# Social Interactions in Multilayered Observational Networks

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**Abstract**

This paper proposes a new method to identify and estimate the parameters of an extension of a linear model of peer effects where individuals form different types of social and professional connections that can affect their outcomes. A stylized model provides a theoretical framework for the peer effects linear specification and the main identifying assumptions, which are used to provide identification results in a setting that allows all layers in the multilayer network to not be strictly exogenous. I show that identifying heterogeneous network effects is possible under the assumption that the dependence between individuals in the population is characterized by a stochastic process where dependence vanishes in the network space. I offer a novel multilayer measure of distance that provides a source of exogenous variation used to form identifying moment conditions. I propose a Generalized Method of Moments estimator that is consistent and asymptotically normal at the standard rate, for which I characterize the asymptotic variance-covariance matrix that considers the intrinsic network dependence among individuals. A Monte Carlo experiment confirms the desirable finite properties of the proposed estimator, and an empirical application finds positive and significant peer effects in citations from a multilayer network of professional connections.

**Keywords**: Multilayer Networks, $\psi$-dependence, Peer Effects, Network Endogeneity, GMM.
**JEL Classifications**: C13, C31, C51

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# 1 Introduction

Economic models where social interactions influence individual behavior are becoming increasingly popular in the literature. The so-called Linear-in-Means (LiM) model (Manski, 1993) is the most widely used tool in applied work to estimate the effects of peers' behaviors and characteristics on individual outcomes (see Section 3.1 in de Paula, 2017, pp. 275-289). The challenges to identify the parameters of the LiM model are widely recognized in the econometrics of networks literature. In particular, the outstanding identification issue in the field is how to address the endogenous network formation problem (Jackson et al., 2017). Recent methods designed to solve this issue are built under the standard assumption that only one social or professional network exists and they generally require explicit network formation models, see, e.g., Johnsson and Moon (2019) for estimation of network effects and Graham (2017, 2020) for estimation of network formation models. However, empirical work has shown that different types of connections such as classmates, neighbors, friends, or coauthors can create peer effects (Miguel and Kremer, 2004; Conley and Udry, 2010; Bursztyn et al., 2014; Ductor et al., 2014; Zacchia, 2019; De Giorgi et al., 2020).

This paper proposes a novel method to employ multilayer network data to identify and estimate the parameters of an extension of the LiM model. The proposed approach can handle both the reflection problem and the issue of correlated effects while at the same time allowing for different types of social connections to generate network effects. The standard LiM model assumes full observability of one network whose potential impacts on individuals' outcomes are summarized by the peer and contextual effects. These parameters capture the effects of the outcomes and characteristics of an individual's peers on her own outcome. I propose an extension where I assume full observability of $M$ different networks (also called layers here) that can produce $M$ potentially different peer and contextual effects. I call this model the *Multilayer Linear-in-Means model* (MLiM hereafter). This generalization is meaningful because it nests standard models relevant to applied work and relaxes the assumption of *monolayer* network effects. I provide a game theoretical stylized model of social interactions in the spirit of Blume et al. (2015) that offers a theoretical foundation for the MLiM model and the main identifying assumptions derived from the multilayer network formation model.

As originally proposed by Kelejian and Prucha (1998), I solve the reflection problem in the MLiM by using the exclusions restrictions generated by a multilayer network data structure that is not fully connected. I provide a set of empirically testable conditions on the architecture of the multilayer network that guarantees the existence of enough exclusion restrictions to identify all the parameters of interest. In particular, I show that in the multilayer network context, there is a linear independence condition between products of different layers' adjacency matrices that resembles the conditions in Bramoullé et al. (2009) but extends the idea to accommodate the existence of multiple types of connections (see e.g., Manta et al., 2022). As in previous literature, the existence exclusion restrictions has a natural interpretation in terms of instrumental variables. Intuitively, powers and products of the layers adjacency matrices can be used to form relevant (though not necessarily valid) instruments for the peer effects regressors creating the reflection issue. The matrix resulting from performing iterative multiplications of a sequence of $r$ different layers' adjacency matrices contains in its $ij$th position the number of paths of length $r$ connecting individuals $i$ and $j$ formed by a succession of edges whose types are given by the layers types in the multiplication sequence. Therefore, the fact that products of adjacency matrices, which contain multilayer paths with different edge types, can be used to form relevant instruments speaks about the importance of considering edge-type changes when constructing identifying moment conditions.

However, the possibility that the multiple layers in the MLiM model are not exogenous makes identification more complex, as the network structure cannot be used directly as an instrument. Instead of imposing explicit structural restrictions on the multilayer network formation process, I assume that a $\psi-$dependent stochastic process characterizes the dependence of individuals in the population (Doukhan and Louhichi, 1999; Kojevnikov et al., 2020). Based on the $\psi-$dependence characterization, I impose the main identifying assumption: Weak Neighborhood Dependence (WND). This assumption guarantees that individuals' dependence dies out when their distance in the multilayer network space increases. Imposing the WND assumption on any network that happens with positive probability allows for the construction of moment conditions to separately identify the potential $M$ different peer and contextual effects after controlling for the presence of correlated effects. The social interactions model I propose provides a set of conditions under which the WND assumption holds. In particular, I offer an information regime based on the existence of local informational clusters, which, under some additional restrictions on the dependence patterns of individual characteristics across clusters, avoids the potential systematic correlation between the observed and unobserved characteristics of individuals far apart in the network space. The proposed network-generating process in the social interactions model is not the only model that can rationalize the WND assumption. Still, it serves as an example to motivate the plausibility of the high-level assumption.

This article proposes an innovation to the idea of $\psi$-dependence to accommodate multilayer network data structures by providing a novel *multilayer network measure of distance* that considers both the standard monolayer geodesic distance and the number of edge-type changes. The idea of a measure of distance that incorporates the complete information provided by the multilayer network data is at the core of my identification, estimation, and inference strategies. It allows taking advantage of local multilayer network structures nearly independent from each other to form moment conditions and incorporate network dependence for inference. In the multilayer network context, edge types refer to the nature of the social or professional relationships connecting two nodes. I interpret the edge-type changes as reducing the dependence between any two nodes faster than standard monolayer paths. Empirical articles have used a similar idea to argue that, for instance, the correlation of individual $i$'s characteristics with her co-worker spouse's characteristics is lower than that between her and her co-worker's co-worker (De Giorgi et al., 2020). Nicoletti et al. (2018) and Nicoletti and Rabe (2019) use similar arguments in the context of a multilayer network composed by friends and neighbors connections.

In addition to the fact that paths containing edge-type changes appear as natural candidates to construct identifying moment conditions given the MLiM structure, there are other reasons to justify the inclusion of edge-type changes when defining the new multilayer network measure of distance. This paper shows that the aggregate moment condition function I propose for identification is a weighted combination of individual moment conditions with heterogeneous identifying power. The heterogeneity comes from different multilayer network structures inducing different multilayer network distances for different individuals. Therefore, for any multilayer network realization, it may be possible to form moment conditions for some individuals but not others. To the extent that some observed multilayer network structures have short average shortest path lengths between nodes but still contain edge-type changes, including the types of edges in the distance measure becomes increasingly relevant. For the empirical application data, I indeed find that by considering edge-type changes, I can significantly increase the number of individuals for whom it is possible to find moment conditions (increasing the effective sample size by 33% and 17% of the total sample size for some examples). Edge-type changes are particularly relevant for special cases of multilayer network data, such as the non-overlapping networks used by De Giorgi et al. (2020). For the non-overlapping network case, not considering edge-type changes would ignore the fact that a path containing a change in edge types necessarily has to connect two clusters that would otherwise be disconnected, which is critical for identification.

Current methods to identify the LiM model's parameters in the presence of correlated effects generally require estimating a monolayer network formation model. This paper abstracts away from network formation estimation and is agnostic about the underlying process generating the network up to the WND assumption. Instead, it uses changes in the characteristics of individuals who are sufficiently far in the multilayer network space as a source of exogenous variation to construct moment conditions for identification and estimation. To the best of my knowledge, this is the first work to formalize the use of multilayer network data structures to identify the network effects parameters of an extension of the linear-in-means model with flexible assumptions on the network formation process.

Based on the moment conditions used for identification, I propose a Generalized Method of Moments (GMM) estimation procedure that is consistent and asymptotically normal. The sample consists of $n$ individuals drawn from an arbitrarily large population characterized by a joint distribution of the errors, the multilayer network, and the regressors. Limiting distributions are studied when $n \to \infty$. The linearity assumption of the MLiM model guarantees that the resulting GMM estimator has a closed-form solution. The asymptotic results show that the variance-covariance matrix of the GMM estimator differs from the standard sandwich formulas because it considers the network dependence between individuals and the heterogeneity in the identifying information they provide in the population. The derived asymptotic variance-covariance matrix formally explains the anecdotal finding that Monte Carlo variability increases with network density (see, e.g., Bramoullé et al., 2009). Intuitively, higher network density reduces the possibility of forming moment conditions which reduces the identification power in the population and increases the variance of the estimator in the sample. These results are new and relevant for correct inference in empirical work estimating network effects. To incorporate the presence of network dependence, I use a HAC estimator of the variance-covariance matrix in the same spirit as Newey and West (1987), Conley (1999) and Kojevnikov et al. (2020). A Monte Carlo experiment based on an exemplary network formation model confirms the desirable finite properties of the proposed estimator when the assumption that individuals' dependence decreases with their distance in the multilayer network space holds.

To show the importance of taking into account network endogeneity when dealing with observational data, I present an empirical application to publication outcomes in Economics. The use of web scraping and existing data on authors' research fields, education, and employment history allows the creation of four types of professional ties among scholars: co-authorship, alumni, advisors, and colleagues connections. The multilayer network data is used to uncover positive and significant peer effects in citations from the co-authorship network among articles published by these scholars. However, peer effects from any of the other types of networks included in the estimating model are not found. This result is interpreted as emphasizing the importance of a network that guarantees a direct communication channel between authors instead of other professional networks that may generate fewer interpersonal interactions. The empirical application also shows that the OLS estimator can be severely biased when trying to estimate network effects without considering the potential network endogeneity of the layers. Positive results of research teams that are gender diverse on the quality of a paper measured in terms of citation outcomes are also found, after controlling for other articles' characteristics such as number of pages, number of bibliographic references, and various network fixed effects.

**Related Literature**

This paper provides new insights on current identification results in the literature. The multilayer network data structure is general enough to cover cases such as the non-overlapping network structure used in De Giorgi et al. (2020) and the multiplex structure in Chan et al. (2022). The proposed framework collapses to the reduced form version of Zacchia's (2019) model in the monolayer case when the researcher is willing to assume that the endogenous peer effect coefficient is zero. In that sense, this paper presents a general theory of using multilayer network data to estimate network effects in linear models. The commonality between the articles mentioned above emerges from the idea of keeping the network formation process unspecified. Recent papers augment the standard linear-in-means model to include specific generating mechanisms for the network formation process, see e.g., Goldsmith-Pinkham and Imbens (2013), Qu and Lee (2015) and Johnsson and Moon (2019). In general, these network formation models are difficult to estimate and involve additional assumptions such as the absence of strategic interactions on individuals' utilities of forming peers. For a complete discussion on the importance of strategic interactions to network formation models' point-identification see Graham (2017), De Paula et al. (2018) and Graham and Pelican (2020).

This work relates with the broad reduced form literature on social interactions. It is most closely associated with observational studies which aim to identify both endogenous peer effects and contextual effects. There have been two recent approaches to deal with endogenous network formation in observational studies: quasi randomization and structural endogeneity (Bramoullé et al., 2020). This article separates from the structural endogeneity approach to a significant extent. For instance, the proposed model does not have to assume a particular source of unobserved heterogeneity, and it does not require network formation estimation. The framework is also distinguished from models found in the literature using natural or artificial experiments to randomize peers because it is explicitly assumed that the networks can form endogenously. Thus, the proposed approach is closer to the literature using random shocks (experimental or quasi-experimental) on the regressors for identification. The reason for this is that the use of individual characteristics of distant nodes as a source of exogenous variation can be interpreted as a partial population experiment, see Moffitt (2001) and Kuhn et al. (2011). Also closely related is the approach proposed by Kuersteiner and Prucha (2020), which extends the standard linear-in-means model to include panel data. The two methods relate in that they present an extension of the standard linear model with additional data to provide new identification and estimation results.

The structure of the paper is as follows. Section 2 introduces the multilayer network data structure and its representation in terms of adjacency matrices. Section 3 introduces the MLiM model and provides conditions under which it has a solution in terms of regressors, errors, and layers. The model section also provides some examples where the multilayer network data structure has been used in empirical work. Section 4 presents conditions for the parameters of the MLiM to be uniquely recovered (point identification) from the joint distribution characterizing the infinite population of interest. Appendix A in the Online Appendix outlines a two-stage game including a formation model and a game of social interactions under which the main identifying assumptions are satisfied. Section 5 describes the proposed GMM estimation procedure, its asymptotic distribution, and how to calculate valid asymptotic standard errors. Section 6 presents a Monte Carlo simulation study, while section 7 presents the empirical application to publication outcomes in economics. Finally, Section 8 provides a summary of the main findings and their implications, while section 9 presents the proofs for the main results. Appendix A, Appendix B, Appendix C, and Appendix D present the model of social interactions, the proofs for intermediate results, proposed algorithms, and data construction and robustness results, respectively.

## 2 Preliminaries

This section introduces the background and notation necessary to develop the framework for identification and estimation. Following Boccaletti et al.'s (2014) notation, a multilayer network is a pair $\mathcal{M} = (\mathcal{G}, \mathcal{C})$ where $\mathcal{G} = \{G_m; m \in \{1, \ldots, M\}\}$ is a set of graphs $G_m = (V_m, E_m)$. For each graph $m$, $V_m$ and $E_m$ represent the set of nodes and edges, respectively. In principle, the graphs in $\mathcal{G}$ are allowed to be directed or undirected, weighted or unweighted but are assumed not to have self cycles. The graphs forming the set $\mathcal{G}$ are known as the multilayer network layers. The set of edges $E_m$ is known as *intralayer connections*. To complete the multilayer network structure's characterization, let $\mathcal{C}$ be the set of interconnections between nodes of different layers $G_m$ and $G_s$ with $m \neq s$ known as *crossed layers* and constructed as $\mathcal{C} = \{E_{m,s} \subseteq V_m \times V_s; m, s \in \{1, \dots, M\}, m \neq s\}$. The elements of each set $E_{m,s}$ are known as the *interlayer connections* of $\mathcal{M}$. To accommodate the multilayer network data structure into the MLiM model, I represent the multilayer network by the adjacency matrix of each layer $G_m$, where each graph $G_m$ is defined on the same set of nodes, i.e., $V_m = V$ for all $m \in \{1, \dots, M\}$. I denote each adjacency matrix by $\mathbf{W}_m = [w_{m;i,j}]$, where $w_{m;i,j} = \rho_{m;i,j}$ if $(v_{m;i}, v_{m;j}) \in E_m$ and 0 otherwise. The constant $\rho_{m;i,j} \in (0, 1]$ represents the weights on the $(i, j)$th connection, which may or may not sum up to one. Using this notation, the next section introduces the MLiM model and provides conditions under which the model has a solution in terms of errors, regressors, and layers' adjacency matrices.

# 3 Multilayer Linear-in-Means (MLiM) Model

The object of study is a MLiM model where agents can create connections in more than one social or professional aspect. The MLiM can be derived as a best response function of a structural game of social interactions with a quadratic utility (Blume et al., 2015). Appendix A provides the theoretical framework motivating the MLiM used as a basis to analyze identification and estimation. The model is composed of a collection $\mathcal{I}_N$ of $N$ economic agents ($N$ is allowed to be arbitrarily large), described by a set of $Q$ characteristics $\mathbf{x}_{N,i}$. Individuals' social interactions can be embedded into a multilayer network $\mathcal{M}_N$ composed by a set $\mathcal{G}_N$ of $M$ graphs, and the choices and characteristics of a person's peers can influence her decision-making process.

In particular, I propose a model of incomplete information where in addition to the observed features, each individual is endowed with a private characteristic not observable to the others. Individuals choose actions to maximize their expected utility featuring strategic complementarities (Calvó-Armengol et al., 2009). In the context of the empirical application, $\mathcal{I}_N$ can be thought a set of scholars trying to maximize their utility of publishing academic articles by choosing their levels of effort, consequently affecting the articles quality. The strategic complementary assumption implies that scholars will allocate more effort to projects if they expect their peers also to put in more effort. The preferences also allow for observable peers' characteristics to affect individual optimal choices. Moreover, the effects of peers behavior and characteristics on a scholar's choices can be heterogeneous. For instance, a coauthor's expected level of effort may significantly impact a scholar's effort more than a colleague's or alumni's expected efforts. From the social interactions model in Appendix A, it is possible to write the optimal choice (outcome) of an individual as

$$y_{N,i} = \alpha^0 + \sum_{m=1}^{M} \sum_{j \neq i} w_{N,m;i,j} y_{N,j} \beta_m^0 + \sum_{m=1}^{M} \sum_{j \neq i} w_{N,m;i,j} \mathbf{x}_{N,j}^\top \boldsymbol{\delta}_m^0 + \mathbf{x}_{N,i}^\top \boldsymbol{\gamma}^0 + \varepsilon_{N,i}, \tag{1}$$

where $j \in \mathcal{I}_N$, $w_{N,m;i,j}$ represents the position $i, j$ in the adjacency matrix of layer $m$,[1] and $\varepsilon_{N,i}$ is an unobserved shock which may include unobserved idiosyncratic characteristics relevant to determine the outcome $y_i$, or believes about others' private types (see Appendix A). Importantly, this model allows for the links between any $i, j \in \mathcal{I}_N$ to differ in strength by allowing for the weights $\rho_{m;i,j}$ to be different across individuals and layers (for example, if two scholars have coauthored more papers together). However, the model requires full observability of those weights. Though providing identification and estimation results under misclassified or unobserved links is out of the scope of this paper, Lewbel et al. (forthcoming) shows that the standard estimation of the LiM parameters remains consistent even in the presence of misclassified connections, as long as the number of such links does not grow too quickly with the sample size. The observable characteristics generating the contextual effects ($\sum_{j \neq i} w_{N,m;i,j} \mathbf{x}_{N,j}^\top$) may differ across different layers and may be different from the features creating direct effects. However, those changes do not affect the identification and estimation results. Therefore, for simplicity, I will keep the assumption that the set of characteristics generating contextual and direct effects are the same and do not vary across different layers. The coefficients $(\beta_{0,m}, \boldsymbol{\delta}_{0,m})$ represent the social effects for network $m \in \{1, \ldots, M\}$, while $\boldsymbol{\gamma}_0$ captures the direct effects. I use the zero superscript to emphasise that $[\beta_1^0, \ldots \beta_M^0, \boldsymbol{\delta}_1^0, \ldots, \boldsymbol{\delta}_M^0, \alpha^0, \boldsymbol{\gamma}^0]$ is the true parameter vector. Equation (1) can be written in matrix form as

$$\mathbf{y}_N = \alpha^0 \boldsymbol{\iota}_N + \left( \sum_{m=1}^{M} \beta_m^0 \mathbf{W}_{N,m} \right) \mathbf{y}_N + \sum_{m=1}^{M} \mathbf{W}_{N,m} \mathbf{X}_N \boldsymbol{\delta}_m^0 + \mathbf{X}_N \boldsymbol{\gamma}^0 + \boldsymbol{\varepsilon}_N, \tag{2}$$

where $\boldsymbol{\iota}_N$ is the $N \times 1$ vector of ones. Let $\mathbf{S}(\boldsymbol{\beta}^0, \mathcal{M}_N) = \mathbf{I}_N - \sum_{m=1}^{M} \beta_m^0 \mathbf{W}_{N,m}$. The model described by equation (2) has a solution for a given $\mathbf{X}_N$, $\boldsymbol{\varepsilon}_N$, and $\mathcal{M}_N$ if the matrix $\mathbf{S}(\boldsymbol{\beta}^0, \mathcal{M}_N)$ has an inverse. Lemma 2 in Appendix B shows that the parametric restrictions in Assumption 1 are sufficient to guarantee the existence of $\mathbf{S}^{-1}(\boldsymbol{\beta}^0, \mathcal{M}_N)$. Regarding the adjacency matrices' characteristics, the invertibility result in Lemma 2 only requires the assumption of no cycles for each adjacency matrix $m$. Thus, it covers cases of directed, undirected, weighted, or unweighted graphs. When the adjacency matrices of the layers are weighted such that $\|\mathbf{W}_{N,m}\|_\infty = 1$ for all $m$, the condition in Assumption 1 reduces to

[1] This article focuses on the representation of the multilayer network as the set of layers' adjacency matrices. This approach is motivated by the fact the proposed MLiM model does not consider interlayer connections to affect individuals' outcomes. The multilayer measure of distance defined in section 4 can handle data structures where the multilayer network could contain interlayer connections. Then, so long as the model in (2) is correctly specified -in the sense that intralayer edges do not generate network effects- the identification idea proposed in this paper still works. It is still possible to use this paper's identification approach in potential settings where interlayer connections exist and can generate network effects. The only required modification is to include the regressors associated with interlayer connections on the right-hand side of the MLiM by using, for instance, an interlayer adjacency matrix.

$|\beta_1^0| + \cdots + |\beta_M^0| < 1$, which is a generalization of a familiar assumption on the peer effects coefficient that is customary in the literature when $m = 1$ (see, e.g., Kelejian and Prucha (1998), Kelejian and Prucha (2001), and Lee (2007a) in spacial econometrics, Lee (2007b) and Bramoullé et al. (2009) in econometrics of networks, to mention some).

**Assumption 1 (Invertibility)** *The adjacency matrix of layer $m$ has no cycles, i.e., $w_{N,m;i,i} = 0$ for all $m \in \{1, \ldots, M\}$ and $i \in \mathcal{I}_N$. The peer effects coefficients for the layers $\mathbf{W}_{N,1}, \mathbf{W}_{N,2}, \ldots, \mathbf{W}_{N,M}$ are such that $|\beta_1^0|\|\mathbf{W}_{N,1}\|_\infty + \cdots + |\beta_M^0|\|\mathbf{W}_{N,M}\|_\infty < 1$, where $\|\mathbf{W}_{N,m}\|_\infty = \sup_{i\in\mathcal{I}_N} \sum_{j=1}^N |w_{N,m;ij}|$ and $m \in \{1, \ldots, M\}$.*

If Assumption 1 is satisfied, it follows that the solution for equation (2) can be written in terms of $\mathbf{X}_N$, $\boldsymbol{\varepsilon}_N$, and $\mathcal{M}_N$ as follows

$$\mathbf{y}_N = \mathbf{S}^{-1}(\boldsymbol{\beta}^0, \mathcal{M}_N) \left( \alpha^0 \boldsymbol{\iota}_N + \sum_{m=1}^M \mathbf{W}_{N,m}\mathbf{X}_N\boldsymbol{\delta}_m^0 + \mathbf{X}_N\boldsymbol{\gamma}^0 + \boldsymbol{\varepsilon}_N \right), \tag{3}$$

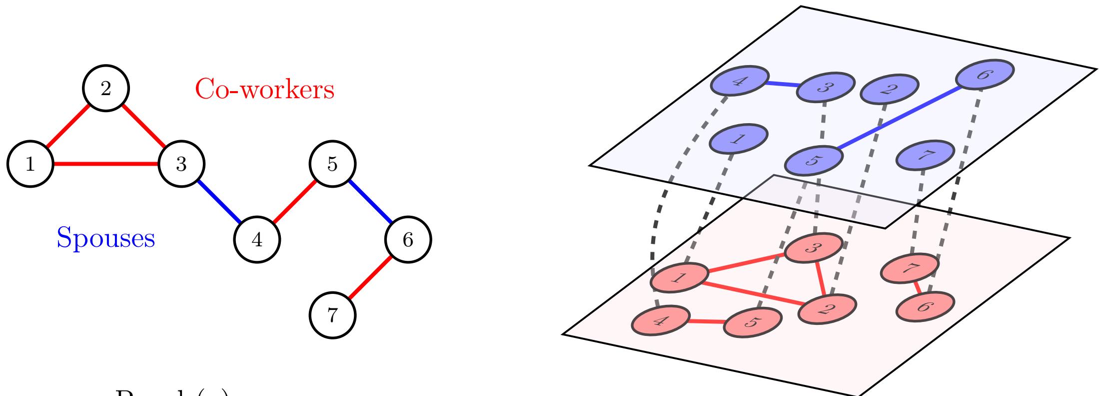
which makes explicit the correlation between $\mathbf{W}_{N,m}\mathbf{y}_N$ and $\boldsymbol{\varepsilon}_N$ for all $m$. The endogeneity of the peer effects regressors is usually known in the literature as the reflection problem (Manski, 1993). In addition to the reflection problem generated by the endogeneity of the variable $\mathbf{W}_{N,m}\mathbf{y}_N$, this article allows for dependence between the observable characteristics and the unobserved shocks given the potential network endogeneity. In practice, endogeneity can arise when individuals form connections in the networks $\{G_{N,m}\}_{m=1}^M$ based on observed and unobserved characteristics correlated with the outcome $\mathbf{y}_N$. Appendix A shows that in the social interactions model with incomplete information, the endogeneity issue arises because of an endogenous network formation process in which individuals care about the private type of others and use all their available knowledge to predict it (see also Section VI in Blume et al., 2015, pp. 474-477). Moreover, the homophily preferences characterizing the network formation could also induce endogeneity as individuals try to match both observed and unobserved characteristics.

One legitimate question regarding the relevance of the MLiM mode is whether multilayer network data are available in empirical contexts. Before detailing the primary identification approach, the following subsection provides examples that showcase both the availability and versatility of the multilayer data structure.

### 3.1 Examples

**Example 1 (non-overlapping networks):** De Giorgi et al. (2020) studies consumption network effects in a context where co-workers are the relevant reference group. The authors propose a consumption model that yields an Euler equation which can be cast as a special case of the model in (1). Acknowledging the potential network endogeneity, the authors use what they define as a non-overlapping network structure to form valid peer consumption instruments. Essentially, the data structure contains two

Figure 1: De Giorgi et al.'s (2020) Data Structure Represented as a Multilayer Network




Panel (a)

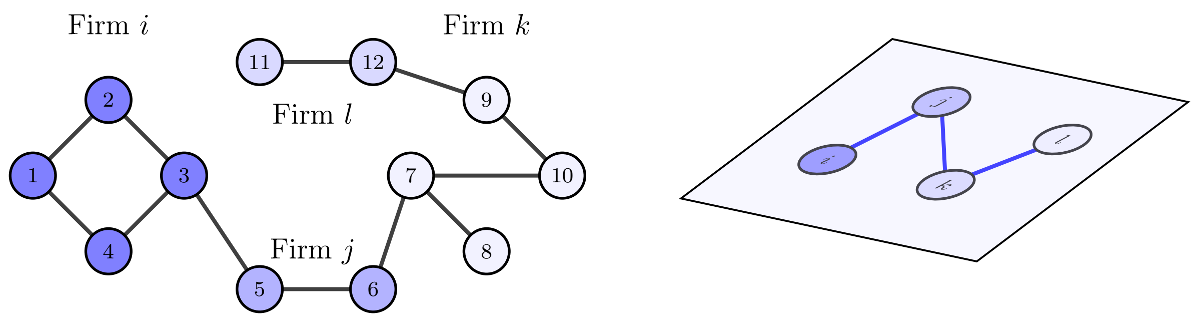
Panel (b)

Note: Panel (a) displays an example of the non-overlapping network data structure in De Giorgi et al. (2020) which is a modification of the example presented in Figure 2 of their article. The blue edges represent co-workers' connections while the red edges illustrate spouses relationships. Panel (b) shows the representation of the same graph as a multilayer network.

types of connections: spouses and co-workers. Figure 3 presents a minimal example of their primary data structure. Panel (a) in Figure 1 shows a flat representation of the network data structure where the connections between co-workers and spouses are depicted in blue and red, respectively. Panel (b) shows the multilayer representation of the network in Panel (a). As mentioned before, intralayer edges are the only relevant type of connection. Interlayer edges do not provide relevant information as they only connect a node in one layer to itself in the other layer. The non-overlapping is particular to De Giorgi et al. (2020). However, the method proposed in this paper can also accommodate links between two people across multiple layers, i.e., if the spouses are also co-workers.

**Example 2 (monolayer network):** Zacchia (2019) studies knowledge spillovers generated by interactions between inventors of different firms. The author proposes a model of R&D investment with knowledge spillovers where the firms' production functions can be cast into a special case of the model in (1). The estimation procedure is based on a monolayer network where two firms are connected if they employ inventors who have collaborated before. Zacchia (2019) is a relevant example because it provides an analytical framework to understand how network endogeneity complicates the identification of contextual effects in a linear model where the endogenous peer effect parameter is not of interest. Figure 2 presents an simplified example of the network data structure in Zacchia (2019). This example explicitly highlights the fact that any monolayer network can be represented as a multilayer network by setting $M = 1$.

Figure 2: Zacchia's (2019) Monolayer Network Data Structure




Panel (a) Panel (b)

Note: Panel (a) displays the monolayer data structure in Zacchia (2019). The nodes represent inventors who are connected by an edge if they have worked together in a project before. Nodes of the same color belong to the same firm. Panel (b) represents the monolayer firms network. Two firms $i$ and $j$ are connected if at least one of the inventors working for $i$ is connected to an inventor working for $j$.

# 4 Identification

I assume that the matrix of observable characteristics $\mathbf{X}_N$, the multilayer network $\mathcal{M}_N$, and the vector of shocks $\boldsymbol{\varepsilon}_N$ are random variables characterized by a joint distribution $\mathcal{F}_{\mathbf{X}_N,\mathcal{M}_N,\boldsymbol{\varepsilon}_N}(\mathbf{X},\mathcal{M},\boldsymbol{\varepsilon})$, where $\mathbf{X}$, $\mathcal{M}$ and $\boldsymbol{\varepsilon}$ are arbitrary realizations. This joint distribution could reflect the potential correlation between $\mathcal{M}_N$ and $\boldsymbol{\varepsilon}_N$ caused by a strategic network formation characterized by unobserved homophily. Correlation between $\mathbf{X}_N$ and $\boldsymbol{\varepsilon}_N$ is also allowed as assortative matching could induce network dependence across observed and unobserved characteristics or by the structural motives of individuals trying to use their information to predict the behavior of others.

The identification approach is based on imposing the restriction that functionals of network data become uncorrelated as observations become distant in the network space characterized by $\mathcal{M}_N$. In particular, I assume that implicit network formation processes or the potential existence of local network shocks induce correlation patterns that decrease with the individuals' distance in the network space. Thus, there exists a level of distance between individuals such that their observed and unobserved characteristics are not correlated.

