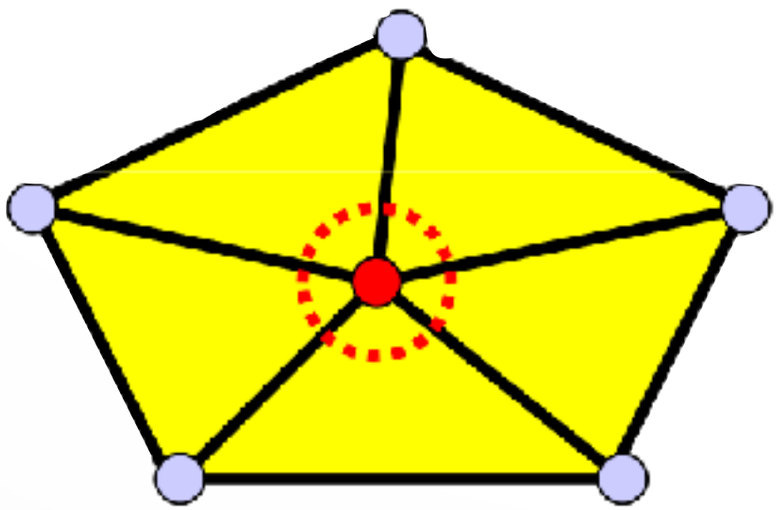




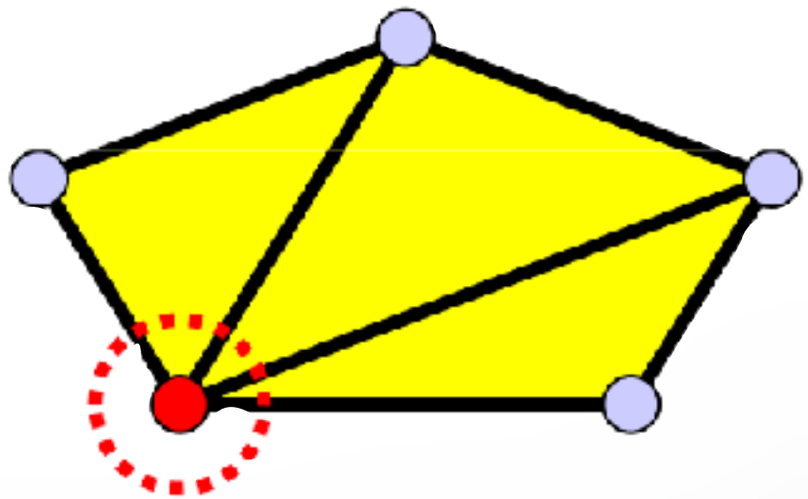


When is a Triangle Mesh a Manifold?

- Every Edge incident to 1 or 2 Triangles
- Faces incident to a vertex form closed or open fan



closed fan



open fan

Outline

- **Surface Representations**
- Explicit Surfaces
- Implicit Surfaces
- Conversion

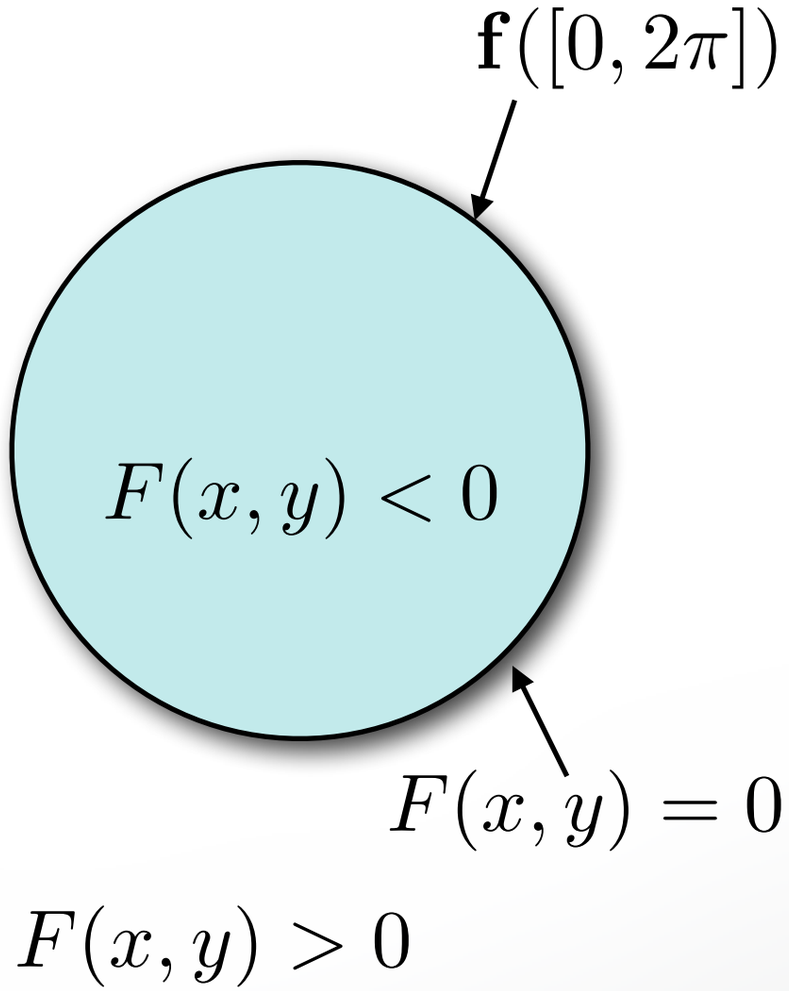
Explicit vs. Implicit

Explicit: $\mathbf{f}(x) = (r\cos(x), r\sin(x))^T$

- Range of parameterization function

Implicit: $F(x, y) = \sqrt{x^2 + y^2} - r$

- Kernel of implicit function



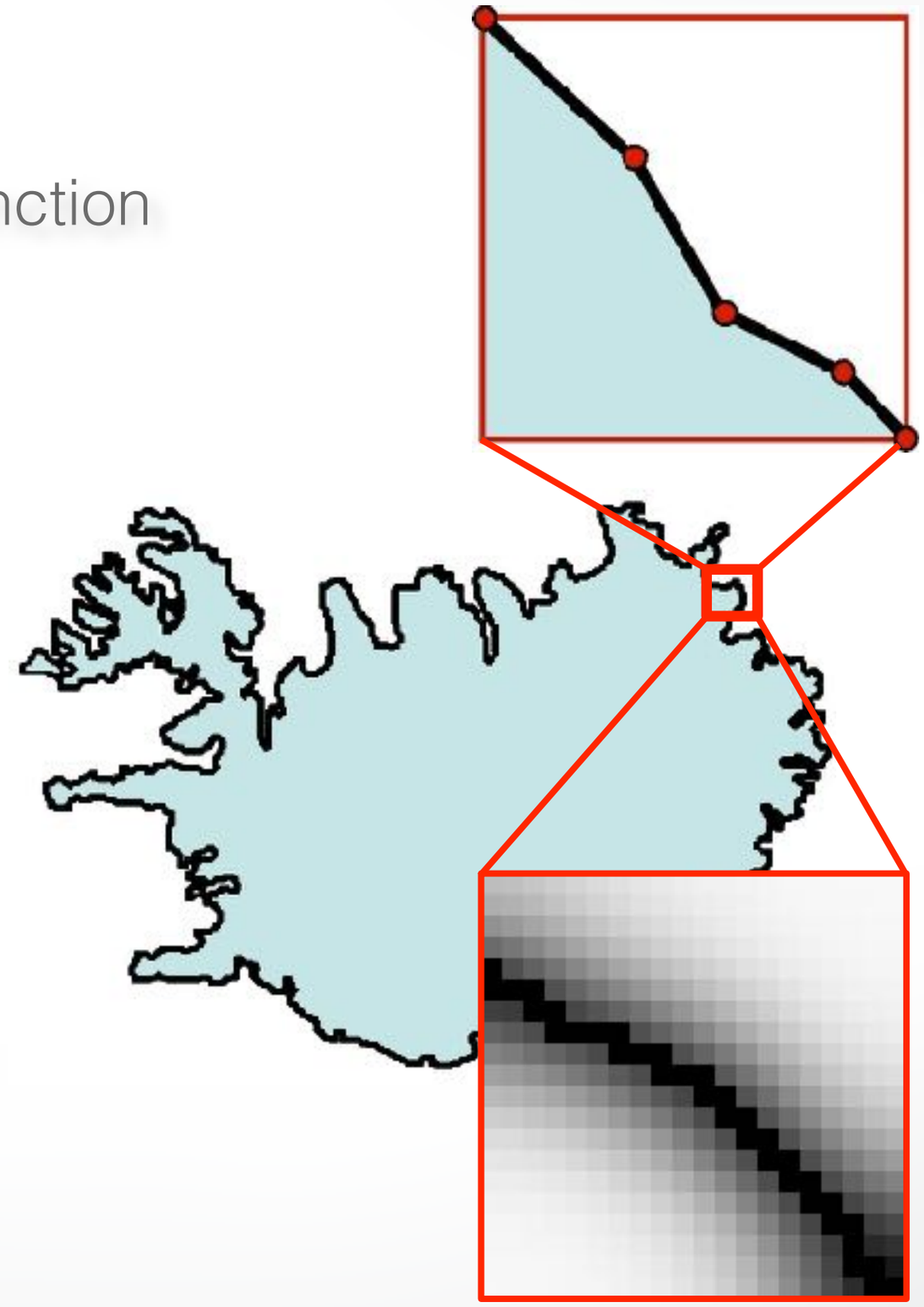
Explicit vs. Implicit

Explicit: $\mathbf{f}(x) =$?

- Range of parameterization function
- Piecewise approximation

Implicit: $F(x, y) =$?

- Kernel of implicit function
- Piecewise approximation



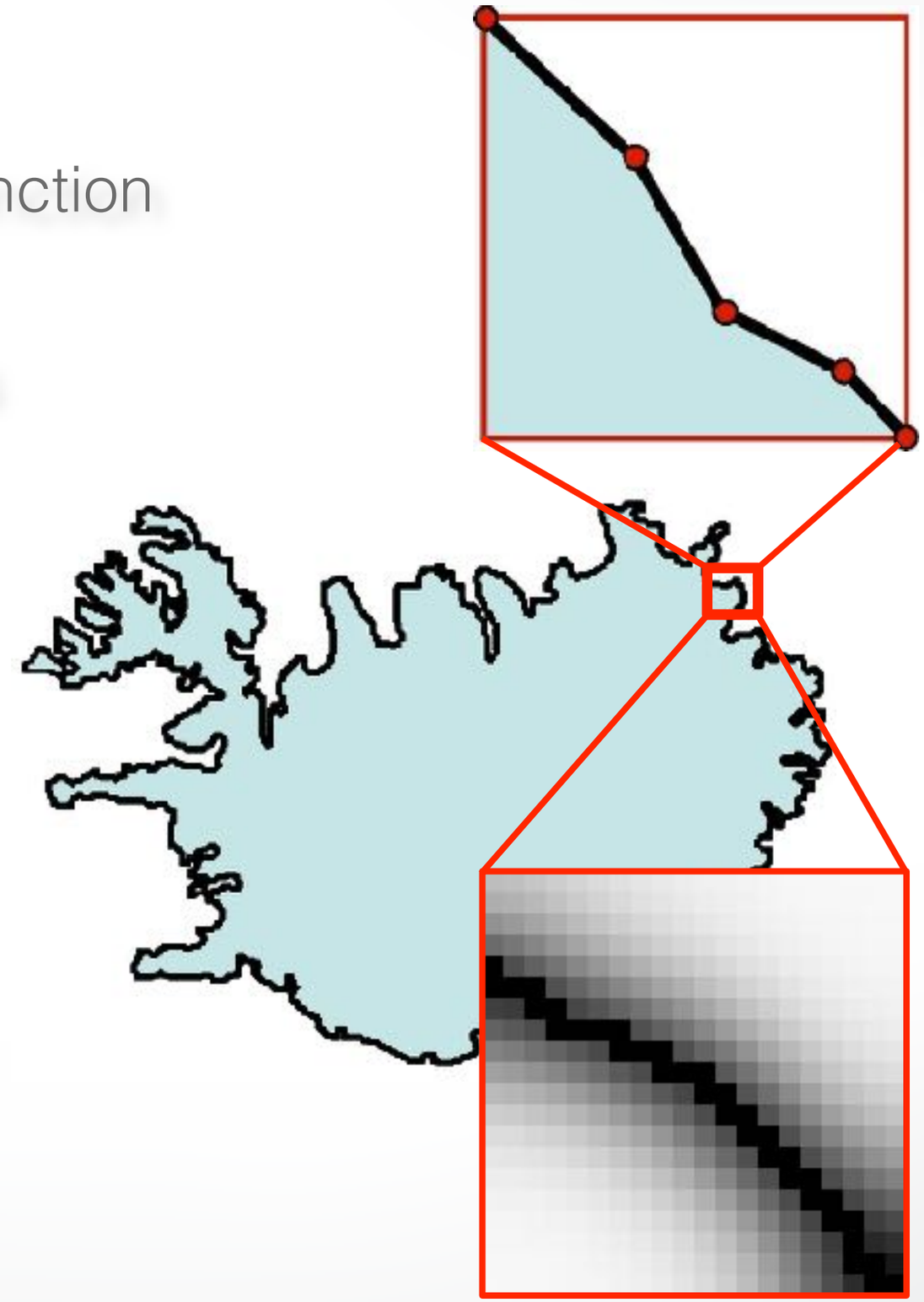
Explicit vs. Implicit

Explicit:

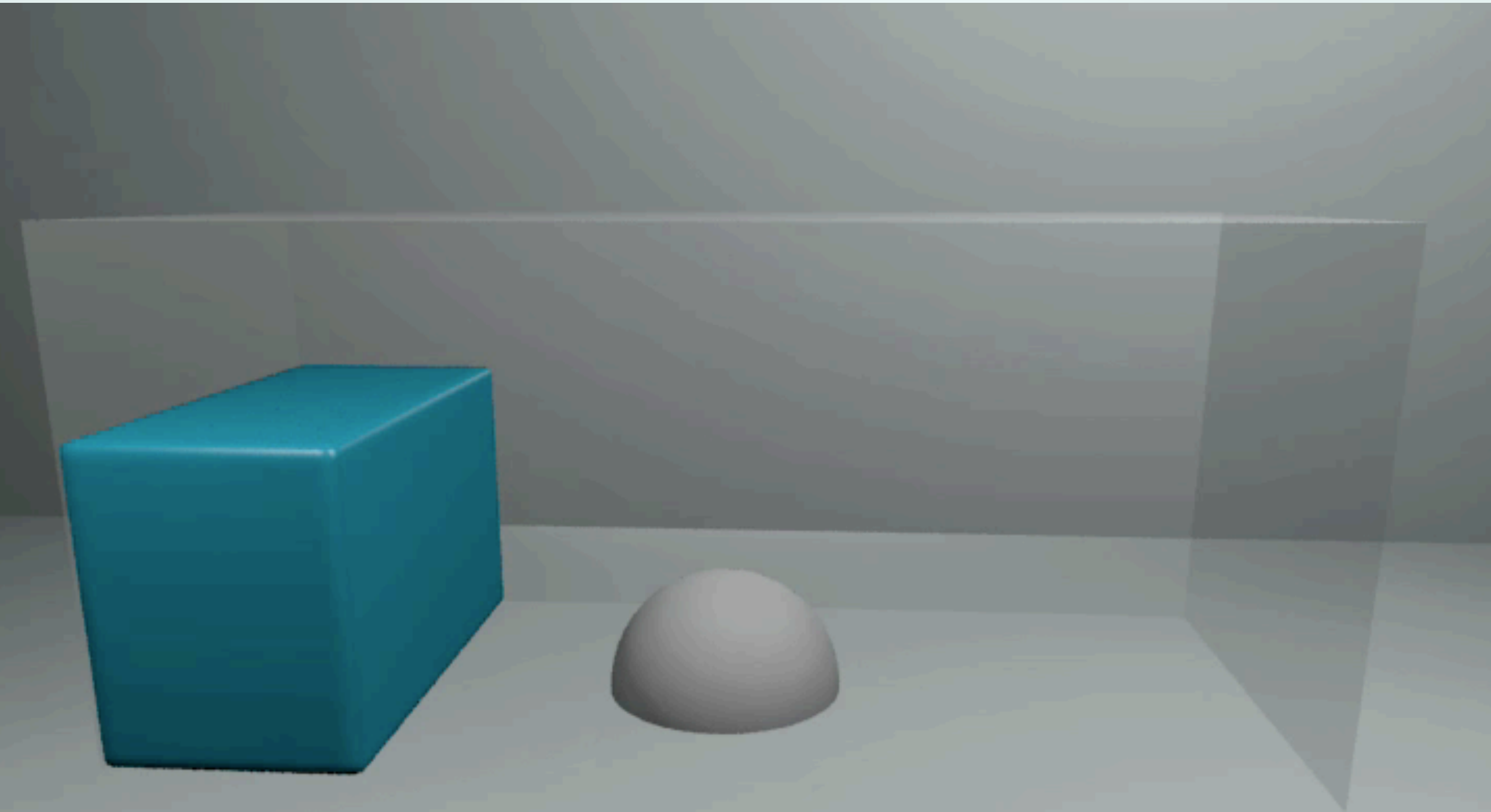
- Range of parameterization function
- Piecewise approximation
- Splines, triangle mesh, points
- Easy enumeration
- Easy geometry modification

Implicit:

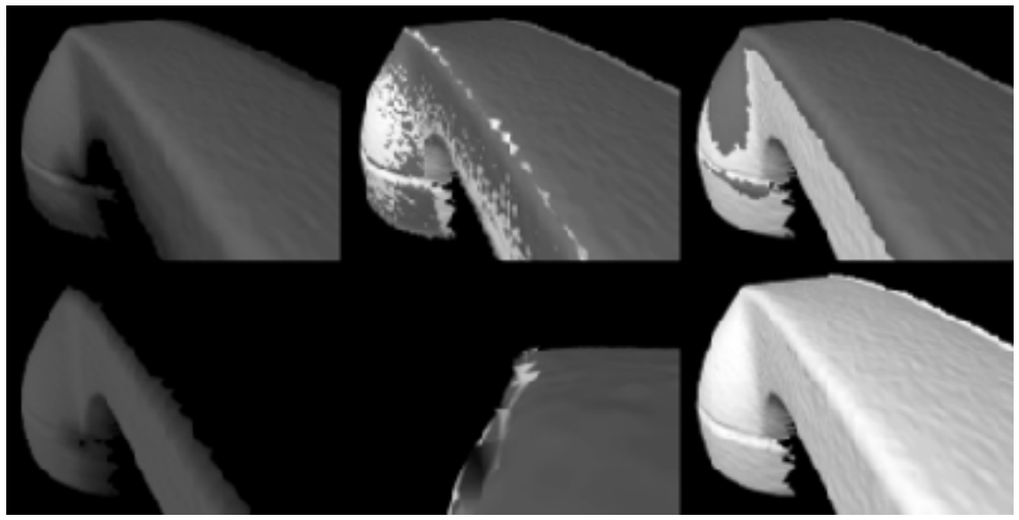
- Kernel of implicit function
- Piecewise approximation
- Scalar-valued 3D grid
- Easy in/out test
- Easy topology modification

