
Descent-based joint optimization

Jean Feng and Noah Simon

Abstract

Tuning regularization parameters in regression problems allows one to control model complexity and induce desired structure. The current method of searching over a k -dimensional grid of parameter values is computationally intractable for $k > 2$. We propose tuning the penalty parameters by treating it as a continuous optimization problem and updating the parameter values using a descent-based approach. Compared to performing cross validation over a grid, our method is significantly more efficient while achieving the same performance. This descent-based approach enables us to test regularizations with many penalty parameters, through which we discover new regularization methods with superior accuracy. Our experiments are performed on simulated and real data.

1 Introduction

Consider the usual regression framework with p features and a response measured on n observations. Let X denote the $n \times p$ design matrix and y the response vector. For ill-posed or high-dimensional problems ($p \gg n$), there can often be an infinite number of solutions that minimize the loss function L but have high generalization error. A common solution is to use regularization, or penalization, to select models with desirable properties, such as smoothness and sparsity.

In recent years, there has been much interest in combining regularization methods to produce models with multiple desired characteristics. Examples include the elastic net (Zou and Hastie 2003), which combines the lasso and ridge penalties, and the sparse group lasso (Simon 2007), which combines the group lasso and lasso penalties. The general form of these regression problems is:

$$\hat{f}(\lambda_1, \dots, \lambda_J) = \arg \min_f L(y, f(X)) + \sum_{i=1}^J \lambda_i P_i(f) \quad (1)$$

where L is the loss function, $\{P_i\}_{i=1, \dots, J}$ are the penalty functions, and $\{\lambda_i\}_{i=1, \dots, J}$ are the regularization parameters.

Regularization parameters control the degree of various facets of model complexity (e.g. amount of sparsity or smoothness). Often, the goal is to set the parameters to minimize the fitted model's generalization error. One usually estimates this using a training/validation approach (or cross-validation). There one fits a model on a training set (X_T, y_T) and measures the model's error on a validation set (X_V, y_V) . The goal then is to choose penalty parameters that minimize the validation error, as formulated in the following optimization problem:

$$\begin{aligned} & \min_{\lambda \in \Lambda} L(y_V, \hat{f}(X_V | \lambda)) \\ & \text{where } \hat{f}(\cdot | \lambda) = \arg \min_f L(y_T, f(X_T)) + \sum_{i=1}^J \lambda_i P_i(f) \end{aligned} \quad (2)$$

The simplest approach to solving the problem above is brute force: one fits models over a grid of parameter values and selects the model with the lowest validation error. As long as the grid is large and fine enough, this method of “grid search” will find a solution close to the global optimum. This approach is the current standard for choosing penalty parameters via train/validation. Unfortunately, it is

computationally intractable in cases with more than two parameters. Many variants of grid search have been proposed to increase efficiency, but their runtimes are all exponential in the number of parameters.

In this paper, we propose leveraging the tools of optimization to solve (2) over the penalty parameter space. We give a gradient descent algorithm for the penalty parameters (to minimize validation error). In contrast to an exhaustive “grid search”, this “descent-based” optimization makes use of the smoothness of our validation-error surface.

In simulation studies we show that our descent-based optimization produces solutions with the same validation error as those from grid search. In addition, we find that our approach is highly efficient and can solve regressions with hundreds of penalty parameters. Finally, we use this method to analyze regularization methods that were previously computationally intractable. Through this, we discover that a variant of sparse group lasso with many more penalty parameters can significantly decrease error and produce more meaningful models.

Lorbert and Ramadge (2010) presented some related work on this topic. They solved linear regression problems by updating regression coefficients and regularization parameters using cyclical coordinate gradient descent. We take a more general approach that allows us to apply joint optimization to both linear and nonlinear regressions. In particular this paper focuses on three examples that demonstrate the wide applicability of our method: elastic net, sparse group lasso, and additive partial linear models.

In Section 2, we describe descent-based optimization in detail and present an algorithm for solving it in example regressions. In Section 3, we show that our method achieves validation errors as low as those achieved by grid search. In Section 4, we explore variants of the example regression problems that have many more regularization parameters and demonstrate that solving (2) is still computationally tractable. Finally, we present results on data predicting colitis status from gene expression in Section 5.

2 Descent-based Joint Optimization

2.1 Definition

Suppose that we have n observations on each of which we have measured p features and a response. Let \mathbf{y} be the n response vector and \mathbf{X} be the $n \times p$ design matrix.