When thinking about the validity of the identification results, a fundamental question is whether the idea of correlation patterns across observed and unobserved characteristics that decrease with individuals' distance in the network space is consistent with the potential existence of network endogeneity. If, for example, individuals form links according to unobserved homophily based on a characteristic correlated with $\mathbf{X}_N$ and $\boldsymbol{\varepsilon}_N$. Then, individuals far apart in the network space are likely to have different observed and unobserved characteristics, which imply a correlation between them. In Appendix

A, I provide a set of informational and distributional assumptions under which an endogenous network formation process based on observed and unobserved homophily still exhibits a pattern of dependence between the characteristics in which correlation decreases with individuals' distance in the network space. Before formalizing the identifying assumption on the joint distribution $\mathcal{F}_{\mathbf{X}_N,\mathcal{M}_N,\boldsymbol{\varepsilon}_N}$, the following remark summarizes the intuition for the main modeling premises guaranteeing that dependence between individual characteristics decreases with their distance in the multilayer network space.

**Remark 1 (Decreasing Dependence Under Endogenous Networks)** An information assumption characterizes the social interactions model that constrains individuals' knowledge of others $\mathbf{X}_N$ only within a cluster that forms *exogenously* during the network formation process. These clusters could represent, for example, initial geographical, economic, or skills locations. The model also includes a distributional assumption requiring people within the same cluster but not across to have correlated characteristics. Moreover, under the model's information regime, individuals do not observe others' private characteristics. Therefore, there should not be systematic correlations across those characteristics for the *difficult-to-predict* individuals in different clusters. Within clusters, however, when deciding whether to connect with $j$, individual $i$ can use all her knowledge of her private characteristics and her intra-cluster observed characteristics to predict $j$'s private type. Given the homophily preference, if those variables are good predictors, that could induce a correlation between the unobserved characteristics of individuals close in the multilayer network space.

It is possible to formalize the previous discussion using the $\psi$-dependence framework proposed by Doukhan and Louhichi (1999) and Kojevnikov et al. (2020). The $\psi$-dependence approach requires the availability of a metric that can characterize the distance between individuals in the multilayer network space. Kojevnikov et al. (2020) uses the *minimum path lengths* (or geodesic distance), which is the standard monolayer measure of distance. However, my proposed identification idea uses multilayer network data. I argue that the change in the data structure from monolayer to multilayer networks benefits from a measure of distance that considers the existence of different types of connections. The next section presents the argument.

### 4.1 Multilayer Measures of Distance

Following Boccaletti et al. (2014), for a given multilayer network $\mathcal{M}_N = (\mathcal{G}_N, \mathcal{C}_N)$, a *multilayer walk* of length $q-1$ can be defined as a sequence of edges $\{v_{1;m_1}, v_{2;m_2}, \ldots, v_{q;m_q}\}$ where $m_1, m_2, \ldots, m_q \in \{1, \ldots, M\}$ and two adjacent nodes in the sequence are connected by an edge belonging to the set $\{E_{N,1}, \ldots, E_{N,M}\} \bigcup \mathcal{C}_N$. In words, a walk connects nodes $v_{1;m_1}$ and $v_{q;m_q}$ (which are allowed to be in different layers) through a sequence of nodes that either intralayer or interlayer edges can connect. For instance, Alice is friends with Bob (layer 1), and Bob knows Cassey because they are neighbors (layer 2). Alice and Cassey are not friends. However, they are at a distance 2 once we consider all the layers in the multilayer network. A *multilayer path* is then defined as a *multilayer walk* where each node is only visited once. A *multilayer minimum path lengths* is the shortest *multilayer path* connecting two nodes, and it is denoted by $d_N^*(i,j)$. The star superscript is used to emphasize that from all the possible paths connecting $i$ and $j$, those at a distance $d_N^*(i,j)$ are the shortest.

In addition to the shortest path length, the proposed measure of distance also considers the number of edge-type changes in a path connecting two individuals. To formalize this idea, let $\mathcal{D}_N(i,j;d)$ be the set of all possible paths $p_N(i,j)$ -which can include paths across multiple layers- of length $d$ connecting individuals $i$ and $j$. Based on the set of all possible paths, the set of all possible shortest paths is defined as $\mathcal{D}_N(i,j) = \arg\min_{d\in\mathbb{N}_+} \mathcal{D}_N(i,j;d)$ (where $\mathbb{N}_+$ is the set of positive natural numbers). For instance, the minimum path length between individuals 1 and 5 in Figure 1 is given by $d_7^*(1,5) = 3$ and the set $\mathcal{D}_7(1,5) = \{(1,3,4,5)\}$ is a singleton in this case. For any path $p_N(i,j) \in \mathcal{D}_N(i,j;d)$, define $c_N(i,j;p)$ as path $p$'s number of edge-type changes. In the example, the total number of edge-type changes associated with the shortest path $(1,3,4,5)$ is given by $c_7(1,5;(1,3,4,5)) = 2$. If any steps on a path $p_N(i,j)$ involve more than one edge-type between any two edges, there is an edge-type change for any two consecutive sets of edges only if there is no overlap on the two sets of edge types. For instance, if Alice and Bob are friends (layer 1), Bob knows Cassey because they are neighbors (layer 2), but Bob and Cassey are not friends, there is an edge-type change in the path connecting Alice and Cassey. However, if instead Bob and Cassey are friends, even when Alice and Cassey are still not friends, there is not an edge-type change in the path connecting Alice and Cassey because of the double friendship links connecting Alice with Bob and Bob with Cassey.

In general, the set of possible shortest paths $\mathcal{D}_N(i,j)$ does not need to be a singleton. Different combinations of nodes could connect two individuals with the minimum possible number of edges. To incorporate the intuition that less edge-type changes are associated with shorter distances, let $c_N^*(i,j) = \min_{p\in\mathcal{D}_N(i,j)} c_N(i,j;p)$ be the minimum number of edge-type changes of the shortest paths connecting $i$ and $j$. Let $\mathcal{D}_N^*(i,j)$ be the set of all paths $p_N(i,j)$ of length $d_N^*(i,j)$ and total edge changes $c_N^*(i,j)$. The sets $\mathcal{D}_N(i,j)$ and $\mathcal{D}_N^*(i,j)$ are such that $\mathcal{D}_N^*(i,j) \subseteq \mathcal{D}_N(i,j)$ as $\mathcal{D}_N^*(i,j)$ only consider the shortest paths with the minimum number of edge-type changes (the set $\mathcal{D}_N^*(i,j)$ does not have to be a singleton either). I call the paths $p_N(i,j) \in \mathcal{D}_N^*(i,j)$ *multilayer shortest paths.* Having described these objects, I now define the *multilayer measure of distance.*

**Definition 1 (Multilayer Shortest Path length)** *Let $d_N^*(i,j)$ be the multilayer minimum path length and $c_N^*(i,j)$ the minimum number of edge-type changes for the shortest path connecting individuals $i,j \in \mathcal{I}_N$. I define the multilayer measure of distance $d_N^{\mathcal{M}}(i,j)$ as*

$$d_N^{\mathcal{M}}(i,j) = d_N^*(i,j) + \tau_{i,j} c_N^*(i,j), \tag{4}$$

*where $\tau_{i,j} \geq 1$ is an integer that captures the intuition that the distance between two individuals $i$ and $j$ (and consequently their levels of dependence) increases (reduces) faster when the shortest path connecting them involves edge-type changes.*

This multilayer measure of distance penalizes edge-type changes more than shortest path lengths. The idea of penalizing the number of edge-type changes in a multilayer measure of distance comes from the physics literature on complex networks. Kivela et al. (2014) suggests that it is natural to hypothesize that paths containing only one edge-type may have different *lengths* than those containing more than one type. In physical systems, changes in edge types may be associated with an increase in the cost of forming the connection. In the context of potential restrictions to the distribution $\mathcal{F}_{\mathbf{X}_N, \boldsymbol{\varepsilon_N}, \mathcal{M}_N}$, I argue that it is reasonable to think that individuals first form less costly links in one social or professional aspect and then form more costly ones in other potentially different environments. These new ties can give individuals access to other potentially disconnected groups characterized by different social or professional connections, see, e.g., the network formation model in Appendix A, and the idea of bridging structural holes in Burt (2004).

Additional relevant reasons justify the inclusion of edge-type changes in the definition of the multilayer shortest path. First, as shown in equation (18) in Section 9, infinite products of the adjacency matrices of different layers can be used as relevant instruments for $\mathbf{W}_{N,m}\mathbf{y}_N$. For any arbitrary sequence of layers $\{1, 2, 3, \ldots, r-1, r\}$ of size $r$, the result of the product of adjacency matrices $\mathbf{W}_{N,1}\mathbf{W}_{N,2}\cdots\mathbf{W}_{N,r}$ contains the paths of size $r$ between any individuals $i$ and $j$ that change edge types from 1 to 2, from 2 to 3, and all the way from $r-1$ to $r$. Therefore, the form of the equation of interest naturally suggests including paths with edge-type changes to form relevant moment conditions. Second, only considering the number of edges when defining a multilayer measure of distance ignores entirely the additional information the edge types provide. Not taking into account the additional information from the edge types can affect the strength of identification. For instance, when considering the multilayer network based on the data for the empirical application, Figure 5 shows that the individuals providing identifying information reduce sharply with the shortest path lengths. In particular, for the example of estimating the coauthorship peer effects, for some choice of the parameters limiting dependence across individuals, Figure 5 implies that not considering edge-type changes in the measure of distance would reduce the number of individuals for whom it is possible to find moment conditions by 33% and 17% of the full sample size for the years 2002 and 2006, respectively. Finally, for special cases of multilayer network data, such as the non-overlapping networks used by De Giorgi et al. (2020), not considering edge-type changes would ignore the fact that a path containing a change in edge types necessarily has to connect two clusters that would otherwise be disconnected. In De Giorgi et al.'s (2020) data, ignoring edge-type changes would imply that the dependence between the characteristics of a coworker's coworker who work for the same company is the same as the dependence between a person's spouse's coworker who works in a different company.

I characterize the value of the penalization parameter $\tau_{ij}$ for the measure of distance in Proposition 1 below. The characterization takes into account an additional measure of distance between any two individuals $i, j \in \mathcal{I}_N$ in the multilayer network space. I call this distance the *second shortest path*. This measure is relevant because it is the foundation to construct a method to guarantee that two individuals seemingly far away in terms of geodesic distance and the number of edge-type changes are not connected by a second path with larger geodesic distance but fewer edge-type changes.

**Definition 2 (Second Shortest Path)** *Let $d^c_N(i,j)$ be the distance associated with the path connecting individuals $i, j \in \mathcal{I}_N$ that has the second shortest number of links, and has a number of edge-type changes lower than $c^*_N(i,j)$. If no path exists with those characteristics, then $d^c_N(i,j) \to \infty$.*

For example, the second shortest path in Figure 1 for individuals 1 and 5 is given by $d_{1,5}(2) \to \infty$. That happens because no path connecting 1 and 5 has less than two edge-type changes. The main intuition for the necessity of the second shortest path and its connection with $\tau_{ij}$ is that the penalization scheme has to trade off the relative importance of edge-type changes with the potential existence of other paths with fewer edge types that could become a shorter path. Based on the pairwise measure of distance, it is possible to define the distance between sets of nodes. Following Kojevnikov et al. (2020), for $a, b \in \mathbb{N}_+$ the distance between two sets $A$ and $B$ with $a$ and $b$ nodes respectively is given by

$$d_N(A,B) = \min_{i\in A} \min_{j \in B} d^{\mathcal{M}}_N(i,j). \tag{5}$$

Through the identification and estimation sections, I will be using different sets that contain groups of nodes that are at different intralayer or multilayer distances. The following definition collects all the distance sets that will be used in the following sections.

**Definition 3 (Distance Sets)** *Consider the distance measures $d^{\mathcal{M}}_N(i,j)$, $d_N(A,B)$ and $d^*_N(i,j,m)$, where $d^*_N(i,j,m)$ is the geodesic distance between individuals $i$ and $j$ only considering connections in layer $m \in \{1,\dots,M\}$. Consider the sets: (i) $\mathcal{P}^+_N(a,b,d) = \{(A,B) : A, B \subset \mathcal{I}_N, |A| = a, |B| = b$, and $d_N(A,B) \geq d\}$ containing groups of nodes at distance of at least $d$ from each other, (ii) $\mathcal{P}^-_N(a,b,d) = \{(A,B) : A, B \subset \mathcal{I}_N, |A| = a, |B| = b$, and $d_N(A,B) \leq d\}$ containing groups of nodes at distance of at most $d$ from each other, (iii) $\mathcal{P}_N(a,b,d) = \{(A,B) : A, B \subset \mathcal{I}_N, |A| = a, |B| = b$, and $d_N(A,B) = d\}$ the set associated with groups of nodes at distance $d$ from each other. The associated set that contain all nodes at a certain distance of node $i$ are $\mathcal{P}^+_n(i,d) = \{j \in \mathcal{I}_n : d^{\mathcal{M}}_n(i,j) \geq d\}$, $\mathcal{P}_n(i,d) = \{j \in \mathcal{I}_n : d^{\mathcal{M}}_n(i,j) = d\}$ and $\mathcal{P}^-_n(i,d) = \{j \in \mathcal{I}_n : d^{\mathcal{M}}_n(i,j) \leq d\}$. The same notation applies for sets based on the interlayer connections of layer $m$ by adding the index: $\mathcal{P}^+_n(i,d,m)$, $\mathcal{P}_n(i,d,m)$ and $\mathcal{P}^+_n(i,d,m)$.*

## 4.2 Network $\psi$-dependence

This section introduces a helpful framework to characterize the levels of network dependence between the regressors and the errors in equation (1). Form the vector $\mathbf{r}_{i,N} = [\mathbf{x}^\top_{i,N}, \varepsilon_{i,N}]^\top \in \mathbb{R}^{Q+1}$. I define $\mathbf{r}_{i,N}$ in terms of the observed and unobserved characteristics to allow for dependence between them. As mentioned before, the network dependence can be generated by a situation where individuals form links based on observed and unobserved homophily, as in Johnsson and Moon (2019). Therefore, the network formation can induce correlation among the vectors $\mathbf{r}_{i,N}$ and $\mathbf{r}_{j,N}$ for some $i,j \in \mathcal{I}_N$ nearby in the multilayer network space and, consequently, on the average outcomes and covariates of their connections. For $Q, a \in \mathbb{N}_+$, endow $\mathbb{R}^{(Q+1)\times a}$ with the distance measure $\boldsymbol{d}_a(\mathbf{x},\mathbf{y}) = \sum_{l=1}^{a} \|x_l - y_l\|_2$ where $\|\cdot\|_2$ denotes the Euclidean norm and $\mathbf{x},\mathbf{y} \in \mathbb{R}^{(Q+1)\times a}$. Let $\mathscr{L}_{Q,a}$ denote the collection of bounded Lipschitz real functions mapping values from $\mathbb{R}^{(Q+1)\times a}$ to $\mathbb{R}$. For each set of nodes $A$, let $\mathbf{r}_{A,N} = (\mathbf{r}_{N,i})_{i\in A}$. The $\psi$-dependence definitions in Doukhan and Louhichi (1999) and Kojevnikov et al. (2020) are accommodated to the proposed multilayer network framework.

**Definition 4 ($\psi$-dependence)** *A triangular array $\mathbf{r}_{n,i}$ for $n \geq 1$ and $\mathbf{r}_{n,i} \in \mathbb{R}^{Q+1}$ is $\psi$-dependent if for each $n \in \mathbb{N}$ there is a sequence $\theta_n = \{\theta_{n,d}\}_{d\geq 0}, \theta_{n,0} = 1$ and a collection of non-random functions $(\psi_{a,b})_{a,b\in\mathbb{N}}, \psi_{a,b} : \mathscr{L}_{v,a} \times \mathscr{L}_{v,b} \to [0,\infty)$ such that for all $A, B \in \mathcal{P}_N^{+}(a,b,d)$ for $d > 0$ and all $f \in \mathscr{L}_{Q+1,a}$ and $g \in \mathscr{L}_{Q+1,b}$,*

$$|\mathrm{Cov}\left(f\left(\mathbf{r}_{n,A}\right), g\left(\mathbf{r}_{n,B}\right)\right)| \leq \psi_{a,b}(f,g)\theta_{n,d}.$$

The sequence $\theta_n$ is called the dependence coefficients of $\mathbf{r}_{n,i}$. I state the definition in terms of triangular arrays because it fits the asymptotic results for the estimator in section 5. Indeed the $\psi$-dependence assumption is used in the literature to establish asymptotic results when the level of network dependence decreases asymptotically. In this paper, I use the $\psi$-dependence assumption to characterize the dependence patterns generated by the joint distribution $\mathcal{F}_{\mathbf{X}_N,\mathcal{M}_N,\varepsilon_N}$ in the population. Therefore, I effectively use the idea of $\psi$-dependence for both identification and estimation. Note that by choosing appropriate functions $f$ and $g$, and appropriate sets $A$ and $B$, Definition 4 bounds the covariance between any pair $\varepsilon_{N,i}$ and $\mathbf{x}_{N,j}$ (and also between any two $\mathbf{x}_{N,i}$ and $\mathbf{x}_{N,j}$). The following assumption guarantees that individuals' dependence decreases with their distance in the multilayer network space.

**Assumption 2 (Weak Neighborhood Dependence (WND))** *Consider the set $\mathscr{M}$ of all possible realizations of $\mathcal{M}_N$ with positive probability mass in $\mathcal{F}_{\mathbf{X}_N,\boldsymbol{\varepsilon_N},\mathcal{M}_N}$. For all multilayer network realizations $\mathcal{M} \in \mathscr{M}$, the conditional distribution $\mathcal{F}_{\mathbf{X}_N,\boldsymbol{\varepsilon_N}|\mathcal{M}_N}(\mathbf{X},\boldsymbol{\varepsilon} \mid \mathcal{M}_N = \mathcal{M})$ is such that:*

*(i) $\{\mathbf{r}_{N,i}\}$ is $\psi$-dependent with dependence coefficients $\theta_N$.*

*(ii) For $C > 0$, $\psi_{a,b}(f,g) \leq C \times ab\left(\|f\|_\infty + \mathrm{Lip}(f)\right)\left(\|g\|_\infty + \mathrm{Lip}(g)\right)$.*

*(iii) $\max_{d\geq 1}\theta_{N,d} < \infty$ and there exists a finite constant $D \in \mathbb{N}_+$ such that if $d > D$, $\theta_{N,d} = 0$.*

Assumption 2 Part *(ii)* states that the functional in the upper bound for the covariance in definition 4 is increasing in the set sizes, the sup-norm of the aggregating functions $f$ and $g$, and their Lipschitz constants, $\mathrm{Lip}(f)$ and $\mathrm{Lip}(g)$, related to the continuity of the functions.[2] Perhaps more importantly, part *(iii)* is crucial for identification, and it imposes a stronger condition than Doukhan and Louhichi

[2]The Lipschitz constant for a function $f : \mathbb{R}^{(Q+1)\times a} \to \mathbb{R}$ is the smallest constant $L$ such that $|f(\mathbf{x}) - f(\mathbf{y})| \leq C\boldsymbol{d}_a(\mathrm{x},\mathrm{y})$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^{(Q+1)\times a}$.

(1999) and Kojevnikov et al. (2020) because the dependence coefficients dissipate to zero after a finite distance $D$, not asymptotically. It is important to clarify that the finite threshold $D$ is required only for identification but not estimation. The reason is that the sharp bound allows the formation of identifying moment conditions based on exact distances (see Proposition 1). As discussed in remark 1, there exists a set of conditions under which it is possible to have dependence patterns that decrease with individuals' distance in the network space even when the network formation process features unobserved homophily. Moreover, by dividing the set of individuals $\mathcal{I}_N$ into a set of finite clusters and imposing the distributional assumptions in A.13, Lemma 1 in Appendix A shows that it is possible to find a finite distance $D$ under which condition *(iii)* holds.

Intuitively, in terms of the empirical application, the implication of the local informational assumption in Appendix A ensuring condition *(iii)* is that initially (before starting a Ph.D. program), students can only observe the characteristics of others in their same level of skills (you can think of also creating more refined clusters such as geographical locations and skill levels). After the vector dividing individuals into clusters is realized, students begin a process where they first create connections based on sharing the same Ph.D. program, then advisory, coauthorship, and coworker ties. Assumption A.12 implies that other characteristics, such as writing styles and reading habits affecting individuals articles quality, are correlated only between those who are connected or belong to the same group. Finally, assumption A.13 guarantees that it is more likely to form connections with others in the same cluster than others in different clusters. In the context of the running example, this means that, for instance, it is more likely that individuals with similar levels of characteristics end up in the same Ph.D. programs or co-authoring together.

Even when it is possible to guarantee that assumption 2 part *(iii)* holds for some $D$ and the specific social interactions model in Appendix A, the value of the constant is generally not identified. Here I use an empirical approach showing that the estimation results are robust to changes in the hyperparameter. Given that the value of $D$ depends on the network formation process, and there is an implicit trade-off between the variance and the bias of the estimated parameters, I conjecture that, for a given network formation model for which the WND assumption holds, there should exist an optimal value for the hyperparameter. However, the optimal choice of $D$ is out of the scope of this paper, and I leave it as an exercise for future research. In addition to the WND assumption, I impose an additional restriction related to the marginal distribution of the errors and their correlation with the regressors for the same individual.

**Assumption 3 (Errors' Moments)** *The unobserved shocks $\varepsilon_{N,i}$ are such that (i) $\mathbb{E}(\varepsilon_{N,i}) = 0$ and (ii) for all $\mathcal{M} \in \mathscr{M}$, $\mathbb{E}\left(\mathbf{x}_{N,i}\varepsilon_{N,i} \mid \mathcal{M}_N = \mathcal{M}\right) = \mathbf{0}_{Q\times 1}$ for all $i \in \mathcal{I}_N$, where $\mathbf{0}_{Q\times 1}$ is the $Q \times 1$ vector of zeros. The first expectation is taken with respect to the marginal distribution of $\varepsilon_{N,i}$ and the second with respect to the conditional distribution $\mathcal{F}_{\mathbf{X}_N, \boldsymbol{\varepsilon}_{\boldsymbol{N}} \mid \mathcal{M}_N}$.*

The first part of Assumption 3 can be viewed as just a normalization. Part *(ii)* is more substantial. It implies that the dependence between observed and unobserved characteristics generated by the underlying network formation process rules out any correlation for the same individual. Assumption A.12 in Appendix A presents a more primitive restriction on the distribution of the social interactions model, which implies assumption 3. This assumption is customary in network effects studies, see, e.g., De Giorgi et al. (2020), Chan et al. (2022), and Zacchia (2019). Articles in the literature that are concerned with endogenous network formation for the monolayer case use assumptions of no correlation between $\varepsilon_{N,i}$ and $\mathbf{x}_{N,i}$ after controlling for the effects of the underlying matching process. These classes of estimators share the spirit of Assumption 3 in the sense that they think of the network formation process as generating the dependence between individuals' observed and unobserved attributes, see e.g., Auerbach (2016) and Johnsson and Moon (2019). The following proposition formalizes how to construct the moment conditions under the WDN assumption.

**Proposition 1** *Let Assumptions 2 and 3 hold for $\mathbf{r}_{N,i}$. Then, there exist two integers $K_c$ and $K_d$ with $K_d > K_c + 1$ such that for all $\mathcal{M} \in \mathscr{M}$, the conditional distribution $\mathcal{F}_{\mathbf{X}_N,\boldsymbol{\varepsilon_N}|\mathcal{M}_N}(\mathbf{X}, \boldsymbol{\varepsilon} \mid \mathcal{M}_N = \mathcal{M})$ is such that for all pairs $i, j \in \mathcal{I}_N$*

$$\mathbb{E}\left(\mathbf{x}_{N,j}\varepsilon_{N,i} \mid c^*_N(i,j) \geq K_c, d^c_N(i,j) \geq K_d\right) = \mathbf{0}_{Q\times 1}, \tag{6}$$

$$\mathbb{E}\left(\mathbf{x}_{N,j}\varepsilon_{N,i} \mid c^*_N(i,j) < K_c, d^*_N(i,j) \geq K_d\right) = \mathbf{0}_{Q\times 1}. \tag{7}$$

Section 9 presents the proof for Proposition 1 and Proposition A.4 in Appendix A provides a micro-founded set of restrictions justifying the use of the conditional moments (6) and (7) as a source for identification. The proof for Proposition 1 shows that if $K_d \leq K_c + 1$, the conditioning set in equation (6) does not provide different information from that in equation (7). Intuitively, the inequality $K_d > K_c + 1$ is required to guarantee that the dependence between individuals decreases faster when the number of edge-type changes increases. Importantly, the proof sets $K_d = D + 1$ and $\tau_{ij} = d^c_N(i,j) - K_c$, so that choosing the values of $K_c$ and $K_d$ completely characterizes the measure of distance $d^{\mathcal{M}}_N(i,j)$. As mentioned before when discussing the choice for the constant $D$, I use a robustness approach in this paper to deal with the selection of the hyperparameters $K_c$ and $K_d$.

Equation (6) can be interpreted in the context of the multilayer network structure. It states that paths involving edge-type changes make the dependence between individuals decrease *faster*. Individuals connected by the same type of social or professional ties will tend to be similar because of the homophily characterizing the network formation process, e.g., Graham (2017). However, when additional types of connections are allowed, similar individuals connected by the same edge-type can be indirectly connected to others who are different, given that they do not belong to that same *local* monolayer network. Equation (7) provides a result that has been used in recent literature for the case when $c^*_N(i,j) = 0$ for any $i$ and $j$ (monolayer network data case). It states that if none of the paths connecting two individuals change edge types enough times, they have to be at a longer distance in the network space for their characteristics to be not correlated. One illustrating case is when the network only includes intralayer edges. In that case, this assumption implies that if two individuals are too far apart *in the same layer*, it is more likely that their characteristics are *not* correlated. It is instructive to see how the examples presented before have implicitly used the results in Proposition 1.

**Example 1 (continuation):** In the context of non-overlapping network structure, De Giorgi et al. (2020) uses distance two coworkers of spouses and firms' shocks to identify peer effects in consumption.[3] Figure 1 helps to visualize the central identifying assumption. The idea is that shocks in firm 1, which are assumed to affect individuals' 1, 2, and 3 consumption levels, are independent of the unobserved characteristics of individual 5 who is indirectly connected to 3 by a coworker of a spouse relationship (see Section 4.1 in De Giorgi et al., 2020, pp. 142-144). In the context of Proposition 1, this means that it is enough to choose the parameter $K_c$ to equal one to guarantee that equation (6) is satisfied when two types of connections are observed ($M = 2$). In general, the non-overlapping network structure guarantees that whenever it is possible to connect individuals $i$ and $j$ with a path containing at least one edge change, their interlayer distance is such that $d^c_N(i, j) \to \infty$. From Figure 1, it is clear that if $K_c = 1$ and $K_d$ is arbitrarily large, it is still possible to find pairs such as $(2, 4)$, $(1, 4)$ or $(4, 6)$ for which equation (6) holds. This property makes non-overlapping network structures particularly useful for identifying peer effects, even more, when combined with credible exogenous shocks as in De Giorgi et al. (2020).

**Example 2 (continuation):** Zacchia (2019) is concerned with the identification of contextual effects. His setting is one where the network structure and some of the characteristics in $\mathbf{x}_{N,i}$ are allowed to be endogenously determined. Observing a monolayer network structure, the author proposes a game theoretical framework to formalize that observed and unobserved attributes are orthogonal for individuals far in the (monolayer) network space. Assumption 1 in Zacchia (2019) resembles the restrictions imposed by equation (7) in this paper (see Section 2.1 in Zacchia, 2019, pp. 1994). Given the monolayer data, Zacchia (2019) only has to choose the value of $K_d$. The author presents empirical evidence to justify the choice of $K_d$. Zacchia (2019) fixes $K_d = 3$ and presents robustness results for $K_d = 2$.

## 4.3 Identification Result

The identification argument combines the rich information provided by the multilayer network structure with the intuition that the strength of the dependence between individuals decays with their distance in the multilayer network space, as suggested by Proposition 1. To form the moment conditions required for identification, I construct the $(N \times N)$ matrices $\mathcal{W}_{N,m,\beta} = [w_{N,m,\beta;i,j}]$ and $\mathcal{W}_{N,m,\delta} = [w_{N,m,\delta;i,j}]$,

[3] In Section 4.1 in (De Giorgi et al., 2020, pp. 142-144), the authors consider distance-3 nodes to be the same as intransitive triads. However, if we assume that each connection's weight is one, the shortest path measure for individuals connected by an intransitive triad is 2.

where $w_{N,m,\beta;i,j}, w_{N,m,\delta;i,j} \in [0,1]$ are weights that are different from zero if (6) or (7) are satisfied for individuals $i$ and $j$ for layer $m$, respectively, and $w_{N,m,\beta;i,j} = w_{N,m,\delta;i,j} = 0$, otherwise. The sum of weights across rows may or may not sum up to one. These conditions apply for some fixed values of $K_c$ and $K_d$. The two matrices are indexed by $m$ because the paths connecting nodes $i$ and $j$ are required to start with an edge-type $m$ representing social effects generated by that layer.

Identification of the model in (1) requires at least $(M+1)(Q+1)$ moment conditions, given that there are $(M+1)(Q+1)$ parameters to estimate. If the $2M$ matrices $\mathcal{W}_{N,m,\beta}$ and $\mathcal{W}_{N,m,\delta}$ provide different information for each $m$, in the sense that their expectations are linearly independent (see assumption 6), then it is possible to construct at least $(2M+1)(Q+1)$ moment conditions based on the $Q$ observable characteristics $\mathbf{x}_{N,i}$. Define the matrix $\mathbf{D}_N = [\mathbf{W}_{N,1}\mathbf{y}_N, \ldots, \mathbf{W}_{N,M}\mathbf{y}_N, \mathbf{W}_{N,1}\mathbf{X}_N, \ldots, \mathbf{W}_{N,M}\mathbf{X}_N, \widetilde{\mathbf{X}}_N]$ associated with the vector $\boldsymbol{\psi} = [\beta_1, \ldots, \beta_M, \boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_M, \widetilde{\boldsymbol{\gamma}}]$, where $\widetilde{\mathbf{X}}_N = [\boldsymbol{\iota}_N, \mathbf{X}_N]$, and $\widetilde{\boldsymbol{\gamma}} = [\alpha, \boldsymbol{\gamma}]$. As before, the true parameters are denoted by $\boldsymbol{\psi}^0 = [\beta_1^0, \ldots, \beta_M^0, \boldsymbol{\delta}_1^0, \ldots, \boldsymbol{\delta}_M^0, \alpha^0, \boldsymbol{\gamma}^0]$. For any given $K_c$ and $K_d$, the matrices $\mathcal{W}_{N,m,\beta}$ and $\mathcal{W}_{N,m,\delta}$ can be used to construct the instrumental matrix associated with the moment conditions given by $\mathbf{Z}_N = [\mathcal{W}_{N,1,\beta}\mathcal{Z}_N, \ldots, \mathcal{W}_{N,M,\beta}\mathcal{Z}_N, \mathcal{W}_{N,1,\delta}\mathcal{Z}_N, \ldots, \mathcal{W}_{N,M,\delta}\mathbf{X}_N, \widetilde{\mathbf{X}}_N]$, where $\mathcal{Z}_N$ is an $N \times R$ matrix containing the variables used as instruments for the peer effect regressors. The matrix $\mathcal{Z}_N$ may or may not have the same variables in $\mathbf{X}_N$. The previously defined matrices $\mathbf{D}_N$ and $\mathbf{Z}_N$ have dimensions $N \times (M+1)(Q+1)$ and $N \times (1 + LM + QM + Q)$, respectively. When $L > 1$, more than one regressor in the matrix $\mathcal{Z}_N$ can be used as an instrument for the peer effects variables. Then, depending on the value of $R$, the system can be over or just identified.

