Advanced Engineering Mathematics

1 SERIES

A sequence is a list of terms that have been arranged in a certain order.

A series is the sum of all the terms in a sequence. However, there has to be a definite relationship between all the terms.

1.1 ARITHMETIC SEQUENCE

A sequence is arithmetic if $d \in \mathbb{R} \ni \forall k \in \mathbb{Z}^+$,

$$a_{k+1} = a_{k+d}$$

where $d = a_{k+1} - a_k$ is the common difference
and $d = a_k + (n-k)d$ is the nth term of the sequence

NOTATION: $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

1.1.1 ARITHMETIC SERIES

Partial Sum:

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

Or

$$S_n = \frac{n}{2}(a_1 + a_n)$$

1.2 GEOMETRIC SEQUENCE

A sequence is geometric if $a_1 \neq 0$ and if $r \in \mathbb{R} \neq 0 \ni \forall k \in \mathbb{Z}$,

$$a_{k+1} = a_k r$$

where $r = \frac{a_{k+1}}{a_k}$ is the common ratio
and $a_n = a_k r^{n-k}$ is the nth term

1.2.1 GEOMETRIC SERIES

$$\sum_{n=1}^{\infty} a^{n-1} = a + ar + ar^2 + ...$$

is convergent if $|r| < 1$
and the sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, |r| < 1$$

1.3 CONVERGENCE

A series converges when the infinite sequence of the partial sums have a finite limit.

Any series in which individual terms approach zero converges.

If $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$

1.4 DIVERGENCE

A series diverges when the infinite sequence of the partial sums does not have a finite limit.

Any series in which individual terms does not approach zero diverges.
Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...,$

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n$$

If $\{S_n\}$ is convergent and $\lim_{n\to\infty} S_n = s$ exists as a real number, then $\sum a_n$ is called convergent and

$$a_1 + a_2 + ... + a_n + ... = s$$

Or

$$\sum_{n=1}^{\infty} a_n = s$$

where s is the sum. Otherwise, the series is divergent.

1.5 TEST FOR DIVERGENCE

If $lim_{n\to\infty} a_n$ does not exist or if $lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

1.6 PROPERTIES OF CONVERGENT SERIES

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are:

(i) $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

1.7 INTEGRAL TEST

Suppose f is a *continuous*, *positive*, *decreasing* function on $[1, \infty)$.
Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

1.8 p-SERIES

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

1.9 COMPARISON TEST

Suppose $\sum a_n$ and $\sum b_n$ are series with *positive* terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n \forall n$, then $\sum a_n$ is convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n \forall n$, then $\sum a_n$ is divergent.

1.10 LIMIT COMPARISON TEST

Suppose $\sum a_n$ and $\sum b_n$ are series with *positive* terms.
If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$
where c is a finite number and $c > 0$, then both series either converge or diverge.

1.11 ALTERNATING SERIES TEST

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + ..., b_n > 0$$

Satisfies

(i) $b_{n+1} \leq b_n \forall n$

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

1.12 ABSOLUTE CONVERGENCE

If $\sum_{n=0}^{\infty}|a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

1.13 CONDITIONAL CONVERGENCE

If $\sum a_n$ converges, but $|a_n|$ does not, $\sum a_n$ converges conditionally.

1.14 RATIO TEST

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive*.

*use another test.

1.15 ROOT TEST

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the test is inconclusive*.

*use another test.

1.16 POWER SERIES

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$$

where x is a variable and the c_n's are the coefficients of the series.
A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + ... + c_n x^2 + ...$$

whose domain is the set of all x for which the series converges.

In general, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ...$$

is called a power series in $(x-a)$ or a power series centered at a or a power series about a.

1.16.1 THEOREM

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

i) The series converges only when $x = a$

ii) The series converges $\forall x$

iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R. The Root and Ratio Tests *always* fail if x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.
We can represent certain types of functions as sums of power series by manipulating geometric series or by differentiating such a series. Expressing a known function as a sum of infinitely many terms is useful for integrating functions that don't have elementary antiderivatives, for solving different equations, and approximating functions by polynomials. Recall that:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\text{inf}} x^n \qquad |x| < 1$$

1.16.2 DIFFERENTIATION AND INTEGRATION OF POWER SERIES

If the power series $C_n(x-a)^n$ has radius of convergence $R > 0$ then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

i) $\frac{d}{dx}[\sum_{n=0}^{\infty} c_n(x-a)^n] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$

ii) $\int[\sum_{n=0}^{\infty} c_n(x-a)^n]dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n dx$

The radii of convergence of the power series in i and ii are both R

1.17 TAYLOR SERIES

If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting c_n back to the series gives

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The above series is called the Taylor series of the function f at a.

