Digital Geometry
-Continuous Geometry of Curves & Surfaces

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http://jjcao.github.io/DigitalGeometry/

Pleasure may come from illusion, but happiness can come only of reality.

3 Representations of Curve

- Explicit: $y = mx + b$

- Explicit Parametric (seen as a kinematic motion):
  - $P = P_0 + t\,(P_1 - P_0)$
  - curve: $r=r(t)$,
  - surface: $r=r(u,v)$

- Implicit: $ax + by + c = 0$

Implicit representation of 3d Curve

- surface: level set of function $f(x,y,z)$: $f(x,y,z)=0$, viz, solution set of $f(x,y,z)=0$.

- curve: solution set of
  - $f(x,y,z)=0$
  - $g(x,y,z)=0$

- point: solution set of
  - $f(x,y,z)=0$
  - $g(x,y,z)=0$
  - $h(x,y,z)=0$

From implicit 2 Parametric representation

- If conditions of implicit function theorem are guaranteed

- Curve =>$r(x)=(x,y(x),z(x))$

- Surface =>$r(x,y)=(x,y,z(x,y))$ (Monge patch)

Parametric Curves

$$\mathbf{x} : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^3$$

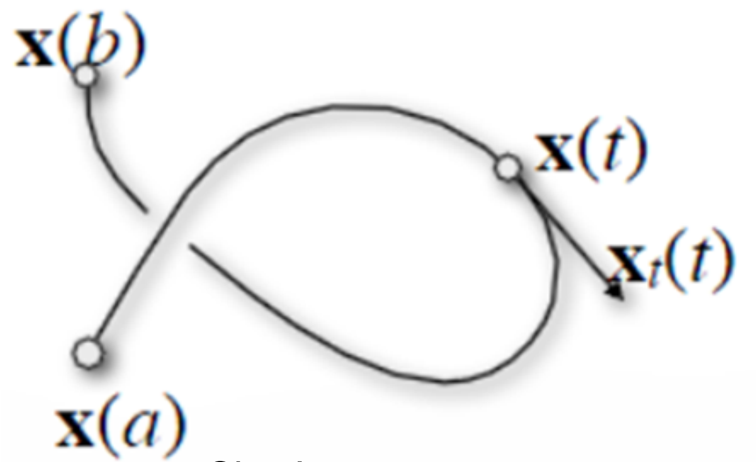
$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad \mathbf{x}_t(t) := \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \begin{pmatrix} \mathrm{d}x(t)\,/\mathrm{d}t \\ \mathrm{d}y(t)\,/\mathrm{d}t \\ \mathrm{d}z(t)\,/\mathrm{d}t \end{pmatrix}$$

Advantages of parametric forms

- More degrees of freedom
- Directly transformable
- Dimension independent
- No infinite slope problems
- Separates dependent and independent variables
- Inherently bounded
- Easy to express in vector and matrix form
- Common form for many curves and surfaces

Simple curve

- A curve is simple if it does not cross itself, i.e., injective



Simple curve


nonsimple curves

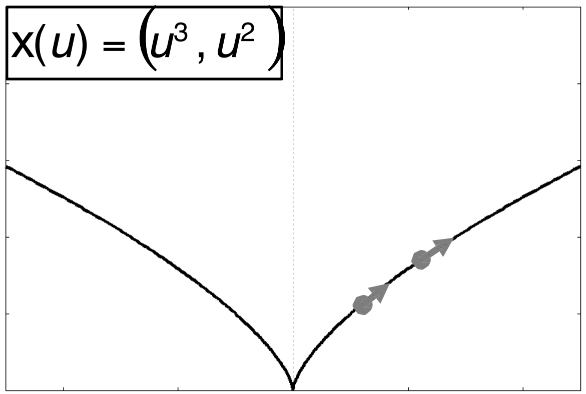
Differentiable Curves

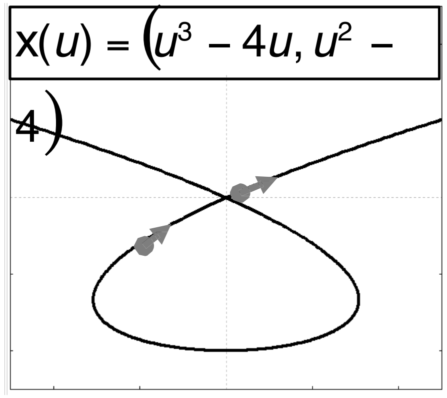
<u>Definition</u>:

A *parameterized differentiable curve* is a differentiable map $\boldsymbol{x}: I \longrightarrow \boldsymbol{R}^2$ of an open interval $I=(a,b)$ of the real line $\boldsymbol{R}$ into $\boldsymbol{R}^2$:

$$\boldsymbol{x}(u)=(x(u),y(u))$$

where $x(u)$ and $y(u)$ are differentiable functions.



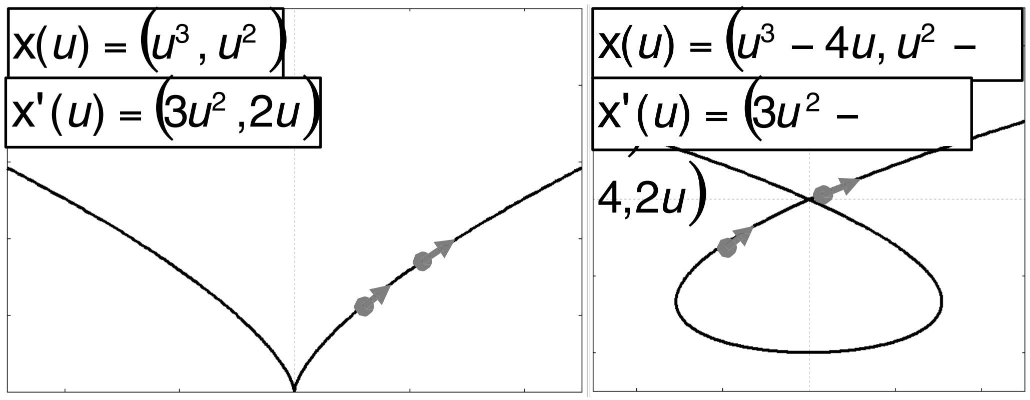


Differentiable Curves - derivative

<u>Definition</u>:

The *derivative* of the curve at $\boldsymbol{x}(u)$ is the vector, tangent to the curve, defined as:

$$\boldsymbol{x}'(u)=(x'(u),y'(u))$$



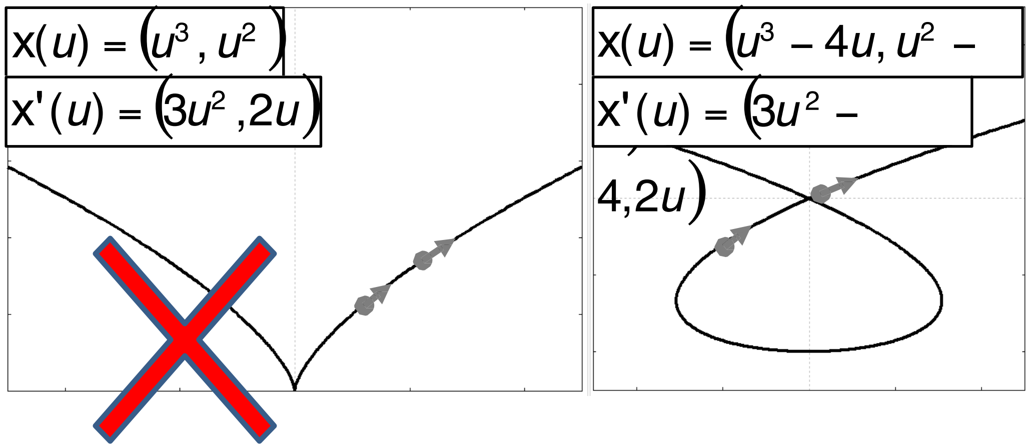
Differentiable Curves - regular

<u>Definition</u>:

The *derivative* of the curve at $\boldsymbol{x}(u)$ is the vector, tangent to the curve, defined as:

$$\boldsymbol{x}'(u)=(x'(u),y'(u))$$

The curve is said to be *regular* if $\boldsymbol{x}'(u)\neq 0$.



Parametric Curves

A parametric curve $\mathbf{x}(t)$ is

- simple: $\mathbf{x}(t)$ is injective (no self-intersections)
- differentiable: $\mathbf{x}_t(t)$ is defined for all $t \in [a, b]$
- regular: $\mathbf{x}_t(t) \neq 0$ for all $t \in [a, b]$

Length of a Curve / Arc length

Polyline chord length

$$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \left\| \frac{\Delta \mathbf{x}_i}{\Delta t} \right\| \Delta t, \quad \Delta \mathbf{x}_i := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$$

norm change

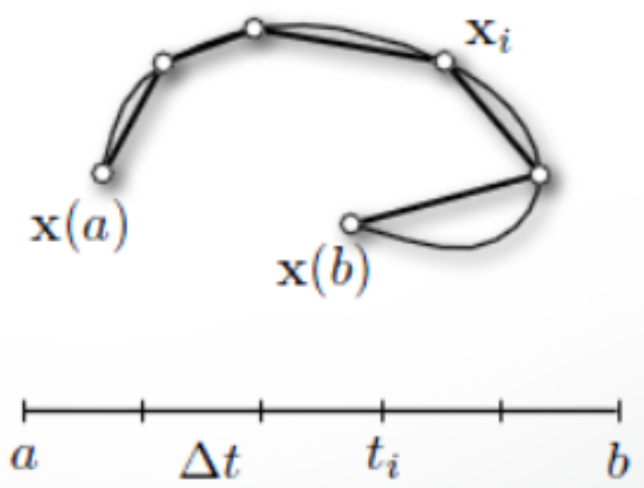
Curve arc length ($\Delta t \to 0$)

$$s = s(t) = \int_a^t \|\mathbf{x}_t\| \, \mathrm{d}t$$

length =

Integration of infinitesimal change

× norm of speed



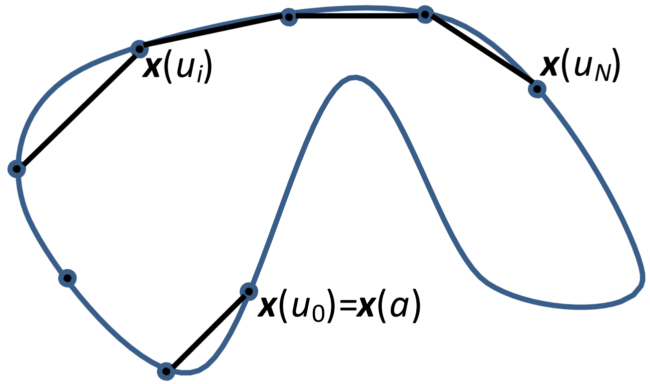
Regular Curves

Given a regular curve $\boldsymbol{x}(u)$, and given, the *arc-length* from *a* to the point *u* is:

$$s(u) = \int_a^u |\mathrm{x}'(v)| \, dv$$

If we partition the interval $[a,u]$ into N sub-intervals, setting $\Delta u=(u-a)/N$ and $u_i=a+i\Delta u$:

$$\begin{aligned} s(u) &= \lim_{N\to\infty} \sum_{i=0}^{N-1} |\mathrm{x}(u_{i+1}) - \mathrm{x}(u_i)| \\ &= \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{|\mathrm{x}(u_{i+1}) - \mathrm{x}(u_i)|}{\Delta u} \Delta u \\ &= \lim_{N\to\infty} \sum_{i=0}^{N-1} |\mathrm{x}'(u_i)| \Delta u \\ &= \int_a^u |\mathrm{x}'(v)| dv \end{aligned}$$



Differentiable Curves

Definition:

We say that a regular curve is *parameterized by arc-length* if:

$$|\boldsymbol{x}'(u)|=1$$

In this case:

$$s(u) = \int_a^u |x'(v)|\,dv = \int_a^u dv = u - a$$

There are various names for such a parameterization ("unit speed", "arc-length", "isometric")

Regular Curves - Tangent

<u>Definition</u>:

The *tangent* to the curve at $\boldsymbol{x}(u)$ is the unit vector pointing in the direction of the derivative:

$$\mathrm{t}(u) = \frac{\mathrm{x}'(u)}{\|\mathrm{x}'(u)\|}$$

If $\boldsymbol{x}$ is parameterized by arc-length: $\mathrm{t}(u) = \mathrm{x}'(u)$

Regular Curves - *Normal*

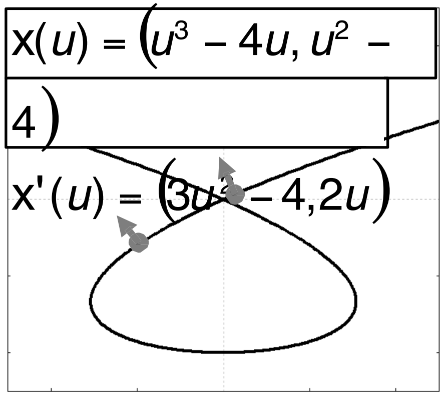
Definition:

The *normal* of the curve at $\boldsymbol{x}(u)$ is the unit vector that is perpendicular to the tangent:

$$\mathrm{n}(u) = \mathrm{t}(u)^{\perp} = \frac{\mathrm{x}'(u)^{\perp}}{\|\mathrm{x}'(u)\|} = \frac{\left(-y'(u), x'(u)\right)}{\sqrt{\left(x'(u)\right)^2 + \left(y'(u)\right)^2}}$$

If $\boldsymbol{x}$ is parameterized by arc-length:

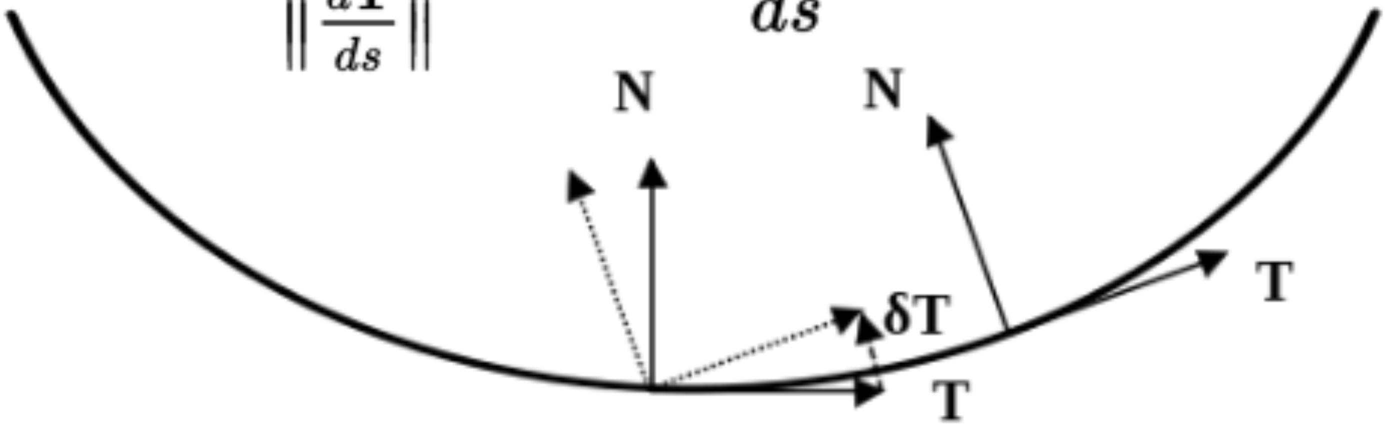
$$\mathrm{n}(u) = \mathrm{x}'(u)^{\perp} = \left(-y'(u), x'(u)\right)$$



Normal

- **N is the normal unit vector, the derivative of T with respect to the arclength parameter of the curve, divided by its length:**

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left\| \frac{d\mathbf{T}}{ds} \right\|} \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds}$$



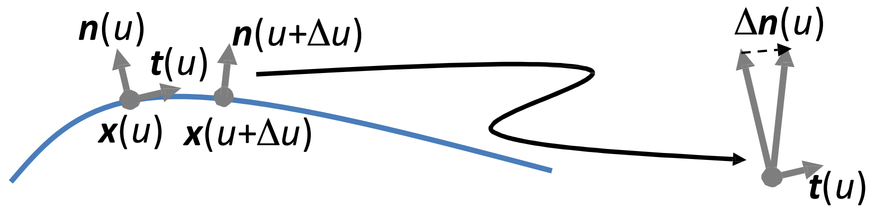
The **T** and **N** vectors at two points on a plane curve, a translated version of the second frame (dotted), and the change in **T**: δ**T'**. δs is the distance between the points. In the limit dT/ds will be in the direction **N**

Change in the normal is aligned with the tangent

Claim:

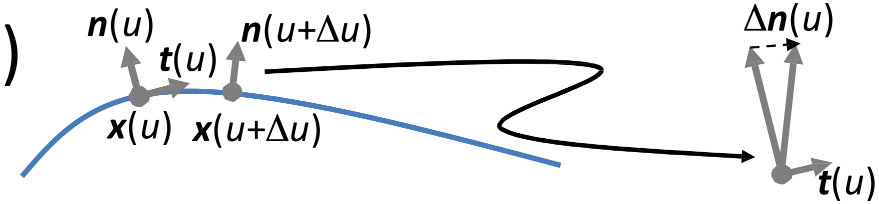
If we look at **how the normal changes along a curve**, we find that for small distances, the change is in the direction of the tangent:

$$\Delta \boldsymbol{n}(u) = \boldsymbol{n}(u+\Delta u) - \boldsymbol{n}(u) \approx \kappa(u)\boldsymbol{t}(\mathrm{u})$$



Change in the normal is aligned with the tangent

$$\Delta\boldsymbol{n}(u)=\boldsymbol{n}(u+\Delta u)-\boldsymbol{n}(u)\approx \kappa(u)\boldsymbol{t}(\mathrm{u})$$



<u>Proof</u>:

Since $\boldsymbol{n}(u)$ is a unit-vector, we know that:

$$1=\langle \mathrm{n}(u),\mathrm{n}(u)\rangle$$

Taking derivatives of both sides, we get:

$$0=\frac{d}{du}\langle \mathrm{n}(u),\mathrm{n}(u)\rangle$$

$$=2\left\langle \frac{d}{du}\mathrm{n}(u),\mathrm{n}(u)\right\rangle$$

Thus, the change in the normal is perpendicular to the normal direction, so it's aligned with the tangent.