Let $\mathbf{z}_{N,i}$ and $\mathbf{d}_{N,i}$ represent the $(1+LM+QM+Q)\times 1$ and $(M+1)(Q+1)\times 1$ column vectors formed by selecting the $i$th row of the matrix $\mathbf{Z}_N$ and $\mathbf{D}_N$, respectively. By Proposition 1 and the definition of $\mathbf{z}_{N,i}$, it follows that $\mathbb{E}[\mathbf{z}_{N,i}\varepsilon_{N,i}(\boldsymbol{\psi})] = \mathbf{0}_{1+LM+QM+Q}$. Therefore, the moment condition function can be defined as $\mathbf{m}_N(\boldsymbol{\psi}) = \sum_{i\in\mathcal{I}_N} \mathbf{z}_{N,i}\varepsilon_{N,i}(\boldsymbol{\psi})$. Identification requires the instrumental variables in $\mathbf{z}_{N,i}$ to be relevant in addition to the restrictions guaranteeing the validity of the moment conditions. The following set of assumptions impose restrictions on the model's parameters and the shape of the multilayer networks that realize with positive probability to guarantee that $\mathbb{E}[\sum_{i\in\mathcal{I}_N} \mathbf{z}_{N,i}\mathbf{d}_{N,i}^\top]$ has full rank, which is sufficient for identification.

**Assumption 4 (Rank Condition)** *For all multilayer network realizations $\mathcal{M} \in \mathscr{M}$, the conditional distribution $\mathcal{F}_{\mathbf{X}_N|\mathcal{M}_N}(\mathbf{X} \mid \mathcal{M}_N = \mathcal{M})$ is such that $\mathbb{E}[\mathbf{X} \mid \mathcal{M}_N = \mathcal{M}]$ has full column rank.*

**Assumption 5 (Exclusion Restrictions)** *For all $\mathcal{M} \in \mathscr{M}$, (i) $\mathbf{W}_{N,m} \neq \mathbf{O}_{N\times N}$, where $\mathbf{O}_{N\times N}$ is the $N \times N$ matrix of zeros, and (ii) there exists a layer $s \in \{1, \ldots, M\}$, such that $\mathbf{I}_N$, $\mathbf{W}_{N,m}$ and $\mathbf{W}_{N,s}\mathbf{W}_{N,m}$ are linearly independent for all $m \in \{1, \ldots, M\}$.*

Assumption 4 is a high-level rank condition that rules out redundant observed characteristics. Assumption 5 is more substantial, and it derives from rewriting equation (3) in terms of an infinite sum of the product of different adjacency matrices (see Corollary 1 in Appendix B). In particular, $\mathbf{S}^{-1}(\boldsymbol{\beta}^0, \mathcal{M}_N)$

can be written as a function of infinite powers of $\mathbf{W}_{N,m}$ and infinite products of $\mathbf{W}_{N,m}\mathbf{W}_{N,s}$ for all $m \in \{1,\dots,M\}$ and $m \neq s$. In the monolayer network case, it is well known that the $(i,j)$th element of the matrix $\mathbf{W}^k_{N,m}$ gives the number of paths of length $k$ from agents $i$ to $j$ (for some layer $m$), see, e.g., Graham (2015). For the multilayer case, under the maintained assumption that $V_m = V$ for all $m$, the $(i,j)$th element of the product of two adjacency matrices $\mathbf{W}_{N,m}$ and $\mathbf{W}_{N,s}$ for layers $m$ and $s$, contains the number of paths of length two between nodes $i$ and $j$, where each path begins with a type $m$ edge and changes to type $s$ after the second node in the sequence. Corollary 1 in Appendix B shows that both interlayer and intralayer indirect connections can be used as relevant instruments for $\mathbf{W}_{N,m}\mathbf{y}_N$ given the conditions on the parameters in assumption 7. The rule for choosing which indirect connections are also valid is given in Proposition 1. This result also motivates the use of edge-type changes when defining the multilayer measure of distance in Definition 1.

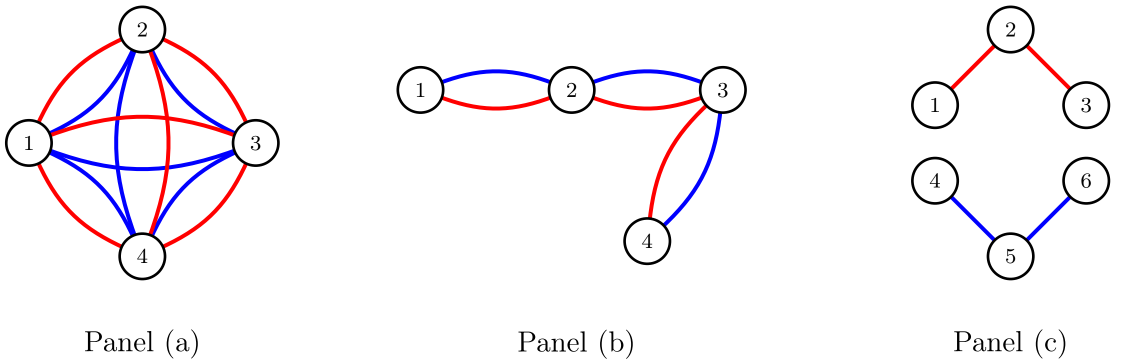
Assumption 5 formalizes the necessary conditions on the multilayer network structure that ensure the existence of enough exclusion restrictions to solve the reflection problem. This assumption generalizes Bramoullé et al.'s (2009) ideas to the multilayer network data structure, and can be verified empirically given an observed multilayer network. Although assumption 5 is more straightforward to fulfill than its monolayer counterpart (because it only requires the existence of one layer for which the condition holds), as mentioned before, because of the endogenous network formation, it is not possible to use the network structure directly for identification. Instead, the following assumption guarantees that for any multilayer network realization with positive probability, for some $K_c$ and $K_d$, under which equations (6) and (7) hold, there is enough identifying variation to identify the parameters of interest.

**Assumption 6 (Identifying Variation)** *For some $K_c$ and $K_d$ with $K_d > K_c + 1$ and all $\mathcal{M} \in \mathscr{M}$, the matrices in the sequence $\{\mathcal{W}_{m,\lambda}\}$ are non-zero and linearly independent for all $m \in \{1,\dots,M\}$ and $\lambda \in \{\beta,\delta\}$. The matrix $\mathcal{W}_{m,\lambda}$ represents the matrix constructed following equations (6) and (7) under the realized multilayer network $\mathcal{M} \in \mathscr{M}$.*

Section 4.4 emphasizes the existence of different levels of identifying power across different individuals in the population for different given realizations of the multilayer network. Assumption 6, rules out cases in which all individuals in the population provide zero identifying power. In particular, there may be too much or not enough sparsity in the intralayer and interlayer connections to guarantee that Assumption 6 holds for given values of $K_c$ with $K_d$.

Figure 3 presents three examples where Assumption 6 fails. Panels (a), (b) and (c) illustrate flat representations of different multilayer networks for $M = 2$. This example's main feature is that the number of edge-type changes is zero for all the presented multilayer networks in all possible shortest paths between two individuals $i$ and $j$, for all $(i,j) \in V_m$, where $m = \{1,2\}$. Having zero changes in edge types for all shortest paths implies that $\mathcal{W}_{n,m,\beta} = \mathbf{O}$ for all $m \in \{1,2\}$ and $n = 4$ (where $\mathbf{O}$ represents the matrix of zeros). Assumption 6 breaks down for the examples in panels (a) and (b) because both layers' adjacency matrices are equal. Therefore, the necessary condition of linear

Figure 3: Failure of Assumption 6




Note: Panels (a), (b) and (c) display examples of flat representations of different multilayer networks for $M = 2$. In panel (a) the multilayer network is dense, meaning that all individuals are connected to each other in both layers. The multilayer network in panel (b) contains intransitive triads in both individual layers, and the adjacency matrices of both layers are equal. Panel (c) shows a multilayer network with intransitive triads in both layers, and linearly independent adjacency matrices for each layer. The number of edge-type changes for the three multilayer networks equals zero for all possible shortest paths. Thus, for the presented examples, $\mathcal{W}_{n,m,\beta} = \mathbf{O}$ for all $m \in \{1, 2\}$ and $n = 4$ or $n = 6$ (where $\mathbf{O}$ represents the matrix of zeros). For the example in panel (a), for any $K_d > 1$, $\mathcal{W}_{m,\delta} = \mathbf{O}$ for all $m \in \{1, 2\}$, as the minimum path length for any nodes $i$ and $j$ is always 1.

independence of the $m$ layers' adjacency matrices $\mathbf{W}_{n,m}$ is not satisfied. Panel (c) presents an example where the two layers' adjacency matrices are different and linearly independent, but still, Assumption 6 fails. The reason is that the two networks are completely disjointed. These examples show that linear independence is only a necessary condition and that the layers also need to have connections in common for Assumption 6 to be satisfied. Notably, as shown in the examples presented in panels (b) and (c), unlike previous work in the monolayer case, intransitive triads in each layer's network are not enough to guarantee identification. See Bramoullé et al. (2009) and De Giorgi et al. (2010) for seminal work on the importance of intransitive triads for identification in the monolayer case. The following assumption rules out the network effects canceling out with each other for all layers.

**Assumption 7 (Relevant Network Effects)** *There exists $m \in \{1, \ldots, M\}$ and $q \in \{1, \ldots, Q\}$ such that $\delta_{m,q} + \gamma_q \beta_m \neq 0$.*

Assumption 7 extends the condition $\gamma\beta + \delta \neq 0$ for the LiM in Bramoullé et al. (2009) for the monolayer network to the multilayer case. Notice that the condition in 7 is less restrictive than that for the monolayer case because it has to hold only some some layer $m \in \{1, \ldots, M\}$. Again, the additional information embodied in the multilayer network structure increases the number of potential instruments, but the network endogeneity constrains the use of all the possible exclusion restrictions. The following result shows that under the previous assumptions, it is feasible to form a vector of instruments containing relevant information sufficient for identification.

**Proposition 2 (Relevance)** *Let 1 to 7 hold for some $K_c$ and $K_d$ such that $K_d > K_c + 1$. Then, the matrix $\mathbb{E}[\sum_{i\in\mathcal{I}_N} \mathbf{z}_{N;i}\mathbf{d}_{N;i}^\top]$ has full column rank.*

Section 9 presents the proof for this proposition. After showing that the WND assumption can be used to form valid moment conditions that are also relevant under Proposition 2, the following theorem formalizes the identification result.

**Theorem 1** *Suppose Assumptions 1 to 7 hold for some $K_c$ and $K_d$ such that $K_d > K_c + 1$. Then, the parameters $\boldsymbol{\psi}^0 = [\beta_1^0, \ldots, \beta_M^0, \boldsymbol{\delta}_1^0, \ldots, \boldsymbol{\delta}_M^0, \alpha^0, \boldsymbol{\gamma}^0]$ are identified by the moment conditions $\mathbb{E}[\mathbf{m}_N(\boldsymbol{\psi})] = 0$, where $\mathbf{m}_N(\boldsymbol{\psi}) = \sum_{i\in\mathcal{I}_N} \mathbf{z}_{N,i}\varepsilon_{N,i}(\boldsymbol{\psi})$.*

**Remark 2** Theorem 1 is based on the existence of a set of regressors for which Assumption 3 applies. Note that only one of such regressors is necessary to form sufficient moment conditions to identify $\beta_1, \ldots, \beta_m$. The parameters $\boldsymbol{\gamma}$ are only identified for the set of regressors for which part *(ii)* of Assumption 3 is satisfied. It is possible to control for other observable characteristics that are not orthogonal to the errors as long as they are uncorrelated with the set of exogenous regressors. The estimated parameters for those regressors do not have a causal interpretation. Identification of $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_M$ requires the existence of at least $Q$ exogenous regressors. The validity of the instruments formed by the $\mathcal{W}_{m,\delta}$ matrices is guaranteed by Proposition 1.

Section 9 presents the proof for Theorem 1. This theorem shows that identification is possible in a general framework with different layers where the network formation process induces a potential correlation between the regressors and errors. This result exploits the additional information provided by the multilayer network structure and depends crucially on the possibility of forming valid paths. The level of generality of this identification approach makes it applicable to different models and data structures proposed in the literature. The following examples describe how Theorem 1 can be applied in the context of the two empirical cases presented in the previous sections.

**Example 1 (continuation):** Given their data structure, De Giorgi et al. (2010) rules out the possibility of peer effects generated by the spouse's network. Assuming that the social effects from the spouses' network equal zero and ignoring the panel data structure for simplicity, the linear model (1) reduces to

$$y_{N,i} = \alpha + \sum_{j\neq i} w_{N,1;i,j} y_j \beta + \sum_{j\neq i} w_{1;i,j} \mathbf{x}_{N,j}^\top \boldsymbol{\delta} + \mathbf{x}_{N,i}^\top \boldsymbol{\gamma} + \varepsilon_{N,i}, \tag{8}$$

where $\mathbf{W}_{N,1}$ is the co-workers network of interest. The authors also observe the network of spouses $\mathbf{W}_{N,2}$. De Giorgi et al. (2020) assumes that $K_c = 1$, and the structure of non-overlapping networks guarantee that equation (6) in Proposition 1 holds for any $K_d$. With this information, it is possible to construct $\mathcal{W}_{N,1,\beta}$, which can be used to identify the co-workers' peer effects $\beta$. By choosing $K_c = 1$ and any arbitrary $K_d$, it is possible to construct $\mathcal{W}_{N,1,\beta}$ for the example in Figure 1:

$$\mathcal{W}_{N,1,\beta} = \begin{array}{c c} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

De Giorgi et al. (2020) uses individuals connected by length-two paths that change their edge type from co-workers to spouses as an instrument for the peer effects endogenous variable. This set of paths can be characterized by the matrix product of the two layers' adjacency matrices $\mathbf{W}_{N,1}\mathbf{W}_{N,2}$, and it is a subset of the valid paths presented in the matrix $\mathcal{W}_{N,1,\beta}$. Assuming only one regressor for the sake of illustrations, note that $\mathbb{E}[\mathcal{W}_{N,1,\beta}\mathbf{x}_N\boldsymbol{\varepsilon}_N]$ produces a vector of expectations with components such as $\mathbb{E}[x_4\varepsilon_1]$, $\mathbb{E}[x_5\varepsilon_1]$, $\mathbb{E}[x_6\varepsilon_1]$, $\mathbb{E}[x_7\varepsilon_1]$, which are all equal to zero under the results in Proposition 1 for $K_c = 1$, and where $d^1(i,j) \to \infty$ follows from the network structure for all $i$ and $j$.

De Giorgi et al. (2020) does not use any instrument to identify contextual effects. In light of Proposition 1, it is possible to understand the absence of an instrument for contextual effects as assuming $K_d = 0$. Under these assumptions, the matrix $\mathcal{W}_{N,1,\delta}$ reduces to $\mathcal{W}_{N,1,\delta} = \mathbf{W}_{N,1}$, and the matrix of instruments is given by $\mathbf{Z}_N = [\mathbf{W}_{N,1}\mathbf{W}_{N,2}\mathbf{X}_N, \mathbf{W}_{N,1}\mathbf{X}_N, \widetilde{\mathbf{X}}_N]$. Theorem 1 can be apply to show that $[\beta, \boldsymbol{\delta}, \alpha, \boldsymbol{\gamma}]$ are indeed point identified.

**Example 2 (continuation):** Zacchia (2019) considers a model where only contextual effects are relevant, and the researcher observes only one network. His model can be written as

$$y_{N,i} = \alpha + \sum_{j\neq i} w_{N,i,j}\mathbf{x}_{N,j}^{\top}\boldsymbol{\delta} + \mathbf{x}_{N,i}^{\top}\boldsymbol{\gamma} + \varepsilon_{N,i},$$

where $\delta$ is the coefficient of interest for Zacchia (2019). The author has access to panel data, but this example considers the cross-section case for simplicity. Because the author is assuming that the peer effects are zero, the only relevant matrix, in this case, is $\mathcal{W}_{N,\delta}$ constructed based on equation 7. Zacchia (2019) chooses $K_d = 2$ but also includes empirical estimations for $K_d = 3$. Assuming the parameter value $K_d = 2$, the matrix to identify contextual effects for the network in Figure 2, this matrix is given by

$$\mathcal{W}_{n,\delta} = \begin{array}{c c} & \begin{array}{cccc} i & j & k & l \end{array} \\ \begin{array}{c} i \\ j \\ k \\ l \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

Based on $\mathcal{W}_{N,\delta}$, the matrix of instruments can be constructed as $\mathbf{Z}_N = [\mathcal{W}_{N,\delta}\mathbf{X}_N, \widetilde{\mathbf{X}}_N]$, and $\mathbf{D}_N = [\mathbf{W}_N\mathbf{X}_N, \widetilde{\mathbf{X}}_N]$. Theorem 1 can also be applied for this case to show that the parameters $[\boldsymbol{\delta}, \alpha, \boldsymbol{\gamma}]$ are point identified.

### 4.4 Heterogeneous Identifying Power

From the definition of the moment condition matrices, it follows that different individuals may provide different identifying power for the parameters of interest. For instance, if $\|\mathbf{w}_{N,m,\lambda,i}\|_1 = 0$ (where $\mathbf{w}_{N,m,\lambda,i}$ represents the $i$th row of $\mathcal{W}_{N,m,\lambda}$ and $\|\cdot\|_1$ represent the $L_1$ norm), then, individual $i$ does not provide any information to identify the parameter $\lambda$, for $\lambda \in \{\beta, \delta\}$. In addition, isolated individuals do not contribute information to identify the network effects parameters in this context. To formalize this idea, notice that the joint distribution $\mathcal{F}_{\mathbf{X}_N,\boldsymbol{\varepsilon_N},\mathcal{M}_N}$ induces a marginal distribution on the multilayer network $\mathcal{M}_N$, which determines the subset of individuals providing variation to identify the parameters of interest.

Let the random variable $\eta_{N,m,i}$ equal one if individual $i$ is non-isolated in layer $m$, and zero otherwise. Note that $\kappa_{N,m,i} = \mathbb{E}[\eta_{N,m,i}]$ gives the unconditional probability that individual $i$ is non-isolated in layer $m$, where the expectation is taken with respect to the marginal distribution of $\mathcal{M}_N$. Additionally, it is possible to construct a set of random variables that contain possible measures of the expected identifying power for each individual $i$. These measures are based on the probability distribution of $\mathcal{W}_{N,m,\beta}$ and $\mathcal{W}_{N,m,\delta}$ induced by the marginal distribution of $\mathcal{M}_N$ and the values of $K_c$ and $K_d$. Let $\eta_{N,m,\lambda,i}$ equals one if $\|\mathbf{w}_{N,m,\lambda,i}\|_1 > 0$ and zero otherwise. The expectation $\kappa_{N,m,\lambda,i} = \mathbb{E}[\eta_{N,m,\lambda,i}]$ represents the unconditional probability that individual $i$ provides information to identify the parameter $\lambda$, where the expectation is taken with respect to the marginal distribution of $\mathbf{w}_{N,m,\lambda,i}$. Thus, $\kappa_{N,m,\lambda,i}$ measures the expected identifying power of individual $i$ for any values of $\mathbf{X}_N$ and $\boldsymbol{\varepsilon}_N$ in their respective support. Importantly, it is possible to write the moment condition $\mathbf{m}_N(\boldsymbol{\psi}) = \sum_{i\in\mathcal{I}_N} \mathbf{z}_{N,i}\varepsilon_{N,i}(\boldsymbol{\psi})$ in terms of the individuals' expected identifying power.

Define $\mathcal{H}_{N,\lambda,i} = \operatorname{diag}(\eta_{N,1,\beta,i}\boldsymbol{\iota}_Q, \ldots, \eta_{N,M,\delta,i}\boldsymbol{\iota}_Q, \boldsymbol{\iota}_{Q+1})$ as a $(1+R+Q)\times(1+R+Q)$ matrix, where $\mathcal{K}_{N,\lambda,i} = \mathbb{E}[\mathcal{H}_{N,\lambda,i}]$. Similarly, define the $(M+1)(Q+1)\times(M+1)(Q+1)$ matrix $\mathcal{H}_{N,i} = \operatorname{diag}(\eta_{N,1,i}\boldsymbol{\iota}_Q, \ldots, \eta_{N,M,i}\boldsymbol{\iota}_Q, \boldsymbol{\iota}_{Q+1})$, where $\mathcal{K}_{N,i} = \mathbb{E}[\mathcal{H}_{N,i}]$. Then, by the law of total expectations if follows that $\mathbb{E}[\mathbf{z}_{N,i}\mathbf{d}_{N,i}^{\top}] = \mathcal{K}_{N,\lambda,i}\mathbb{E}[\mathbf{z}_{N,i}\mathbf{d}_{N,i}^{\top} \mid \mathcal{H}^*_{N,\lambda,i}, \mathcal{H}^*_{N,i} \neq \mathbf{O}_{R\times R}]\mathcal{K}_{N,i}$, where $\mathcal{H}^*_{N,\lambda,i}$ contains the left top upper matrix of $\mathcal{H}_{N,\lambda,i}$, $\mathbf{O}_{R\times R}$ is the $(R \times R)$ matrix of zeros, and $\mathcal{H}^*_{N,i}$ is defined analogously to $\mathcal{H}^*_{N,\lambda,i}$. By the law of total expectation, $\mathbb{E}[\mathbf{z}_{N,i}\varepsilon_{N,i}(\boldsymbol{\psi})] = \mathcal{K}_{N,\lambda,i}\mathbb{E}[\mathbf{z}_{N,i}\varepsilon_{N,i}(\boldsymbol{\psi}) \mid \mathcal{H}^*_{N,\lambda,i} \neq \mathbf{O}_{R\times R}] + (\mathbf{I}_{1+R+Q} - \mathcal{K}_{N,\lambda,i})\mathbb{E}[\mathbf{z}_{N,i}\varepsilon_{N,i}(\boldsymbol{\psi}) \mid \mathcal{H}^*_{N,\lambda,i} = \mathbf{O}_{R\times R}]$. Note that when $\mathcal{H}^*_{N,\lambda,i} = \mathbf{O}_{R\times R}$ the upper left matrix of the conditional expectation is trivially zero, and the $(Q+1)$ lower right component of $(\mathbf{I}_{1+R+Q} - \mathcal{K}_{N,\lambda,i})$ is also a matrix of zeros. Therefore, $\mathbb{E}[\mathbf{z}_{N,i}\varepsilon_{N,i}(\boldsymbol{\psi})] = \mathcal{K}_{N,\lambda,i}\mathbb{E}[\mathbf{z}_{N,i}\varepsilon_{N,i}(\boldsymbol{\psi}) \mid \mathcal{H}^*_{N,\lambda,i} \neq \mathbf{O}_{R\times R}]$. From the definition of the moment condition, it follows that $\mathbb{E}[\mathbf{m}_N(\boldsymbol{\psi})]$ can be written as $\sum_{i\in\mathcal{I}_N} \mathbb{E}[\mathbf{z}_{N,i}\varepsilon_{N,i}(\boldsymbol{\psi})] = \sum_{i\in\mathcal{I}_N} \mathcal{K}_{N,\lambda,i}\mathbb{E}[\mathbf{z}_{N,i}\varepsilon_{N,i}(\boldsymbol{\psi}) \mid \mathcal{H}^*_{N,\lambda,i} \neq \mathbf{O}_{R\times R}]$. Intuitively, the moment condition is a weighted sum of conditional expectations where the weights are the unconditional probability that $\mu_{N,m,\lambda,i} = 1$. This weighting scheme gives more importance to individuals for whom the probability of finding moment conditions is higher in the population.

## 5 Estimation

This section provides details for the construction of a GMM estimator for the vector of parameters $\boldsymbol{\psi}^0$. The empirical counterparts of the moment conditions used for identification in Theorem 1 are the basis for estimation. Consider the joint distribution $\mathcal{F}_{\mathbf{X}_N,\mathcal{M}_N,\boldsymbol{\varepsilon}_N}$ that characterizes the values of $\mathbf{x}_{N,i}$, $\varepsilon_{N,i}$, and $\mathcal{M}_N$ in an arbitrarily large population where outcomes follow the model in equation (1). Assume that a practitioner observes a random sample of size $n < N$ from that population that preserves the network structure in $\mathcal{M}_N$.[4] This article interprets the sampling mechanism as in Graham (2020). The sample schema is a thought experiment useful to derive limiting distributions convenient in practice.

The analyst observes a sample $\{y_i, \mathbf{x}_i^\top, \{\{w_{m;i,j}\}_{j=1,j\neq i}^n\}_{m=1}^M\}_{i=1}^n$. Then, it is possible to construct the $n \times (M+1)(Q+1)$ matrix of regressors $\mathbf{D}_n$, and the $n \times (1+R+Q)$ matrix of instruments $\mathbf{Z}_n$. The errors are a function of the unknown parameters, $\boldsymbol{\varepsilon}_n(\boldsymbol{\psi}) = \mathbf{y}_n - \mathbf{D}_n\psi$ according to equation (1). The construction of the matrix of instrument $\mathbf{Z}_n$ requires the computation of the matrices $\mathcal{W}_{n,m,\beta}$ and $\mathcal{W}_{n,m,\delta}$ for $m = 1\ldots,M$, which involves evaluating equations (6) and (7) for all the possible dyads $i,j \in \mathcal{I}_n \times \mathcal{I}_n$ (where $\mathcal{I}_n$ represents the set of individuals in the sample). This problem could entail a computational complexity in the order of $n^{\binom{n}{2}}$. I use two algorithms that are based on the idea behind Dijkstra's shortest path algorithm for monolayer graphs. The modified algorithms compute the first and second shortest paths with a minimum number of edge types in polynomial time (Balasubramanian et al., 2022). The algorithms are described in the Online Appendix C and may be of independent interest for those interested in path problems.

The GMM estimator is formed based on the moments $J_N(\boldsymbol{\psi}) = \mathbb{E}[\mathbf{m}_N(\boldsymbol{\psi})]^\top(\mathbf{A}_N^\top\mathbf{A}_N)\mathbb{E}[\mathbf{m}_N(\boldsymbol{\psi})]$, with sample analog given by $J_n(\boldsymbol{\psi}) = [n^{-1}\mathbf{Z}_n^\top\boldsymbol{\varepsilon}_n(\boldsymbol{\psi})]^\top(\mathbf{A}_n^\top\mathbf{A}_n)[\mathbf{Z}_N^\top\boldsymbol{\varepsilon}_n(\boldsymbol{\psi})]$, where $\mathbf{A}_n$ is a fixed $(M+1)(Q+1) \times (1+R+Q)$ full row-rank matrix, assumed to converge to a constant full row-rank matrix $\mathbf{A}_N$. The linearity assumption of the model in (1) guarantees that the GMM estimator has a closed

[4] A sample preserving the network structure in $\mathcal{M}_N$ is one where the sample networks are subgraphs of the population networks. In other words, all sample individuals maintain their edges in the population for that subset of nodes.

form given by

$$\widehat{\boldsymbol{\psi}}_{GMM} = [\mathbf{D}_n^\top \mathbf{Z}_n(\mathbf{A}_n^\top \mathbf{A}_n)\mathbf{Z}_n^\top \mathbf{D}_n]^{-1}[\mathbf{D}_n^\top \mathbf{Z}_n(\mathbf{A}_n^\top \mathbf{A}_n)\mathbf{Z}_n^\top \mathbf{y}_n].$$

The GMM estimator $\widehat{\boldsymbol{\psi}}_{GMM}$ is constructed based on a random sample from the joint distribution $\mathcal{F}_{\mathbf{X}_N,\mathcal{M}_N,\boldsymbol{\varepsilon}_N}$ which allows for correlation between the errors, the regressors, and the multilayer network. As discussed in section 4.1, the weak dependence Assumption in 2 controls the levels of dependence between individuals based on their distance in the multilayer network space. Given that the data used to calculate $\widehat{\boldsymbol{\psi}}_{GMM}$ are assumed to come from a random sample of a weak-dependent population, they will inherit that property for all finite samples of size $n$. In addition to the weak dependence condition, the following assumptions are necessary to characterize the asymptotic behavior of the linear GMM estimator in a context where network dependence is allowed.

**Assumption 8 (Existence of Moments)** *Let the functions $f_{q,\ell}$ and $g_{q',\ell'}$ mapping $\mathbb{R}^{(Q+1)\times 2}$ to $\mathbb{R}$ be such that $f_{q,\ell}(\mathbf{r}_{n,\{i,j\}}) = r^q_{n,i} r^\ell_{n,j}$ and $g_{q',\ell'}(\mathbf{r}_{n,\{h,s\}}) = r^{q'}_{n,h} r^{\ell'}_{n,s}$ for $i,j,h,s \in \mathcal{I}_n$, $i \neq j$, $h \neq s$, $q \neq \ell$ and $q' \neq \ell'$. Assume that $\|f_{q,\ell}(\mathbf{r}_{n,\{i,j\}})\|_{p_f^*} + \|g_{q',\ell'}(\mathbf{r}_{n,\{h,s\}})\|_{p_g^*} < \infty$ for all $q,\ell$ where $p_f^* = \max\{p_{f,i}, p_{f,j}\}$ (analogous for $p_g^*$) and $1/p_{f,i} + 1/p_{f,j} + 1/p_{g,h} + 1/p_{g,s} < 1$.*

This assumption imposes the existence of moments for non-linear functions of $\psi$-dependent random variables. These moments are required to guarantee that the covariances between the transformed random variables for two groups of individuals are bounded for a large enough distance in the network space. Importantly, this assumption does not impose any differentiating restrictions on the correlation structure between the regressors and the errors compared to any two sets of regressors. The following assumption guarantees that the sum of the weights for any adjacency matrix $\mathbf{W}_{n,m}$, and any matrix of moment conditions $\mathcal{W}_{n,\lambda,m}$ does not grow faster than the sample size. Similar to the notation introduced in Definition 3, let $\mathcal{P}_n(i,1,m,\lambda) = \{i : i,j \in \mathcal{I}_N\text{, and } w_{n,m\lambda;i,j} > 0\}$ be the set of all individuals $j$ that are far enough from $i$ given $K_c$ and $K_d$ in a path starting by edge $m$.

