Part 1: Rayleigh-Taylor Instability with Surface Tension Linear Analysis

1. **Neglecting Viscosity**. To neglect viscosity, it must be the case that the viscous time scale is much longer than the time scale of the instability we are looking at. To compare these scales, we can consider $\tau = ul$ and $\tau_\nu = l^2/\nu$, the instability time scale and viscous time scale, respectively. We want $\frac{\tau_\nu}{\tau} \gg 1$, so this means that $\frac{ul^3}{\nu} \gg 1$.

2. **Governing Equations**. To find the system of equations, we consider irrotational, inviscid, incompressible flow. First consider the interface elevation, $z = \zeta(x, y, t)$ when the flow is disturbed. Then, we consider a velocity potential ϕ on each side of the interface, where $\mathbf{u} = \nabla\phi_i$, denoted by $\phi_i = \phi_1, \phi_2$ for $z < \zeta, z > \zeta$, respectively. The equation of continuity for an incompressible fluid gives $\nabla \cdot \mathbf{u} = 0$. Therefore, $\nabla \cdot \nabla\phi_i = \Delta\phi_i = 0$ where Δ is the Laplacian operator in this case.

   Boundary Conditions. Since the initial disturbance will occur only in a finite region, then as z gets far enough from the disturbance, the basic flow will resume. So, for all time, $\nabla\phi \to \mathbf{U}$ as $z \to \pm\infty$. Also, no vertical velocity at $\pm\infty$, so $\frac{\partial\phi}{\partial z} = 0$ as $z \to \pm\infty$ (which is as similar a condition as the previous one if you consider that $\mathbf{U} = 0$, which we will in the next question). Fluid particles at the interface must move with the interface without creating cavities or overlaps. To satisfy this, we impose a kinematic boundary condition at the interface, given by $z = \zeta, \frac{\partial\phi}{\partial z} = \frac{\mathrm{D}\zeta}{\mathrm{d}t} = \frac{\partial\zeta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial z}\frac{\partial\zeta}{\partial z}$. Note that we are in the xz-plane, so any derivatives involving y are gone. We will need two equations for the interface, so one will be written for ϕ_1 and one for ϕ_2. The last boundary condition deals with the normal stress of the fluid at the interface, which means that pressure is continuous at $z = \zeta$. We will write this out relating Bernoulli's theorem and the Young-Laplace equation, which states that $\Delta p = -\gamma\nabla \cdot \mathbf{n}$.

   The full system of equations:

$$\Delta\phi_1 = 0 \text{ for } z < \zeta$$
$$\Delta\phi_2 = 0 \text{ for } z > \zeta$$

   With boundary conditions:

$$\frac{\partial\phi_i}{\partial x} = U_i = 0, \text{ and } \frac{\partial\phi_i}{\partial z} = 0 \text{ as } z \to \mp\infty \text{ for } i = 1,2$$
$$\frac{\partial\phi_i}{\partial z} = \frac{\partial\zeta}{\partial t} + \frac{\partial\phi_i}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi_i}{\partial z}\frac{\partial\zeta}{\partial z} \text{ at } z = \zeta_{\mp} \text{ for } i = 1,2$$
$$\rho_2\left(\frac{\partial\phi_2}{\partial t} + g\zeta + \frac{1}{2}(\nabla\phi_2)^2\right) - \rho_1\left(\frac{\partial\phi_1}{\partial t} + g\zeta + \frac{1}{2}(\nabla\phi_1)^2\right) = -\gamma\nabla \cdot \mathbf{n} \text{ at } z = \zeta \approx 0$$

   The hypothesis here is that if we apply a "small" perturbation to the system, we can determine a stability by analyzing under what conditions this perturbation will either grow or decay exponentially.

3. **Basic Flow**. Considering no initial velocity, the basic flow is $\mathbf{U}(z) = \begin{cases} U_1 = 0, & z < \zeta \\ U_2 = 0, & z > \zeta \end{cases}$ for velocity, $\bar{\rho}(z) = \begin{cases} \rho_1 & z < \zeta \\ \rho_2 & z > \zeta \end{cases}$ for density, $P(z) = \begin{cases} p_0 - g\rho_1 z, & z < \zeta \\ p_0 - g\rho_2 z, & z > \zeta \end{cases}$ for pressure, and $\zeta = 0$ for the interface.