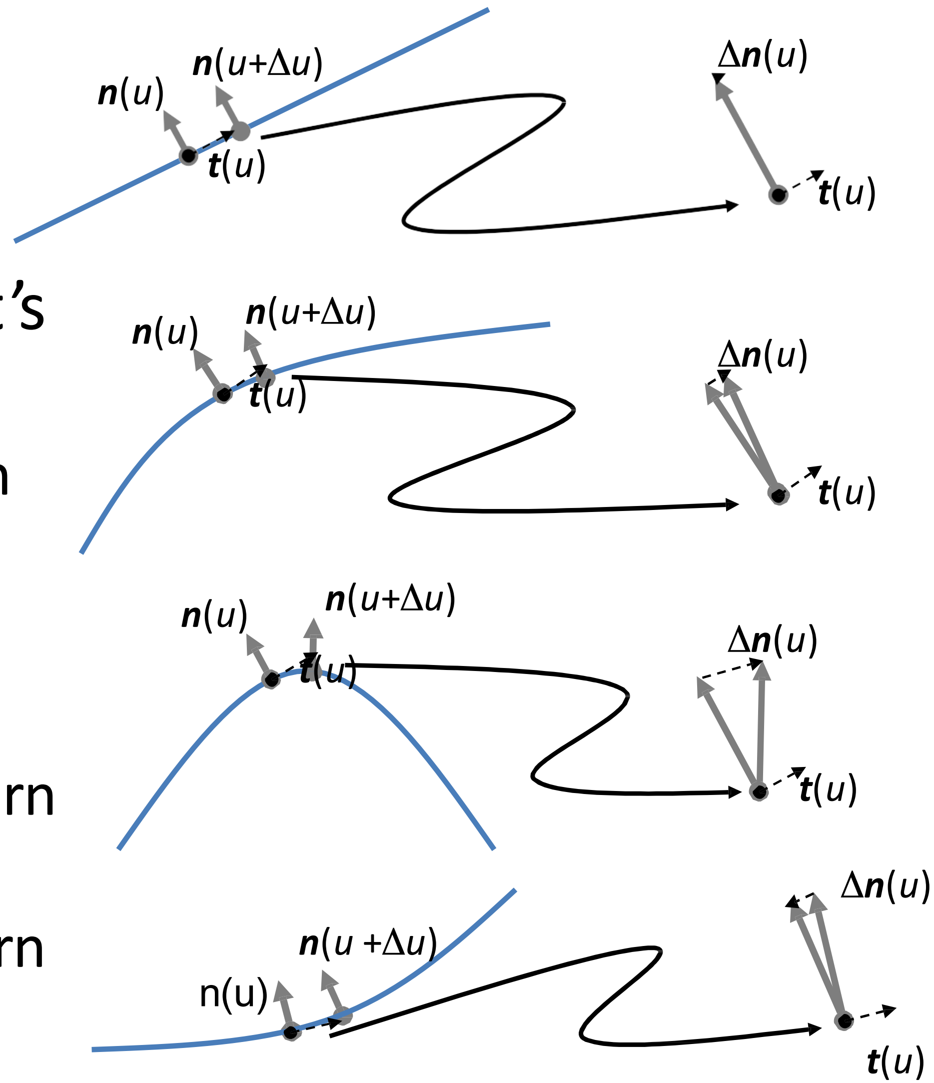
Change in the normal is aligned with the tangent

$\Delta\boldsymbol{n}(u)=\boldsymbol{n}(u+\Delta u)-\boldsymbol{n}(u)\approx \kappa(u)\boldsymbol{t}(\mathrm{u})$

<u>Note</u>:

If we look at the value of κ we see that it's

- zero for straight curves
- small/positive for convex curves that turn slowly
- large/positive for convex curves that turn quickly
- small/negative for concave curves that turn slowly
- large/negative for concave curves that turn quickly

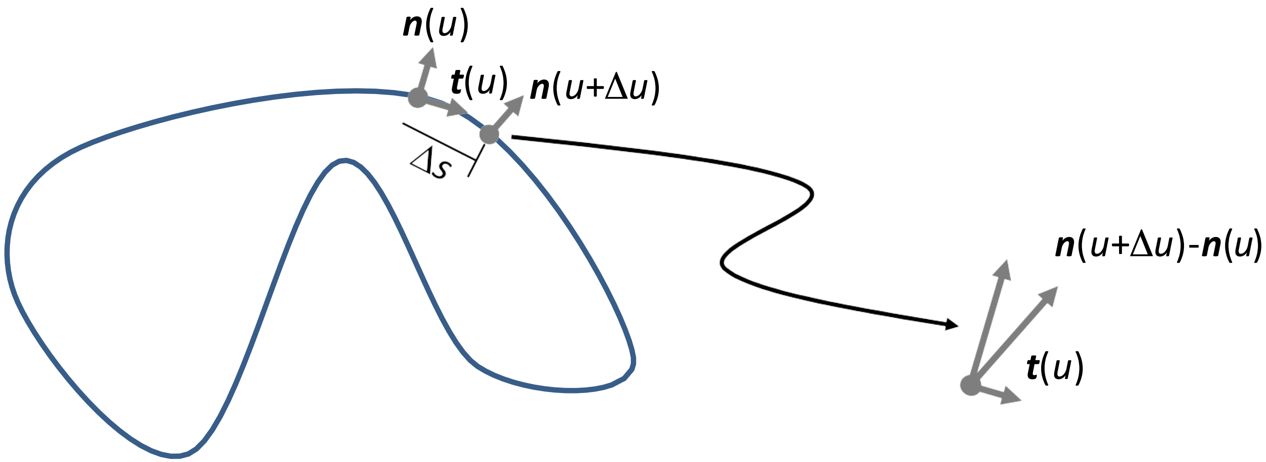


Regular Curves - *curvature*

Definition:

The *curvature* at $\boldsymbol{x}(u)$ is the change in normal vector along the tangent direction relative to change in distance along the curve:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\mathrm{n}(u+\Delta u) - \mathrm{n}(u)}{\Delta s}, \mathrm{t}(u) \right\rangle$$



Regular Curves

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\mathbf{n}(u+\Delta u) - \mathbf{n}(u)}{\Delta s}, \mathbf{t}(u) \right\rangle$$

<u>Note</u>:

If ***x*** is parameterized by arc-length, then $\Delta s = \Delta u$ so the curvature becomes:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\mathbf{n}(u+\Delta u) - \mathbf{n}(\Delta u)}{\Delta u}, \mathbf{t}(u) \right\rangle = \left\langle \mathbf{n}'(u), \mathbf{t}(u) \right\rangle$$

Otherwise, we have $\Delta s/\Delta u = |\boldsymbol{x}'(\mathrm{u})|$, so that:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\mathbf{n}(u+\Delta u) - \mathbf{n}(u)}{\Delta u \cdot |\mathbf{x}'(u)|}, \mathbf{t}(u) \right\rangle = \frac{\left\langle \mathbf{n}'(u), \mathbf{t}(u) \right\rangle}{|\mathbf{x}'(u)|}$$

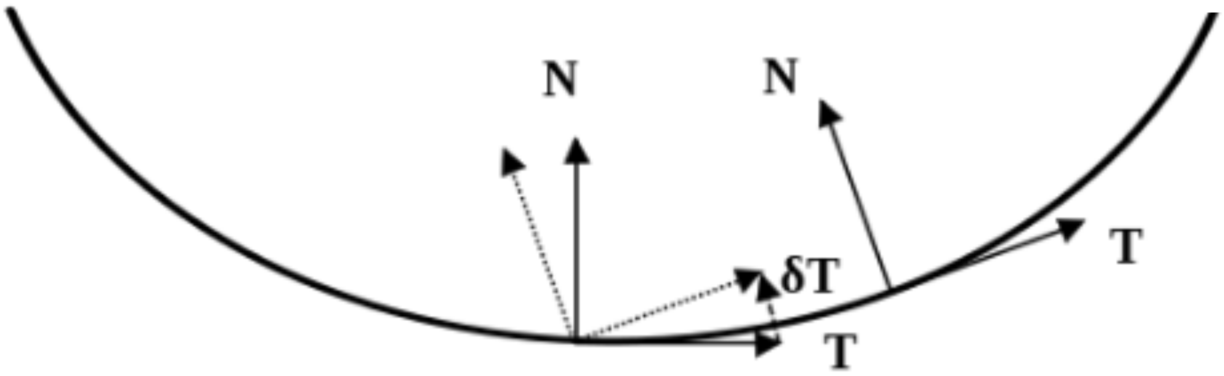
Curvature

- Suppose that a particle moves along the curve with unit speed.
- Tangent **T**: velocity vector
- dT/ds: acceleration vector
  - Curvature: magnitude of it
  - Normal: direction of it

$$\mathbf{T} = \frac{d\mathbf{r}}{ds}$$

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$$

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left\| \frac{d\mathbf{T}}{ds} \right\|}$$



The Frenet Frame & formula

- The tangent unit vector **T** is defined as

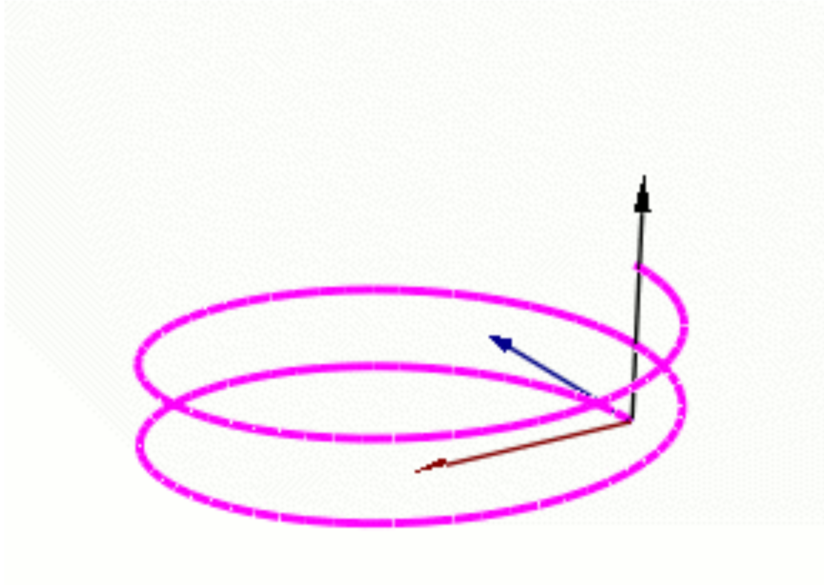
$$\mathbf{T} = \frac{d\mathbf{r}}{ds}. \qquad (1)$$

- The normal unit vector **N** is defined as

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left\| \frac{d\mathbf{T}}{ds} \right\|}. \qquad (2)$$

- The binormal unit vector **B** is defined as the cross product of **T** and **N**:

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}. \qquad (3)$$



$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

Torsion (deviation from planarity)

$$\tau = \frac{1}{\kappa^2} \det([\mathbf{x}_s, \mathbf{x}_{ss}, \mathbf{x}_{sss}])$$

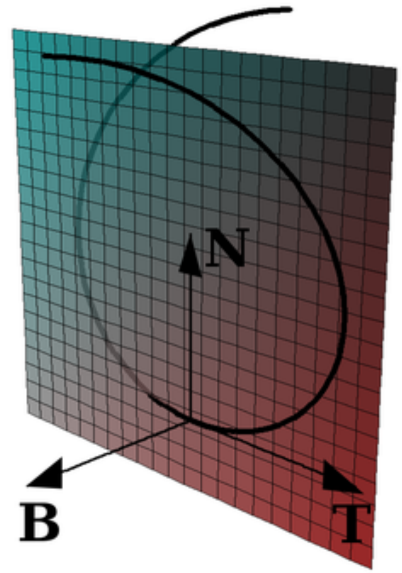
Curvature & Osculating circle

Planes defined by $\mathbf{x}$ and two vectors

- osculating plane: vectors $\mathbf{t}$ and $\mathbf{n}$
- normal plane: vectors $\mathbf{n}$ and $\mathbf{b}$
- rectifying plane: vectors $\mathbf{t}$ and $\mathbf{b}$

Osculating circle

- second order contact with curve
- center $\mathbf{c} = \mathbf{x} + (1/\kappa)\mathbf{n}$
- radius $1/\kappa$



- The tangent unit vector **T** is define

$$\mathbf{T} = \frac{d\mathbf{r}}{ds}. \quad (1)$$

- The normal unit vector **N** is define

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left\| \frac{d\mathbf{T}}{ds} \right\|}. \quad (2)$$

- The binormal unit vector **B** is defin

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}. \quad (3)$$

Curvature and Torsion

- **Curvature**: Deviation from straight line
- **Torsion**: Deviation from planarity

- Independent of parameterization
  - **intrinsic** properties of the curve

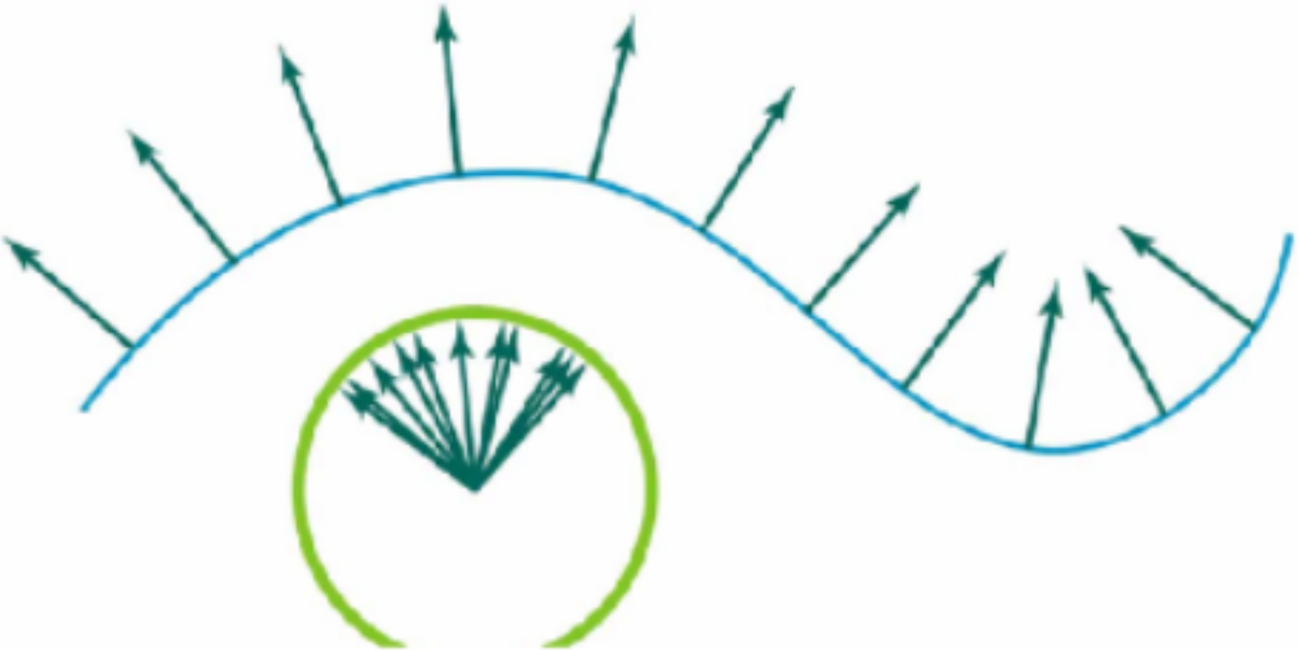
- Euclidean invariants
  - **invariant** under rigid motion