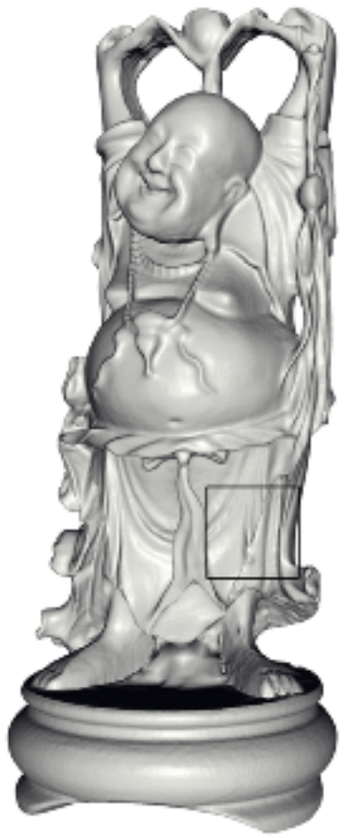


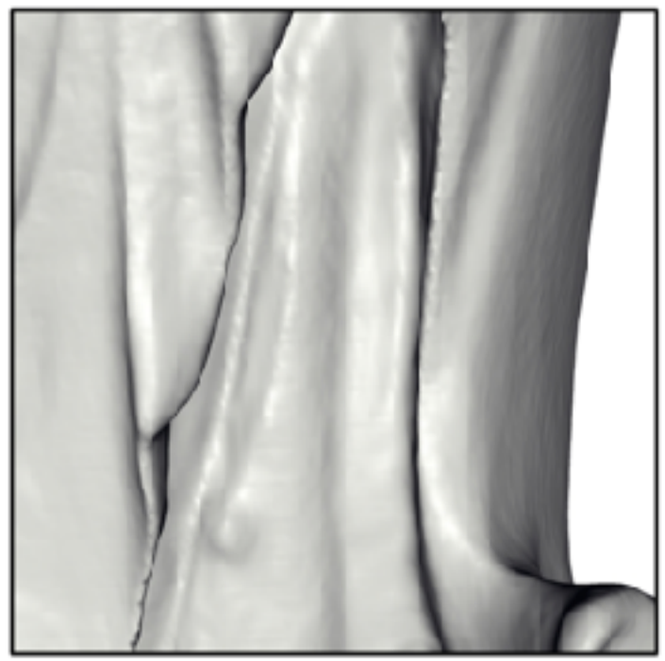
Examples: Fluid Simulation

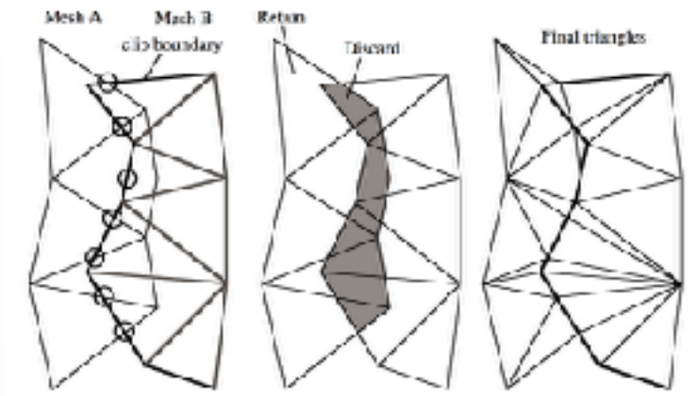


Examples: 3D Reconstruction

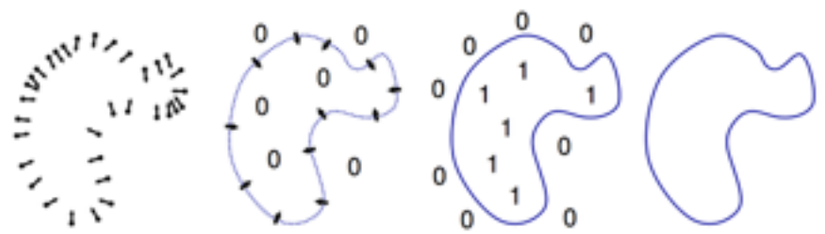








Zippering



Poisson Reconstruction

Examples: Kinect Fusion

1. Capture
2. Align
3. Fuse

http://msdn.microsoft.com/en-us/library/dn188670.aspx

Outline

- Surface Representations
- **Explicit Surfaces**
- Implicit Surfaces
- Conversion

Polynomial Approximation

Polynomials are computable functions

$$f(t) \;=\; \sum_{i=0}^{p} c_i\, t^i \;=\; \sum_{i=0}^{p} \tilde{c}_i\, \phi_i(t)$$

Taylor expansion up to degree p

$$g(h) \;=\; \sum_{i=0}^{p} \frac{1}{i!}\, g^{(i)}(0)\; h^i \;+\; O\big(h^{p+1}\big)$$

Error for approximation g by polynomial f

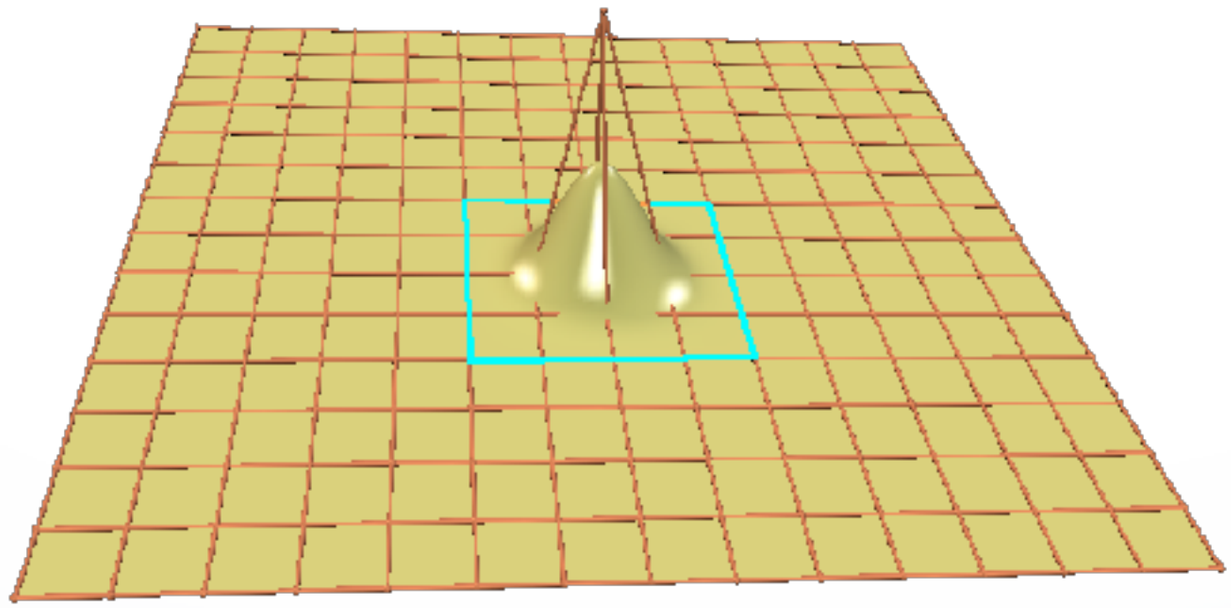
$$f(t_i) = g(t_i)\,, \quad 0 \le t_0 < \cdots < t_p \le h$$

$$|f(t) - g(t)| \;\le\; \frac{1}{(p+1)!}\, \max f^{(p+1)} \prod_{i=0}^{p} (t - t_i) \;=\; O\Big(h^{(p+1)}\Big)$$

Spline Surfaces

Piecewise polynomial approximation

$$\mathbf{f}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{c}_{ij} N_i^n(u) N_j^m(v)$$

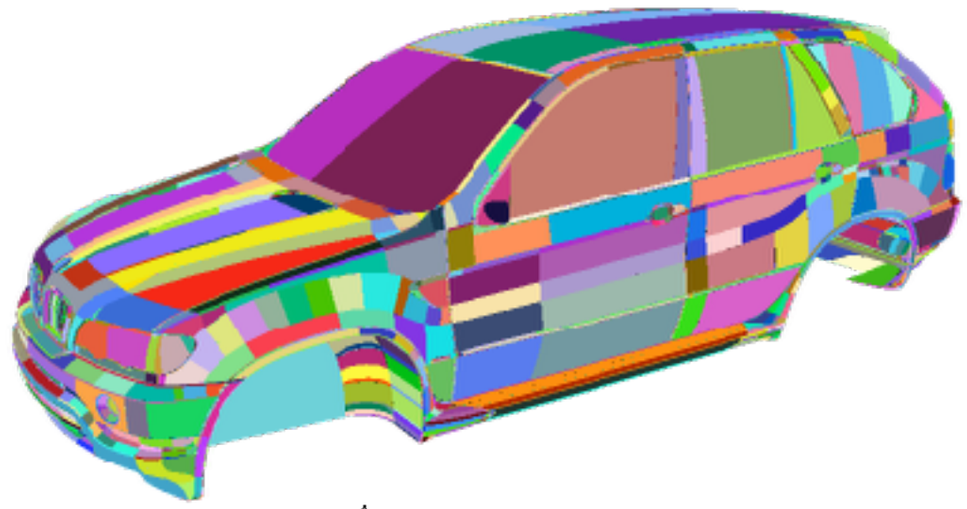


Spline Surfaces

Piecewise polynomial approximation

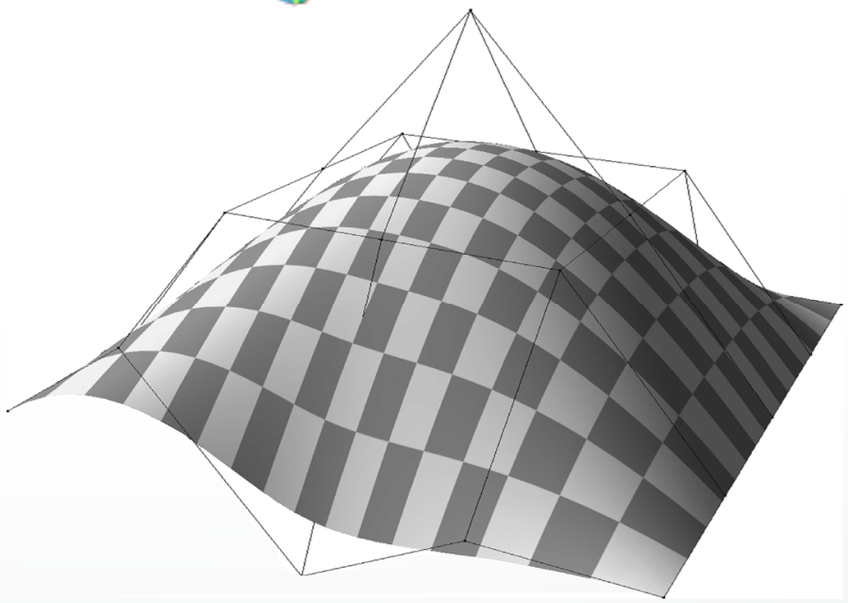
Geometric constraints

- Large number of patches
- Continuity between patches
- Trimming



Topological constraints

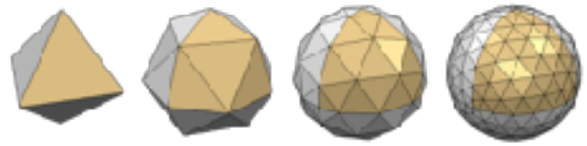
- Rectangular patches
- Regular control mesh

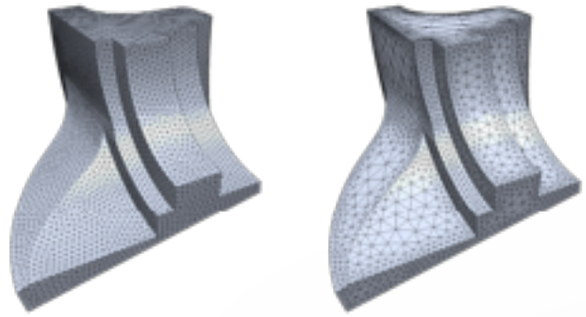


Polygon Meshes

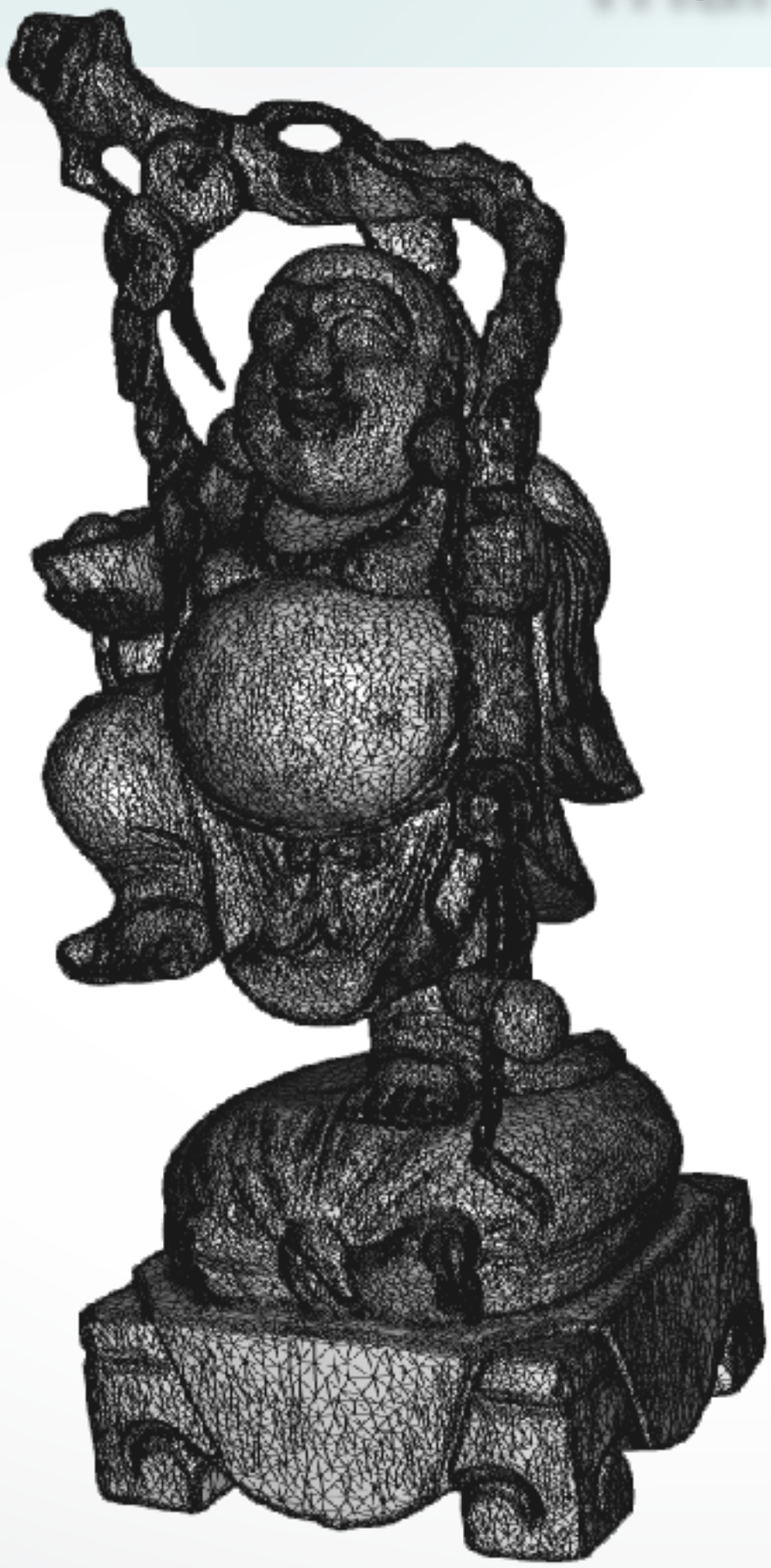
Polygonal meshes are a good compromise

- Piecewise linear approximation → error is $O(h^2)$
- Error inversely proportional to #faces
- Arbitrary topology surfaces
- Piecewise smooth surfaces
- Adaptive sampling
- Efficient GPU-based rendering/processing





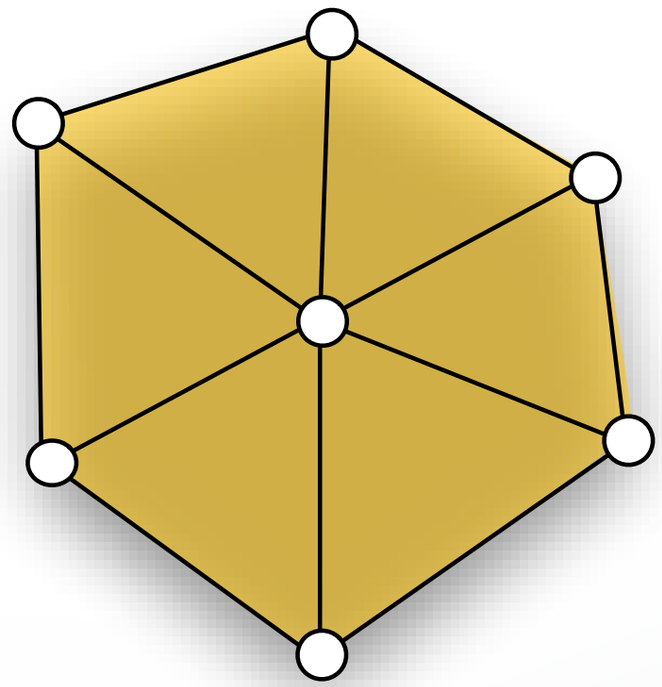
Triangle Meshes



$$\mathcal{M} = (\{\mathbf{v}_i\}, \{e_j\}, \{f_k\})$$

geometry $\mathbf{v}_i \in \mathbb{R}^3$

topology $e_i, f_i \subset \mathbb{R}^3$

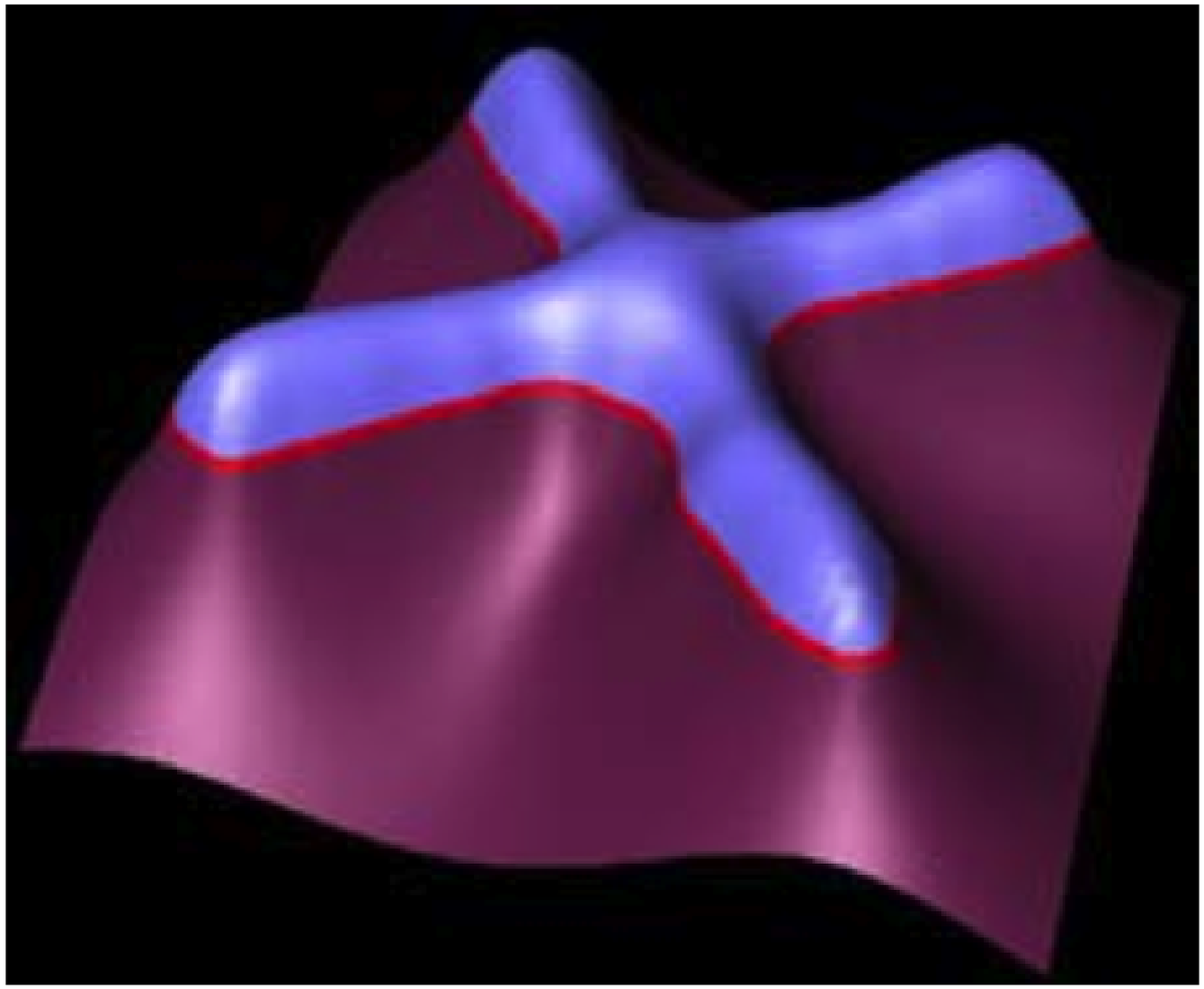


Outline

- Surface Representations
- Explicit Surfaces
- **Implicit Surfaces**
- Conversion

