Homogeneous linear dynamic system

$$\dot{x}(t) = \frac{dx}{dt} = F(t)x(t)$$

solution to homogeneous linear dynamic system:

$$x(t) = \Phi(t)x(0)$$

Nonhomogeneous linear dynamic system

$$\dot{x}(t) = F(t)x(t) + C(t)u(t)$$

Properties of state transition matrix (STM):

1. $\Phi^{-1}(t)x(t) = x(0), \qquad \Phi(t)\Phi^{-1}(t)x(t) = x(t), \qquad \Phi(\tau, t) = \Phi(\tau)\Phi^{-1}(t), \qquad \Phi(\tau, 0) = \Phi(\tau)$
2. $\frac{d}{dt}(\Phi(t)) = F(t)\Phi(t), \qquad \frac{d}{dt}(\Phi^{-1}(t)) = -F(t)\Phi^{-1}(t)$
3. $\Phi(\tau, \tau) = \Phi(0) = I$
4. $\Phi^{-1}(\tau, t) = \Phi(t, \tau)$
5. $\Phi(\tau, \sigma)\Phi(\sigma, t) = \Phi(\tau, t)$
6. $(\partial/\partial\tau)\Phi(\tau, t) = F(\tau)\Phi(\tau, t)$
7. $(\partial/\partial t)\Phi(\tau, t) = -\Phi(\tau, t)F(t)$

Solution to nonhomogeneous linear dynamic system

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)C(\tau)u(\tau)\, d\tau = \Phi(t)\Phi^{-1}(t_0)x(t_0) + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(\tau)C(\tau)u(\tau)\, d\tau \qquad (\text{note } t_0 \text{ does not mean } t_0 = 0)$$

Proof:

differentiate $x(t)$ and use properties of STM:

$$\dot{x}(t) = \frac{d}{dt}(x(t)) = \frac{d}{dt}(\Phi(t))\,\Phi^{-1}(t_0)x(t_0) + \frac{d}{dt}\left(\Phi(t)\int_{t_0}^{t} \Phi^{-1}(\tau)C(\tau)u(\tau)\, d\tau\right)$$

$$= F(t)\Phi(t)\Phi^{-1}(t_0)x(t_0) + \frac{d}{dt}(\Phi(t))\int_{t_0}^{t} \Phi^{-1}(\tau)C(\tau)u(\tau)\, d\tau + \Phi(t)\frac{d}{dt}\left(\int_{t_0}^{t} \Phi^{-1}(\tau)C(\tau)u(\tau)\, d\tau\right)$$

$$= F(t)x(t) + F(t)\Phi(t)\int_{t_0}^{t}\Phi^{-1}(\tau)C(\tau)u(\tau)\,d\tau + \Phi(t)\Phi^{-1}(t)C(t)u(t)$$

$$= F(t)x(t) + C(t)u(t) + F(t)\Phi(t)\underbrace{\int_{t_0}^{t}\Phi^{-1}(\tau)C(\tau)u(\tau)\,d\tau}_{=0} = F(t)x(t) + C(t)u(t)$$

The final term is zero because it follows from:

$$\dot{x}(t) - F(t)x(t) = C(t)u(t)$$
$$\Phi^{-1}(t)\dot{x}(t) - F(t)\Phi^{-1}(t)x(t) = \Phi^{-1}(t)C(t)u(t)$$
$$\frac{d}{dt}\big(\Phi^{-1}(t)x(t)\big) = \Phi^{-1}(t)C(t)u(t)$$
$$\int_{t_0}^{t}\frac{d}{d\tau}\big(\Phi^{-1}(\tau)x(\tau)\big)\;d\tau = \int_{t_0}^{t}\Phi^{-1}(\tau)C(\tau)u(\tau)\,d\tau$$
$$\underbrace{\Phi^{-1}(t)x(t)}_{=x(0)} - \underbrace{\Phi^{-1}(t_0)x(t_0)}_{=x(0)} = \int_{t_0}^{t}\Phi^{-1}(\tau)C(\tau)u(\tau)\,d\tau$$

so

$$\int_{t_0}^{t}\Phi^{-1}(\tau)C(\tau)u(\tau)\,d\tau = x(0) - x(0) = 0$$