- Define curve **uniquely** up to a rigid motion

Curvature: Some Intuition

Gauß map $\hat{\mathrm{n}}(\mathrm{x})$

Point on curve maps to point on unit circle



Curvature: Some Intuition

Shape operator (Weingarten map)

Change in normal as we slide along curve

negative directional derivative D of Gauß map

$$\boxed{\mathrm{S}(\mathbf{v}) = -D_{\mathbf{v}}\hat{\mathbf{n}}}$$

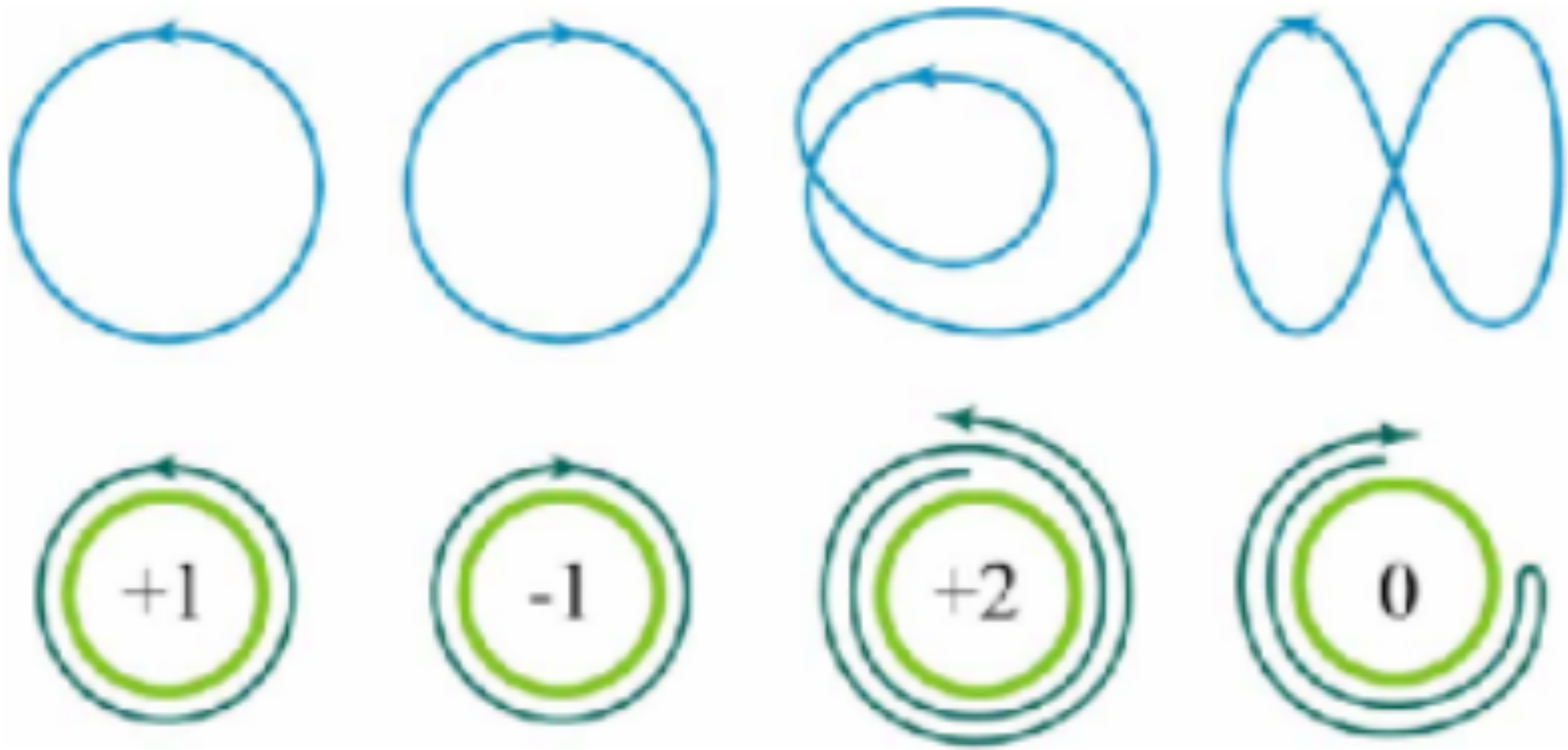
describes directional curvature

using normals as degrees of freedom

→ accuracy/convergence/implementation (discretization)

Turning number

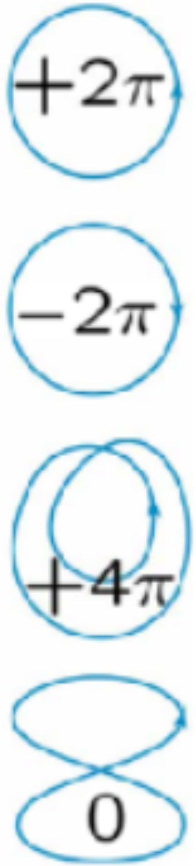
- **Turning number,** *k*
- Number of orbits in Gaussian image



Turning number theorem

- For a closed curve, the integral of curvature is an integer multiple of 2π

$$\boxed{\int_{\Omega} \kappa ds = 2\pi k}$$



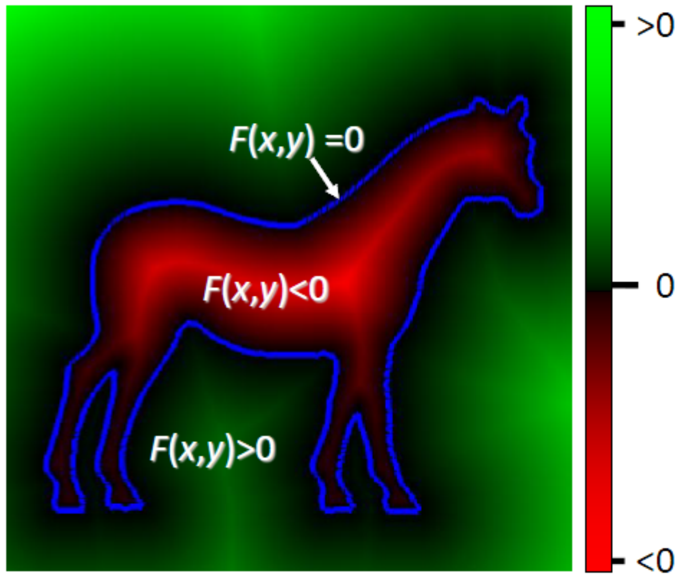
Differential Geometry

- Curves
- Surfaces

Shape Representations

- Parametric
  - Represent a surface as (continuous) injective function from a domain $\Omega \subset \boldsymbol{R}^2$ to $S \subset \boldsymbol{R}^3$
- Implicit
  - Represent a surface as the zero set of a scalar-valued function defined in R^3.

$$K = g^{-1}(0) = \{\mathbf{p} \in \mathrm{R}^3 : g(\mathbf{p}) = 0\}$$



Implicit Surfaces

Gradient

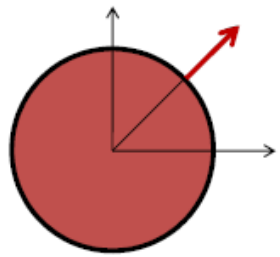
- Represent a surface as the zero set of a (regular) function defined in R^3.
- The normal vector to the surface is given by the gradient of the (scalar) implicit function

$$\nabla g(x,y,z) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right)^{\mathrm{T}}$$

  - Example

$$g(x,y,z) = x^2 + y^2 + z^2 - r^2$$

$$\nabla g(x,y,z) = (2x, 2y, 2z)^{\mathrm{T}}$$



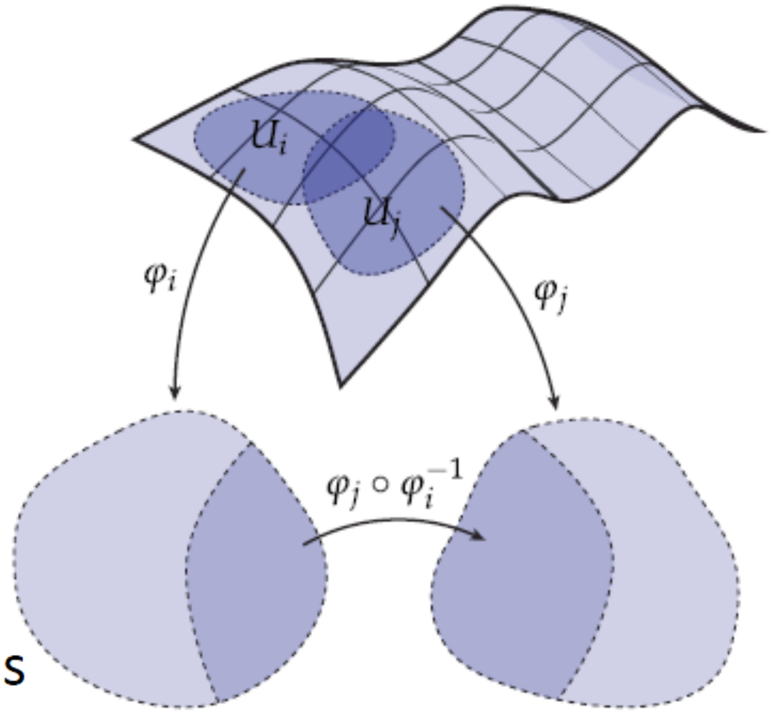
$$\nabla g(x,y,z) = (2,2,0)^{\mathrm{T}}$$

- Why is the condition that the function be regular (i.e. have non-vanishing derivative) necessary?
- How smooth is the surface?

Normal field is a gradient field of a scalar function g(x,,y,z)=r!

Mesh Representations

- Parametric
  - Represent a surface as (continuous) injective function from a domain $\Omega \subset \boldsymbol{R}^2$ to $S \subset \boldsymbol{R}^3$



- In practice, it's not easy to find a single function that parameterizes the surface.
- So instead, we represent a surface as a collection of functions (charts) from (simple) 2D domains into 3D.

Given a set of charts, we say that the manifold S is "smooth" if for any two charts $\phi_1:\Omega_1 \to S$ and $\phi_2:\Omega_2 \to S$, the map $\phi_2^{-1} \circ \phi_1$ is smooth.

Surfaces

- **What characterizes shape?**
  - shape does not depend on Euclidean motions
  - metric and curvatures

Metric on Surfaces

- **Measure Stuff**
  - angle, length, area
  - requires an inner product

- we have:
  - Euclidean inner product in domain

- we want to turn this into:
  - inner product on surface

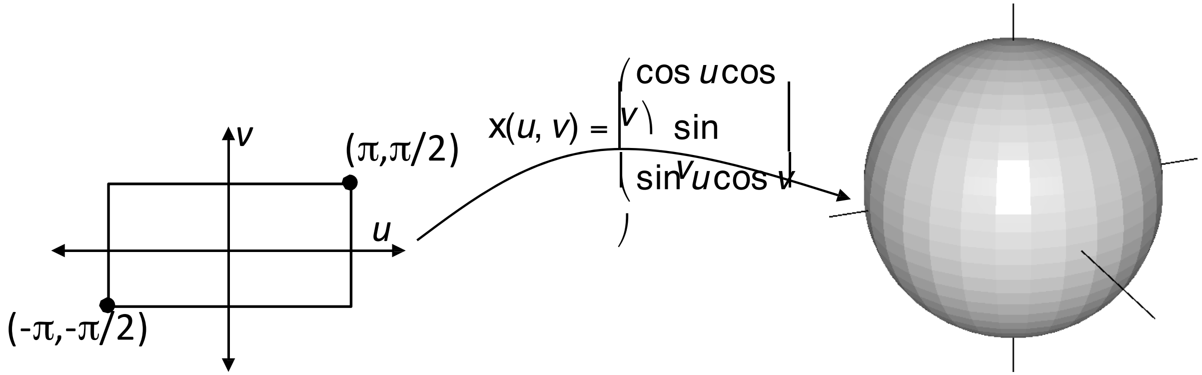
Differentiable Surfaces

<u>Definition</u>:

A *parameterized* ***differentiable*** *surface* is a differentiable map $\boldsymbol{x}:\Omega \to \boldsymbol{R}^3$ of an open domain $\Omega \subset \boldsymbol{R}^2$ into $\boldsymbol{R}^3$:

$$\boldsymbol{x}(u,v)=(x(u,v),y(u,v),z(u,v))$$

where $x(u,v)$, $y(u,v)$, and $z(u,v)$ are differentiable functions.

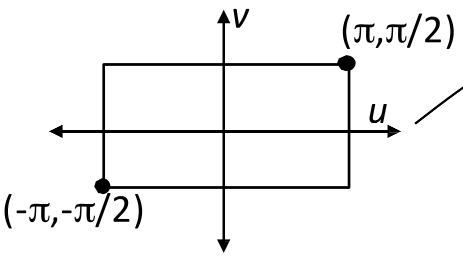


Differentiable Surfaces

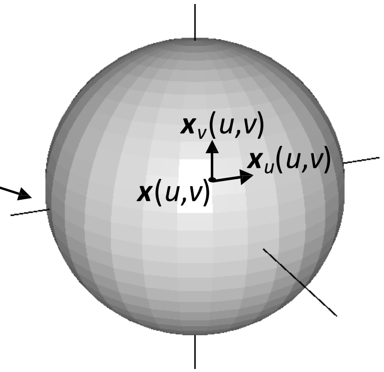
Definition:

The ***derivatives*** of the surface at $\boldsymbol{x}(u,v)$ are the vectors:

$$x_u(u,v) = \frac{\partial x(u,v)}{\partial u} = \begin{pmatrix} \partial x/\partial u \\ \partial y/\partial u \\ \partial z/\partial u \end{pmatrix} \qquad x_v(u,v) = \frac{\partial x(u,v)}{\partial v} = \begin{pmatrix} \partial x/\partial v \\ \partial y/\partial v \\ \partial z/\partial v \end{pmatrix}$$



$$x(u,v) = \begin{pmatrix} \cos u \cos v \\ \sin v \\ \sin u \cos v \end{pmatrix}$$



Differentiable Surfaces

$$x_u(u, v) = \frac{\partial x(u, v)}{\partial u} \qquad x_v(u, v) = \frac{\partial x(u, v)}{\partial v}$$

Definition:

The surface is said to be ***regular*** if at each point (u,v) the derivatives/tangents $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are linearly independent.

This is equivalent to the statement:

$$x_u \times x_v \neq 0$$

i.e. that a normal (line) can be defined everywhere.

