HW1

Problem 1

(1)

$$\sup_{\|x\|\neq 0} \frac{\|Ax\|_*}{\|x\|_*} \geq \sup_{\|u_i\|\neq 0} \frac{\|Au_i\|_*}{\|u_i\|_*} = \sup_i \lambda_i = \rho(A)$$

$$Au_i = \lambda_i u_i$$

(2)

$$\begin{pmatrix} 9 & 10 \\ 3 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 4 \\ 8 & 2 \end{pmatrix}$$

$\quad \overset{\shortparallel}{A} \qquad\qquad \overset{\shortparallel}{B}$

$\rho(A) \approx 12$
$\rho(B) \approx 7.17$
$\rho(A+B) = 13.7$

$$A = \begin{pmatrix} 6 & 9 \\ 5 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 3 \\ 9 & 3 \end{pmatrix}$$

$\rho(A) \approx 13.22$
$\rho(B) \approx 9.4 \qquad \rho(A+B) = 24$

(3)

$$\|A^{-1}\|_* = \sup_{\|x\|_*\neq 0} \frac{\|A^{-1}x\|_*}{\|x\|_*} = \sup_{\|x\|_*\neq 0} \frac{\|A^{-1}x\|_*}{\|AA^{-1}x\|_*} = \sup_{\|y\|_*\neq 0} \frac{\|y\|_*}{\|Ay\|_*} = \left(\inf_{\|y\|_*\neq 0} \frac{\|Ay\|_*}{\|y\|_*} \right)^{-1} =$$

$$= \frac{1}{\inf_{\|y\|=1} \|Ay\|_*}$$

(4)

$$\|A^{-1}\|_2 = \frac{1}{\inf_{\|x\|_2=1} \|Ax\|_2} = \frac{1}{\inf_{\|x\|_2=1} \sqrt{(A^TAx, x)}} = \frac{1}{\inf_{\sqrt{c_i^2}=1} \sqrt{\sum (c_i\lambda_i u_i, c_i u_i)}} =$$

$$= \frac{1}{\inf_{\sqrt{\sum c_i^2}=1} \sqrt{\sum c_i^2 \lambda_i}} = \frac{1}{\inf \sqrt{\lambda_i}} = \frac{1}{\min \xi_i}$$

where $A^TA u_i = \lambda_i u_i$
$A w_i = \xi_i w_i$

Problem 2

$$\begin{cases} Ax = b \\ (A+\Delta A)(x+\Delta x) = b \end{cases}$$

perturbation to matrix A, ΔA can be expressed as a perturbation to the RHS b, Δb. Since $C(A)$ is large this gives a large difference to the solution Δx.

Problem 3 $\omega_{opt} \simeq 1$ according to the plot.

HW2

Problem 1

iteration matrix

$$(D+\omega L)^{-1}((1-\omega)D-\omega U) \quad \text{must have } \lambda_i < 1$$

Using hint $\left|\det\left((D+\omega L)^{-1}((1-\omega)D-\omega U)\right)\right| < 1 \rightarrow$

$$\rightarrow \left(\det(D+\omega L)\right)^{-1}\left((1-\omega)I-\omega UD^{-1}\right)\det D = |1-\omega|^N < 1$$

$$|1-\omega| < 1 \rightarrow \quad \underline{0<\omega<2}$$

HW3

Problem 1

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0 \quad , v>0$$

$$u(t+k,x) = u(t,x) + k\frac{\partial u}{\partial t}(t,x) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(t,x) + O(k^3)$$

$$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x} = -v\,\frac{u_{j+1}^n - u_{j-1}^n}{2h}$$

$$\frac{\partial^2 u}{\partial t^2} = v^2\frac{\partial^2 u}{\partial x^2} = v^2\,\frac{u_{j+1}^{n+1} - 2u_j^n + u_{j-1}^n}{h^2}$$

$$u_j^{n+1} = u_j^n + k(-v)(u_{j+1}^n - u_{j-1}^n)\frac{1}{2h} + v^2\frac{k^2}{h}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)\frac{1}{h^2}$$

$$c = \frac{k}{h}v$$

$$u_j^{n+1} = u_j^n - \frac{c}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{c^2}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