We are interested in finding the optimal model f in some model class F that minimizes a penalized criterion of the form:

$$\min_{f \in F} L(\mathbf{y}, f(\mathbf{X})) + \sum_{i=1}^J \lambda_i P_i(f) \quad (3)$$

where $L : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is the loss function, $P_i : \mathbb{R}^{m_i} \rightarrow \mathbb{R}$ is the i th penalty function, and λ_i are the corresponding penalty parameters. Since the optimization problem in (3) only requires fitted values at the points in the design matrix \mathbf{X} and the penalty functions take in finite-dimensional inputs, the optimization problem reduces to optimizing over some finite-dimensional space Θ . If F is the space of parametric models, then Θ would be the space of possible model parameters. If F is the space of nonparametric models, then Θ is an $n + k$ -dimensional space where the first n dimensions are the fitted values for the points in the design matrix and the last k dimensions are any additional parameters required for specifying the nonparametric model. We can therefore rewrite the optimization problem in the following form instead:

$$\min_{\theta \in \Theta} L(\mathbf{y}, f_{\theta}(\mathbf{X})) + \sum_{i=1}^J \lambda_i P_i(\theta) \quad (4)$$

Suppose that we use a training/validation split to select penalty parameters $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_J)^T$. Let the data be partitioned into a training set $(\mathbf{y}_T, \mathbf{X}_T)$ and validation set $(\mathbf{y}_V, \mathbf{X}_V)$. We choose our



Figure 1. Left: darker points mean lower validation loss. Descent-based optimization descends in the most direct path towards the point producing the lowest validation loss. Right: The 3D version. We can tune regularization parameters using a grid search... or just descend opposite of the gradient.

penalty parameters to minimize the validation error by solving the following optimization problem:

$$\begin{aligned} & \arg \min_{\boldsymbol{\lambda} \in \mathbb{R}^k} L(\mathbf{y}_V, f_{\hat{\theta}(\boldsymbol{\lambda})}(\mathbf{X}_V)) \\ & \text{where } \hat{\theta}(\boldsymbol{\lambda}) = \arg \min_{\theta \in \Theta} L(\mathbf{y}_T, f_{\theta}(\mathbf{X}_T)) + \sum_{i=1}^J \lambda_i P_i(\theta) \end{aligned} \quad (5)$$

(5) is the explicit, though often unstated, criterion that training/validation methods attempt to minimize to choose penalty parameters. The current standard is to minimize this using an exhaustive grid-search. Grid-based methods solve the joint optimization problem by fitting models over a J -dimensional grid G in the penalty parameter space — the computational runtime of grid-based methods grows exponentially with the number of parameters. While the approach is simple and powerful for a single penalty parameter, optimizing even moderate dimensional functions (3+) via exhaustive grid search is inefficient (and quickly becomes completely intractable). In addition, (5) is generally a continuous, piecewise-smooth problem. Using an exhaustive search ignores information available from the smoothness of the surface.

We propose leveraging the tools of smooth optimization to solve (5). In particular we discuss iterative methods, based on walking in a descent direction until convergence to a local minimum. In the simple case where the criterion is differentiable with respect to the penalty parameters, it is straightforward to use gradient descent or some variant thereof. We show that, with some slight tweaks, gradient descent can be applied in situations where the penalty is only differentiable when restricted to directions involving an active set.

Figure 1 illustrates the differences between the two approaches. Grid-based method fits a model at every grid point, even though many of these grid points are not close to the global or local minima. We can save significant computational time if we avoid those points unlikely to yield good models. By incorporating information about the shape of the local neighborhood, descent-based methods choose an intelligent descent direction and explore the space more efficiently.

Of course, the joint optimization problem is non-convex and therefore our method provides no guarantees. The major benefit of using our method is that it opens up the possibility of using regularization methods that combine multiple penalty terms.

To ease exposition, we will assume throughout the remainder of the manuscript that $L(\mathbf{y}_V, f_{\theta}(\mathbf{X}_V))$ is differentiable in θ . This assumption is met if both 1) $f_{\theta}(\mathbf{X}_V)$ is continuous as a function of θ ; and 2) $L(\mathbf{y}_V, \cdot)$ is smooth. Examples include the squared-error, logistic, and poisson loss functions, though not the hinge loss.

2.2 Smooth Training Criterion

Let us denote the training criterion as follows

$$L_T(\theta, \boldsymbol{\lambda}) \equiv L(\mathbf{y}_T, f_\theta(\mathbf{X}_T)) + \sum_{i=1}^J \lambda_i P_i(\theta) \quad (6)$$

First we consider the simple case where $L_T(\theta, \boldsymbol{\lambda})$ is smooth as a function of $(\theta, \boldsymbol{\lambda})$. As shown later, the validation loss is differentiable as a function of $\boldsymbol{\lambda}$. So, we can directly apply gradient descent to solve the joint optimization problem:

Initialize $\boldsymbol{\lambda}^{(0)}$.
 For each iteration k :

$$\boldsymbol{\lambda}^{(k+1)} := \boldsymbol{\lambda}^{(k)} - \alpha^{(k)} \nabla_{\boldsymbol{\lambda}} L(\mathbf{y}_V, f_{\hat{\theta}(\boldsymbol{\lambda})}(\mathbf{X}_V)) \Big|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}^{(k)}} \quad (7)$$