Normal Vectors

- Continuous surface

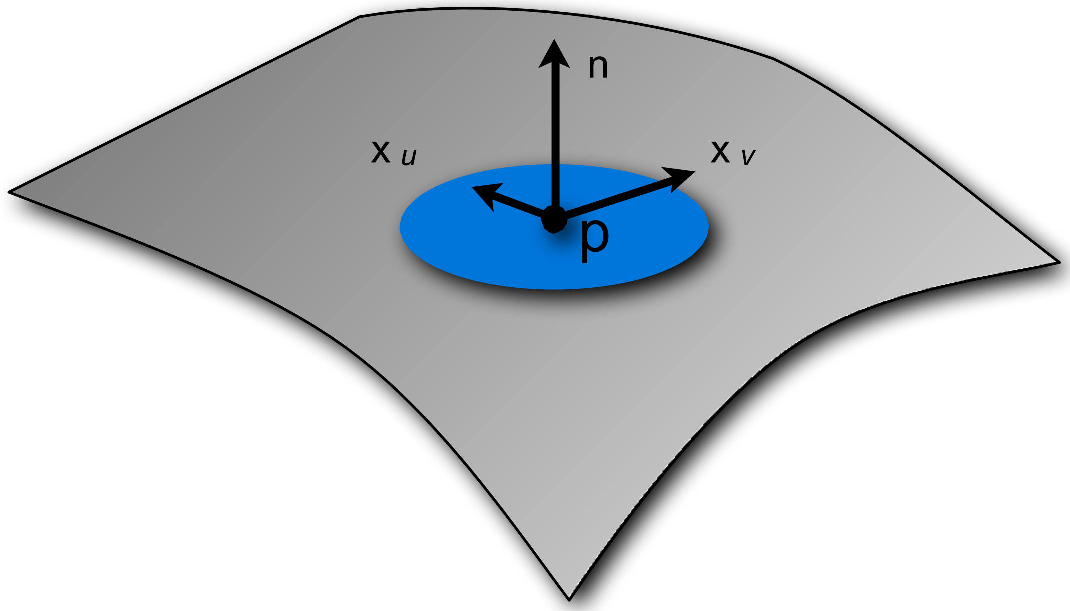
$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

Assume *regular* parameterization

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$ normal exists



Riemannian Metric & first fundamental form

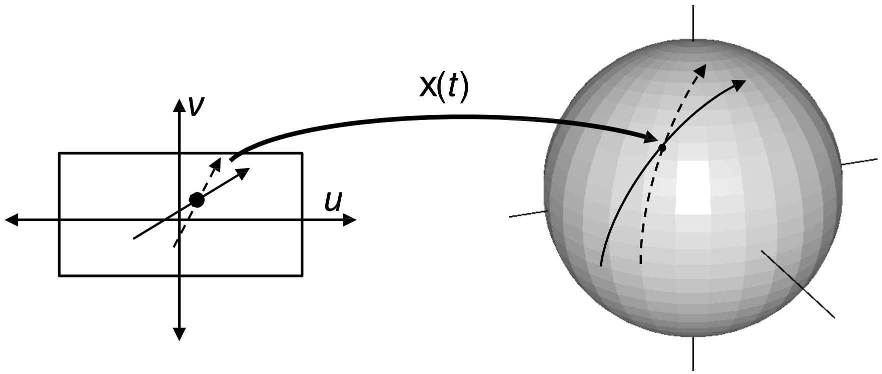
Curve in parameter domain => curve on surface

$$x_u(u,v) = \frac{\partial x(u,v)}{\partial u} \qquad x_v(u,v) = \frac{\partial x(u,v)}{\partial v}$$

<u>Definition</u>:

Given a point $p_0=(u_0,v_0)\in\Omega$ and given a **direction $w=(u_w,v_w)$** in the parameter space, we can define the (3D) curve:

$$x(t) = x(p_0 + tw)$$



Directional derivatives

$$x_u(u,v) = \frac{\partial x(u,v)}{\partial u} \qquad x_v(u,v) = \frac{\partial x(u,v)}{\partial v}$$

Definition:

$$x(t) = x(p_0 + tw)$$

Taking the derivative at t=0, we get:

$$x'(0) = w_u x_u + w_v x_v = J(w)$$

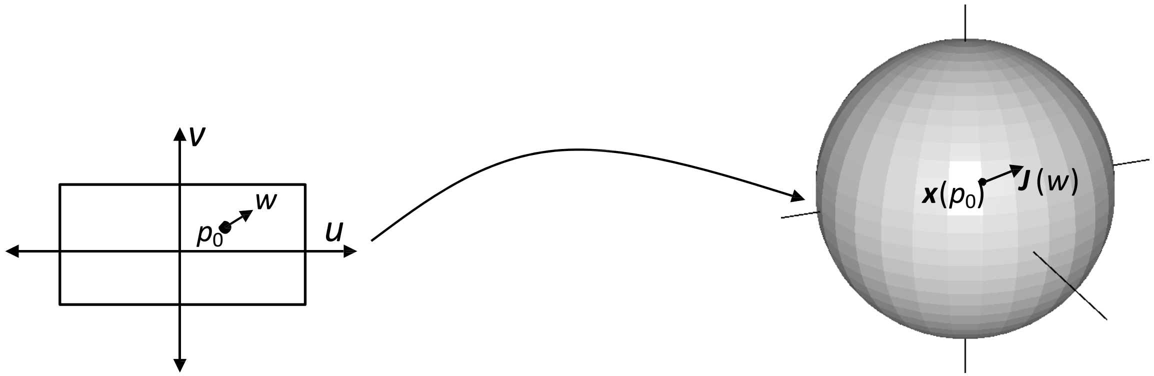
where $\boldsymbol{J}$ is the Jacobian matrix taking directions in Ω to tangent vectors on the surface:

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

Metric Properties - length

Thus, given a point $p_0=(u_0,v_0)\in\Omega$ and given a direction $w=(u_w,v_w)$, we can use the Jacobian to compute the length of the corresponding tangent vector over $\boldsymbol{x}(p_0)$:

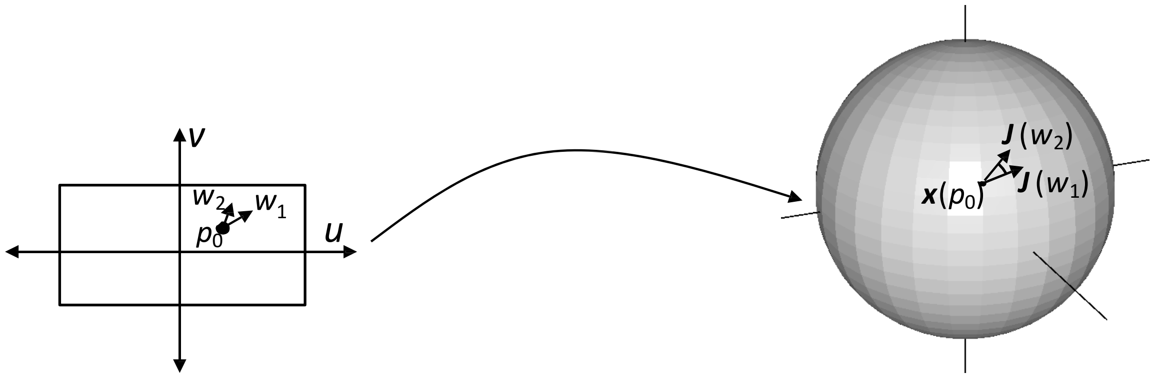
$$length^2 = \left\|\boldsymbol{J}w\right\|^2 = w^t \boldsymbol{J}^t \boldsymbol{J}w$$



Metric Properties - angle

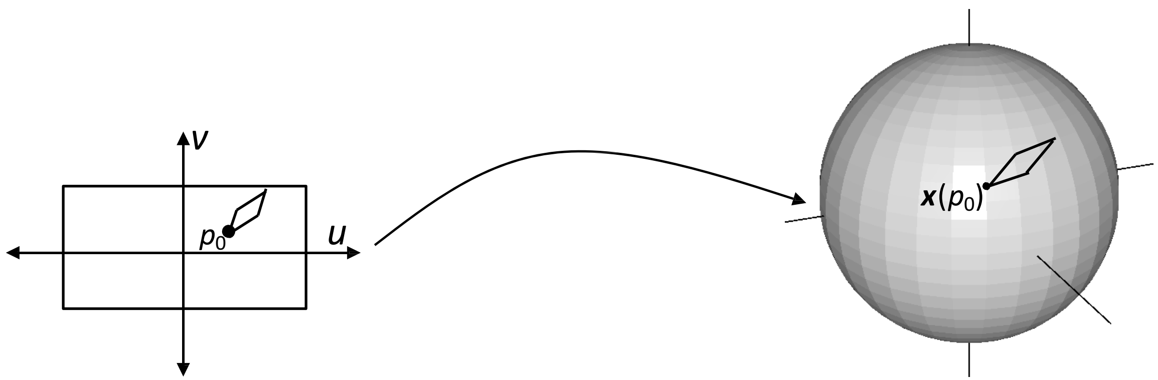
- Similarly, given a point $p_0=(u_0,v_0)\in\Omega$ and given directions $w_1=(u_1,v_1)$ and $w_2=(u_2,v_2)$ we can use the Jacobian to compute the angle of the corresponding tangent vectors over $\boldsymbol{x}(p_0)$:

$$\cos(angle) = \frac{\langle \boldsymbol{J}w_1, \boldsymbol{J}w_2 \rangle}{\|\boldsymbol{J}w_1\| \|\boldsymbol{J}w_2\|} = \frac{w_1^t \boldsymbol{J}^t \boldsymbol{J}w_2}{\sqrt{w_1^t \boldsymbol{J}^t \boldsymbol{J}w_1}\sqrt{w_2^t \boldsymbol{J}^t \boldsymbol{J}w_2}}$$



Metric Properties - area

- Finally, given a point $p_0=(u_0,v_0)\in\Omega$ and given directions $w_1=(u_1,v_1)$ and $w_2=(u_2,v_2)$ we can use the Jacobian to compute the area of the corresponding parallelogram in the tangent space:
  - $area = length_1 \cdot length_2 \cdot \sin(angle)$



Metric Properties - area

<u>Note</u>:

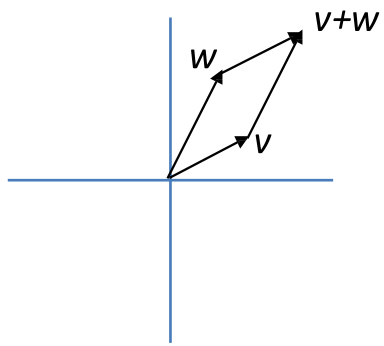
Given vectors *v* and *w* in $\boldsymbol{R}^n$, the area of the parallelogram spanned by *v* and *w* is:

$$Area(v,w) = |v| \cdot |w| \cdot \sin\big(Angle(v,w)\big)$$

$$= |v| \cdot |w| \cdot \sqrt{1-\cos^2 Angle(v,w)}$$

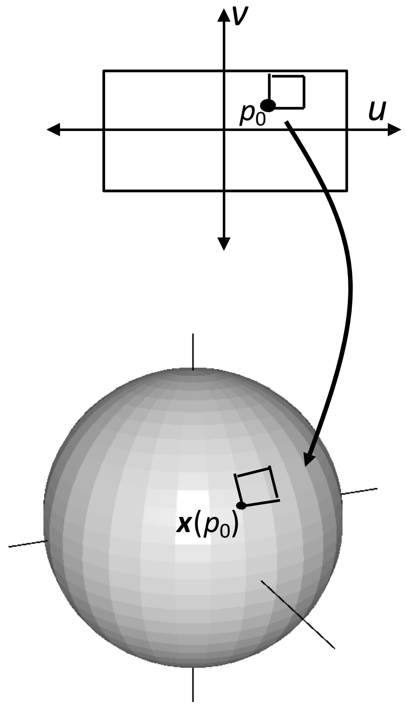
$$= |v| \cdot |w| \cdot \sqrt{1 - \frac{\langle v, w\rangle^2}{|v|^2 |w|^2}}$$

$$= \sqrt{|v|^2 |w|^2 - \langle v, w\rangle^2}$$



Metric Properties - area

$$Area(v,w) = \sqrt{|v|^2 |w|^2 - \langle v, w \rangle}$$



Note:

Since the first fundamental form is defined as:

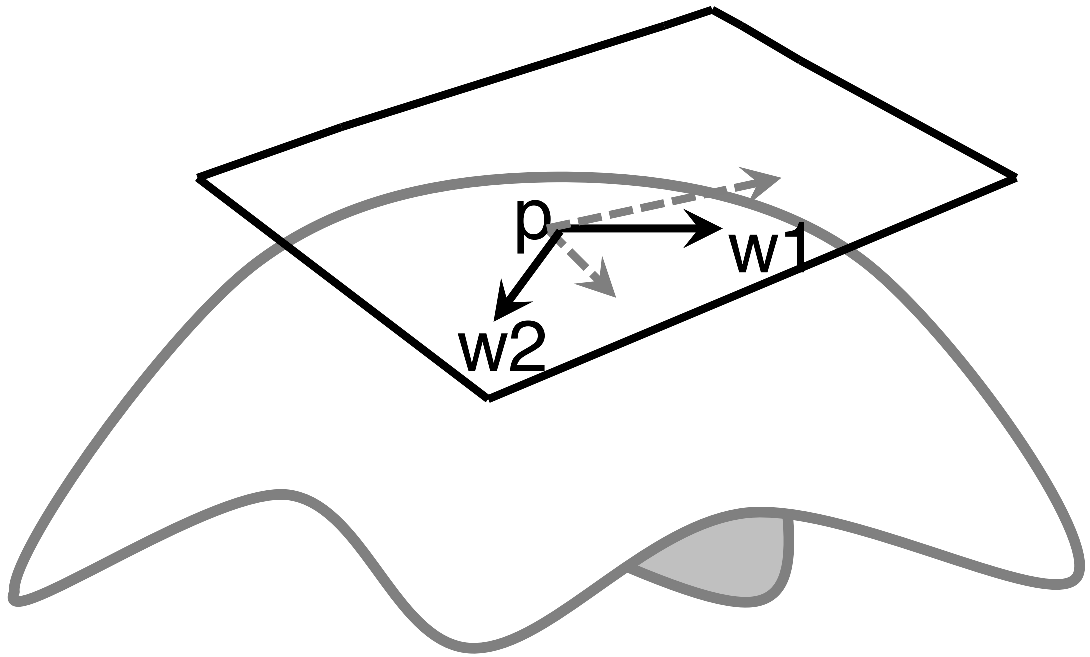
$$I = J J^t = \begin{pmatrix} \langle x_u, x_u \rangle & \langle x_u, x_v \rangle \\ \langle x_v, x_u \rangle & \langle x_v, x_v \rangle \end{pmatrix}$$

in mapping from Ω to the surface, the area of a tiny patch of surface gets scaled by:

$$\sqrt{\|x_u\|^2 \|x_v\|^2 - \langle x_u, x_v \rangle^2} = \sqrt{\det I}$$

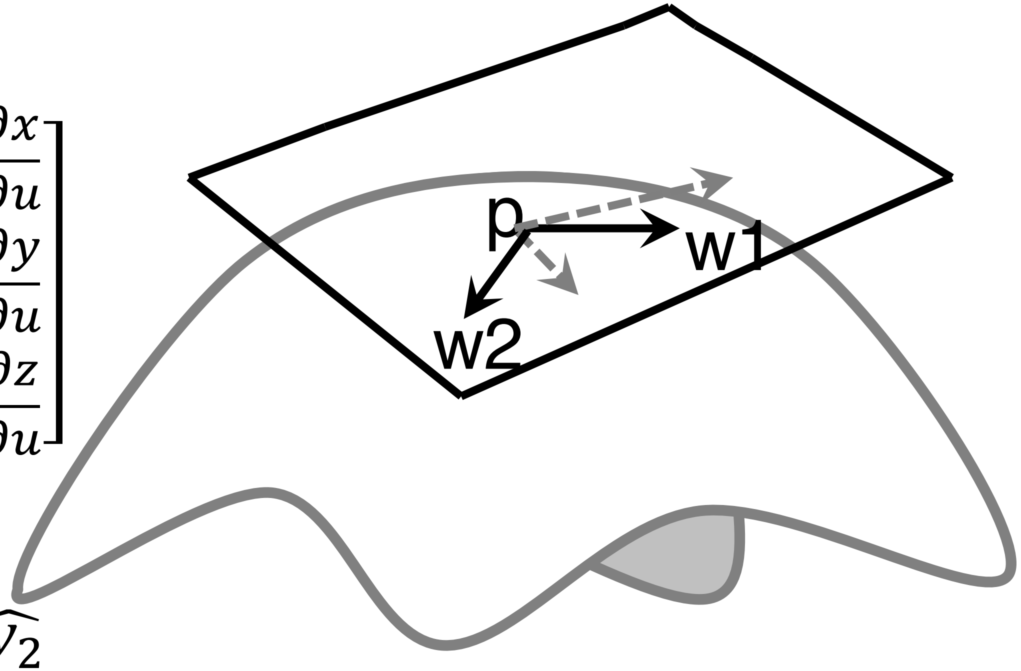
First fundamental form

- It is the inner product on the tangent space of a surface in three-dimensional Euclidean space which is induced canonically from the dot product of $\mathbf{R}^3$



First Fundamental Form I_S

- **Riemannian metric**, Metric Tensor, Fundamental Tensor
- S(u,v)=(x(u,v), y(u,v),z(u,v))
- Jacobian matrix $J = [S_u, S_v] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial u} \end{bmatrix}$
- $w = J\widehat{w} = [S_u, S_v]\begin{bmatrix} u \\ v \end{bmatrix}$
- $< w_1, w_2 >= (J\widehat{w_1})^T(J\widehat{w_2})$=$\widehat{w_1}^T(J^TJ)\widehat{w_2}$
- $I = J^TJ = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$



First Fundamental Form

First fundamental form $\mathbf{I}$ allows to measure

(w.r.t. surface metric)

Angles $\quad \mathbf{t}_1^\top \mathbf{t}_2 = \langle(\alpha_1, \beta_1), (\alpha_2, \beta_2)\rangle$