1.18 MACLAURIN SERIES

The special case $a = 0$ of the Taylor series, such series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

This case arises frequently enough and is called the Maclaurin series.

1.18.1 BINOMIAL SERIES

If $k \in \mathbb{R}$ and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} (\,^{k}_{n}) x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots$$

1.18.2 MACLAURIN SERIES AND THEIR RADII OF CONVERGENCE

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad R = \infty$$

$$sinx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad R = \infty$$

$$cosx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad R = \infty$$

$$tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad R = 1$$

$$ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} (\,^{k}_{n}) x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \dots \qquad R = 1$$

2 SERIES SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

2.1 ANALYTIC AT A POINT

A function f is analytic at a point a if it can be represented by a power series in $x - a$ with a positive radius of convergence.

Examples:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

for $|x| < \infty$. these Maclaurin series are analytic at $x = 0$.

2.2 SHIFTING THE SUMMATION INDEX

Combining two or more summations as a single summation often requires reindexing, that is, a shift in the index of summation.

EXAMPLE

Write

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1}$$

As one power series.

SOLUTION

Write the first term of the first summation:

at $n = 2$:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = (2)(2-1)c_2 x^{2-2}$$

the expression then becomes:

$$(2)(2-1)c_2 x^{2-2} + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$= 2c_2 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1}$$

For the first summation, create a dummy variable $k = n - 2$ and thus $n = k + 2$. The first summation becomes:

$$\sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2} x^k$$

For the second summation, create a dummy variable $k = n + 1$ and thus $n = k - 1$. The second summation becomes:

$$\sum_{k=1}^{\infty} c_{k-1} x^k$$

both summations now *start at the same index* and have *the same exponent of x*. You can now combine them:

$$2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2} x^k - \sum_{k=1}^{\infty} c_{k-1} x^k$$

$$= 2c_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} - c_{k-1}] x^k$$

2.3 SOLUTIONS ABOUT ORDINARY POINTS

Suppose the linear second order differential equation:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

divide by the leading coefficient $a_2(x)$:

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = 0$$

Let $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$; thus the standard form:

$$y'' + P(x)y' + Q(x)y = 0$$

2.3.1 ORDINARY POINTS

A point x_0 is said to be the **ordinary point** of a differential equation if both $P(x)$ and $Q(x)$ are analytic at x_0. A point that is not an ordinary point is said to be a **singular point** of the equation.

Ordinary points are extracted from the values of x in the expressions of the numerators, while **singular points** can be extracted from the expressions in the denominators.

2.3.2 POWER SERIES SOLUTIONS

If $x = x_0$ is an ordinary point of a differential equation, we can *always find two linearly independent solutions* in the form of a power series centered at x_0;

that is,

$$y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$$

A series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

Power series solutions can only be used if a differential equation has an **ordinary point**.

2.4 SOLUTIONS ABOUT SINGULAR POINTS

A singular point is a **regular singular point** when the expression in the denominator of $P(x)$ is *at most* to the first degree and the expression in the denominator of $Q(x)$ is *at most* to the second degree. Otherwise, it is an **irregular singular point**.

2.4.1 FROBENIUS METHOD

If $x = x_0$ is a regular point, then there exists at least one non-zero solution in the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n(x - x_0)^n = \sum_{n=0}^{\infty} c_n(x - x_0)^{n+r}$$

where r is a constant to be determined. The series will converge at least on some given interval defined by $0 < x - x_0 < R$.
Assume that $c_0 \neq 0$.

2.4.2 GENERAL INDICIAL EQUATION

To determine the values of r_1 and r_2, we use the **general indicial equation** given by:

$$r(r-1) + a_0 r + b_0 = 0$$
$$r^2 - r + a_0 r + b_0 = 0$$

rearranging,

$$r^2 + (a_0 - 1)r + b_0 = 0$$

which is in the form of a quadratic equation in standard form $ar^2 + br + c = 0$, where $a = 1$, $b = a_0 - 1$ and $c = b_0$.