Note: $\alpha^{(k)}$ is the step size

Algorithm 1

To calculate the gradient, we first apply the chain rule:

$$\nabla_{\boldsymbol{\lambda}} L(\mathbf{y}_V, f_{\hat{\theta}(\boldsymbol{\lambda})}(\mathbf{X}_V)) = \left[\frac{\partial}{\partial \theta} L(\mathbf{y}_V, f_\theta(\mathbf{X}_V)) \Big|_{\theta=\hat{\theta}(\boldsymbol{\lambda})} \right]^T \frac{\partial}{\partial \boldsymbol{\lambda}} \hat{\theta}(\boldsymbol{\lambda}) \quad (8)$$

The first term, $\frac{\partial}{\partial \theta} L(\mathbf{y}_V, f_\theta(\mathbf{X}_V))$, is problem specific, but generally straightforward to calculate. To calculate the second term, $\frac{\partial}{\partial \boldsymbol{\lambda}} \hat{\theta}(\boldsymbol{\lambda})$, we note that $\hat{\theta}(\boldsymbol{\lambda})$ minimizes (6). Since (6) is smooth,

$$\nabla_{\theta} \left(L(\mathbf{y}_T, f_\theta(\mathbf{X}_T)) + \sum_{i=1}^J \lambda_i P_i(\theta) \right) \Big|_{\theta=\hat{\theta}(\boldsymbol{\lambda})} = \mathbf{0}. \quad (9)$$

Taking the derivative of both sides of (9) in $\boldsymbol{\lambda}$ and solving for $\frac{\partial}{\partial \boldsymbol{\lambda}} \hat{\theta}(\boldsymbol{\lambda})$, we get:

$$\frac{\partial}{\partial \boldsymbol{\lambda}} \hat{\theta}(\boldsymbol{\lambda}) = - \left[\nabla_{\theta}^2 \left(L(\mathbf{y}_T, f_\theta(\mathbf{X}_T)) + \sum_{i=1}^J \lambda_i P_i(\theta) \right) \right]^{-1} \begin{pmatrix} \nabla_{\theta} P_1(\theta) & \nabla_{\theta} P_2(\theta) & \dots & \nabla_{\theta} P_J(\theta) \end{pmatrix} \Big|_{\theta=\hat{\theta}(\boldsymbol{\lambda})} \quad (10)$$

We can plug this back into (8) to get our gradient. Note that because $\frac{\partial}{\partial \boldsymbol{\lambda}} \hat{\theta}(\boldsymbol{\lambda})$ is defined in terms of $\hat{\theta}(\boldsymbol{\lambda})$, each gradient step requires solving the penalized regression problem on the training data.

2.3 Nonsmooth Training Criterion

When the training criterion in the penalized regression is not smooth, a common approach is to use slower but more general optimization methods. However, for many of these problems, the solution $\hat{\theta}(\boldsymbol{\lambda})$ is still smooth at almost every $\boldsymbol{\lambda}$ (eg. Lasso, Group Lasso, Trend Filtering). Thus we can calculate $\nabla_{\boldsymbol{\lambda}} L(\mathbf{y}_V, f_{\hat{\theta}(\boldsymbol{\lambda})}(\mathbf{X}_V))$ using (8) and apply gradient descent in practice. To provide a general approach to solving such problems, we discuss two primary difficulties: the characterization of problems for which $\hat{\theta}(\boldsymbol{\lambda})$ is almost everywhere smooth and the actual calculations for $\frac{\partial}{\partial \boldsymbol{\lambda}} \hat{\theta}(\boldsymbol{\lambda})$.

To characterize problems that are almost everywhere smooth, we begin with two definitions:

Definition 1. *The differentiable space of a real-valued function L and a point η in its domain is the set of vectors $\Omega^L(\eta)$ such that the directional derivative of L at η along $u \in \Omega^L(\eta)$ exists.*

$$\Omega^L(\eta) = \left\{ \delta \mid \lim_{\epsilon \rightarrow 0} \frac{L(\eta + \epsilon \delta) - L(\eta)}{\epsilon} \text{ exists} \right\} \quad (11)$$

Definition 2. S is a “local optimality space” for a real-valued function $L(\cdot, \lambda)$ at λ_0 if there exists a neighborhood W containing λ_0 such that for every $\lambda \in W$,

$$\arg \min_{\theta} L(\theta, \lambda) = \arg \min_{\theta \in S} L(\theta, \lambda) \quad (12)$$

does this definition work for local minimas?

Using these two definitions we can now give three conditions which together are sufficient for differentiability of $L(\mathbf{y}_V, f_{\hat{\theta}(\lambda)}(\mathbf{X}_V))$.