Implicit Representations

Level set of 2D function defines 1D curve



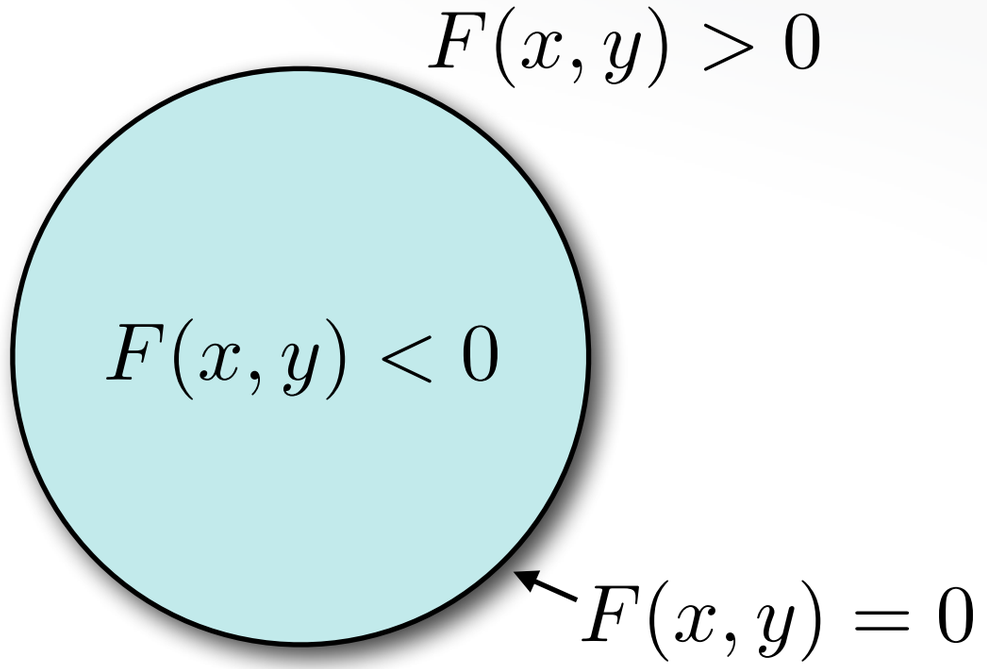
Implicit Representations

Level set of 3D function defines 2D surface

Implicit Representations

General implicit function:

- Interior: $F(x, y, z) < 0$
- Exterior: $F(x, y, z) > 0$
- Surface: $F(x, y, z) = 0$



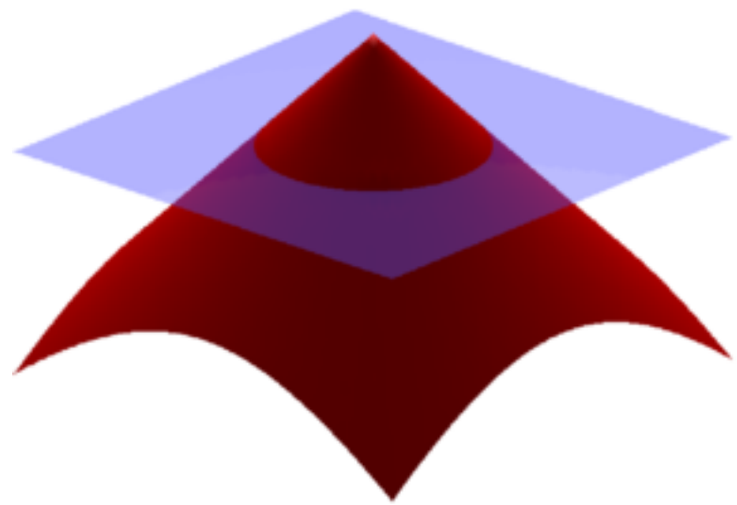
Gradient ∇F is orthogonal to level set

Special case

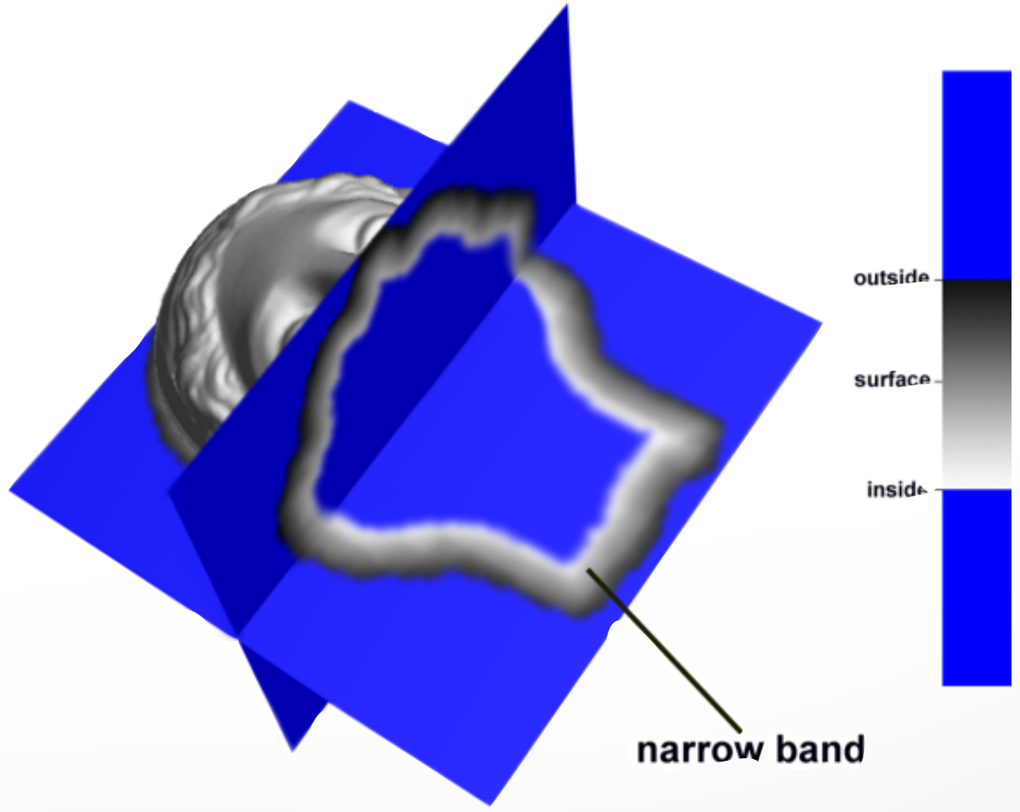
- Signed distance function (SDF)
- Gradient ∇F is unit surface normal

Signed Distance Function

SDF of a circle?



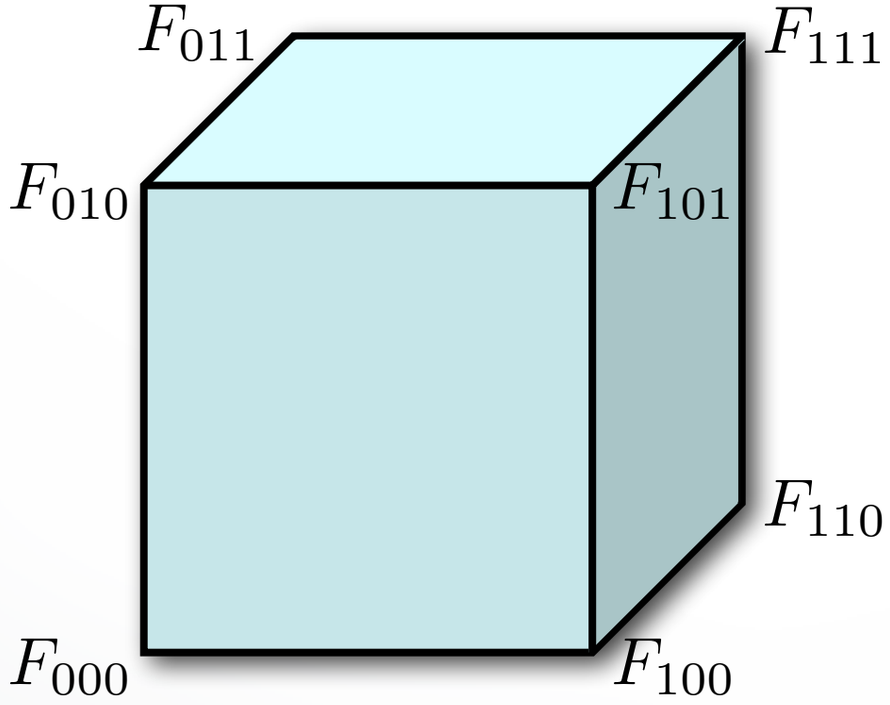
General shapes



SDF Discretization

Regular cartesian 3D grid

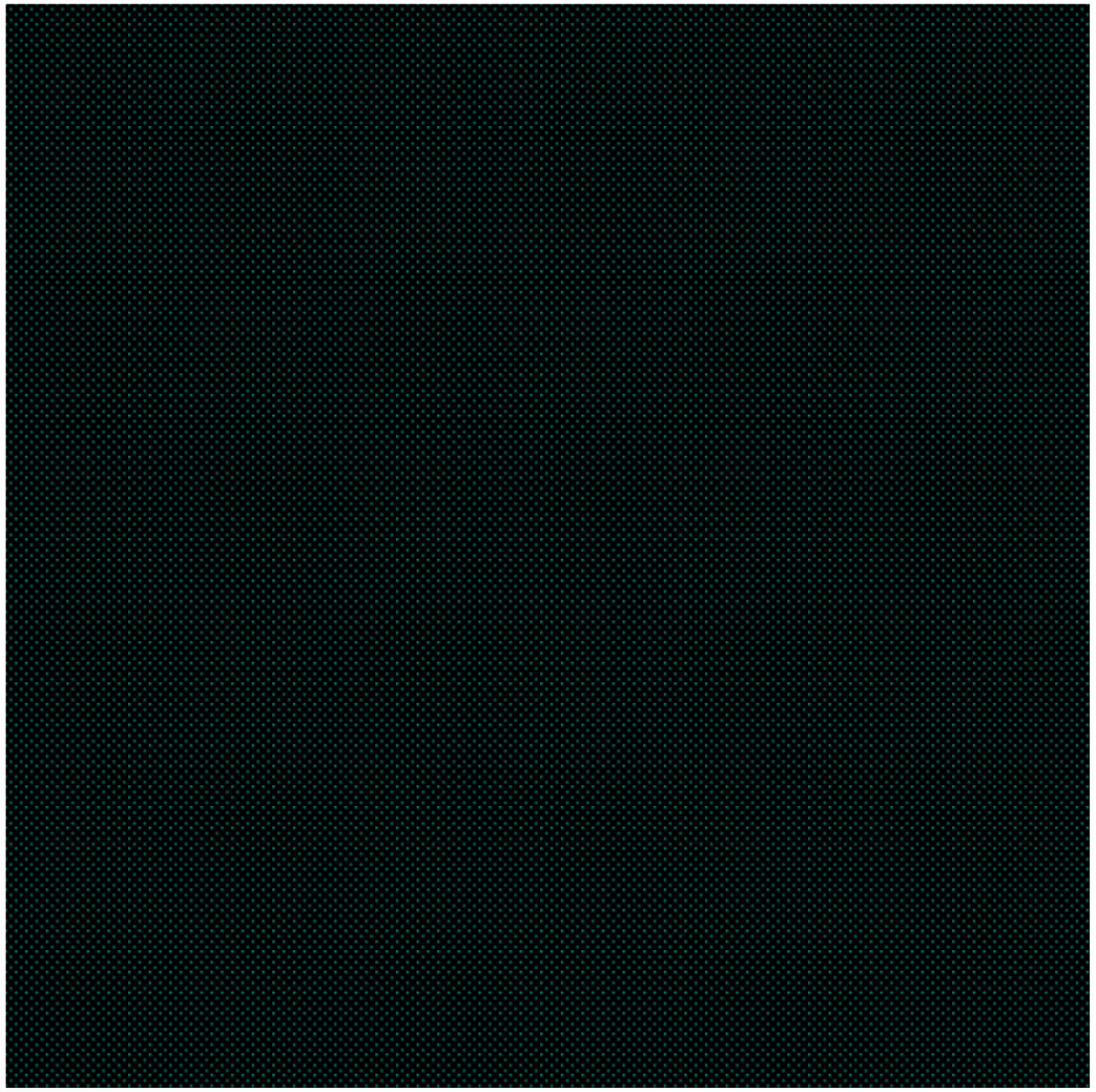
- Compute signed distance at nodes
- Tri-linear interpolation within cells



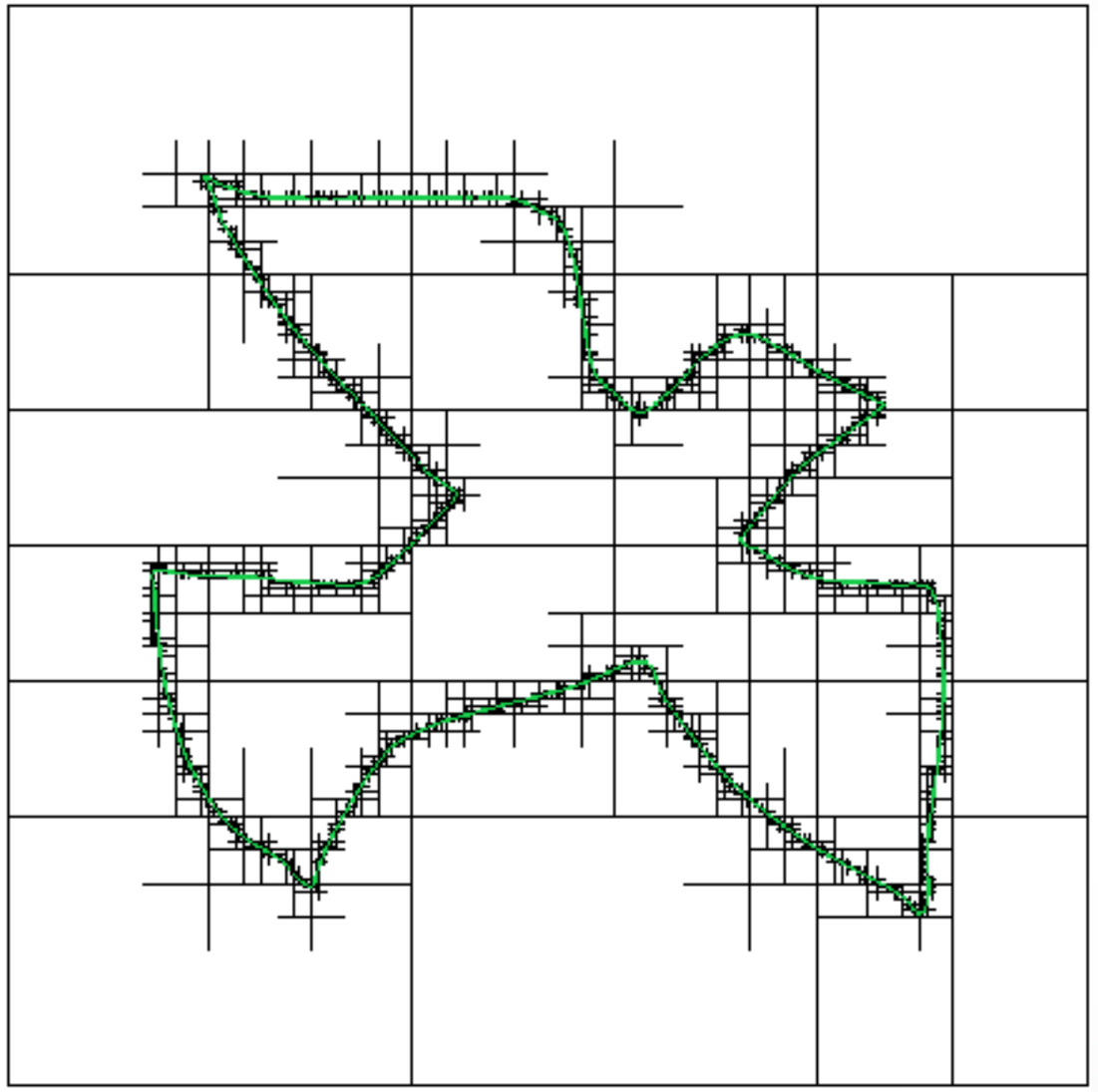
$$
\begin{array}{llllll}
F_{000} & (1-u) & (1-v) & (1-w) & + \\
F_{100} & u & (1-v) & (1-w) & + \\
F_{010} & (1-u) & v & (1-w) & + \\
F_{001} & (1-u) & (1-v) & w & + \\
\vdots & & & & \\
F_{111} & u & v & w &
\end{array}
$$