And thus the quadratic formula can be used to solve for r_1 and r_2:

$$r_{1,2} = \frac{-(a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4(1)(b_0)}}{2(1)}$$

2.4.3 CASE I

If r_1 and r_2 are distinct (their difference is between 0 and 1), there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

2.4.4 CASE II

If $r_1 - r_2 = N$, where N is a positive integer, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \qquad , c_0 \neq 0$$

$$y_2(x) = C y_1(x) ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2} \qquad , b_0 \neq 0$$

Where C is a constant that could be zero.

2.4.5 CASE III

If r1 = r2, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^n + r_1 \qquad , c_0 \neq 0$$

$$y_2(x) = y_1(x) ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

if $r_1 - r_2 = 0$, the method fails to give a series solution. However, if $y_1(x)$ is a known solution, we can obtain the second solution using:

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

3 Vector Analysis

3.1 Vectors in $\mathbb{R}^n$

3.1.1 Vectors in $\mathbb{R}^2$

Scalars vs Vectors:

Scalar: A quantity with a *magnitude* (in simplest terms, just a number)
Vector: A quantity with both a *magnitude* and *direction* (weight, force)

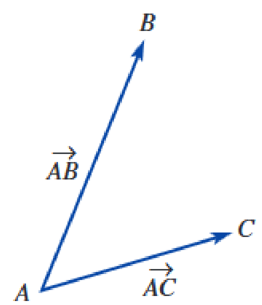
3.1.2 Vector Notation and Terminology

$\overrightarrow{AB}$: a vector with start point at **A** and terminal point at **B**
$\left\|\overrightarrow{\mathbf{AB}}\right\|$: magnitude of a vector
$\overrightarrow{AB} = \overrightarrow{CD}$: two vectors that have the same magnitude and direction
$-\overrightarrow{AB}$: negative of a vector
$k\overrightarrow{AB}$: scalar multiple of a vector
$0\overrightarrow{AB} = 0$: the zero vector

Vectors are *free*— they can be moved to any new position if their magnitudes and directions are not changed.

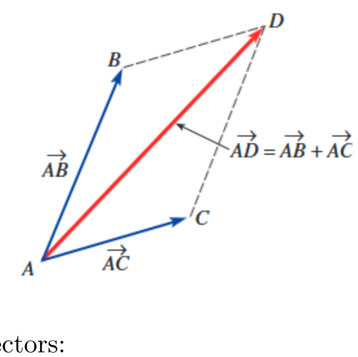
if $k > 0$, $k\overrightarrow{AB}$ has the same direction as $\overrightarrow{AB}$
if $k < 0$, $k\overrightarrow{AB}$ has the same direction, but opposite of $\overrightarrow{AB}$
Two vectors are *parallel* if and only if they are nonzero scalar multiples of each other.

3.1.3 Vector Addition and Subtraction



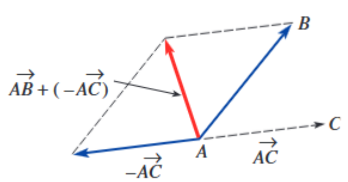
If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are on the sides of a parallelogram, $\overrightarrow{AD}$ is the sum of $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}$$



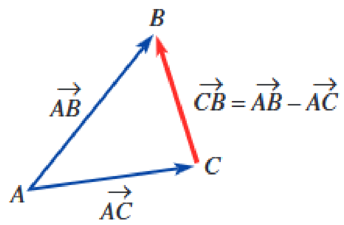
The difference between two vectors:

$$\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{AB} + (-\overrightarrow{AC})$$



The same vector difference can be interpretted as the third side of a triangle with sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

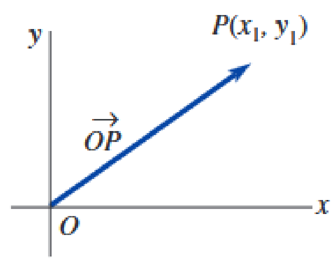
$$\overrightarrow{CB} = \overrightarrow{AB} - \overrightarrow{AC}$$