**Assumption 9 (Bounded Weights)** *For all realizations $\mathcal{M} \in \mathscr{M}$ the following hold: (i) for all layers $m$, coefficient $\lambda$, and all individuals $i \in \mathcal{I}_n$, $\sum_{j\in\mathcal{P}_n(i,1,m,\lambda)} w_{n,m,\lambda;i,j} = o(n)$ and $\sum_{j\in\mathcal{P}_n(i,1,m)} w_{n,m;i,j} = o(n)$, (ii) the set of individuals formed by the intersection of non-isolated nodes in layer $m$ and $\lambda \in \{\beta,\delta\}$ denoted by $\nu_{m,\lambda}$, and those from the the product of $\zeta$ adjacency matrices organized in a sequence $\phi$, denoted by $\nu_{\zeta,\phi,\lambda,m_1}$, is such that $\sum_{j\in\eta_{i,\mu,\phi}} w_{\mu,\phi;i,j} = o(n)$ for any $j \in \mathcal{I}_n$, all $(i,s) \in \eta_{\mu,\phi}$, any product of adjacency matrices $\zeta$ and any sequence $\phi$.*

Part *(i)* of Assumption 9 guarantees that the weighted sum of individual connections and nodes that can be used as instruments does not grow faster than the sample size. This condition is generally satisfied. In particular, it is immediately satisfied when the matrix $\mathcal{W}_{n,m,\lambda}$ and $\mathcal{W}_{n,m}$ are row-normalized for $\lambda \in \{\beta,\delta\}$ and any $m$. The reason is that $\sum_{j\in\mathcal{P}_n(i,m,\lambda,1)} w_{m,\lambda;i,j} = 1$ for any individual $i$ (and the same is true for the adjacency matrices of the layers). The use of row-normalized adjacency matrices is common in the literature on the econometrics of networks, see, e.g., de Paula (2017). This assumption is related to the relevant condition for identification in section 3, where I pointed out that too dense or too sparse population networks can break out identification. Part *(ii)* guarantees that the number of paths of order $\zeta$ does not grow faster than $n$. This is a technical requirement necessary for the moments of the outcome $\mathbf{y}_n$ to exist. The next assumption imposes a global measure of sparsity on the asymptotic multilayer network $\mathcal{M}_N$.

**Assumption 10 (Dependence Rate of Decay)** *Let $\bar{D}_n(d) = n^{-1}\sum_{i\in\mathcal{I}_n}|\mathcal{P}_n(i,d)|$ be the average number of distance-d connections on the multilayer network network $\mathcal{M}_n$. Then, for all realizations $\mathcal{M} \in \mathscr{M}$, $n^{-1}\sum_{d\geq 1}\bar{D}_n(d)\theta_{n,d} \to 0$ as $n \to \infty$ in probability.*

Assumption 10 captures the trade-off between the network density and the level of dependence that the model can allow via the dependence coefficients $\theta_{n,d}$. Intuitively, it guarantees that, on average, the level of sparsity captured by $\bar{D}_n(d)$ does not increase faster than the dependence between individuals at a distance $d$ for all possible distances. Assumption 10 extends Assumption 3.2 in Kojevnikov et al. (2020) for the case where the distance between individuals is defined in the multilayer network space. In addition to characterizing the rate of decay for the dependence coefficients $\theta_{n,d}$, the central limit theorem results in Kojevnikov et al. (2020) require to impose sparsity restrictions based on average neighborhood sizes and average neighborhood shell sizes. Following Kojevnikov et al. (2020), define a measure for the average neighborhood size as $\delta_n(d;k) = n^{-1}\sum_{i\in\mathcal{I}_n}|\mathcal{P}_n(i,d)|^k$, and a measure for the average neighborhood shell size as

$$\delta_n^-(d,m;k) = \frac{1}{n}\sum_{i\in\mathcal{I}_n}\max_{j\in\mathcal{P}_n(i,d)}\left|\mathcal{P}_n^-(i,m)\setminus\mathcal{P}_n^-(j,d-1)\right|^k,$$

were $\mathcal{P}_n^-(j,d-1) =$ when $d=0$. With these two measures of average density, construct the combined quantity

$$c_n(d,m,k) = \inf_{\alpha>1}\left[\delta_n^-(d,m,k\alpha)\right]^{\frac{1}{\alpha}}\left[\delta_n\left(d;\frac{\alpha}{\alpha-1}\right)\right]^{1-\frac{1}{\alpha}}. \tag{9}$$

The measure of average density in equation (9) is crucial to impose a set of assumptions that are sufficient for Kojevnikov et al.'s (2020) central limit theorem to apply. For an arbitrary position $q$ in $\mathbf{Z}_{n,i}$, define $S_n = \sum_{i\in\mathcal{I}_n} z_{n,i,q}\varepsilon_{n,i}$. Defining $\sigma_n^2 = \text{Var}(S_n)$, the following assumption guarantees the existence of higher order moments, imposes asymptotic sparsity, and bounds the long-run variance.

**Assumption 11 (Average Sparsity)** *For all realizations $\mathcal{M} \in \mathscr{M}$, (i) for some $p > 4$, it follows that $\sup_{n\geq 1}\max_{i\in\mathcal{I}_n}\|z_{n,i,q}\varepsilon_{n,i}\|_p < \infty$. There exists a sequence $m_n \to \infty$ such that for $k = 1,2$, (ii) $\frac{n_q}{\sigma_n^{2+k}}\sum_{d\geq 0} c_n(d,m_n,k)\,\theta_{n,d}^{1-\frac{2+k}{p}} \to 0$ as $n\to\infty$, (iii) $\frac{n^2\theta_{n,m_n}^{1-(1/p)}}{\sigma_n} \to 0$ as $n\to\infty$ w.p.1.*

Lemmas 6 and 7 in the Online Appendix B show that the sample averages $n^{-1}\mathbf{D}_n^\top \mathbf{Z}_n$ and $n^{-1}\mathbf{Z}_n^\top \boldsymbol{\varepsilon}_n$ converge to their population counterparts $\mathbb{E}[\sum_{i\in\mathcal{I}_N} \mathbf{z}_{N,i}\mathbf{d}_{N,i}^\top]$ and $\mathbb{E}[\sum_{i\in\mathcal{I}_N} \mathbf{z}_{N,i}\varepsilon_{N,i}(\boldsymbol{\psi})]$, respectively. Moreover, Lemma 9 shows that $\boldsymbol{\Omega}_n = \mathrm{Var}(\mathbf{Z}_n^\top \boldsymbol{\varepsilon}_n)$ converges to the population variance

$$\boldsymbol{\Omega}_N = \lim_{n\to\infty} n^{-1}\left[\sum_{i=1}^{n} \mathrm{var}(\mathbf{z}_{n,i}\varepsilon_{n,i}) + \sum_{i\neq j} \mathrm{cov}(\mathbf{z}_{n,i}\varepsilon_{n,i}, \mathbf{z}_{n,j}\varepsilon_{n,j})\right] < \infty, \tag{10}$$

which is necessary for the multivariate central limit theorem result in Lemma 10. Summing over all individuals is analogous to summing over all possible distances. By definition, $\mathbb{E}[\mathbf{z}_{n,i}\varepsilon_{n,i}] = 0$ for all $i$ and $n$. Then, the limiting measure of covariance in equation (10) can be written in terms of a generic distance $d$ as

$$\boldsymbol{\Gamma}_N(d) = \sum_{i\in\mathcal{I}_N} \sum_{j\in\mathcal{P}_N(i,d)} \mathbb{E}\left[\mathbf{z}_{N,i}\varepsilon_{N,i}\varepsilon_{N,j}\mathbf{z}_{N,j}^\top\right], \tag{11}$$

which implies that the population variance-covariance matrix $\boldsymbol{\Omega}_N$ can be constructed as the sum of the covariance estimators in equation (11) for all possible distances $d \geq 0$ (equality is allowed to account for the variance),

$$\boldsymbol{\Omega}_N = \sum_{d\geq 0} \boldsymbol{\Gamma}_N(d). \tag{12}$$

After characterizing the variance-covariance matrix $\boldsymbol{\Omega}_N$ both in terms of individual and distances sums, Theorem 2 shows that the optimal GMM estimator is consistent and asymptotically normal. The optimal estimator is given by $\widehat{\boldsymbol{\psi}}^\star_{GMM} = (\mathbf{D}_n^\top \mathbf{Z}_n \boldsymbol{\Omega}_N^{-1} \mathbf{Z}_n^\top \mathbf{D}_n)^{-1}(\mathbf{D}_n^\top \mathbf{Z}_n \boldsymbol{\Omega}_N^{-1} \mathbf{Z}_n^\top \mathbf{y}_n)$.

**Theorem 2** *Let Assumptions 1-10 hold, then* $\widehat{\boldsymbol{\psi}}^\star_{GMM} = \boldsymbol{\psi} + o_p(1)$ *and* $\sqrt{n}(\widehat{\boldsymbol{\psi}}^\star_{GMM} - \boldsymbol{\psi}) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_N^\star)$, *where* $\boldsymbol{\Sigma}_N^\star = (\mathbb{E}[\sum_{i\in\mathcal{I}_N} \mathbf{z}_{N,i}\mathbf{d}_{N,i}^\top]^\top \boldsymbol{\Omega}_N^{-1} \mathbb{E}[\sum_{i\in\mathcal{I}_N} \mathbf{z}_{N,i}\mathbf{d}_{N,i}^\top])^{-1}$.

Section 9 presents the proof for Theorem 2. As discussed before, the expectation $\mathbb{E}[\sum_{i\in\mathcal{I}_N} \mathbf{z}_{N,i}\mathbf{d}_{N,i}^\top]$ can be written as $\sum_{i\in\mathcal{I}_N} \mathcal{K}_{N,\lambda,i}\mathbb{E}[\mathbf{z}_{N,i}\mathbf{d}_{N,i}^\top \mid \mathcal{H}^*_{N,\lambda,i}, \mathcal{H}^*_{N,i} \neq \mathbf{O}_{R\times R}]\mathcal{K}_{N,i}$, where $\mathcal{K}_{N,\lambda,i}$ and $\mathcal{K}_{N,i}$ contain the probabilities of finding moment conditions and the probability of being isolated for individual $i$ in all possible layers $m$. These probabilities implicitly depend on the network formation model and are a function of the network's multilayer connectivity. The results in this theorem are interpreted as the network counterpart of the results by Lee (2007b) in the context of group structures. Lee (2007b) shows that the network parameters of peer and contextual effects have a slower convergence rate than the direct effect coefficients and that they slow down when the groups' size increases. Intuitively, for larger groups, social interactions are more challenging, which could be interpreted as more isolated individuals reducing social interactions in the network framework. Instead of affecting the rate of convergence, this paper shows that, in the context of multilayer networks, the existence of isolated individuals directly affects the variance-covariance matrix of the peer and contextual effects estimators. Moreover, $\mathbf{\Sigma}_N^\star$ depends on the population probabilities of an individual providing identification information. When that probability approaches zero, the $2Q$ upper right sub-matrix of $\mathcal{K}_{N,\lambda,i}$ approaches the zero matrix, and the variance-covariance matrix could grow arbitrarily large. In the extreme case of non-identification, the matrix $\widehat{\boldsymbol{\psi}}_{GMM}^\star$ approaches infinity. This result provides a theoretical justification for the simulation results in Bramoullé et al. (2009), showing that an increase in the graph's density decreases the precision of the peer and contextual effects estimators. Under this framework, denser networks provide fewer opportunities to form moment conditions (see Figure 3 for an example). Theorem 2 exposes a relationship between the network parameters' convergence rate, precision, and network sparsity. To my knowledge, this result is new in the literature on the econometrics of networks.

## 5.1 Covariance Matrix Estimation

The estimation of the asymptotic variance-covariance matrix $\mathbf{\Sigma}_N^\star$ follows the network HAC estimator proposed by Kojevnikov et al. (2020). This class of covariance matrix estimators are formed by taking weighted averages of the network autocovariance terms with weights that are zero if the distance between two nodes is greater than $D$ (see Assumption 2). In principle, as in Kojevnikov et al. (2020), the constant $D$ can be a function of the sample size. The covariance matrix estimator is given by

$$\widehat{\mathbf{\Omega}}_N = \sum_{d\geq 0} K(d/D_n)\frac{1}{n}\sum_{i=1}^{n}\sum_{j\in\mathcal{P}_n(i,d)} \mathbf{z}_{N,i}\widehat{\varepsilon}_{N,i}\widehat{\varepsilon}_{N,j}\mathbf{z}_{N,j}^\top.$$

where $\widehat{\varepsilon}_{n,i} = y_{n,i} - \mathbf{d}_{n,i}^\top\widehat{\boldsymbol{\psi}}_{GMM}^\star$, and $K(d)$ represents the wight associated with the size of the correlation between observed and unobserved characteristics of individuals $i$ and $j$ who are at distance $d$. The kernel function is such that $K(0) = 1$ and $K(x) = 0$ for $x > 1$. Kojevnikov et al. (2020) suggests a set of possible kernel functions that guarantee the expected properties for the covariance matrix estimator. The choice of the bandwidth $D_n$, and whether or not it depends on the sample size, is directly connected to the choices of $K_c$ and $K_d$, and whether or not they depend on the sample size. From this estimator, it follows that the efficient variance-covariance matrix $\mathbf{\Sigma}_n^\star$ can be estimated by

$$\widehat{\mathbf{\Sigma}}_n^\star = \left[n^{-1}\mathbf{Z}_n^\top\mathbf{D}_n\widehat{\mathbf{\Omega}}_n^{-1}n^{-1}\mathbf{Z}_n^\top\mathbf{D}_n\right]^{-1}. \tag{13}$$

### Standard Errors Calculation

The standard errors for the coefficient of interest can be computed by taking the squared-root of the main diagonal elements of the matrix $\widehat{\mathbf{\Sigma}}_n^\star$ after dividing them by $n$. The estimation of the efficient weighting matrix in (13) requires a consistent estimator of $\boldsymbol{\psi}$. A standard two-step GMM approach is used, where the one-step estimator for $\boldsymbol{\psi}$ is two stage least squares.

With this method, it is possible to efficiently estimate the parameters of the MLiM model while at the same time adjusting for multilayer network dependence. These estimation results show that in general, when estimating peer and contextual effects, the researcher should expect larger variances when the network density or sparsity increases. These results are novel and relevant for empirical work concerning network effects.

# 6 Monte Carlo Simulations

This section presents the simulation experiments designed to test the finite sample properties of the GMM estimator proposed in Theorem 2 and its robustness to endogenous multilayer network formation. I use a data generating process that mirrors the endogenous multilayer formation rule described in Appendix A. In particular, the number of regressors in equation (1) are defined as $Q = 1$. I assume that the same observed characteristic $x_i$ affects both the outcome in (1) and the network formation rule in (A-25). The number of exogenous clusters are set to be $K = 10$. I randomly separate nodes into the ten clusters. For each cluster $k \in \{1, \ldots, 10\}$, the unobserved random vector $\mathbf{g}_k$ of size $n_k$ is drawn from a multivariate normal with mean $\mu_k$, variance 1, and inter-cluster correlation of 0.9 for all clusters $k$. I draw the $10 \times 1$ vector of cluster means $\boldsymbol{\mu}$ from a multivariate normal with mean 5, variance 2, and correlation 0.6. The idea of randomly drawing the cluster means is to generate some variation in how close the clusters are to each other. Finally, the vector of observed characteristics for cluster $k$ is generated as $\mathbf{x}_k = \mathbf{g}_k + \boldsymbol{\epsilon}_{1,k}$, where $\boldsymbol{\epsilon}_{1,k}$ follows a multivariate normal standard normal with correlation 0.6 for all clusters.

I assume that the total number of layers is $M = 3$, and I form the network following the rule described in equation (A-25). Based on the resulting network, the outcome vector $\mathbf{y}$ is constructed following equation (3) with parameters $[\beta_1^0, \beta_2^0, \beta_3^0, \delta_1^0, \delta_2^0, \delta_3^0, \gamma^0, \alpha^0] = [0.1, 0.2, 0.3, 1, 2, 3, 2, 1]$, where $\varepsilon_i = g_i + \epsilon_{2,i}$, and $\epsilon_{2,i}$ follows a standard normal distribution. The fact that both $\varepsilon_i$ and $x_i$ depend on $g_i$ induces endogeneity in the model. Given that the cluster membership within the data generating process (DGP) are known, and the ten clusters created are independent to each other, it is possible to exactly calculate the values of $K_c$ and $K_d$ that make equations (6) and (7) hold. Therefore, I can form the correctly specified matrices of moment conditions given by $\mathcal{W}_{m,\beta}$ and $\mathcal{W}_{m,\delta}$ for $m = \{1, 2, 3\}$. Note that the values of $K_c$ and $K_d$ depend on the realization of the random variables presented before, and therefore change from iteration to iteration. There are cases where given $K_c$ and $K_d$, there are not enough identifying variation to estimate the parameters of interest. To solve this issue, the DGP is simulated until I reached a total of 1,000 data sets.

Based on the 1,000 data sets $\left\{y_{n;i}, x_{n;i}, \{w_{n,1;i,j}\}_{j=1,j\neq i}^{n}, \{w_{n,2;i,j}\}_{j=1,j\neq i}^{n}, \{w_{n,3;i,j}\}_{j=1,j\neq i}^{n}\right\}_{i=1}^{n}$ with $n \in \{50, 100, 200\}$, the parameters of interest $[\beta_1^0, \beta_2^0, \beta_3^0, \delta_1^0, \delta_2^0, \delta_3^0, \gamma^0, \alpha^0]$ are estimated by using the proposed GMM estimator. Given that $Q = 1$, in this case, the system is just identified, and concerns about efficiency can be disregarded. Figure 4 presents the results for the Monte Carlo simulation for the parameters $[\beta_2^0, \delta_2^0, \gamma^0]$. The result for the other parameters can be found in the Online Appendix D. The first row in Figure 4 displays the performance of the GMM where the box plots are shown with whiskers displaying the 5% and 95% empirical Monte Carlo quantiles. Across the board, for all parameters, the proposed GMM estimator performs well in terms of bias and sampling variability for sample sizes as small as 50 nodes. As expected, the estimation variability decreases when increasing the sample size. Similarly, the second row of Figure 4 displays the corresponding Q-Q plots for the GMM based on the standardized version of the Monte Carlo replications for sample sizes $n = 50$ (light gray), $n = 100$ (gray), and $n = 200$ (black). The blue dashed line depicts the 45 degree line. This plot shows that the asymptotic normal approximation in Theorem 2 works well even with small samples. The approximation improves when the sample size increases. These results confirm that the proposed GMM estimator is robust to multilayer network formation process, under the assumption that individuals' dependence decreases with their distance in the multilayer network space.

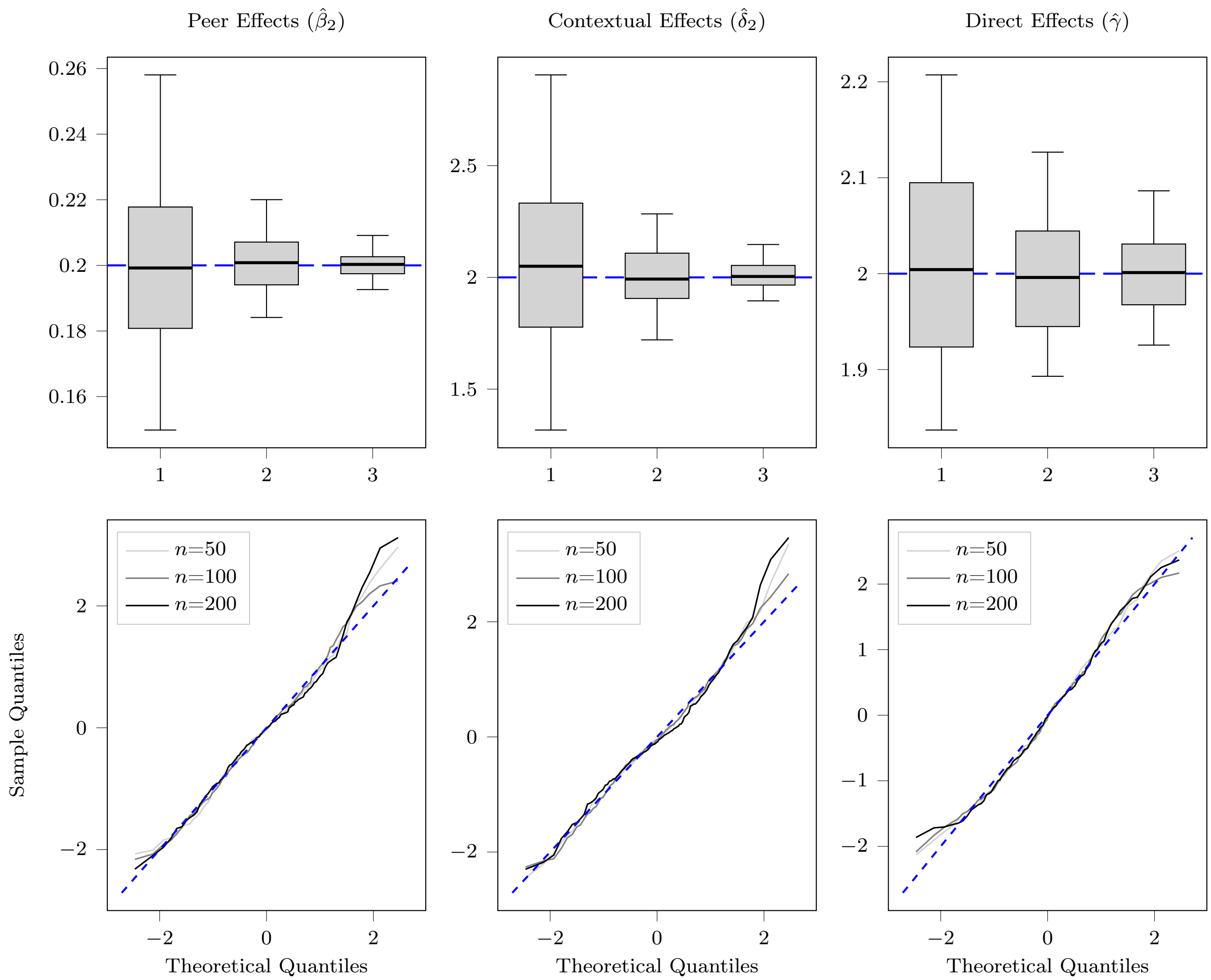
# 7 Application to Publication Outcomes in Economics

To illustrate the behavior of the efficient GMM estimator presented in Theorem 2, I use data of 1,628 peer-reviewed articles published between 2000-2006 in the top-four general-interest journals in Economics. The availability of online scientific research repositories has resulted in a stable source of data to uncover scholars' professional connections. Additionally, when linked with scholars' biographical public information, other type of professional connections such as Ph.D. alumni, colleagues and advisorship can be uncovered (Colussi, 2018). A major challenge when performing causal inference analysis with these data is that the professional networks that scholars form are inherently correlated with publications outcomes such as citation counts. Therefore, this is an ideal scenario where the proposed method can be used to uncover network effects and attach them causal interpretation using observational data.

## 7.1 Data on Publication Outcomes

The data consists of all peer-reviewed articles published in the top 4 general-interest journals in Economics, namely the *American Economics Review* (AER), *Econometrica* (ECA), the *Journal of Political Economy* (JPE), and the *Quarterly Journal of Economics* (QJE) between 2000 and 2006. This information was web scrapped from three different sources: Ideas RePEc, Scopus, and the journal websites themselves. This allows to correctly identify editors' names and tenure, as well as research articles' titles, page numbers, total number of bibliographic references, complete authors' names and affiliation at the time of publication, authors-provided Journal of Economic Literature (JEL) codes (AER, QJE), and keywords (ECA). After excluding editorial reports, conference announcements and proceedings, corrigendums, comments, replies, special issues, and Nobel prize lectures reprints, a total of 1,628 articles are used. This roughly coincides with the 1,657 articles compiled by Card and DellaVigna (2013), and the 1,620 papers identified by Colussi (2018) for the same journals and time period. The total number

Figure 4: Box Plots and Q-Q plots of the GMM Estimator




Note: Box plots in the first row depict the Monte Carlo performance of the proposed GMM estimator. The boxplots are based on 1,000 for sample sizes $n \in \{50, 100, 200\}$. The whiskers display the 5% and 95% empirical quantiles. Q-Q plots in the second row are based on the standardized sample of 1,000 Monte Carlo replications of the proposed GMM estimator of the parameters in (1) and sample sizes $n = 50$ (light gray), $n = 100$ (gray), and $n = 200$ (black). The blue dashed line shows the 45 degree line.

of citations 8 years post publication for each article was then extracted from Scopus and completed with that of Colussi (2018). Similarly, having identified 1,988 unique authors and 42 unique editors (37 of which also published papers in these journals in this time period), information regarding their gender, research interests (coded as a JEL code), education, and employment history was obtained from online profiles and online curricula vitae by means of web scrapping and text mining. This information was then merged with similar attributes already collected by Colussi (2018) for 1,882 of them.

Table 1 presents descriptive statistics. It shows that the number of peer-reviewed research articles published in these journals is relatively stable in this time period, and that the AER has the largest share of about 40% of the total number of articles for each year. There was an average of 1.8 authors per paper in this time period, and the average number of pages per article has slightly increased from 25 in 2002 to 27 in 2006. All these findings coincide overall with the facts presented by Card and DellaVigna (2013) in a larger time period including ours. The total number of citations up to 8 years post publication allows direct comparisons of the impact of papers published at different points in time. The average citation per article is relatively constant with an average of 60 citations per paper. The QJE had a higher-than-average citation per article relatively to the other journals in 2002 and 2003. Authors' gender allows to identified articles written by same-gender authors (males or females only) as well as different-gender authors (males and females). Different-gender articles represent about 14% of the total number of articles and this proportion does not show any particular trend in this time period.

### 7.2 Multilayer Network Data

The multilayer network is composed of four different types of connections among authors and editors (scholars hereafter), and it can be constructed using their co-authorship information, research interests, education, and employment history. In particular, the *Co-author* layer is constructed by defining connections (edges) between scholars $l$ and $k$ if they co-authored a paper together. The *Alumni* layer is constructed by creating edges between scholars $l$ and $k$ if both obtained their Ph.D. from the same institution during the same time window. The *Ph.D. Advisor* connection is constructed by creating a link between two scholars if one of them held the rank of Assistant, Associate, or Professor at the same university and in the same year in which the other obtained his or her Ph.D, and they share least one research field. Finally, two scholars are said to be *Colleagues* if they worked in the same institution in the same time period. Details of how these types of academic ties among scholars are constructed can be found in the Online Appendix D.

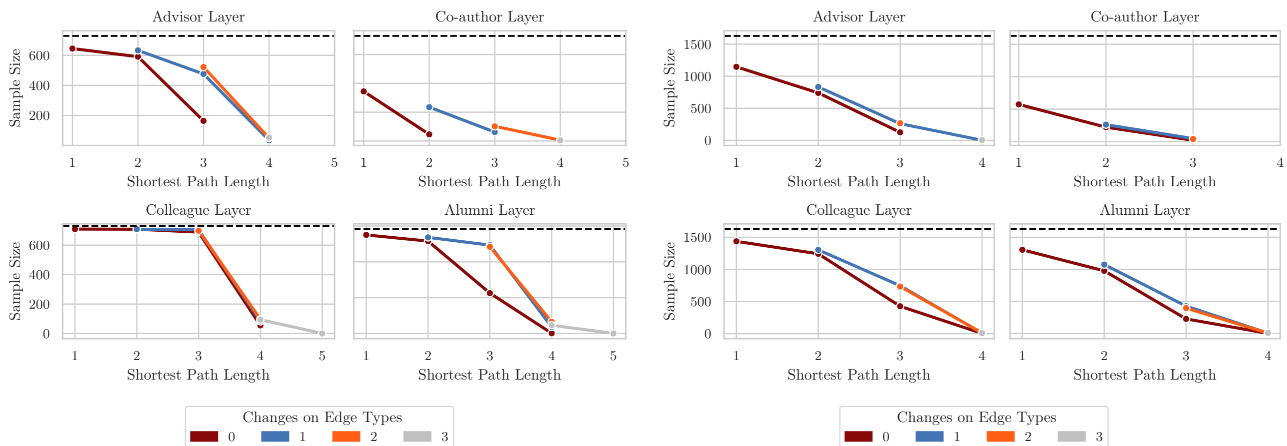
Since articles' authors are observed, these four types of professional ties can then be collapsed at the article level instead, i.e., articles $i$ and $j$ are connected in networks $\mathbf{W}_{n,m}$ if at least one of the authors of article $i$ shares a $m$-type connection with at least one of the authors of article $j$ respectively. These networks are formed in a cumulative way, e.g., the 459 articles (nodes) in 2001 include the 221 articles published in 2000 and so on till all 1,628 articles are accounted for in 2006. The change in

Table 1: Descriptive Statistics for Research Articles

| **Variable** | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 |
|---|---|---|---|---|---|---|---|
| **Number of Authors** | 376 | 410 | 486 | 411 | 395 | 410 | 411 |
| AER | 136 | 166 | 189 | 168 | 140 | 169 | 172 |
| ECA | 93 | 111 | 153 | 107 | 107 | 111 | 108 |
| JPE | 87 | 79 | 93 | 80 | 98 | 76 | 71 |
| QJE | 74 | 80 | 73 | 80 | 80 | 82 | 84 |
| **Number of Articles** | 221 | 238 | 270 | 232 | 226 | 225 | 216 |
| AER | 79 | 88 | 92 | 90 | 79 | 89 | 88 |
| ECA | 51 | 64 | 90 | 59 | 61 | 55 | 51 |
| JPE | 49 | 44 | 48 | 43 | 46 | 41 | 37 |
| QJE | 42 | 42 | 40 | 40 | 40 | 40 | 40 |
| **Number of Citations** | 10,856 | 12,088 | 14,161 | 14,079 | 14,564 | 14,861 | 13,103 |
| AER | 4,140 | 4,833 | 4,476 | 4,586 | 5,354 | 5,606 | 5,289 |
| ECA | 2,530 | 2,773 | 3,346 | 3,379 | 2,811 | 2,724 | 2,914 |
| JPE | 1,692 | 1,446 | 2,240 | 1,954 | 2,787 | 3,420 | 1,852 |
| QJE | 2,494 | 3,036 | 4,099 | 4,160 | 3,612 | 3,111 | 3,048 |
| **Number of Editors** | 16 | 20 | 23 | 16 | 18 | 24 | 14 |
| AER | 3 | 2 | 7 | 6 | 4 | 8 | 9 |
| ECA | 4 | 9 | 5 | 3 | 8 | 2 | 3 |
| JPE | 4 | 2 | 7 | 4 | 3 | 7 | 1 |
| QJE | 5 | 7 | 4 | 3 | 3 | 7 | 1 |
| **Number of Pages** | 5,486 | 5,984 | 6,870 | 6,327 | 6,356 | 6,529 | 5,979 |
| AER | 1,409 | 1,537 | 1,557 | 1,636 | 1,589 | 1,702 | 1,807 |
| ECA | 1,297 | 1,566 | 2,341 | 1,735 | 1,772 | 1,877 | 1,559 |
| JPE | 1,337 | 1,379 | 1,424 | 1,396 | 1,429 | 1,397 | 1,131 |
| QJE | 1,443 | 1,502 | 1,548 | 1,560 | 1,566 | 1,553 | 1,482 |
| **Number of References** | 7,042 | 7,425 | 8,426 | 7,633 | 7,645 | 8,115 | 7,141 |
| AER | 2,420 | 2,783 | 2,729 | 2,896 | 2,576 | 3,032 | 2,888 |
| ECA | 1,388 | 1,695 | 2,598 | 1,772 | 2,013 | 1,886 | 1,467 |
| JPE | 1,285 | 1,311 | 1,452 | 1,391 | 1,503 | 1,445 | 1,166 |
| QJE | 1,949 | 1,636 | 1,647 | 1,574 | 1,553 | 1,752 | 1,620 |
| **Different Gender** | 24 | 34 | 37 | 34 | 29 | 27 | 41 |
| AER | 7 | 15 | 18 | 16 | 9 | 15 | 21 |
| ECA | 7 | 6 | 8 | 5 | 5 | 2 | 9 |
| JPE | 4 | 6 | 5 | 5 | 7 | 0 | 3 |
| QJE | 6 | 7 | 6 | 8 | 8 | 10 | 8 |

Note: Descriptive statistic for articles published in the *American Economics Review* (AER), *Econometrica* (ECA), the *Journal of Political Economy* (JPE), and the *Quarterly Journal of Economics* (QJE) between 2000 and 2006. 'Number of Authors' counts the total number of unique authors that participated in the writing of the articles presented in the variable 'Number of Papers.' 'Number of Citations' refers to the total citations up to 8 years post publication. 'Number of Editors' shows the total number editors in these journals with at least 1 year tenure in the time period. 'Number of References' counts the total number of items that each paper cites in its bibliographic references section, and 'Different Gender' counts the total number of articles written by co-authors of different gender.