Condition 1. For almost every λ , $\Omega^{L_T(\cdot, \lambda)}(\hat{\theta}(\lambda))$ is a local optimality space for $L_T(\cdot, \lambda)$ at λ .

Condition 2. For almost every λ , $L_T(\cdot, \cdot)$ restricted to $\Omega^{L_T(\cdot, \lambda)}(\hat{\theta}(\lambda), \lambda)$ is twice continuously differentiable within some neighborhood of λ .

Condition 3. For almost every λ , the Hessian of $L_T(\cdot, \lambda)$ at $\hat{\theta}(\lambda)$ with respect to a basis of $\Omega^{L_T(\cdot, \lambda)}(\hat{\theta}(\lambda))$ is invertible.

Putting all of these together we get the following theorem which shows that the gradient exists almost everywhere; and gives a recipe for calculating it.

Theorem 1. Suppose our optimization problem is of the form in (5), with $L_T(\theta, \lambda)$ defined as in (6).

Suppose that $L(\mathbf{y}_V, f_{\theta}(\mathbf{X}_V))$ is continuously differentiable in θ , and conditions 1, 2, and 3, defined above, hold.

Then the validation loss $L(\mathbf{y}_V, f_{\hat{\theta}(\lambda)}(\mathbf{X}_V))$ is continuously differentiable almost everywhere with respect to λ . Furthermore, the gradient of $L(\mathbf{y}_V, f_{\hat{\theta}(\lambda)}(\mathbf{X}_V))$, where it is defined, is

$$\nabla_{\lambda} L(\mathbf{y}_V, f_{\hat{\theta}(\lambda)}(\mathbf{X}_V)) = \left[\frac{\partial}{\partial \theta} L(\mathbf{y}_V, f_{\theta}(\mathbf{X}_V)) \Big|_{\theta=\hat{\theta}(\lambda)} \right]^T \frac{\partial}{\partial \lambda} \tilde{\theta}(\lambda) \quad (13)$$

where

$$\tilde{\theta}(\lambda) = \arg \min_{\theta \in \Omega^{L_T(\cdot, \lambda)}(\hat{\theta}(\lambda))} L_T(\theta, \lambda) \quad (14)$$

We can therefore construct a gradient descent procedure based on the key idea in (14). At each iteration, we define a smooth, locally equivalent function $\tilde{\theta}(\lambda)$ and use $\frac{\partial}{\partial \lambda} \tilde{\theta}(\lambda)$ to calculate the gradient. The gradient descent procedure is given in more detail below.

Initialize $\lambda^{(0)}$.

For each iteration k until stopping criteria is reached:

Solve for $\hat{\theta}(\lambda^{(k)}) = \arg \min_{\theta \in \Theta} L_T(\theta, \lambda^{(k)})$.

Let $\tilde{\theta}(\lambda^{(k)})$ be the optimal model parameters restricted to the space $\Omega^{L_T(\cdot, \lambda)}(\hat{\theta}(\lambda^{(k)}))$.

$$\tilde{\theta}(\lambda^{(k)}) = \arg \min_{\theta \in \Omega^{L_T(\cdot, \lambda)}(\hat{\theta}(\lambda^{(k)}))} L_T(\theta, \lambda^{(k)}) \quad (15)$$

Calculate the gradient

$$\nabla_{\lambda} L(\mathbf{y}_V, f_{\hat{\theta}(\lambda)}(\mathbf{X}_V)) \Big|_{\lambda=\lambda^{(k)}} = \left[\frac{\partial}{\partial \theta} L(\mathbf{y}_V, f_{\theta}(\mathbf{X}_V)) \Big|_{\theta=\hat{\theta}(\lambda^{(k)})} \right]^T \frac{\partial}{\partial \lambda} \tilde{\theta}(\lambda) \Big|_{\lambda=\lambda^{(k)}} \quad (16)$$

Perform the gradient update

$$\lambda^{(k+1)} := \lambda^{(k)} - \alpha^{(k)} \nabla_{\lambda} L(\mathbf{y}_V, f_{\hat{\theta}(\lambda)}(\mathbf{X}_V)) \Big|_{\lambda=\lambda^{(k)}}$$

Algorithm 2

2.4 Joint optimization for example regressions

To better understand the proposed gradient descent procedure, we present example joint optimization problems and their corresponding gradient calculations. We start with ridge regression where the training criterion is smooth. Then we consider the elastic net and sparse group lasso, in which the training criterion is smooth almost everywhere. Finally, we discuss doing descent-based joint optimization for an additive partial linear model, an example of a semi-parametric regression with nonsmooth penalties.