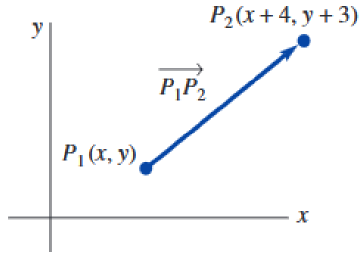
3.1.4 Vectors in a Coordinate Plane

Suppose vectors are on a 2D plane.
A position vector of the point P is a vector with an initial point at origin O and a terminal point $P(x_1, y_1)$:

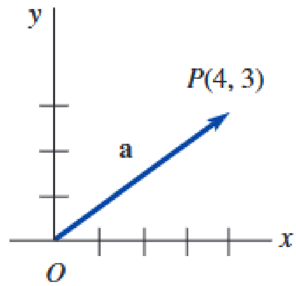
$$\overrightarrow{OP} = \langle x_1, y_1 \rangle$$



EXAMPLE 1
The displacement from an initial point $P_1(x, y)$ to a terminal point $P_2(x+4, y+3)$ is 4 units to the right and 3 units up.



The position vector $a = \langle 4, 3 \rangle$ emanating from the origin is the equivalent to the displacement vector $\overrightarrow{P_1P_2}$ from $P_1(x, y)$ to $P_2(x+4, y+3)$.



In general, a vector a in $\mathbb{R}^2$ is any ordered pair of real numbers

$$a = \langle a_1, a_2 \rangle$$

The numbers a_1 and a_2 are components of vector a.

Addition, Scalar Multiplication, Equality
Let $a = \langle a_1, a_2 \rangle$ and $b = \langle b_1, b_2 \rangle$ be vectors in $\mathbb{R}^2$.

i Addition: $a + b = \langle a_1 + b_1, a_2 + b_2 \rangle$

ii Scalar Multiplication: $ka = \langle ka_1, kb_1 \rangle$

iii Equality: $a = b$ if and only if $a_1 = b_1, a_2 = b_2$

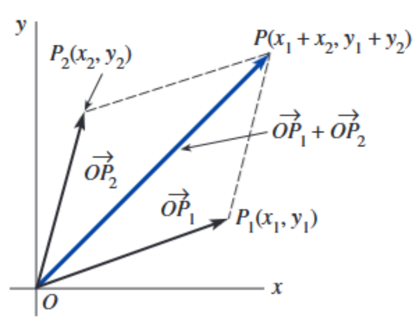
Subtraction
The *negative* of a vector b is defined by

$$-b = (-1)b = \langle -b_1, -b_2 \rangle$$

Subtraction can then be defined by

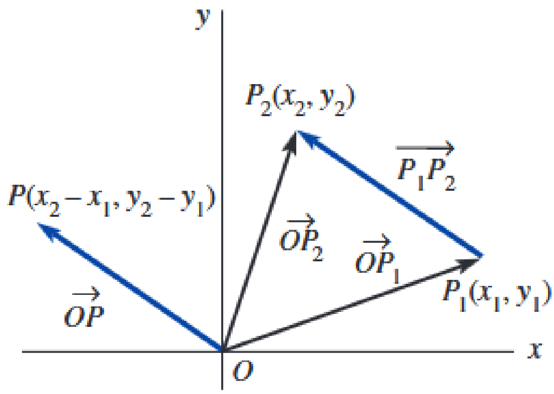
$$a - b = a + (-b) = \langle a_1 - b_1, a_2 - b_2 \rangle$$

The sum of two vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$



The vector $\overrightarrow{P_1P_2}$ with initial point P_1 and terminal point P_2 is the difference of position vectors

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \langle x_2 - x_1, y_2 - Y_1 \rangle$$



The vector $\overrightarrow{P_1P_2}$ can be drawn either starting from the terminal point of OP_1 to the terminal point of OP_2, or as the position vector $\overrightarrow{OP}$ whose terminal point has coordinates $(x_2 - x_1, y_2 - y_1)$

$\overrightarrow{OP}$ and $\overrightarrow{P_1P_2}$ are considered equal, since the have the same magnitude and direction.

