1 Binomial series

The **binomial series** is the Maclaurin series for the function f given by $f(x) = (1+x)^\alpha$ where $\alpha \in \mathbb{C}$ is an arbitrary complex number. Explicitly,

$$\begin{aligned}(1+x)^\alpha &= \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \qquad (1)\\ &= 1 + \alpha x + \tfrac{\alpha(\alpha-1)}{2!}x^2 + \cdots ,\end{aligned}$$

and the binomial series is the power series on the right hand side of (1), expressed in terms of the (generalized) binomial coefficients

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$$

1.1 Special cases

If α is a nonnegative integer n, then the $(n+2)^{th}$ term and all later terms in the series are 0, since each contains a factor $n-n$; thus in this case the series is finite and gives the algebraic binomial formula.

The following variant holds for arbitrary complex β, but is especially useful for handling negative integer exponents in (1):

$$\frac{1}{(1-z)^{\beta+1}} = \sum_{k=0}^{\infty} \binom{k+\beta}{k} z^k.$$

To prove it, substitute $x = -z$ in (1) and apply a binomial coefficient identity, which is,

$$\binom{-\beta-1}{k} = (-1)^k \binom{k+\beta}{k}.$$

1.2 Summation of the binomial series

The usual argument to compute the sum of the binomial series goes as follows. Differentiating term-wise the binomial series within the convergence disk $|x| < 1$ and using formula (1), one has that the sum of the series is an analytic function solving the ordinary differential equation $(1+x)u'(x) = \alpha u(x)$ with initial data $u(0) = 1$. The unique solution of this problem is the function $u(x) = (1+x)^\alpha$, which is therefore the sum of the binomial series, at least for $|x| < 1$. The equality extends to $|x| = 1$ whenever the series converges, as a consequence of Abel's theorem and by continuity of $(1+x)^\alpha$.