4. **Linearization**. We linearize the system of equations for a perturbation of the basic flow, seeking for solution of the form $(\zeta, \phi_1', \phi_2') = \left(\hat{\zeta}, \hat{\phi}_1(z), \hat{\phi}_2(z)\right)\exp(st + \mathrm{i}kx)$. See attached work for the linear stability analysis and derivation of the generalized dispersion relation, $s(k)$.

   The linearized system:

$$\Delta\phi'_1 = 0 \text{ for } z < \zeta$$
$$\Delta\phi'_2 = 0 \text{ for } z > \zeta$$

   With linearized boundary conditions:

$$\frac{\partial\phi'_i}{\partial x} = U_i = 0, \text{ and } \frac{\partial\phi'_i}{\partial z} = 0 \text{ as } z \to \mp\infty \text{ for } i = 1,2$$
$$\frac{\partial\phi'_i}{\partial z} = \frac{\partial\zeta}{\partial t} \text{ at } z = \zeta_\mp \text{ for } i = 1,2$$
$$\rho_2\left(\frac{\partial\phi_2}{\partial t} + g\zeta + \frac{1}{2}(\nabla\phi_2)^2\right) - \rho_1\left(\frac{\partial\phi_1}{\partial t} + g\zeta + \frac{1}{2}(\nabla\phi_1)^2\right) = \gamma\nabla\cdot\mathbf{n} \text{ at } z = \zeta \approx 0$$

   After using the method of normal mode, the following dispersion relation is obtained:

$$s = \left(\frac{gk(\rho_2 - \rho_1) - \gamma k^3}{\rho_1 + \rho_2}\right)^{\frac{1}{2}}$$

5. **General Stability Condition**. We can find when this system will be stable or unstable by investigating whether s will be real or imaginary. According to the dispersion relation, when the top fluid of this system has a higher density than the bottom fluid *plus another term*, then the system will be unstable. Mathematically this is $\rho_2 \geq \rho_1 + \frac{\gamma k^2}{g}$. This means that this system can be stable even when $\rho_2 > \rho_1$ in some cases, due to the resistance of surface tension at the interface.

   In terms of k, we can take the instability condition and say whenever $k \leq \left(\frac{g(\rho_2 - \rho_1)}{\gamma}\right)^{\frac{1}{2}}$, then this system is unstable. Again, this allows us to have some wavenumbers where the system is stable when $\rho_2 > \rho_1$.

6. **Gravity Waves**. In deep water, we can say that the phase velocity of a gravity wave is given by $\omega = \sqrt{gk}$, where $c = \frac{\omega}{k}$. We can solve for $\sqrt{gk}$ in our dispersion relation above, but the k^3 term makes this hard to do so. We can reduce surface tension to where this term is negligible, giving $\omega = \sqrt{gk}\left(\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\right)^{\frac{1}{2}}$. This then leads to the phase velocity of a linear gravity wave, $c = \sqrt{\frac{g}{k}}\left(\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\right)^{\frac{1}{2}}$. But we will only get back to gravity waves when surface tension is negligible, where gravity will be the restoring force instead.

   Unstable flow when $\boldsymbol{\rho_1 < \rho_2}$. According to the dispersion relation, s will always be negative when $\rho_1 < \rho_2$, giving an oscillatory, but stable, solution. The only other way flow could be unstable is if another force was added, such as horizontal velocity of the fluids, or convective effects. But as this Rayleigh-Taylor system is configured now, it is not possible to have unstable flow when $\rho_1 < \rho_2$.

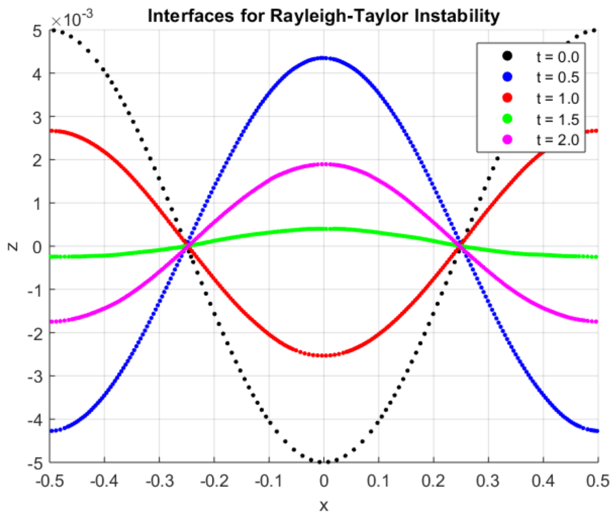
   Effects of Surface Tension. Surface tension will resist instability when $\rho_2 > \rho_1$. Mathematically this was shown above, because ρ_2 had to be greater than the sum of ρ_1 and another term involving surface tension. If surface tension were zero, the term $\frac{\gamma k^2}{g}$ would be zero, so the system would only need $\rho_2 > \rho_1$ to be unstable. But now, since there is surface tension, this term will resist the impact of the density difference. Physically, this makes sense. Surface tension works to minimize surface area when possible, so since the instability will expand and curve the interface, surface tension will work to resist these high-curvature perturbations.