EXAMPLE 2
if $a = \langle 1, 4\rangle$ and $b = \langle -6, 3\rangle$, find:

- $\mathbf{a} + \mathbf{b}$
  $a + b = \langle 1 + (-6), 4 + 3\rangle = \langle -5, 7\rangle$
- $\mathbf{a} - \mathbf{b}$
  $a - b = \langle 1 - (-6), 4 - 3\rangle = \langle 7, 1\rangle$
- $\mathbf{2a} + \mathbf{3b}$
  $2a + 3b = \langle 2(1) + 3(-6), 2(4) + 3(3)\rangle = \langle 2 + (-18), 8 + 9\rangle = \langle -16, 17\rangle$

Properties of Vectors

(i) *Commutative:* $a + b = b + a$

(ii) *Associative:* $a + (b + c) = (a + b) + c$

(iii) *Additive Identity:* $a + 0 = a$

(iv) *Additive Inverse:* $a + (-a) = 0$

(v) $k(a + b) = ka + kb$, k a scalar

(vi) $(k_1 + k_2)a = k_1a + k_2a$, k_1 and k_2 scalars

(vii) $k_1(k_2a) = (k_1k_2)a$, k_1 and k_2 scalars

(viii) $1a = a$

(ix) *Zero Vector:* $0a = 0$

the zero vector 0 in properties (iii), (iv), and (ix) is defined as

$$0 = \langle 0, 0\rangle$$

Magnitude

The **magnitude**, **length** or **norm** of a vector a is denoted by $\|a\|$
By pythagorean theorem,

$$\|a\| = \sqrt{a_1^2 + a_2^2}$$

$\|a\| \geq 0$ for any vector a, and $\|a\| = 0$ if and only if $a = 0$. For example, if $a = \langle 6, -2 \rangle$,

$$\|a\| = \sqrt{6^2 + -2^2} = \sqrt{36 + 4} = \sqrt{40} = 2\sqrt{10}$$

Unit Vectors

A vector of magnitude 1 is called a *unit vector.*

$$\|u\| = \left\| \frac{1}{\|a\|} a \right\| = \frac{1}{\|a\|} \|a\| = 1$$

write $u = \frac{1}{\|a\|} a$ as

$$u = \frac{a}{\|a\|}$$

EXAMPLE 3

Given $\mathbf{a} = \langle 2, -1 \rangle$, form a unit vector in the same direction as $\mathbf{a}$. In the oppposited direction of $\mathbf{a}$.

SOLUTION

$$\|\mathbf{a}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$\mathbf{u} = \frac{1}{\sqrt{5}} \mathbf{a} = \frac{1}{\sqrt{5}} \langle 2, -1 \rangle = \langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle$$

In the opposite direction of $\mathbf{a}$:

$$-\mathbf{u} = \langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$$

If $\mathbf{a}$ and $\mathbf{b}$ are vectors and c_1 and c_2 are scalars, then $c_1\mathbf{a} + c_2\mathbf{b}$ is called a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

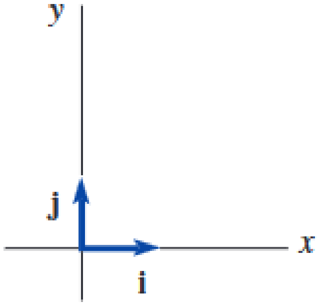
The i, j Vectors

Any vector $\mathbf{a} = \langle a_1, a_2 \rangle$ can be written as a sum:

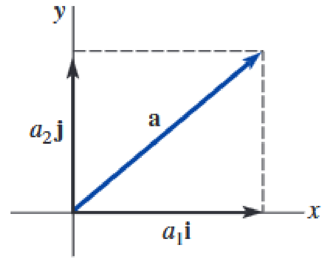
$$\langle a_1, a_2 \rangle = \langle a_1, 0 \rangle + \langle 0, a_2 \rangle = a_1 \langle 1, 0 \rangle + a_2 \langle 0, 1 \rangle$$

The unit vectors $\langle a_1, a_2 \rangle$ and $\langle 0, 1 \rangle$ are usually given the special symbols $\mathbf{i}$ and $\mathbf{j}$.

$$i = \langle 1, 0 \rangle \text{ and } j = \langle 0, 1 \rangle$$



thus $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$



The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are referred to as the **standard basis** for the system of 2D vectors. the scalar a_1 is called the *horizontal component* and a_2 is called the *vertical components* of $\mathbf{a}$.

