AERO 424 Homework 2

By Jonathan Jermstad (9/10/2024)

```
import sympy as sy
from sympy.physics.vector import ReferenceFrame,dynamicsymbols
from sympy.physics.vector import Point

import IPython
from IPython.display import display

import matplotlib.pyplot as plt

import numpy as np

from scipy.integrate import odeint

def displayH(a1,a2='', a3='', a4='', a5='', a6='', a7='',):
    latex_a1 = sy.latex(a1)
    latex_a2 = sy.latex(a2)
    latex_a3 = sy.latex(a3)
    latex_a4 = sy.latex(a4)
    latex_a5 = sy.latex(a5)
    latex_a6 = sy.latex(a6)
    latex_a7 = sy.latex(a7)
    display( IPython.core.display.Math(latex_a1 + latex_a2 + latex_a3 + latex_a4 + latex_a
```

1) Particle dynamics

Consider a particle of mass m moving in a horizontal plane as shown in the figure. The particle is attached to a linear spring with spring constant k and unstretched length l. Assume no gravity. Perform the following tasks:

1. Derive the equations of motion for all degrees of freedom of the system.

Solution:

$$\vec{r} = r\hat{e}_r$$

$$\vec{v} = \frac{^N d\vec{r}}{dt} = \frac{^E d\vec{r}}{dt} + \omega \times \vec{r} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{a} = \frac{^N d\vec{v}}{dt} = \frac{^E d\vec{v}}{dt} + \omega \times \vec{v} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e}_\theta$$

$$\vec{F} = -k(r-l)$$

Equations of Motion:

$$\begin{aligned}
\hat{e}_r &: m(\ddot{r} - r\dot{\theta}^2) = -k(r-l) \\
\hat{e}_\theta &: m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \\
\hat{e}_z &: 0 = 0
\end{aligned}$$

```
theta = dynamicsymbols("theta")
R = dynamicsymbols("r")

m = sy.Symbol("m")
k = sy.Symbol("k")
l = sy.Symbol("l")
t = sy.Symbol("t")

# Define inrtl reference frame (sun)
N = ReferenceFrame('N')
O = Point('O')
# Define intermediate reference frame (earth) rotated about N.z by phi
E = N.orientnew('E','Axis',[theta,N.z])
```

```
r = O.locatenew("pos_moon",R*E.x).pos_from(O)
v = sy.diff(r,t,N)
a = sy.diff(v,t,N)

F = -k*(R-l)*E.x

displayH(sy.Symbol(r"\vec{r} ="),r)
displayH(sy.Symbol(r"\vec{v} ="),v.express(E).simplify())
displayH(sy.Symbol(r"\vec{a} ="),a.express(E).simplify())

displayH(sy.Symbol(r"\vec{F} ="),F.express(E).simplify())
displayH(sy.Matrix([sy.Symbol(r"\vec{e}_{r}"),sy.Symbol(r"\vec{e}_{\theta}"),sy.Symbol(r"\
```

$\vec{r} = r(t)\hat{\mathbf{e}}_{\mathbf{x}}$

$\vec{v} = \frac{d}{dt}r(t)\hat{\mathbf{e}}_{\mathbf{x}} + r(t)\frac{d}{dt}\theta(t)\hat{\mathbf{e}}_{\mathbf{y}}$

$\vec{a} = (-r(t)\left(\frac{d}{dt}\theta(t)\right)^2 + \frac{d^2}{dt^2}r(t))\hat{\mathbf{e}}_{\mathbf{x}} + (r(t)\frac{d^2}{dt^2}\theta(t) + 2\frac{d}{dt}r(t)\frac{d}{dt}\theta(t))\hat{\mathbf{e}}_{\mathbf{y}}$