3-Color Octree

interior,exterior,boundary

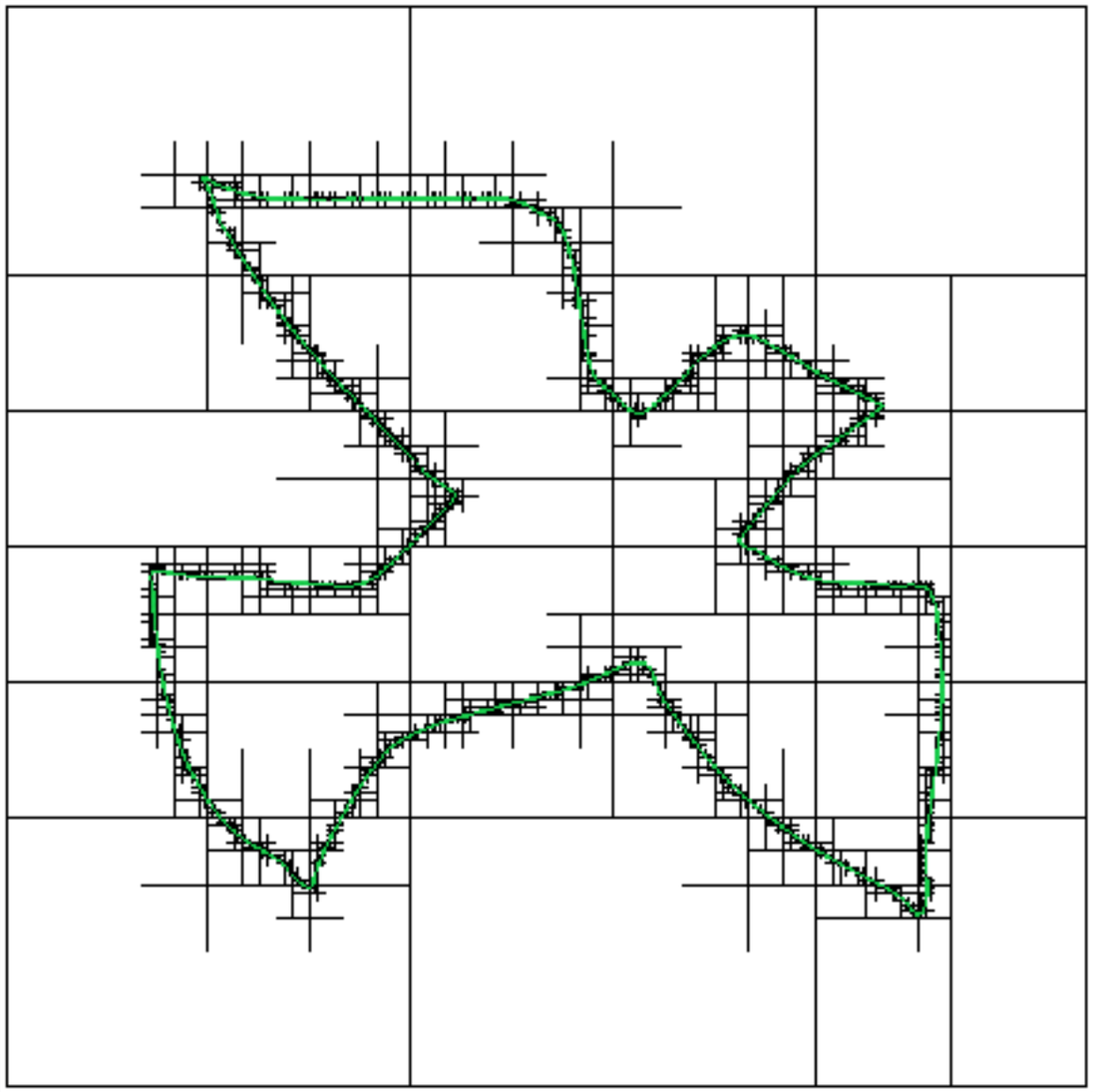


1048576 cells

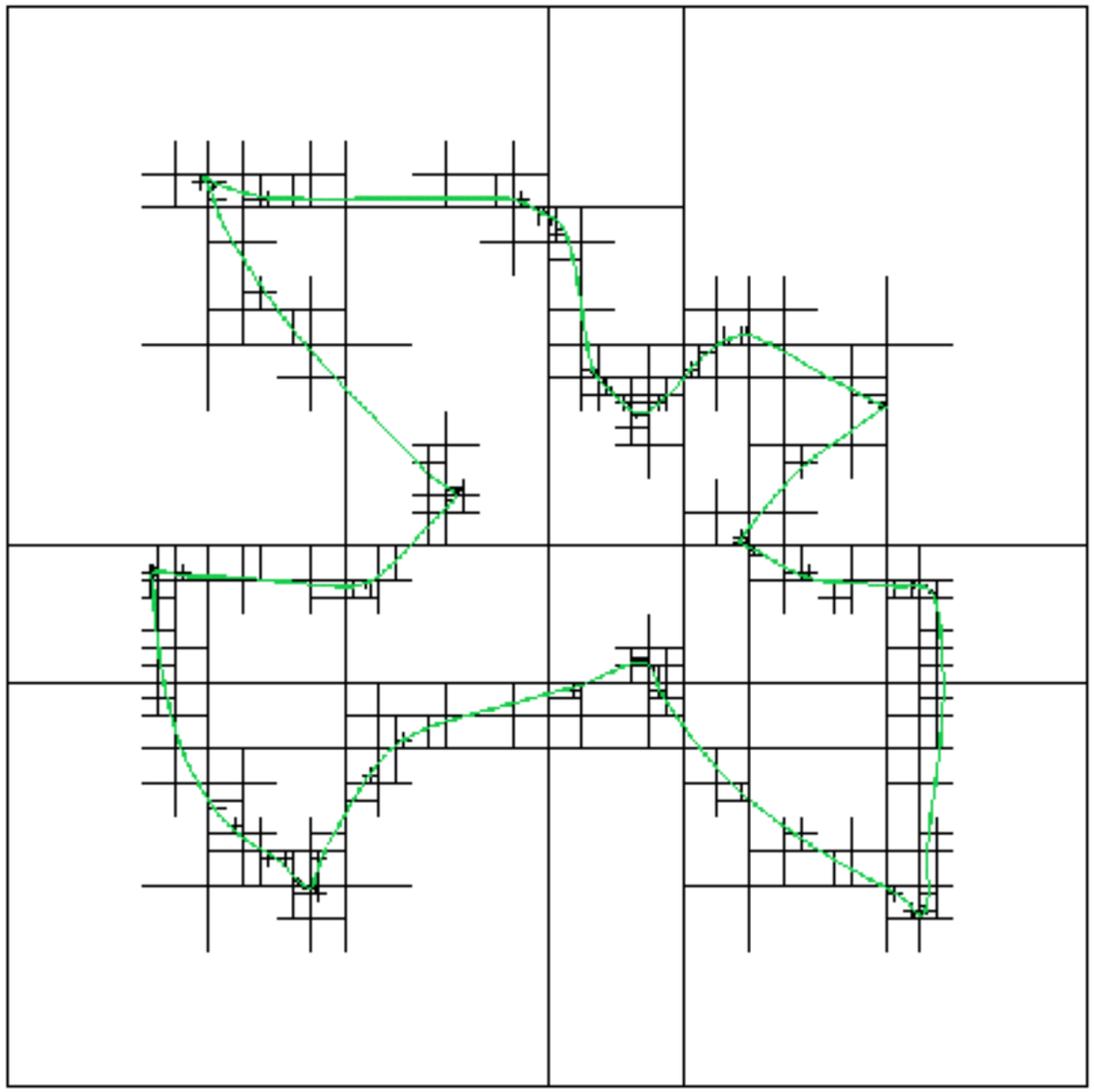


12040 cells

Adaptively Sampled Distance Fields

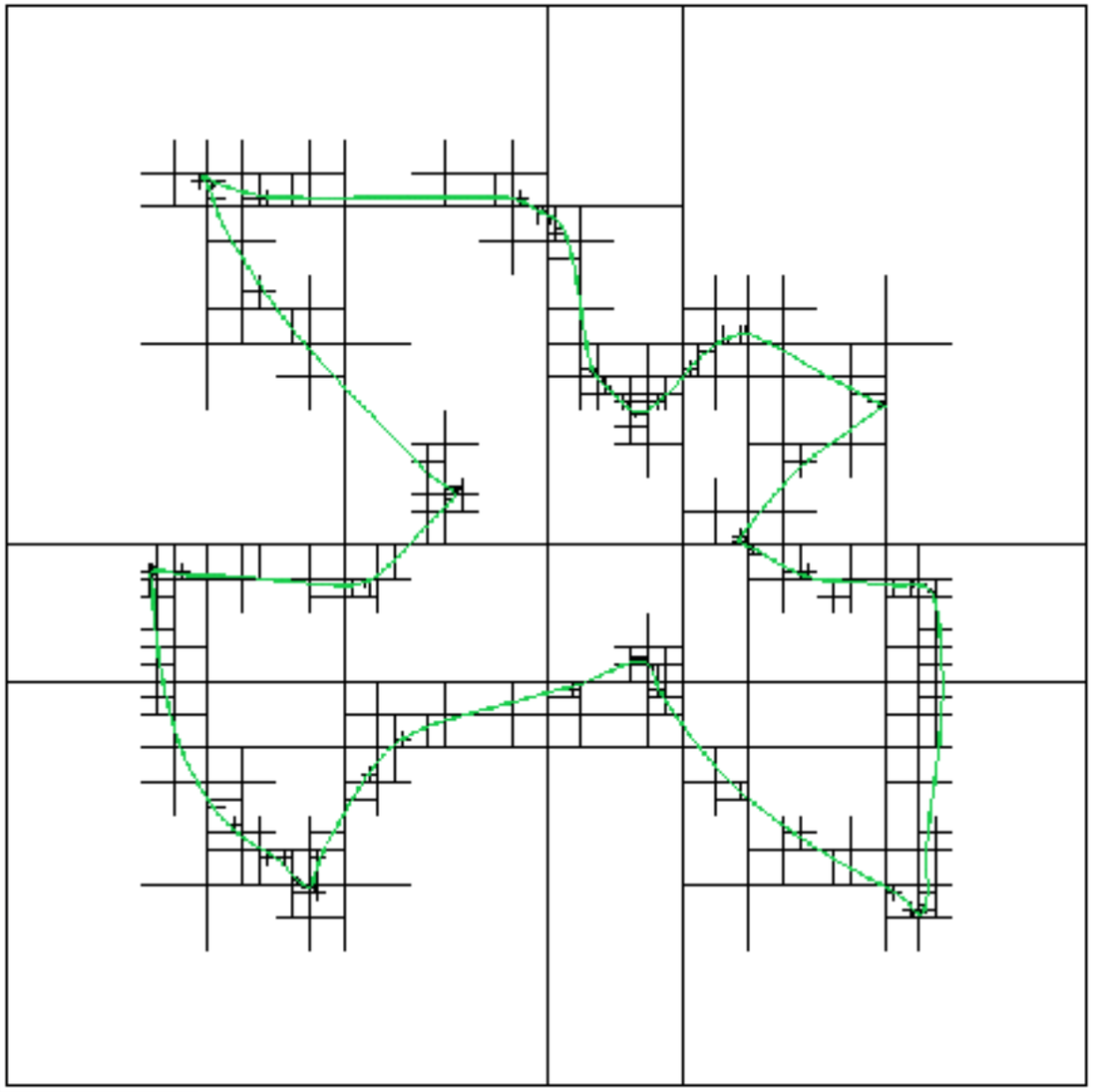


12040 cells

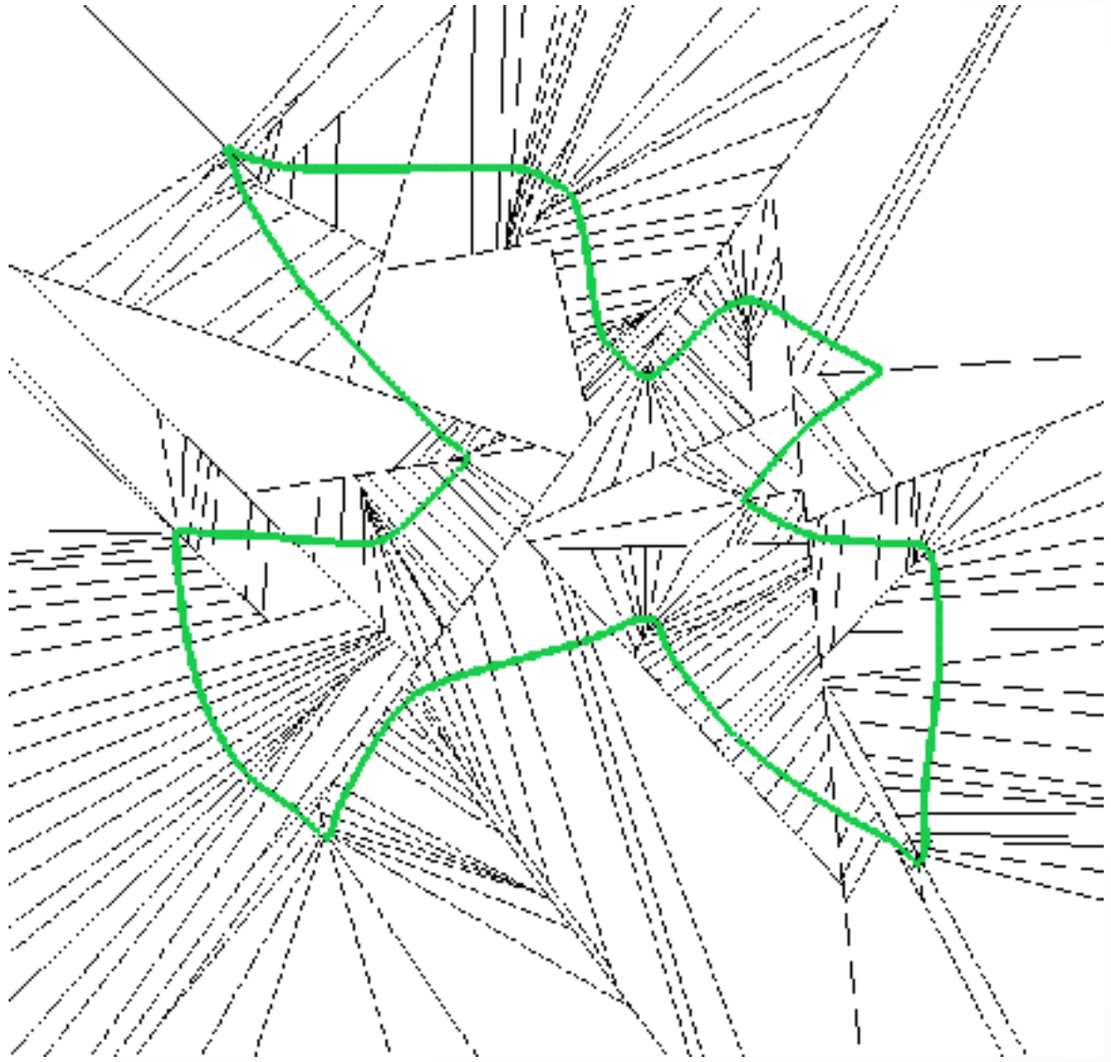


895 cells

Binary Space Partitions



895 cells



254 cells

Regularity vs. Complexity

Implicit surface discretizations

- Uniform, regular voxel grids $O(h^{-3})$
- Adaptive, 3-color octrees
  - Surface-adaptive refinement $O(h^{-2})$
  - Feature-adaptive refinement $O(h^{-1})$
- Irregular hierarchies
  - Binary space partition (BSP) $O(h^{-1})$

Literature

- Frisken et al., “Adaptively Sampled Distance Fields: A general representation of shape for computer graphics”, SIGGRAPH 2000
- Wu & Kobbel, “Piecewise Linear Approximation of Signed Distance Fields”, VMV 2003

Implicit Representations

- Natural representation for **volumetric data**: CT scans, density fields, etc.

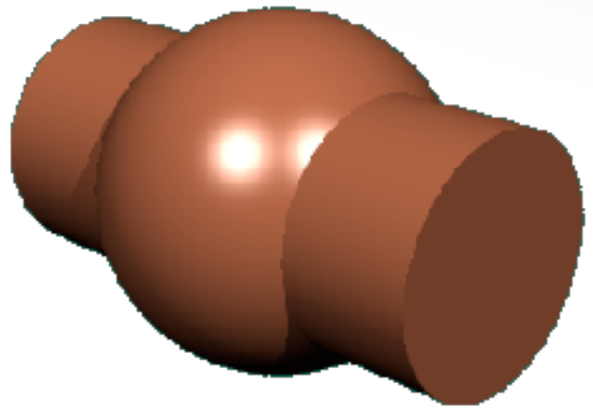
- Advantageous when modeling shapes with **complex and/or changing topology** (e.g., fluids)