EXAMPLE 4: Vector Operations Using i and j

(a) $\langle 4, 7\rangle = 4\mathbf{i} + 7\mathbf{j}$

(b) $(2\mathbf{i} - 5\mathbf{j}) + (8\mathbf{i} + 13\mathbf{j}) = 10\mathbf{i} + 8\mathbf{j}$

(c) $\|\mathbf{i} + \mathbf{j}\| = \sqrt{2}$

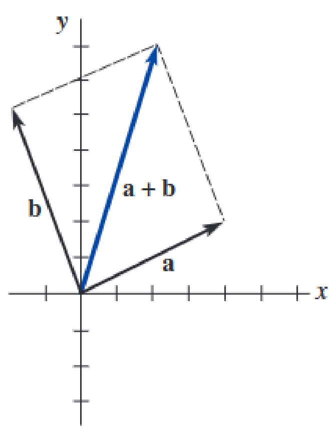
(d) $10(3\mathbf{i} - \mathbf{j}) = 30\mathbf{i} - 10\mathbf{j}$

(e) $\mathbf{a} = 6\mathbf{i} + 4\mathbf{j}$ and
$\mathbf{b} = 9\mathbf{i} + 6\mathbf{j}$ are parallel, since $\mathbf{b}$ is a scalar multiple of $\mathbf{a}$. We see that $\mathbf{b} = \frac{3}{2}\mathbf{a}$.

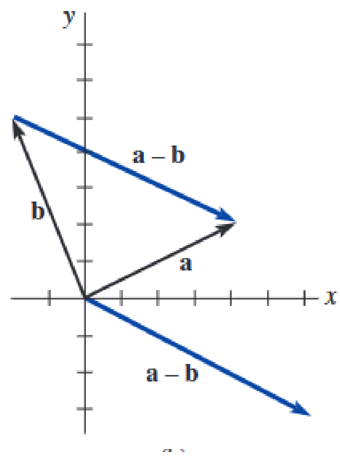
EXAMPLE 5: Graphs of Vector Sum/Vector Difference
Let $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = -2\mathbf{i} + 5\mathbf{j}$. Graph $\mathbf{a} + \mathbf{b}$ and $\mathbf{a}$ - $\mathbf{b}$.

SOLUTION

$$\mathbf{a} + \mathbf{b} = 2\mathbf{i} + 7\mathbf{j}$$

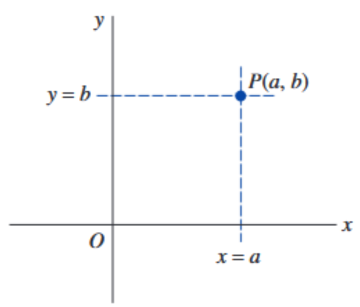


$$\mathbf{a} - \mathbf{b} = 6\mathbf{i} - 3\mathbf{j}$$

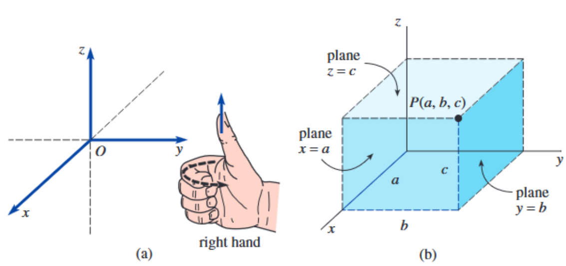


3.1.5 Vectors in $\mathbb{R}^3$

If P is the point of intersection of the line $x = a$, and the line $y = b$, then the *ordered pair* (a, b) is said to be the *rectangular* or *Cartesian coordinates* of the point.



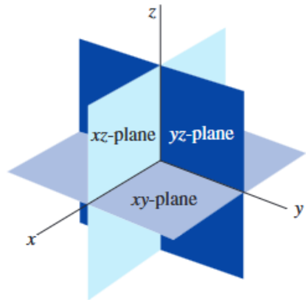
Rectangular Coordinate System in 3-Space



The dashed lines figure (a) represent the negative axes. Now if:

$$x = a, \quad y = b, \quad z = c$$

are planes perpendicular to the x-axis, y-axis, and z-axis, then the point P at which these points intersect can be represented as an *ordered triple* of numbers (a, b, c) said to be the **rectangular** or **Cartesian coordinates** of the point.
The numbers a, b, and c are called the x-, y-, and z-**coordinates** of $P(a, b, c)$.
Each pair of coordinate axes determines a **Coordinate pair**.