$\vec{F} = k\left(l - r(t)\right)\hat{\mathbf{e}}_{\mathbf{x}}$

$$\begin{bmatrix}\vec{e}_r\\ \vec{e}_\theta\\ \vec{e}_z\end{bmatrix} \rightarrow \begin{bmatrix}-r(t)\left(\frac{d}{dt}\theta(t)\right)^2 + \frac{d^2}{dt^2}r(t)\\ r(t)\frac{d^2}{dt^2}\theta(t) + 2\frac{d}{dt}r(t)\frac{d}{dt}\theta(t)\\ 0\end{bmatrix} = \begin{bmatrix}k\left(l - r(t)\right)\\ 0\\ 0\end{bmatrix}$$

2. What is the angular momentum of the mass about point O. Is it conserved? Prove this mathematically.

Solution:

$$\vec{H}_O = \vec{r} \times m\vec{r} = mr^2\dot{\theta}\hat{e}_z$$

$$\dot{\vec{H}}_O = \frac{d\vec{H}_O}{dt} = mr\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e}_z$$

Using the Equations of Motion ($\hat{e}_\theta$), we can simplify this expression to a trivial expression.

$$\dot{\vec{H}}_O = 0$$

With the rate of change of angular momentum being zero, we know that angular momentum is conserved.

```
omega = E.ang_vel_in(N)

H_O = r.cross(m*v)
Hprime_O = sy.diff(H_O,t,N)

displayH(sy.Symbol(r"\vec{H}_{O} ="),H_O.express(E).simplify())
displayH(sy.Symbol(r"\vec{\dot{H}}_{O} ="),Hprime_O.express(E).simplify())
```

$\vec{H}_O = mr^2(t)\frac{d}{dt}\theta(t)\hat{\mathbf{e}}_\mathbf{z}$

$\dot{\vec{H}}_O = m\left(r(t)\frac{d^2}{dt^2}\theta(t) + 2\frac{d}{dt}r(t)\frac{d}{dt}\theta(t)\right)r(t)\hat{\mathbf{e}}_\mathbf{z}$

3. Is the total energy of the system conserved? Answer "yes" or "no" and explain your reasoning.

Solution:

Yes! The total energy of the system is conserved because the only force acting on this system is a spring force, which is a conservative force. As the system moves, the system's energy is traded between Kinetic Energy and Spring Potential Energy.

2) Moment of Inertia.

The moment of inertia matrix in the current coordinate frame is,

$$I = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

Perform the following tasks,

1. Find the Directions Cosines Matrix (DCM) [C] that will transform the current coordinate frame to a new frame F that diagonalizes the inertia matrix.

Solution:

$$v = eigenvectors(I)\,[C] = \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix}$$

```
I = np.array([[3,1,1],
              [1,5,2],
              [1,2,4]])

E,C = np.linalg.eig(I)
displayH(sy.Symbol("C ="),sy.Matrix(C.tolist()))
```

$$C = \begin{bmatrix} -0.327985277605682 & -0.736976229099577 & -0.591009048506106 \\ -0.736976229099578 & 0.591009048506104 & -0.32798527760568 \\ -0.591009048506104 & -0.327985277605683 & 0.736976229099577 \end{bmatrix}$$

2. Perform the frame transformation of matrix I to frame F using matrix C. What are the principal moments of inertia?

Solution:

$$I^* = [C][I][C]^T$$

```
principal_inertia = C@I@C.T
displayH(sy.Symbol("I^{*} ="),sy.Matrix(principal_inertia.tolist()))
```

```
for idx,z in enumerate(["xx","yy","zz"]):
    displayH(sy.Symbol(f"I_{z} ="),E[idx])
```

$$I^* = \begin{bmatrix} 7.04891733952232 & -9.32587340685131 \cdot 10^{-15} & 9.32587340685131 \cdot 10^{-15} \\ -8.88178419700125 \cdot 10^{-15} & 2.64310413210779 & 1.4432899320127 \cdot 10^{-15} \\ 8.65973959207622 \cdot 10^{-15} & 1.55431223447522 \cdot 10^{-15} & 2.3079785283699 \end{bmatrix}$$

$I_{xx=}$7.0489173395223

$I_{yy=}$2.64310413210779

$I_{zz=}$2.3079785283699

3) Numerical integration of Kepler's Two-body Equation

The Kepler's two-body equation is given as follows,

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$

where $\vec{r} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ is the inertial position vector of the orbiting body and μ is the gravitational parameter of the primary body.