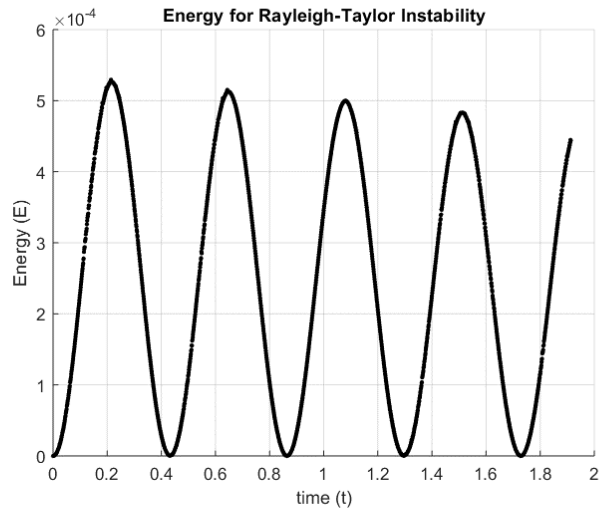
7. **Finite Depth in Both Fluids**. If there is finite depth in both fluids, two scenarios could occur, and that could depend on if the depth makes the instability a shallow-water wave or a deep-water wave. If the depth is deep enough to be a deep-water wave, then it is likely that nothing would change in the problem. Put another way, this means that if the instability is far enough away from the fluid wall top and bottom, then these finite depths won't have a large effect on the instability. However, if the depth is close enough to the instability such that the initial perturbation might be a shallow-water wave, there may be different effects on the system. Thinking mathematically and numerically, another equation would also have to be added to the boundary conditions, to simulate what would happen if a fluid particle hit the bottom or top of the box.

   Effect of Viscosity. If viscosity was involved in this problem, for starters, the equations would change. The Navier-Stokes equation would have to include the viscous term $\mu\nabla^2\mathbf{u}$ in the kinematic boundary condition. Since viscosity is essentially a measure of resistance to gradual deformation of a fluid, including viscosity in the system might slow down the growth of different instabilities or "spikes" in the fluid. Because of this fact, viscous diffusion might enhance the low-wavenumber modes and suppress the high wavenumber modes.

Part 2: Rayleigh-Taylor Instability with Numerical Analysis

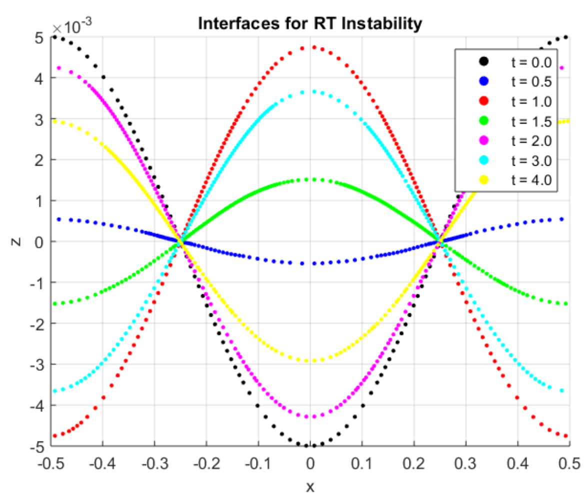
8. **Case where $\boldsymbol{\rho_2 < \rho_1}$.** For this case, $\rho_2 = 1, \rho_1 = 10, \gamma = 0.1, g = 10$. Below are two plots, one of the interface data and the other as energy as a function of time.

