Digital Geometry
-Continuous Geometry of Curves

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http://jjcao.github.io/DigitalGeometry/

Pleasure may come from illusion, but happiness can come only of reality.

3 Representations of Curve

- Explicit: $y = mx + b$
- Explicit Parametric (seen as a kinematic motion):
  - $P = P_0 + t\,(P_1 - P_0)$
  - curve: $r=r(t)$,
  - surface: $r=r(u,v)$
- Implicit: $ax + by + c = 0$

Implicit representation of 3d Curve

- surface: level set of function $f(x,y,z)$: $f(x,y,z)=0$, viz, solution set of $f(x,y,z)=0$.

- curve: solution set of
  - $f(x,y,z)=0$
  - $g(x,y,z)=0$

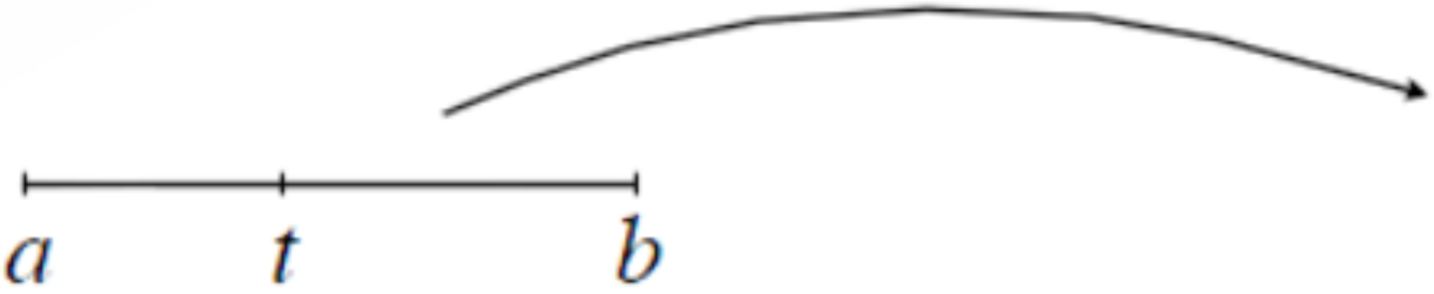
- point: solution set of
  - $f(x,y,z)=0$
  - $g(x,y,z)=0$
  - $h(x,y,z)=0$

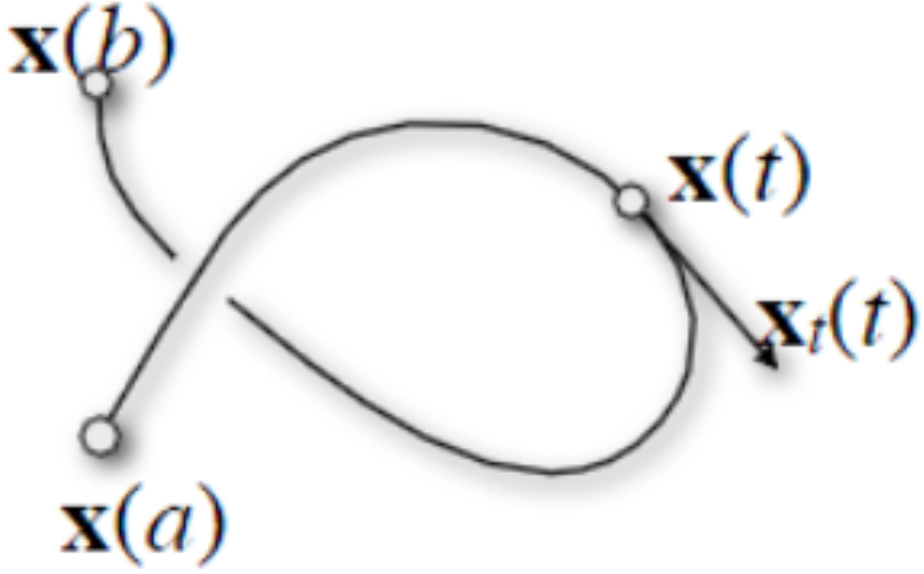
From implicit 2 Parametric representation

- If conditions of implicit function theorem are guaranteed
- Curve =>$r(x)=(x,y(x),z(x))$
- Surface =>$r(x,y)=(x,y,z(x,y))$ (Monge patch)

Parametric Curves

$$\mathbf{x} : [a, b] \subset \mathbb{R} \to \mathbb{R}^3$$





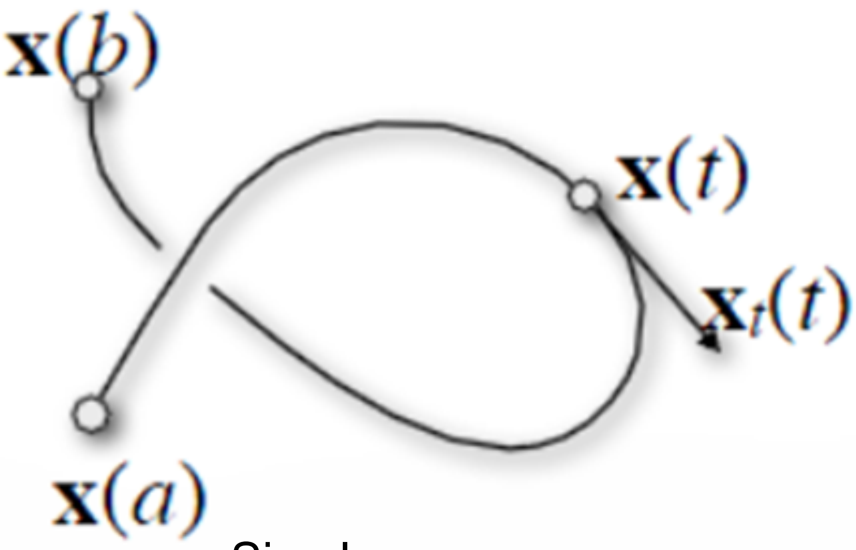
Advantages of parametric forms

1. Dimension independent
2. Easy to express in vector and matrix form

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad \mathbf{x}_t(t) := \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \begin{pmatrix} \mathrm{d}x(t)/\mathrm{d}t \\ \mathrm{d}y(t)/\mathrm{d}t \\ \mathrm{d}z(t)/\mathrm{d}t \end{pmatrix}$$

A parametric curve $\mathbf{x}(t)$ is

- simple: $\mathbf{x}(t)$ is injective (no self-intersections)
- differentiable: $\mathbf{x}_t(t)$ is defined for all $t \in [a, b]$
- regular: $\mathbf{x}_t(t) \neq 0$ for all $t \in [a, b]$



Simple curve


nonsimple curves

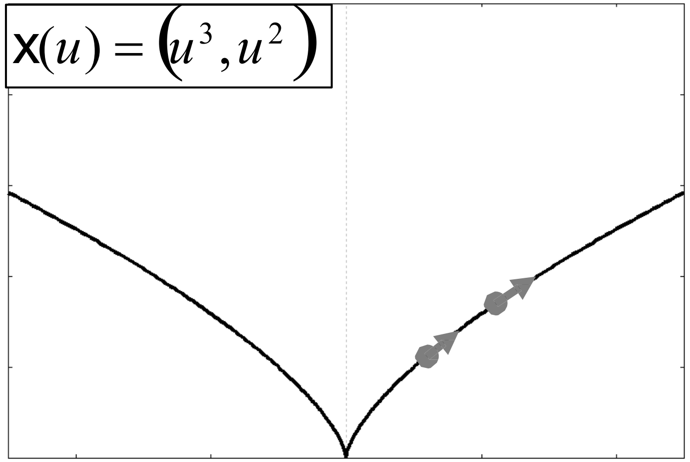
Differentiable Curves

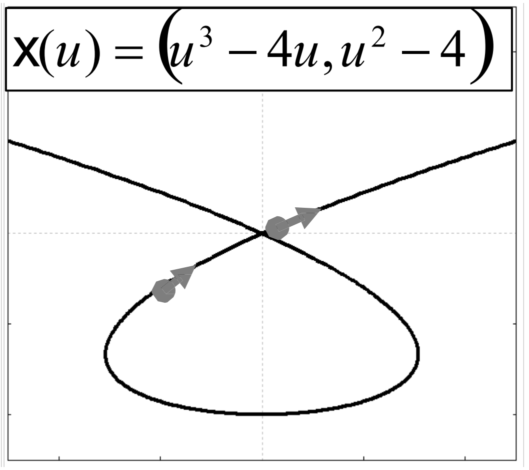
<u>Definition</u>:

A *parameterized differentiable curve* is a differentiable map $\boldsymbol{x}: I \rightarrow \boldsymbol{R}^2$ of an open interval $I=(a,b)$ of the real line $\boldsymbol{R}$ into $\boldsymbol{R}^2$:

$$\boldsymbol{x}(u)=(x(u),y(u))$$

where $x(u)$ and $y(u)$ are differentiable functions.