- Very suitable representation for **Constructive Solid Geometry** *(CSG)*

CSG Example

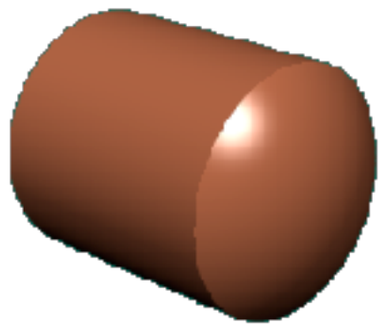
Union

$$F_{C \cup S}(\cdot) = \min \{F_C(\cdot), F_S(\cdot)\}$$



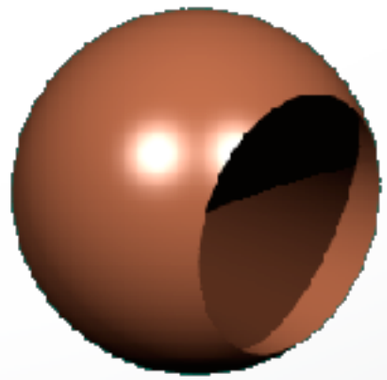
Intersection

$$F_{C \cap S}(\cdot) = \max \{F_C(\cdot), F_S(\cdot)\}$$

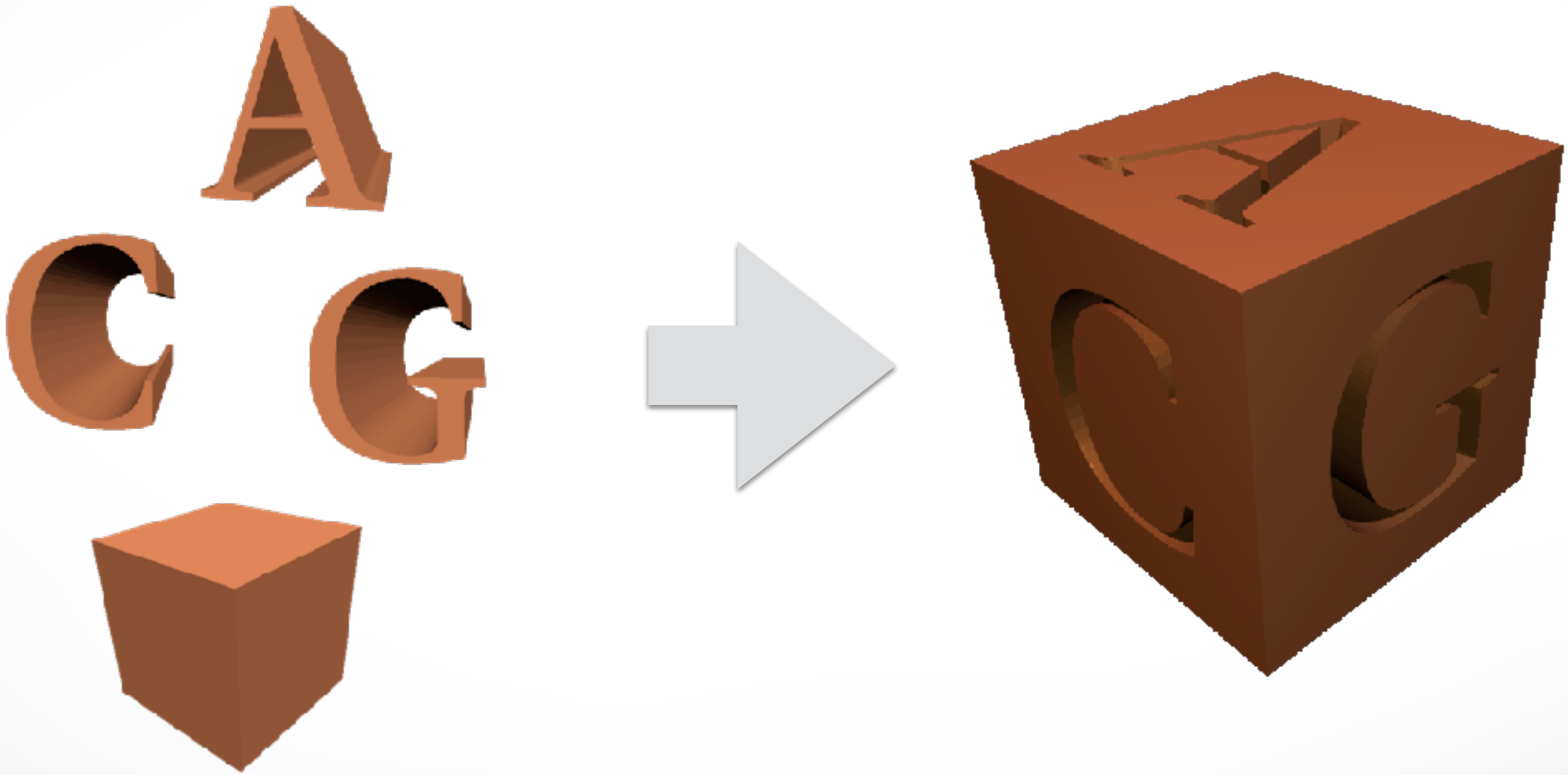


Difference

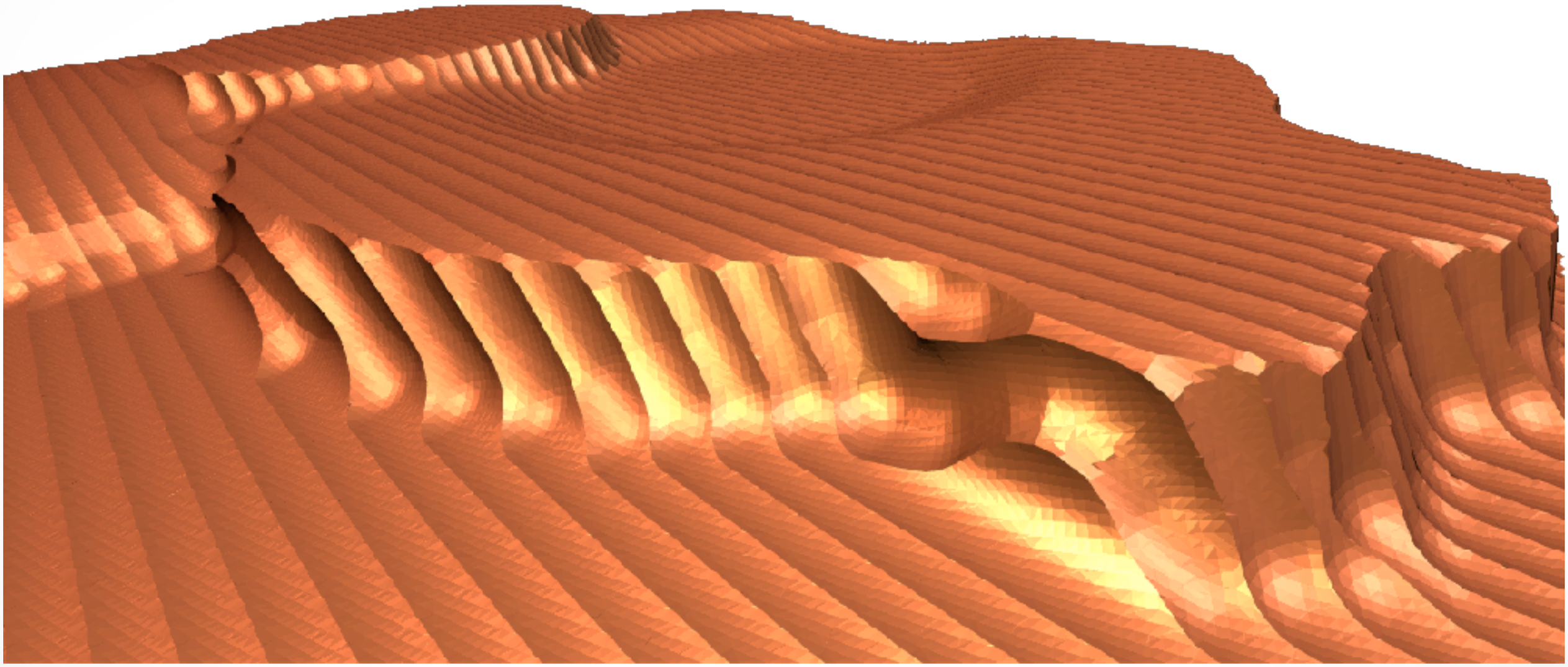
$$F_{S \setminus C}(\cdot) = \max \{-F_C(\cdot), F_S(\cdot)\}$$



CSG Example



CSG Example: Milling



Outline

- Surface Representations
- Explicit Surfaces
- Implicit Surfaces
- **Conversion**

Conversion

Explicit to Implicit

- Compute signed distance at grid points
- Compute distance point-mesh
- Fast marching

Implicit to Explicit

- Extract zero-level iso-surface $F(x, y, z) = 0$
- Other iso-surfaces $F(x, y, z) = C$
- Medical imaging, simulations, measurements, …

Signed Distance Computation

Find closest mesh triangle

- Use spatial hierarchies (octree, BSP tree)

Distance point-triangle

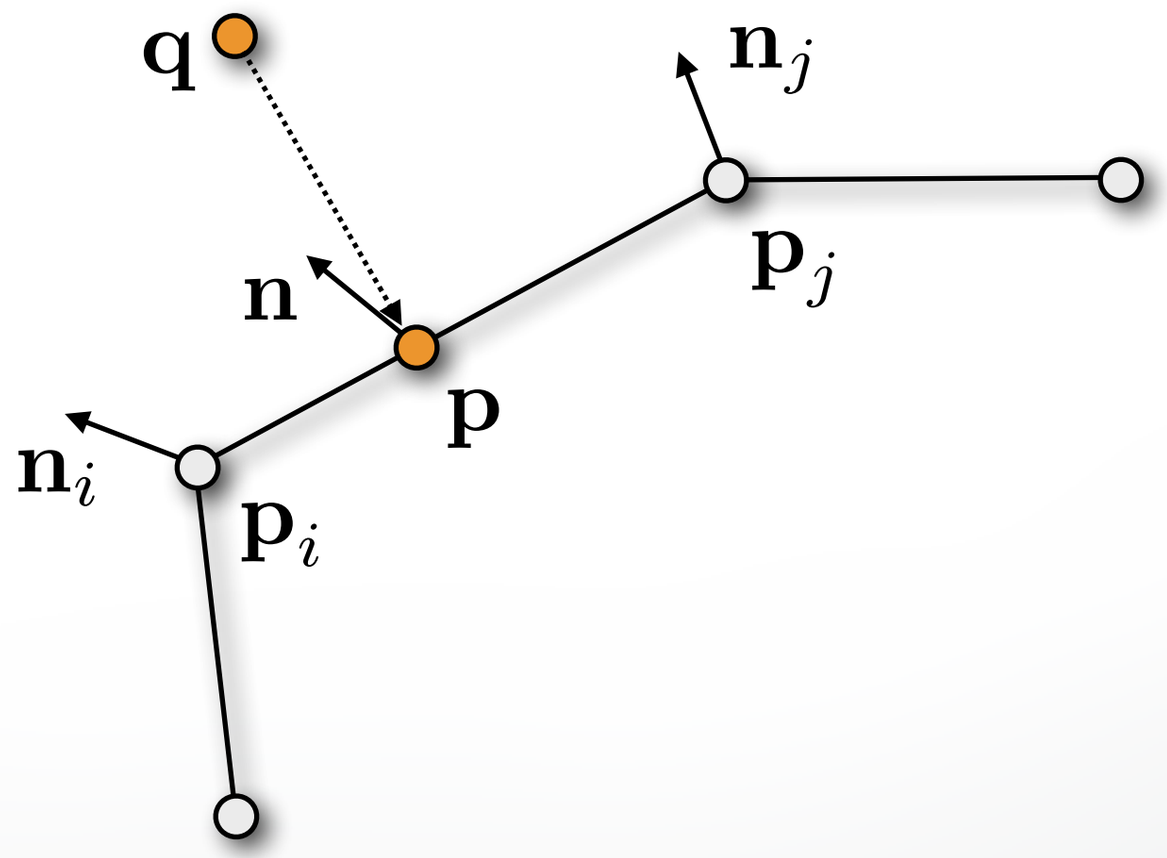
- Distance to plane, edge, or vertex
- http://www.geometrictools.com

Inside or outside?

- Based on interpolated surface normals

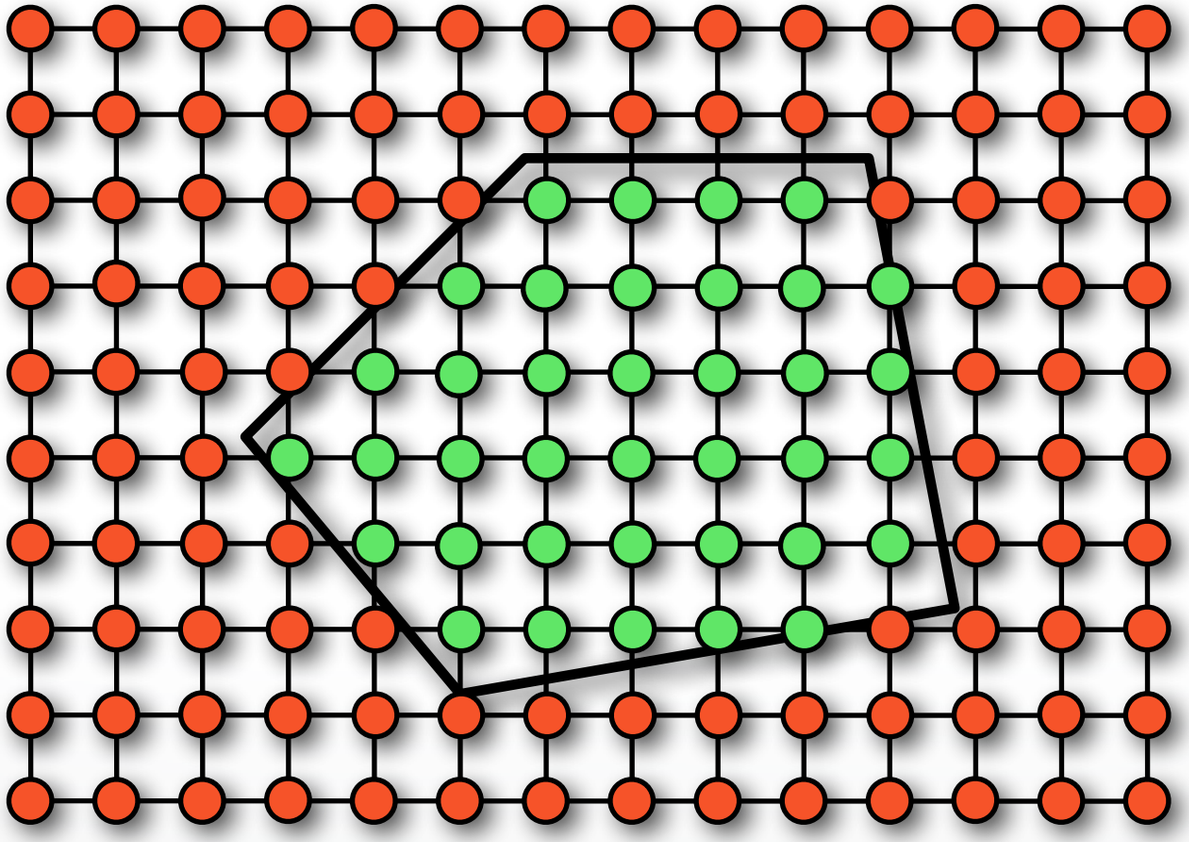
Signed Distance Computation

- Closest point $\mathbf{p} = \alpha \mathbf{p}_i + (1 - \alpha)\mathbf{p}_j$
- Interpolated normal $\mathbf{n} = \alpha \mathbf{n}_i + (1 - \alpha)\mathbf{n}_j$
- Inside if $(\mathbf{q} - \mathbf{p})^\top \mathbf{n} < 0$



Fast Marching Techniques

- Initialize with exact distance in mesh's vicinity
- Fast-march outwards
- Fast-march inwards



Literature

- Schneider, Eberly, “Geometric Tools for Computer Graphics”, Morgan Kaufmann, 2002
- Sethian, “Level Set and Fast Marching Methods”, Cambridge University Press, 1999

Conversion

Explicit to Implicit

- Compute signed distance at grid points
- Compute distance point-mesh
- Fast marching

Implicit to Explicit

- Extract zero-level iso-surface $F(x, y, z) = 0$
- Other iso-surfaces $F(x, y, z) = C$
- Medical imaging, simulations, measurements, …

2D: Marching Square

1. Classify grid nodes as inside/outside

- Is $F(\mathbf{x}_{i,j}) > 0$ or < 0 ?

2. Classify cell: 2^4 configurations

- In/out for each corner

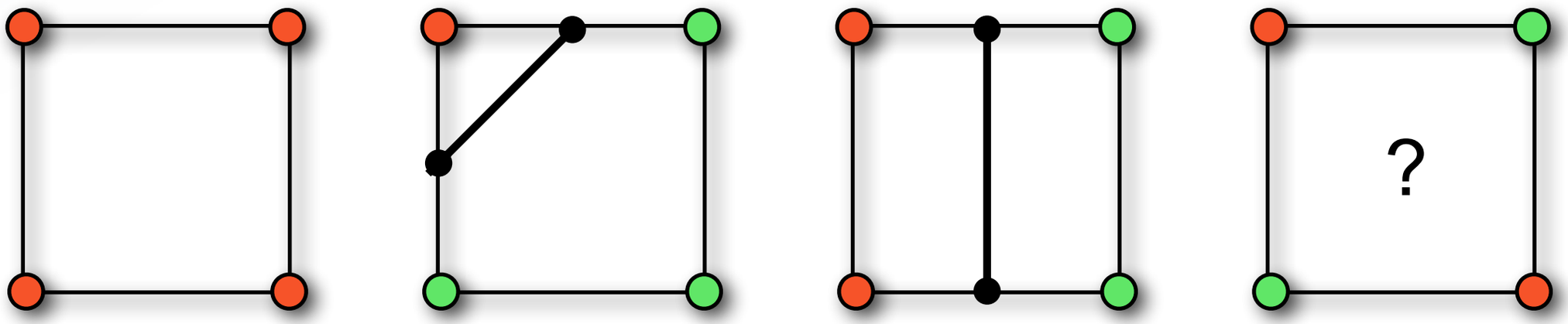
3. Compute intersection points

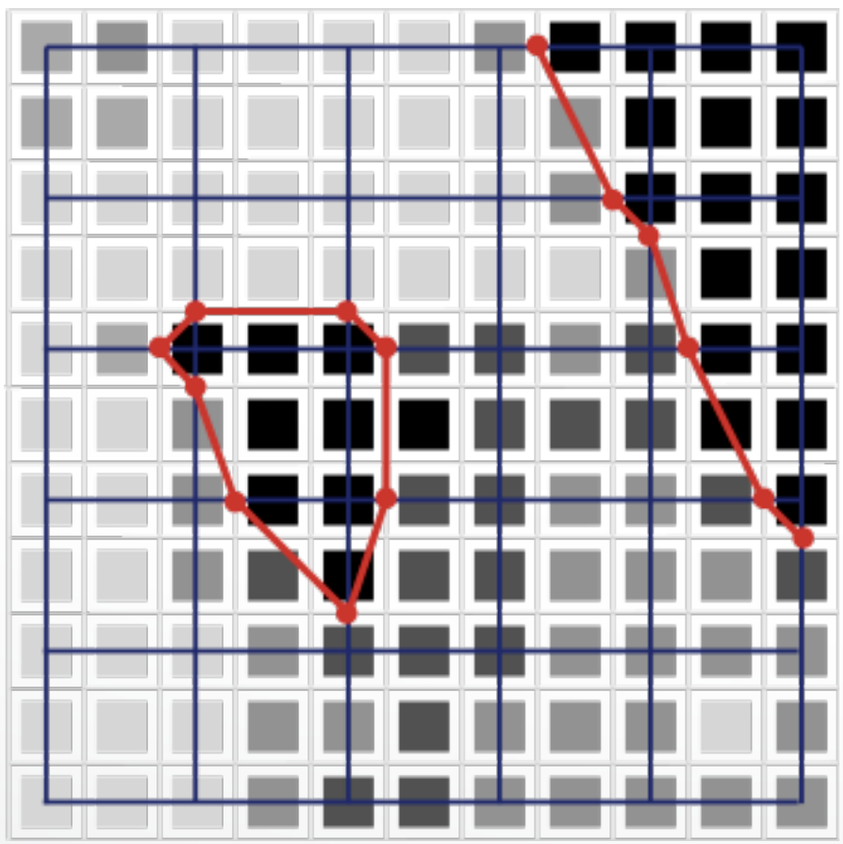
- Linear interpolation along edges

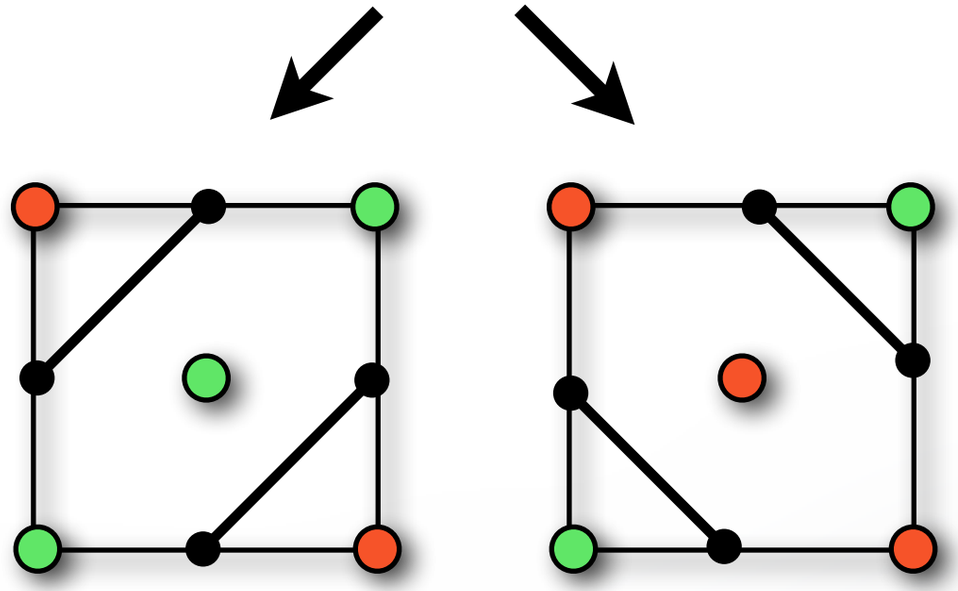
4. Connect them by edges

- Look-up table for edge configuration

2D: Marching Square







3D: Marching Cubes

1. Classify grid nodes as inside/outside

- Is $F(\mathbf{x}_{i,j,k}) > 0$ or < 0

2. Classify cell: 2^8 configurations

- In/out for each corner

3. Compute intersection points

- Linear interpolation along edges

4. Connect them by edges

- Look-up table for path configuration

- Disambiguation by modified table [Montani ’94]