2.4.1 Ridge Regression

In ridge regression, the training criterion is smooth so applying gradient descent is straightforward. The joint optimization problem for ridge regression is:

$$\begin{aligned} & \min_{\lambda \in \mathbb{R}} \frac{1}{2} \|\mathbf{y}_V - \mathbf{X}_V \hat{\boldsymbol{\theta}}(\lambda)\|_2^2 \\ & \text{where } \hat{\boldsymbol{\theta}}(\lambda) = \arg \min_{\boldsymbol{\theta}} \frac{1}{2} \|\mathbf{y}_T - \mathbf{X}_T \boldsymbol{\theta}\|_2^2 + \frac{1}{2} \lambda \|\boldsymbol{\theta}\|_2^2 \end{aligned} \quad (17)$$

The closed-form solution for $\hat{\boldsymbol{\theta}}(\lambda)$ is

$$\hat{\boldsymbol{\theta}}(\lambda) = (\mathbf{X}_T^T \mathbf{X}_T + \lambda \mathbf{I})^{-1} \mathbf{X}_T^T \mathbf{y}_T \quad (18)$$

The gradient of the validation loss can be easily derived by differentiating the above equation with respect to λ and then using the chain rule.

$$\nabla_{\lambda} L(\mathbf{y}_V, f_{\hat{\boldsymbol{\theta}}(\lambda)}(\mathbf{X}_V)) = (\mathbf{X}_V (\mathbf{X}_T^T \mathbf{X}_T + \lambda \mathbf{I})^{-1} \hat{\boldsymbol{\theta}}(\lambda))^T (\mathbf{y}_V - \mathbf{X}_V \hat{\boldsymbol{\theta}}(\lambda)) \quad (19)$$

2.4.2 Elastic Net

The elastic net, a linear combination of the lasso and ridge penalties, is an example of a regularization method that is not smooth. We are interested in finding the optimal regularization parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^T$, which is formulated as follows:

$$\begin{aligned} & \min_{\boldsymbol{\lambda} \in \mathbb{R}^2} \frac{1}{2} \|\mathbf{y}_V - \mathbf{X}_V \hat{\boldsymbol{\theta}}(\boldsymbol{\lambda})\|_2^2 \\ & \text{where } \hat{\boldsymbol{\theta}}(\boldsymbol{\lambda}) = \arg \min_{\boldsymbol{\theta}} \frac{1}{2} \|\mathbf{y}_T - \mathbf{X}_T \boldsymbol{\theta}\|_2^2 + \lambda_1 \|\boldsymbol{\theta}\|_1 + \frac{1}{2} \lambda_2 \|\boldsymbol{\theta}\|_2^2 \end{aligned} \quad (20)$$

This joint optimization problem satisfies all three conditions in theorem 1. For the first condition, note that $\Omega^{L_T(\cdot, \boldsymbol{\lambda})}(\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda}))$ is the space spanned by the nonzero elements of $\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda})$. Furthermore, for the elastic net penalty, it is known that the nonzero indices of the fitted regression coefficients stay locally constant for almost every $\boldsymbol{\lambda}$. Therefore, $\Omega^{L_T(\cdot, \boldsymbol{\lambda})}(\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda}))$ is a local optimality space. The second condition is also satisfied since the ℓ_1 penalty is smooth when restricted to $\Omega^{L_T(\cdot, \boldsymbol{\lambda})}(\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda}))$. Finally, to check the third condition, let $\tilde{\mathbf{X}}(\boldsymbol{\lambda})$ be the rows of \mathbf{X} projected onto $\Omega^{L_T(\cdot, \boldsymbol{\lambda})}(\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda}))$. The Hessian of $L_T(\cdot, \boldsymbol{\lambda})$ with respect to the standard basis of $\Omega^{L_T(\cdot, \boldsymbol{\lambda})}(\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda}))$ is $\tilde{\mathbf{X}}_T(\boldsymbol{\lambda})^T \tilde{\mathbf{X}}_T(\boldsymbol{\lambda}) + \lambda_2 \mathbf{I}$ is always invertible.

To calculate the gradient, we consider the minimization problem restricted to $\Omega^{L_T(\cdot, \boldsymbol{\lambda})}(\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda}))$, the space spanned by the non-zero elements of $\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda})$:

$$\tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda}) = \arg \min_{\{\boldsymbol{\theta}_i | \hat{\theta}_i(\boldsymbol{\lambda}) \neq 0\}} \|\mathbf{y}_T - \mathbf{X}_T \boldsymbol{\theta}\|_2^2 + \lambda_1 \|\boldsymbol{\theta}\|_1 + \lambda_2 \|\boldsymbol{\theta}\|_2^2 \quad (21)$$

From (10), we get that

$$\frac{\partial}{\partial \lambda_1} \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda}) = (\tilde{\mathbf{X}}_T(\boldsymbol{\lambda})^T \tilde{\mathbf{X}}_T(\boldsymbol{\lambda}) + \lambda_2 \mathbf{I})^{-1} \text{sgn}(\tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda})) \quad (22)$$

$$\frac{\partial}{\partial \lambda_2} \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda}) = (\tilde{\mathbf{X}}_T(\boldsymbol{\lambda})^T \tilde{\mathbf{X}}_T(\boldsymbol{\lambda}) + \lambda_2 \mathbf{I})^{-1} \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda}) \quad (23)$$