Figure 5: Number of Individuals Providing Identifying Information




Panel (a): 2002

Panel (b): 2006

Note: Panels (a) and (b) present four plots showing the total number of articles for which it is possible to find paths at different geodesic distances and edge-type changes using each of the four layers in the data as the starting point and for the years 2002 and 2006, respectively. For each plot, the dashed line represents the total number of scholars in the sample. Given the cumulative nature of the network construction, the total number of articles was 725 in 2002 and 1,628 in 2006. As is usually the case in observed networks, finding paths containing larger geodesic distances and more edge-type changes becomes increasingly more challenging.

the unit of analysis from authors to articles facilitates the definition of the quality outcome variable to be a function of the directly observable number of citations, instead of a more complicated proxy for the authors unobserved quality. Moreover, it avoids potential mechanical correlations caused by two authors coauthoring the same quality articles.

Considering the multilayer network at the article level, Figure 5 displays the total number of nodes for which it is possible to find paths at different geodesic distances and edge-type changes using the Advisor, Co-author, Colleague, and Alumni layers as starting points. Panels (a) and (b) show the results for the years 2002 and 2006 with total sample sizes of 725 and 1,628, respectively. Compared with the full sample sizes depicted by the dashed lines, these figures show that finding paths containing larger geodesic distances and more edge-type changes becomes increasingly less likely. Moreover, note that the effective sample sizes for all combinations of shortest path lengths and edge-type changes become more distant from the full sample sizes when the number of nodes increases from 2002 to 2006. Figure 5 reinforces the idea that even when it is more likely that individuals at larger multilayer network distances have uncorrelated characteristics, extending the zero correlation parameters $K_c$ and $K_d$ comes at the cost of larger standard errors generated by the decreasing number of individuals for whom it is possible to form moment conditions (see equation (22) in the proof for Theorem 2).

The patterns highlighted in Figure 5 also show the importance of considering the edge-type changes when defining the multilayer distance measure in (4). Consider, for example, the objective of estimating the peer effects for the co-author's layer by only considering the multilayer minimum path length $d_n^*(i,j)$ and a hyperparameter. Set $K_d$ to be, for instance, $K_d = 3$. It is only possible to form moment conditions for the number of individuals with shortest path length values of 3, 4, and 5. For the 2002 data, the proportion of nodes with moment conditions is only about 24%, reducing to about 6% for the 2006 data. This trend could be problematic when considering arbitrarily large populations. However, by including the number of edge-type changes in the multilayer measure of distance, it is possible to add a new set of moment conditions by following equation (6) in Proposition 1 for some additional hyperparameter $K_c$. By considering $K_c$ to be fixed at, say, $K_c = 1$, the total number of individuals with moment conditions increases to approximately 57% for 2002 and 23% for 2006, a significant improvement over only using shortest path lengths.

### 7.3 Estimation of Network Effects in Publication Outcomes

This section aims at quantifying the potential existence of human capital externalities (peer effects) among scholars publishing in the 4 top general-interest journals described above. Specifically, the running hypothesis is that if a paper's authors are connected with other scholars who produce good quality articles (measured in terms of citations, see, e.g., Card et al. 2020), the quality of their own article will increase because of the existence of strategic complementarities, see., e.g., Boucher and Fortin (2016). The previously defined professional connections of co-authorship ($m = 1$), alumni ($m = 2$), advisorship ($m = 3$), and colleagues ($m = 4$) are taken into account in the following specific case of the estimating equation (1):

$$y_{i,r,t} = \alpha + \sum_{m=1}^{3} \sum_{j \neq i} w_{m,i,j,t} y_{j,r,t} \beta_m + \sum_{m=1}^{3} \sum_{j \neq i} w_{m,i,j,t} \widetilde{\mathbf{x}}_{j,r,t}^{\top} \boldsymbol{\delta}_m + \mathbf{x}_{i,r,t}^{\top} \boldsymbol{\gamma} + \lambda_r + \lambda_t + \lambda_0 + \varepsilon_{i,r,t}, \qquad (14)$$

where $y_{i,r,t}$ represents the natural logarithm of the total number of citations up to 8 years post publication of article $i$, in journal $r$, at time $t$. The scalar $w_{m,i,j,t}$ represents the $(i,j)$ entry of the $\mathbf{W}_m$ adjacency matrix for the *co-authorship*, *alumni*, *advisorship* and *colleagues* networks at time $t$. The controls in $\mathbf{x}_{i,r,t}$ include dummies for whether current or previous editors of journal $r$ at time $t$ are authors of article $i$ (`Editor`), and for whether all the authors of article $i$ have different gender (`Different Gender`). The latter is coded as zero for single-authored publications. Other articles characteristics are also included such as the total number of pages (`Number of Pages`), total number of authors (`Number of Authors`), total number of bibliographic references (`Number of References`), and another dummy that equals one for isolated articles in the three different networks (`Isolated`). Contextual effects are only calculated for the `Editor` and `Different Gender` covariates, i.e., $\widetilde{\mathbf{x}}_{j,r,t}$. Model (14) is estimated in a rolling-regression setting for $t = 2002$, 2003, 2004, 2005, and 2006, i.e., the estimating sample each year includes those from previous years. Results for the year 2000 and 2001 are not included because they suffer degrees-of-freedom problems given specification (14). Given that the estimator in this paper is designed for cross-sectional data, I pull all the years together and add year fixed effects ($\lambda_t$). Estimating equation (14) also includes journal ($\lambda_r$) fixed effects. As mentioned before, the scalar structural error $\varepsilon_{i,r,t}$ is such that $\mathbb{E}(\boldsymbol{\varepsilon}|\mathbf{X}, \mathbf{W}_1, \ldots, \mathbf{W}_4) \neq 0$ because of the potential endogeneity of the professional networks included in the estimation. A potential reason to be concerned about network endogeneity is that authors producing high quality papers may be connected with each other just because they are similar in their labels of skills, i.e., peer effects can be confounded with unobserved heterogeneity or homophily.

For this empirical application, based on the distribution of individuals with moment conditions, I the constants $K_c$ and $K_d$ to be $K_c = 1$ and $K_d = 3$. the number of individuals with identifying moment conditions reduces to near the point of non-identification. This choice of parameters implies that individuals who are connected by paths with one edge type change or at least three edges are assumed to be uncorrelated. Therefore, changes in their exogenous characteristics can be used as exogenous variation to identify the parameters of interest. In particular, I use the `Different Gender`, `Number of Pages`, `Number of Authors` and `Number of References` as the exogenous characteristics to form the moment conditions in equations (6) and (7). Table 2 presents the results for the Efficient GMM estimator characterized in Theorem 2. For comparison, the results for the first stage GMM estimator and the OLS estimator are presented in Tables 3 and 4 in the Online Appendix D. I perform the same analysis by changing the constants $K_c$ and $K_d$ to be $K_c = 2$ and $K_d = 4$. The magnitude and direction of the estimated parameters in Table 2 are robust to changes in these constants. However, as suggested by the identification and asymptotic theory, the standard errors of the estimated coefficients increase. The reason is that the amount of information that can be used to identify and estimate the parameters of interest is decreasing in $K_c$ and $K_d$. When trying to perform the analysis for values of $K_c = 3$ and $K_d = 5$, all the standard errors greatly surpass the value of the estimated parameters, this presents empirical evidence of the remarks in Theorem 2 arguing that when the probability of finding moment conditions approaches zero, the parameters' variance-covariance matrix could grow arbitrarily large.

Regarding the estimators' behavior, Tables 2 and 3 confirms the results predicted by the estimation theory. Across the board, the estimated standard errors of the Efficient GMM estimator are lower than those of the first stage GMM. The coefficients estimated by the first stage GMM and the Efficient GMM are relatively consistent, which shows empirical evidence of the consistency results in Theorem 2. Additionally, the OLS estimated peer and contextual effects parameters in Table 4 differ from those estimated by the proposed GMM method. This result is expected. As argued before, the layers included in the estimation are likely to be formed endogenously.

In terms of empirical findings, Table 2 provides different meaningful results. First, building upon Ductor et al. (2014), peer effects are found to be positive and statistically significant for articles' quality coming from the *Co-authors* network for all years. This positive result can potentially reflect human capital spillovers that could act through a variety of mechanisms, e.g., scholars provide feedback on each other's work, serve as referees, and work directly together. These instances of collaboration are often paramount to the extension of existing research agendas and can bring to light novel ideas. However, the peer effects estimators from the other professional networks do not significantly affect the publication outcomes. This result is new to the literature of scholars' research productivity. It emphasizes the importance of a network that guarantees a direct channel of communication between authors instead of other professional networks that may generate fewer interpersonal interactions.

The `Editor` contextual effects are, in general, insignificant for all the networks included in the regression. For each article, this variable counts the number of authors who are or have been editors of at least one of the 4 journals till the relevant year. Then, the contextual effect for article $i$ measures the influence on quality of the average number of editors that have written papers that are connected to $i$ in one of the networks of interest. The intuition here is that editors may have superior information of the publication process, they have greater recognition, and therefore they can potentially produce highly cited papers. However, after controlling for peer effects and other article characteristics, the editor contextual effects are not significantly different from zero across the two networks. This is an interesting result because there has been evidence suggesting that authors connected to editors are more likely to publish their work (see, e.g., Laband and Piette 1994), however, when it comes to citations, the results suggest that articles connected to an editor do not benefit from such a connection. The same is true for potential gender contextual effects, except for the advisor network between the years 2003 and 2005. Having advisors that work in projects with gender diverse groups, seems to have a positive impact on their students publications' outcomes.

Similarly, a larger number of bibliographic references is associated with greater impact in terms of citations for estimators. Finally, the direct effects of `Different Gender` on the articles' quality are all positive and significant for most years. Articles written by authors of different gender have a significantly higher number of citations than articles written by same-gender authors. On average, different-gender articles will have somewhere between 24% and 26% more citations than same-gender articles holding everything else constant. This finding is robust across all estimators and presents evidence of an improvement in the quality of the papers when the research teams are gender diverse after controlling for peer effects, editorial participation, article characteristics, and a complete set of fixed effects. This finding is in line with recent research investigating differences in publication outcomes by gender, see, e.g., Card et al. 2020.

# 8 Conclusion

This paper provides a novel approach to show how multilayer network data structures can be used to identify and consistently estimate network effects. The proposed method applies to an extension of

Table 2: Efficient GMM Estimation Results for Social and Direct Effects

| | 2002 | 2003 | 2004 | 2005 | 2006 |
|---|---|---|---|---|---|
| Peer Effects ($\{\widehat{\beta}\}_{m=1}^{3}$) | | | | | |
| `Co-authors` | 0.425 | 0.453 | 0.574 | 0.534 | 0.523 |
| | (0.117) | (0.107) | (0.111) | (0.103) | (0.271) |
| `Alumni` | 0.125 | 0.128 | 0.195 | 0.148 | 0.369 |
| | (0.237) | (0.201) | (0.237) | (0.164) | (0.381) |
| `Advisor` | -0.133 | 0.192 | -0.655 | -0.369 | -0.105 |
| | (0.229) | (0.253) | (0.316) | (0.227) | (0.360) |
| `Colleagues` | 0.514 | -0.233 | -0.054 | 0.051 | 0.092 |
| | (0.406) | (0.111) | (0.096) | (0.096) | (0.155) |
| Contextual Effects ($\{\widehat{\boldsymbol{\delta}}\}_{m=1}^{3}$) | | | | | |
| `Co-authors: Editor in Charge` | 0.059 | -0.281 | -0.326 | -0.465 | 0.879 |
| | (0.657) | (0.589) | (0.655) | (0.542) | (0.952) |
| `Alumni: Editor in Charge` | -0.031 | 0.055 | 0.019 | 0.051 | 0.417 |
| | (0.201) | (0.150) | (0.136) | (0.125) | (0.312) |
| `Advisor: Editor in Charge` | 0.678 | 0.257 | 1.585 | 0.975 | 0.431 |
| | (0.662) | (0.710) | (0.726) | (0.653) | (0.774) |
| `Colleagues: Editor in Charge` | 0.348 | 0.006 | -0.225 | -0.271 | -0.208 |
| | (0.258) | (0.179) | (0.169) | (0.182) | (0.394) |
| `Co-authors: Different Gender` | -0.462 | -0.541 | -1.548 | -1.089 | 3.001 |
| | (1.508) | (1.117) | (1.496) | (0.827) | (1.776) |
| `Alumni: Different Gender` | -0.439 | -0.207 | -0.329 | 0.039 | -1.041 |
| | (0.401) | (0.363) | (0.372) | (0.288) | (0.866) |
| `Advisor: Different Gender` | 0.238 | 4.096 | 6.062 | 4.224 | 1.268 |
| | (2.735) | (1.566) | (1.740) | (1.443) | (1.764) |
| `Colleagues: Different Gender` | 0.435 | 0.169 | -0.073 | -0.382 | -2.710 |
| | (0.476) | (0.543) | (0.554) | (0.712) | (1.636) |
| Contextual Effects ($\widehat{\boldsymbol{\gamma}}$) | | | | | |
| `Editor in Charge` | -0.099 | -0.081 | -0.042 | -0.029 | 0.011 |
| | (0.125) | (0.111) | (0.133) | (0.118) | (0.138) |
| `Different Gender` | 0.247 | 0.238 | 0.182 | 0.127 | 0.080 |
| | (0.115) | (0.097) | (0.089) | (0.078) | (0.092) |
| `Number of Pages` | 0.023 | 0.022 | 0.019 | 0.017 | 0.016 |
| | (0.004) | (0.003) | (0.003) | (0.003) | (0.003) |
| `Number of Authors` | 0.067 | 0.088 | 0.046 | 0.072 | 0.069 |
| | (0.053) | (0.045) | (0.043) | (0.038) | (0.035) |
| `Number of References` | 0.009 | 0.009 | 0.007 | 0.009 | 0.011 |
| | (0.002) | (0.002) | (0.002) | (0.002) | (0.002) |
| `Co-authors: Isolated` | 1.427 | 1.477 | 1.789 | 1.642 | 1.661 |
| | (0.414) | (0.388) | (0.398) | (0.372) | (1.001) |
| $n$ | 729 | 961 | 1187 | 1412 | 1628 |

Note: Standard errors are in parenthesis and are calculated using the network HAC estimator of the covariance matrix in equation (5.1) where the function $K$ is the Parzen kernel and the bandwidth $D_n = 1.8 \times [\log(\text{ avg.deg } \vee (1.05))]^{-1} \times \log n$. All specifications include indicator variables for Journal and Year. The indicator for isolated nodes in the Alumni, Advisor and Colleagues networks are also included but are not statistically significant.

the linear-in-means model which relaxes the assumption that only one type of network can generate peer and contextual effects. This paper's results provide a tool to identify multilayer network effects in settings with endogenous network formation and network dependence between observed and unobserved individual characteristics. I show that it is possible to identify the model by imposing mild conditions on the dependence structure of the multilayer network space. In particular, I impose $\psi-$weak dependence to model the correlation structure. This work provides a simple linear GMM estimator and characterize its limiting distribution. The asymptotic covariance matrix accounts for the multilayer network dependence and incorporates the possibility that too dense or too sparse networks could provide weak identifying information, increasing the estimator's variance. These results regarding the asymptotic covariance matrix allow constructing correct standard errors for inference.

The proposed method will be helpful to empirical researchers aiming to estimate network effects using observational data. This framework can handle both monolayer and multilayer data structures. However, it is more useful when the analyst observes at least two layers. The reason is that the proposed method is designed to use all the additional information provided by including additional types of connections. In addition, this method is best suited for studies where the researcher has access to a credible exogenous shock. The shocks do not need to be strictly exogenous but rather exogenous from the perspective of other individuals far away in the multilayer network space. This characteristic of the method resembles the partial population identification ideas (Moffitt, 2001).

To showcase the applicability of the proposed method, this work presents an empirical application where the tools of web scraping and text mining are used to construct a data set consisting of all peer-reviewed research articles published in 4 top general-interest economics journals between 2000 and 2006. Using publicly available information on where the authors of these publications obtained their Ph.D. degree from, along with their working history, I construct the *Alumni*, *Ph.D. Advisor*, and *Colleagues* layers, in addition to the *Co-authorship* layer. Results show the existence of positive peer effects in terms of citations among peer-reviewed research articles connected through co-authorship connections of their authors as well as significant positive effects of research teams that are gender diverse on the quality of a paper measured in terms of citation outcomes. I do not find evidence of network effects from any of the other layers. I interpret this result as emphasizing the importance of a network that guarantees a direct communication channel between authors instead of other professional networks that may generate fewer interpersonal interactions. I find evidence of bias in the OLS estimator and increasing standard errors corresponding to increases in the values of $K_c$ and $K_d$ (as predicted by the asymptotic theory).

While this paper provides new results in the econometrics of networks literature, it also opens the possibility for new research. Future work could include the explicit characterization of the relationship between the values of the matrices $\mathcal{K}_{N,\lambda,i}$ and $\mathcal{K}_{N,i}$ and the denseness or sparsity of the multiyear network by potentially assuming a general multilayer network formation process. Another research avenue can be to investigate how to optimally choose the hyperparameters controlling the network dependence or the weights used in the moment conditions.

# 9 Proofs of Main Results

**Proof of Proposition 1.** Choose $f$ such that it selects an arbitrary position $q$ from the vector $\mathbf{r}_{N,j}$, i.e., $f(\mathbf{r}_{N,j}) = V_q \mathbf{r}_{N,j} = x_{N,j,q}$ where $x_{N,j,q}$ denotes the $q$th regressor in $\mathbf{x}_{N,j}$. Similarly, choose $g$ to select the $Q+1$th position of $\mathbf{r}_{N,i} = V_{Q+1}\mathbf{r}_{N,i} = \varepsilon_{N,i}$. Note that $\|f\|_\infty = |x_{N,j,q}| < \infty$, $\|g\|_\infty = |\varepsilon_{N,i}| < \infty$, $\mathrm{Lip}(f) = \mathrm{Lip}(g) = 1$, so that from Assumption 2 *(i)* and *(ii)*:

$$|\mathrm{Cov}\left(x_{N,j,q}, \varepsilon_{N,i}\right)| \leq C\left(|x_{N,j,q}|+1\right)\left(|\varepsilon_{N,i}|+1\right)\theta_{N,d_N^{\mathcal{M}}(i,j)}.$$

By Assumption 2 part *(iii)*, it follows that if $d_N^{\mathcal{M}}(i,j) > D$, then $\theta_{N,d_N^{\mathcal{M}}(i,j)} = 0$. Set $K_d, K_c \in \mathbb{N}_+$ such that $K_d > K_c+1$. Note that if $K_d \leq K_c+1$ the condition in equation (7) does not provide any additional information. The reason is that by definition, the shortest path $d_N^*(i,j)$ and the number of edge type changes $c_N^*(i,j)$ are such that $d_N^*(i,j) \geq c_N^*(i,j)+1$. Let $K_d \leq K_c+1$, then $c_N^*(i,j) \geq K_c$ directly implies $d_N^*(i,j) \geq K_d$. Thus, the inequality $c_N^*(i,j) \geq K_c$ does not provide additional information. Let $K_d = D+1$ and $\tau_{i,j} = d_N^c(i,j) - K_c$ if $d_N^c(i,j) > K_c$ and $\tau_{i,j} = 1$, otherwise. Let the inequalities in the conditioning set of equation (7) hold. Then, $c_N^*(i,j) < K_c$ and $d_N^*(i,j) \geq K_d$. The smallest possible value that $d_N^{\mathcal{M}}(i,j)$ can take is $D+1 > D$. Therefore, $\theta_{n,d_N^{\mathcal{M}}(i,j)} = 0$. Now let the inequalities in (6) hold. If $c_N^*(i,j) \geq K_c$ and $d_N^c(i,j) \geq K_d$, the smallest possible value $d_N^{\mathcal{M}}(i,j)$ can take is $K_c+1+(K_d-K_c)K_c$. The inequality $K_c+1+(K_d-K_c)K_c < D < D+1 = K_d$ holds if and only if $(K_d-K_c)(K_c-1)+1 > 0$, which follows from the restriction $K_d > K_c+1$. By Assumption 3, $\mathrm{Cov}(x_{N,i,q}, \varepsilon_{N,i}) = \mathbb{E}(x_{N,i,q}, \varepsilon_{N,i})$, so that equations (6) and (6) hold for $x_{N,j,q}$. Because $q$ was chosen arbitrarily, the result holds for all $q$ in $\mathbf{x}_i$ completing the proof. ■

**Proof of Proposition 2.** I show that in expectation, the columns of the matrix $\mathbf{z}_{N;i}\mathbf{d}_{N;i}^\top$ are linearly independent for all $i$, implying the matrix $\mathbb{E}[\sum_{i\in\mathcal{I}_N}\mathbf{z}_{N;i}\mathbf{d}_{N;i}^\top]$ has full column rank. Fix $K_c$ and $K_d$ such that $K_d > K_c+1$, and construct the matrices $\mathcal{W}_{N,m,\beta}$ and $\mathcal{W}_{N,m,\delta}$ as in section 4 following equations (6) and (7). Let $\mathbf{w}_{N,m,\lambda,i}$ and $\mathbf{w}_{N,s,i}$ be the $i$th row of the matrices $\mathcal{W}_{N,m,\lambda}$ and $\mathbf{W}_{N,m}$, respectively, where $m, s \in \{1,\dots,M\}$ and $\lambda \in \{\beta,\delta\}$. Similarly, let $\mathbf{x}_{N,q}$ and $\mathcal{Z}_{N,\ell}$ be the $N\times 1$ vectors generated by extracting the $q$th and $\ell$th columns of the matrices $\mathbf{X}_N$ and $\mathcal{Z}_N$, where $\mathcal{Z}_N$ contains a subset of columns from the matrix $\mathbf{X}_N$. Finally, define $x_{N,q,i}$ to be the value of the regressor $q$ for individual $i$. There are three types of generic rows in the matrix $\mathbb{E}[\mathbf{z}_{N;i}\mathbf{d}_{N;i}^\top]$ of the form

$$[\mathbb{E}[\mathbf{w}_{N,m,\beta,i}\mathcal{Z}_{N,\ell}\mathbf{w}_{N,1,i}\mathbf{y}_N],\dots,\mathbb{E}[\mathbf{w}_{N,m,\beta,i}\mathcal{Z}_{N,\ell}\mathbf{w}_{N,M,i}\mathbf{x}_{N,Q}],\dots,\mathbb{E}[\mathbf{w}_{N,m,\beta,i}\mathcal{Z}_{N,\ell}x_{N,Q,i}]], \tag{15}$$

$$[\mathbb{E}[\mathbf{w}_{N,m,\delta,i}\mathbf{x}_{N,q}\mathbf{w}_{N,1,i}\mathbf{y}_N],\dots,\mathbb{E}[\mathbf{w}_{N,m,\delta,i}\mathbf{x}_{N,q}\mathbf{w}_{N,M,i}\mathbf{x}_{N,Q}],\dots,\mathbb{E}[\mathbf{w}_{N,m,\delta,i}\mathbf{x}_{N,q}x_{N,Q,i}]], \tag{16}$$

$$[\mathbb{E}[x_{N,q,i}\mathbf{w}_{N,1,i}\mathbf{y}_N],\dots,\mathbb{E}[x_{N,q,i}\mathbf{x}_{N,Q}],\dots,x_{N,Q,i}^2]]. \tag{17}$$

I check the linear independence across columns for the three generic rows in equations (15), (16) and (17). From Corollary 1, it follows that the peer effect regressor for an arbitrary layer $m$ can be written as

$$\mathbf{w}_{N,m,i}\mathbf{y}_N = \sum_{q=1}^{Q}\sum_{m=1}^{M}\pi_{m,q}\sum_{\zeta=0}^{\infty}\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\left(\prod_{\ell\in\phi}\beta_\ell\right)\mathbf{w}_{N,m,i}\mathbf{A}_N(\zeta,\phi)\mathbf{W}_{N,m}\mathbf{x}_{N,q} \\ +\mathbf{w}_{N,m,i}\mathbf{X}_N\boldsymbol{\gamma}+\mathbf{e}_N, \tag{18}$$

where $\mathbf{e}_N = \sum_{\zeta=0}^{\infty}\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\left(\prod_{\ell\in\phi}\beta_\ell\right)\mathbf{w}_{N,m,i}\mathbf{A}_N(\zeta,\phi)\boldsymbol{\varepsilon}_N$, $\pi_{m,q}=\delta_{m,q}+\gamma_q\beta_m$, and to facilitate notation and without loosing generality I set $\alpha=0$. Replacing the $\mathbf{w}_{N,m,i}\mathbf{y}_N$ for the expression in 18, the vector in 15 can be written as

$$\Bigg[\sum_{q=1}^{Q}\sum_{s=1}^{M}\pi_{m,q}\sum_{\zeta=0}^{\infty}\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\left(\prod_{\ell\in\phi}\beta_\ell\right)\mathbb{E}[\mathbf{w}_{N,m,\beta,i}\mathcal{Z}_{N,\ell}\mathbf{w}_{N,1,i}\mathbf{A}_N(\zeta,\phi)\mathbf{W}_{N,s}\mathbf{x}_{N,q}] \\ +\mathbb{E}[\mathbf{w}_{N,m,\beta,i}\mathcal{Z}_{N,\ell}\mathbf{w}_{N,1,i}\mathbf{X}_N]\boldsymbol{\gamma},\ldots,\mathbb{E}[\mathbf{w}_{N,m,\beta,i}\mathcal{Z}_{N,\ell}\mathbf{w}_{N,M,i}\mathbf{x}_{N,Q}],\ldots,\mathbb{E}[\mathbf{w}_{N,m,\beta,i}\mathcal{Z}_{N,\ell}x_{N,Q,i}]\Bigg], \tag{19}$$

where $\mathbb{E}[\mathbf{e}_N]=\mathbf{0}_N$ from assumption 3 and the law of iterated expectations. The components of the vector in (19) are *linearly dependent* if and only if there exists three sequence of constants $\{v_l\}$ for $l\in\{1,\ldots,(M+1)(Q+1)\}$ all different from zero, such that

$$\mathbb{E}\Bigg[\sum_{l=1}^{M}\sum_{q=1}^{Q}\sum_{s=1}^{M}v_l\pi_{m,q}\sum_{\zeta=0}^{\infty}\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\left(\prod_{\ell\in\phi}\beta_\ell\right)\mathbf{w}_{N,m,\beta,i}\mathcal{Z}_{N,\ell}\mathbf{w}_{N,l,i}\mathbf{A}_N(\zeta,\phi)\mathbf{W}_{N,s}\mathbf{x}_{N,q} \\ +\sum_{l=1}^{M}\sum_{q=1}^{Q}(\gamma_q v_l+v_{M+l+q})\mathbf{w}_{N,m,\beta,i}\mathcal{Z}_{N,\ell}\mathbf{w}_{N,l,i}\mathbf{x}_{N,q}+\sum_{q=1}^{Q}v_{M+QM+q}\mathbf{w}_{N,m,\beta,i}\mathcal{Z}_{N,\ell}x_{N,q,i}\Bigg]=0. \tag{20}$$

Assumptions 4 implies that there exists some individual $i\in\mathcal{I}_N$ for which the linear combination of the elements in the vector $\mathbb{E}[\mathbf{x}_{N,i}\mid\mathcal{M}_N=\mathcal{M}]$ is different from zero. Choose an arbitrary individual $i$ for which that is the case. Therefore, assumption 6 implies that $\sum_{q=1}^{Q}v_{M+QM+q}\mathbb{E}[\mathbf{w}_{N,m,\beta,i}\mathcal{Z}_{N,\ell}x_{N,q,i}]\neq 0$. Moreover, assumptions 4 and 6 guarantee that the rows formed by considering different values of $\mathbf{w}_{N,m,\beta,i}\mathcal{Z}_{N,\ell}$ in expectation for different $m$ and $\ell$ are linearly independent. Thus, the sub-matrix created by taking the last $Q$ columns of $\mathbb{E}[\mathbf{z}_{N;i}\mathbf{d}_{N;i}^\top]$ is composed by linearly independent columns.

