Advanced Engineering Mathematics

1 SERIES

A sequence is a list of terms that have been arranged in a certain order.

A series is the sum of all the terms in a sequence. However, there has to be a definite relationship between all the terms.

1.1 ARITHMETIC SEQUENCE

A sequence is arithmetic if $d \in \mathbb{R} \ni \forall k \in \mathbb{Z}^+$,

$$a_{k+1} = a_{k+d}$$

where $d = a_{k+1} - a_k$ is the common difference
and $d = a_k + (n-k)d$ is the nth term of the sequence

NOTATION: $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

1.1.1 ARITHMETIC SERIES

Partial Sum:

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

Or

$$S_n = \frac{n}{2}(a_1 + a_n)$$

1.2 GEOMETRIC SEQUENCE

A sequence is geometric if $a_1 \neq 0$ and if $r \in \mathbb{R} \neq 0 \ni \forall k \in \mathbb{Z}$,

$$a_{k+1} = a_k r$$

where $r = \frac{a_{k+1}}{a_k}$ is the common ratio
and $a_n = a_k r^{n-k}$ is the nth term

1.2.1 GEOMETRIC SERIES

$$\sum_{n=1}^{\infty} a^{n-1} = a + ar + ar^2 + ...$$

is convergent if $|r| < 1$
and the sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, |r| < 1$$

1.3 CONVERGENCE

A series converges when the infinite sequence of the partial sums have a finite limit.

Any series in which individual terms approach zero converges.

If $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$

1.4 DIVERGENCE

A series diverges when the infinite sequence of the partial sums does not have a finite limit.

Any series in which individual terms does not approach zero diverges.
Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...,$

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n$$

If $\{S_n\}$ is convergent and $\lim_{n\to\infty} S_n = s$ exists as a real number, then $\sum a_n$ is called convergent and

$$a_1 + a_2 + ... + a_n + ... = s$$

Or

$$\sum_{n=1}^{\infty} a_n = s$$

where s is the sum. Otherwise, the series is divergent.

1.5 TEST FOR DIVERGENCE

If $lim_{n\to\infty} a_n$ does not exist or if $lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

1.6 PROPERTIES OF CONVERGENT SERIES

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are:

i) $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$

ii) $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

iii) $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

1.7 INTEGRAL TEST

Suppose f is a *continuous*, *positive*, *decreasing* function on $[1, \infty)$.
Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

1.8 *p*-SERIES

The *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

1.9 COMPARISON TEST

Suppose $\sum a_n$ and $\sum b_n$ are series with *positive* terms.

i) If $\sum b_n$ is convergent and $a_n \leq b_n \forall n$, then $\sum a_n$ is convergent.

ii) If $\sum b_n$ is divergent and $a_n \geq b_n \forall n$, then $\sum a_n$ is divergent.

1.10 LIMIT COMPARISON TEST

Suppose $\sum a_n$ and $\sum b_n$ are series with *positive* terms.
If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$
where c is a finite number and $c > 0$, then both series either converge or diverge.

1.11 ALTERNATING SERIES TEST

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + ..., b_n > 0$$

Satisfies

(i) $b_{n+1} \leq b_n \forall n$

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

1.12 ABSOLUTE CONVERGENCE

If $\sum_{n=0}^{\infty}|a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

1.13 CONDITIONAL CONVERGENCE

If $\sum a_n$ converges, but $|a_n|$ does not, $\sum a_n$ converges conditionally.

1.14 RATIO TEST

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive*.

*use another test.

1.15 ROOT TEST

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the test is inconclusive*.

*use another test.

1.16 POWER SERIES

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$$

where x is a variable and the c_n's are the coefficients of the series.
A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + ... + c_n x^2 + ...$$

whose domain is the set of all x for which the series converges.

In general, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ...$$

is called a power series in $(x-a)$ or a power series centered at a or a power series about a.

1.16.1 THEOREM

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

i The series converges only when $x = a$

ii The series converges $\forall x$

iii There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R. The Root and Ratio Tests *always* fail if x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.
We can represent certain types of functions as sums of power series by manipulating geometric series or by differentiating such a series. Expressing a known function as a sum of infinitely many terms is useful for integrating functions that don't have elementary antiderivatives, for solving different equations, and approximating functions by polynomials. Recall that:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\text{inf}} x^n \qquad |x| < 1$$

1.16.2 DIFFERENTIATION AND INTEGRATION OF POWER SERIES

If the power series $C_n(x-a)^n$ has radius of convergence $R > 0$ then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

i $\frac{d}{dx}[\sum_{n=0}^{\infty} c_n(x-a)^n] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$

ii $\int[\sum_{n=0}^{\infty} c_n(x-a)^n]dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n dx$

The radii of convergence of the power series in i and ii are both R

1.17 TAYLOR SERIES

If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting c_n back to the series gives

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The above series is called the Taylor series of the function f at a.