$u_j^n \leftrightarrow w^n e^{ij\theta}$

$$w^{n+1} e^{ij\theta} = w^n e^{ij\theta} - \frac{c}{2} w^n \left(e^{i(j+1)\theta} - e^{i(j-1)\theta}\right) + w^n \frac{c^2}{2} \cdot$$
$$\cdot \left(e^{i(j+1)\theta} - 2e^{ij\theta} + e^{i(j-1)\theta}\right)$$

$$w^{n+1} = w^n\left(1 - ic\sin\theta + c^2(\cos\theta - 1)\right)$$

$$A(\theta) = 1 - ic\sin\theta + c^2(\cos\theta - 1)$$

$$|A(\theta)|^2 = 1 + 4c^2\left(1 - \sin^2\frac{\theta}{2}\right) + 4c^4\left(1 - \sin^2\frac{\theta}{2}\right)^2 + c^2\sin^2\theta$$

$$|A(\theta)|^2 = 1 - 4c^2\sin^2\frac{\theta}{2} + 4c^4\sin^4\frac{\theta}{2} + 4c^2\sin^2\frac{\theta}{2} - 4c^2\sin^4\frac{\theta}{2} =$$

$$= 1 - 4c^4\sin^4\frac{\theta}{2} - 4c^2\sin^4\frac{\theta}{2} = 1 - 4(c^2 - c^4)\sin^4\frac{\theta}{2}$$

$$|A(\theta)|^2 = 1 - 4(c^2 - c^4)\sin^4\frac{\theta}{2}$$

$$\frac{u_j^{n+1} - u_j}{k} = -\frac{v}{2h}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{kv^2}{2h^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$c = \frac{k}{h}v$$

$$u_j^{n+1} = u_j^n + k\frac{\partial u}{\partial t}\Big|_i^n + \frac{1}{2}k^2\frac{\partial^2 u}{\partial t^2} + O(k^3)$$

$$\frac{\partial u}{\partial t}\Big|_i^n + \frac{k}{2}\frac{\partial^2 u}{\partial t^2}\Big|_i^n + \frac{k^2}{6}\frac{\partial^3 u}{\partial x^3} + O(k^3) = -\frac{v}{2}\left(\frac{\partial u}{\partial x}\Big|_i^n + \frac{h^2}{3}\frac{\partial^3 u}{\partial x^3}\Big|_i^n + O(h^3)\right) +$$
$$+ \frac{kv^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\Big|_i^n + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}\Big|_i^n + O(h^3)\right)$$

since $\frac{\partial^2 u}{\partial t^2} = v^2\frac{\partial^2 u}{\partial x^2}$ the k^* term cancels out and we get.

$$\frac{\partial u}{\partial t}\Big|_i^n + \frac{k^2}{6}\frac{\partial^3 u}{\partial t^3} + O(k^3) = -v\frac{\partial u}{\partial x} - \frac{vh^2}{3}\frac{\partial^3 u}{\partial x^3}\Big|_i^n + \frac{h^2}{12}k\frac{\partial^4 u}{\partial x^4}\Big|_i^n + O(h^3)$$

consistency $O(t^2 + h^2)$

Problem 3:

$$u_j^{n+1} = u_j^n - \frac{c}{2}\left(u_{j+1}^n - u_{j-1}^n\right)$$

$$w^{n+1} e^{ij\theta} = w^n e^{ij\theta} - \frac{c}{2} w^n e^{i(j+1)\theta} + \frac{c}{2} w^n e^{i(j-1)\theta}$$

$$w^{n+1} = w^n (1 - ic\sin\theta)$$

$$A(\theta) = 1 - ic^2\sin^2\theta = 1 - ic\sin\theta$$

$$|A(\theta)|^2 = 1 + c^2\sin^2\theta$$

$\exists\theta$: $A(\theta) > 1$ - unstable unconditionally

$$u_j^{n+1} = \frac{u_{j+1}^n + u_{j-1}^n}{2} - \frac{c}{2}\left(u_{j+1}^n - u_{j-1}^n\right)$$