Under assumption 7, there exists some $m\in\{1,\ldots,M\}$ and $q\in\{1,\ldots,Q\}$ for which

$$\sum_{l=1}^{M}v_l\pi_{m,q}\sum_{\zeta=0}^{\infty}\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\left(\prod_{\ell\in\phi}\beta_\ell\right)\mathbb{E}[\mathbf{w}_{N,m,\beta,i}\mathcal{Z}_{N,\ell}\mathbf{w}_{N,l,i}\mathbf{A}_N(\zeta,\phi)\mathbf{W}_{N,s}\mathbf{x}_{N,q}]$$

can be different from zero. Moreover, assumptions 4, 6 and 5$(i)$ guarantee that the infinite sum is not zero. Given that, without considering the constants, the first and second components of the sum in (20) share the same variables except for $\mathbf{I}_N$ and $\sum_{s=1}^{M}\sum_{\zeta=0}^{\infty}\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\mathbf{A}_N(\zeta,\phi)\mathbf{W}_{N,s}$, assumption 5(*ii*) ensures at least three linear independent components in the sum for each layer $m\in\{1,\ldots,M\}$. Therefore, the sub-matrix created by taking the first $LM+QM$ columns of $\mathbb{E}[\mathbf{z}_{N;i}\mathbf{d}_{N;i}^{\top}]$ is composed by linearly independent columns. Finally, given that the matrix $\mathbb{E}[\mathbf{X}\mid\mathcal{M}_N=\mathcal{M}]$ is full rank the first $LM+QM$ columns are linearly independent of the second $Q$ columns. The proof for the linear Independence for rows of the type (16) and (17) uses the same arguments. ∎

**Proof of Theorem 1.** First note that Assumption 1 guarantees that the solution for model (1) exists. Fix $K_c$ and $K_d$ with $K_d>K_c+1$, and let Assumptions 2 and 3 holds such that Proposition 1 follows for any realization of $\mathcal{M}\in\mathscr{M}$. Combining the results of Proposition 1 with the law of iterated expectations, it follows that $\mathbb{E}[\mathbf{m}(\boldsymbol{\psi}^0)]=0$. Choose an arbitrary vector of parameters $\boldsymbol{\psi}\in\boldsymbol{\Psi}$ such that $\mathbb{E}[\mathbf{m}(\boldsymbol{\psi}^0)]=0$. Notice that $\mathbb{E}[\sum_{i\in\mathcal{I}_N}\mathbf{z}_{N,i}(y_{N,i}-\mathbf{d}_{N,i}^{\top}\boldsymbol{\psi})]=0$ implies $\mathbb{E}[\sum_{i\in\mathcal{I}_N}\mathbf{z}_{N,i}\mathbf{d}_{N,i}^{\top}](\boldsymbol{\psi}^0-\boldsymbol{\psi})+\mathbb{E}[\sum_{i\in\mathcal{I}_N}\mathbf{z}_{N,i}\varepsilon_{N,i}]=0$, and $\mathbb{E}[\sum_{i\in\mathcal{I}_N}\mathbf{z}_{N,i}\mathbf{d}_{N,i}^{\top}]\left(\boldsymbol{\psi}^0-\boldsymbol{\psi}\right)=0$. Under the Assumptions 1 to 7, from Proposition 2, it follows that $\mathbb{E}[\mathbf{m}(\boldsymbol{\psi})]=0$ if and only if $\boldsymbol{\psi}^0=\boldsymbol{\psi}$. ∎

**Proof of Theorem 2.** The GMM estimator from Section 5 in the main text can be written as

$$\widehat{\boldsymbol{\psi}}_{GMM}=\boldsymbol{\psi}+(n^{-1}\mathbf{D}_n^{\top}\mathbf{Z}_n\left(\mathbf{A}_n^{\top}\mathbf{A}_n\right)n^{-1}\mathbf{Z}_n^{\top}\mathbf{D}_n)^{-1}n^{-1}\mathbf{D}_n^{\top}\mathbf{Z}_n(\mathbf{A}_n^{\top}\mathbf{A}_n)n^{-1}\mathbf{Z}_n^{\top}\boldsymbol{\varepsilon}_n \tag{21}$$

By construction, the matrix $\mathbf{A}_n$ is assumed to converge to $\mathbf{A}_N$, so that $(\mathbf{A}_n^{\top}\mathbf{A}_n)\to(\mathbf{A}_N^{\top}\mathbf{A}_N)$ as $n\to\infty$, which is assumed to be finite and full rank. From Lemma 6, $n^{-1}\mathbf{Z}_n^{\top}\mathbf{D}_n$ converges to the population quantity $\mathbb{E}[\sum_{i\in\mathcal{I}_N}\mathbf{z}_{N,i}\mathbf{d}_{N,i}^{\top}]$, which is finite given Proposition 2. Finally, Lemma 7 shows that $n^{-1}\mathbf{Z}_n^{\top}\boldsymbol{\varepsilon}_n(\boldsymbol{\psi})$ converges to $\mathbb{E}[\sum_{i\in\mathcal{I}_N}\mathbf{z}_{N,i}\varepsilon_{N,i}(\boldsymbol{\psi})]=0$. It follows that $\widehat{\boldsymbol{\psi}}_{GMM}=\boldsymbol{\psi}+o_p(1)$. For asymptotic normality, note that, from equation (21)

$$\sqrt{n}(\widehat{\boldsymbol{\psi}}_{GMM}-\boldsymbol{\psi})=(n^{-1}\mathbf{D}_n^{\top}\mathbf{Z}_n\left(\mathbf{A}_n^{\top}\mathbf{A}_n\right)n^{-1}\mathbf{Z}_n^{\top}\mathbf{D}_n)^{-1}n^{-1}\mathbf{D}_n^{\top}\mathbf{Z}_n(\mathbf{A}_n^{\top}\mathbf{A}_n)n^{-1/2}\mathbf{Z}_n^{\top}\boldsymbol{\varepsilon}_n.$$

Let $\mathbf{Q}_{zx}=\mathbb{E}[\sum_{i\in\mathcal{I}_N}\mathbf{z}_{N,i}\mathbf{d}_{N,i}^{\top}]$. From Lemmas 6 and 8 it follows that

$$\sqrt{n}\left(\widehat{\psi}_{GMM}-\psi\right)\xrightarrow{d}\left[\mathbf{Q}_{zx}^{\top}\left(\mathbf{A}^{\top}\mathbf{A}\right)\mathbf{Q}_{zx}\right]^{-1}\mathbf{Q}_{zx}^{\top}\left(\mathbf{A}^{\top}\mathbf{A}\right)N(\mathbf{0},\boldsymbol{\Omega}_N),$$

The efficient weighing matrix is given by $\mathbf{A}_N=\boldsymbol{\Omega}_N^{-1/2}$ so that $\mathbf{A}_N^{\top}\mathbf{A}_N=\boldsymbol{\Omega}_N^{-1}$. With that choice of weighting matrix, the asymptotic variance-covariance matrix is given by

$$\boldsymbol{\Sigma}_N^{\star}=[\mathbf{Q}_{zx}^{\top}\boldsymbol{\Omega}_N^{-1}\mathbf{Q}_{zx}]^{-1}$$

By the remarks presented in section 4 about conditional expectation interpretation of $\mathbf{Q}_{zx}$, it follows that $\boldsymbol{\Sigma}_N^{\star}$ can also be written as

$$\left[\left(\sum_{i\in\mathcal{I}_N}\mathcal{K}_{N,\lambda,i}\mathbb{E}[\mathbf{z}_{N,i}\mathbf{d}_{N,i}^\top \mid \mathcal{H}^*_{N,\lambda,i},\mathcal{H}^*_{N,i}\neq\mathbf{O}_{R\times R}]\mathcal{K}_{N,i}\right)^\top \mathbf{\Omega}_N^{-1}\left(\sum_{i\in\mathcal{I}_N}\mathcal{K}_{N,\lambda,i}\mathbb{E}[\mathbf{z}_{N,i}\mathbf{d}_{N,i}^\top \mid \mathcal{H}^*_{N,\lambda,i},\mathcal{H}^*_{N,i}\neq\mathbf{O}_{R\times R}]\mathcal{K}_{N,i}\right)\right]^{-1}, \tag{22}$$

Equation (22) shows that the efficient asymptotic variance-covariance matrix of the coefficient vector $\widehat{\psi}_{GMM}$ can grow arbitrarily large if the matrices $\mathcal{K}^*_{N,\lambda,i}$ and $\mathcal{K}^*_{N,i}$ (containing the upper right sub-matrices related with the values of $\kappa_{N,m,\lambda,i}$ and $N,m,i$) are too close to the zero matrix. I interpret this result as week identification of the peer and contextual effects parameters when the probabilities of finding moment conditions are low. Those probabilities are linked with the density/sparsity of the population network. The efficient GMM estimator is given by $\widehat{\boldsymbol{\psi}}^\star_{GMM} = (\mathbf{D}_n^\top\mathbf{Z}_n\mathbf{\Omega}_n^{-1}\mathbf{Z}_n^\top\mathbf{D}_n)^{-1}(\mathbf{D}_n^\top\mathbf{Z}_n\mathbf{\Omega}_n^{-1}\mathbf{Z}_n^\top\mathbf{y}_n)$. The previous arguments imply that $\sqrt{n}(\widehat{\boldsymbol{\psi}}^\star_{GMM}-\boldsymbol{\psi}) \xrightarrow{d} \mathcal{N}(\mathbf{0},\mathbf{\Sigma}^\star_N)$. Note that the consistency argument also applies to $\widehat{\boldsymbol{\psi}}^\star_{GMM}$ given that by Lemma 9, $\mathbf{\Omega}_n \to \mathbf{\Omega}_N < \infty$, as $n\to\infty$. ∎

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# Social Interactions in Multilayered Observational Networks – Additional Appendix –

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## Appendix A Bayesian Game of Social Interactions

**Players.** I consider a set of $\mathcal{I}_N$ individuals who, for simplicity, are characterized by only one individual feature $x_{N,i} \in \mathbb{R}$, instead of $Q$ characteristics, as in section 3. I also assume the existence of an additional *exogenously given* characteristic $\zeta_{N,i} \in \mathbb{R}$ that agents observe only at the outset of the first stage. The vector $\boldsymbol{\zeta}_N \in \mathbb{R}^N$ is used to partition individuals into $K$ disjoint clusters $\{C_1, \ldots, C_K\}$. The partition groups the closer individuals in terms of the characteristic $\zeta_{N,i} \in \mathbb{R}$ into the same cluster. Finally, individuals are also endowed with $b_{N,i} \in \mathbb{R}$, which is a private characteristic observable only to individual $i$.

**Actions.** In the first stage individuals choose a set of connections in the multilayer network $\mathcal{M}_N$. From Assumption 1, the multilayer network has no cycles, $w_{N,m;i,i} = 0$ for all $m$, and I assume the network $\mathcal{M}_N$ is directed. Therefore, in the first stage, individual $i$ chooses actions $\mathbf{w}_{N,m;i} = [w_{N,m;i,1}, \ldots, w_{N,m;i,N}] \in \{0,1\}^N$, for each layer $m \in \{1, \ldots, M\}$. In the second stage, individuals choose an action (outcome) $y_{N,i} \in \mathbb{R}$ that can be interpreted, for example, as an effort level.

**Information.** The characteristic $\zeta_{N,i} \in \mathbb{R}$ is common knowledge only at the outset of the first stage. Let $\{C_1, \ldots, C_K\}$ be the sequence of $K$ disjoint clusters separating individuals based on their distance in the vector of characteristics $\boldsymbol{\zeta}_N$. Then, (i) individual $i$ only observes $x_j$ if both individuals $i$ and $j$ belong to the same cluster $C_k$ for any arbitrary cluster $k$. Moreover, individual $i$ observes $x_j$ if they share a connection in any layer, i.e., $i, j \in E_m$ for any $m \in M$, and (ii) for all $i$, the realization of $b_{N,i}$ is private information. Intuitively, this informational assumption imposes restrictions on individuals' knowledge based on distances along the vector of characteristics $\boldsymbol{\zeta}_N$. The idea of this distance can include, for example, geographical distances, as discussed in Kuersteiner (2019), or economic distances, as in Zacchia (2019).

**Distributions** there is a prior exogenous distribution of private characteristics $b_i$ denoted by $\rho_b$, which is common knowledge. I also assume the existence of a conditional distribution characterizing the random vector $\mathbf{x}_N$ given $\boldsymbol{\zeta}_N$. This conditional distribution is denoted by $\rho_{x,\zeta}$. The following assumption imposes restrictions on $\rho_b$ and $\rho_{x,\zeta}$.

**Assumption A.12 (Characterization of $\rho_b$ and $\rho_{x,\zeta}$)** *(i) Second moments of $\rho_b$ exist. (ii) The conditional distribution $\rho_{x,\zeta}$ allows for correlation of $x_i$ and $x_j$ if both individuals $i$ and $j$ belong to the same cluster $C_k$, for some $k \in K$, and they are independent otherwise. (iii) For all $i \in \mathcal{I}_N$, $x_i$ and $b_i$ are uncorrelated.*

**Equilibrium and Timing.** The subgame perfect Nash equilibrium of this two-stage game is characterized by applying backward induction. Therefore, I start by describing the Bayesian social interaction game and characterizing its unique equilibrium. Then, conditional on the equilibrium expected payoffs, I define and characterize the solution of the network formation process.

## Appendix A.1 Bayesian Game of Social Interactions

In this game, individuals maximize their expected utility given their beliefs by choosing an action (outcome) $y_{N,i} \in \mathbb{R}$. A quadratic utility function with strategic complementarities is assumed as in Calvó-Armengol et al. (2009). I extend the utility function to include heterogeneous peer and contextual effects for the $M$ possible layers in the multilayer network $\mathcal{M}_N$:

$$U_i\left(\mathbf{x}_N, \mathbf{y}_N, \mathcal{M}_N, b_{N,i}\right) = \left(\gamma^0 x_{N,i} + b_{N,i} + \sum_{m=1}^{M}\sum_{j\neq i} w_{N,m;i,j} x_{N,j}\delta_m^0\right) y_{N,i} + \sum_{m=1}^{M}\sum_{j\neq i} w_{N,m;i,j} y_{N,i} y_{N,j}\beta_m^0 - \frac{1}{2}y_{N,i}^2.$$

Note that the informational assumptions guarantee that any individual $i$ can evaluate the value of her utility for all the necessary peers characteristics $x_{N,j}$. As in Blume et al. (2015), I look for a Bayes-Nash equilibrium where individuals maximize their expected payoffs given their types. Individuals' types are constructed as follows. Let $\mathbf{x}_{N,i}$ be the vector of characteristics for all individuals $j \in \mathcal{I}_N$ such that $i \neq j$, and either $i, j \in C_k$ or $i, j \in E_m$ for any $m \in M$. Let $N_i$ be the number of individuals in that vector. Therefore, individual $i$'s type is a vector $[\mathbf{x}_{N,i}, b_{N,i}] \in \mathbb{R}^{N_i+1}$, and the vector of all players' types is $[\mathbf{x}_N, \mathbf{z}_N] \in \mathbb{R}^{2N} \equiv T$. Equilibrium beliefs are constructed from the individuals' strategy functions and the common prior distributions. Formally, an individual $i$'s strategy is a function $s_i : \mathbb{R}^{N_i+1} \to \mathbb{R}$ mapping possible types $[\mathbf{x}_{N,i}, b_{N,i}]$ to actions $y_i$.

A Bayesian-Nash equilibrium is a vector of strategy profiles $s(\mathbf{x}_N, \mathbf{b}_N)$ where each $s_i$ maximizes $\mathbb{E}[U_i\left(y_{N,i}, y_{N,-i}\right) \mid \mathbf{x}_{N,i}, b_{N,i}]$, and the expectation is taken with respect to $s_{-i}$ and $\rho_b$ given $\rho_{x,\zeta}$ (Blume et al., 2015). In addition to Assumption 1, the existence of a Bayesian-Nash equilibrium requires Assumption A.12$(i)$ to hold. This assumption is necessary to ensure that the expected utility is well defined. The second moment is required because of the quadratic form in actions in the strategic complementarities payoffs. The following Proposition characterizes the Bayes-Nash equilibrium of the social interactions game conditional on the multilayer network from the first stage.

**Proposition A.3 (Bayes-Nash Equilibrium Social Interactions Game)** *If the Bayesian game of social interaction satisfies Assumptions 1 and A.12(i), it has a unique Bayes-Nash equilibrium. The equilibrium strategy profile can be written as*

$$s(\mathbf{x}_N, \mathbf{b}_N) = \left(\mathbf{I}_N - \sum_{m=1}^{M} \beta_m^0 \mathbf{W}_{N,m}\right)^{-1} \left(\gamma^0 x_{N,i} + \sum_{m=1}^{M} \delta_m^0 \mathbf{W}_{N,m} x_{N,j}\right) + \mu(\mathbf{x}_N, \mathbf{b}_N) + \mathbf{b}_N \qquad \text{(A-23)}$$

*where $\mu_i(\mathbf{x}_N, \mathbf{b}_N)$ depends only on $\mathbf{x}_{N,i}$ and $b_{N,i}$.*

**Proof.** Define $\psi_{N,i} = \gamma x_{N,i} + b_{N,i} + \sum_{m=1}^{M} \sum_{j \neq i} w_{N,m;ij} x_{N,j} \delta_m$, and let $\boldsymbol{\psi}_N = [\psi_{N,1}, \ldots \psi_{N,N}]$. With this definition, the utility function can be written as

$$U_i\left(y_{N,i}, y_{N,-i}\right) = \psi_{N,i} y_{N,i} + \sum_{m=1}^{M} \sum_{j \neq i} w_{N,m;i,j} y_{N,i} y_{N,j} \beta_m - \frac{1}{2} y_{N,i}^2. \qquad \text{(A-24)}$$

From Assumption A.12($i$), the expected utilities based on equation (A-24) are well defined. Let $\mathcal{S}$ be the space of all possible vectors of strategy profiles such that $s \in \mathcal{S}$. Endow $\mathcal{S}$ with the $L_p^2$ norm, $\|s\| = \max_{i \in \mathcal{I}_N} \|s_i\|_2$, where $\|\cdot\|_2$ is the $L^2$ norm for a function. Since the strategies are in $L_p^2$, the expected payoff to any i of any strategy profile is finite (Blume et al., 2015). The first-order conditions for expected utility maximization are that for each $i$, and given the strategy profile $s_{-i}$ of the other individuals and type $(\mathbf{x}_{N,i}, z_{N,i}) \in \mathcal{T}$,

$$\psi_{N,i} - y_{N,i} + \sum_{m=1}^{M} \sum_{j \neq i} w_{N,m;ij} y_{N,j} \beta_m = 0.$$

Since the utility function is concave in $y_{N,i}$, the first-order conditions are sufficient for expected utility maximization. To prove the existence of an equilibrium, define the mapping $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ such that

$$\mathcal{T}(s) = \boldsymbol{\psi}_N + \sum_{m=1}^{M} \beta_m \mathbf{W}_{N,m} \mathbb{E}[s \mid \mathbf{x}_N, \mathbf{b}_N],$$

where the conditional expectation for individual $i$ with respect to strategy profile of another individual $j$ is given by $\mathbb{E}[s_j \mid \mathbf{x}_{N,i}, b_{N,i}]$ and the conditioning set given by the vector $[\mathbf{x}_{N,i}, b_{N,i}] \in \mathbb{R}^{N_i+1}$ includes all the information available to that individual. A fixed point of $\mathcal{T}$ satisfies the first-order condition for all $i$, $\psi_{N,i}$, and $b_{N,i}$. Therefore, a fixed point of $\mathcal{T}$ is a Bayes-Nash equilibrium profile. I now show that under Assumption 1, this map is a contraction with contraction constant $\sum_{m=1}^{M} |\beta_m| \|\mathbf{W}_{N,m}\|_\infty$. Let $s' \in \mathcal{S}$ be any arbitrary strategy profile. Then,

$$\begin{aligned}\|\mathcal{T}(s)-\mathcal{T}(s')\| &= \left\|\sum_{m=1}^{M}\beta_m \mathbf{W}_{N,m}\mathbb{E}\left[s-s' \mid \mathbf{x}_N,\mathbf{b}_N\right]\right\| \\ &\leq \sum_{m=1}^{M}\left\|\beta_m \mathbf{W}_{N,m}\mathbb{E}[s-s' \mid \mathbf{x}_N,\mathbf{b}_N]\right\| \\ &\leq \sum_{m=1}^{M}|\beta_m|\|\mathbf{W}_{N,m}\left\|_{\infty}\mathbb{E}[s-s' \mid \mathbf{x}_N,\mathbf{b}_N]\right\| \\ &= \left(\sum_{m=1}^{M}|\beta_m|\|\mathbf{W}_{N,m}\|_{\infty}\right)\left\|\mathbb{E}[s-s' \mid \mathbf{x}_N,\mathbf{b}_N]\right\| \\ &\leq \left\|s-s'\right\|,\end{aligned}$$

where the first inequality follows from the triangle inequality; the second inequality follows from dividing the norm between the two maximums $\max_{i\in\mathcal{I}_N}\sum_{j=1}^{N}|w_{N,m;ij}|$ and $\max_{i\in\mathcal{I}_N}\|s_i\|_2$; and the last inequality follows from the contraction constant $\sum_{m=1}^{M}|\beta_m|\|\mathbf{W}_{N,m}\| < 1$, and the fact that the maximum of the expected difference between any two strategy profiles should be not larger than the maximum of the difference of any two strategy profiles. Thus, it follows that $\mathcal{T}$ is a contraction mapping in $(\mathcal{F},\|\cdot\|)$, and so a fixed point exists and is unique. The rest of the argument follows the proof in Blume et al. (2015). Under the information assumptions, individuals can only use the local information intracluster and across connections to infer any private types. Therefore, the unobserved component $\mu_i(\mathbf{x}_N,\mathbf{b}_N)$ in the equilibrium strategy profile depends only on the local vector of observable characteristics $\mathbf{x}_{N,i}$ and the unobserved individual private type $b_{N,i}$. ■

Equation (A-23) presents a theoretical foundation for the reduced form equation for the MLiM model in (3). Moreover, the fact that the unobserved component $\mu_i(\mathbf{x}_N,\mathbf{b}_N)$ is correlated with the vector of characteristics $\mathbf{x}_{N,i}$ for each individual $i$, provides a theoretical argument for the local endogeneity issue caused by the network formation. The next step in solving the game is to consider the expected utility of the unique second stage equilibrium as an input for the utility function in the first stage network formation process.

### Appendix A.2 Multilayer Network Formation Model

A critical difference between the existing network formation models and the setup I propose is the multilayer network data structure. Following Joshi et al. (2020), I assume a sequential multilayer formation process where individuals choose their set of connections $\mathbf{w}_{N,m;i}$ one layer at a time. Differing from Joshi et al. (2020), for simplicity, I do not assume that the process repeats exhaustively, but it ends when the players decide their connections in the last layer $M$. This structure generates a sequence of $\{1,\ldots,M\}$ conditional link choices. To further simplify the problem, I assume that individuals behave myopically in the sense that they consider only the structure of the multilayer network at the

$m-1$ stage when deciding connections in layer $m$. To formalize this behavioral assumption into the payoff function, let $\mathbf{W}_{N,m-1}$ be the adjacency matrix of the multilayer that has emerged up to layer $m-1$, where $w_{N,m-1;i,j}=1$ if individuals $i$ and $j$ have a link in any of the layers in the sequence of layers $\{1,2,\dots,m-1\}$. Finally, payoff externalities (Miyauchi, 2016; Mele, 2017; De Paula et al., 2018; Christakis et al., 2020; Sheng, 2020), or degree heterogeneity (Graham, 2017) are ruled out for the sake of this characterization. I impose this assumption for simplicity, and the model could be extended to contain payoff externalities and degree heterogeneity as long as there exists an equilibrium selection choosing the multilayer network that realizes in the second stage when there are multiple equilibria, see Leung (2015) for the incomplete information setting.

To complete the characterization of the payoff function, I assume that there is an additional utility component such that individuals experience higher marginal utilities when they connect otherwise disconnected components in different clusters. This assumption is motivated by a well-established feature in empirical social structures: networks are characterized by clusters of dense pairwise connections linked by occasional relations between groups. This regularity on observed networks is known in the literature as bridging structural holes (Burt, 2004). Let individual $i$ be part of an arbitrary cluster $k$, her utility from forming connections in layer $m$ is given by

$$\begin{aligned}U_{i,m}\left(\mathbf{x}_i,\mathbf{v}_i,\mathbf{W}_{N,m-1}\right)=\sum_{j=1}^{N}w_{N,m;i,j}(&\pi_{x,1}|x_{N,i}-\mathbb{E}[x_{N,j}\mid\boldsymbol{\zeta}_N,\mathbf{x}_{N,i},b_{N,i}]|+\pi_{x,2}\mathbb{E}[x_{N,j}\mid\boldsymbol{\zeta}_N,\mathbf{x}_{N,i},b_{N,i}]\\&+\pi_{b,1}|b_{N,i}-\mathbb{E}[b_{N,j}\mid\boldsymbol{\zeta}_N,\mathbf{x}_{N,i},b_{N,i}]|+\pi_{b,2}\mathbb{E}[b_{N,j}\mid\boldsymbol{\zeta}_N,\mathbf{x}_{N,i},b_{N,i}]\\&+\pi_C\mathbf{1}\{j\in C_\ell\}\mathbf{1}\{i\in C_k\}\sum_i w_{N,m-1;i,j}-v_{N,m;i,j}),\end{aligned}\tag{A-25}$$

where $v_{N,m;i,j}$ are mutually independent and identically distributed with marginal distribution $\rho_v$, for all $i,j\in\mathcal{I}_N$ and $m\in\{1,\dots,M\}$. From Assumption A.12($i$), it follows that $\mathbb{E}[x_{N,j}\mid\boldsymbol{\zeta}_N,\mathbf{x}_{N,i},b_{N,i}]=x_{N,j}$ if individuals $i$ and $j$ belong to the same cluster or if they are connected in any layer of the multilayer network. The components $|x_{N,i}-\mathbb{E}[x_{N,j}\mid\boldsymbol{\zeta}_N,\mathbf{x}_{N,i},b_{N,i}]|$ and $|b_{N,i}-\mathbb{E}[b_{N,j}\mid\boldsymbol{\zeta}_N,\mathbf{x}_{N,i},b_{N,i}]|$ represent the taste for homophily on observed and unobserved characteristics, while the additional components $\mathbb{E}[x_{N,j}\mid\boldsymbol{\zeta}_N,\mathbf{x}_{N,i},b_{N,i}]$, and $\mathbb{E}[b_{N,j}\mid\boldsymbol{\zeta}_N,\mathbf{x}_{N,i},b_{N,i}]$ capture the effect on the expected utility that a connection with $j$ will bring individual $i$ in the second stage via the direct value of the characteristics $x_{N,j}$ and $b_{N,j}$. For two clusters $k\neq\ell$, where $\mathbf{1}\{\cdot\}$ represents the indicator function, the component $\mathbf{1}\{j\in C_\ell\}\mathbf{1}\{i\in C_k\}\sum_j w_{N,m-1;i,j}$ captures the idea of bridging structural holes. Individual $i$'s utility of creating a connection with $j$ on layer $m$ increases if $j$ has more connections in layers $\{1,2,\dots,m-1\}$, and individuals $i$ and $j$ belong to two different clusters, i.e., $i\in C_k$ and $j\in C_\ell$ with $k\neq\ell$.

**Equilibrium:** given the simplifying assumption that individuals' payoffs do not depend on the structure of the contemporaneous network and they make choices myopically, individual $i$'s optimal decision of choosing links to maximize utility does not depend on other individuals' decisions. Therefore, conditional on the realizations of the shocks $v_{N,m;i,j}$, the realization of the characteristics $x_{N,i}$ and $b_{N,i}$, and the conditional expectations of other individuals' characteristics taken with respect to the common knowledge distributions $\rho_b$ and $\rho_{x,\zeta}$, the solution for the multilayer network formation process exists and is unique. Network formation models without externalities have the desirable property of producing unique equilibrium networks and have been explored in the dyadic regression literature (Graham, 2017, 2020).

Now I impose restrictions on the distributions $\rho_b$, $\rho_{x,\zeta}$ and $\rho_v$ that are sufficient to producing multilayer network solutions where the dependence between individuals' characteristics disappears after a finite path length or a finite number of edge type changes. Define $p_{in}^{min}$ and $p_{out}^{max}$ as the *minimum* and *maximum* probabilities induced by the distributions $\rho_b$, $\rho_{x,\zeta}$, and $\rho_v$ that an individual forms a connection with another *inside* and *outside* her cluster, respectively. Additionally, let $N_k$ be the number of individuals in cluster $k \in \{1, \ldots, K\}$. The following assumptions are imposed on these objects.

**Assumption A.13 (Linking Probabilities)** *The conditional distribution $\rho_{x,\zeta}$ and the probability distribution $\rho_b$ and $\rho_v$ are such that for some constants $c_{in}$ and $c_{out}$, and any $k \in \{1, \ldots, K\}$, (i) $p_{in}^{min} \geq c_{in} \log N_k/N_K$, (ii) $p_{out}^{max} K \geq c_{out} > 1$, and (iii)*

$$\left\lceil \frac{\log[(33c_{in}^2/400)\, N_k \log N_k]}{\log(N_k p_{in}^{min})} \right\rceil + 2\left\lfloor \frac{1}{c} \right\rfloor + 2 < \frac{\log K}{\log p_{out}^{max} K}.$$

Parts *(i)* and *(ii)* in Assumption A.13 impose restrictions on the minimum and maximum probabilities of links within and across clusters. Following Chung and Lu (2001), part *(i)* guarantees that the diameter of the subnetworks inside an arbitrary cluster $k$ will be bounded by the left-hand side of the inequality on part *(iii)* for the case of a constant probability of connection given by $p_{in}^{min}$. For the links across clusters, I consider an auxiliary network where each cluster $k \in \{1, \ldots, K\}$ represents a node, and inter-cluster connections are formed with a constant probability $p_{out}^{max}$. It is important to note that part 2 imposes a condition that allows the expected number of connections on the auxiliary network to be smaller than the condition implied by *(i)*. The reason is twofold. First, the number of clusters is smaller than the sample size, which means a lower expected number of connections in the auxiliary network of clusters. Second, the probability of links across clusters should be slower than within clusters given the assumptions on the probability distributions in A.12 and the homophily preferences characterizing the network formation process in (A-25). Therefore, under A.13 part *(ii)*, the results in Chung and Lu (2001) guarantee that the diameter of the auxiliary inter-cluster network has a lower bound given by the right-hand-side of the inequality on part *(iii)*.

Intuitively, condition *(iii)* requires the maximum inter-cluster probability of connections to be smaller than the minimum itra-cluster probability of connections. For instance, setting $N = 2000$, $K = 20$, and assuming the same number of individuals per cluster such that $N_k = 100$ for all $k \in \{1, \ldots, K\}$, fixing $c_{in} = 3$ (which implies $c_{in} \log N_k/N_K = 0.06$), would require $p_{in}^{min} \geq 0.4$ and $0.051 \leq p_{out}^{max} \leq 0.086$ (which implies $1.02 \leq p_{out}^{max} K \leq 1.72$) to guarantee that the auxiliary layer has a diameter of at least 6, while the inter-cluster networks have a diameter of at most 5 for any $k \in \{1, \ldots, K\}$. As the previous example illustrates, the conditions on assumption A.13 guarantee that it is always possible to find a path with a larger geodesic distance when considering inter-cluster instead of only intra-cluster links. Moreover, the previous conditions also trivially imply that it is possible to find two individuals from different clusters that will have at most as many edge-type changes than any path connecting two individuals in the same cluster. Importantly, these results represent worst-case-scenarios using a *minimax* approach where the largest possible intra-cluster upper bound is compared with the smallest possible inter-cluster lower bound. The following Lemma summarizes the previous discussion.

**Lemma 1 (Diameter Comparison)** *Let individuals $i, j \in C_k \subset \mathcal{I}_N$ have the largest shortest path length for an arbitrary $k \in \{1, \ldots, K\}$. If Assumption A.13 holds, then (i) there exists two individuals $r, s \in \mathcal{I}_N$ such that $r \in C_k$ and $s \notin C_k$ which are connected by a path with a shortest path length larger than that between individuals $i$ and $j$, and (ii) there exist a path connecting two individuals from different clusters that will have at most as many edge-type changes than any path connecting two individuals in the same cluster.*

Again, it is important to highlight that condition *(iii)* in Assumption A.13 is a worst-case scenario because it assumes that only one edge connects any two clusters. Given that each cluster can have more than one individual, there is likely more than one intra-cluster connection in the path connecting the two clusters. Therefore, the paths connecting individuals across clusters are likely to be larger than the right-hand side of the inequality in *(iii)*, making the assumption easier to fulfill.

