2D FTCS

$$\frac{T^{n+1}_{a,b} - T^{n}_{a,b}}{\Delta t} = \frac{\alpha_x}{\Delta x^2}\left(T^{n}_{a-1,b} - 2T^{n}_{a,b} + T^{n}_{a+1,b}\right) + \frac{\alpha_y}{\Delta y^2}\left(T^{n}_{a,b-1} - 2T^{n}_{a,b} + T^{n}_{a,b+1}\right)$$

Assume: $T^{n}_{a,b} = G^{n} e^{ij\Delta x a} e^{ik\Delta y b}$ where i is $\sqrt{-1}$, j and k are terms of Fourier expansion.

a and b are dimensions.

$$\frac{G^{n+1}\cancel{e^{ij\Delta x a}}\cancel{e^{ik\Delta y b}} - G^{n}\cancel{e^{ij\Delta x a}}\cancel{e^{ik\Delta y b}}}{\Delta t} = G^{n}\cancel{e^{ij\Delta x a}}\cancel{e^{ik\Delta y b}}\left[\frac{\alpha_x}{\Delta x^2}\left(e^{-ij\Delta x} - 2 + e^{ij\Delta x}\right) + \frac{\alpha_y}{\Delta y^2}\left(e^{-ik\Delta y} - 2 + e^{ik\Delta y}\right)\right]$$

$$\frac{G^{n+1}}{G^{n}} = 1 + \frac{\alpha_x \Delta t}{\Delta x^2}\left(e^{-ij\Delta x} - 2 + e^{ij\Delta x}\right) + \frac{\alpha_y \Delta t}{\Delta y^2}\left(e^{-ik\Delta y} - 2 + e^{ik\Delta y}\right)$$

$$= 1 + 2\cdot\frac{\alpha_x \Delta t}{\Delta x^2}\left(\cos(j\Delta x) - 1\right) + 2\frac{\alpha_y \Delta t}{\Delta y^2}\left(\cos(k\Delta y) - 1\right)$$

$$= 1 - 4\frac{\alpha_x \Delta t}{\Delta x^2}\sin^2\left(\frac{j\Delta x}{2}\right) - 4\frac{\alpha_y \Delta t}{\Delta y^2}\sin^2\left(\frac{k\Delta y}{2}\right)$$

Stable as long as $\frac{\alpha_x \Delta t}{\Delta x^2} + \frac{\alpha_y \Delta t}{\Delta y^2} \leq \frac{1}{2}$ (similar to reasoning in FTCS 1D).

3D FTCS

$$\frac{T^{n+1}_{a,b,c} - T^{n}_{a,b,c}}{\Delta t} = \frac{\alpha_x}{\Delta x^2}\left(T^n_{a-1,b,c} - 2T^n_{a,b,c} + T^n_{a+1,b,c}\right) + \frac{\alpha_y}{\Delta y^2}\left(T^n_{a,b-1,c} - 2T^n_{a,b,c} + T^n_{a,b+1,c}\right)$$

$$+ \frac{\alpha_z}{\Delta z^2}\left(T^n_{a,b,c-1} - 2T^n_{a,b,c} + T^n_{a,b,c+1}\right)$$

Assume $T^n_{a,b,c} = G^n e^{ij\Delta x a} e^{ik\Delta y b} e^{il\Delta z c}$ j,k,l are terms of 3D fourier expansion.

$$\frac{G^{n+1} e^{ij\Delta x a} e^{ik\Delta y b} e^{il\Delta z c} - G^n e^{ij\Delta x a} e^{ik\Delta y b} e^{il\Delta z c}}{\Delta t} = G^n e^{ij\Delta x a} e^{ik\Delta y b} e^{il\Delta z c}\left[\frac{\alpha_x}{\Delta x^2}\left(e^{-ij\Delta x} - 2 + e^{ij\Delta x}\right)\right.$$

$$\left. + \frac{\alpha_y}{\Delta y^2}\left(e^{-ik\Delta y} - 2 + e^{ik\Delta y}\right) + \frac{\alpha_z}{\Delta z^2}\left(e^{-il\Delta z} - 2 + e^{il\Delta z}\right)\right]$$

