Inverse Function Theorem Tutorial

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1 Introduction

Inverse functions arise whenever there is a need to undo a function. For example, logarithms and inverse trigonometric functions arose out of the necessity to solve exponential and trigonometric equations. Oftentimes, inverse functions will appear as the solution to a physical problem, but the inverse function will not be able to be expressed in terms of elementary functions. In this case, **information about the inverse function can only be extracted indirectly by studying the original function**. One example is the Lambert W function which is the inverse to the function $f(x) = xe^x$

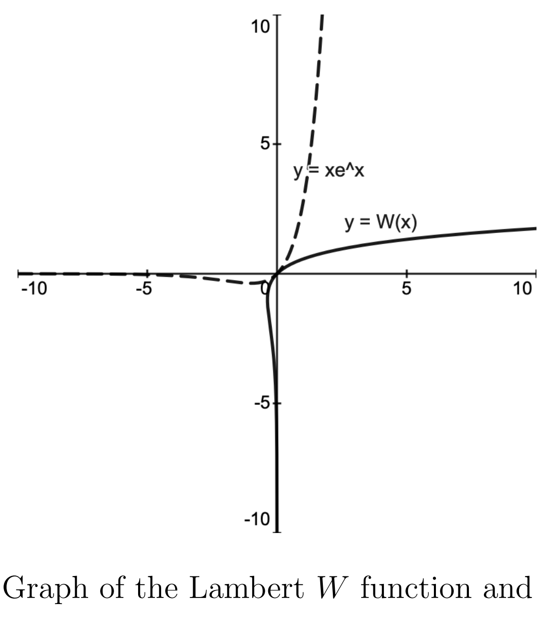

Figure 1: Graph of the Lambert W function and its inverse.

This function has many applications to biology, chemistry, and physics. For example, it is used to find the flight time of a projectile experiencing small air resistance. The **inverse function theorem** is a tool used to study the derivatives of an inverse function when we understand the derivatives of the original function.

2 Inverse Function Example

Consider the curve

$$y = x^2 + 1$$

The derivative with respect to x is

$$\frac{dy}{dx} = 2x$$

To find the curve representing the inverse, solve for x in terms of y. This curve is represented by the equation

$$\sqrt{y-1} = x$$

Instead of switching the variables x and y, it is better to **think of the inverse as a function taking in y values and outputting x values.** The derivative with respect to y is

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y-1}} = \frac{1}{2x}$$

Looking back at the derivative of the original function, it is apparent that

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

This relationship holds in general and is called the **Inverse Function Theorem** in one variable. Observe that the relationship between the derivative of the function and the derivative of the inverse function fails when the derivative is 0.

3 Inverse Function Theorem

Let (a, b) be a point on the curve $y = f(x)$ such that $f'(a) \neq 0$. The inverse function will take the form $x = f^{-1}(y)$. Then,

$$\frac{df^{-1}}{dy}(b) = \frac{1}{\frac{df}{dx}(a)}$$

If $g(y) = f^{-1}(y)$, then the Inverse Function Theorem can be reformulated as

$$g'(b) = \frac{1}{f'(a)}$$

Note that the derivative of the inverse function is evaluated at the y coordinate while the derivative of the original function is being evaluated at the x coordinate.

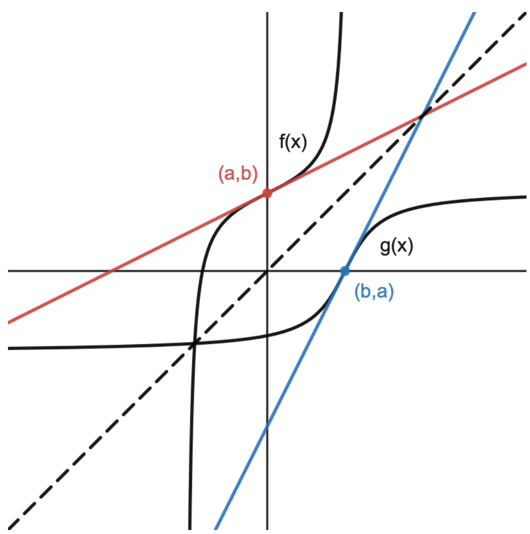

Figure 2: Function and Inverse Function with Corresponding Tangent Lines.

Proof. Figure 2 depicts the function $f(x)$ and its inverse $g(x) = f^{-1}(x)$. The tangent line to the curve $y = g(x)$ at $x = b$ can be derived in two different ways. The first way is to use the fact that $g'(b)$ is the slope of the tangent line at the point (b, a) Therefore,

$$y - a = g'(b)(x - b) \tag{3.1}$$

The second way is to use the properties of inverses. Since the inverse function can be determined geometrically through reflection by the line $y = x$, the tangent line of the inverse function will be the inverse of the tangent line to the original function. The tangent line of the original functions has slope $f'(a)$ and passes through (a, b). Therefore,

$$y - b = f'(a)(x - a) \tag{3.2}$$

Solving for the inverse relation

$$y - a = \frac{1}{f'(a)}(x - b) \tag{3.3}$$

Since equation 3.1 and equation 3.3 are both descriptions of the same line, it must be the case that

$$g'(b) = \frac{1}{f'(a)}$$

completing the proof. □

4 Applying the Inverse Function Theorem

x	$f(x)$	$f'(x)$
0	2	3
1	4	0
2	1	2
3	6	5

Consider the function $f(x)$ and its derivative $f'(x)$ described by the table above. If $g(x) = f^{-1}(x)$, the inverse function theorem implies

$$g'(1) = \frac{1}{f'(2)} = \frac{1}{2}$$

Similarly, $g'(2) = 1/3$ and $g'(6) = 1/5$. However, the inverse function theorem cannot be used to find $g'(4)$ since $f'(1) = 0$ Moreover, $g'(3)$ cannot be found since there is not a corresponding x value on the table such that $f(x) = 3$.