3D: Marching Cubes

Classify grid nodes $\mathbf{x}_{i,j,k}$ based on $F_{i,j,k} = F(\mathbf{x}_{i,j,k})$

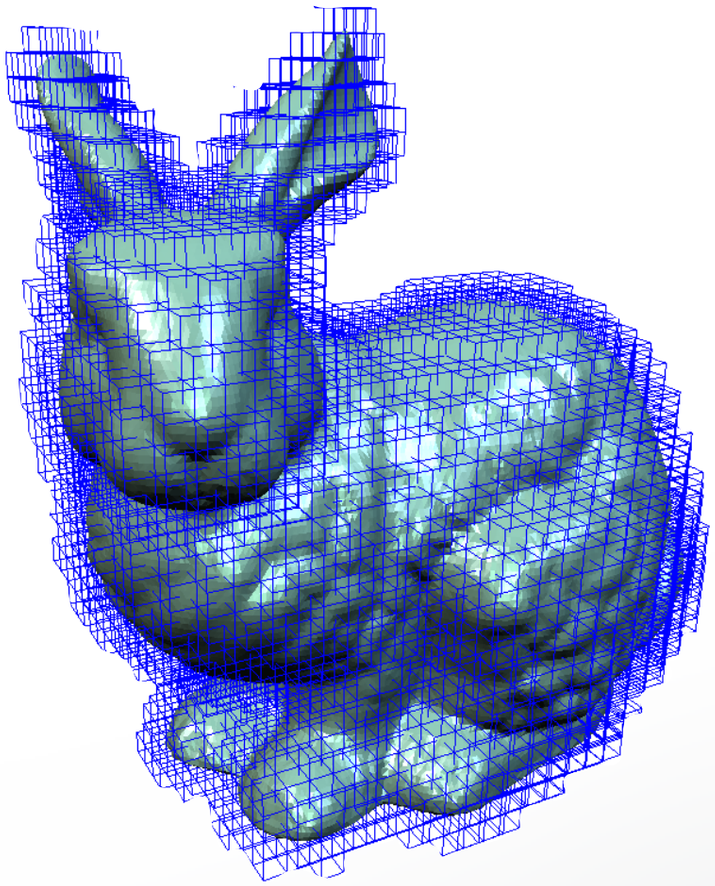
- Inside or outside

Classify all cubes based on $F_{i,j,k}$

- Inside, outside, or intersecting

Refined only intersected cells

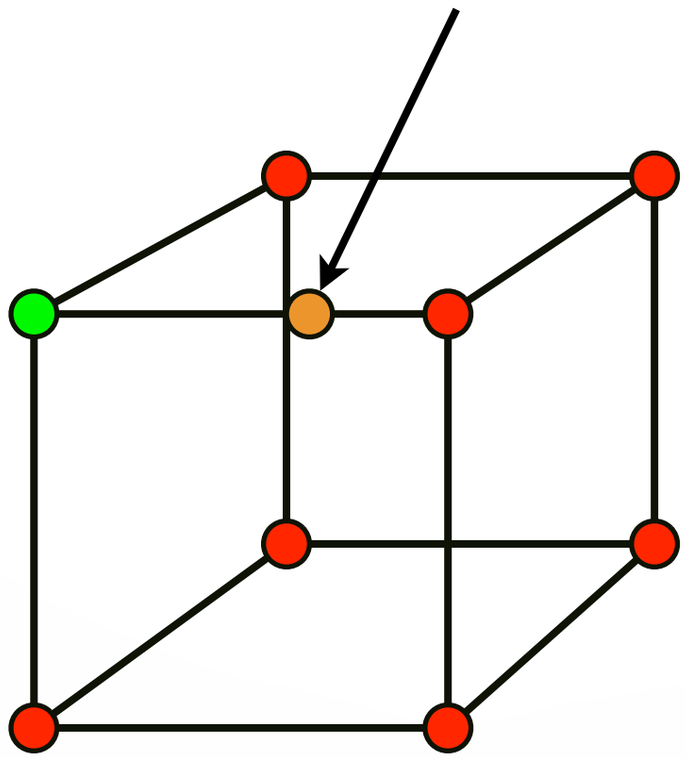
- 3-color adaptive octree
- $O(h^{-2})$ complexity



Intersection Points

Linear interpolation along edges

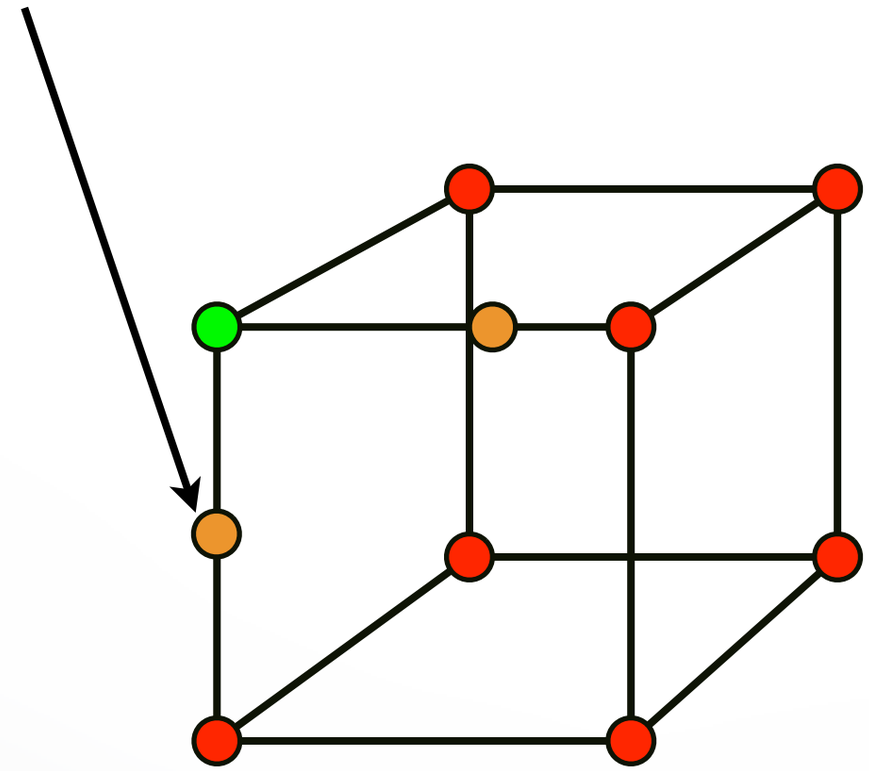
$$\frac{\mathbf{x}_{i,j,k} \cdot |F_{i+1,j,k}| \; + \; \mathbf{x}_{i+1,j,k} \cdot |F_{i,j,k}|}{|F_{i,j,k}| + |F_{i+1,j,k}|}$$



Intersection Points

Linear interpolation along edges

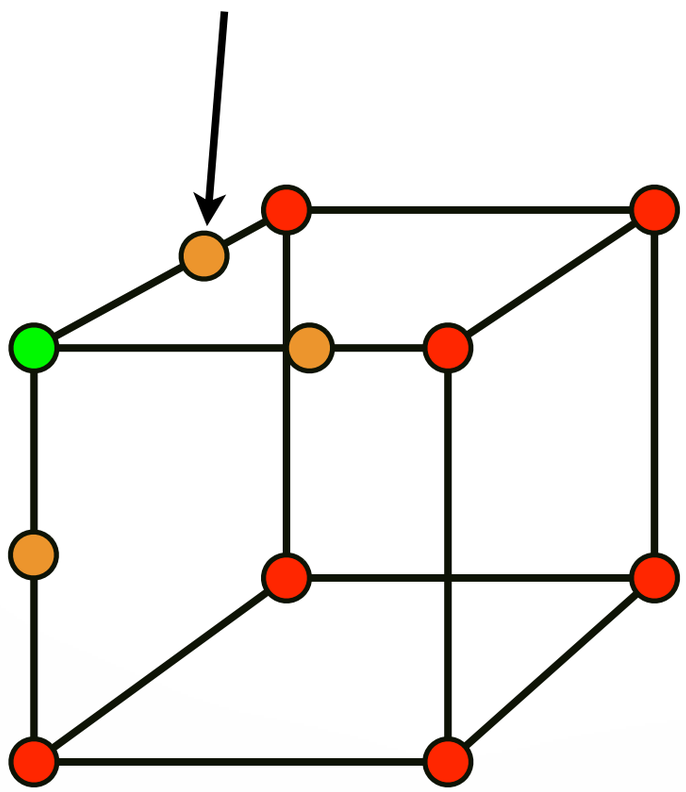
$$\frac{\mathbf{x}_{i,j,k} \cdot |F_{i,j+1,k}| \; + \; \mathbf{x}_{i,j+1,k} \cdot |F_{i,j,k}|}{|F_{i,j,k}| + |F_{i,j+1,k}|}$$



Intersection Points

Linear interpolation along edges

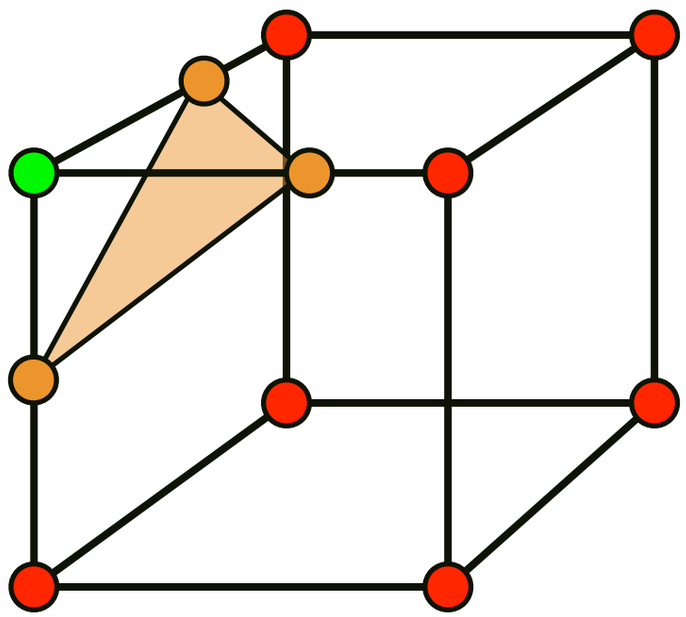
$$\frac{\mathbf{x}_{i,j,k} \cdot |F_{i,j,k+1}| \; + \; \mathbf{x}_{i,j,k+1} \cdot |F_{i,j,k}|}{|F_{i,j,k}| + |F_{i,j,k+1}|}$$

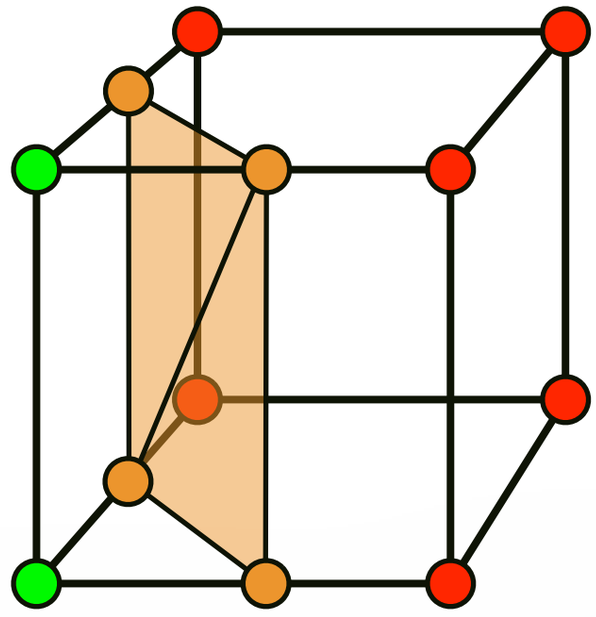


Intersection Points

Linear interpolation along edges

Lookup table for patch configuration

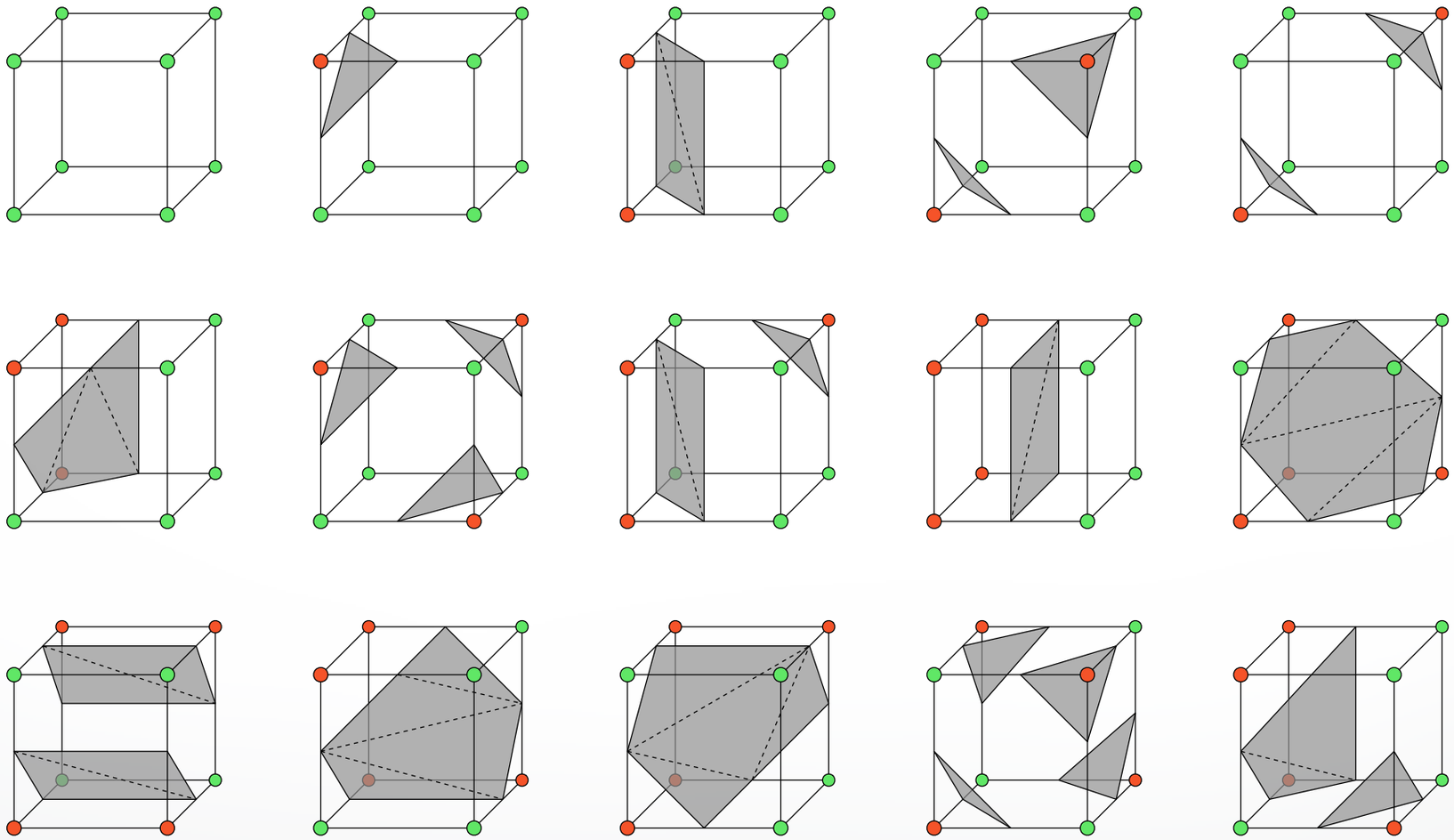




Marching Cubes

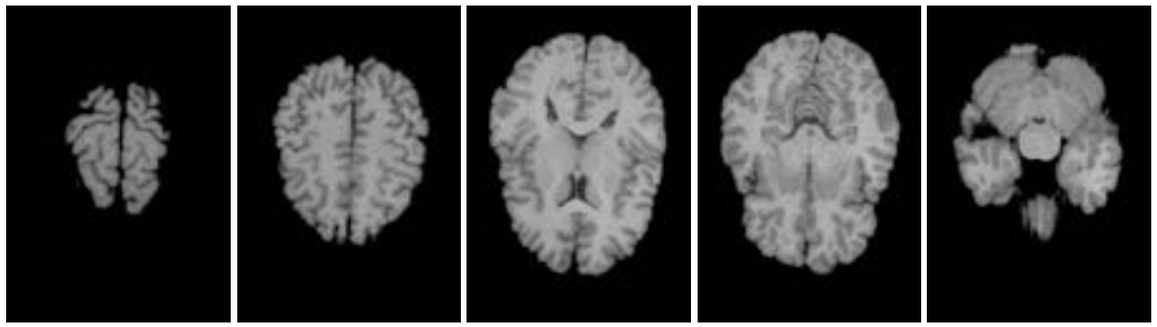
Look-up table with 2^8 entries

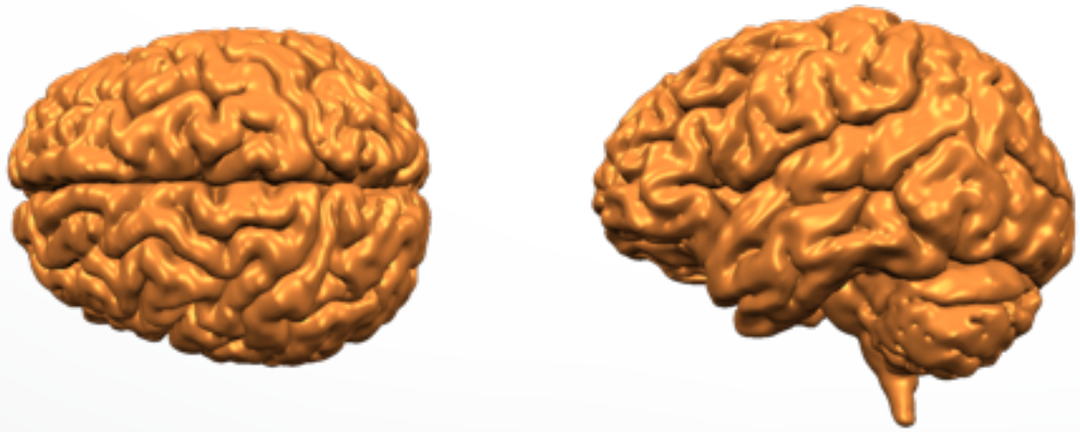
- 15 representative cases shown
- Others follow by symmetry