1.18 MACLAURIN SERIES

The special case $a = 0$ of the Taylor series, such series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

This case arises frequently enough and is called the Maclaurin series.

1.18.1 BINOMIAL SERIES

If $k \in \mathbb{R}$ and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} (\,^k_n) x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots$$

1.18.2 MACLAURIN SERIES AND THEIR RADII OF CONVERGENCE

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad R = \infty$$

$$sinx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad R = \infty$$

$$cosx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad R = \infty$$

$$tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad R = 1$$

$$ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} (\,^k_n) x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \dots \qquad R = 1$$

2 SERIES SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

2.1 ANALYTIC AT A POINT

A function f is analytic at a point a if it can be represented by a power series in $x - a$ with a positive radius of convergence.

Examples:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

for $|x| < \infty$. these Maclaurin series are analytic at $x = 0$.

2.2 SHIFTING THE SUMMATION INDEX

Combining two or more summations as a single summation often requires reindexing, that is, a shift in the index of summation.

EXAMPLE

Write

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1}$$

As one power series.

SOLUTION

Write the first term of the first summation:

at $n = 2$:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = (2)(2-1)c_2 x^{2-2}$$

the expression then becomes:

$$(2)(2-1)c_2 x^{2-2} + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$= 2c_2 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1}$$

For the first summation, create a dummy variable $k = n - 2$ and thus $n = k + 2$. The first summation becomes:

$$\sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2} x^k$$

For the second summation, create a dummy variable $k = n + 1$ and thus $n = k - 1$. The second summation becomes:

$$\sum_{k=1}^{\infty} c_{k-1} x^k$$

both summations now *start at the same index* and have *the same exponent of x*. You can now combine them:

$$2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2} x^k - \sum_{k=1}^{\infty} c_{k-1} x^k$$

$$= 2c_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} - c_{k-1}]x^k$$

2.3 SOLUTIONS ABOUT ORDINARY POINTS

Suppose the linear second order differential equation:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

divide by the leading coefficient $a_2(x)$:

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = 0$$

Let $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$; thus the standard form:

$$y'' + P(x)y' + Q(x)y = 0$$

2.3.1 ORDINARY POINTS

A point x_0 is said to be the **ordinary point** of a differential equation if both $P(x)$ and $Q(x)$ are analytic at x_0. A point that is not an ordinary point is said to be a **singular point** of the equation.

Ordinary points are extracted from the values of x in the expressions of the numerators, while **singular points** can be extracted from the expressions in the denominators.

2.3.2 POWER SERIES SOLUTIONS

If $x = x_0$ is an ordinary point of a differential equation, we can *always find two linearly independent solutions* in the form of a power series centered at x_0;

that is,

$$y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$$

A series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

Power series solutions can only be used if a differential equation has an **ordinary point**.

2.4 SOLUTIONS ABOUT SINGULAR POINTS

A singular point is a **regular singular point** when the expression in the denominator of $P(x)$ is *at most* to the first degree and the expression in the denominator of $Q(x)$ is *at most* to the second degree. Otherwise, it is an **irregular singular point**.

2.4.1 FROBENIUS METHOD

If $x = x_0$ is a regular point, then there exists at least one non-zero solution in the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n(x - x_0)^n = \sum_{n=0}^{\infty} c_n(x - x_0)^{n+r}$$

where r is a constant to be determined. The series will converge at least on some given interval defined by $0 < x - x_0 < R$.
Assume that $c_0 \neq 0$.

2.4.2 GENERAL INDICIAL EQUATION

To determine the values of r_1 and r_2, we use the **general indicial equation** given by:

$$r(r-1) + a_0 r + b_0 = 0$$

$$r^2 - r + a_0 r + b_0 = 0$$

rearranging,

$$r^2 + (a_0 - 1)r + b_0 = 0$$

which is in the form of a quadratic equation in standard form $ar^2 + br + c = 0$, where $a = 1$, $b = a_0 - 1$ and $c = b_0$.

And thus the quadratic formula can be used to solve for r_1 and r_2:

$$r_{1,2} = \frac{-(a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4(1)(b_0)}}{2(1)}$$

2.4.3 CASE I

If r_1 and r_2 are distinct (their difference is between 0 and 1), there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

2.4.4 CASE II

If $r_1 - r_2 = N$, where N is a positive integer, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \qquad , c_0 \neq 0$$

$$y_2(x) = Cy_1(x)ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2} \qquad , b_0 \neq 0$$

Where C is a constant that could be zero.

2.4.5 CASE III

If r1 = r2, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^n + r_1 \qquad , c_0 \neq 0$$

$$y_2(x) = y_1(x)ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

if $r_1 - r_2 = 0$, the method fails to give a series solution. However, if $y_1(x)$ is a known solution, we can obtain the second solution using:

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$