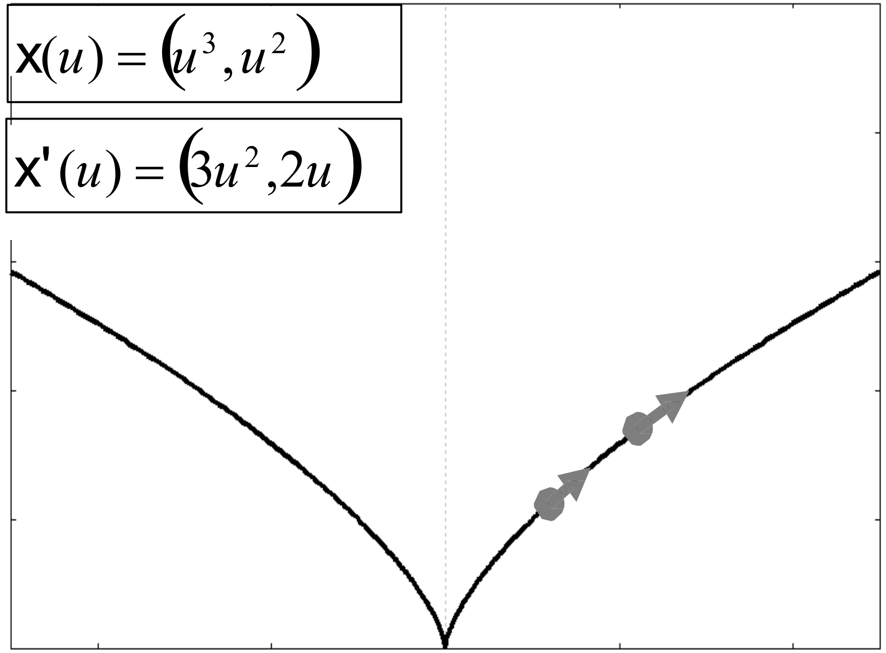


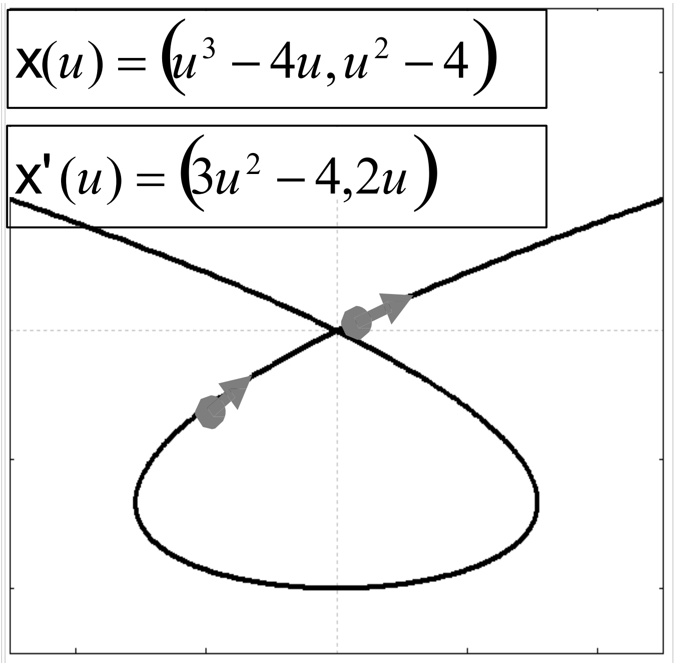
Differentiable Curves - derivative

Definition:

The *derivative* of the curve at $\boldsymbol{x}(u)$ is the vector, tangent to the curve, defined as:

$$\boldsymbol{x}'(u)=(x'(u),y'(u))$$





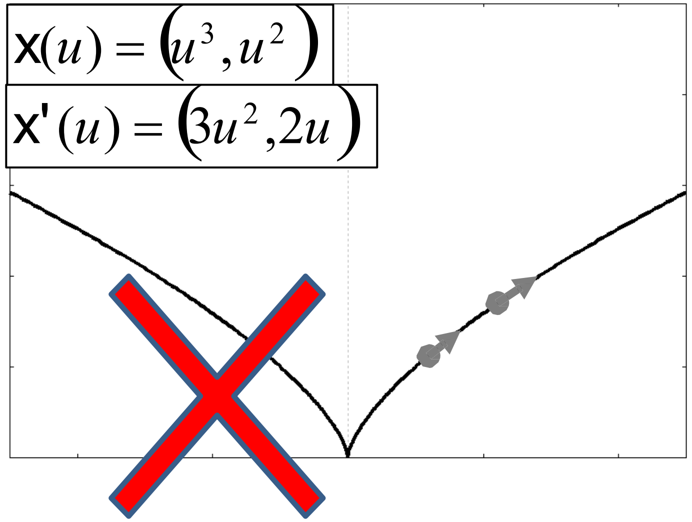
Differentiable Curves - regular

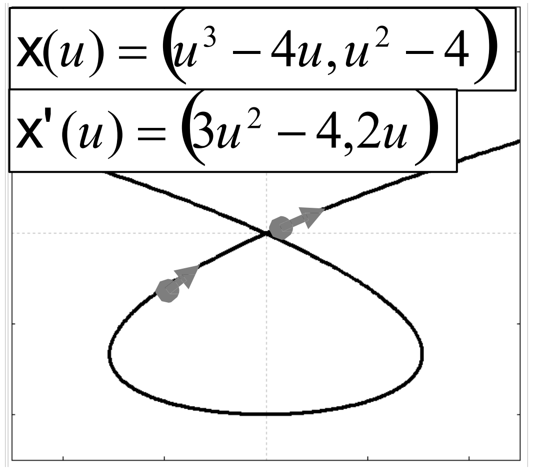
Definition:

The *derivative* of the curve at $\boldsymbol{x}(u)$ is the vector, tangent to the curve, defined as:

$$\boldsymbol{x}'(u)=(x'(u),y'(u))$$

The curve is said to be *regular* if $\boldsymbol{x}'(u)\neq 0$.





Length of a Curve / Arc length

Polyline chord length

$$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \left\| \frac{\Delta \mathbf{x}_i}{\Delta t} \right\| \Delta t\,, \quad \Delta \mathbf{x}_i := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$$

norm change

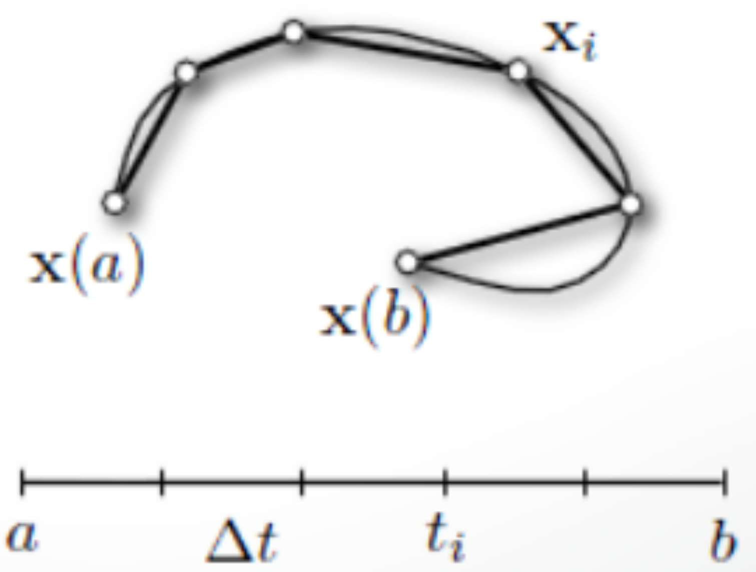
Curve arc length ($\Delta t \to 0$)

$$s = s(t) = \int_a^t \|\mathbf{x}_t\|\, \mathrm{d}t$$

length =

integration of infinitesimal change

× norm of speed



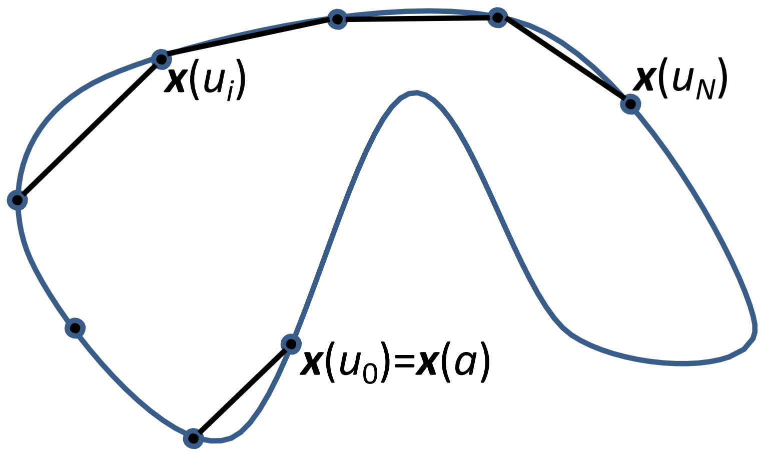
Regular Curves

Given a regular curve **x**(u), and given, the *arc-length* from a to the point u is:

$$s(u) = \int_a^u |\mathrm{x}'(v)| dv$$

If we partition the interval [a,u] into N sub- intervals, setting $\Delta u=(u-a)/N$ and $u_i=a+i\Delta u$:

$$\begin{aligned}
s(u) &= \lim_{N\to\infty} \sum_{i=0}^{N-1} \left| \mathrm{x}(u_{i+1}) - \mathrm{x}(u_i) \right| \\
&= \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{\left| \mathrm{x}(u_{i+1}) - \mathrm{x}(u_i) \right|}{\Delta u} \Delta u \\
&= \lim_{N\to\infty} \sum_{i=0}^{N-1} \left| \mathrm{x}'(u_i) \right| \Delta u \\
&= \int_a^u \left| \mathrm{x}'(v) \right| dv
\end{aligned}$$



Differentiable Curves

Definition:

We say that a regular curve is *parameterized by arc-length* if:

$$|\boldsymbol{x}'(u)|=1$$

In this case:

$$s(u)=\int_a^u |\mathrm{x}'(v)|\,dv=\int_a^u dv=u-a$$

There are various names for such a parameterization ("unit speed", "arc-length", "isometric")

Regular Curves - Tangent

<u>Definition</u>:

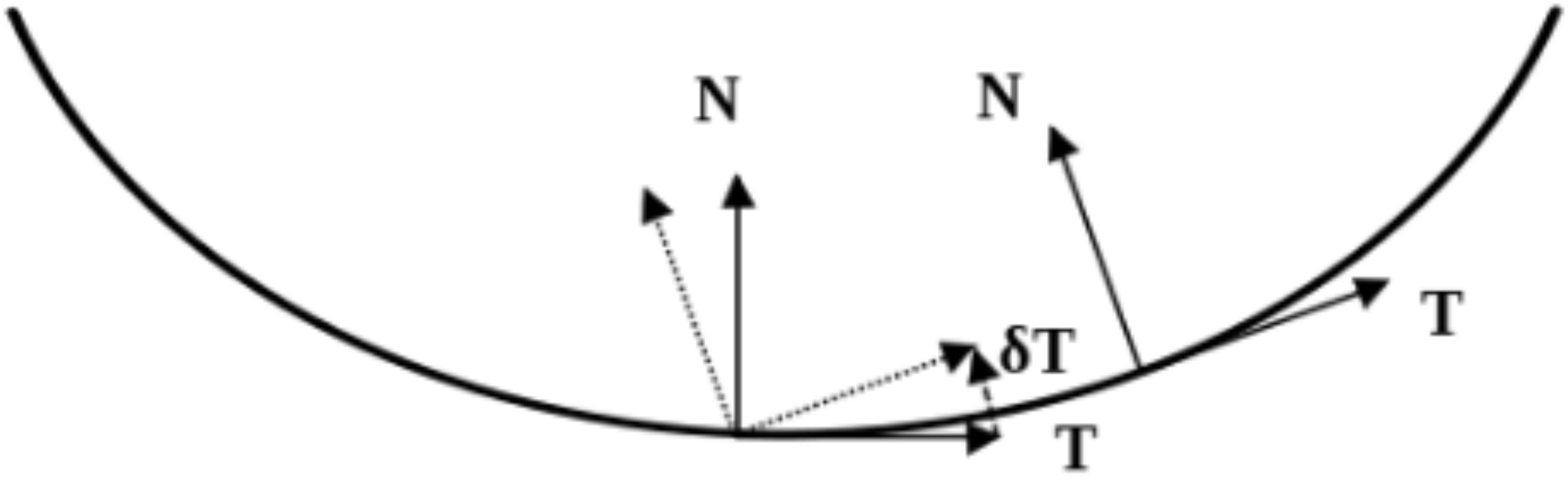
The ***tangent*** to the curve at $\boldsymbol{x}(u)$ is the **unit vector** pointing in the direction of the derivative:

$$\mathbf{t}(u) = \frac{\mathbf{x}'(u)}{\|\mathbf{x}'(u)\|}$$

If $\boldsymbol{x}$ is parameterized by arc-length: $\mathbf{t}(u) = \mathbf{x}'(u)$

Normal N: a unit vector

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} \qquad \mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left\| \frac{d\mathbf{T}}{ds} \right\|} \qquad dT/ds = \kappa N$$



The **T** and **N** vectors at two points on a plane curve, a translated version of the second frame (dotted), and the change in **T**: δ**T**. δs is the distance between the points. In the limit dT/ds will be in the direction **N**

T and N are always orthogonal. Why?

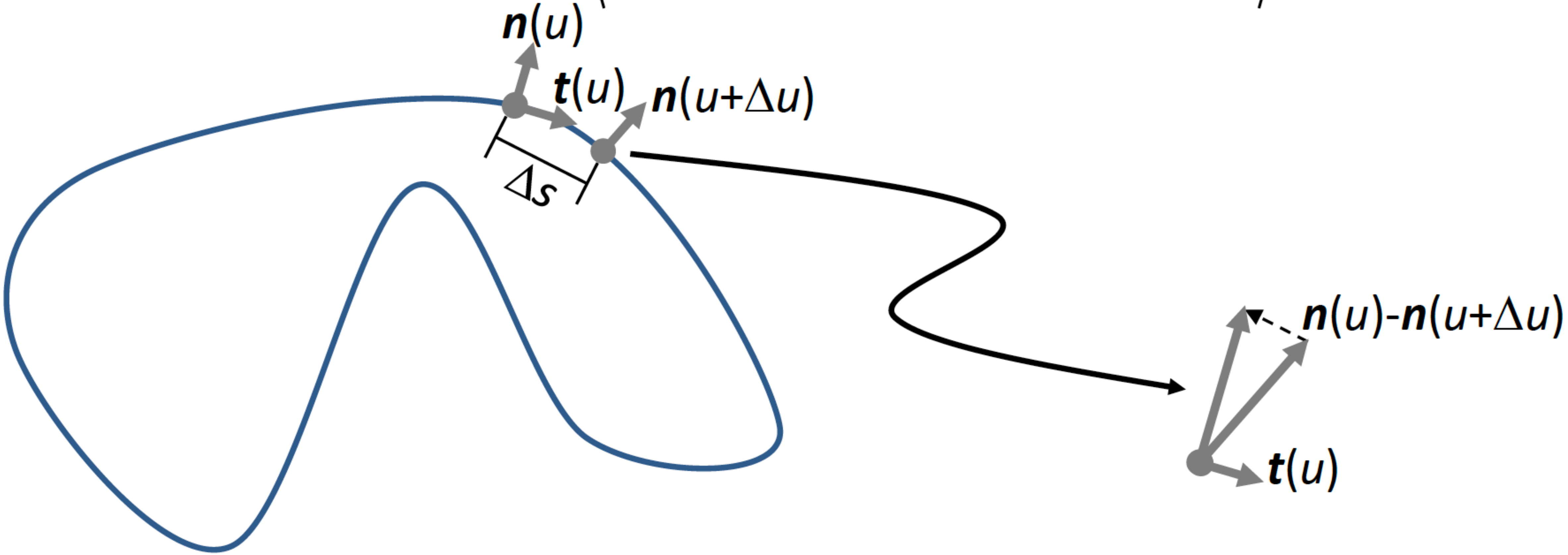
- Because if the change in T were parallel to T, then it would cease to have unit length!
  - (This argument is a good one to keep in mind any time you work with unit vector fields.)

- By convention, N is a quarter turn in the counter-clockwise direction from T.

$$\mathrm{n}(u) = \mathrm{t}(u)^{\perp} = \frac{\mathrm{x}'(u)^{\perp}}{\|\mathrm{x}'(u)\|} = \frac{(-y'(u), x'(u))}{\sqrt{(x'(u))^2 + (y'(u))^2}}$$

- Curvature is the (negative) change in the normal is aligned with the tangent direction relative to change in distance along the curve:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{n}(u) - \boldsymbol{n}(u+\Delta u)}{\Delta s}, \boldsymbol{t}(u) \right\rangle$$



$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{n}(u) - \boldsymbol{n}(u+\Delta u)}{\Delta s}, \boldsymbol{t}(u) \right\rangle$$

If ***x*** is parameterized by arc-length, then Δ*s*=Δ*u* so the curvature becomes:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{n}(\Delta u) - \boldsymbol{n}(u+\Delta u)}{\Delta u}, \boldsymbol{t}(u) \right\rangle = -\langle \boldsymbol{n}'(u), \boldsymbol{t}(u) \rangle$$

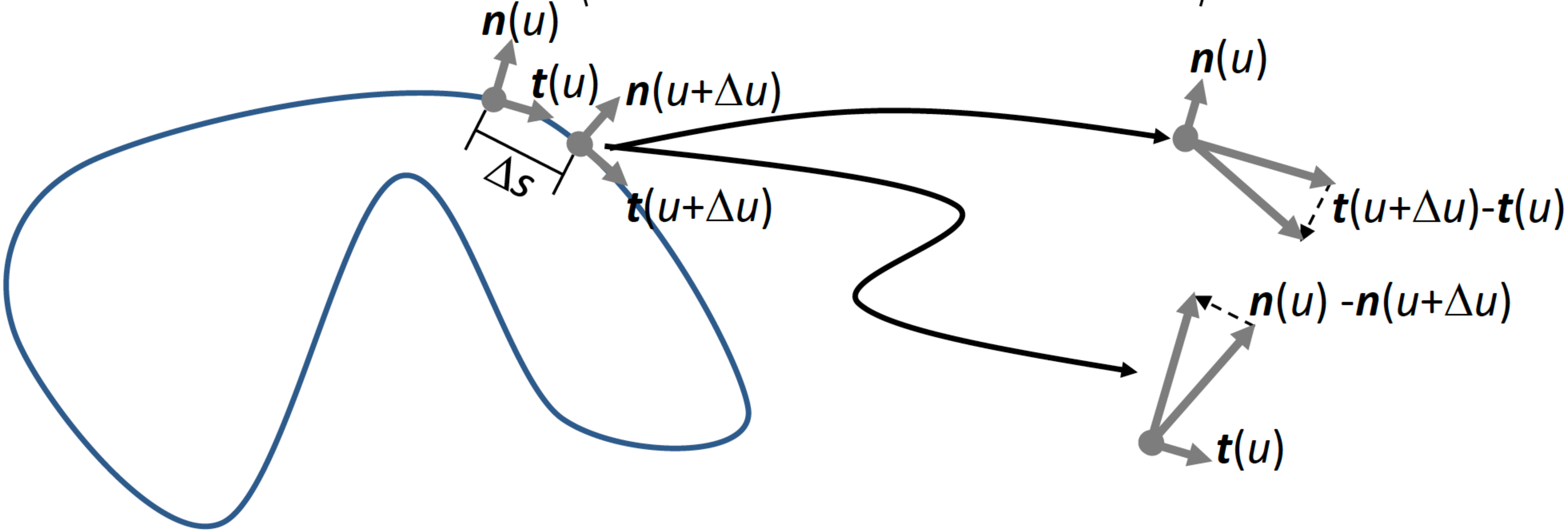
Otherwise, we have Δ*s*/Δ*u*=|***x'***(u)|, so that:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{n}(u) - \boldsymbol{n}(u+\Delta u)}{\Delta u \cdot |\boldsymbol{x}'(u)|}, \boldsymbol{t}(u) \right\rangle = -\frac{\langle \boldsymbol{n}'(u), \boldsymbol{t}(u) \rangle}{|\boldsymbol{x}'(u)|} = -\frac{\langle \boldsymbol{n}'(u), \boldsymbol{x}'(u) \rangle}{\langle \boldsymbol{x}'(u), \boldsymbol{x}'(u) \rangle}$$

Alternate Interpretation:

Curvature is the (positive) change in the tangent vector along the normal direction relative to change in distance along the curve:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{t}(u+\Delta u) - \boldsymbol{t}(u)}{\Delta s}, \boldsymbol{n}(u) \right\rangle$$