Marching Cubes

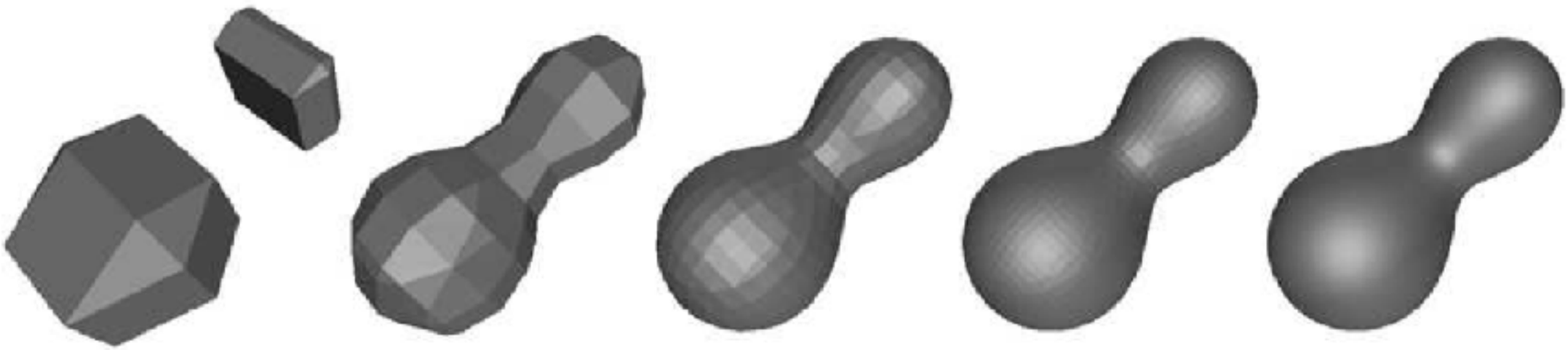
Algorithm for isosurface extraction from medical scans (CT, MRI)





Marching Cubes

Effect of grid size



Grid size=10
70 Facets

Grid size=5
220 Facets

Grid size=2
1700 Facets

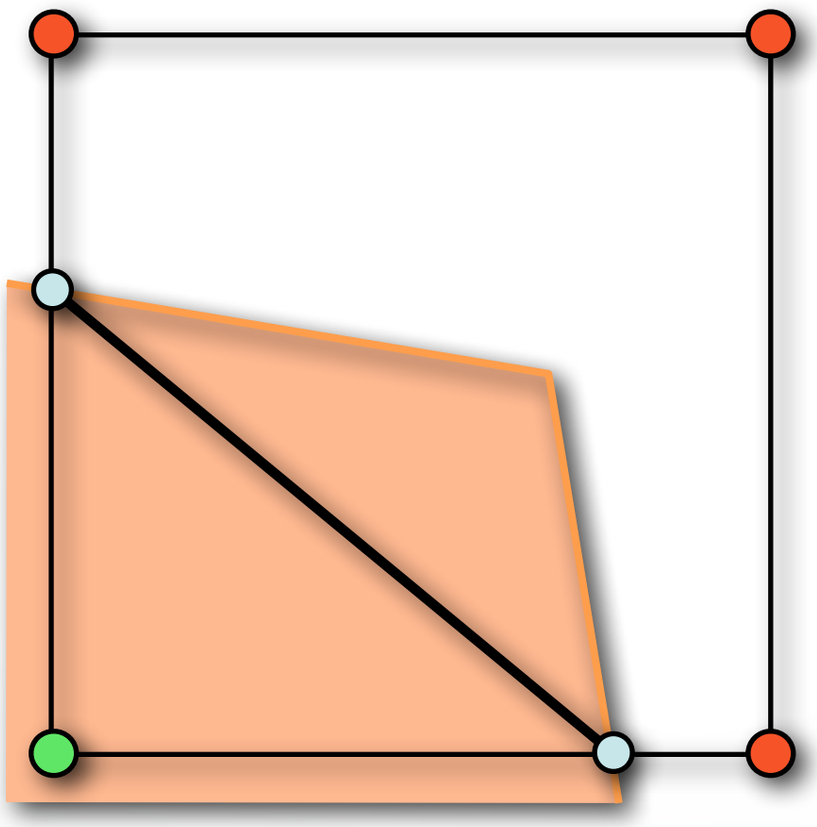
Grid size=1
6800 Facets

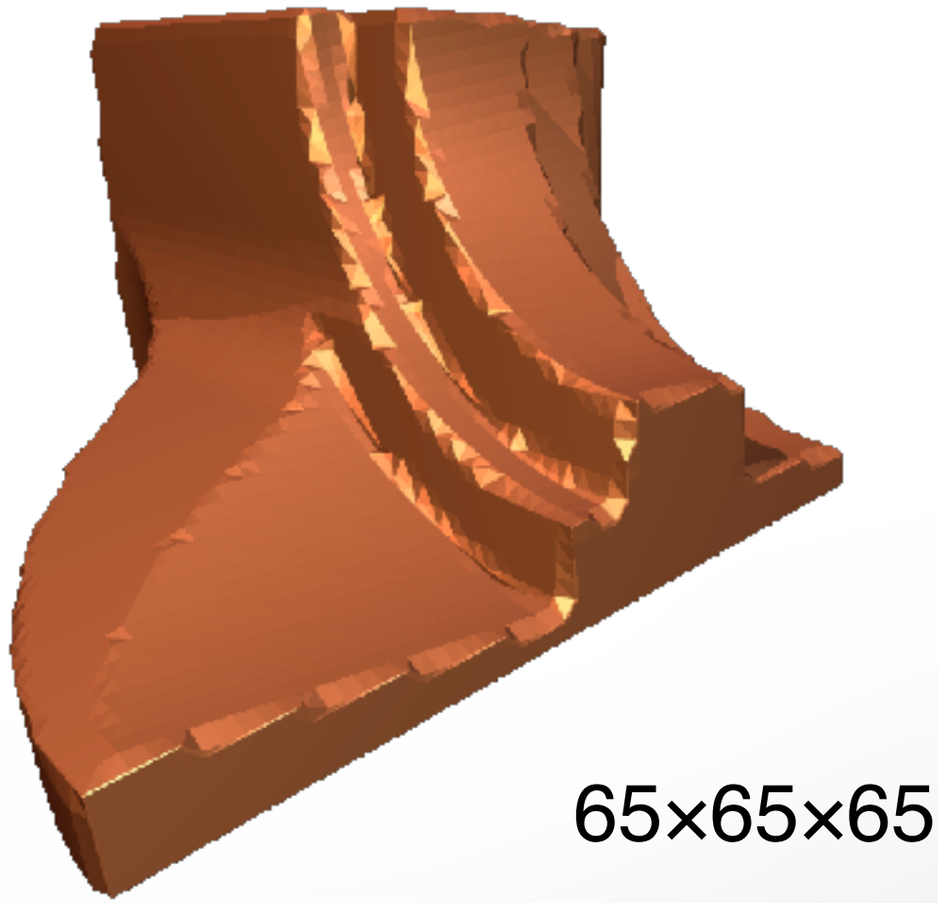
Grid size=0.5
27000 Facets

Marching Cubes

Sample points restricted to edges of regular grid

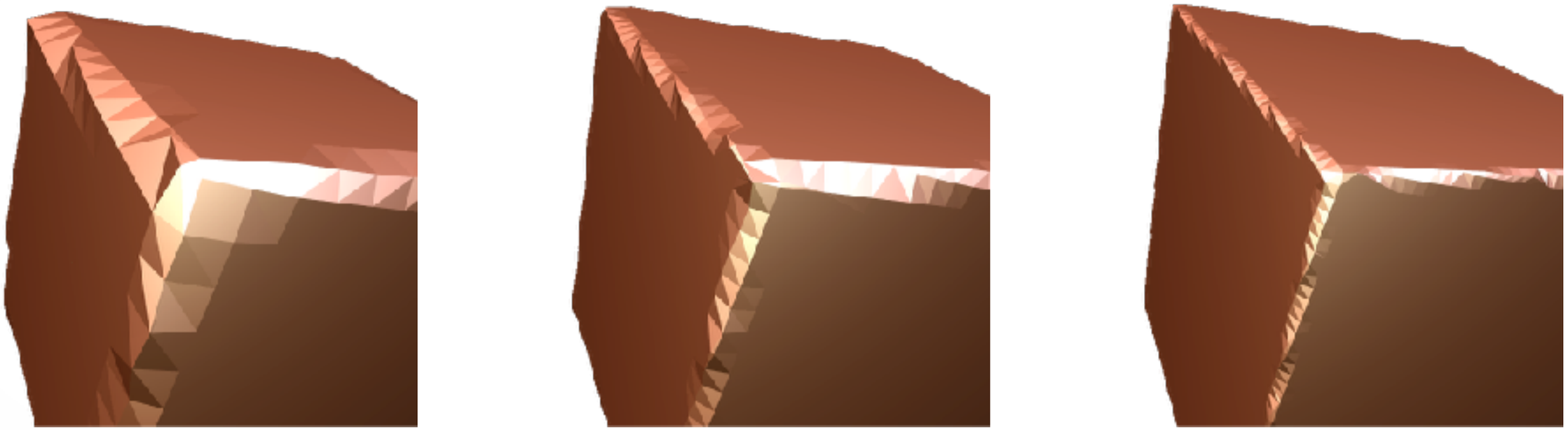
Alias artifacts at sharp features





65×65×65

Increasing Resolution

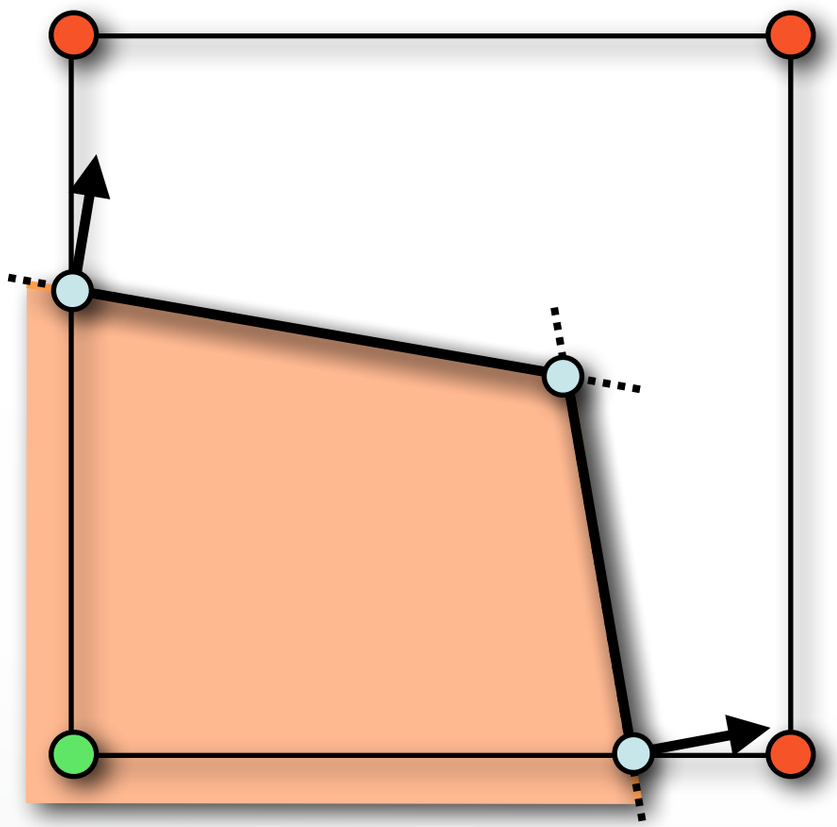


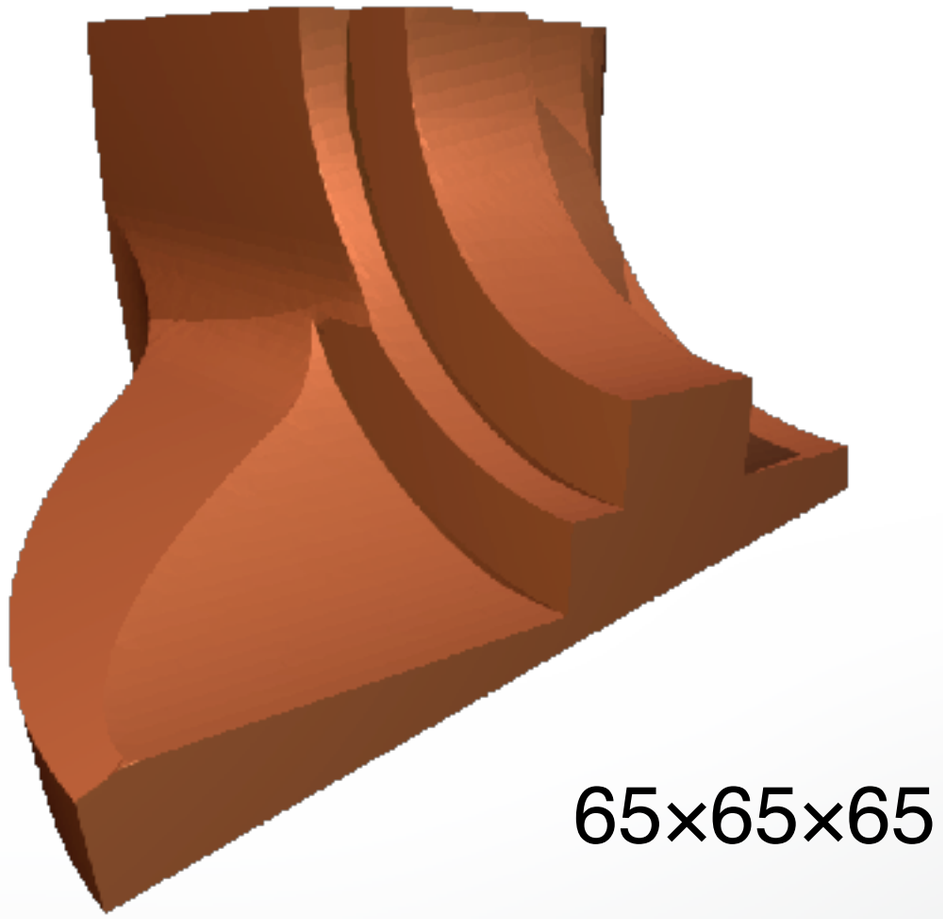
Does not remove alias problems!