The conclusion that there exists at least one inter-cluster path with larger geodesic distance and edge type changes than any intra-cluster path is fundamental to characterizing the constants from proposition 1. Under this simple network formation process, I can define the constants $K_c$ and $K_d$ as follows: let $K_c$ be the minimum inter-cluster number of edge-type changes larger than the maximum intra-cluster number of edge-type changes across all possible clusters. I can define $K_d$ similarly, but using the geodesic distance measure instead. Characterizing $K_c$ and $K_d$ in this way, guarantees that any path connecting two individuals $i$ and $j$ with either more than $K_c$ edge-type changes or more than $K_d$ geodesic distance will guarantee that if $i \in C_k$ then $j \notin C_k$ for any $k \in \{1, \ldots, K\}$. Given the informational assumptions imposed on the clusters $\{C_1, \ldots, C_K$ and the distributional assumptions in A.12, the unobserved component generating endogeneity in the Bayesian Social Interactions Game for individual $i$ is independent of the characteristics of individual $j$. The following proposition formalizes the result.

**Proposition A.4 (Statistical Properties of the Social Interactions Game)** *Let the Assumptions A.12 and A.13 hold. Define $K_c$ as one plus the maximum number of inter-cluster edge-type changes across all possible clusters. Define $K_d$ analogously, but use the geodesic distance measure instead. Then,* *if for any $i$ and $j$, $c^*_N(i,j)$ and $d^*_N(i,j)$ are such that $c^*_N(i,j) \geq K_c$ or $d^*_N(i,j) \geq K_d$, then $\mu_i(\mathbf{x}_N, \mathbf{b}_N)$ in equation (A-23) is independent of $x_j$.*

As discussed before, this proposition follows directly from the statistical and informational assumptions discussed in this section. The main implication of proposition A.4 is that, conditional on the equilibrium multilayer network, the observed characteristics of an individual $j$ who is far away in the multilayer network space are not correlated with the unobserved characteristics of individual $i$. This result motivates the WND assumption, which justifies the use of the conditional moments (6) and (7) as a source for identification. In the context of Blume et al. (2015), the main insight driving the result, is that an individuals $i$ cannot use the characteristics $x_j$ to predict the private types when they do not belong to the same cluster.

**Remark 3 (Network Endogeneity)** The equilibrium strategy profile in (A-23) can be mapped to the econometric model in (3) by defining $\boldsymbol{\varepsilon}_N = \mu(\mathbf{x}_N, \mathbf{b}_N) + \mathbf{b}_N$. In this context, the endogeneity in the "reduced form" outcome in equation (3) is generated by the correlation between $\sum_{m=1}^{M} \mathbf{W}_{N,m} x_{N,j}$ and $\boldsymbol{\varepsilon}_N$, which in terms of the structural model is induced by $\mu(\mathbf{x}_N, \mathbf{b}_N)$. In particular, the endogeneity issue is caused by an endogenous network formation process in which individuals care about the private type of others and use all their available information to predict it. This process induces correlation between $\sum_{m=1}^{M} \mathbf{W}_{N,m} x_{N,j}$ and $\mu(\mathbf{x}_N, \mathbf{b}_N)$. Moreover, the homophily preferences characterizing the utility function in (A-25) could create correlation between $\sum_{m=1}^{M} \mathbf{W}_{N,m} x_{N,j}$ and $b_{N,i}$, as individuals are trying to match in both observed and unobserved characteristics.

**Remark 4 (Exogenous Variation)** The natural question is, how does an endogenous network formation process allow for exogenous variation based on the multilayer measure of distance in definition 1? The key comes from the information assumption defining the Bayesian Game of Social Interactions and the distributional assumptions A.12 and A.13. The information assumption constrains individuals' knowledge of others' "observed" characteristics only within a cluster that forms *exogenously* during the network formation process. These clusters could represent initial geographical, economic, or skills locations. In the context of the empirical application in section 7, for instance, you can think of $\boldsymbol{\zeta}_N$ as an initial endowment of skills exogenous to future individuals' research efforts. The informational assumption implies that initially (before starting a Ph.D. program) students can only observe the characteristics of others in their same level of skills (you can think of also creating more refined clusters such as geographical locations and skill levels). After the vector $\boldsymbol{\zeta}_N$ is realized, students begin a process where they first create connections based on sharing the same Ph.D. program, then advisory, coauthorship, and coworker ties. Assumption A.12 implies that other characteristics, such as writing styles and reading habits, are correlated only between individuals who are connected or belong to the same group. People with similar skills who went to the same schools or worked together tend to have more similar characteristics. Finally, assumption A.13 guarantees that the distributions determining the levels of observed and unobserved characteristics are such that it is more likely to form connections with others in the same cluster than others in different clusters. In the context of the running example, this means that, for instance, it is more likely that individuals with similar levels of characteristics end up in the same Ph.D. programs or co-authoring together.

Because individuals do not directly observe others' private characteristics, there should not be systematic correlations across those characteristics for the *difficult* to predict individuals in different clusters. Within clusters, however, when deciding whether to connect with $j$, individual $i$ can use $\mathbf{x}_{N,i}$ and $b_{N,i}$ to predict $b_{N,j}$. Given the homophily preference, if $\mathbf{x}_{N,i}$ and $b_{N,i}$ improve the prediction of $b_{N,j}$, that could induce correlation between the unobserved characteristics of individuals close in the multilayer network space. Intuitively, under the information regime I have described, the existence of local informational clusters avoids the potential systematic correlation between the observed and unobserved characteristics of individuals far apart in the network space. Therefore, under the previous conditions, Proposition A.3 shows that for any individual $i \in \mathcal{I}_N$, $\mu(\mathbf{x}_N, \mathbf{b}_N)$ depends only on $\mathbf{x}_{N,i}$ and $b_{N,i}$, which guarantees the existence of exogenous variation that can be used for identification.

## Appendix B Auxiliary Results

**Lemma 2 (Invertibility of $\mathbf{S}(\boldsymbol{\beta}, \mathcal{M}_N)$)** *Let Assumption 1 hold, then* $\mathbf{S}(\boldsymbol{\beta}, \mathcal{M}_N) = \mathbf{I}_N - \sum_{m=1}^{M} \beta_m \mathbf{W}_{N,m}$ *is invertible.*

**Proof.** In the trivial case where $\beta_m = 0$ for all $m$, it follows that $\mathbf{S}(\beta, \mathcal{M}_N) = \mathbf{I}_N$ which is invertible. For the non-trivial case, let $\theta = \sum_{m=1}^{M} |\beta_m|$ and note that $\theta = 0$ if only if $\beta_m = 0$ for all $m$. For $\theta \neq 0$, $\mathbf{S}(\boldsymbol{\beta}, \mathcal{M}_N)$ can be written as

$$\mathbf{S}(\boldsymbol{\beta}, \mathcal{M}_N) = \mathbf{I} - \theta \left( \frac{1}{\theta} \sum_{m=1}^{m} \beta_m \mathbf{W}_{N,m} \right) \equiv I - \theta \mathbf{A},$$

where $\mathbf{A} \equiv 1/\theta \sum_{m=1}^{m} \beta_m \mathbf{W}_{N,m}$. To show that $\mathbf{S}(\boldsymbol{\beta})$ has an inverse, it is enough to show that $\det(I - \theta \mathbf{A}) \neq 0$. Note that the Gerschgorin's disk of $\mathbf{A}$ is given by $R_i = \sum_{j=1, i \neq j}^{N} |a_{i,j}|$. By assumption 1, all the matrices forming $\mathbf{A}$ have zeros in the main diagonal, thus $R_i = \sum_{j=1, i \neq j}^{N} |a_{i,j}| = \sum_{j=1}^{N} |a_{i,j}|$. Let $\lambda_i$ be the $i$th eigenvalue of $\mathbf{A}$, then by Gerschgorin's (1931) Circe Theorem, $\lambda_i$ lies within at least one of the Gershgorin discs centered in zero with radius $R_i$. Given that all the circles are centered in zero, it follows that $|\lambda_i| \leq \sup_i \sum_{j=1}^{N} |a_{i,j}| = \|\mathbf{A}\|_\infty$ for all $i$. Note that $\sup_i \sum_{j=1}^{N} |a_{i,j}| = \sup_i \sum_{j=1}^{N} \left| \frac{1}{\theta} \sum_{m=1}^{M} \beta_m w_{N,m;i,j} \right| \leq \left| \frac{1}{\theta} \right| \sum_{m=1}^{M} |\beta_m| \sup_i \sum_{j=1}^{N} |w_{N,m;i,j}| = \left| \frac{1}{\theta} \right| \sum_{m=1}^{M} |\beta_m| \, \|\mathbf{W}_{N,m}\|_\infty$, where the second inequality follows from the triangle inequality and the supremum of the sum being at most the sum of the supremum. Therefore, $|\lambda_i| \leq |1/\theta| \sum_{m=1}^{M} \beta_m \|\mathbf{W}_{N,m}\|_\infty$. Note that if $\lambda_i$ is an eigenvalue of $\mathbf{A}$, then $(1 - \theta \lambda_i)$ is an eigenvalue of $I - \theta \mathbf{A}$. Moreover, given that $I - \theta \mathbf{A}$ is a $N \times N$ matrix, its determinant is given by the product of its eigenvalues, i.e., $\det(I - \theta \mathbf{A}) = \Pi_i (1 - \theta \lambda_i)$. From the discussion before, $|\theta\lambda_i| \leq \sum_{m=1}^{M}\beta_m\|\mathbf{W}_{N,m}\|_\infty < 1$ for all $i$, where the second inequality comes from Assumption 1. Thus, $\Pi_i(1-\theta\lambda_i) \neq 0$, completing the proof. ■

**Corollary 1 (Solution for the Outcome Equation)** *Let Assumption 1 hold. Define the $N \times N$ matrix $\mathbf{A}_N(\zeta,\phi)$ as the product of $\zeta$ adjacency matrices containing a possible combination of edge types given by the sequence $\phi$. Then, the solution for equation (2) can be written as an infinite sum of the product of different adjacency matrices given by*

$$\begin{aligned}\mathbf{y}_N = {} & \alpha\sum_{r=0}^{\infty}\left(\sum_{m=1}^{M}\beta_m\mathbf{W}_{N,m}\right)^r\boldsymbol{\iota}_N + \mathbf{X}_N\boldsymbol{\gamma} \\ & + \sum_{q=1}^{Q}\sum_{m=1}^{M}(\delta_{m,q}+\gamma_q\beta_m)\sum_{\zeta=0}^{\infty}\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\left(\prod_{\ell\in\phi}\beta_\ell\right)\mathbf{A}_N(\zeta,\phi)\mathbf{W}_{N,m}\mathbf{x}_{N,q} \\ & + \sum_{\zeta=0}^{\infty}\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\left(\prod_{\ell\in\phi}\beta_\ell\right)\mathbf{A}_N(\zeta,\phi)\boldsymbol{\varepsilon}_N,\end{aligned} \tag{B-26}$$

*where $\mathbb{P}\big(\{1,\ldots,M\},\zeta\big)$ represents the sequence of all possible $\zeta$-permutations with repetition from the set of possible layers $\{1,\ldots,M\}$, $\mathbf{x}_{N,q}$ is the $q$th variable from the matrix $\mathbf{X}_N$, and $\delta_{m,q}$ and $\gamma_q$ are the $q$th coefficients from the vectors $\boldsymbol{\delta}_m$ and $\boldsymbol{\gamma}$ for $m \in \{1,\ldots,M\}$. By convention, the matrix $\mathbf{A}_N(0,0) = \mathbf{I}_N$ and $\beta_0 = \beta_m^0 = 1$ for all $m \in \{1,\ldots,M\}$.*

**Proof.** Lemma 2 in Appendix B shows that $\mathbf{S}^{-1}(\boldsymbol{\beta},\mathcal{M}_N)$ exists. Therefore, under the conditions in assumption 1, it is possible to write the inverse as an infinite series given by $\mathbf{S}(\boldsymbol{\beta},\mathcal{M}_N)^{-1} = \sum_{r=0}^{\infty}\left(\sum_{m=1}^{M}\beta_m\mathbf{W}_{N,m}\right)^r$. By the definition of the matrix $\mathbf{A}_N(\zeta,\phi)$, we can write the the solution for equation (2) as

$$\begin{aligned}\mathbf{y}_N = {} & \alpha\sum_{r=0}^{\infty}\left(\sum_{m=1}^{M}\beta_m\mathbf{W}_{N,m}\right)^r\boldsymbol{\iota}_N + \sum_{\zeta=0}^{\infty}\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\left(\prod_{\ell\in\phi}\beta_\ell\right)\sum_{m=1}^{M}\mathbf{A}_N(\zeta,\phi)\mathbf{W}_{N,m}\mathbf{X}_N\boldsymbol{\delta}_m \\ & + \sum_{\zeta=0}^{\infty}\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\left(\prod_{\ell\in\phi}\beta_\ell\right)\mathbf{A}_N(\zeta,\phi)\mathbf{X}_N\boldsymbol{\gamma} + \sum_{\zeta=0}^{\infty}\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\left(\prod_{\ell\in\phi}\beta_\ell\right)\mathbf{A}_N(\zeta,\phi)\boldsymbol{\varepsilon}_N,\end{aligned}$$

where the sequence $\psi$ is generated by taking the powers of the sum $\left(\sum_{m=1}^{M}\beta_m\mathbf{W}_{N,m}\right)^r$ for different values of $r$. Notice that when $\zeta = 0$, it follows that $\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\left(\prod_{\ell\in\phi}\beta_\ell\right)\mathbf{A}_N(\zeta,\phi)\mathbf{X}_N\boldsymbol{\gamma} = \gamma\mathbf{X}_N$, which represents the second component of the first line in equation (B-26). Because I extracted the first component of the sum $\sum_{\zeta=0}^{\infty}\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\left(\prod_{\ell\in\phi}\beta_\ell\right)\mathbf{A}_N(\zeta,\phi)\mathbf{X}_N\boldsymbol{\gamma}$, if I want the infinite sum to keep its starting point at zero, I need to multiply the expression by $\beta_m\mathbf{W}_{N,m}$ and add the terms up for all $m \in \{1,\ldots,M\}$. Writing the matrix multiplications $\mathbf{X}_N\boldsymbol{\delta}_m$ and $\mathbf{X}_N\boldsymbol{\gamma}$ as $\sum_{q=1}^{Q}\mathbf{x}_{N,q}\delta_{m,q}$ and $\sum_{q=1}^{Q}\mathbf{x}_{N,q}\gamma_q$ and taking common factor $\sum_{\zeta=0}^{\infty}\sum_{\phi\in\mathbb{P}\left(\{1,\ldots,M\},\zeta\right)}\left(\prod_{\ell\in\phi}\beta_\ell\right)\mathbf{A}_N(\zeta,\phi)\mathbf{W}_{N,m}\mathbf{x}_{N,q}$ for all $m \in \{1,\ldots,M\}$ yields equation B-26 ■

**Remark 5 (Edge-type Changes as a Relevant Instrument)** This notation in terms of $\mathbf{A}_N(\zeta,\phi)$ simplifies the representation of large multiplications of adjacency matrices. For instance, for any arbitrary layers $m_1, m_2, m_3 \in \{1,\ldots,M\}$, the matrix $\mathbf{W}_{N,m_1}\mathbf{W}_{N,m_2}\mathbf{W}_{N,m_3}$ can be represented by $\mathbf{A}_N(3,\{m_1,m_2,m_3\})$. Importantly, note that the $(i,j)$th element of the matrix $\mathbf{W}_{N,m}^{k}$ gives the number of paths of length $k$ from agents $i$ to $j$ (for some layer $m$), see e.g., Graham (2015), while the $(i,j)$th element of the product of two adjacency matrices $\mathbf{W}_{N,m}$ and $\mathbf{W}_{N,s}$ for layers $m$ and $s$, contains the number of paths of length two between nodes $i$ and $j$ where each path begins with a type $m$ edge and changes to type $s$ after the second node in the sequence. In general, the $(i,j)$th position of the matrix formed by $k$ products of adjacency matrices from different layers gives the number of $k$-paths between individuals $i$ and $j$ that change edge types $k$ times. Therefore, (B-26) shows that both interlayer and intralayer indirect connections can be used as relevant instruments for $\mathbf{W}_{N,m}\mathbf{y}_N$.

**Lemma 3** *Let Assumptions 2 hold for $\{\mathbf{r}_{n,i}\}_{n\geq 1}$, $i \in \mathcal{I}_n$. Define $R_{i,j} = f_{q,\ell}(\mathbf{r}_{n,\{i,j\}}) = r_{n,i,q}r_{n,j,\ell}$ and $R_{h,s} = g_{q',\ell'}(\mathbf{r}_{n,\{h,s\}}) = r_{n,h,q'}r_{n,s,\ell'}$ for $i,j,h,s \in \mathcal{I}_n$, where $q$, $q'$, $\ell$, and $\ell'$ are components of the vector $\mathbf{r}_{n,i}$. Let Assumption 8 hold for $R_{i,j}$ and $R_{h,s}$. Then*

$$|\mathrm{Cov}\left(R_{i,j},R_{h,s}\right)| \leq 2\bar{\theta}_{n,s}(C+16)\times 4\left(\pi_1+\tilde{\gamma}_1\right)\left(\pi_2+\tilde{\gamma}_2\right)\underline{\theta}_{n,s}^{1-p_f-p_g}, \tag{B-27}$$

*where $\underline{\theta}_{n,s} = \theta_{n,s}\wedge 1$, $\bar{\theta}_{n,s} = \theta_{n,s}\vee 1$, $\pi_1 = \|\mathbf{r}_{n,i}\|_{p_{f,i}}\|\mathbf{r}_{n,j}\|_{p_{f,j}}$, $\pi_2 = \|\mathbf{r}_{n,h}\|_{p_{f,h}}\|\mathbf{r}_{n,s}\|_{p_{f,s}}$, $\widetilde{\gamma}_1 = \max\{\|\mathbf{r}_{n,i}\|_{p_{f,i}+p_{f,j}}, \|\mathbf{r}_{n,j}\|_{p_{f,i}+p_{f,j}}\}$, $\widetilde{\gamma}_2 = \max\{\|\mathbf{r}_{n,h}\|_{p_f}, \|\mathbf{r}_{n,s}\|_{p_g}\}$ where $p_f = 1/p_{f,i}+1/p_{f,j}$ and $p_g = 1/p_{g,h}+1/p_{g,s}$. The constant $C$ is the same as in Assumption 2, the indexes $i,j,h,s$, and components $q$, $q'$, $\ell$ and $\ell'$ may or may not be the same.*

**Proof.** Define the increasing continuous functions $h_1(x)$ and $h_2(x)$ in (Appendix A in Kojevnikov et al., 2020, pp. 899-907) Theorem A.2 to be $h_1(x) = h_2(x) = x$ . Note that the functions $f_{q,\ell}$ and $g_{q',\ell'}$ are continuous, and their truncated version of the form $\varphi_{K_1}\circ f\circ\varphi_{h_1}(K_2)$ and $\varphi_{K_1}\circ g\circ\varphi_{h_1}(K_2)$ for all $K \in (0,\infty)^2$ are in $\mathscr{L}_{Q+1,2}$. Assumption 8 guarantees the existence of the moments defining $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$. Then, Theorem A.2 in (Appendix A in Kojevnikov et al., 2020, pp. 899-907) applies to this setting (see also Corollary A.2.in Appendix Kojevnikov et al., 2020, pp. 899-907). ■

**Lemma 4 (LLN for Products of $\psi$-dependent Random Variables)** *Let Assumptions 2, 8, 9 and 10 hold. Define $R_{n,i,j} = r_{n,i,q}r_{n,j,\ell}$ and form $\{R_{n,i,j}\}_{i\in\mathcal{I}_n, j\in\mathcal{I}_i}$, where $\mathcal{I}_n$ is the set of all individuals in the sample of size $n$, while $\mathcal{I}_i$ is a set of indexes define for each $i \in \mathcal{I}_n$. Defining the weights $w_{i,j}\in[0,1]$, as $n\to\infty$, in probability*

$$\left\| \frac{1}{n} \sum_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{I}_i} w_{i,j} \left( R_{n,i,j} - \mathbb{E}\left[ R_{n,i,j} \right] \right) \right\|_1 \to 0$$

**Proof.** For simplicity, in this proof, it is assumed that $\mathrm{Var}(R_{n,i,j}) \leq a$ for all $i$ and $j$, and a generic constant $a$. However, this assumption is not necessary. Without the finite variance assumption, the proof proceeds similarly but separating $R_{n,i,j} = R^k_{n,i,j} + \widetilde{R}^k_{n,i,j}$, where $\widetilde{R}^k_{n,i,j} = \varphi_k(R_{n,i,j})$, and $\varphi_k(x) = (-K) \vee (K \wedge x_i)$ is a censoring function, see Jenish and Prucha's (2009) proof of Theorem 3 for more details, and Kojevnikov et al.'s (2020) proof for Theorem 3.1. Define the $k$-norm for a random variable $X$ as $\|X\|_k = (\mathbb{E}[|X|^k])^{1/k}$ for $k \in [1, \infty)$. Thus, by Lyapunov's inequality and the definition of the $k$-norm it follows that

$$\left\| \frac{1}{n} \sum_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{I}_i} w_{i,j} \left( R_{n,i,j} - \mathbb{E}\left[ R_{n,i,j} \right] \right) \right\|_1 \leq \left\| \frac{1}{n} \sum_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{I}_i} w_{i,j} \left( R_{n,i,j} - \mathbb{E}\left[ R_{n,i,j} \right] \right) \right\|_2 \tag{B-28}$$

where (B-28) is an expression for the standard deviation of $\sum_{i \in \mathcal{I}_{n,m,\lambda}} \sum_{j \in \mathcal{I}_i} w_{i,j} R_{n,i,j}$ normalized by the sample size $n$. Moreover, note that the variance of that quantity can be written as

$$\mathrm{Var}\left( \sum_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{I}_i} w_{i,j} R_{n,i,j} \right) = \sum_{i \in \mathcal{I}_n} \mathrm{Var}\left( \sum_{j \in \mathcal{I}_i} w_{i,j} R_{n,i,j} \right) + \sum_{i \neq h \in \mathcal{I}_n} \mathrm{Cov}\left( \sum_{j \in \mathcal{I}_i} w_{i,j} R_{n,i,j}, \sum_{s \in \mathcal{I}_h} w_{h,s} R_{n,h,s} \right)$$

Let us start by analyzing first the variance component. Given the network dependence between individuals, it follows that

$$\begin{aligned} \mathrm{Var}\left( \sum_{j \in \mathcal{I}_i} w_{i,j} R_{n,i,j} \right) &= \sum_{j \in \mathcal{I}_i} w_{i,j}^2 \mathrm{Var}(R_{n,i,j}) + \sum_{j \neq s \in \mathcal{I}_i} w_{i,j} w_{i,s} \mathrm{Cov}(R_{n,i,j}, R_{n,i,s}) \\ &\leq a \sum_{j \in \mathcal{I}_i} w_{i,j}^2 + \sum_{j \in \mathcal{I}_i} \sum_{d \geq 1} \sum_{s \in \mathcal{P}_n(j,d) \cap \mathcal{I}_i} \left| \mathrm{Cov}(R_{n,i,j} R_{n,i,s}) \right| \\ &\leq a \sum_{j \in \mathcal{I}_i} w_{i,j}^2 + b \sum_{d \geq 1} \theta_{n,d} \sum_{j \in \mathcal{I}_i} \left| \mathcal{P}_n(j,d) \right|, \end{aligned} \tag{B-29}$$

where the second inequality follows from $w_{i,j}, w_{i,s} \in [0,1]$, and $a$ is a generic constant that follows from the assumption that $\mathrm{Var}(R_{n,i,j})$ is finite (or the fact that after partitioning $R_{n,i,j}$ it is possible to bound its variance). The third inequality comes from the fact that under Assumptions 2 and 8, and Lemma 3, the covariances are bounded by $|\mathrm{Cov}(R_{n,i,j}, R_{i,s})| \leq b\theta_{n,d}$, where $b < \infty$ contains the constants from Lemma 3, and includes the possibility that $\theta_{n,d}$ could have exponents of either 1 or $1 - p_f - p_g$. Focusing now on the covariance component of equation (B-28), by the properties of the covariance, the expression inside the summation can be written as

$$\mathrm{Cov}\left(\sum_{j\in\mathcal{I}_i} w_{i,j}R_{n,i,j}, \sum_{s\in\mathcal{I}_h} w_{h,s}R_{n,h,s}\right) = \sum_{j\in\mathcal{I}_i}\sum_{s\in\mathcal{I}_h} w_{i,j}w_{h,s}\mathrm{Cov}(R_{n,i,j}, R_{n,h,s}) \tag{B-30}$$
$$\leq \sum_{j\in\mathcal{I}_i}\sum_{d\geq 1}\sum_{s\in\mathcal{P}_n(j,d)\cap\mathcal{I}_h} |\mathrm{cov}(R_{n,i,j}R_{n,h,s})|$$
$$\leq b\sum_{d\geq 1}\theta_{n,d}\sum_{j\in\mathcal{I}_i}|\mathcal{P}_n(j,d)|,$$

where the inequalities in (B-30) follow from the same arguments discussed before. It follows from equations and (B-30) that the total variance can be bounded by

$$\mathrm{Var}\left(\sum_{i\in\mathcal{I}_{n,m,\lambda}}\sum_{j\in\mathcal{I}_i} w_{i,j}R_{n,i,j}\right) \leq a\sum_{j\in\mathcal{I}_i} w_{i,j}^2 + 2b\sum_{d\geq 1}\theta_{n,d}\sum_{j\in\mathcal{I}_i}|\mathcal{P}_n(j,d)| \tag{B-31}$$
$$= a\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{I}_i} w_{i,j}^2 + 2b\sum_{d\geq 1}\theta_{n,d}\sum_{i\in\mathcal{I}_n}|\mathcal{P}_n(j,d)|$$
$$\leq n\left(a\sum_{i\in\mathcal{I}_n} n^{-1}\sum_{j\in\mathcal{I}_i} w_{i,j} + 2b\sum_{d\geq 1}\theta_{n,d}\bar{D}_n(d)\right),$$

where the last inequality follows from $w_{i,j}\in[0,1]$. Define $\bar{\mathcal{I}}_i^w = n^{-1}\sum_{j\in\mathcal{I}_i} w_{i,j}$. It follows that combining equations B-28 and B-31,

$$\left\|\frac{1}{n}\sum_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{I}_i} w_{i,j}\left(R_{n,i,j} - \mathbb{E}\left[R_{n,i,j}\right]\right)\right\|_1 \leq \left(\frac{a}{n}\sum_{i\in\mathcal{I}_n}\bar{\mathcal{I}}_i^w + \frac{2b}{n}\sum_{d\geq 1}\theta_{n,d}\bar{D}_n(d)\right)^{1/2} \tag{B-32}$$

Depending on the component of interest from the matrix $\mathbf{Z}_n^\top\mathbf{D}_n$ and the vector $\mathbf{Z}_n^\top\boldsymbol{\varepsilon}$, the set $\mathcal{I}_i$ can be either: (1) $\mathcal{I}_i=\varnothing$, (2) $\mathcal{I}_i = \mathcal{P}(i,1,m,\lambda)\times\mathcal{P}(i,1,s)$ where $\mathcal{P}(i,1,m,\lambda)$ is the set of individual $i$'s neighbors in the implicit network formed by the weighted adjacency matrix $\mathcal{W}_{n,m,\lambda}$ (set of nodes that can be used to for moment conditions for $i$), and $\mathcal{P}(i,1,s)$ is the set of $i$'s neighbors in layer $s$, (3) $\mathcal{I}_i = \mathcal{P}(i,1,m,\lambda)$, or (4) $\mathcal{P}(i,1,m)$. For any of the four cases, Assumption 9 guarantees that $\bar{\mathcal{I}}_i^w = o(1)$ for all $i$, and by the algebra of stochastic orders $\sum_{i\in\mathcal{I}_n}\bar{\mathcal{I}}_i^w = o(1)$. Moreover, Assumption 10 guarantees that $n^{-1}\sum_{d\geq 1}\theta_{n,d}\bar{D}_n(d)\to 0$. It follows that the right hand side of equation (B-32) is $o(1)$, completing the proof. ■

**Lemma 5 (LLN for Outcomes and Regressors)** *Let Assumptions 1, 2, 8, 9 and 10 hold hold. Define $\mathbf{x}_{n,q}$ as the qth column of the matrix $\mathbf{X}_n$. As in the main text $\mathbf{w}_{n,m,\lambda,i}$ and $\mathbf{w}_{n,s,i}$ represent the ith row of the matrices $\mathcal{W}_{n,m,\lambda}$ and $\mathbf{W}_{n,s}$, respectively. Then, as $n\to\infty$, in probability*

$$\left\|\frac{1}{n}\sum_{i\in\mathcal{I}_n}\mathbf{w}_{n,m,\lambda,i}\mathbf{x}_{n,q}\mathbf{w}_{n,s,i}\mathbf{y}_n - \mathbb{E}[\mathbf{w}_{N,m,\lambda,i}\mathbf{x}_{N,q}\mathbf{w}_{N,s,i}\mathbf{y}]\right\|_1 \to 0 \tag{B-33}$$

*and*

$$\left\| \frac{1}{n} \sum_{i \in \mathcal{I}_n} \mathbf{w}_{n,m,i} \mathbf{y}_n \mathbf{x}_{n,q} - \mathbb{E}[\mathbf{w}_{N,m,i} \mathbf{y} x_{N,q,i}] \right\|_1 \to 0 \tag{B-34}$$

**Proof.** From Assumption 1, as shown in Corollary 1, it follows that for any three arbitrary layers $m_1$, $m_2$, and $m_3$ from $\{1, \ldots, M\}$, two arbitrary characteristics $q$ and $l$ from $\{1 \ldots, Q\}$, an arbitrary number of products of adjacency matrices $\zeta$, and an arbitrary sequence $\phi$, one single summation from the first component $\mathbf{A}(\zeta, \phi)_n \mathbf{W}_{n,m_2} \mathbf{X}_n$ of the vector $(\mathcal{W}_{n,m_1,\lambda} \mathbf{x}_{n,q})^\top \mathbf{W}_{n,m_2} \mathbf{y}_n$ can be written as:

$$\begin{aligned} &\left( \prod_{\ell \in \phi} \beta_\ell \right) (\mathcal{W}_{n,m_1,\lambda} \mathbf{x}_{n,q})^\top \mathbf{W}_{n,m_2} \mathbf{A}(\zeta, \phi) \mathbf{W}_{n,m_3} \mathbf{x}_{n,l} \delta_{m_3,l} \\ &\qquad = \left( \prod_{\ell \in \phi} \beta_\ell \right) (\mathcal{W}_{n,m_1,\lambda} \mathbf{x}_{n,q})^\top \mathbf{A}(\zeta + 2, \{m_2, \phi, m_3\}) \mathbf{x}_{n,l} \delta_{m_3,l}, \end{aligned} \tag{B-35}$$

where the equality follows by the definition of $\mathbf{A}(\zeta, \phi)_n$. For simplicity, define $\mathbf{A}(\zeta + 2, \{m_2, \phi, m_3\})_n \equiv \mathbf{A}(\zeta', \phi')_n$. Let $\mathcal{I}_i = \eta_{i,\lambda,m} \times \eta_{i,\zeta',\phi'}$ represent the Cartesian product of $\eta_{i,\lambda,m}$ and $\eta_{i,\zeta',\phi'}$, which are the set of individual $i$'s neighbors in the implicit networks induced by $\mathcal{W}_{n,m,\lambda}$ and $\mathbf{A}(\zeta', \phi')_n$, respectively. Therefore, the right hand side of equation (B-35) can be written as

$$\frac{1}{n} \sum_{\ell \in \mathcal{I}_n} \sum_{i,j \in \mathcal{I}_\ell} w_{i,j} R_{n,i,j}, \tag{B-36}$$

where $w_{i,j} = w_{\lambda;\ell,i} w_{\ell,j}$ and $R_{n,i,j} = x_{n,q} x_{n,l}$. Thus, from Lemma 4, it follows that (B-36) converges to the population expectation. Because the characteristics $q$ and $l$, the number of products of adjacency matrices $\zeta$, and the sequence $\phi$ were chosen arbitrarily, this convergence process applies for all the components in the infinite sum in $\zeta$. Given that each component of the sum converges to a finite expectation, the infinite sum of finite expectations is also finite given the restriction on the parameters $\beta_m$ from Assumption 1. This completes the proof for the convergence of the first component of the outcome equation $\mathbf{y}_n$ in (B-26).