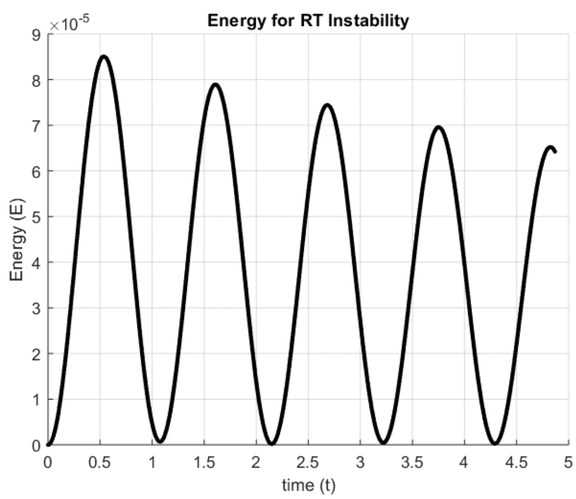




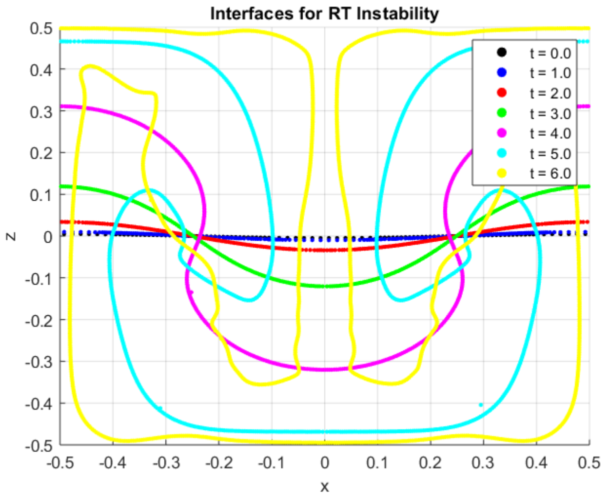
The period of oscillation seems to be ~0.4 seconds. The decay rate is slow, and since s is imaginary (due to the negative in the square root term), it is decaying over time.

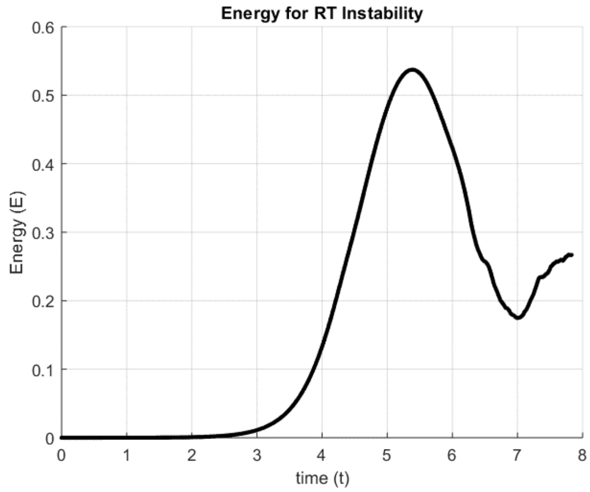
9. **Case where $\boldsymbol{\rho_2 > \rho_1}$: stable flow.** For this case, $\rho_2 = 11, \rho_1 = 10, \gamma = 1, g = 1$. Below are two plots, one of the interface data and the other as energy as a function of time. Note that they are like the case in Q8, meaning even though the density on top is greater than the density on the bottom, the surface tension works to keep this flow from becoming unstable. Also, note that even though these plots look like the ones in Q8, the axis for the time is more than double, meaning that the stable oscillation was much slower in the case where the top fluid was denser than the case where the bottom fluid was denser.



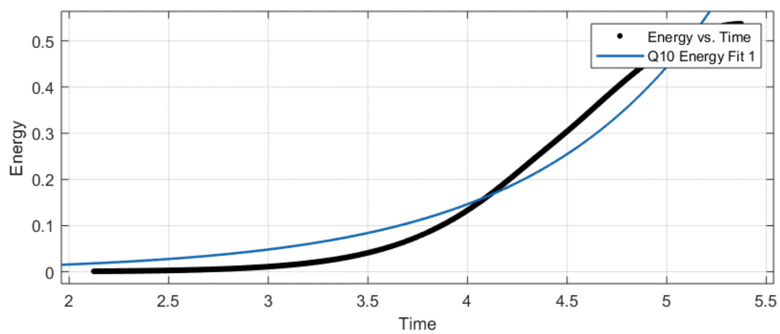


10. **Case where $\boldsymbol{\rho_2 > \rho_1}$: unstable flow.** For this case, $\rho_2 = 11, \rho_1 = 10, \gamma = 0.1, g = 10$. Below are two plots of the interface data and energy as a function of time.