$$\frac{G^{n+1}}{G^n} = 1 + \frac{\alpha_x \Delta t}{\Delta x^2}\left(e^{-ij\Delta x} - 2 + e^{ij\Delta x}\right) + \frac{\alpha_y \Delta t}{\Delta y^2}\left(e^{-ik\Delta y} - 2 + e^{ik\Delta y}\right) + \frac{\alpha_z \Delta t}{\Delta z^2}\left(e^{-il\Delta z} - 2 + e^{il\Delta z}\right)$$

$$= 1 - 4\frac{\alpha_x \Delta t}{\Delta x^2}\sin^2\left(\frac{j\Delta x}{2}\right) - 4\frac{\alpha_y \Delta t}{\Delta y^2}\sin^2\left(\frac{k\Delta y}{2}\right) - 4\frac{\alpha_z \Delta t}{\Delta z^2}\sin^2\left(\frac{l\Delta z}{2}\right)$$

stable as long as:

$$\frac{\alpha_x \Delta t}{\Delta x^2} + \frac{\alpha_y \Delta t}{\Delta y^2} + \frac{\alpha_z \Delta t}{\Delta z^2} \le \frac{1}{2}$$

(similar to reasoning in FTCS 2D).

BECS 1D

$$\frac{T_a^{n+1} - T_a^n}{\Delta t} = \frac{\alpha}{\Delta x^2}\left(T_{a-1}^{n+1} - 2T_a^{n+1} + T_{a+1}^{n+1}\right)$$

Assume $T_a^n = G^n e^{ika\Delta x}$

$$\frac{G^{n+1} e^{ika\Delta x} - G^n e^{ika\Delta x}}{\Delta t} = G^{n+1} e^{ika\Delta x} \cdot \frac{\alpha}{\Delta x^2}\left(e^{-ik\Delta x} - 2 + e^{ik\Delta x}\right)$$

$$\frac{G^{n+1}}{G^n} = \frac{1}{1 - \frac{\alpha\Delta t}{\Delta x^2}\left(e^{-ik\Delta x} - 2 + e^{ik\Delta x}\right)} = \frac{1}{1 - \frac{2\alpha\Delta t}{\Delta x^2}\left(\cos(k\Delta x) - 1\right)}$$

For all $k\Delta x$, $\cos(k\Delta x) \leq 1$

$\cos(k\Delta x) - 1 \leq 0$

Since $\frac{\alpha\Delta t}{\Delta x^2} > 0$, $\frac{2\alpha\Delta t}{\Delta x^2}\left(\cos(k\Delta x) - 1\right) \leq 0$

$$1 - 2\frac{\alpha\Delta t}{\Delta x^2}\left(\cos(k\Delta x) - 1\right) \geq 1$$

So denominator of growth rate always ≥ 1.

$$\boxed{0 < \frac{G^{n+1}}{G^n} \leq 1}$$

BECS 2D

$$\frac{T^{n+1}_{a,b} - T^{n}_{a,b}}{\Delta t} = \cancel{t^{n+1}} \frac{\alpha_x}{\Delta x^2}\left(T^{n+1}_{a-1,b} - 2T^{n+1}_{a,b} + T^{n+1}_{a+1,b}\right) + \frac{\alpha_y}{\Delta y^2}\left(T^{n+1}_{a,b-1} - 2T^{n+1}_{a,b} + T^{n+1}_{a,b+1}\right)$$

Assume $T^n_{a,b} = G^n e^{ij\Delta x a} e^{ik\Delta y b}$

$$\frac{G^{n+1} e^{ij a\Delta x} e^{ik\Delta y b} - G^n e^{ij\Delta x a} e^{ik\Delta y b}}{\Delta t} = G^{n+1} e^{ij\Delta x a} e^{ik\Delta y b} \cdot \left[\frac{\alpha_x}{\Delta x^2}\left(e^{-ij\Delta x} - 2 + e^{ij\Delta x}\right) + \frac{\alpha_y}{\Delta y^2}\left(e^{-ik\Delta y} - 2 + e^{ik\Delta y}\right)\right]$$

$$\frac{G^{n+1}}{G^n} = \frac{1}{1 - \frac{\alpha_x \Delta t}{\Delta x^2}\left(e^{-ij\Delta x} - 2 e^{ij\Delta x}\right) - \frac{\alpha_y \Delta t}{\Delta y^2}\left(e^{-ik\Delta y} - 2 + e^{ik\Delta y}\right)} = \frac{1}{1 - \frac{2\alpha_x \Delta t}{\Delta x^2}\left(\cos(j\Delta x) - 1\right) - \frac{2\alpha_y \Delta t}{\Delta y^2}\left(\cos(k\Delta y) - 1\right)}$$