For the second component in (B-26), given by $\mathbf{A}(\zeta, \phi)_n \widetilde{\mathbf{X}}_n$, the proof works analogously by substituting $\mathbf{A}(\zeta + 2, \{m_2, \phi, m_3\})_n$ for $\mathbf{A}(\zeta + 1, \{m_2, \phi\})_n$. Finally, for the third component in (B-26) given by $\mathbf{A}(\zeta, \phi)_n \boldsymbol{\varepsilon}_n$, because $\varepsilon_{n,i}$ is just another component of $\mathbf{r}_{n,i}$ for all $i$, the results in Lemma 4 also applies for this case. This completes the proof for equation (B-33). The proof for (B-34) is analogous. ■

**Lemma 6 (LLN for Instruments and Regressors)** *Let Assumptions 2, 8, 9 and 10 hold. Then as $n \to \infty$, in probability*

$$\left\| \frac{1}{n} \sum_{i \in \mathcal{I}_n} \mathbf{z}_{n,i} \mathbf{d}_{n,i}^\top - \mathbb{E}[\mathbf{z}_{N,i} \mathbf{d}_{N,i}^\top] \right\|_1 \to 0. \tag{B-37}$$

**Proof.** There are four different types of components in the matrix $\mathbf{Z}_n^\top \mathbf{D}_n$ formed by summations of products of (1) non-network regressors of the form $x_{n,i,q} x_{n,i,\ell}$, (2) network regressors of the form $\mathbf{w}_{n,m,\lambda,i}\mathbf{x}_{n,q}\mathbf{w}_{n,s,i}\mathbf{x}_{n,\ell}$, (3) network and non-network regressors of the form $\mathbf{w}_{n,m\lambda,i}\mathbf{x}_{n,q}\mathbf{x}_{n,i,\ell}$, and (4) network regressors and network outcomes of the form $\mathbf{w}_{n,m,\lambda,i}\mathbf{x}_{n,q}\mathbf{w}_{n,s,i}\mathbf{y}_n$. Thus, to proof the convergence result in (B-37), it suffices to show the convergence for each of the four components described before. Firs, from Lemma 5, it follows that the network regressors and network outcomes component converges to its population mean. The other three results follow by appropriately choosing $R_{n,i,j}$, $\mathcal{I}_i$, and $w_{i,j}$ to apply the results from Lemma 4. For (1), choose $R_{n,i} = x_{n,i,q} x_{n,i,\ell}$ for two arbitrary regressors $q$ and $\ell$, $\mathcal{I}_i = \varnothing$ so that there is not a second summation over $j$ and $w_i = 1$. For (2) and (3), choose $R_{n,i,j} = x_{q,i} x_{\ell,j}$ for arbitrary regressors $q$ and $\ell$. Regarding the set of indexes, for (2), choose $\mathcal{I}_i = \mathcal{P}_n(i,1,m,\lambda) \times \mathcal{P}_n(i,1,m)$, and for (3) choose $\mathcal{I}_i = \mathcal{P}_n(i,1,m,\lambda)$ and corresponding weights if the relevant network is $\mathcal{W}_{n,m,\lambda}$, or $\mathcal{I}_i = \mathcal{P}_n(i,1,m)$ and corresponding weights if the relevant network is $\mathbf{W}_m$. Therefore, applying Lemma 4 component-wise for $\sum_{i\in\mathcal{I}_n} \mathbf{z}_{n,i}\mathbf{d}_{n,i}^\top$ implies the result. ■

**Lemma 7 (LLN for Instruments and Errors)** *Let Assumptions 2, 8, 9 and 10 hold. Then as $n \to \infty$, in probability*

$$\left\| \frac{1}{n} \sum_{i\in\mathcal{I}_n} \mathbf{z}_{n,i}\varepsilon_{n,i}^\top - \mathbb{E}[\mathbf{z}_{N,i}\varepsilon_{N,i}^\top] \right\|_1 \to 0. \tag{B-38}$$

**Proof.** Given that $\mathbf{r}_{n,i} = [\mathbf{x}_i, \varepsilon_i]$ and $\mathbf{z}_i$ can be divided into network and non-network components, the proof of this result is analogous for that of Lemma 6 parts (1) and (3). ■

**Lemma 8 (Central Limit Theorem)** *Let Assumptions 2, and 8 to 11 hold. Define the sum $S_n = \sum_{i\in\mathcal{I}_n} z_{n,i,q}\varepsilon_{n,i}$, where $z_{n,i,q}$ is the qth entrance of the vector $\mathbf{z}_{n,i}$, $\mathcal{I}_{n,q}$ is the set of individuals with non-zero values in column $q$ of the matrix $\mathbf{Z}_n$. By definition of $\mathbf{z}_i$, $\mathbb{E}[z_{n,i,q}\varepsilon_{n,i}] = 0$. Define $\sigma_n = Var(S_n)$. Then, as $n \to \infty$*

$$\sup_{t\in\mathbf{R}} \left| \mathbf{P}\left\{ \frac{S_n}{\sigma_n} \le t \mid \mathcal{C}_n \right\} - \Phi(t) \right| \to 0,$$

*where $\Phi$ denotes the distribution function of a $\mathcal{N}(0,1)$,*

**Proof.** Let $Y_{n,i} = z_{n,i,q}\varepsilon_{n,i}$, from Lemma 3, the covariance of any two $Y_{n,i}$ and $Y_{n,j}$ is bounded. Then, the proof follows from applying Lemmas A.2 and A.3 in (Appendix A in Kojevnikov et al., 2020, pp. 899-907) to $Y_{n,i} = z_{n,i,q}$ and $S_n/\sigma_n$, respectively. ■

**Lemma 9 (Finite Variance)** *Define $\mathbf{S}_n = \mathbf{Z}_n^\top \varepsilon_n$ and $\mathbf{\Omega}_n = \mathrm{Var}(n^{-1/2}\mathbf{S}_n)$. Let Assumptions 2, and 8 to 10 hold, then $\mathbf{\Omega}_n \to \mathbf{\Omega}_N < \infty$ as $n \to \infty$.*

**Proof.** As defined before $n^{-1/2}\mathbf{S}_n = n^{-1/2}\sum_{i=1}^{n}\mathbf{z}_{n,i}\varepsilon_{n,i}$. The bounded covariance assumptions from Lemma 3 combined with the arguments in Lemma 4 guarantee that $\lim_{n\to\infty} n^{-1}\text{Var}\left(\sum_{i=1}^{n}\mathbf{z}_{n,i}\varepsilon_{n,i}\right)$ is finite. In particular, from equation (B-31), using the appropriate values for $R_{n,i,j}$, $\mathcal{I}_{n,m,\lambda}$, $\mathcal{I}_i$, $n_{m,\lambda}$, and $w_{i,j}$ (see Lemma 6), it follows that $\text{var}(\sum_{i=1}^{n}\mathbf{z}_{n,i}\varepsilon_{n,i}) = \mathcal{O}_p(1)$. Given that $\mathbf{\Omega}_n$ converges to a finite quantity, it follows that $\mathbf{\Omega}_n \to \mathbf{\Omega}_N$, where

$$\mathbf{\Omega}_N = \lim_{n\to\infty} n^{-1}\left[\sum_{i=1}^{n}\text{var}(\mathbf{z}_i\varepsilon_i) + \sum_{i\neq j}\text{cov}(\mathbf{z}_i\varepsilon_i, \mathbf{z}_j\varepsilon_j)\right] < \infty.$$

■

**Lemma 10 (Multivariate Central Limit Theorem)** *Let Assumption 2, and 8 to 11 hold. Then,*

$$n^{-1/2}\sum_{i=1}^{n}\mathbf{z}_{n,i}\varepsilon_i \xrightarrow{d} \mathcal{N}(0,\Omega_N) \quad as \quad n\to\infty$$

**Proof.** From Lemma 8, $n^{-1/2}\sum_{i=1}^{n} z_{n,i,q}\varepsilon_{n,i} \xrightarrow{\text{d}} \mathcal{N}(0,\sigma_n^2)$. From Lemma 9, $\mathbf{\Omega}_N$ exists. Therefore, the result follows from the Cramér-Wold device. ■

# Appendix C Algorithms for the Moment Condition Matrices

This section describes the computation process required to calculate the empirical analog of the moment condition matrices based on Balasubramanian et al. (2022). First, I use Algorithm 1 to calculate the multilayer shortest paths between a source node $s$ and all the other nodes $v \in \bigcup_{m\in M} V_m$. The time complexity of this algorithm is $\mathcal{O}(V + E\log E)$, where $V \equiv \bigcup_{m\in M} V_m$ and $E \equiv \bigcup_{m\in M} E_m \bigcup \mathcal{C}$. I then use parallel computation to repeat the process for all possible sources. Thus, the described procedure provides all shortest path lengths and edge type changes for the sampled nodes in the multilayer network. With this information, it is feasible to evaluate equation 7 for all possible dyads efficiently.

**Algorithm 1:** Multilayer Colored Shortest Path

**Input:** (1) a multilayer graph $\mathcal{M} = (\mathcal{G}, \mathcal{C})$ with non-negative edge weights, and (2) a source vertex $s \in \bigcup_{m \in M} V_m$.

**Output:** shortest paths and color changes for all nodes $v \neq s$ in $\mathcal{M}$.

Initialization $Q = \bigcup_{m \in M} V_m$, $D = \infty$ for all nodes in $Q$, $P = CP = CC = \varnothing$ Define $D[s] = 0$;

**while** *$Q$ is not empty* **do**

- $u$= node in $Q$ with the minimum distance to $s$; Remove $u$ from $Q$ and add it to $P$; **foreach** *$v$ directly connected to $u$* **do**
  - distance = $D[u]$+ weight of the edge from $u$ to $v$;
  - **if** *distance <= $D[v]$* **then**
    - $D[v]$ =distance
    - **if** $s = v$ **then**
      - $CP$=layer where the edge exists and $CC = 0$;
    - **else**
      - $CP$=$CP[u]$+layer where the edge exists and $CC$ = number of edge-layer changes

The calculation of the second shortest path involves a recursive execution of the Multilayer Colored Shortest Path algorithm. The basic idea is that for each node, I replace weighs of the edges in the shortest path for any arbitrary node $v$ for infinite and then run Algorithm 1 again. Algorithm 2 details the process. This algorithm has the same complexity as the one in 1. The construction of the matrices $\mathcal{W}_{m,\beta}$ and $\mathcal{W}_{m,\delta}$ for all $m \in 1, \ldots, M$ is then only a matter of filtering the dyads that fulfill the requirements of equations 6 and 7.

**Algorithm 2:** Multilayer Colored Second Shortest Path

**Input:** (1) a multilayer graph $\mathcal{M} = (\mathcal{G}, \mathcal{C})$ with non-negative edge weights, and (2) a source vertex $s \in \bigcup_{m \in M} V_m$.

**Output:** second shortest paths and color changes for all nodes $v \neq s$ in $\mathcal{M}$.

**foreach** $v$ $s \in \bigcup_{m \in M} V_m$ **do**

- Multilayer Colored Shortest Path for $v$;
- store shortest path;
- **foreach** *$u$ in the shortest path* **do**
  - replace the edge weights for $\infty$;
- **do** Multilayer Colored Shortest Path for $v$ again;

# Appendix D Data and Additional Empirical Results

## Appendix D.1 Scholars Network Constructions

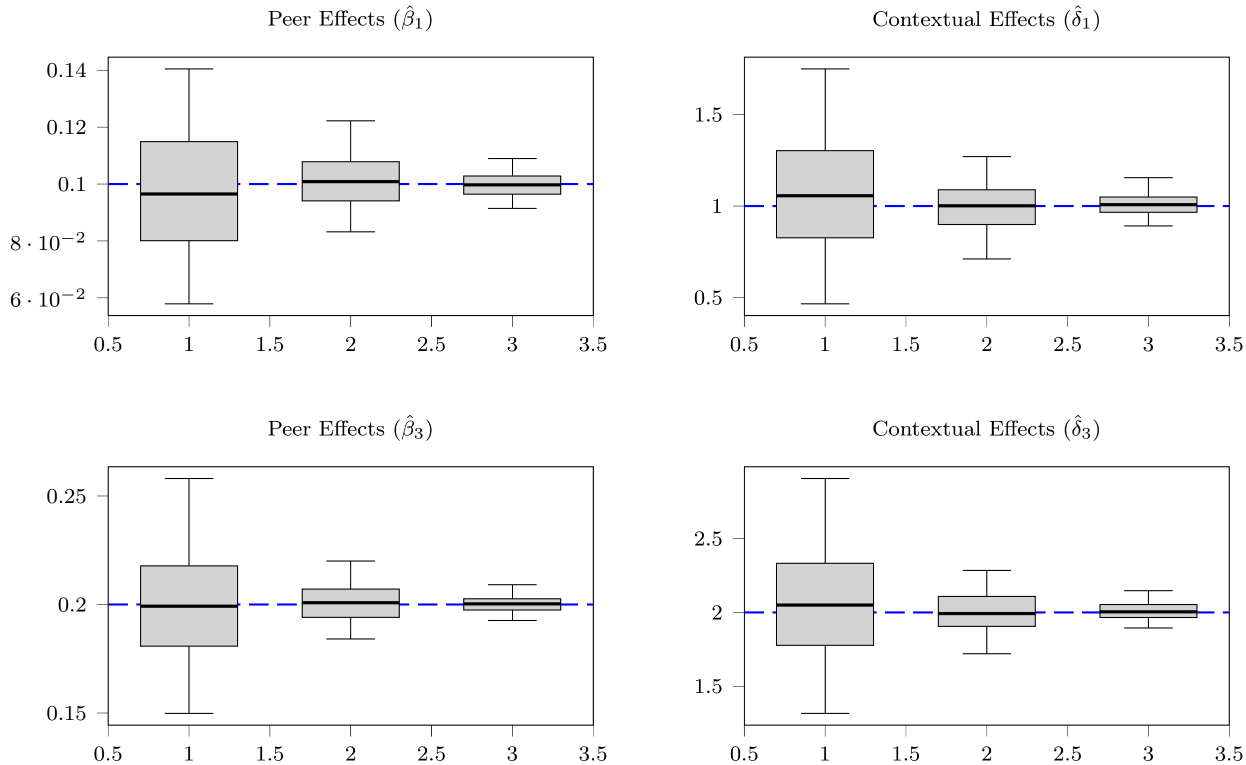
***Co-authors*** – This type of connection happens when 1) a scholar publishes a paper (alone or with other authors) in any of the four journals under consideration 2) in any year of interest, 3) also publishes other papers (with the same or other co-authors) in any of the four journals and in the seven-year timeframe considered here. Then a co-autorship connection is created between this scholar and all of his/her co-authors in these multiple publications. For example, scholar 5 published a paper in JPE with scholars 424, 436 and 1,041 in 2001. Additionally, the same scholar 5 published another article in the AER in 2005, this time in collaboration with scholar 1,108. Moreover, in addition to the article in JPE with authors 5, 436 and 1,041, scholar 424 wrote another paper in ECA along with authors 847 and 889. Thus, scholar 5 is said to have a co-authorship connection with 424, 436, 1,041, and 1,108, as long as the year of interest is 2005 or 2006 because in those year all the publications obtained during the previous years are also considered. Similarly, scholars 424, 436, and 1,041 are connected between them, and scholar 424 is also connected with 847 and 889, who are also co-authors themselves.

***Alumni*** – This type of connection happens when two scholars got their Ph.D. degrees from the same university and within a maximum of a three-year gap. For example, scholars 1 and 1,699 have an alumni connection equal to one using this criteria because both completed their Ph.D. degrees at Princeton University in 1995.

***Ph.D. Advisor*** – This type of connection happens when a scholar studying in an institution obtains his or her Ph.D. in a particular year when another scholar held the position of assistant, associate, or full professor in the same institution and also shares at least one common JEL code. Of the 2,057 scholars, 46% are full professors, 16% are associate professors, while 23% are assistant professors. 15% of scholars held other positions such as post doc, visiting, etc. For example, scholar 2 was employed by University of California Berkeley as a full professor between 2000-2006, working in Law and Economics (JEL code: K) and Industrial Organization (JEL code: L). In 2001, author 1,035 finished her Ph.D. in Industrial Organization (JEL code: L) and Public Economics (JEL code: H). Then author 2 and 1,035 are said to have a Ph.D. Advisor connection. There are 19 scholars who held the title Assistant Professor in the same institution they obtained their Ph.D. in the same year. Their Ph.D. Advisor connection with the remaining 2,038 scholars and among themselves are set to zero.

***Colleagues*** – This link happens when two scholars are considered each other's colleague, i.e., they ever worked in the same institution in the same time period. For instance, scholars 1 and 103 are connected because both were working at the University of Illinois Urbana-Champaign between 2000 and 2002. As in *Same Ph.D.*, 136 scholars are omitted from the estimating sample because institution information for them is missing.

Figure 6: Box Plots for the GMM Estimator

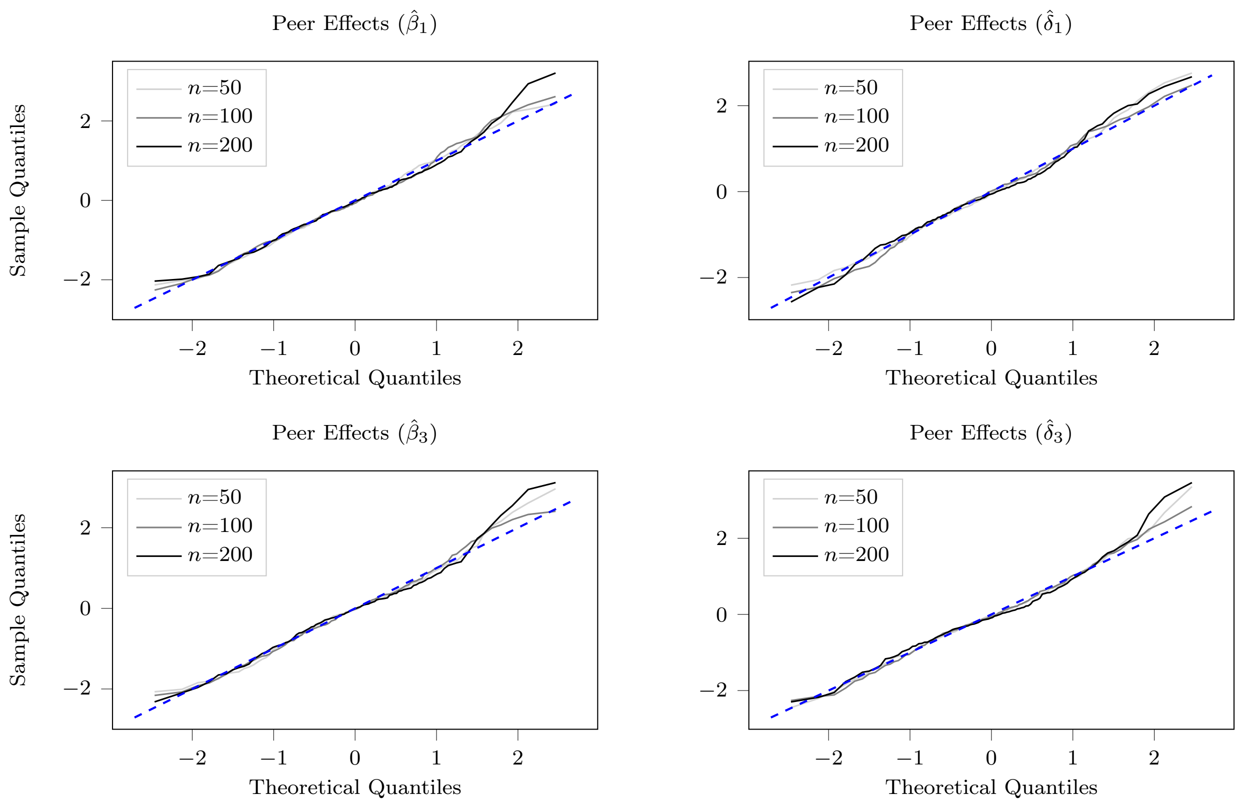



Note: Box plots in the first row depict the Monte Carlo performance of the proposed GMM estimator. The boxplots are based on 1,000 for sample sizes $n \in \{50, 100, 200\}$. The whiskers display the 5% and 95% empirical quantiles.

## Appendix D.2 Additional Simulation and Estimation Results

This section contains additional simulation and estimation results. Plots 6 and 7 show the results from the Monte Carlo simulation for the additional network effects $[\beta_1^0, \beta_3^0, \delta_1^0, \delta_3^0]$. The results for the other parameters are presented in Section 6 in the main text. The results here are consistent with those for the parameters $[\beta_2^0, \delta_2^0]$. The simulation confirms the desirable properties of the estimator in finite sample. Similarly, Tables 3 and 4 present the empirical estimation results using the publications data described in section 7. These results are used as a benchmark to compare the behavior of the efficient GMM estimator.

Figure 7: Q-Q plots of the GMM Estimator




Note: Q-Q plots in the second row are based on the standardized sample of 1,000 Monte Carlo replications of the proposed GMM estimator of the parameters in (1) and sample sizes $n = 50$ (light gray), $n = 100$ (gray), and $n = 200$ (black). The blue dashed line shows the 45 degree line.

Table 3: GMM Estimation Results for Social and Direct Effects

| | 2002 | 2003 | 2004 | 2005 | 2006 |
|---|---|---|---|---|---|
| Peer Effects ($\{\widehat{\beta}\}_{m=1}^{3}$) | | | | | |
| `Co-authors` | 0.485 | 0.428 | 0.566 | 0.536 | 0.468 |
| | (0.123) | (0.112) | (0.114) | (0.106) | (0.279) |
| `Alumni` | 0.120 | 0.113 | 0.165 | 0.175 | 0.374 |
| | (0.262) | (0.216) | (0.263) | (0.172) | (0.387) |
| `Advisor` | 0.033 | 0.197 | -0.606 | -0.437 | -0.086 |
| | (0.259) | (0.272) | (0.330) | (0.238) | (0.369) |
| `Colleagues` | 0.132 | -0.283 | -0.042 | 0.081 | 0.086 |
| | (0.471) | (0.1421) | (0.103) | (0.098) | (0.159) |
| Contextual Effects ($\{\widehat{\boldsymbol{\delta}}\}_{m=1}^{3}$) | | | | | |
| `Co-authors: Editor in Charge` | 0.215 | -0.288 | -0.317 | -0.501 | 0.885 |
| | (0.699) | (0.639) | (0.681) | (0.550) | (0.993) |
| `Alumni: Editor in Charge` | -0.069 | 0.069 | 0.029 | 0.060 | 0.402 |
| | (0.216) | (0.156) | (0.138) | (0.128) | (0.316) |
| `Advisor: Editor in Charge` | 0.417 | 0.105 | 1.589 | 1.071 | 0.337 |
| | (0.724) | (0.744) | (0.740) | (0.676) | (0.802) |
| `Colleagues: Editor in Charge` | 0.145 | 0.001 | -0.204 | -0.293 | -0.321 |
| | (0.285) | (0.185) | (0.174) | (0.187) | (0.396) |
| `Co-authors: Different Gender` | -0.929 | -0.753 | -1.627 | -1.215 | 2.689 |
| | (1.695) | (1.253) | (1.606) | (0.856) | (1.825) |
| `Alumni: Different Gender` | -0.385 | -0.284 | -0.301 | -0.014 | -0.951 |
| | (0.428) | (0.379) | (0.424) | (0.307) | (0.892) |
| `Advisor: Different Gender` | 1.875 | 4.991 | 5.954 | 3.595 | 1.382 |
| | (3.116) | (1.734) | (1.884) | (1.515) | (1.869) |
| `Colleagues: Different Gender` | 0.606 | 0.253 | 0.015 -0.357 | -2.146 | |
| | (0.522) | (0.552) | (0.567) | (0.732) | (1.633) |
| Contextual Effects ($\widehat{\boldsymbol{\gamma}}$) | | | | | |
| `Editor in Charge` | -0.069 | -0.094 | -0.042 | -0.017 | 0.061 |
| | (0.135) | (0.115) | (0.139) | (0.123) | (0.141) |
| `Different Gender` | 0.253 | 0.229 | 0.189 | 0.143 | 0.069 |
| | (0.122) | (0.102) | (0.091) | (0.081) | (0.093) |
| `Number of Pages` | 0.025 | 0.023 | 0.020 | 0.017 | 0.016 |
| | (0.004) | (0.003) | (0.003) | (0.003) | (0.003) |
| `Number of Authors` | 0.064 | 0.084 | 0.052 | 0.074 | 0.065 |
| | (0.056) | (0.047) | (0.043) | (0.038) | (0.035) |
| `Number of References` | 0.009 | 0.009 | 0.007 | 0.008 | 0.011 |
| | (0.003) | (0.002) | (0.002) | (0.002) | (0.002) |
| `Co-authors: Isolated` | 1.566 | 1.367 | 1.775 | 1.629 | 1.452 |
| | (0.437) | (0.404) | (0.409) | (0.384) | (1.032) |
| $n$ | 729 | 961 | 1187 | 1412 | 1628 |

Note: Standard errors are in parenthesis and are calculated using the network HAC estimator of the covariance matrix in equation (5.1) where the function $K$ is the Parzen kernel and the bandwidth $D_n = 1.8 \times [\log(\text{ avg.deg } \vee (1.05))]^{-1} \times \log n$. All specifications include indicator variables for Journal and Year. The indicator for isolated nodes in the Alumni, Advisor and Colleagues networks are also included but are not statistically significant.

Table 4: OLS Estimation Results for Social and Direct Effects

| | 2002 | 2003 | 2004 | 2005 | 2006 |
|---|---|---|---|---|---|
| Peer Effects ($\{\widehat{\beta}\}_{m=1}^{3}$) | | | | | |
| `Co-authors` | 0.359 | 0.397 | 0.461 | 0.452 | 0.453 |
| | (0.061) | (0.049) | (0.047) | (0.045) | (0.042) |
| `Alumni` | 0.026 | 0.088 | 0.113 | 0.154 | 0.139 |
| | (0.087) | (0.076) | (0.064) | (0.064) | (0.063) |
| `Advisor` | -0.123 | -0.117 | -0.075 | -0.055 | -0.045 |
| | (0.07) | (0.066) | (0.061) | (0.054) | (0.056) |
| `Colleagues` | 0.266 | 0.096 | 0.138 | 0.156 | 0.028 |
| | (0.122) | (0.105) | (0.072) | (0.068) | (0.048) |
| Contextual Effects ($\{\widehat{\boldsymbol{\delta}}\}_{m=1}^{3}$) | | | | | |
| `Co-authors: Editor in Charge` | -0.219 | -0.141 | -0.186 | 0.055 | 0.086 |
| | (0.215) | (0.202) | (0.178) | (0.180) | (0.172) |
| `Alumni: Editor in Charge` | -0.063 | 0.076 | -0.012 | 0.025 | 0.001 |
| | (0.193) | (0.139) | (0.117) | (0.117) | (0.108) |
| `Advisor: Editor in Charge` | 0.519 | 0.278 | 0.202 | -0.065 | -0.002 |
| | (0.373) | (0.308) | (0.299) | (0.296) | (0.279) |
| `Colleagues: Editor in Charge` | 0.207 | 0.038 | -0.027 | -0.092 | -0.072 |
| | (0.235) | (0.169) | (0.158) | (0.142) | (0.161) |
| `Co-authors: Different Gender` | -0.453 | -0.265 | -0.319 | -0.359 | -0.048 |
| | (0.421) | (0.388) | (0.368) | (0.387) | (0.432) |
| `Alumni: Different Gender` | -0.181 | -0.162 | -0.035 | -0.122 | -0.214 |
| | (0.27) | (0.229) | (0.219) | (0.204) | (0.211) |
| `Advisor: Different Gender` | 1.434 | 2.120 | 0.753 | 0.939 | 1.290 |
| | (0.786) | (0.742) | (0.971) | (0.974) | (0.964) |
| `Colleagues: Different Gender` | 0.386 | 0.325 | 0.012 | -0.118 | 0.112 |
| | (0.405) | (0.393) | (0.379) | (0.974) | (0.964) |
| Contextual Effects ($\widehat{\boldsymbol{\gamma}}$) | | | | | |
| `Editor in Charge` | -0.017 | -0.057 | -0.047 | 0.009 | 0.044 |
| | (0.126) | (0.112) | (0.131) | (0.114) | (0.107) |
| `Different Gender` | 0.260 | 0.206 | 0.179 | 0.133 | 0.121 |
| | (0.113) | (0.094) | (0.082) | (0.078) | (0.069) |
| `Number of Pages` | 0.026 | 0.023 | 0.019 | 0.016 | 0.016 |
| | (0.004) | (0.003) | (0.003) | (0.003) | (0.003) |
| `Number of Authors` | 0.054 | 0.078 | 0.071 | 0.084 | 0.063 |
| | (0.055) | (0.044) | (0.038) | (0.035) | (0.031) |
| `Number of References` | 0.009 | 0.009 | 0.009 | 0.009 | 0.011 |
| | (0.002) | (0.002) | (0.002) | (0.002) | (0.002) |
| `Co-authors: Isolated` | 1.129 | 1.258 | 1.383 | 1.328 | 1.312 |
| | (0.236) | (0.196) | (0.183) | (0.178) | (0.164) |
| $n$ | 729 | 961 | 1187 | 1412 | 1628 |

Note: Standard errors are in parenthesis and are calculated using the network HAC estimator of the covariance matrix in equation (5.1) where the function $K$ is the Parzen kernel and the bandwidth $D_n = 1.8 \times [\log(\text{ avg.deg } \vee (1.05))]^{-1} \times \log n$. All specifications include indicator variables for Journal and Year. The indicator for isolated nodes in the Alumni, Advisor and Colleagues networks are also included but are not statistically significant.