Consider a satellite orbiting the Earth ($\mu = 398600.4418 \frac{km^3}{s^2}$), with inertial position and velocity vector at time t_0 given by,

$$\vec{r}_0 = \begin{bmatrix} 5.411843 \\ 3.431758 \\ 0.969722 \end{bmatrix} [10^3 km], \dot{\vec{r}}_0 = \begin{bmatrix} 4.4146087 \\ 5.62748695 \\ 4.760006 \end{bmatrix} [km/s]$$

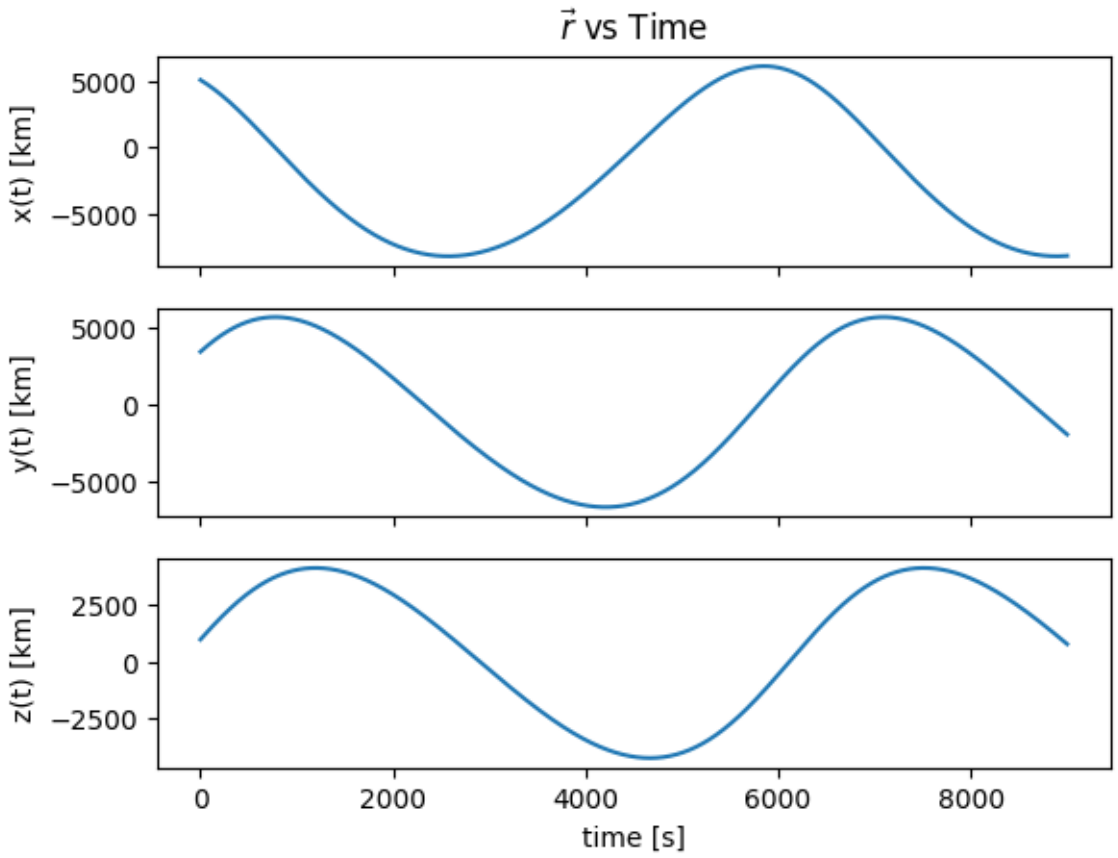
Using MATLAB's "ode45" function (or an equivalent numerical integration tool from other software), numerically integrate the two-body equation over a period of 2.5 hours. Plot the following quantities,

