# 1 Vector Analysis

## 1.1 Vector Algebra

### 1.1.1 Vector Geometry

**Definition** (Vector)**.** A **vector** is a quantity that has both *magnitude* and *direction.*

A vector is represented geometrically as an arrow with the length of the arrow determined by the magnitude. Vectors will be identified with **math boldface** notation. The magnitude of a vector $\mathbf{A}$ is written $|\mathbf{A}|$. A **scalar** is a quantity with magnitude but no direction. Vectors have magnitude and direction but no *location.* Arrow diagrams can be slid at will as long as there is no change in direction or magnitude.

**Definition** (Vector Addition)**.** Place the tail of $\mathbf{B}$ at the head of $\mathbf{A}$. The sum $\mathbf{A}+\mathbf{B}$ is the vector from the tail of $\mathbf{A}$ to the tail of $\mathbf{B}$.

Addition is *commutative* and *associative*

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$$

$$\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$$

Minus $\mathbf{A}$ denoted $-\mathbf{A}$ is a vector facing in the opposite direction. To subtract two vectors add the minus of the second vector

$$\mathbf{A}-\mathbf{B}=\mathbf{A}+(-\mathbf{B})$$

**Definition** (Scalar Multiplication)**.** Multiplication of a vector by a positive scalar $a$ multiplies the magnitude but leaves the direction unchanged. If $a$ is negative, the direction will be reversed.

Scalar multiplication is *distributive*

$$a(\mathbf{A}+\mathbf{B})=a\mathbf{A}+a\mathbf{B}$$

**Definition** (Dot Product)**.** The dot product of two vectors is defined by

$$\mathbf{A}\cdot\mathbf{B}\equiv|\mathbf{A}|\,|\mathbf{B}|\cos\theta \tag{1.1}$$

where $\theta$ is the angle between two vectors when placed tail-to-tail. $\mathbf{A}\cdot\mathbf{B}$ is a *scalar.*

The dot product is *commutative* and *distributive*

$$\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A}$$

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C} \tag{1.2}$$

If $\mathbf{A}$ and $\mathbf{B}$ are perpendicular, then $\mathbf{A}\cdot\mathbf{B}=0$. If the two vectors are parallel, then $\mathbf{A}\cdot\mathbf{B}=|\mathbf{A}|\,|\mathbf{B}|$. In particular,

$$\mathbf{A}\cdot\mathbf{A}=|\mathbf{A}|^2 \tag{1.3}$$

**Definition** (Cross Product)**.** The cross product of two vectors is defined by

$$\mathbf{A}\times\mathbf{B}\equiv|\mathbf{A}|\,|\mathbf{B}|\sin\theta\;\hat{\mathbf{n}} \tag{1.4}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. The direction is chosen according to the right hand rule. $\mathbf{A}\times\mathbf{B}$ is a vector.

The cross product is *distributive* and *anticommutative*

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{1.5}$$

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B} \tag{1.6}$$

Geometrically, $|\mathbf{A} \times \mathbf{B}|$ is the area of a parallelogram generated by $\mathbf{A}$ and $\mathbf{B}$. If the vectors are parallel, their cross product is zero. In particular,

$$\mathbf{A} \times \mathbf{A} = 0$$

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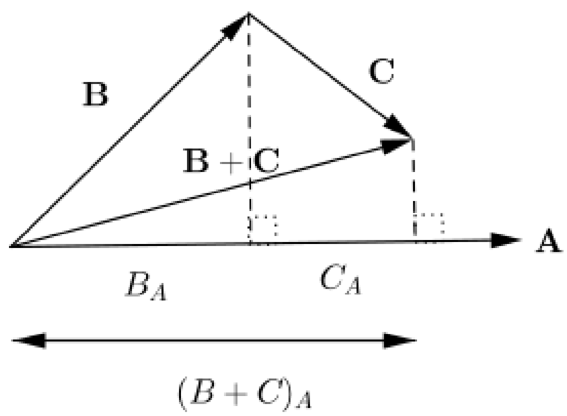
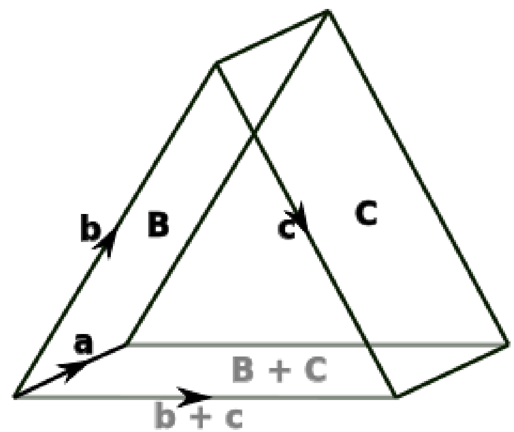
**Problem 1.1.** Using the definitions in equations 1.1 and 1.4, and appropriate diagrams, show that the dot product and cross product are distributive.