$$w^{n+1} e^{ij\theta} = \frac{1}{2} w^n\left(e^{i(j+1)\theta} + e^{i(j-1)\theta}\right) - \frac{c}{2}\left(e^{i(j+1)\theta} - e^{i(j-1)\theta}\right) w^n$$

$$w^{n+1} = w^n(\cos\theta - ic\sin\theta)$$

$$A(\theta) = \cos\theta - ic\sin\theta$$

$$|A(\theta)|^2 = 1 - \sin^2\theta(1 + c^2)$$

$$|A(\theta)|^2 = 1 - \sin^2\theta(1 + c^2)$$

$$|A(\theta)|^2 < |1 - (1 - c^2)| = |c^2| < 1 \quad \cancel{\text{prov}}$$

Stable provided $c < 1$

$$U_j^{n+1} = \frac{U_{j+1}^n - U_{j-1}^n}{2} - c\,\frac{U_{j+1}^n - U_{j-1}^n}{2}$$

$$u_j^{n+1} = u_j^n + k\frac{\partial u}{\partial t}(x_j, t^n) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_j, t^n) + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3}(x_j, \eta^n)$$

$$u_{j+1}^n = u_j^n + h\frac{\partial u}{\partial x}(x_j, t^n) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t^n) + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(\xi_j, t^n)$$

$$u_{j-1}^n = u_j^n - h\frac{\partial u}{\partial x}(x_j, t^n) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t^n) - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(\xi_j, t^n)$$

$$\frac{u_{j+1}^n - u_{j-1}^n}{2h} = \frac{\partial u}{\partial x}(x_j, t^n) + O(h^2)$$

$$\frac{u_{j+1} - u_{j-1}}{2} = u_j^n + O(h^2)$$

consistency $O(k+h^2)$

<u>HW4</u>

<u>Problem 1</u>

$$\frac{U_j^{n+1} - U_j^n}{k} = \theta\,\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{h^2} + (1-\theta)\,\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$$

$$w^{n+1} - w^n = 2\theta r\, w^{n+1}(\cos\varphi - 1) + 2(1-\theta)\, r(\cos\varphi - 1)\, w^n$$

$$r = \frac{k}{h^2}$$

$$A(\theta,\varphi) = \frac{1 - 4(1-\theta) r \sin^2\frac{\varphi}{2}}{1 + 4\theta r \sin^2\frac{\varphi}{2}} = 1 - \frac{4 r \sin^2\frac{\varphi}{2}}{1 + 4\theta r \sin^2\frac{\varphi}{2}}$$

$$|A(\theta,\varphi)|^2 = \Bigg(1 - \underbrace{\frac{4 r \sin^2\frac{\varphi}{2}}{1 + 4\theta r \sin^2\frac{\varphi}{2}}}_{>0}\Bigg)^2$$

~~Unconditionally stable.~~ See below for further analysis.

if $\theta = \frac{1}{2}$ we get Crank-Nicholson scheme and accuracy $O(k^2 + h^2)$

$$|A(\theta,\varphi)|^2 = \left(1 - \frac{4r\sin^2\frac{\varphi}{2}}{1+4\theta r\sin^2\frac{\varphi}{2}}\right)^2$$

$$0 < \frac{4r\sin^2\frac{\varphi}{2}}{1+4\theta r\sin^2\frac{\varphi}{2}} \leq 2 \quad \text{- stable}$$

$$2r\sin^2\frac{\varphi}{2} \leq 1+4\theta r\sin^2\frac{\varphi}{2}$$

$$2(1-2\theta)r\sin^2\frac{\varphi}{2} < 1$$

$$\theta \geq \frac{1}{2} - \frac{1}{4r} \quad \text{- stable}$$

For $\theta \geq \frac{1}{2}$ unconditionally stable

Problem 2:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{1}{Pe}\frac{\partial^2 u}{\partial x^2} = 0$$