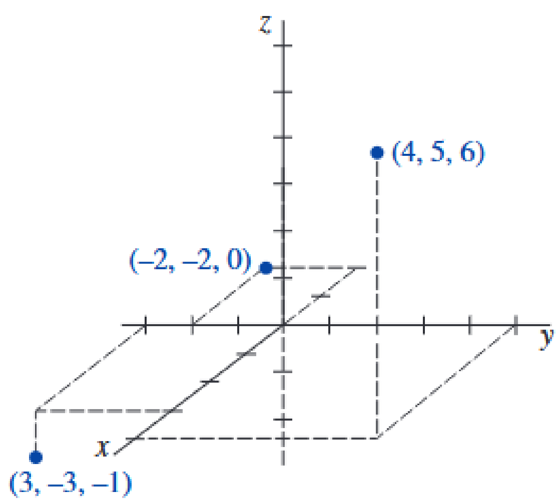


The coordinate planes divide 3-space into eight parts known as *octants*. the octant in which all three coordinates of a point are positive is called the *first octant*.

Axes	Coordinates	Plane	Coordinates
x	$(a, 0, 0)$	xy	$(a, b, 0)$
y	$(0, b, 0)$	xz	$(a, 0, c)$
z	$(0, 0, c)$	yz	$(0, b, c)$

EXAMPLE 1: Graphs of Three Points

Graph the points $(4, 5, 6)$, $(3, -3, -1)$, and $(-2, -2, 0)$.



Distance Formula

To find the *distance* between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in 3-space, consider their projection onto the xy-plane. The distance between $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

If the coordinates for P_3 are (x_2, y_2, z_1), then the Pythagorean theorem applied to $P_1P_2P_3$ yields

$$[d(P_1, P_2)^2] = [\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}]^2 + |z_2 - z_1|^2$$

or

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

EXAMPLE 2: Distance Between Two Points

Find the distance between $(2, -3, 6)$ and $(-1, -7, 4)$

SOLUTION
P_2 as $(2, -3, 6)$ and P_1 as $(-1, -7, 4)$

$$d = \sqrt{(2-(-1))^2 + (-3-(-7))^2 + (6-4)^2} = \sqrt{29}$$

Midpoint Formula

$$(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2})$$

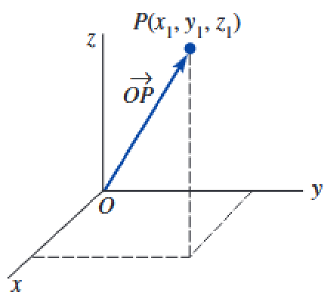
EXAMPLE 3: Coordinates of a Midpoint

Using the coordinates from example 3:

$$(\frac{2 + (-1)}{2}, \frac{-3 + (-7)}{2}, \frac{6 + 4}{2})\text{or}(\frac{1}{2}, -5, 5)$$

Vectors in 3-Space
A **vector a in 3-space** is any order triple of real numbers

$$a = \langle a_1, a_2, a_3 \rangle$$



Component Definitions in 3-space

Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ be vectors in $\mathbb{R}^3$.

(i) *Addition:* $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

(ii) *Scalar multiplication:* $k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$

(iii) *Equality:* $\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1, a_2 = b_2, a_3 = b_3$

(iv) *Negative:* $-b = (-1)\mathbf{b} = \langle -b_1, -b_2, -b_3 \rangle$

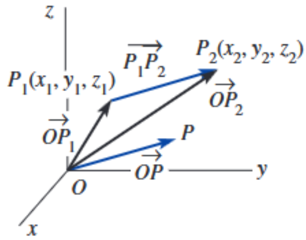
(v) *Subtraction:* $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$

(vi) *Zero vector:* $\mathbf{0} = \langle 0, 0, 0 \rangle$

(vii) *Magnitude:* $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^3}$

If $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are the position points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ then the vector $\overrightarrow{P_1P_2}$ is given by

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$



EXAMPLE 4: Vector Between Two Points

Find the vector $\overrightarrow{P_1P_2}$ if the points P_1 and P_2 are given by $P1(4, 6, -2)$ and $P2(1, 8, 3)$, respectively.

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \langle 1-4, 8-6, 3-(-2) \rangle = \langle -3, 2, 5 \rangle$$

EXAMPLE 5: A Unit Vector

Find a unit vector in the direction of $\mathbf{a} = \langle -2, 3, 5 \rangle$