Hence, the gradient descent direction at $\boldsymbol{\lambda}$ is

$$\nabla_{\boldsymbol{\lambda}} L(\mathbf{y}_V, f_{\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda})}(\mathbf{X}_V)) = \begin{bmatrix} \text{sgn}(\tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda})) \\ \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda}) \end{bmatrix}^T (\tilde{\mathbf{X}}_V(\boldsymbol{\lambda})^T (\tilde{\mathbf{X}}_T(\boldsymbol{\lambda})^T \tilde{\mathbf{X}}_T(\boldsymbol{\lambda}) + \lambda_2 \mathbf{I})^{-1})^T (\mathbf{y}_V - \tilde{\mathbf{X}}_V(\boldsymbol{\lambda}) \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda})) \quad (24)$$

2.4.3 Sparse Group Lasso

The sparse group lasso combines the $\|\cdot\|_2$ and $\|\cdot\|_1$ penalties, both of which are not smooth. This regularization method is particularly well-suited to modeling problems in which the grouped covariates are thought to have a sparse effect on a group level and within groups.

The problem setup is as follows. Given M covariate groups, let \mathbf{X} and $\boldsymbol{\theta}$ be split into $\mathbf{X}^{(m)}$ and $\boldsymbol{\theta}^{(m)}$ for groups $m = 1, \dots, M$. We are interested in finding the optimal regularization parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^T$. The joint optimization problem is formulated as follows.

$$\min_{\boldsymbol{\lambda} \in \mathbb{R}^2} \frac{1}{2n} \|\mathbf{y}_V - \mathbf{X}_V \hat{\boldsymbol{\theta}}(\boldsymbol{\lambda})\|_2^2 \quad (25)$$

where $\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda}) = \arg \min_{\boldsymbol{\theta}} \frac{1}{2n} \|\mathbf{y}_T - \mathbf{X}_T \boldsymbol{\theta}\|_2^2 + \lambda_1 \sum_{m=1}^M \|\boldsymbol{\theta}^{(m)}\|_2 + \lambda_2 \|\boldsymbol{\theta}\|_1$

First, we note that this problem satisfies the three conditions given in theorem 1. The reasoning is very similar to that in the elastic net example so we do not repeat them here. Since the points of nondifferentiability for $\|\cdot\|_2$ and $\|\cdot\|_1$ are the same, $\Omega^{L_T(\cdot, \boldsymbol{\lambda})}(\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda}))$ is the space spanned by the nonzero elements of $\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda})$.

To calculate the gradient, we solve for $\tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda})$ which allows us to define the locally equivalent function $\tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda})$:

$$\tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda}) = \arg \min_{\{\boldsymbol{\theta}_i | \tilde{\boldsymbol{\theta}}_i(\boldsymbol{\lambda}) \neq 0\}} \frac{1}{2n} \|\mathbf{y}_T - \mathbf{X}_T \boldsymbol{\theta}\|_2^2 + \lambda_1 \sum_{m=1}^M \|\boldsymbol{\theta}^{(m)}\|_2 + \lambda_2 \|\boldsymbol{\theta}\|_1 \quad (26)$$

From (9), (10), and the chain rule, we get that the gradient of the validation loss is

$$\nabla_{\boldsymbol{\lambda}} L(\mathbf{y}_V, f_{\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda})}(\mathbf{X}_V)) = \frac{1}{n} \begin{bmatrix} \frac{\tilde{\boldsymbol{\theta}}^{(1)}}{\|\tilde{\boldsymbol{\theta}}^{(1)}\|_2} \\ \vdots \\ \frac{\tilde{\boldsymbol{\theta}}^{(M)}}{\|\tilde{\boldsymbol{\theta}}^{(M)}\|_2} \\ \text{sgn}(\tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda})) \end{bmatrix}^T \left(\tilde{\mathbf{X}}_V(\boldsymbol{\lambda}) \left(\frac{1}{n} \tilde{\mathbf{X}}_T(\boldsymbol{\lambda})^T \tilde{\mathbf{X}}_T(\boldsymbol{\lambda}) - \lambda_1 \mathbf{M}_1 + \lambda_1 \mathbf{M}_2 \right)^{-1} \right)^T (\mathbf{y}_V - \tilde{\mathbf{X}}_V(\boldsymbol{\lambda}) \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda})) \quad (27)$$

where \mathbf{M}_1 is the block diagonal matrix with components $\|\tilde{\boldsymbol{\theta}}^{(m)}(\boldsymbol{\lambda})\|_2^{-3} \tilde{\boldsymbol{\theta}}^{(m)}(\boldsymbol{\lambda}) \tilde{\boldsymbol{\theta}}^{(m)}(\boldsymbol{\lambda})^T$ and \mathbf{M}_2 is the block diagonal matrix with components $\|\tilde{\boldsymbol{\theta}}^{(k)}(\boldsymbol{\lambda})\|_2^{-1} \mathbf{I}$ for $m = 1, \dots, M$ from top left to bottom right.