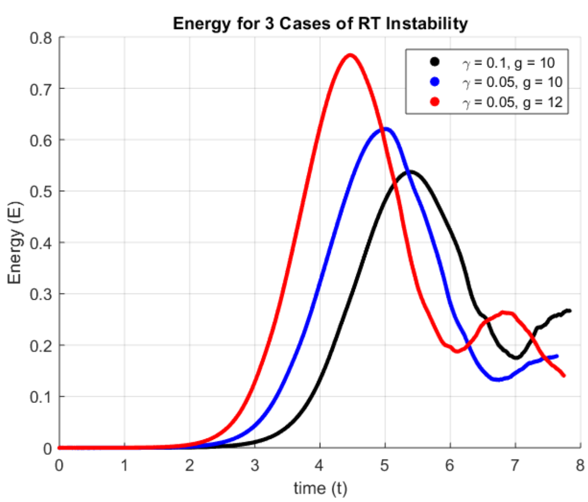


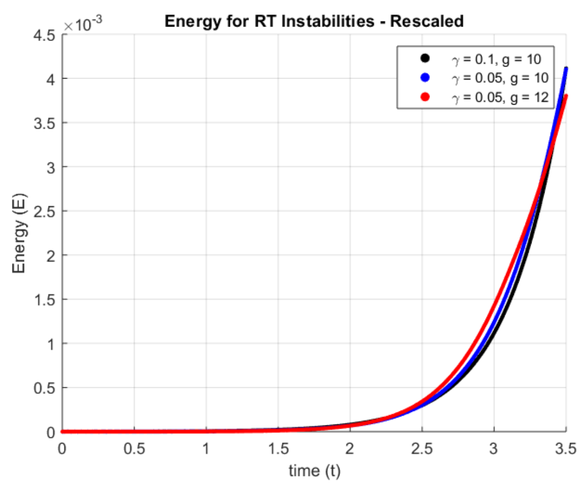


For the energy plot on the right, we will try to fit an exponential at the beginning part of the curve.



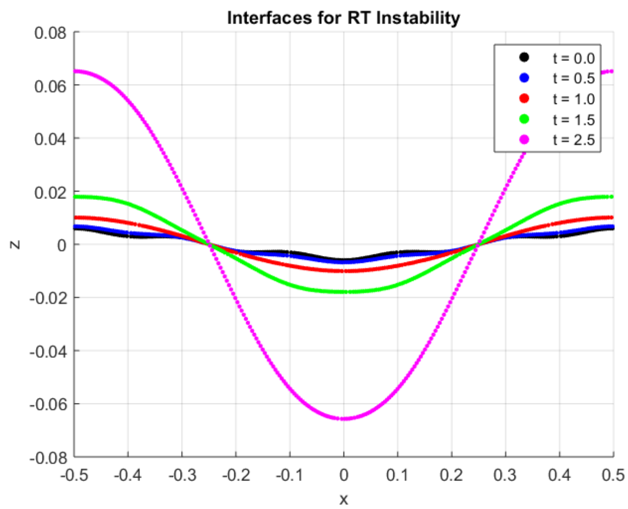
Now we will compare the energies of 3 different cases. The density difference is the same in all 3 cases ($\rho_2 = 11, \rho_1 = 10$), but the surface tension and gravity forces vary. Below, to the left, are the three different growth rates with three different parameters.

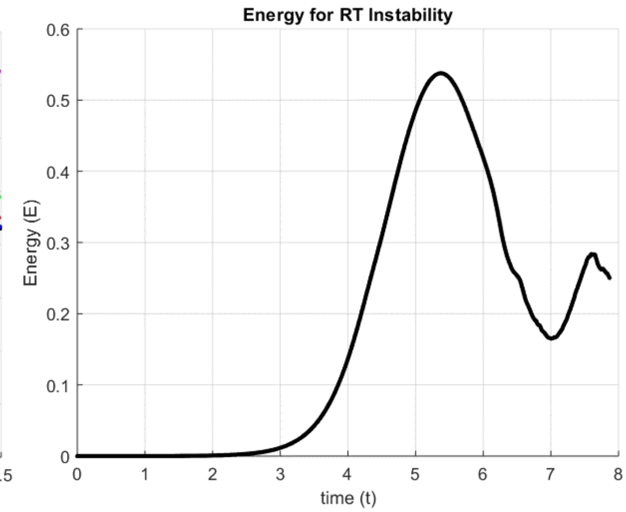




The figure on the right shows that if we rescale the energy curves, we can see the beginning stages of the curve collapse on itself.

