Assignment 1

1 Problem 1

- Show that $\rho(A) \leq ||A||$ for any subordinate norm $||.||$.
- Show that the spectral radius of $A \in R^{n\times n}$ is not a norm by finding two matrices A and B such that $\rho(A+B) > \rho(A) + \rho(B)$.
- Show that $1/||A^{-1}|| = \inf_{||x||=1} ||Ax||$ for any subordinate norm $||.||$.
- Let A be s.p.d. with smallest eigenvalue λ_{min}. Show that $||A^{-1}||_2 = 1/\lambda_{min}$.

2 Problem 2

Devise a compact storage mode suitable for symmetric band matrices. Write an algorithm to compute the Cholesky decomposition using this storage scheme and implement it into a code.

Consider the so-called *Hilbert matrix* given by

$$\mathbf{A}_{ij} = \frac{1}{i+j-1}, \quad i,j = 1, ..., n. \tag{1}$$

It is a notorious example of a badly conditioned matrix. Take $n = 5$ and solve the system $\mathbf{Ax} = \mathbf{q}$ for $\mathbf{q} = (5, 3.55, 2.81428571428571, 2.34642857142857, 2.01746031746032)^T$ using the Cholesky code that you wrote. Note that this is a symmetric band matrix with a bandwidth of 9 (trivial case of a band matrix). Slightly perturb $\mathbf{A}_{51}, \mathbf{A}_{15}$, to be 0.20001. Compute the solution of the new system. Compare the results and explain. Note that the condition number $k(\mathbf{A}) = O(10^5)$.

3 Problem 3

Using the same compact storage strategy as in problem 2 (considering a symmetric matrix), implement the SOR method in a subroutine. Apply it for the solution of a system with a 10×10 band matrix with a bandwidth 5, containing 7 on its main diagonal and 1 on the two diagonals above and below the main one. The rest of the matrix entries are 0. The right-hand side vector is: $(9, 10, 11, ..., 11, 10, 9)^T$. Use at least 5 different values of $\omega \in (0, 2)$ and check which one gives the best convergence rate to an accuracy $\epsilon = 10^{-6}$.