2.4.4 Additive Partial Linear Models

Finally, we present an example of a penalized, semi-parametric regression: an additive partial linear model (APLM) combined with the lasso and trend-filtering penalties. This regression method models the response as the sum of linear and nonlinear components and selects model fits that have sparse linear effects and smooth nonparametric estimates.

The setup for this semi-parametric problem is slightly different. Let $\mathbf{X} \in \mathbb{R}^{n \times p}$ be the design matrix with p linear predictors, $\mathbf{z} \in \mathbb{R}^n$ be the design matrix with the nonlinear predictor, and $\mathbf{y} \in \mathbb{R}^n$ be the response variable. We assume that the observations are ordered by increasing z_i . Let \mathbf{M} a diagonal matrix with elements randomly assigned to 1 or 0. \mathbf{M} partitions the data into the training set $\mathbf{X}_T = \mathbf{M}\mathbf{X}$ and the validation set $\mathbf{X}_V = (\mathbf{I} - \mathbf{M})\mathbf{X}$.

The response y_i in this APLM is modeled as the sum of a linear function of x_i and a nonlinear function of z_i

$$y_i = \mathbf{x}_i^T \boldsymbol{\beta} + g(z_i) + \epsilon \quad (28)$$

where $\boldsymbol{\beta}$ is an unknown vector and g is an unknown nonlinear function. We want to solve for the optimal $\boldsymbol{\beta}$ and the trend-filtering estimates $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ for $(g(z_1), \dots, g(z_n))^T$. Since these values depend on the regularization parameters, we are ultimately interested in finding the optimal $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^T$. The joint optimization problem is as follows:

$$\min_{\boldsymbol{\lambda} \in \mathbb{R}^2} \frac{1}{2} \|\mathbf{y}_V - \mathbf{X}_V \hat{\boldsymbol{\beta}}(\boldsymbol{\lambda}) - (\mathbf{I} - \mathbf{M}) \hat{\boldsymbol{\theta}}(\boldsymbol{\lambda})\|_2^2 \quad (29)$$

where $\hat{\boldsymbol{\beta}}(\boldsymbol{\lambda}), \hat{\boldsymbol{\theta}}(\boldsymbol{\lambda}) = \arg \min_{\boldsymbol{\beta}, \boldsymbol{\theta}} \frac{1}{2} \|\mathbf{y}_T - \mathbf{X}_T \boldsymbol{\beta} - \mathbf{M} \boldsymbol{\theta}\|_2^2 + \lambda_1 \|\boldsymbol{\beta}\|_1 + \frac{1}{2} \lambda_2 \|\mathbf{D}(\mathbf{z}) \boldsymbol{\theta}\|_2^2$

The third component in the training criterion is the trend-filtering penalty $\|\mathbf{D}(\mathbf{z})\boldsymbol{\theta}\|_2^2$. We use the second-order $\|\cdot\|_2^2$ trend-filtering penalty in this example, which penalizes the norm of the second-order discrete derivatives over the input points. The second-order difference matrix $\mathbf{D}(\mathbf{z})$ is defined as

$$\mathbf{D}(\mathbf{z}) = \mathbf{D}^{(1)} \cdot \text{diag}\left(\frac{1}{z_2 - z_1}, \frac{1}{z_3 - z_2}, \dots, \frac{1}{z_n - z_{n-1}}, 0\right) \cdot \mathbf{D}^{(1)} \quad (30)$$

where

$$\mathbf{D}^{(1)} = \begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n} \quad (31)$$

We can solve this joint optimization problem using gradient descent since the conditions in theorem 1 are satisfied. The reasoning is same as before, but the differentiable space is slightly different. Let training criterion be denoted as $L_T(\boldsymbol{\beta}(\boldsymbol{\lambda}), \boldsymbol{\theta}(\boldsymbol{\lambda}), \boldsymbol{\lambda})$. Then $\Omega^{L_T(\cdot, \cdot, \boldsymbol{\lambda})}(\hat{\boldsymbol{\beta}}(\boldsymbol{\lambda}), \hat{\boldsymbol{\theta}}(\boldsymbol{\lambda}))$ is the space spanned by the nonzero elements of $\hat{\boldsymbol{\beta}}(\boldsymbol{\lambda})$ and all dimensions $\hat{\boldsymbol{\theta}}(\boldsymbol{\lambda})$.