Regular Curves

- Proof of Equivalence:
  - To show equivalence, we need to show that:

$$-\langle \boldsymbol{n}'(u), \boldsymbol{t}(u) \rangle = \langle \boldsymbol{n}(u), \boldsymbol{t}'(u) \rangle$$

Taking the derivative of both sides:

$$0 = \langle \boldsymbol{n}(u), \boldsymbol{t}(u) \rangle$$

we get:

$$0 = \frac{d}{du} \langle \boldsymbol{n}(u), \boldsymbol{t}(u) \rangle = \langle \boldsymbol{n}'(u), \boldsymbol{t}(u) \rangle + \langle \boldsymbol{n}(u), \boldsymbol{t}'(u) \rangle$$

Regular Curves

- Thus, we can also express the curvature as:

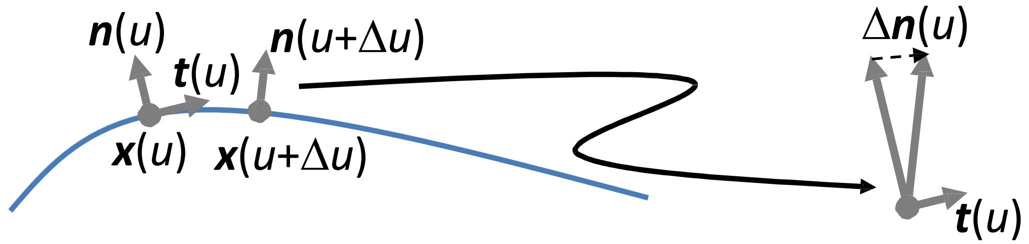
$$\kappa(u) = -\frac{\langle \boldsymbol{n}'(u), \boldsymbol{t}(u) \rangle}{|\boldsymbol{x}'(u)|} = \frac{\langle \boldsymbol{n}(u), \boldsymbol{t}'(u) \rangle}{|\boldsymbol{x}'(u)|} = \ldots = \frac{\langle \boldsymbol{n}(u), \boldsymbol{x}''(u) \rangle}{\langle \boldsymbol{x}'(u), \boldsymbol{x}'(u) \rangle}$$

Curvature is the (negative) change in the normal is aligned with the tangent

<u>Claim</u>:

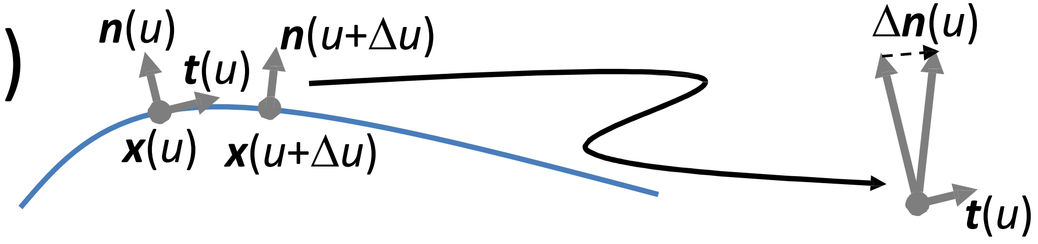
If we look at **how the normal changes along a curve**, we find that for small distances, the change is in the direction of the tangent:

$$\Delta\boldsymbol{n}(u)=\boldsymbol{n}(u+\Delta u)-\boldsymbol{n}(u)\approx \kappa(u)\boldsymbol{t}(\mathrm{u})$$



Change in the normal is aligned with the tangent

$$\Delta\boldsymbol{n}(u)=\boldsymbol{n}(u+\Delta u)-\boldsymbol{n}(u)\approx \kappa(u)\boldsymbol{t}(\mathrm{u})$$



Proof:

Since $\boldsymbol{n}(u)$ is a unit-vector, we know that:

$$1=\langle \mathrm{n}(u),\mathrm{n}(u)\rangle$$

Taking derivatives of both sides, we get:

$$\begin{aligned}0&=\frac{d}{du}\langle \mathrm{n}(u),\mathrm{n}(u)\rangle\\&=2\left\langle \frac{d}{du}\mathrm{n}(u),\mathrm{n}(u)\right\rangle\end{aligned}$$

Thus, the change in the normal is perpendicular to the normal direction, so it's aligned with the tangent.

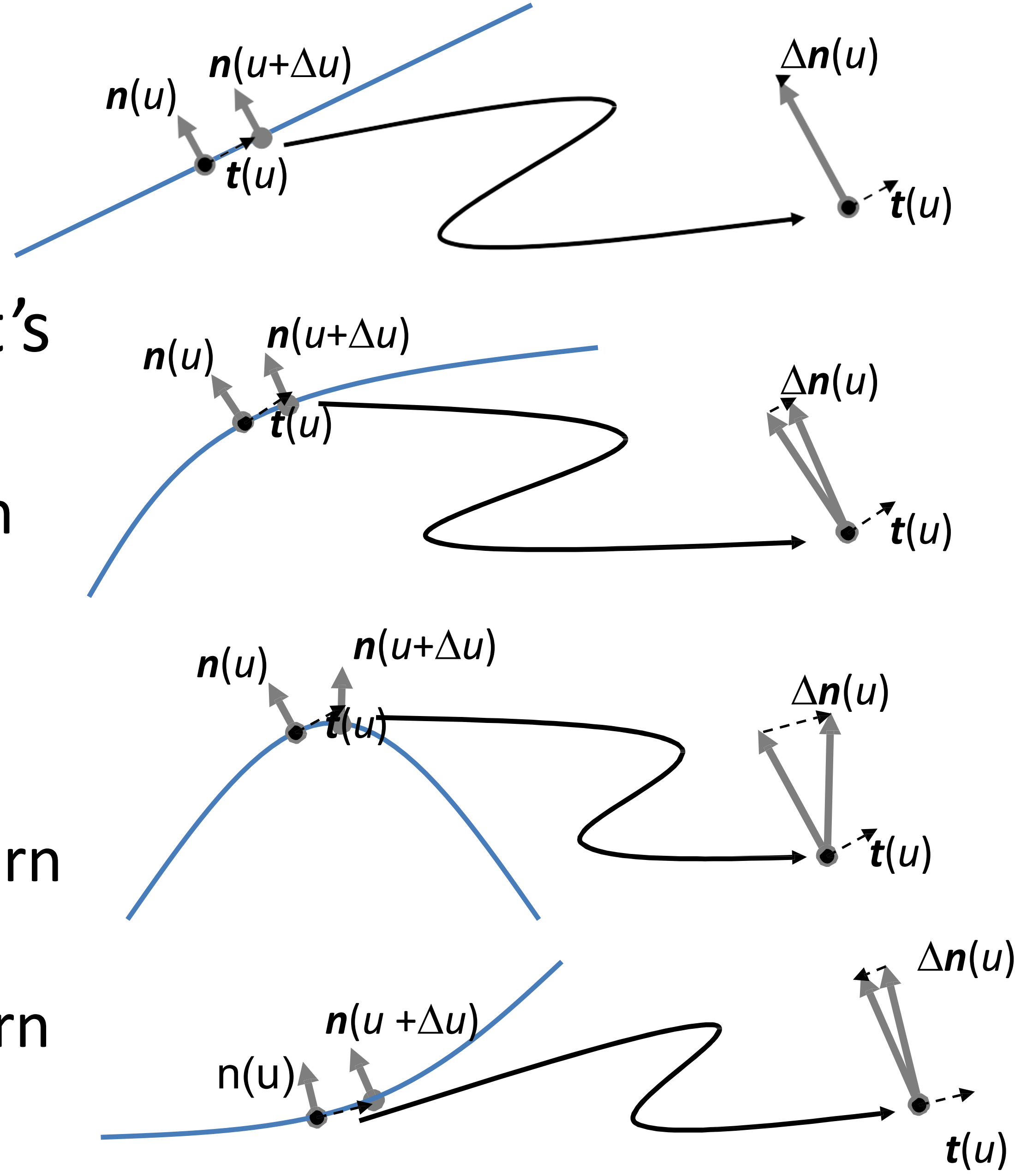
Change in the normal is aligned with the tangent

$$\Delta\boldsymbol{n}(u)=\boldsymbol{n}(u+\Delta u)-\boldsymbol{n}(u)\approx \kappa(u)\boldsymbol{t}(\mathrm{u})$$

<u>Note</u>:

If we look at the value of κ we see that it's

- zero for straight curves
- small/positive for convex curves that turn slowly
- large/positive for convex curves that turn quickly
- small/negative for concave curves that turn slowly
- large/negative for concave curves that turn quickly

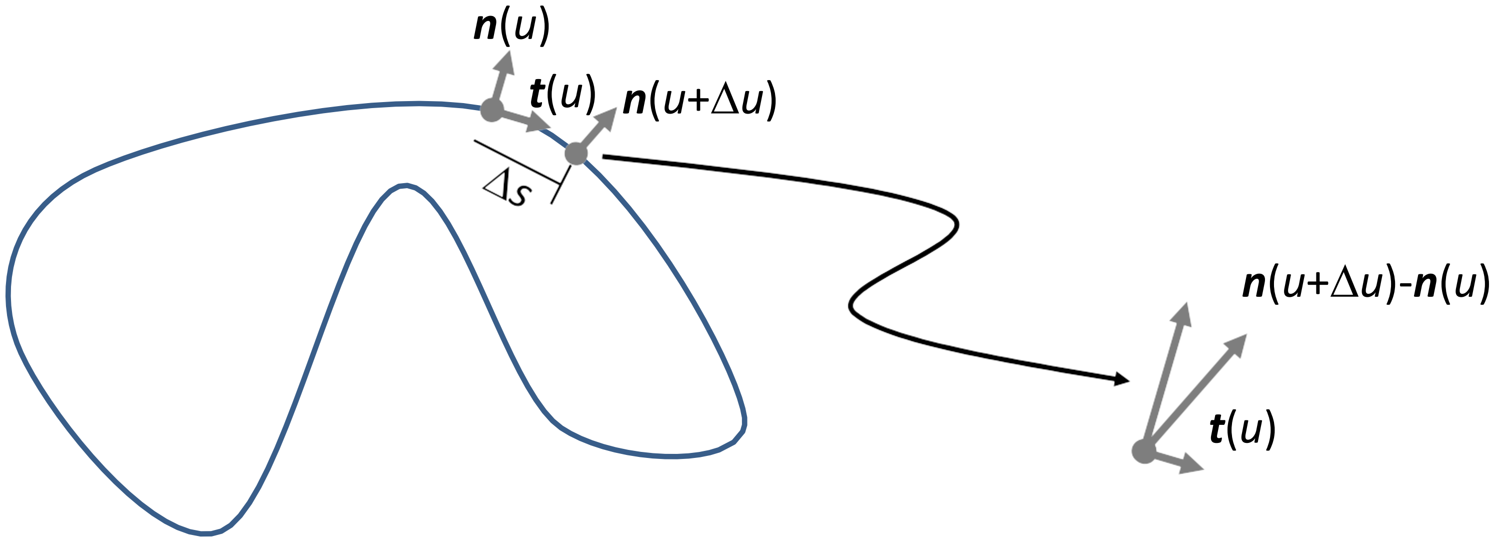


Regular Curves - *curvature*

<u>Definition</u>:

The *curvature* at $\boldsymbol{x}(u)$ is the change in normal vector along the tangent direction relative to change in distance along the curve:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\mathrm{n}(u+\Delta u) - \mathrm{n}(u)}{\Delta s}, \mathrm{t}(u) \right\rangle$$



Regular Curves

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\mathrm{n}(u+\Delta u) - \mathrm{n}(u)}{\Delta s}, \mathrm{t}(u) \right\rangle$$

<u>Note</u>:

If ***x*** is parameterized by arc-length, then $\Delta s = \Delta u$ so the curvature becomes:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\mathrm{n}(u+\Delta u) - \mathrm{n}(\Delta u)}{\Delta u}, \mathrm{t}(u) \right\rangle = \langle \mathrm{n}'(u), \mathrm{t}(u) \rangle$$

Otherwise, we have $\Delta s/\Delta u = |\boldsymbol{x}'(\mathrm{u})|$, so that:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\mathrm{n}(u+\Delta u) - \mathrm{n}(u)}{\Delta u \cdot |\mathrm{x}'(u)|}, \mathrm{t}(u) \right\rangle = \frac{\langle \mathrm{n}'(u), \mathrm{t}(u) \rangle}{|\mathrm{x}'(u)|}$$

Curvature

- Suppose that a particle moves along the curve with unit speed.
- Tangent **T**: velocity vector
- dT/ds: acceleration vector
  - Curvature: magnitude of it
  - Normal: direction of it

$$\mathbf{T} = \frac{d\mathbf{r}}{ds}$$

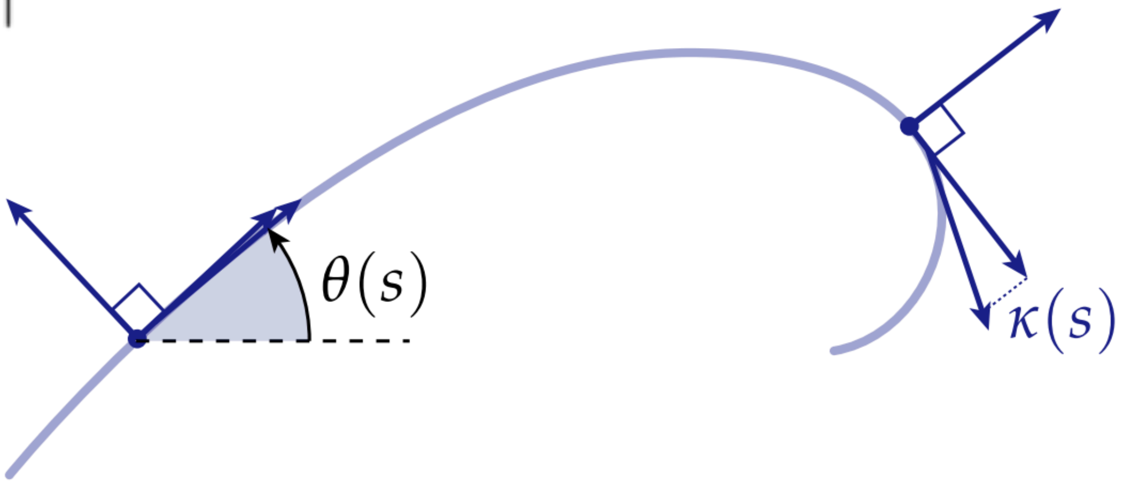
$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$$

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left\| \frac{d\mathbf{T}}{ds} \right\|}$$

$$\kappa(s) := \langle N(s), \tfrac{d}{ds} T(s) \rangle$$
$$= \langle N(s), \tfrac{d^2}{ds^2} \gamma(s) \rangle$$

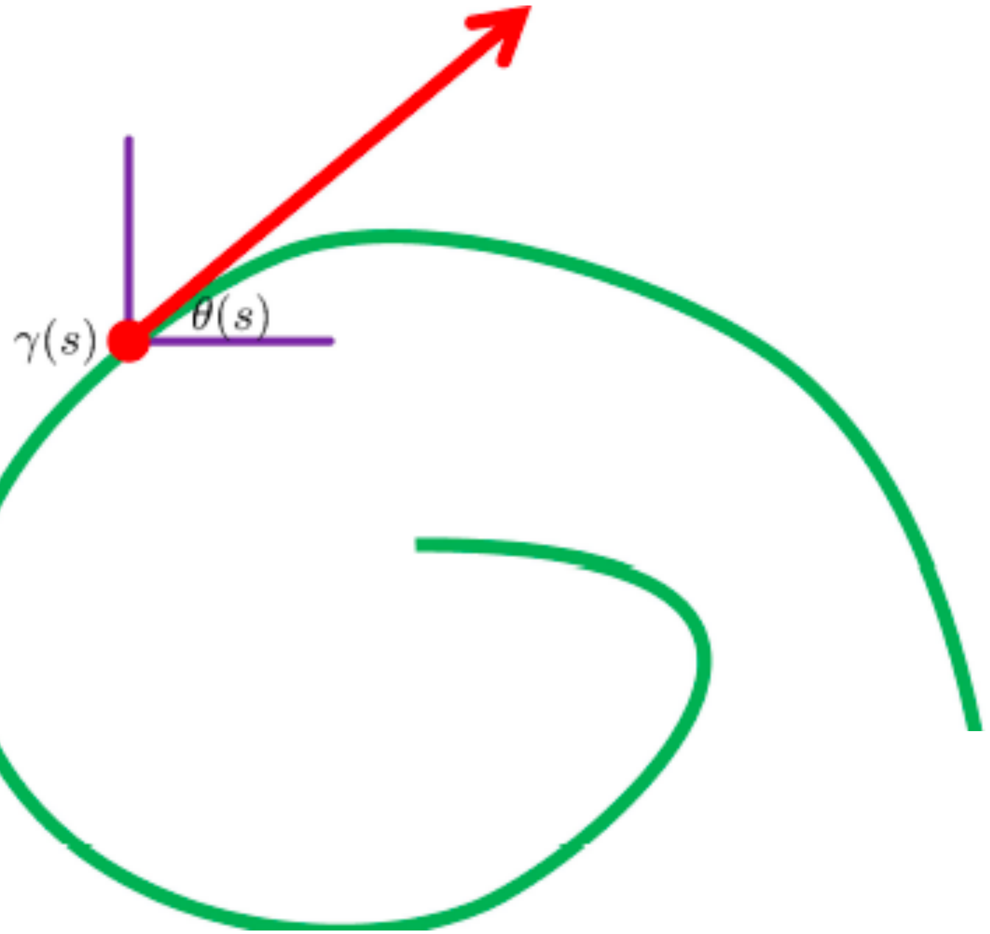
Equivalently:

$$\kappa(s) = \frac{d}{ds}\theta(s)$$



$$T(s) = (\cos\theta(s), \sin\theta(s))$$

$$\begin{aligned} T'(s) &= \theta'(s)(-\sin\theta(s), \cos\theta(s)) \\ &:= \kappa(s)N(s) \end{aligned}$$



Fundamental Theorem of Plane Curves

- Fact. Up to rigid motions, an arc-length parameterized plane curve is uniquely determined by its curvature.
- Q: Given only the curvature function, how can we recover the curve?

A: Just "invert" the two relationships $\frac{d}{ds}\theta = \kappa, \; \frac{d}{ds}\gamma = T$