1. The time histories of the position vector components, i.e., plot X(t), Y (t), and Z(t) as functions of time.

Solution:

```
r0 = np.array([5.11843,
               3.431758,
               0.969722])*10**3
v0 = np.array([-4.4146087,
               5.62748695,
               4.760006])

mu = 398600.4418

def EOM(x,t):
    r = x[:3]
    a = -mu * r / np.linalg.norm( r ) ** 3
    return np.array([x[3],x[4],x[5],a[0],a[1],a[2]])
```

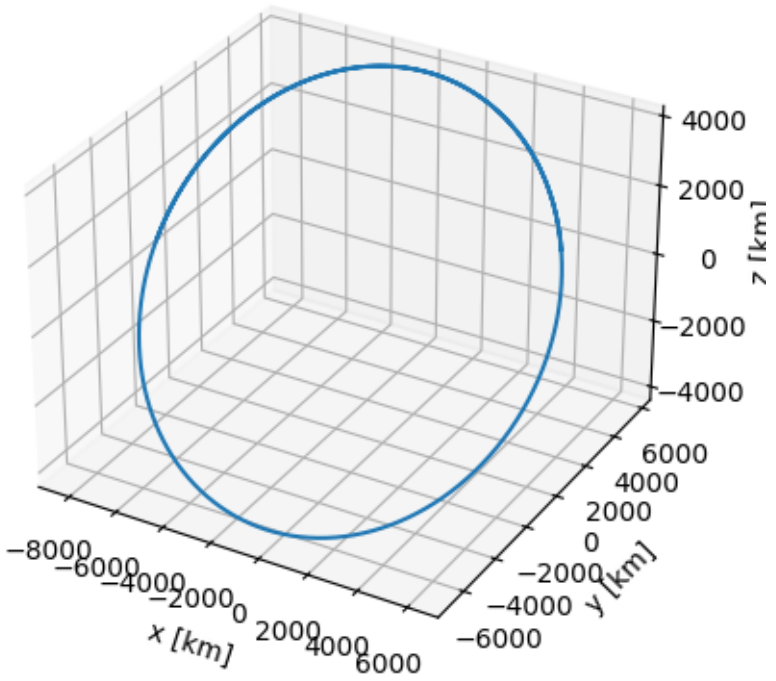
```
x0 = np.hstack([r0,v0])
times = np.linspace(0,3600*2.5,3601)
tol = 10**-12

x = odeint(EOM,x0,times,rtol=tol,atol=tol)
r = x[:,:3]
v = x[:,3:]

fig,axes = plt.subplots(3,sharex=True)
axes[0].set_title(r"$\vec{r}$ vs Time")
for idx in range(3):
    axes[idx].plot(times,r[:,idx])
axes[0].set_ylabel("x(t) [km]")
axes[1].set_ylabel("y(t) [km]")
axes[2].set_ylabel("z(t) [km]")
axes[2].set_xlabel("time [s]")
plt.show()
```



2. The 3D orbit of the satellite, i.e., plot X(t), Y (t), and Z(t) in the inertial frame.

Solution:

```
plt.figure()
ax = plt.axes(projection="3d")
ax.plot(r[:,0],r[:,1],r[:,2])
ax.set_xlabel("x [km]")
ax.set_ylabel("y [km]")
ax.set_zlabel("z [km]")
plt.show()
```


4000
2000
0
−2000
−4000
z [km]
6000
4000
2000
0
−2000
−4000
−6000
y [km]
−8000
−6000
−4000
−2000
0
2000
4000
6000
x [km]