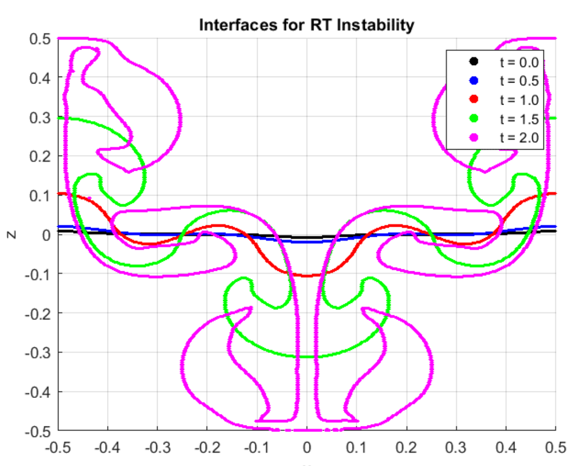
11. **Most Unstable Mode: High frequency stable, low frequency unstable.** Keeping the first configuration of Q10 ($\rho_2 = 10, \rho_1 = 11, \gamma = 0.1, g = 10$), we will run a simulation where we start with a superposition of several modes. Below are the two plots.

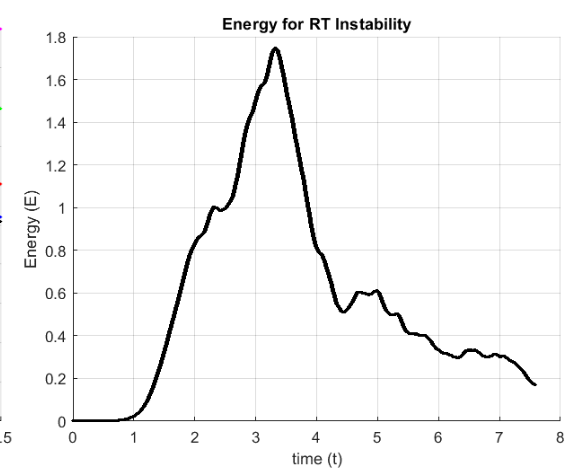




Here, the high frequency modes are stable, while the low frequency modes are unstable. This can be seen by noticing that the first two interface curves (the black and blue curves) have two modes present, which is seen by the multiple crests in the initial perturbation. However, as time goes on, only one mode seems to "take over" in a sense – the low frequency modes. This is evident by noticing that later curves (i.e. the green and magenta curves) have only one mode, which is the dominating mode. In this case, the dominating mode is the low frequency wavenumber, which makes the system unstable.

12. **Most Unstable Mode: High Frequency More Unstable than Low Frequency.** Now, we want to make a higher frequency unstable, by changing the physical variables of the problem. We know that k_i is based on $\rho_1, \rho_2, \gamma,$ and g, so by changing these variables, we can make it such that values like $2k_0$ or $4k_0$ are more unstable than $k_0 = 2\pi$. I have changed the following parameters as follows: $\rho_2 = 12, \rho_1 = 10, \gamma = 0.05, g = 20$. Below are the two plots of the interface data and energy as a function of time.





13. **Unstable Modes.** Let's consider $\zeta_{high} = a\cos(8k_0x)$, a stable mode, while $\zeta_{low} = a\cos(k_0x)$ unstable, although it is not present initially. Regarding numerical simulations, we can set it such that ζ_{high} is the first mode to appear, being stable (which means that any higher modes are also stable), and any lower modes (ζ_{low}, for example) are not present initially. We would expect that the higher mode will just oscillate and decay until the perturbation is gone – but this is not what happens. Numerically, any noise produced from the oscillation of the higher (but stable) mode will trigger lower modes that *are* in the range of k_0's that are unstable modes. Once these modes are triggered, an instability will arise.

14. **Resolution and Numerical Accuracy.** I ran the same exact case as I did in question 8 and ran it with MAXLEVEL=7 and MAXLEVEL=9. As can be seen below, the results are not the same. Since this is not an analytical solution, and is instead a numerical solution, the results will be different even though it is the same mathematical problem. Other variables, such as the grid spacing used, will change the answer for the same analytical problem. It is also notable that at MAXLEVEL=7, there are fewer data points than at MAXLEVEL=9, meaning there is a higher resolution the higher the MAXLEVEL is set to be.