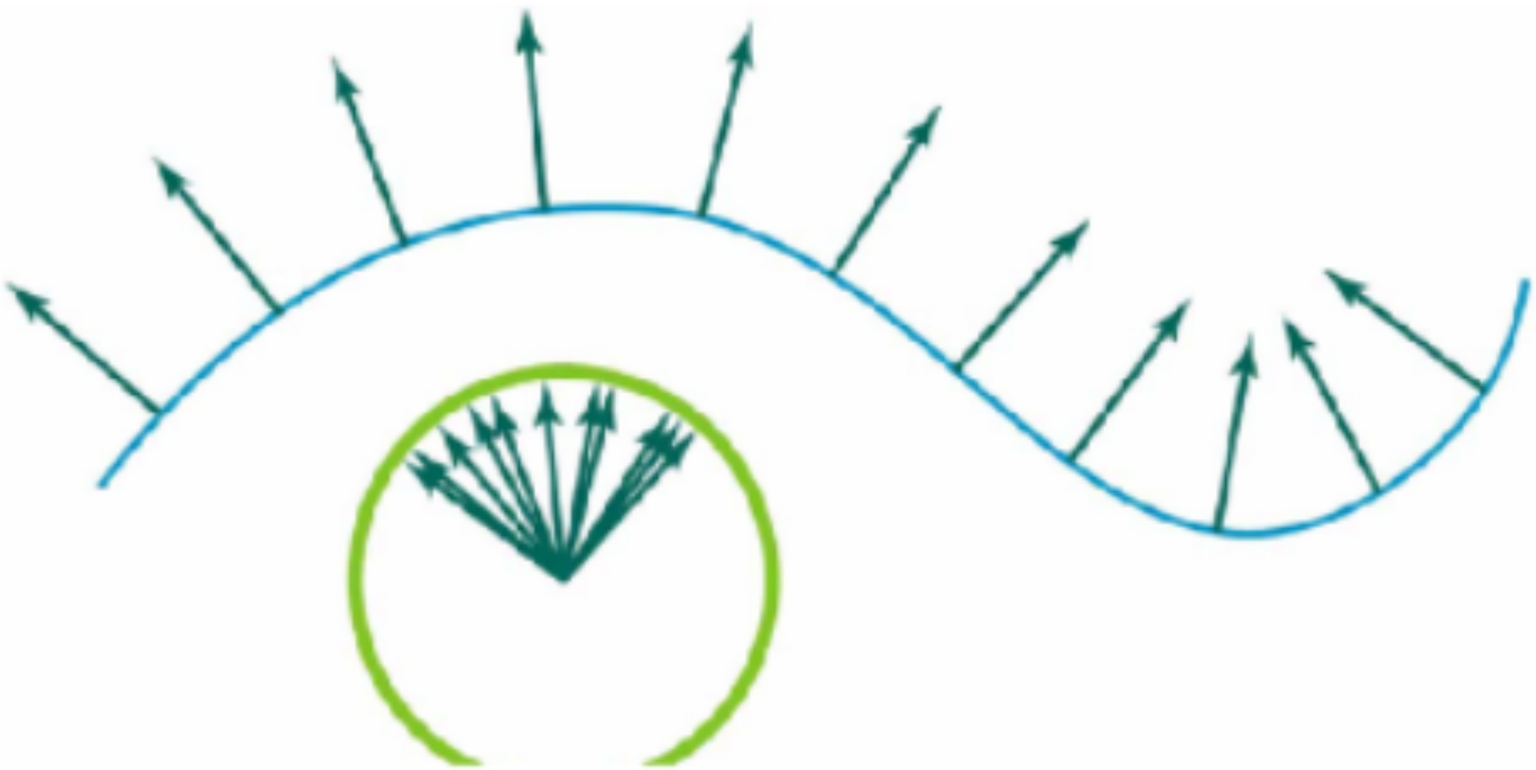
$$\textit{First integrate curvature to get angle:} \quad \theta(s) := \int_0^s \kappa(t)\,dt$$

$$\textit{Then evaluate unit tangents:} \quad T(s) := (\cos(\theta), \sin(\theta))$$

$$\textit{Finally, integrate tangents to get curve:} \quad \gamma(s) := \int_0^s T(t)\,dt$$

Gauß map $\hat{n}(x)$

Point on curve maps to point on unit circle



Curvature: Some Intuition

Shape operator (Weingarten map)

Change in normal as we slide along curve

negative directional derivative D of Gauß map

$$\boxed{\mathrm{S}(\mathrm{v}) = -D_{\mathrm{v}}\hat{\mathrm{n}}}$$

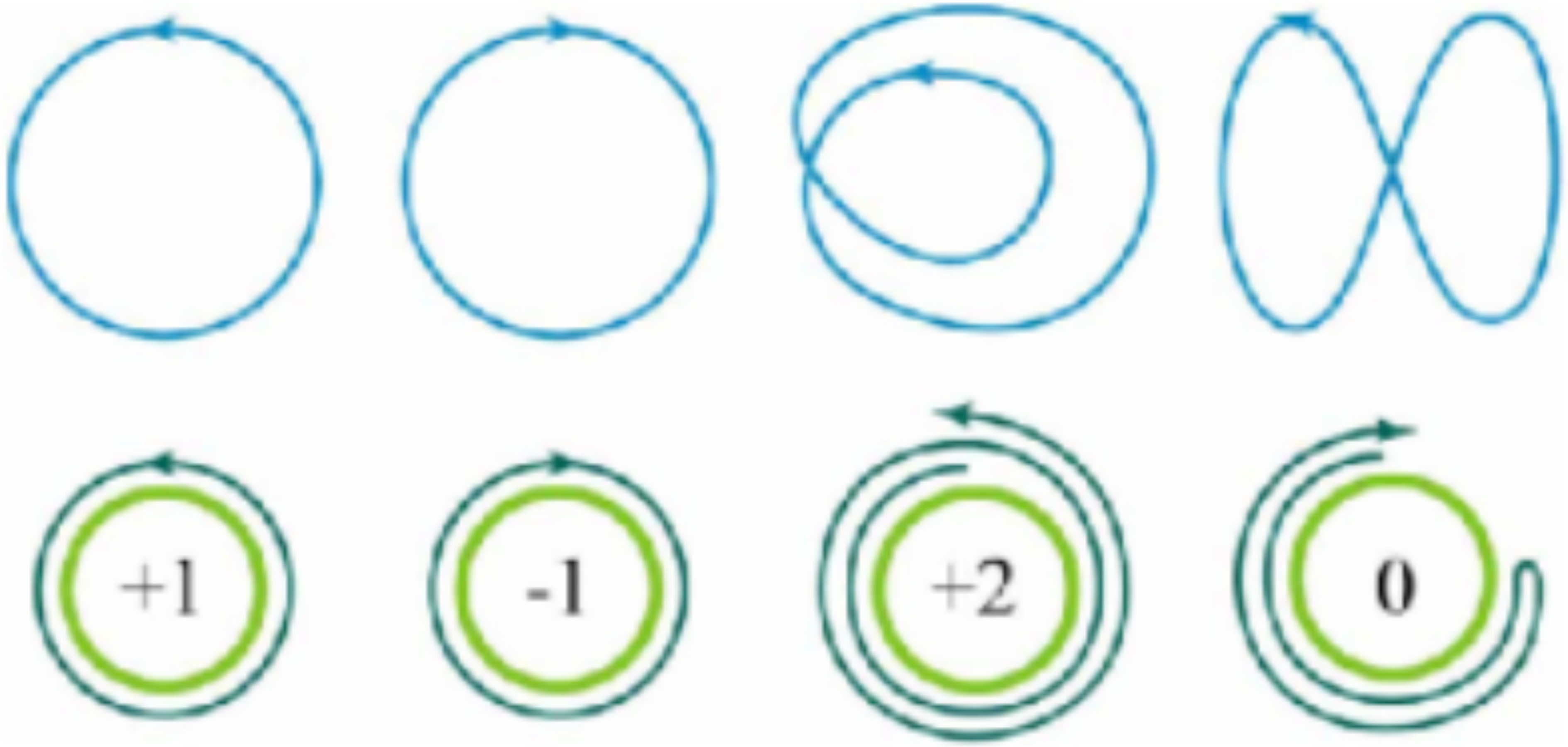
describes directional curvature

using normals as degrees of freedom

→ accuracy/convergence/implementation (discretization)

Turning number

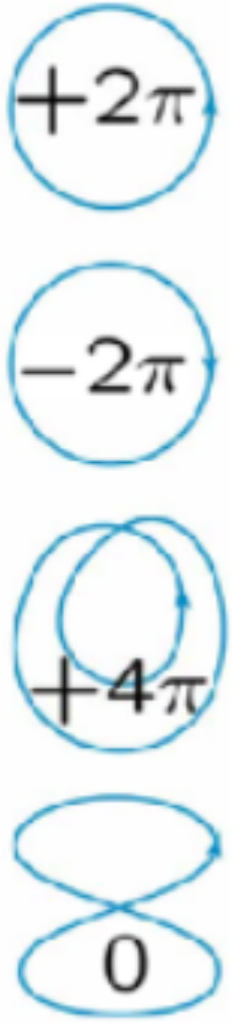
- **Turning number,** *k*
- Number of orbits in Gaussian image



Turning number theorem

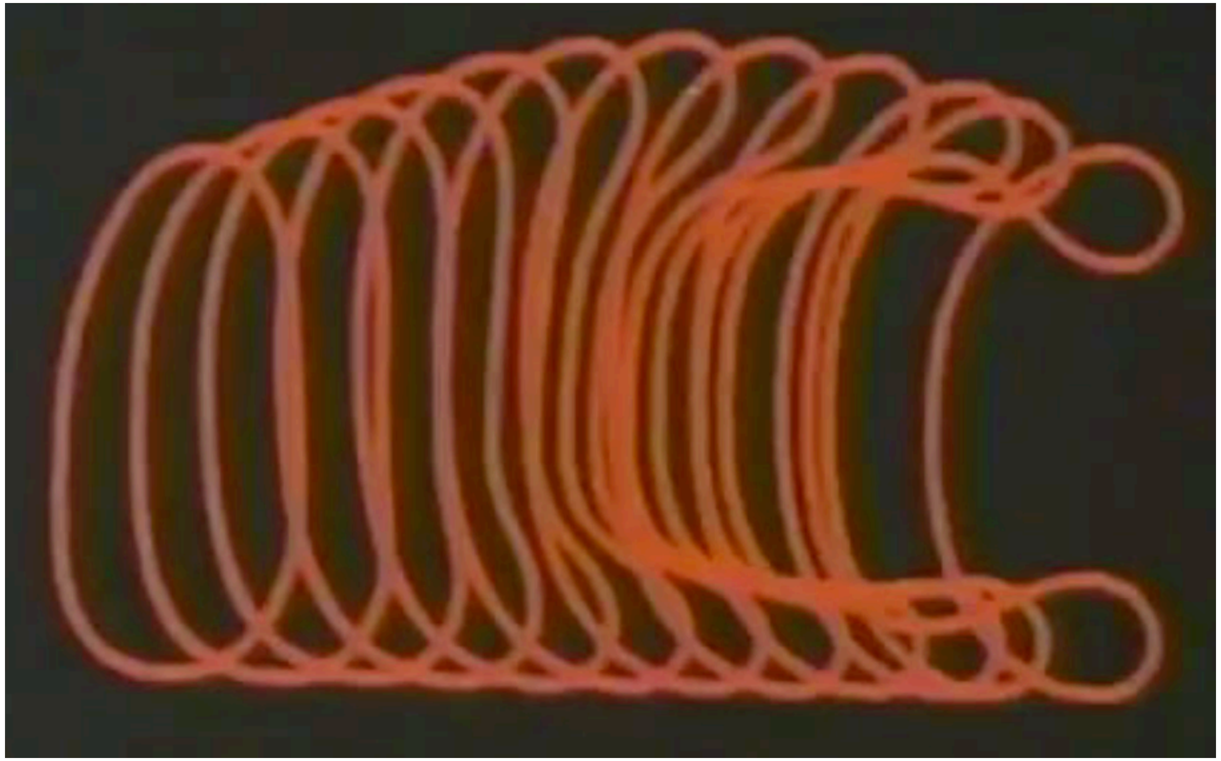
- For a closed curve, the integral of curvature is an integer multiple of 2π