Length
$$\begin{aligned} \mathrm{d}s^2 &= \langle(\mathrm{d}u, \mathrm{d}v), (\mathrm{d}u, \mathrm{d}v)\rangle \\ &= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 \end{aligned}$$

squared infinitesimal length

Area
$$\begin{aligned} \mathrm{d}A &= \|\mathbf{x}_u \times \mathbf{x}_v\| \, \mathrm{d}u \, \mathrm{d}v \\ &= \sqrt{\mathbf{x}_u^T \mathbf{x}_u \cdot \mathbf{x}_v^T \mathbf{x}_v - (\mathbf{x}_u^T \mathbf{x}_v)^2} \, \mathrm{d}u \, \mathrm{d}v \\ &= \sqrt{EG - F^2} \mathrm{d}u \, \mathrm{d}v \end{aligned}$$

infinitesimal Area

cross product → determinant with unit vectors → area

- curve length

$$L = l(a,b) = \int_a^b \|\mathbf{x}'(u)\| \mathrm{d}u$$

$$l(a,b) = \int_a^b \sqrt{(u_t, v_t)\mathbf{I}(u_t, v_t)^T} \mathrm{d}t$$

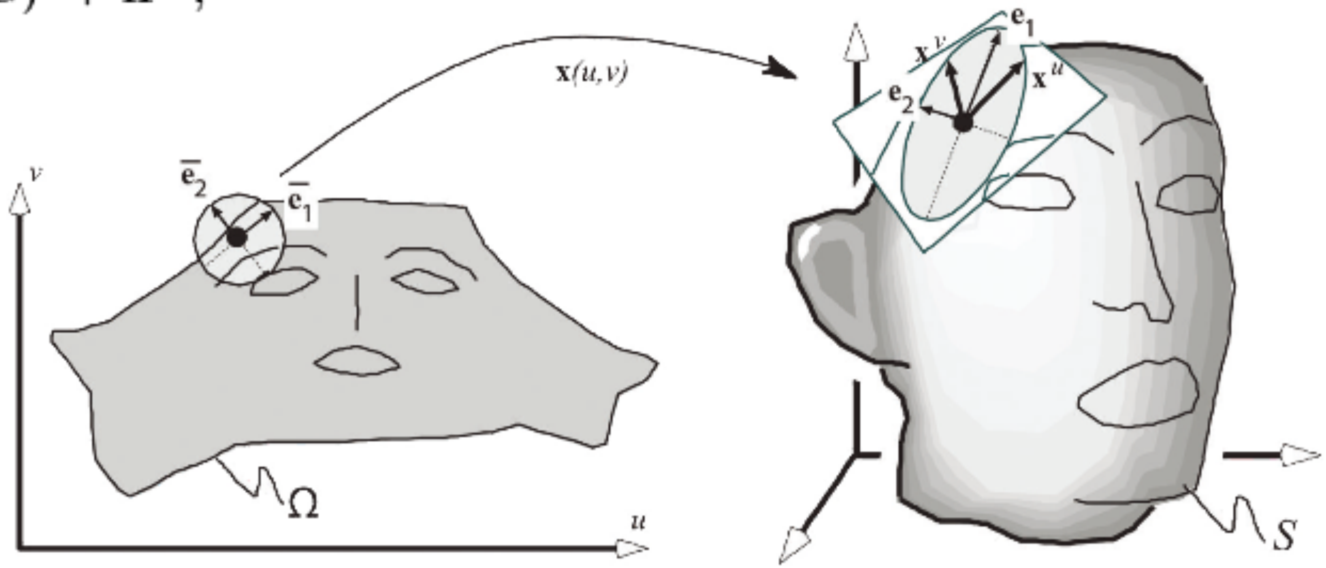
$$= \int_a^b \sqrt{Eu_t^2 + 2Fu_tv_t + Gv_t^2} \mathrm{d}t.$$

- Surface area
- $\mathrm{A} = \mathrm{A}(\mathrm{X}) = \iint_{\mathrm{U}} |\mathrm{X_u} \times \mathrm{X_v}| \mathrm{dudv} = \iint_{\mathrm{U}} \sqrt{\mathrm{EG} - \mathrm{F}^2} \mathrm{dudv} = \iint_{\mathrm{U}} \sqrt{\det(\mathrm{I_X})} \mathrm{dudv} = \iint_{\mathrm{U}} \det(\mathrm{J_X}) \mathrm{dudv} = \iint_{\mathrm{U}} \mathrm{Jacobian}(\mathrm{X}) \mathrm{dudv}$

Anisotropy

- the axes of the anisotropy ellipse are $\mathbf{e}_1 = \mathbf{J}\bar{\mathbf{e}}_1$ and $\mathbf{e}_2 = \mathbf{J}\bar{\mathbf{e}}_2$;
- the lengths of the axes are $\sigma_1 = \sqrt{\lambda_1}$ and $\sigma_2 = \sqrt{\lambda_2}$.

$$\sigma_1 = \sqrt{1/2(E+G) + \sqrt{(E-G)^2 + 4F^2}},$$
$$\sigma_2 = \sqrt{1/2(E+G) - \sqrt{(E-G)^2 + 4F^2}},$$

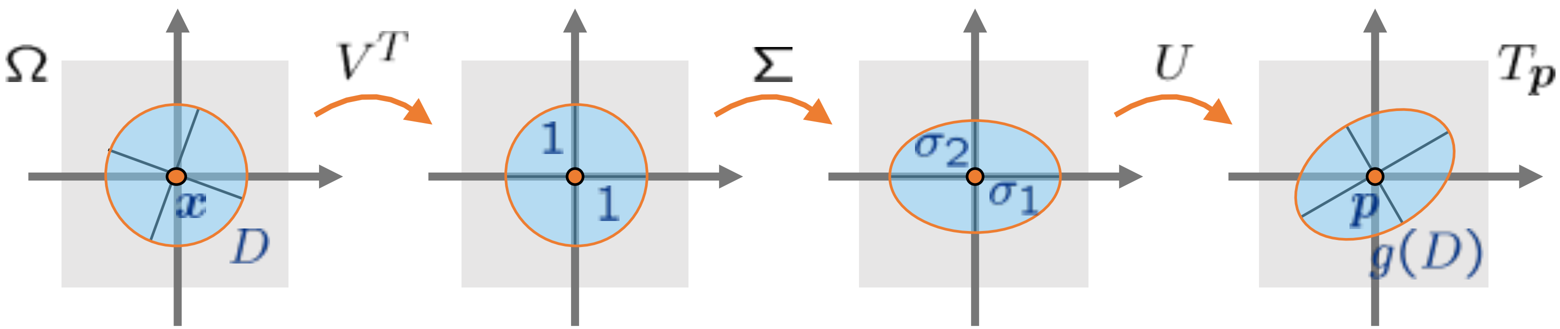


Linear Map Surgery

- **Singular Value Decomposition** (SVD) of J_f

$$J_f = U \Sigma V^T = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} V^T$$

with **rotations** $U \in \mathbb{R}^{3\times 3}$ and $V \in \mathbb{R}^{2\times 2}$
and **scale factors** (singular values) $\sigma_1 \geq \sigma_2 > 0$



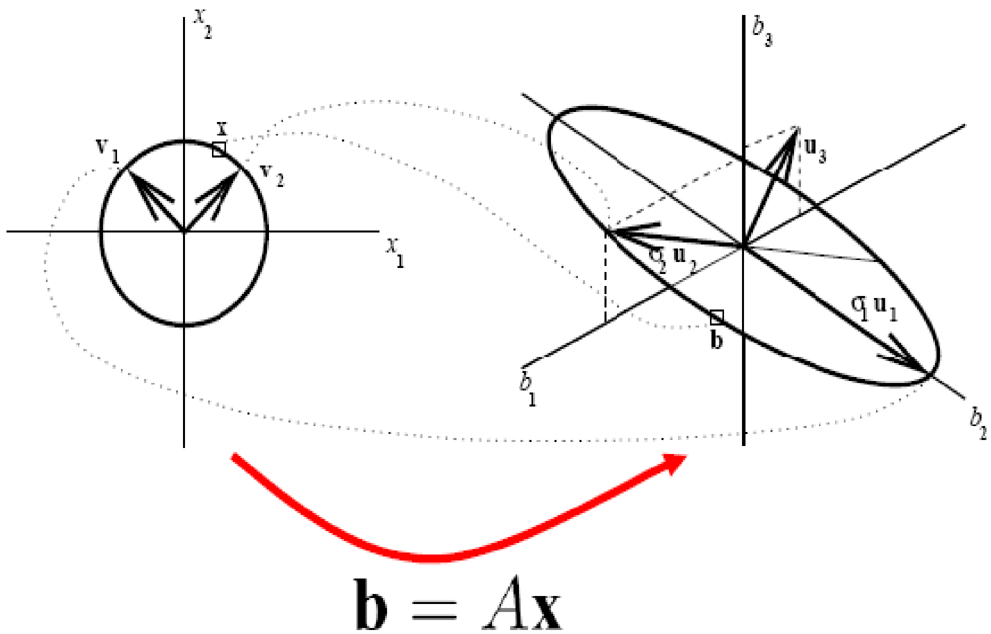
SVD

- Each matrix can be treated as a linear map or Jacobian Matrix of a map. Each owns a SVD decomposition, i.e. can be **described as an aligner followed by a stretch followed by a hanger.** (can be represented by a concatenation of rotation and scale.**)**

$J_f = (f_u \quad f_v)$ is a matrix of 3 by 2.

$J_f = U\Sigma V^T = (U_1 \quad U_2 \quad U_3)\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\\ 0 & 0\end{pmatrix}(V_1 \quad V_2)^T$, V_i are eigenvectors of ${J_f}^T J_f$, U_i are eigenvectors of $J_f {J_f}^T$.

(Note: $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \lambda_1, \lambda_2$ are eigenvalues of ${J_f}^T J_f$, not $J_f {J_f}^T$)



Notion of Distortion

- **isometric** or **length**-preserving

$$\sigma_1 = \sigma_2 = 1$$

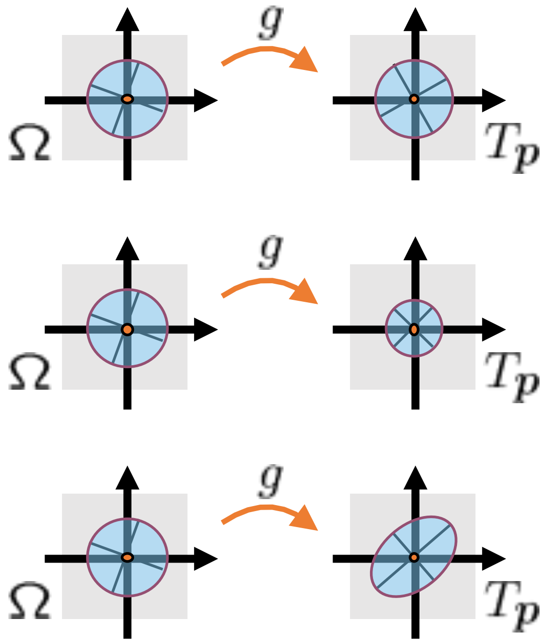
- **conformal** or **angle**-preserving

$$\sigma_1 = \sigma_2$$

- **equiareal** or **area**-preserving

$$\sigma_1 \cdot \sigma_2 = 1$$

- everything defined **pointwise** on Ω

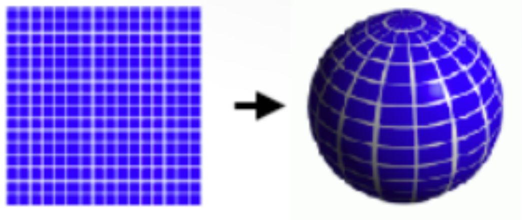


Theorem 4. *Every isometric mapping is conformal and equiareal, and every conformal and equiareal mapping is isometric, i.e.,*

$$\text{isometric} \Leftrightarrow \text{conformal} + \text{equiareal}.$$

Sphere Example

Spherical parameterization



$$\mathbf{x}(u,v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}, \quad (u,v) \in [0, 2\pi) \times [0, \pi)$$

Tangent vectors

$$\mathbf{x}_u(u,v) = \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix} \quad \mathbf{x}_v(u,v) = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{pmatrix}$$

First fundamental Form

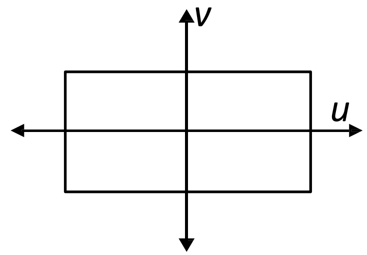
$$\mathbf{I} = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

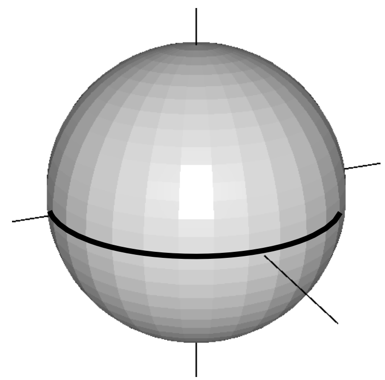
Metric Properties

$$x(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t \qquad \mathrm{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the length of the equator?





Metric Properties

$$\mathrm{x}(u,v) = (\cos u \cos v \; \sin v \; \sin u \cos v)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

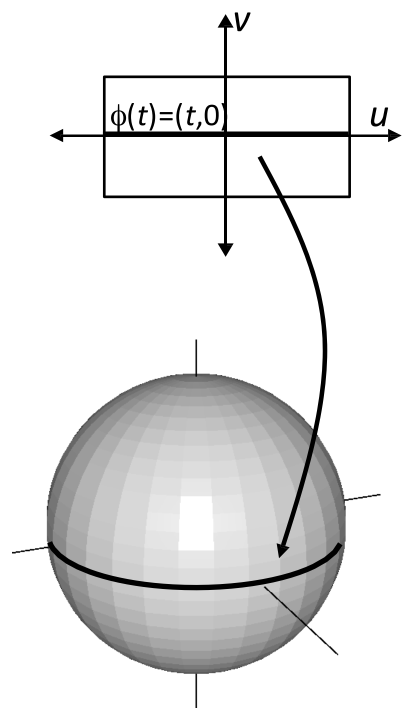
<u>Example (Sphere)</u>:

- What is the length of the equator?

The equator is the image of:

$$\phi(t)=(t,0) \quad \text{with } t\in[-\pi, \pi]$$

under the parameterization.



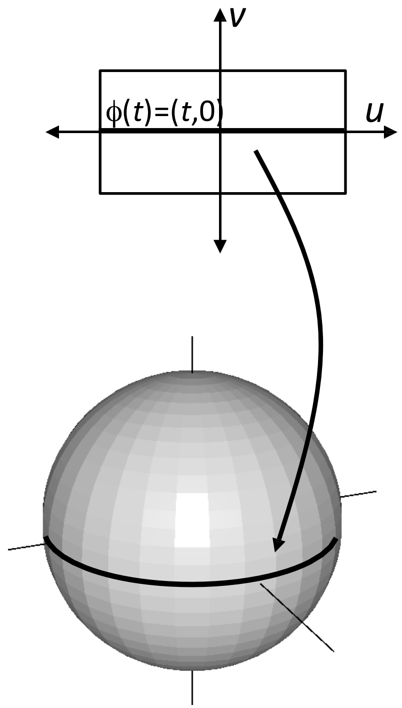
Metric Properties

$x(u,v) = (\cos u \cos v \;\; \sin v \;\; \sin u \cos v)^t$ $\qquad I(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$

<u>Example (Sphere)</u>:

- What is the length of the equator?

$$
\begin{aligned}
length(\boldsymbol{x} \circ \phi) &= \int_{-\pi}^{\pi} \sqrt{\phi'(t)^t \boldsymbol{I} \phi'(t)}\, dt \\
&= \int_{-\pi}^{\pi} \sqrt{(1,0)^t \begin{pmatrix} \cos^2(0) & 0 \\ 0 & 1 \end{pmatrix} (1,0)}\, dt \\
&= \int_{-\pi}^{\pi} dt \\
&= 2\pi
\end{aligned}
$$

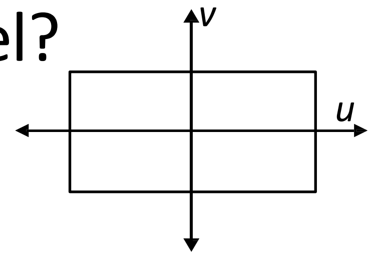


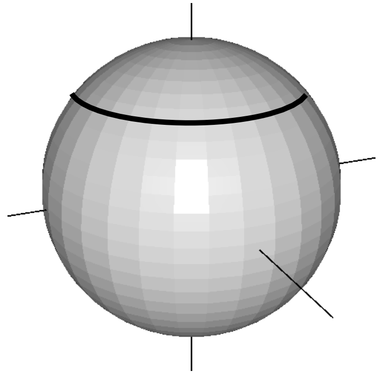
Metric Properties

$$x(u, v) = \left(\cos u \cos v \;\; \sin v \;\; \sin u \cos v\right)^t \qquad I(u, v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

Example (Sphere):

- What is the length of the w^{th} parallel?