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**Problem 1.2.** Is the cross product associative?

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \stackrel{?}{=} \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$$

The cross product is not associative in general. Suppose $\mathbf{A}$ and $\mathbf{B}$ are orthogonal unit vectors. Then,

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{B} = -\mathbf{A}$$

whereas

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{B}) = \mathbf{0}$$

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### 1.1.2 Vector Components

**Definition** (Components)**.** In rectangular coordinates, the **basis vectors** $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors pointing in the directions of the $x$, $y$, and $z$ axes. An arbitrary vector $\mathbf{A}$ can be expressed as a linear combination of these basis vectors

$$\mathbf{A} = a_1\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}} = \langle a_1, a_2, a_3 \rangle$$

The coefficients $a_1$, $a_2$, and $a_3$ are called **components** of $\mathbf{A}$.

Vector operations can be restated in terms of components.

$$\mathbf{A} + \mathbf{B} = (a_1 + b_1)\hat{\mathbf{x}} + (a_2 + b_2)\hat{\mathbf{y}} + (a_3 + b_3)\hat{\mathbf{z}} \tag{1.7}$$

$$\alpha\mathbf{A} = (\alpha a_1)\hat{\mathbf{x}} + (\alpha a_2)\hat{\mathbf{y}} + (\alpha a_3)\hat{\mathbf{z}} \tag{1.8}$$

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1.10}$$

$$\mathbf{A} \times \mathbf{B} = (a_2 b_3 - a_3 b_2)\hat{\mathbf{x}} + (a_3 b_1 - a_1 b_3)\hat{\mathbf{y}} + (a_1 b_2 - a_2 b_1)\hat{\mathbf{z}} \tag{1.13}$$

*Remark.* The cross product can be calculated as the determinant of a matrix whose first row is $[\hat{\mathbf{x}} \quad \hat{\mathbf{y}} \quad \hat{\mathbf{z}}]$, whose second row is the components $[a_1 \quad a_2 \quad a_3]$, and whose third row is the components $[b_1 \quad b_2 \quad b_3]$.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1.14}$$

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**Problem 1.3.** Find the angle between the body diagonals of a cube

Without loss of generality, consider the unit cube with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, and $(1,1,1)$. The body diagonals are given by the vectors

$$\begin{aligned} \mathbf{A}_1 &= \langle 1,1,1 \rangle \\ \mathbf{A}_2 &= \langle -1,1,1 \rangle \\ \mathbf{A}_3 &= \langle 1,-1,1 \rangle \\ \mathbf{A}_4 &= \langle 1,1,-1 \rangle \end{aligned}$$

For, $i, j$ in the range $\{1,2,3,4\}$,

$$\mathbf{A}_i \cdot \mathbf{A}_j = \pm 1, \quad |\mathbf{A}_i| = \sqrt{3}$$

The angle between the vectors representing the diagonals satisfies

$$cos(\theta) = \frac{\mathbf{A}_i \cdot \mathbf{A}_j}{|\mathbf{A}_i|\,|\mathbf{A}_j|} = \pm\frac{1}{3}, \quad \theta \approx 70.53^\circ$$

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**Problem 1.4.** Use the cross product to find the components of the unit vector $\hat{\mathbf{n}}$ perpendicular to the plane passing through the points $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$.