$$\boxed{\int_{\Omega} \kappa ds = 2\pi k}$$



Whitney-Graustein Theorem

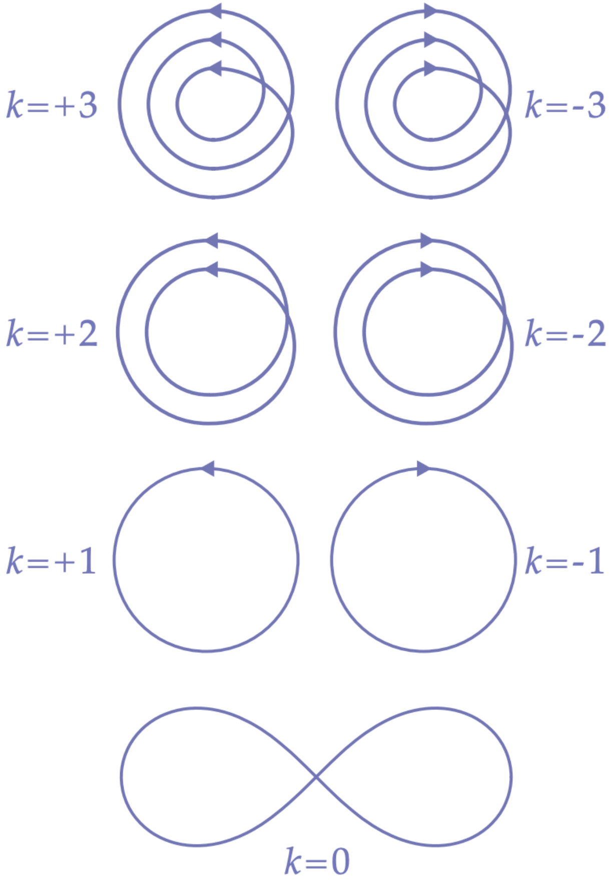
- (Whitney-Graustein) Two curves have the same turning number k if and only if they are related by regular homotopy, i.e., if one can continuously “deform” into the other while remaining regular (immersed).

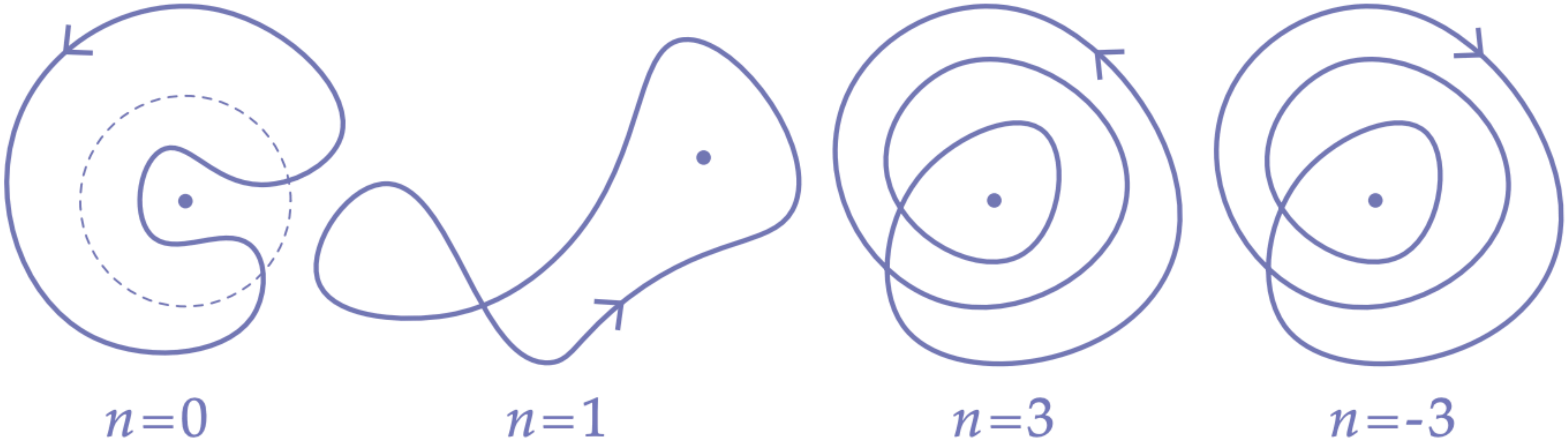


“Regular Homotopies in the Plane” — https://youtu.be/fKFH3c7b57s

Turning and Winding Numbers

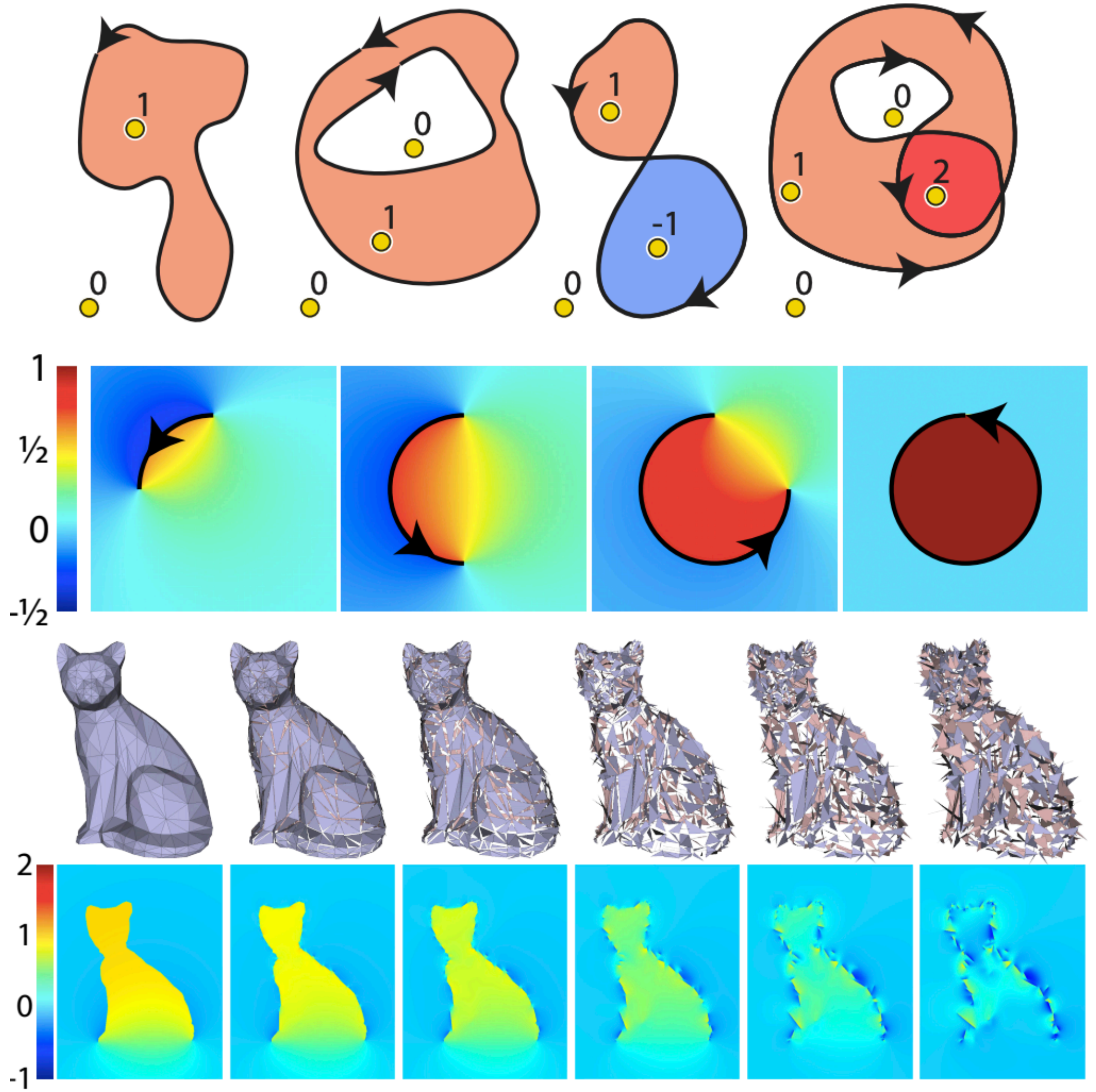
- For a closed regular curve in the plane…
- The turning number k is the number of counterclockwise turns made by the tangent
- The winding number n is the number of times the curve goes around a particular point p
  - can also be viewed as the total signed length of the projection of the curve onto a unit-length circle around p





Application: Generalized Winding Numbers

- winding number gives good indication of which points are inside/ outside
- Useful for a wide variety of practical tasks: extracting "watertight" mesh, tetrahedral meshing, constructive solid geometry (booleans), …