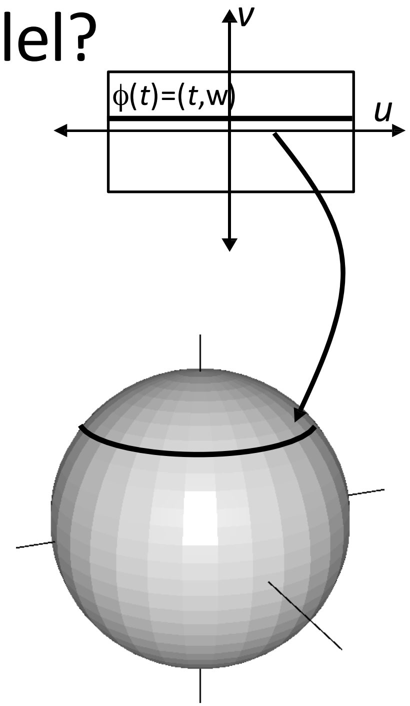


Metric Properties

$$x(u,v) = (\cos u \cos v \;\; \sin v \;\; \sin u \cos v)^t \qquad I(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

Example (Sphere):

- What is the length of the w^{th} parallel?



The w^{th} parallel is the image of:

$$\phi(t)=(t,w) \quad \text{with } t\in[-\pi, \pi]$$

under the parameterization.

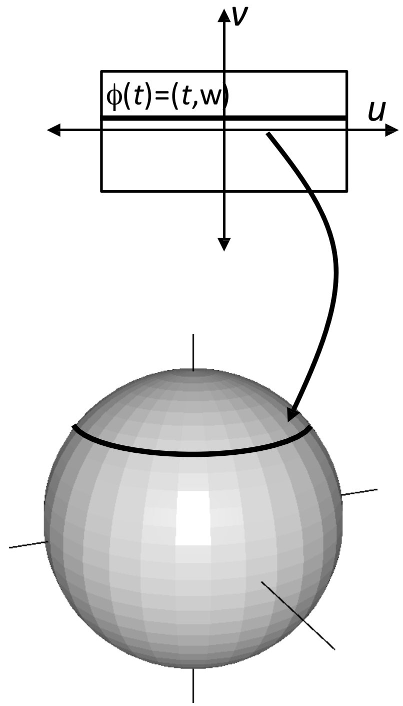
Metric Properties

$$x(u,v) = (\cos u \cos v \sin v \quad \sin u \cos v)^t \qquad I(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the length of the *w*th parallel?

$$
\begin{aligned}
length(\boldsymbol{x} \circ \phi) &= \int_{-\pi}^{\pi} \sqrt{\phi'(t)^t \boldsymbol{I} \phi'(t)}\, dt \\
&= \int_{-\pi}^{\pi} \sqrt{(1,0)^t \begin{pmatrix} \cos^2 w & 0 \\ 0 & 1 \end{pmatrix} (1,0)}\, dt \\
&= \int_{-\pi}^{\pi} \cos w \, dt \\
&= 2\pi \cos w
\end{aligned}
$$

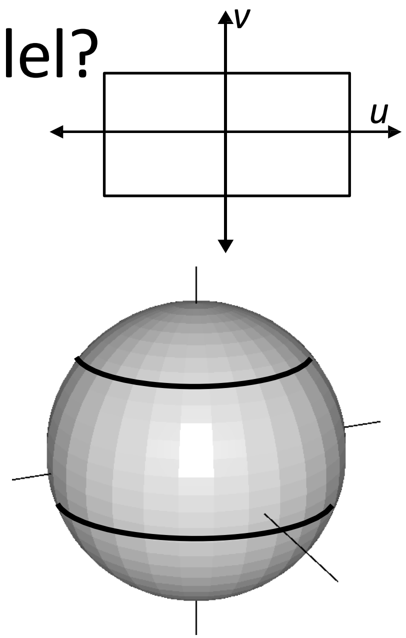


Metric Properties

$$x(u,v) = \begin{pmatrix} \cos u \cos v & \sin v & \sin u \cos v \end{pmatrix}^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

Example (Sphere):

- What is the area of the band between the w_1^{th} parallel and the w_2^{th} parallel?

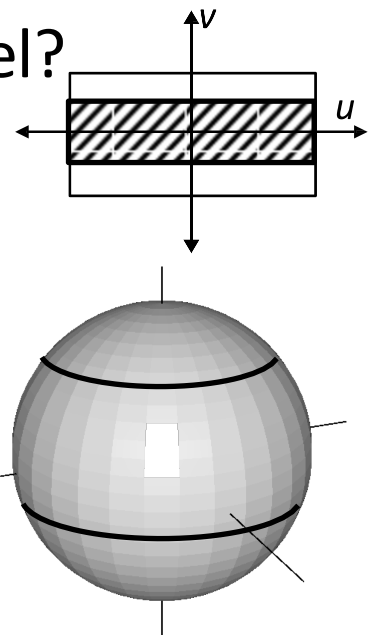


Metric Properties

$$\mathrm{x}(u,v) = (\cos u \cos v \;\; \sin v \;\; \sin u \cos v)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example (Sphere):</u>

- What is the area of the band between the w_1th parallel and the w_2th parallel?



The band is the image of:

$\phi(s,t)=(s,t)$ with $s\in[-\pi,\pi]$, $t\in[w_1,w_2]$

under the parameterization.

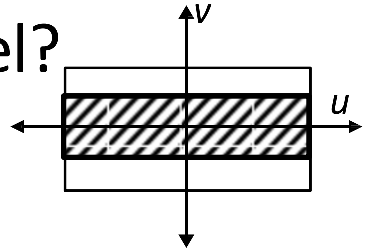
Metric Properties

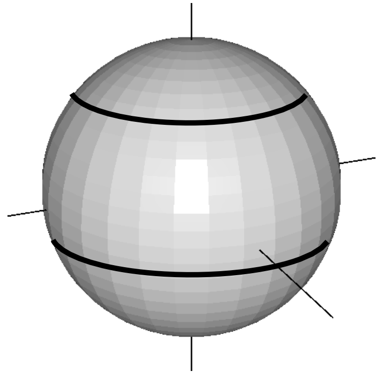
$$\mathrm{x}(u,v) = (\cos u \cos v \sin v \;\; \sin u \cos v)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

~~Example (Sphere):~~

- What is the area of the band between the w_1^th parallel and the w_2^th parallel?

$$\begin{aligned} area(\mathrm{x} \circ \phi) &= \int_{w_1}^{w_2} \int_{-\pi}^{\pi} \sqrt{\det \mathrm{I}}\, ds\, dt \\ &= \int_{w_1}^{w_2} \int_{-\pi}^{\pi} \cos t \; ds\, dt \\ &= 2\pi \int_{w_1}^{w_2} \cos t \; dt \\ &= 2\pi(\sin w_2 - \sin w_1) \end{aligned}$$



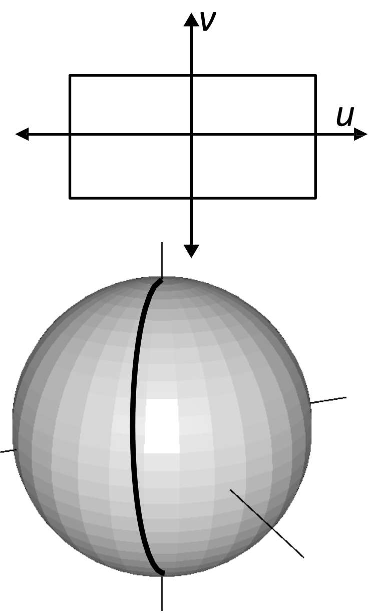


Metric Properties

$$\mathrm{x}(u,v) = (\cos u \cos v \;\; \sin v \;\; \sin u \cos v)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example (Sphere)</u>:

- What is the area of the band between the w_1^{th} and the w_2^{th} meridians?



Metric Properties

$$x(u,v) = (\cos u \cos v \;\; \sin v \;\; \sin u \cos v)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

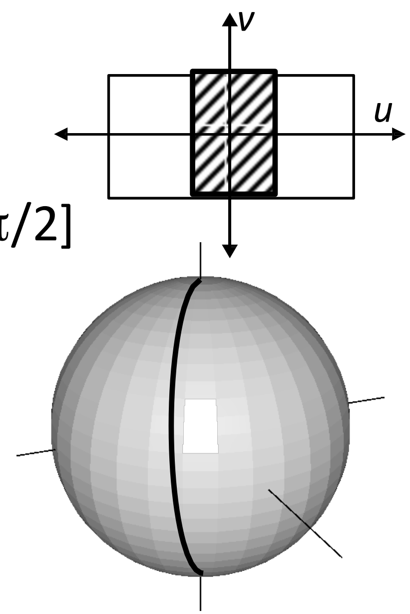
<u>Example (Sphere)</u>:

- What is the area of the band between the w_1th and the w_2th meridians?

The band is the image of:

$$\phi(s,t)=(s,t) \quad \text{with } s\in[w_1, w_2],\ t\in[-\pi/2, \pi/2]$$

under the parameterization.



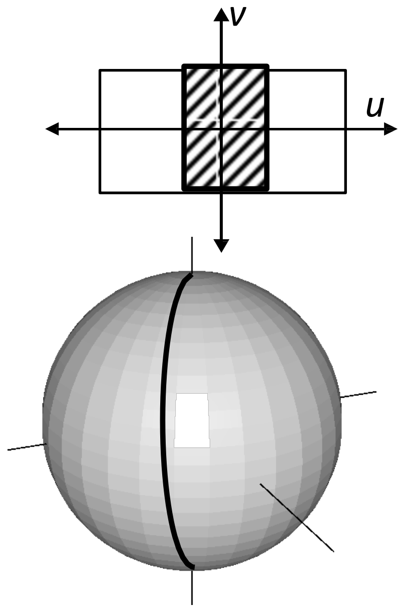
Metric Properties

$$x(u,v) = \begin{pmatrix} \cos u \cos v & \sin u \cos v & \sin v \end{pmatrix}^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

Example (Sphere):

- What is the area of the band between the w_1th and the w_2th meridians?

$$\begin{aligned}
area(x \circ \phi) &= \int_{-\pi/2}^{\pi/2} \int_{w_1}^{w_2} \sqrt{\det \mathrm{I}}\, ds\, dt \\
&= \int_{-\pi/2}^{\pi/2} \int_{w_1}^{w_2} \cos t\, ds\, dt \\
&= (w_2 - w_1) \int_{-\pi/2}^{\pi/2} \cos t\, dt \\
&= (w_2 - w_1)\big(\sin(\pi/2) - \sin(-\pi/2)\big) \\
&= 2(w_2 - w_1)
\end{aligned}$$

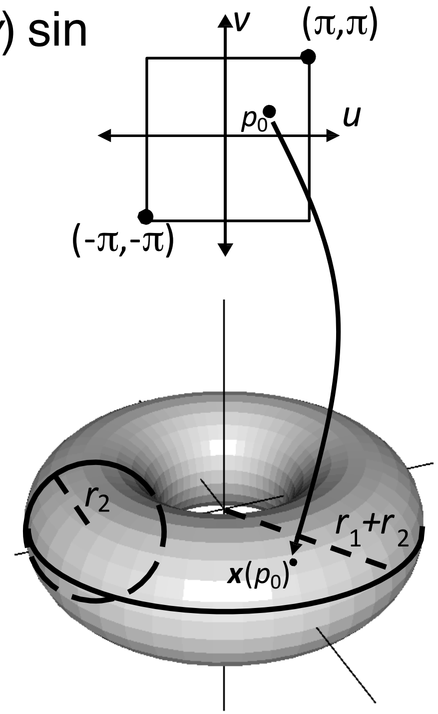


Metric Properties

Example (Torus):

We can parameterize the torus by the map:

$$\mathbf{x}(u,v) = \big((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\big)^t$$



Metric Properties

Example (Torus):

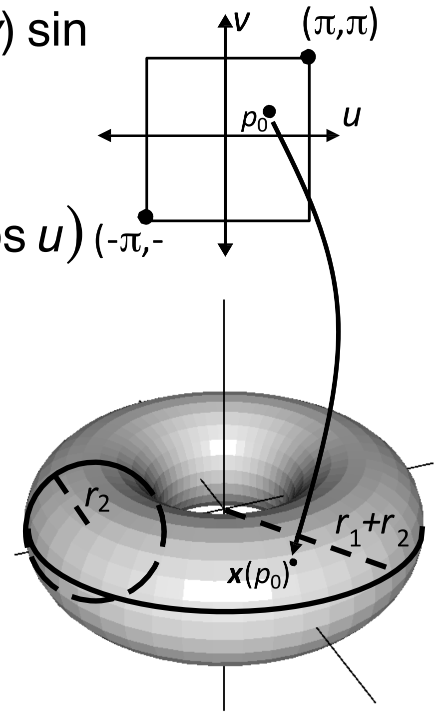
We can parameterize the torus by the map:

$$x(u,v) = \big((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\big)^t$$

Then the partial derivatives are:

$$x_u(u,v) = \big(-(r_1 + r_2 \sin v)\sin u \quad 0 \quad (r_1 + r_2 \sin v)\cos u\big)$$

$$x_v(u,v) = \big(r_2 \cos v \cos u \quad -r_2 \sin v \quad r_2 \cos v \sin u\big)$$



Metric Properties

<u>Example (Torus)</u>:

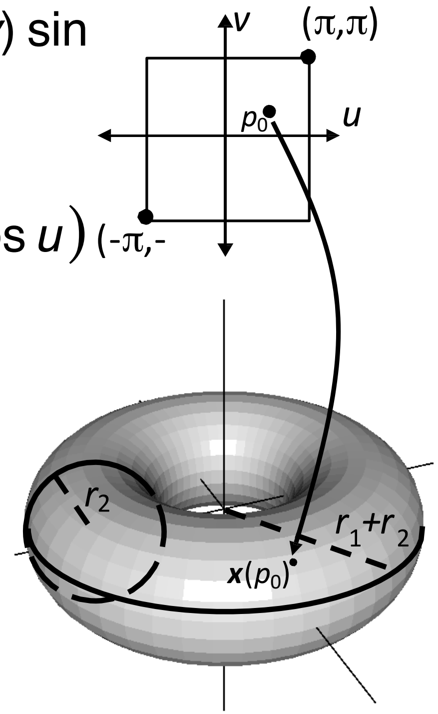
We can parameterize the torus by the map:

$$\mathrm{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t$$

Then the partial derivatives are:

$$\mathrm{x}_u(u,v) = \left(-(r_1 + r_2 \sin v)\sin u \quad 0 \quad (r_1 + r_2 \sin v)\cos u\right)$$

$$\mathrm{x}_v(u,v) = \left(r_2 \cos v \cos u \quad -r_2 \sin v \quad r_2 \cos v \sin u\right)$$



Which gives:

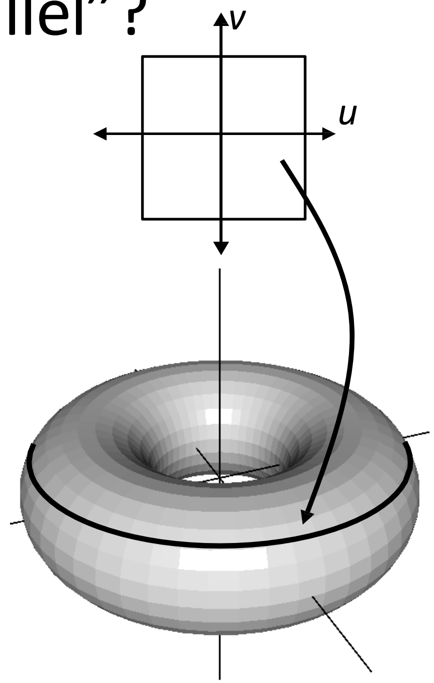
$$\mathrm{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

Metric Properties

$$\mathrm{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t \quad \mathrm{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

Example (Torus):

- What is the length of the w^{th} "parallel"?



Metric Properties

$$\mathbf{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \;\; r_2 \cos v \;\; (r_1 + r_2 \sin v)\sin u\right)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

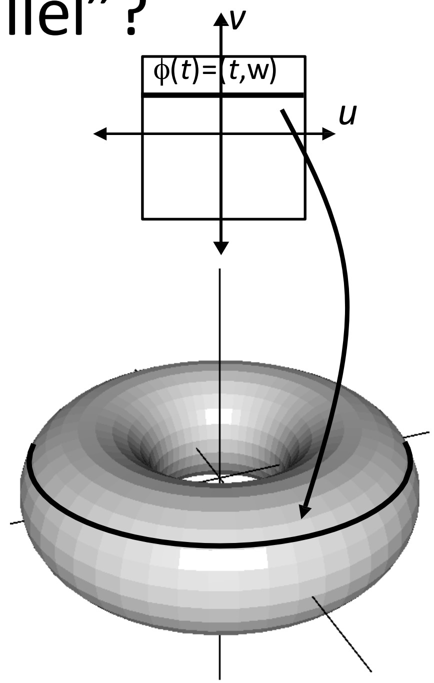
Example (Torus):

- What is the length of the w^{th} "parallel"?

The w^{th} parallel is the image of:

$\phi(t)=(t,w)$ with $t\in[-\pi, \pi]$

under the parameterization.