$\mathbf{A} = \langle -1,2,0 \rangle$ and $\mathbf{B} = \langle -1,0,3 \rangle$ are vectors inside the plane since they are displacement vectors from $(1,0,0)$ to $(0,2,0)$ and $(0,0,3)$ respectively. Since,

$$\mathbf{A} \times \mathbf{B} = \langle 6,3,2 \rangle, \quad |\langle 6,3,2 \rangle| = 7$$

Therefore,

$$\hat{\mathbf{n}} = \left\langle \frac{6}{7}, \frac{3}{7}, \frac{2}{7} \right\rangle$$

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### 1.1.3 Triple Products

**Definition** (Scalar Triple Product)**.** Geometrically, $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$ represents the area of a parallelpiped generated by the $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \tag{1.15}$$

In component form,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{1.16}$$

**Definition** (Vector Triple Product)**.** The vector triple product follows the **BAC-CAB** rule:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{1.17}$$

All higher order vector products can be simplified into an expression with no more than one cross product per term. For instance,

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \\ (\mathbf{A} \times (\mathbf{B} \times (\mathbf{C} \times \mathbf{D}))) = \mathbf{B}(\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})) - (\mathbf{A} \cdot \mathbf{B})(\mathbf{C} \times \mathbf{D}) \end{aligned} \tag{1.18}$$

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**Problem 1.5.** Prove the **BAC-CAB** rule by writing out both sides in component form.

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) =& \langle a_1, a_2, a_3 \rangle \times \langle b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1 \rangle \\ =& \langle a_2b_1c_2 - a_2b_2c_1 + a_3b_1c_3 - a_3b_3c_1, \\ & a_3b_2c_3 - a_3b_3c_2 + a_1b_2c_1 - a_1b_1c_2, \\ & a_1b_3c_1 - a_1b_1c_3 + a_2b_3c_2 - a_2b_2c_3 \rangle \\ =& \langle a_2b_1c_2 - a_2b_2c_1 + a_3b_1c_3 - a_3b_3c_1 + a_1b_1c_1 - a_1b_1c_1, \\ & a_3b_2c_3 - a_3b_3c_2 + a_1b_2c_1 - a_1b_1c_2 + a_2b_2c_2 - a_2b_2c_2, \\ & a_1b_3c_1 - a_1b_1c_3 + a_2b_3c_2 - a_2b_2c_3 + a_3b_3c_3 - a_3b_3c_3 \rangle \\ =& \langle b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3), \\ & b_2(a_1c_1 + a_2c_2 + a_3c_3) - c_2(a_1b_1 + a_2b_2 + a_3b_3), \\ & b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3(a_1b_1 + a_2b_2 + a_3b_3), \\ =& \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \end{aligned}$$

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**Problem 1.6.** Prove the Jacobi identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$$

Under what conditions does $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$?

Using equation (1.17) and commutativity of the dot product

$$\begin{aligned}&\mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{B}\times(\mathbf{C}\times\mathbf{A})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B})\\&=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})+\mathbf{C}(\mathbf{B}\cdot\mathbf{A})-\mathbf{A}(\mathbf{B}\cdot\mathbf{C})+\mathbf{A}(\mathbf{C}\cdot\mathbf{B})-\mathbf{B}(\mathbf{C}\cdot\mathbf{A})=0\end{aligned}$$

Suppose that the vector triple product of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ were associative.

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$$

Since the cross product is anticommutative,

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=0$$

Combining this observation with the Jacobi identity yields $\mathbf{B}\times(\mathbf{C}\times\mathbf{A})=0$. Hence, the cross product is commutative if and only if $\mathbf{A}$ is parallel to $\mathbf{C}$ or $\mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{C}$.

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### 1.1.4 Position and Displacement

**Definition** (Position)**.** The **position vector** is a vector pointing from the origin to its Cartesian coordinates.

$$\mathbf{r}\equiv x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}} \tag{1.19}$$

with magnitude

$$r=\sqrt{x^2+y^2+z^2} \tag{1.20}$$

Converting to a unit vector, the position vector points out radially

$$\hat{\mathbf{r}}=\frac{\mathbf{r}}{r}=\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}} \tag{1.21}$$

**Definition** (Infintesimal Displacement Vector)**.** The **Infintesimal Displacement Vector** is a vector pointing from a position $(x,y,z)$ to $(x+dx,y+dy,z+dz)$ denoted by

$$d\mathbf{r}=dx\hat{\mathbf{x}}+dy\hat{\mathbf{y}}+dz\hat{\mathbf{z}} \tag{1.22}$$

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**Problem 1.7.** Find the separation vector from source point $(2,8,7)$ to the field point $(4,6,8)$. Determine its magnitude, and construct the unit separation vector.

