1 Vector Spaces

Linear algebra is the study of linear maps on finite dimensional vector spaces. In linear algebra, complex numbers are necessary for classifying the basic types of linear transformations in terms of their eigenvalues. Eigenvalues arise as solutions to certain polynomial equations.

Definition 1.1. complex numbers

A **complex number** is an ordered pair (a, b) of real numbers. Complex numbers are often written as $a + bi$. The set of all complex numbers is denoted $\mathbb{C}$.

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

Addition and **multiplication** on $\mathbb{C}$ are defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

The complex numbers $a + 0i$ can be identified with the real numbers $\mathbb{R}$. Using multiplication as defined above, it is easy to see that $i^2 = -1$. The complex numbers inherit almost all the properties from the real numbers.

Commutativity $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$

Associativity $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for all $\alpha, \beta, \gamma \in \mathbb{C}$

Identities $\alpha + 0 = \alpha$ and $\alpha 1 = \alpha$ for all $\alpha \in \mathbb{C}$

Additive Inverse for every $\alpha \in \mathbb{C}$, there is a unique $-\alpha \in \mathbb{C}$ such that $\alpha + (-\alpha) = 0$

Multiplicative Inverse for every $\alpha \in \mathbb{C}$, there is a unique $\alpha^{-1} \in \mathbb{C}$ such that $\alpha\alpha^{-1} = 1$

Distributive Property $\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$ for all $\alpha, \beta, \gamma \in \mathbb{C}$

These properties for addition and multiplication on the complex numbers define a field. Both $\mathbb{C}$ and $\mathbb{R}$ are fields. Most theorems in linear algebra hold for fields, so all further statements will use the terminology $\mathbb{F}$ to refer to either $\mathbb{C}$ or $\mathbb{R}$.

Definition 1.2. n-tuples

Let n be a nonnegative integer. An **n-tuple** is an ordered list of n elements.

$$(x_1, x_2, \ldots, x_n)$$

Two lists are equal if and only if they have the same length and the same elements in the same order.