For all $j\Delta x$ and $k\Delta y$, $\cos(j\Delta x)$ and $\cos(k\Delta y) \le 1$

So $\cos(j\Delta x) - 1 \le 0$
$\cos(k\Delta y) - 1 \le 0$.

since $\frac{\alpha_x \Delta t}{\Delta x}$ and $\frac{\alpha_y \Delta t}{\Delta y} > 0$, $\frac{2\alpha_x \Delta t}{\Delta x^2}\left(\cos(j\Delta x) - 1\right)$ and $\frac{2\alpha_y \Delta t}{\Delta y^2}\left(\cos(k\Delta y) - 1\right)$ are both negative.

So for all choices of $\alpha_x, \alpha_y, \Delta t, \Delta x, \Delta y$, denominator is positive and growth rate is positive and smaller than one. (or equal to)

$$\boxed{0 < \frac{G^{n+1}}{G^n} \le 1}$$

BECS 3D

$$\frac{T^{n+1}_{a,b,c}-T^{n}_{a,b,c}}{\Delta t} = \frac{\alpha_x}{\Delta x^2}\left(T^{n+1}_{a-1,b,c}-2T^{n+1}_{a,b,c}+T^{n+1}_{a+1,b,c}\right)+\frac{\alpha_y}{\Delta y^2}\left(T^{n+1}_{a,b-1,c}-2T^{n+1}_{a,b,c}+T^{n+1}_{a,b+1}\right.$$

$$+\frac{\alpha_z}{\Delta z^2}\left(T^{n+1}_{a,b,c-1}-2T^{n+1}_{a,b,c}+T_{a,b,c+1}\right)$$

Assume $T^n_{a,b,c} = G^n e^{ij\Delta x a} e^{ik\Delta y b} e^{il\Delta z c}$

$$\frac{G^{n+1} e^{ij\Delta x a} e^{ik\Delta y b} e^{il\Delta z c} - G^n e^{ij\Delta x a} e^{ik\Delta y b} e^{il\Delta z c}}{\Delta t} = G^{n+1} e^{ij\Delta x a} e^{ik\Delta y b} e^{il\Delta z c}\Big[$$

$$\frac{\alpha_x}{\Delta x^2}\left(e^{-ij\Delta x}-2+e^{ij\Delta x}\right)+\frac{\alpha_y}{\Delta x^2}\left(e^{-ik\Delta y}-2+e^{ik\Delta y}\right)+\frac{\alpha_z}{\Delta z^2}\left(e^{-il\Delta z}-2+e^{il\Delta z}\right)\Big]$$

$$\frac{G^{n+1}}{G^n} = \frac{1}{1-\frac{\alpha_x\Delta t}{\Delta x^2}\left(e^{-ij\Delta x}-2+e^{ij\Delta x}\right)-\frac{\alpha_y\Delta t}{\Delta y^2}\left(e^{-ik\Delta y}-2+e^{ik\Delta y}\right)-\frac{\alpha_z\Delta t}{\Delta z^2}\left(e^{-il\Delta z}-2+e^{il\Delta z}\right)}$$

By similar reasoning as BECS 2D, denominator always ≥ 1.

$$\boxed{0 < \frac{G^{n+1}}{G^n} \leq 1}$$

CN 1D

$$\frac{T_a^{n+1} - T_a^n}{\Delta t} = \frac{\alpha}{2\Delta x^2}\left[\left(T_{a+1}^{n+1} - 2T_a^{n+1} + T_{a-1}^{n+1}\right) + \left(T_{a+1}^n - 2T_a^n + T_{a-1}^n\right)\right]$$