The separation vector is given by $\langle 4-2, 6-8, 8-7\rangle=\langle 2,-2,1\rangle$. It has magnitude

$$\sqrt{2^2+(-2)^2+1^2}=3$$

The unit separation vector is $\left\langle\frac{2}{3},\frac{-2}{3},\frac{1}{3}\right\rangle$.

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### 1.1.5 Transforming Vectors

Vectors are not defined solely as objects that have *magnitude* and *direction*. Their components also transform in a specific way under a change of coordinates. Consider the rotation transformation
In the $yz$-coordinate system,

$$\mathbf{A} = \langle |\mathbf{A}|\cos\theta, |\mathbf{A}|\sin\theta\rangle = \langle a_y, a_z\rangle$$

In the $\bar{y}\bar{z}$-coordinate system

$$\mathbf{A} = \left\langle |\mathbf{A}|\cos\bar{\theta}, |\mathbf{A}|\sin\bar{\theta}\right\rangle = \langle \bar{a}_y, \bar{a}_z\rangle$$

The angles are related by $\bar{\theta} = \theta - \phi$. Applying the angle addition formulas for sine and cosine yields the transformation law

$$\begin{bmatrix} \bar{a}_y \\ \bar{a}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & cos\phi \end{bmatrix} \begin{bmatrix} a_y \\ a_z \end{bmatrix} \tag{1.29}$$

For a rotation abount an arbitrary axis in three dimensions, the transformation law becomes

$$\begin{bmatrix} \bar{a}_x \\ \bar{a}_y \\ \bar{a}_z \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \tag{1.30}$$

or more compactly

$$\bar{a}_i = \sum R_{ij} a_j \tag{1.31}$$

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**Problem 1.8.**

(a) Prove that the two-dimensional rotation matrix (1.29) preserves dot products.

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= \bar{a}_y\bar{b}_y + \bar{a}_z\bar{b}_z \\
&= (cos(\phi)a_y + sin(\phi)a_z)(cos(\phi)b_y + sin(\phi)b_z) \\
&\quad + (-\sin(\phi)a_y + \cos(\phi)a_z)(-\sin(\phi)b_y + \cos(\phi)b_z) \\
&= \cos^2(\phi)a_yb_y + \sin(\phi)\cos(\phi)a_zb_y + \cos(\phi)\sin(\phi)a_yb_z + \sin^2(\phi)a_zb_z \\
&\quad + \sin^2(\phi)a_yb_y - \sin(\phi)\cos(\phi)a_zb_y - \cos(\phi)\sin(\phi)a_yb_z + \cos^2(\phi)a_zb_z \\
&= a_yb_y + a_zb_z
\end{aligned}$$

(b) What constraints must elements ($R_{ij}$) of the three-dimensional rotation matrix (1.30) satisfy in order to preserve the length of $\mathbf{A}$?

If the magnitude is preserved then the squared magnitude will also be preserved. Consequently,

$$\begin{aligned}
|\mathbf{A}|^2 &= \sum_{i=1}^{3} \bar{a}_i^2 \\
&= \sum_{i=1}^{3} \left( \sum_{j=1}^{3} R_{ij} a_j \right) \left( \sum_{k=1}^{3} R_{ik} a_k \right) \\
&= \sum_{i=1}^{3} \sum_{j,k} R_{ij} R_{ik} a_j a_k \\
&= \sum_{j,k} \sum_{i=1}^{3} R_{ij} R_{ik} a_j a_k \\
&= a_1^2 + a_2^2 + a_3^2
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{3} R_{ij} R_{ik} = \delta_{jk} \quad \forall j, k$$

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**Problem 1.9.** Find the transformation $R$ that describes a clockwise rotation by $120°$ about an axis from the origin through the point $(1, 1, 1)$.