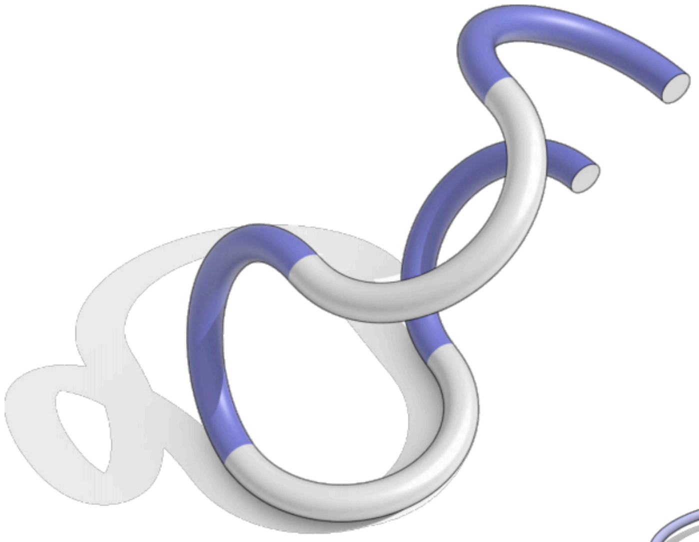


Jacobson et al, "Robust Inside-Outside Segmentation using Generalized Winding Numbers" (2013)

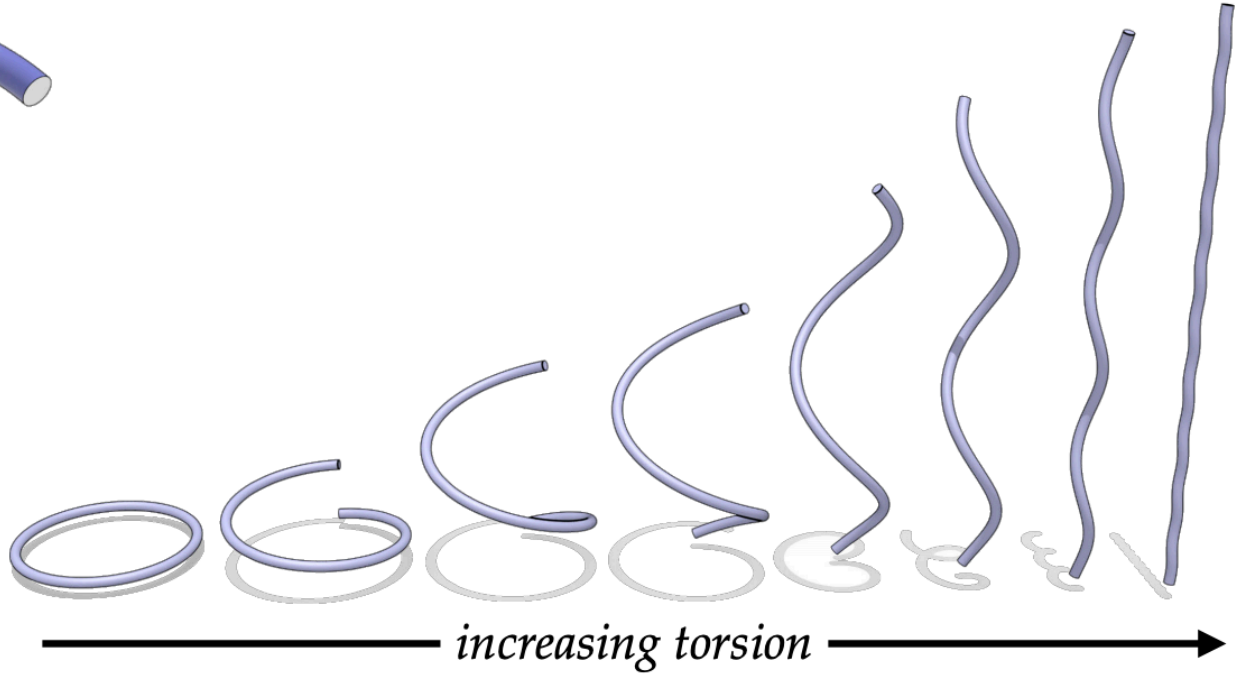
Space curves

Curvature and Torsion of a Space Curve

- Euclidean invariants, i.e. **invariant** under rigid motion
  - **Curvature**: Deviation from straight line, “bending”
  - **Torsion**: Deviation from planarity, “twisting”
- **Intrinsic** properties of the curve
  - Independent of parameterization
- Define curve **uniquely** up to a rigid motion



Intuition: torsion is “out of plane bending”



The Frenet Frame & formula

- The tangent unit vector **T** is defined as

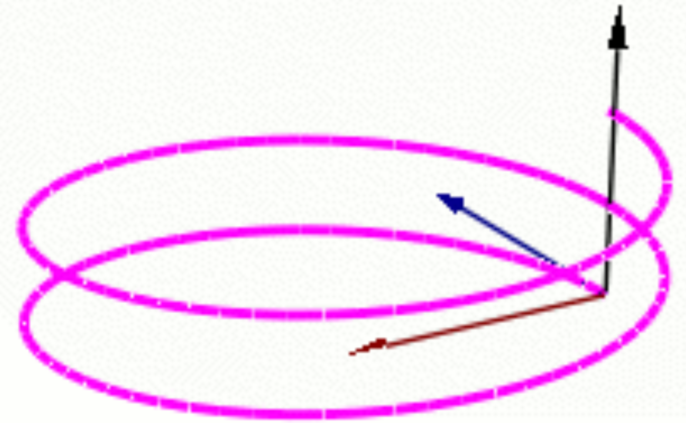
$$\mathbf{T} = \frac{d\mathbf{r}}{ds}. \qquad (1)$$

- The normal unit vector **N** is defined as

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left\| \frac{d\mathbf{T}}{ds} \right\|}. \qquad (2)$$

- The binormal unit vector **B** is defined as the cross product of **T** and **N**:

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}. \qquad (3)$$



$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

Torsion (deviation from planarity)

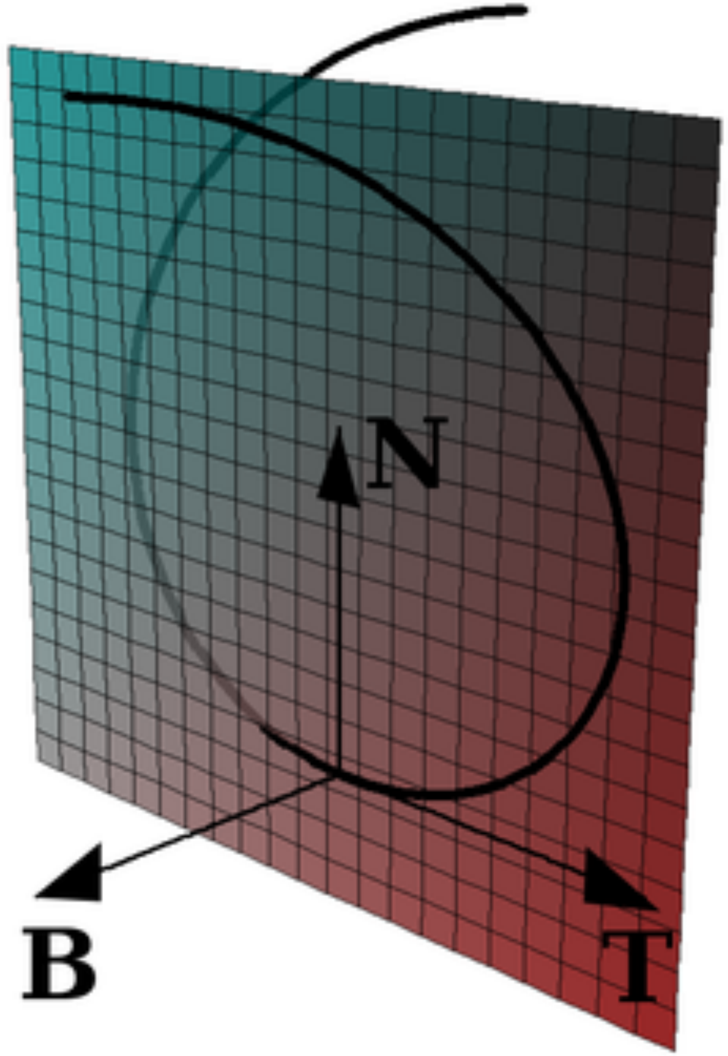
$$\tau = \frac{1}{\kappa^2} \det([\mathbf{x}_s, \mathbf{x}_{ss}, \mathbf{x}_{sss}])$$

Curvature & Torsion

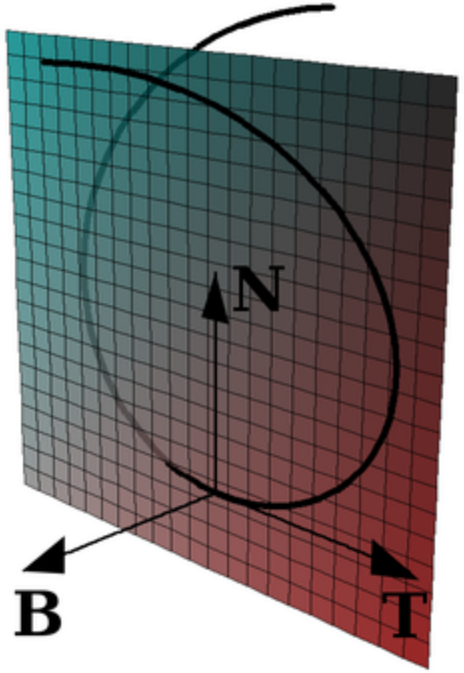
- Change in the tangent describes bending (curvature);
- Change in binormal describes twisting (torsion)

$$\kappa = -\langle N, \tfrac{d}{ds} T \rangle$$

$$\tau = \quad \langle N, \tfrac{d}{ds} B \rangle$$



Curvature & Osculating circle



Planes defined by $\mathbf{x}$ and two vectors

- osculating plane: vectors $\mathbf{t}$ and $\mathbf{n}$
- normal plane: vectors $\mathbf{n}$ and $\mathbf{b}$
- rectifying plane: vectors $\mathbf{t}$ and $\mathbf{b}$

Osculating circle

- second order contact with curve
- center $\mathbf{c} = \mathbf{x} + (1/\kappa)\mathbf{n}$
- radius $1/\kappa$

Thanks