Metric Properties

$$\mathbf{x}(u,v) = \big((r_1 + r_2 \sin v)\cos u,\; r_2 \cos v,\; (r_1 + r_2 \sin v)\sin u\big)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

<u>Example (Torus)</u>:

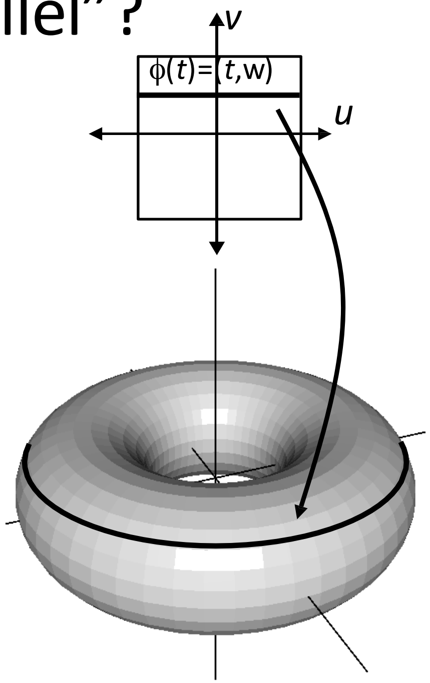
- What is the length of the w^{th} "parallel"?

$$length(\mathbf{x}\circ\phi) = \int_{-\pi}^{\pi} \sqrt{\phi'(t)^t\, \mathrm{I}\, \phi'(t)}\, dt$$

$$= \int_{-\pi}^{\pi} \sqrt{(1,0)^t \begin{pmatrix} (r_1 + r_2 \sin w)^2 & 0 \\ 0 & r_2^2 \end{pmatrix} (1,0)}\, dt$$

$$= \int_{-\pi}^{\pi} r_1 + r_2 \sin w \, dt$$

$$= 2\pi (r_1 + r_2 \sin w)$$

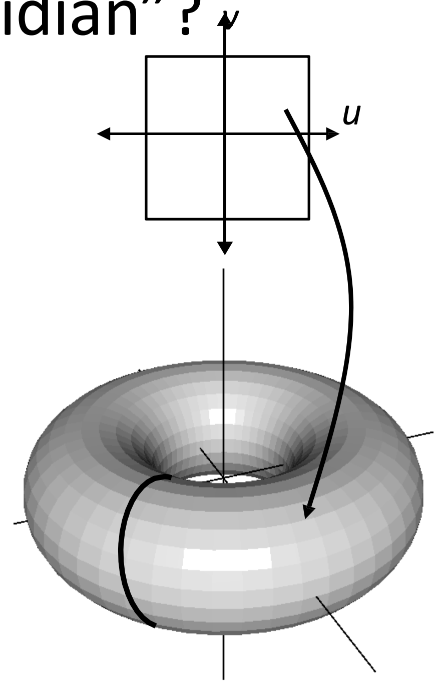


Metric Properties

$$\mathrm{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

<u>Example (Torus)</u>:

- What is the length of the w^{th} "meridian"?



Metric Properties

$$x(u,v) = \left((r_1 + r_2 \sin v)\cos u,\ r_2 \cos v,\ (r_1 + r_2 \sin v)\sin u\right)^t \qquad I(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

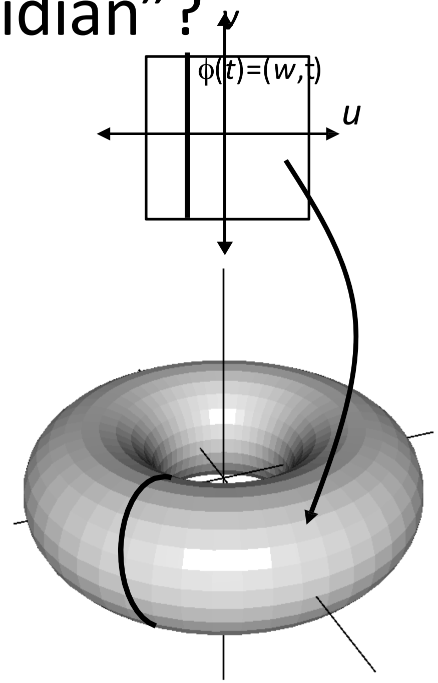
Example (Torus):

- What is the length of the w^{th} "meridian"?

The w^{th} meridian is the image of:

$$\phi(t)=(w,t) \quad \text{with } t\in[-\pi, \pi]$$

under the parameterization.



Metric Properties

$$\mathrm{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

<u>Example (Torus)</u>:

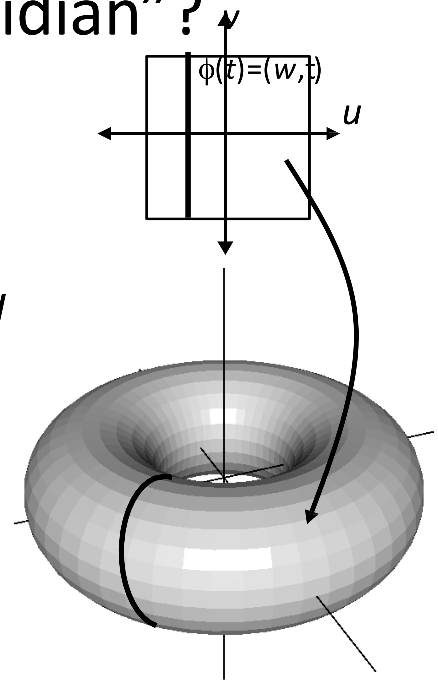
- What is the length of the w^{th} "meridian"?

$$length(\mathrm{x} \circ \phi) = \int_{-\pi}^{\pi} \sqrt{\phi'(t)^t \, \mathrm{I} \, \phi'(t)} \, dt$$

$$= \int_{-\pi}^{\pi} \sqrt{(0,1)^t \begin{pmatrix} (r_1 + r_2 \sin w)^2 & 0 \\ 0 & r_2^2 \end{pmatrix} (0,1)} \, dt$$

$$= \int_{-\pi}^{\pi} r_2 \, dt$$

$$= 2\pi r_2$$

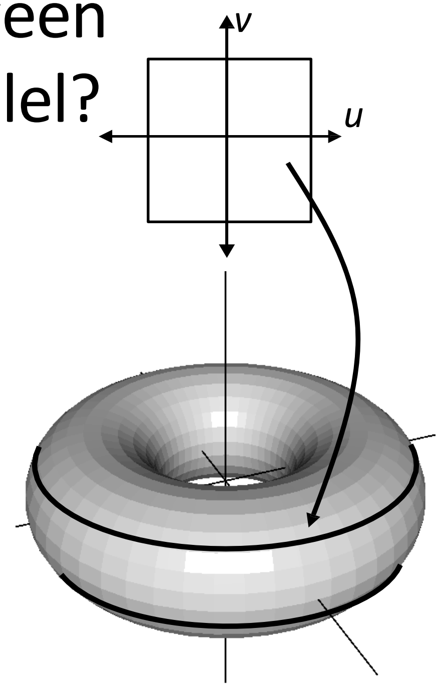


Metric Properties

$$\mathrm{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

Example (Torus):

- What is the area of the band between the w_1th parallel and the w_2th parallel?

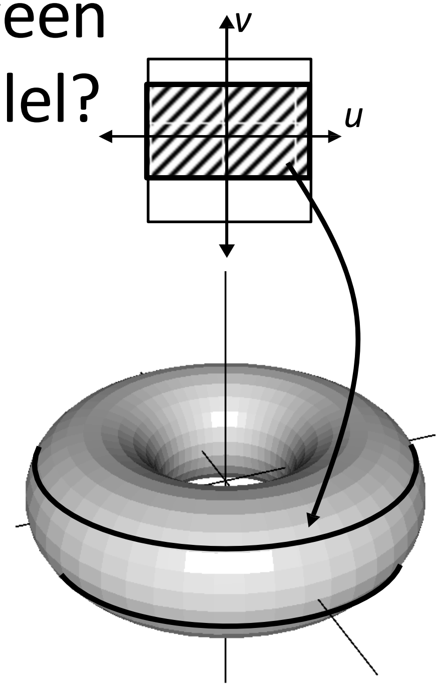


Metric Properties

$$\mathrm{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

<u>Example (Torus)</u>:

- What is the area of the band between the w_1 th parallel and the w_2 th parallel?



The band is the image of:

$\phi(s,t)=(s,t)$ with $s\in[-\pi,\pi]$, $t\in[w_1,w_2]$

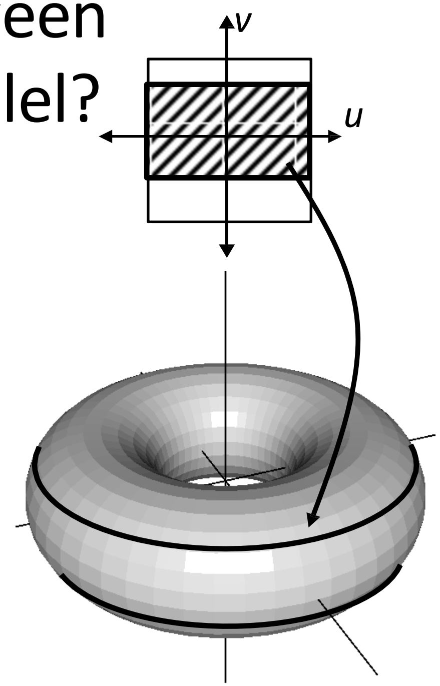
under the parameterization.

Metric Properties

$$\mathrm{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

<u>Example (Torus)</u>:

- What is the area of the band between the w_1th parallel and the w_2th parallel?



$$area(\mathrm{x} \circ \phi) = \int_{w_1}^{w_2} \int_{-\pi}^{\pi} \sqrt{\det \mathrm{I}}\, ds\, dt$$

$$= \int_{w_1}^{w_2} \int_{-\pi}^{\pi} \left(r_1 + r_2 \sin t\right) r_2 \; ds\, dt$$

$$= 2\pi \int_{w_1}^{w_2} \left(r_1 + r_2 \sin t\right) r_2 \; dt$$

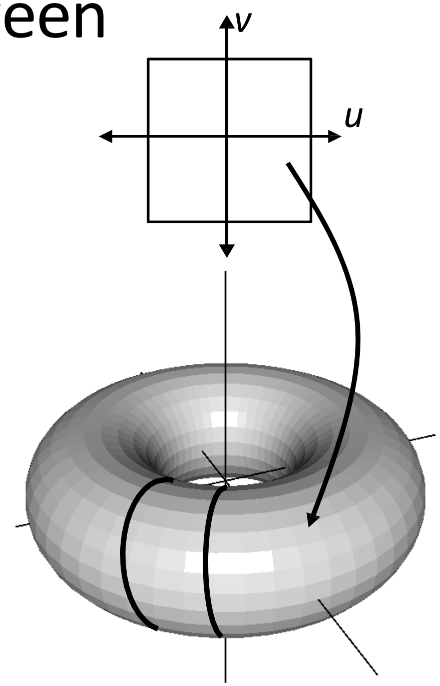
$$= 2\pi \left(r_1 (w_2 - w_1) + r_2 (\cos w_1 - \cos w_2 \right.$$

Metric Properties

$$x(u,v) = \left((r_1 + r_2 \sin v)\cos u \;\; r_2 \cos v \;\; (r_1 + r_2 \sin v)\sin u \right)^t \quad I(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

<u>Example (Torus)</u>:

- What is the area of the band between the w_1^{th} and the w_2^{th} meridian?



Metric Properties

$$x(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t \qquad I(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

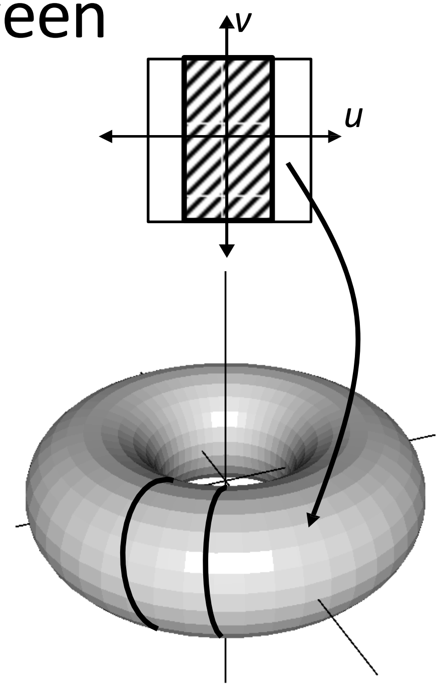
<u>Example (Torus)</u>:

- What is the area of the band between the w_1th and the w_2th meridian?

The band is the image of:

$\phi(s,t)=(s,t)$ with $s\in[w_1,w_2]$, $t\in[-\pi,\pi]$

under the parameterization.



Metric Properties

$$\mathrm{x}(u,v) = \big((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\big)^t \qquad \mathrm{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

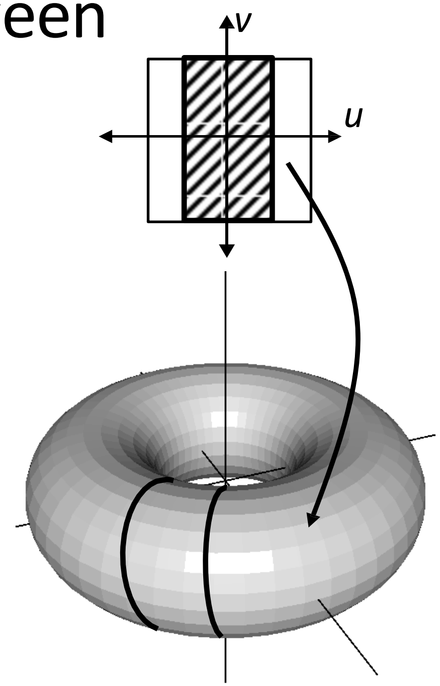
<u>Example (Torus)</u>:

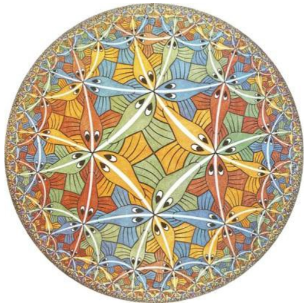
- What is the area of the band between the w_1th and the w_2th meridian?

$$area(\mathrm{x} \circ \phi) = \int_{-\pi}^{\pi} \int_{w_1}^{w_2} \sqrt{\det \mathrm{I}}\, ds\, dt$$

$$= \int_{-\pi}^{\pi} \int_{w_1}^{w_2} (r_1 + r_2 \sin t) r_2 \, ds\, dt$$

$$= (w_2 - w_1) \int_{-\pi}^{\pi} (r_1 + r_2 \sin t) r_2 \, dt$$





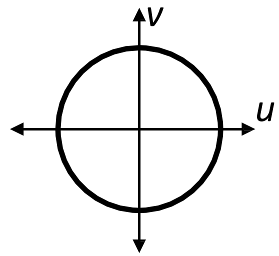
Metric Properties

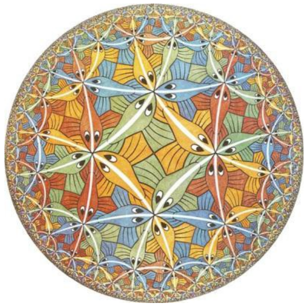
Example (Hyperbolic Plane):

If we are given the first fundamental form, we can ignore the embedding of the surface in 3D, and integrate directly.

Consider the domain $\Omega=\{u,v \mid (u^2+v^2<1)\}$, with the first fundamental form:

$$\mathrm{I}(u,v)=\begin{pmatrix}\frac{1}{1-u^2-v^2} & 0\\ 0 & \frac{1}{1-u^2-v^2}\end{pmatrix}$$



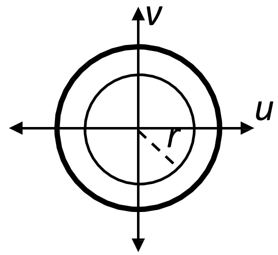


Metric Properties

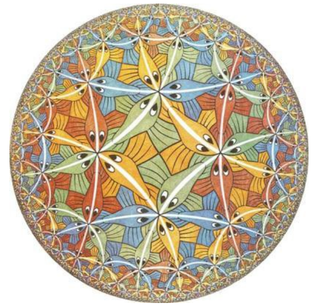
$$\Omega = \left\{(u,v) \mid u^2 + v^2 < 1\right\} \qquad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the length of the circle with radius *r*?



Metric Properties



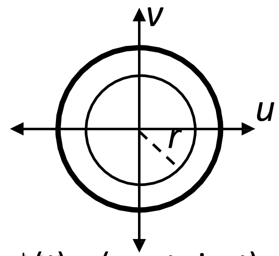
$$\Omega = \left\{ (u,v) \mid u^2 + v^2 < 1 \right\} \qquad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the length of the circle with radius r?