This rotation sends $\hat{\mathbf{x}} \to \hat{\mathbf{z}}$, $\hat{\mathbf{y}} \to \hat{\mathbf{x}}$, and $\hat{\mathbf{z}} \to \hat{\mathbf{y}}$. Therefore, the rotation is represented by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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**Problem 1.10.**

(a) How do the components of a vector transform under a **translation** of coordinates?

Since a vector is not constrained to a specific location, the vector can be moved to be centered at the origin of the new coordinate system. Therefore, the components of the vector are invariant under translation, $\mathbf{A} \to \mathbf{A}$.

(b) How do the components of a vector transform under an **inversion** of coordinates ($\hat{\mathbf{x}} \to -\hat{\mathbf{x}}$, $\hat{\mathbf{y}} \to -\hat{\mathbf{y}}$, $\hat{\mathbf{z}} \to -\hat{\mathbf{z}}$)?

The components of a vector will change sign under an inversion of coordinates. Therefore, $\mathbf{A} \to -\mathbf{A}$.

(c) How does the cross product of two vectors transform under an inversion?

Since the cross product is bilinear, $\mathbf{C} = \mathbf{A} \times \mathbf{B} \to (-\mathbf{A}) \times (-\mathbf{B}) = \mathbf{A} \times \mathbf{B} = \mathbf{C}$. Therefore, the cross product is invariant under an inversion.

(d) How does the scalar triple product of three vectors transform under inversions.

Since the dot product and cross product are both bilinear,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (-\mathbf{A}) \cdot ((-\mathbf{B}) \times (-\mathbf{C})) = -(\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}))$$

Therefore, the scalar triple product changes sign under inversion.

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## 1.2 Differential Calculus

### 1.2.1 Gradient

In single variable calculus, the derivative the function $f(x)$ tells how rapidly a $f$ varies when we change the argument $x$ by a tiny amount $dx$:

$$df = \left(\frac{df}{dx}\right) dx \tag{1.33}$$

In the case of three variables, **partial derivatives** record how much a function $f(x, y, z)$ changes in each coordinate direction. The total change in $f(x, y, z)$ is summarized by the relation

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{1.34}$$

This equation can be rewritten using a dot product as

$$df = \left(\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial x}\hat{\mathbf{x}}\right) \cdot (dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}) = (\nabla f) \cdot (d\mathbf{r}) \tag{1.35}$$

**Definition** (Gradient)**.** The vector quantity

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial x}\hat{\mathbf{x}} \tag{1.36}$$

is called the gradient of $f$.

The gradient $\nabla f$ points in the direction of maximum increase of the function. Moreover, the magnitude of the gradient $|\nabla f|$ gives the rate of increase along this maximal direction.

**Definition** (Stationary point)**.** A point $(x, y, z)$ where the gradient vanishes is called a stationary point.

A stationary point could be a local maximum, local minimum, or a saddle point. These are the analagous to the maximum, minimum, and inflection point encountered in single variable calculus.

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**Problem 1.11.** Find the gradient of the following functions:

(a) $f(x, y, z) = x^2 + y^3 + z^4$.

$$\nabla f = \left\langle 2x, 3y^2, 4z^3 \right\rangle$$

(b) $f(x, y, z) = x^2y^3z^4$.

$$\nabla f = \left\langle 2xy^3z^4, 3x^2y^2x^4, 4x^2y^3z^3 \right\rangle$$

(c) $f(x, y, z) = e^x \sin(y) \ln(z)$

$$\nabla f = \left\langle e^x \sin(y) \ln(z), e^x \cos(y) \ln(z), e^x \sin(y) z^{-1} \right\rangle$$

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**Problem 1.12.** The height of a certain hill is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

(a) Where is the top of the hill located?

The top of the hill will be located at a stationary point of the function,

(b) How high is the hill?

(c) How steep is the slope at the point $(1, 1)$? In what direction is the slope steepest at that point?

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**Problem 1.13.** Show that

(a) $\nabla(r^2) = 2\mathbf{r}$

(b) $\nabla(r^{-1}) = -r^{-2}\hat{\mathbf{r}}$

(c) What is the general formula for $\nabla(r^n)$?

## 1.3 Divergence

## 1.4 Curl

## 1.5 Product Rules

## 1.6 Second Derivatives