Extended Marching Cubes

Locally extrapolate distance gradient

Place samples on estimated features



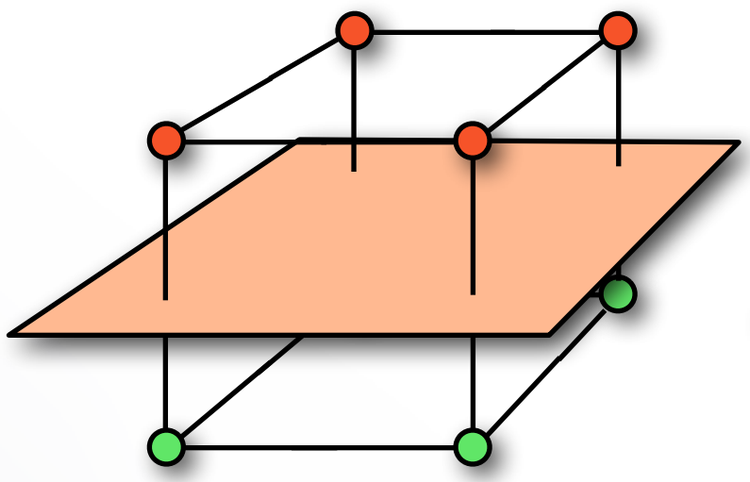


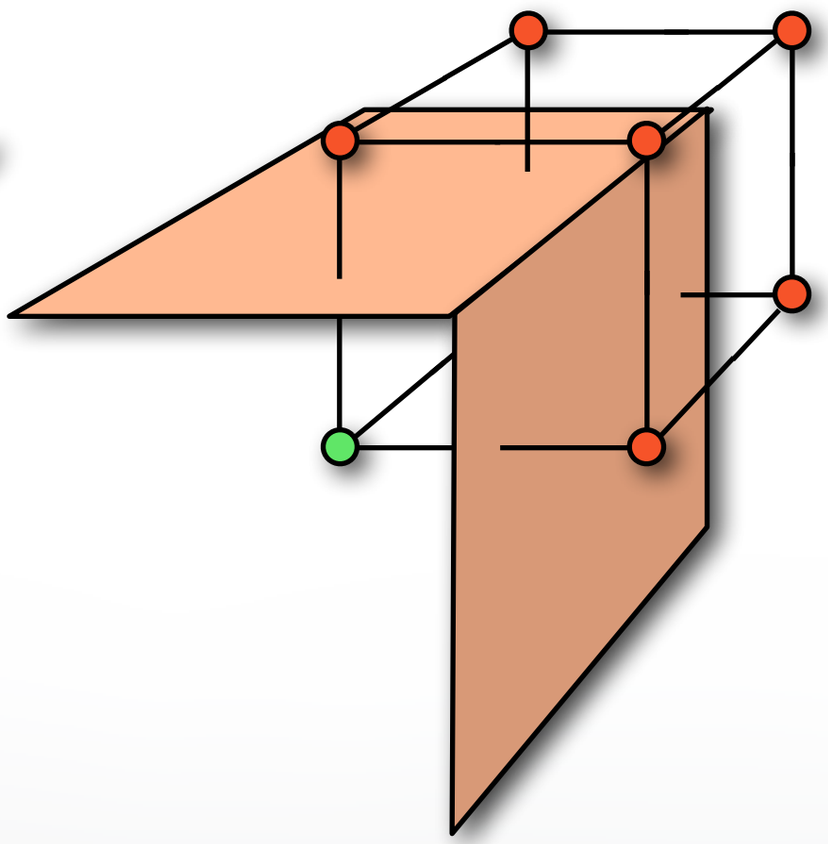
65×65×65

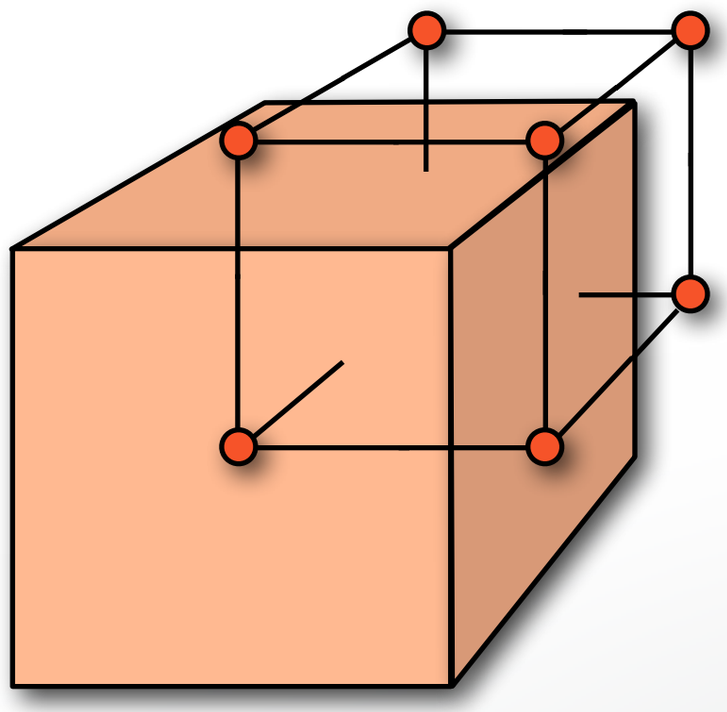
Extended Marching Cubes

Feature detection

- Based on angle between normals $\mathbf{n}_i$
- Classify into edges / corners







Extended Marching Cubes

Feature sampling

- Intersect tangent planes $(\mathbf{s}_i, \mathbf{n}_i)$

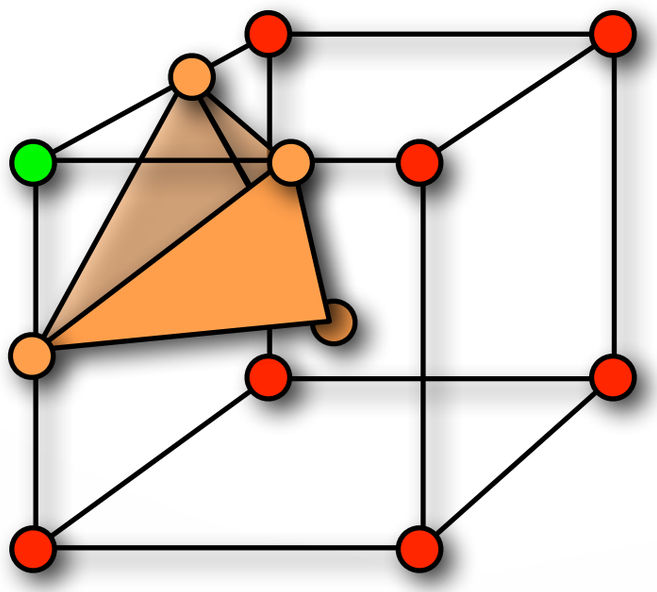
$$\begin{pmatrix} \vdots \\ \mathbf{n}_i \\ \vdots \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathbf{n}_i^T \mathbf{s}_i \\ \vdots \end{pmatrix}$$

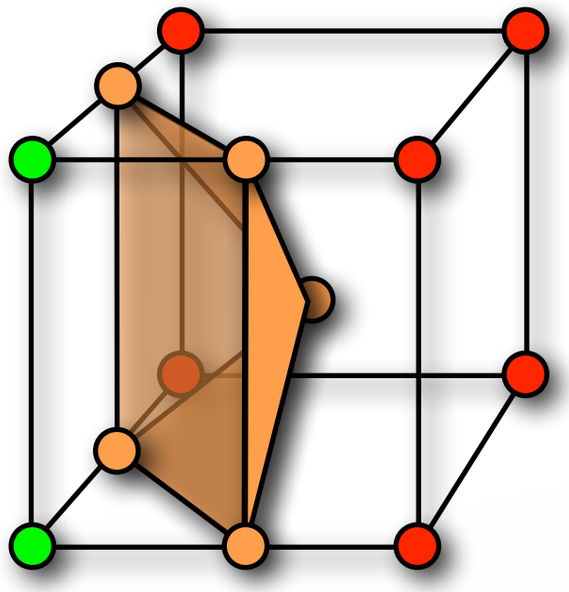
- Over- or under-determined system
- Solve by SVD pseudo-inverse

Extended Marching Cubes

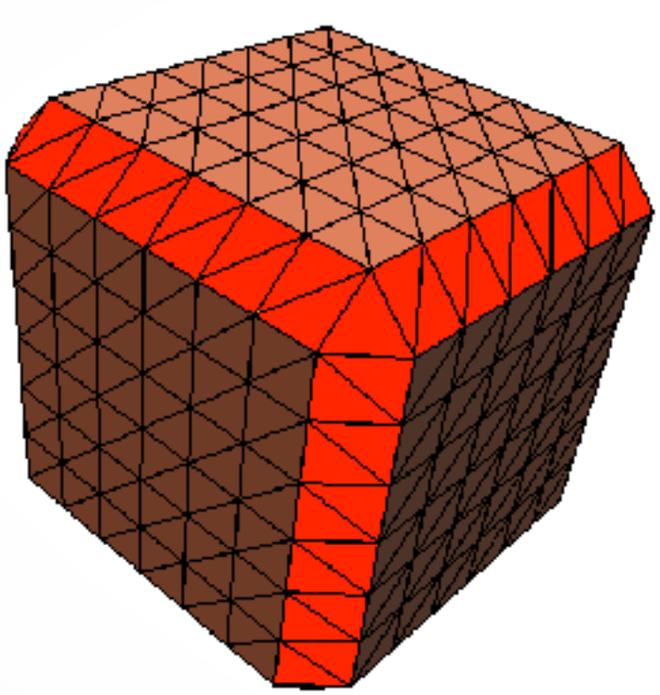
Feature sampling

- Intersect tangent planes $(\mathbf{s}_i, \mathbf{n}_i)$
- Triangle fans centered at feature point

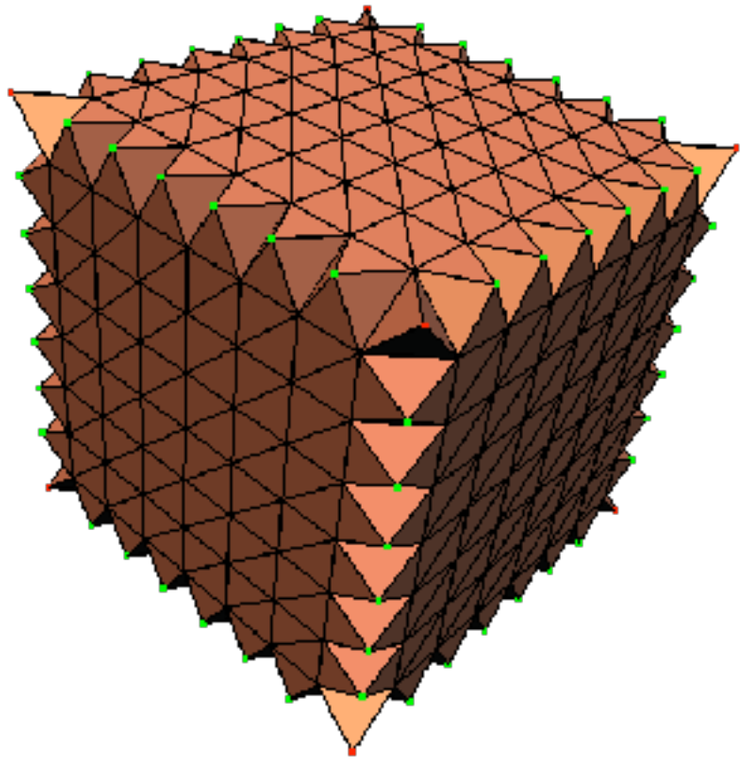




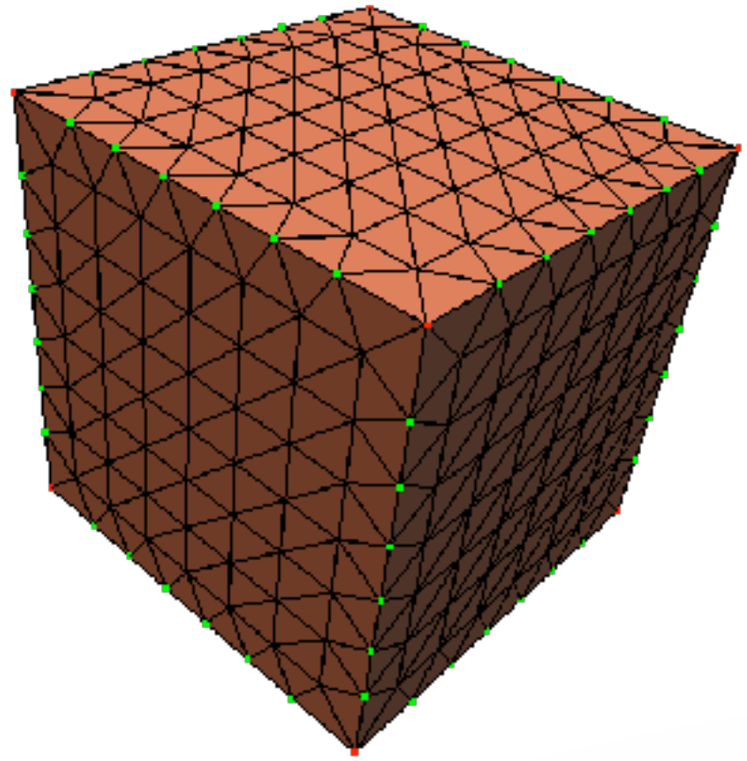
Extended Marching Cubes



Feature Detection

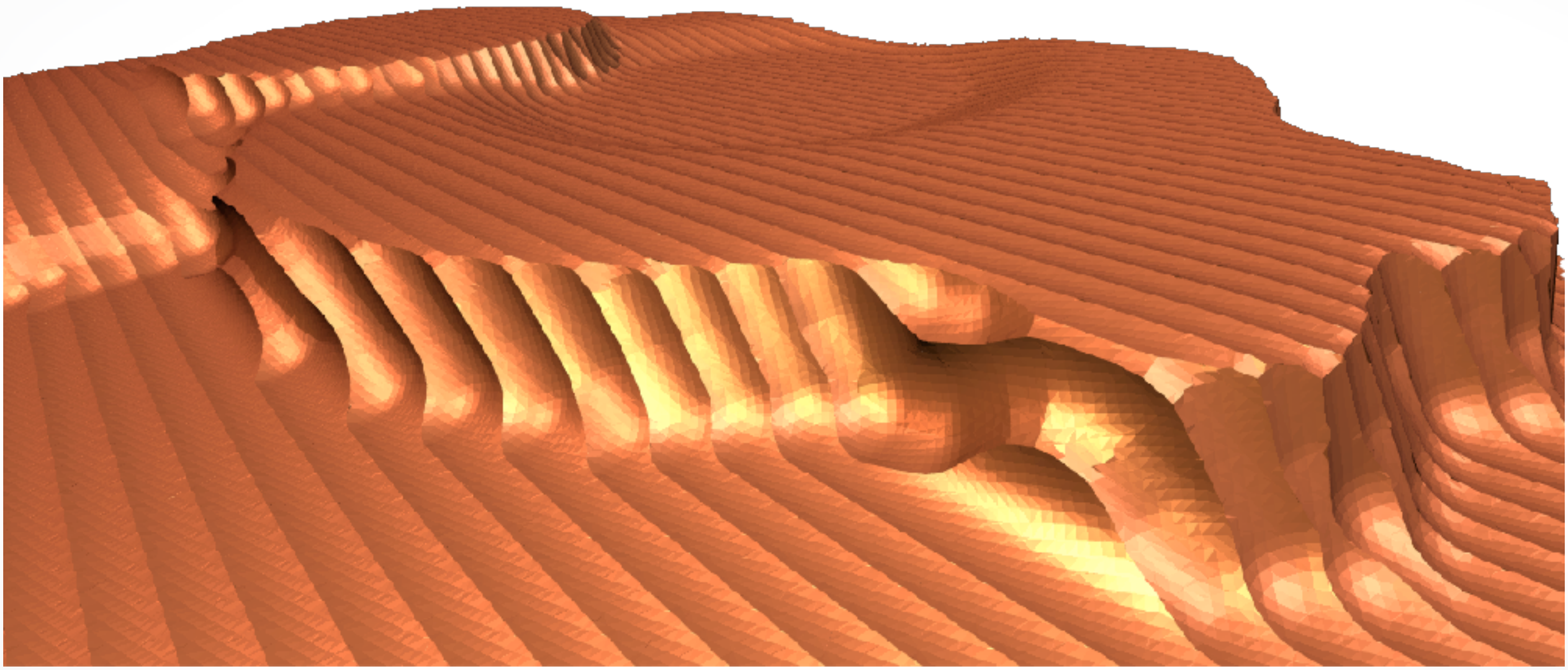


Feature Sampling



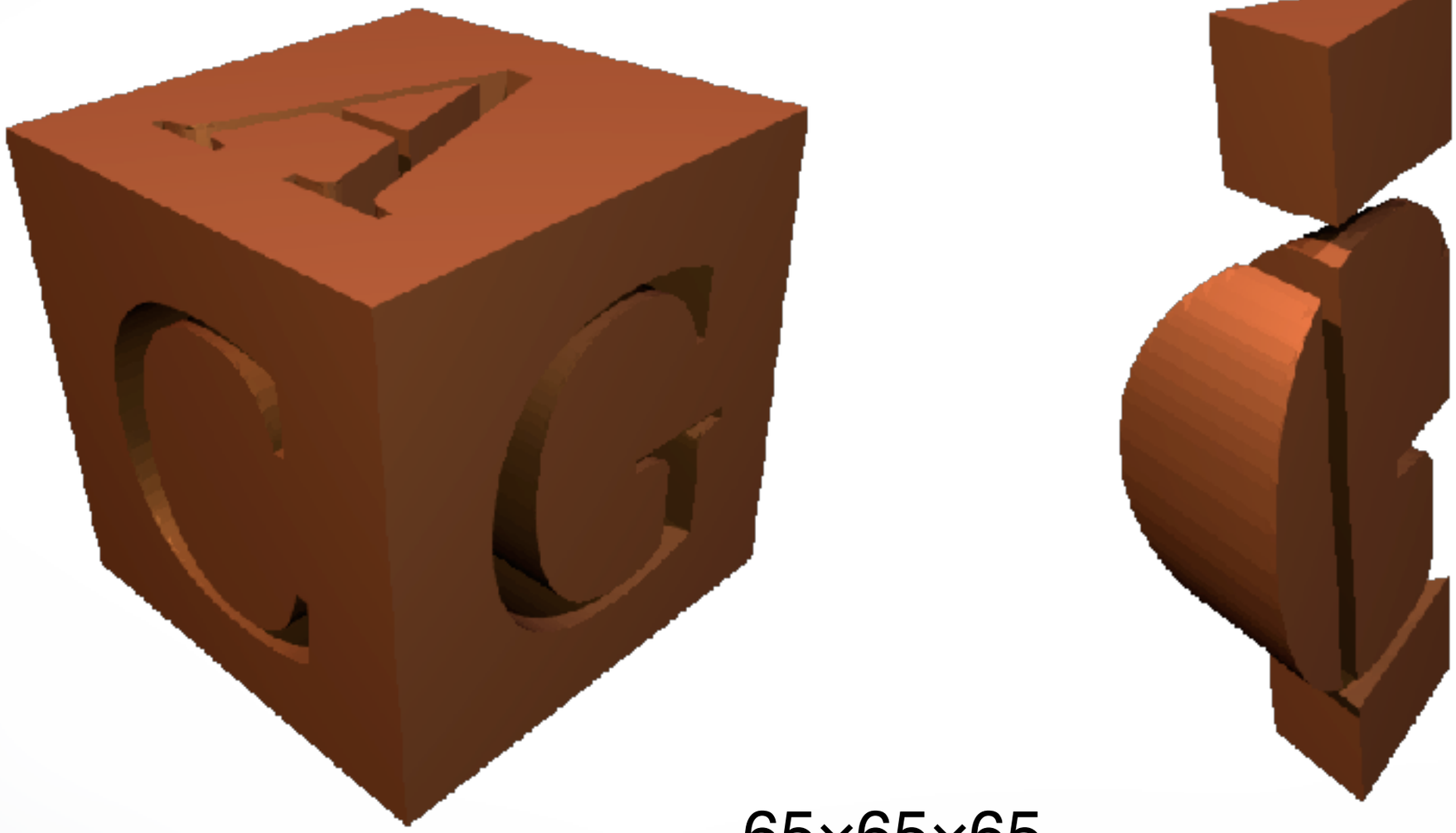
Edge Flipping

Milling Simulation



257×257×257

CSG Modeling



65×65×65

Marching Cubes

+ Result is watertight, closed 2-manifold surface!

+ Easy to parallelize

- Uniform (over-) sampling (→ mesh decimation)

- Degenerate triangles (→ remeshing)

- MC does not preserve features

+ EMC preserves features, but…

  about 10% more triangles

  20-40% computational overhead

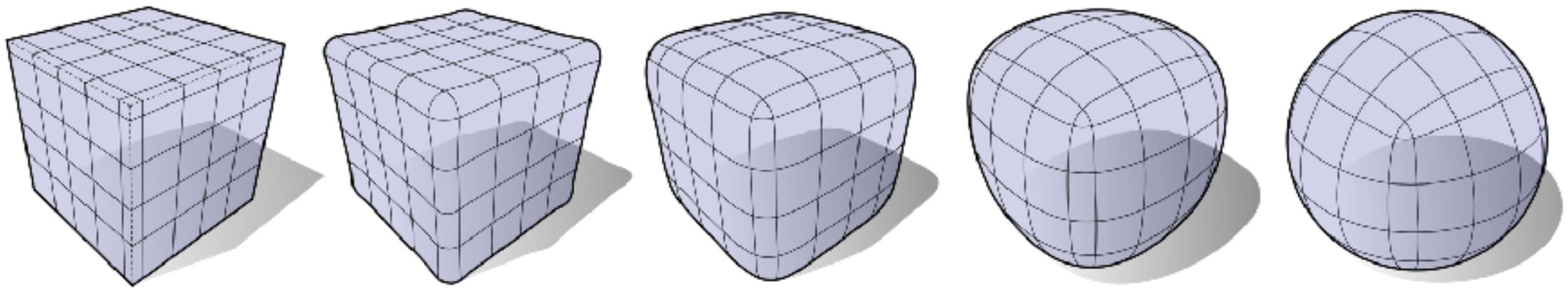
Literature

- Lorensen & Cline, “Marching Cubes: A High Resolution 3D Surface Construction Algorithm”, SIGGRAPH 1987

- Montani et al., “A modified look-up table for implicit disambiguation of Marching Cubes”, Visual Computer 1994

- Kobbelt et al., “Feature Sensitive Surface Extraction from Volume Data”, SIGGRAPH 2001

Next Time



Discrete Differential Geometry

http://cs621.hao-li.com

Thanks!