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial t\partial x} + \frac{\partial^2 u}{\partial x^2} - \frac{1}{Pe}\frac{\partial^3 u}{\partial x^3} = 0 \\ &\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t\partial x} - \frac{1}{Pe}\frac{\partial^3 u}{\partial t\partial x^2} = 0 \end{aligned}\right\} \rightarrow (-)\downarrow \rightarrow$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \frac{1}{Pe}\left(\frac{\partial^3 u}{\partial x^3} - \frac{\partial^3 u}{\partial t\partial x^2}\right)$$

$$\delta_t^- u_j^{n+1} = \frac{u_j^{n+1}-u_j^n}{k} = \frac{\partial u}{\partial t}(j,n+1) + \underbrace{\frac{\partial^2 u}{\partial t^2}(j,n+1)}\cdot k + \cdots$$

$$\underbrace{\frac{\partial^2 u}{\partial x^2}} + \frac{1}{Pe}\left(\frac{\partial^3 u}{\partial x^2\partial t} - \frac{\partial^3 u}{\partial x^3}\right)$$

diffusion-like effect.

HW5

Problem 1:

$$-u'' + c(x)u = 1 \qquad c \in L^2(0,1)$$
$$u(0) = u(1) = 0 \qquad 0 \le c(x) \le c_1$$

$$\int_0^1 (-u'' + c(x)u)\, w\, dx = \int_0^1 w\, dx$$

$w \in H_0^1(0,1)$

$$\int_0^1 (-u'' + cu)\, w\, dx = \int_0^1 w\, dx$$

$$-u'w\Big|_0^1 + \int_0^1 w'u'\, dx + \int_0^1 cuw\, dx = \int_0^1 w\, dx$$

$$\int_0^1 (u'w' + cuw)\, dx = \int_0^1 w\, dx \qquad \forall\, w \in H_0^1(0,1)$$

$$a(w,u) = \int_0^1 (w'u' + cuw)\, dx$$

$$G(w) = \int_0^1 w\, dx$$

Obviously $a(w,u)$ ~~and~~ is bilinear, $G(w)$ - linear and bounded

1) $a(u,u) \ge \beta \|u\|_0^2$

$$\int_0^1 (u'u' + cuu)\, dx \ge \int_0^1 cuu\, dx \quad \text{or} \quad \int_0^1 (u'u' + cuu)\, dx \ge \int_0^1 u'u'\, dx$$

$$\int_0^1 u'u'\, dx = \|u'\|_0^2 \ge \|u\|_0^2 \qquad \text{Using the Poincare inequality}$$

2) $|a(u,v)| \le \beta \|u\|_0 \|v\|_0$

$$\int_0^1 (u'v' + cuv)\, dx \le \int_0^1 (u'v' + c_1 uv)\, dx \le \|u'\|_0 \|v'\|_0 + c_1 \|u\|_0 \|v\|_0 \le$$

$\leq \|u\|_0 \|v\|_0 (1+C_1)$ Using poincare ineq

Using Lax-Milgram we get that this problem has a unique sln.

Problem 2:

Lemma

if A - a nonsingular $n \times n$ matrix then $\exists$ unique sln for problem:

$$\text{find } u \in V: \quad a(u,v) = G(v), \ \forall v \in V \quad (1)$$

where $a(u,v) = u^T A v$

$G(v) = f^T v$.

Proof: Since (1) holds $\forall v \in V$ then we can chose $\{v_i\}_{i=1}^n$ - a basis of V $v_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ - i row this way we

will obtain a system of n linear eqns:

$$u^T A v_i = f^T v_i, \quad i = 1 \dots n$$

Since A is non-singular $n \times n$ matrix, we get a system of n linear eqns which must have a unique sln.

A need not be positive - definite for this.

Problem 3:

in L^2 norm the error is bounded by $C \cdot h^2$

so the plot should be a line with slope 2.