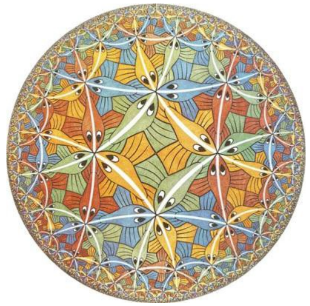
The circle is described by:

$\phi(s)=r(\cos s, \sin s)$ with $s \in [0,2\pi]$.



$\phi(t)=r(\cos t,\sin t)$

Metric Properties

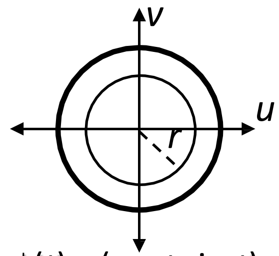


Example (Hyperbolic Plane):

$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \qquad I(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

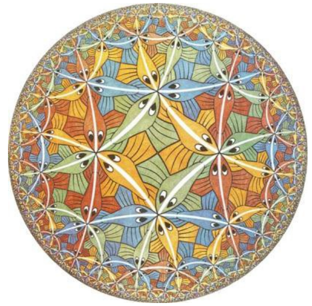
- What is the length of the circle with radius r?

$$
\begin{aligned}
length(\phi) &= \int_0^{2\pi} \sqrt{\phi'(t)^t I \phi'(t)}\, dt \\
&= \int_0^{2\pi} \sqrt{r(-\sin t, \cos t)^t \begin{pmatrix} \frac{1}{1-r^2} & 0 \\ 0 & \frac{1}{1-r^2} \end{pmatrix} r(-\sin t, \cos t)}\, dt \\
&= \int_0^{2\pi} \sqrt{\frac{r^2}{1-r^2}}\, dt \\
&= 2\pi r \sqrt{\frac{1}{1-r^2}}
\end{aligned}
$$



$\phi(t)=r(\cos t,\sin t)$

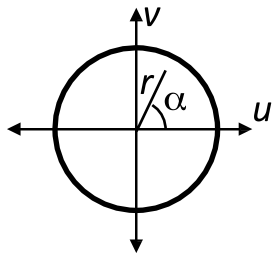
Metric Properties



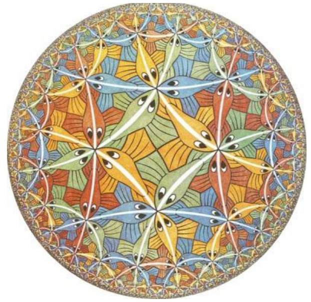
$$\Omega = \left\{(u,v) \mid u^2 + v^2 < 1\right\} \qquad I(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

Example (Hyperbolic Plane):

- What is the length of the segment with angle α and radius r?



Metric Properties



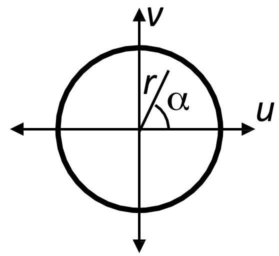
$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \qquad g(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the length of the segment with angle α and radius r?

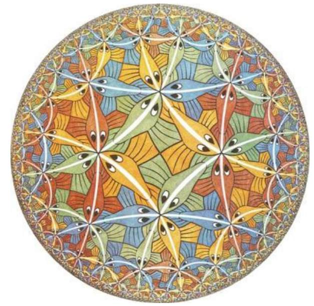
The segment is described by:

$$\phi(s)=s(\cos\alpha, \sin\alpha) \quad \text{with } s\in[0,r].$$



$\phi(s)=s(\cos\alpha,\sin\alpha)$

Metric Properties



$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \qquad I(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

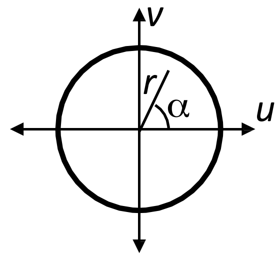
<u>Example (Hyperbolic Plane)</u>:

- What is the length of the segment with angle α and radius r?

$$length(\phi) = \int_0^r \sqrt{\phi'(s)^t I \phi'(s)}\, ds$$

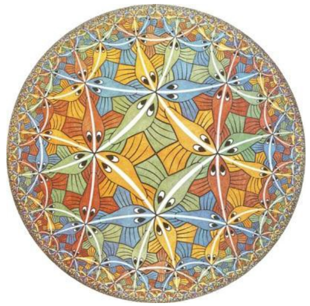
$$= \int_0^r \sqrt{(\cos\alpha, \sin\alpha)^t \begin{pmatrix} \frac{1}{1-s^2} & 0 \\ 0 & \frac{1}{1-s^2} \end{pmatrix} (\cos\alpha, \sin\alpha)}\, ds$$

$$= \int_0^r \frac{1}{1-s^2}\, ds = \frac{1}{2} \log \frac{1+r}{1-r}$$



$\phi(s)=s(\cos\alpha, \sin\alpha)$

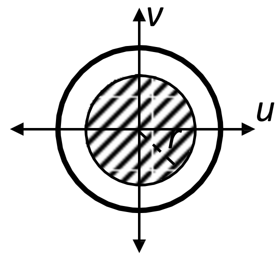
Metric Properties



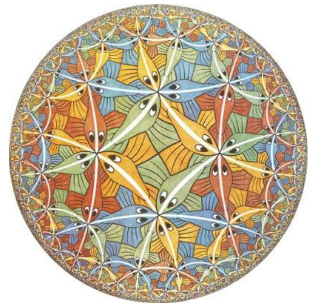
$$\Omega = \left\{ (u,v) \mid u^2 + v^2 < 1 \right\} \qquad I(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the area of the region with radius less than *r*?



Metric Properties



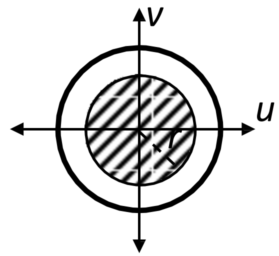
$$\Omega = \{(u,v) \mid u^2 + v^2 < 1\} \qquad I(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

Example (Hyperbolic Plane):

- What is the area of the region with radius less than r?

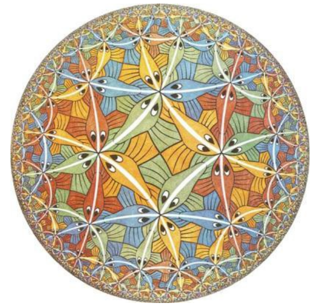
The region is the image of:

$\phi(s,t)=s(\cos t, \sin t)$ with $s\in[0,r]$, $t\in[-\pi,\pi]$.



$\phi(s,t)=s(\cos t, \sin t)$

Metric Properties

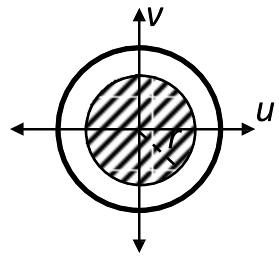


$$\Omega = \left\{(u,v) \mid u^2 + v^2 < 1\right\} \qquad \mathrm{I}(u,v) = \begin{pmatrix} \frac{1}{1-u^2-v^2} & 0 \\ 0 & \frac{1}{1-u^2-v^2} \end{pmatrix}$$

<u>Example (Hyperbolic Plane)</u>:

- What is the area of the region with radius less than r?

$$\begin{aligned} area(\phi) &= \int_{-\pi}^{\pi}\int_0^r \sqrt{\det \mathrm{I}}\, s\, ds\, dt \\ &= \int_{-\pi}^{\pi}\int_0^r \frac{s}{1-s^2}\, ds\, dt \\ &= 2\pi \int_0^r \frac{s}{1-s^2}\, ds \\ &= -\pi \ln\left(1-r^2\right) \end{aligned}$$



$\phi(s,t)=s(\cos t, \sin t)$

Metric on Surfaces

- From p10 of 12-Differential Geometry-curve surface-curvature.pdf

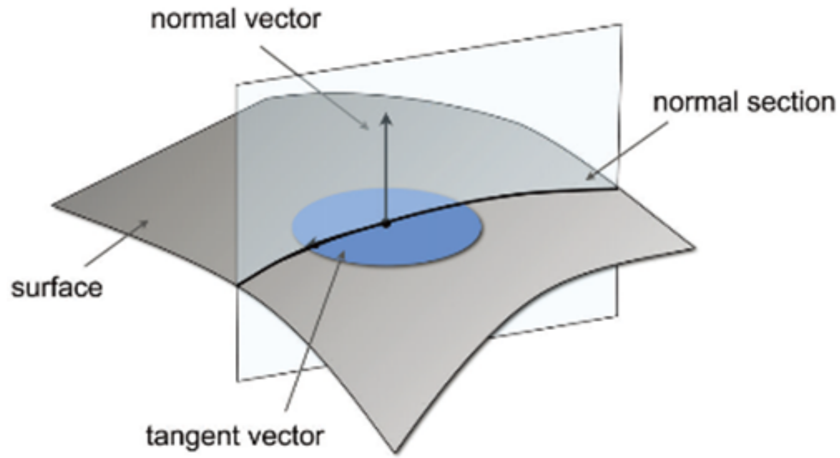
normal curvature $\kappa_n(\bar{\mathbf{t}})$ at $\mathbf{p}$

curvature of curves embedded in the surface. Let $\mathbf{t} = u_t\mathbf{x}_u + v_t\mathbf{x}_v$ be a tangent vector at a surface point $\mathbf{p} \in \mathcal{S}$ represented as $\bar{\mathbf{t}} = (u_t, v_t)^T$ in Parameter space

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{eu_t^2 + 2fu_tv_t + gv_t^2}{Eu_t^2 + 2Fu_tv_t + Gv_t^2},$$

where $\mathbf{II}$ denotes the *second fundamental form* defined as

$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T\mathbf{n} & \mathbf{x}_{uv}^T\mathbf{n} \\ \mathbf{x}_{uv}^T\mathbf{n} & \mathbf{x}_{vv}^T\mathbf{n} \end{bmatrix}.$$



Surfaces Curvatures

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{eu_t^2 + 2fu_tv_t + gv_t^2}{Eu_t^2 + 2Fu_tv_t + Gv_t^2}$$

- Principal curvatures
  - Maximal curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
  - Minimal curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$
- Mean curvature: $k_H = \frac{k_1 + k_2}{2} = \frac{1}{2\pi}\int_0^{2\pi} k_n(\theta) d\,\theta = \lim_{diam(A) \to 0} \frac{\nabla A}{A}$
- Gaussian curvature: $k_G = k_1 \cdot k_2 = \lim_{diam(A) \to 0} \frac{A^G}{A}$
- Curvature tensor: $C = PDP^{-1}$, with P=[t1, t2, n] and D=diag(k1, k2, 0)

Discrete Differential-Geometry Operators for triangulated 2-manifolds_02

高斯曲率 反应了曲面的弯曲程度。在给出高斯曲率的几何解释之前，首先引入高斯映射的定义，设 A 是曲面上包含 p 点的一小片曲面（其面积仍用 A 表示），把 A 上的每点的单位法向量 n 平移到原点 O 处，那么 n 的终点轨迹是以 O 为中心的单位球面 $S2$ 上的一块区域 A^*。这个对应称为高斯映射。则 p 点的高斯曲率可以表示为：

$$\kappa_G(p) = \lim_{A \to 0} \frac{A^*}{A}$$

其中高斯曲率 κ_G 和平均曲率 κ_H 都反映局部曲面的几何特征。

Lagrange注意到 $\kappa_H = 0$ 是极小曲面的Lagrange方程，于是就给出了一个极小曲面与平均曲率的直接关系：

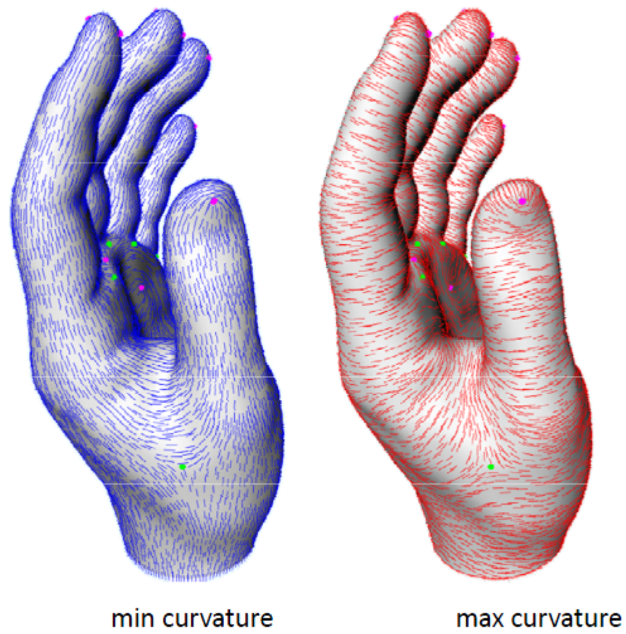
$$2\kappa_H n = \lim_{diam(A)\to 0} \frac{\nabla A}{A}$$

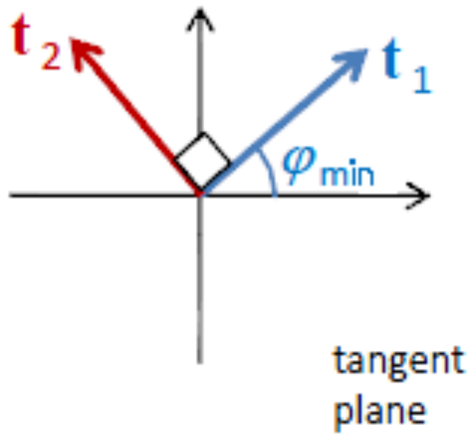
其中，A 是点 p 处无穷小区域的面积，$diam(A)$是它的直径，∇ 是关于点 $p(x, y, z)$ 坐标的梯度，因此，定义算子 $K(p) = 2\kappa_H(p)n(p)$ 这就是著名的Laplace-Beltrami算子。

Euler theorem $\kappa_n(\bar{\mathrm{t}}) = \kappa_1 \cos^2 \psi + \kappa_2 \sin^2 \psi,$

- ψ is the angle between $\bar{t}$ and t1, t1 is the
- Principal directions: tangent vectors corresponding to φ_{max} & φ_{min}

- any normal curvature is a convex combination of the minimum and maximum curvature
- principal directions are orthogonal to each other

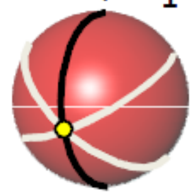




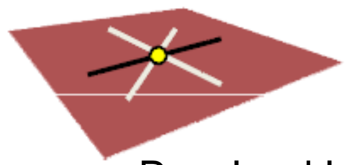
Classification

A point p on the surface is called

Isotropic: all directions are principle directions



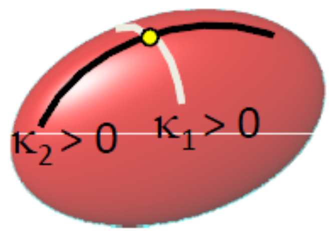
spherical (umbilical)



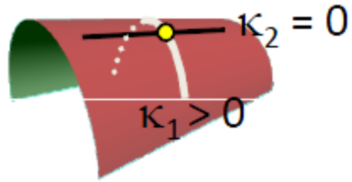
Developable surface ⇔ K=0

planar

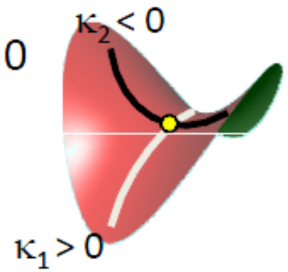
Anisotropic: 2 distinct principle directions



elliptic



parabolic



hyperbolic

Laplace & Laplace-Beltrami Operator

Laplace Operator: $\mathrm{divF} = \nabla \cdot F$

- $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
- $f = f(x, y, z)$, $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$
- $F = (U(x, y, z), V(x, y, z), W(x, y, z))$
- $\mathrm{divF} = \nabla \cdot F = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}$

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \mathrm{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

function in Euclidean space

divergence operator

Cartesian coordinates

Laplace-Beltrami Operator: $\Delta_S \mathrm{f} = \mathrm{div}_S \nabla_S f$

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_S \mathbf{x} = \mathrm{div}_S \, \nabla_S \mathbf{x} = 2H\mathbf{n}$$

coordinate function

divergence operator

surface normal

For researchers in CG (for differential coordinates), $\Delta_s = -2Hn$

For mathematician, $\Delta_s = 2Hn$

The only difference is the sign.

curvature tensor

$$\kappa_p(w) = \kappa_1(p)\cos^2\alpha + \kappa_2(p)\sin^2\alpha$$

Given the principal curvatures/directions, k1/***J****w*1 and k2/***J****w*2, the *curvature tensor* is a 3x3 symmetric matrix associated to each point on the surface, defined by:

$$\mathbf{C}(\mathbf{x}(p)) = \kappa_1 \mathbf{J}w_1 \mathbf{J}w_1^t + \kappa_2 \mathbf{J}w_2 \mathbf{J}w_2^t$$

Thanks