Assume $T_a^n = G^n e^{ik\Delta x a}$

$\cancel{G_a^{n+1} + G}$

$$G_a^{n+1} \cancel{e^{ik\Delta x a}} - G^n \cancel{e^{ik\Delta x a}} = \frac{\alpha \Delta t}{2\Delta x^2}\left(G^{n+1} \cancel{e^{ik\Delta x a}}\right)\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$
$$+ \frac{\alpha \Delta t}{2\Delta x^2}\left(G^n \cancel{e^{ik\Delta x a}}\right)\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$

$$G^{n+1}\left[1 - \frac{\alpha\Delta t}{2\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\right] = G^n\left[1 + \frac{\alpha\Delta t}{2\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\right]$$

$$\frac{G^{n+1}}{G^n} = \frac{1 + \frac{\alpha\Delta t}{2\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)}{1 - \frac{\alpha\Delta t}{2\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)} = \frac{1 + \frac{\alpha\Delta t}{2\Delta x^2}\left(2\cos(ka\Delta x) - 2\right)}{1 - \frac{\alpha\Delta t}{2\Delta x^2}\left(2\cos(ka\Delta x) - 2\right)}$$

$$\frac{G^{n+1}}{G^n} = \frac{1 + C\left(\cos(ka\Delta x) - 1\right)}{1 - C\left(\cos(ka\Delta x) - 1\right)} \quad \text{where } C = \frac{\alpha\Delta t}{\Delta x^2}$$

since $C > 0$ and $\cos(ka\Delta x) \leq 1$, denominator > numerator.

$\left|\frac{G^{n+1}}{G^n}\right| < 1$, so CN 1D is unconditionally stable.

CN 2D

$$\frac{T^{n+1}_{a,b}-T^{n}_{a,b}}{\Delta t} = \frac{\alpha_x}{2\Delta x^2}\left[\left(T^{n+1}_{a+1,b}-2T^{n+1}_{a,b}+T^{n+1}_{\cancel{a+1,b}\;a-1,b}\right)+\left(T^{n}_{a+1,b}-2T^{n}_{a,b}+T^{n}_{\cancel{a+1,b}\;a-1,b}\right)\right]$$

$$+\frac{\alpha_y}{2\Delta y^2}\left[\left(T^{n+1}_{a,b+1}-2T^{n+1}_{a,b}+T^{n+1}_{a,b-1}\right)+\left(T^{n}_{a,b+1}-2T^{n}_{a,b}+T^{n}_{a,b-1}\right)\right]$$

Assume $T^{n}_{a,b} = G^{n} e^{ij\Delta x a} e^{ik\Delta y b}$

$$G^{n+1}e^{ij\Delta x a}e^{ik\Delta y b} - G^{n}e^{ij\Delta x a}e^{ik\Delta y b} = \frac{\alpha_x \Delta t}{2\Delta x^2}\left(G^{n+1}e^{ij\Delta x a}e^{ik\Delta y b}\right)\left(e^{ij\Delta x}-2+e^{-ij\Delta x}\right)$$

$$+\frac{\alpha_x \Delta t}{2\Delta x^2}\left(G^{n}e^{ij\Delta x a}e^{ik\Delta y b}\right)\left(e^{ij\Delta x}-2+e^{-ij\Delta x}\right)$$

$$+\frac{\alpha_y \Delta t}{2\Delta y^2}\left(G^{n+1}e^{ij\Delta x a}e^{ik\Delta y b}\right)\left(e^{ik\Delta y}-2+e^{-ik\Delta y}\right)$$

$$+\frac{\alpha_y \Delta t}{2\Delta y^2}\left(G^{n}e^{ij\Delta x a}e^{ik\Delta y b}\right)\left(e^{ik\Delta y}-2+e^{-ik}\right.$$

$$G^{n+1}\left[1-\frac{\alpha_x \Delta t}{2\Delta x^2}\left(e^{ij\Delta x}-2+e^{-ij\Delta x}\right)-\frac{\alpha_y \Delta t}{\Delta y^2}\left(e^{ik\Delta y}-2+e^{-ik\Delta y}\right)\right]$$

$$= G^{n}\left[1+\frac{\alpha_x \Delta t}{2\Delta x^2}\left(e^{ij\Delta x}-2+e^{-ij\Delta x}\right)+\frac{\alpha_y \Delta t}{\Delta y^2}\left(e^{ik\Delta y}-2+e^{-ik\Delta y}\right)\right]$$

$$\frac{G^{n+1}}{G^{n}} = \frac{1+\frac{\alpha_x \Delta t}{2\Delta x^2}\left(e^{ij\Delta x}-2+e^{-ij\Delta x}\right)+\frac{\alpha_y \Delta t}{\Delta y^2}\left(e^{ik\Delta y}-2+e^{-ik\Delta y}\right)}{1-\frac{\alpha_x \Delta t}{2\Delta x^2}\left(e^{ij\Delta x}-2+e^{-ij\Delta x}\right)-\frac{\alpha_y \Delta t}{\Delta y^2}\left(e^{ik\Delta y}-2+e^{-ik\Delta y}\right)}$$

$$= \frac{1+C_x\left(\cos(ja\Delta x)-1\right)+C_y\left(\cos(kb\Delta y)-1\right)}{1-C_x\left(\cos(ja\Delta x)-1\right)-C_y\left(\cos(kb\Delta y)-1\right)}, \quad C_x=\frac{\alpha_x \Delta t}{\Delta x^2}, \quad C_y=\frac{\alpha_y \Delta t}{\Delta y^2}$$

CN 2D (continued)

similar to reasoning for CN 1D, denominator > numerator.

so $\left|\frac{G^{n+1}}{G^n}\right| < 1$, so CN 2D is unconditionally stable

CN 3D

$$\frac{T^{n+1}_{a,b,c} - T^{n}_{a,b,c}}{\Delta t} = \frac{\alpha_x}{2\Delta x^2}\left[\left(T^{n+1}_{a+1,b,c} - 2T^{n+1}_{a,b,c} + T^{n+1}_{a-1,b,c}\right) + \left(T^{n}_{a+1,b,c} - 2T^{n}_{a,b,c} + T^{n}_{a-1,b,c}\right)\right]$$
$$+ \frac{\alpha_y}{2\Delta y^2}\left[\left(T^{n+1}_{a,b+1,c} - 2T^{n+1}_{a,b,c} + T^{n+1}_{a,b-1,c}\right) + \left(T^{n}_{a,b+1,c} - 2T^{n}_{a,b,c} + T^{n}_{a,b-1,c}\right)\right]$$
$$+ \frac{\alpha_z}{2\Delta z^2}\left[\left(T^{n+1}_{a,b,c+1} - 2T^{n+1}_{a,b,c} + T^{n+1}_{a,b,c-1}\right) + \left(T^{n}_{a,b,c+1} - 2T^{n}_{a,b,c} + T^{n}_{a,b,c-1}\right)\right]$$

Assume $T^n_{a,b,c} = G^n e^{ij\Delta x a} e^{ik\Delta y b} e^{il\Delta z c}$

... similar algebra as CN 2D ...

$$\frac{G^{n+1}}{G^n} = \frac{1 + \frac{\alpha_x \Delta t}{2\Delta x^2}\left(e^{ij\Delta x} - 2 + e^{-ij\Delta x}\right) + \frac{\alpha_y \Delta t}{2\Delta y^2}\left(e^{ik\Delta y} - 2 + e^{-ik\Delta y}\right) + \frac{\alpha_z \Delta t}{\Delta z^2}\left(e^{il\Delta z} - 2 + e^{-il\Delta z}\right)}{1 - \frac{\alpha_x \Delta t}{2\Delta x^2}\left(e^{ij\Delta x} - 2 + e^{-ij\Delta x}\right) - \frac{\alpha_y \Delta t}{2\Delta y^2}\left(e^{ik\Delta y} - 2 + e^{-ik\Delta y}\right) - \frac{\alpha_z \Delta t}{\Delta z^2}\left(e^{il\Delta z} - 2 + e^{-il\Delta z}\right)}$$

$$\frac{G^{n+1}}{G^n} = \frac{1 + C_x(\cos(ja\Delta x) - 1) + C_y(\cos(kb\Delta y) - 1) + C_z(\cos(lc\Delta z) - 1)}{1 - C_x(\cos(ja\Delta x) - 1) - C_y(\cos(kb\Delta y) - 1) - C_z(\cos(lc\Delta z) - 1)}$$

where $C_x = \frac{\alpha_x \Delta t}{\Delta x^2}$, $C_y = \frac{\alpha_y \Delta t}{\Delta y^2}$, $C_z = \frac{\alpha_z \Delta t}{\Delta z^2}$

since denominator > numerator, unconditionally stable since $\left|\frac{G^{n+1}}{G^n}\right| < 1$