To calculate the gradient, we solve for the $\tilde{\boldsymbol{\beta}}(\boldsymbol{\lambda})$, and define $\tilde{\boldsymbol{\beta}}(\boldsymbol{\lambda}), \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda})$ as

$$\tilde{\boldsymbol{\beta}}(\boldsymbol{\lambda}), \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda}) = \arg \min_{\{\boldsymbol{\beta}_i | \hat{\boldsymbol{\beta}}_i(\boldsymbol{\lambda}) \neq 0\}, \boldsymbol{\theta}} \frac{1}{2} \|\mathbf{y}_T - \mathbf{X}_T \boldsymbol{\beta} - \mathbf{M} \boldsymbol{\theta}\|_2^2 + \lambda_1 \|\boldsymbol{\beta}\|_1 + \frac{1}{2} \lambda_2 \|\mathbf{D}(\mathbf{z}) \boldsymbol{\theta}\|_2^2 \quad (32)$$

Using the same procedure, we see that the gradient of the validation loss is

$$\nabla_{\boldsymbol{\lambda}} L_V(\boldsymbol{\lambda})_i = - \left(\tilde{\mathbf{X}}_V(\boldsymbol{\lambda}) \frac{\partial \tilde{\boldsymbol{\beta}}(\boldsymbol{\lambda})}{\partial \lambda_i} + (\mathbf{I} - \mathbf{M}) \frac{\partial \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda})}{\partial \lambda_i} \right)^T \left(\mathbf{y}_V - \tilde{\mathbf{X}}_V(\boldsymbol{\lambda}) \tilde{\boldsymbol{\beta}}(\boldsymbol{\lambda}) - (\mathbf{I} - \mathbf{M}) \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda}) \right) \text{ for } i = 1, 2$$

where the partial derivatives are

$$\begin{aligned} \frac{\partial}{\partial \lambda_1} \tilde{\boldsymbol{\beta}}(\boldsymbol{\lambda}) &= - \left(\tilde{\mathbf{X}}_T(\boldsymbol{\lambda})^T \tilde{\mathbf{X}}_T(\boldsymbol{\lambda}) - \tilde{\mathbf{X}}_T(\boldsymbol{\lambda})^T \mathbf{M} (\mathbf{M}^T \mathbf{M} + \lambda_2 \mathbf{D}^T \mathbf{D})^{-1} \mathbf{M}^T \tilde{\mathbf{X}}_T(\boldsymbol{\lambda}) \right)^{-1} \text{sgn}(\tilde{\boldsymbol{\beta}}(\boldsymbol{\lambda})) \\ \frac{\partial}{\partial \lambda_1} \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda}) &= - \left(\mathbf{M}^T \mathbf{M} + \lambda_2 \mathbf{D}^T \mathbf{D} \right)^{-1} \mathbf{M}^T \tilde{\mathbf{X}}_T(\boldsymbol{\lambda}) \frac{\partial}{\partial \lambda_1} \tilde{\boldsymbol{\beta}}(\boldsymbol{\lambda}) \\ \frac{\partial}{\partial \lambda_2} \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda}) &= - \left(\mathbf{M}^T \mathbf{M} - \mathbf{M}^T \tilde{\mathbf{X}}_T(\boldsymbol{\lambda}) (\tilde{\mathbf{X}}_T(\boldsymbol{\lambda})^T \tilde{\mathbf{X}}_T(\boldsymbol{\lambda}))^{-1} \tilde{\mathbf{X}}_T(\boldsymbol{\lambda})^T \mathbf{M} + \lambda_2 \mathbf{D}^T \mathbf{D} \right) \mathbf{D}^T \mathbf{D} \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda}) \\ \frac{\partial}{\partial \lambda_2} \tilde{\boldsymbol{\beta}}(\boldsymbol{\lambda}) &= - \left(\tilde{\mathbf{X}}_T(\boldsymbol{\lambda})^T \tilde{\mathbf{X}}_T(\boldsymbol{\lambda}) \right)^{-1} \tilde{\mathbf{X}}_T(\boldsymbol{\lambda})^T \mathbf{M} \frac{\partial}{\partial \lambda_2} \tilde{\boldsymbol{\theta}}(\boldsymbol{\lambda}) \end{aligned}$$

3 Results: validation error minimization

The purpose of this simulation study is to compare the performance and efficiency of descent-based and grid-based joint optimization in minimizing the validation loss for different regularization methods. Experiments were run for the example regressions elastic net, sparse group lasso, and APLMs. For each experiment, we implemented three methods to tune the regularization parameters: cross validation over a grid of parameter values, gradient descent with a constant step size, and Nesterov's gradient descent with adaptive restarts. Below, we describe the simulation settings, followed by a discussion of the results.

3.1 Elastic net

We generated 30 datasets with 80 training and 20 validation observations each and 250 predictors. The predictors were randomly generated such that the pairwise correlation between predictors x_i and x_j was $\text{corr}(i, j) = 0.5^{|i-j|}$. The response vector y was constructed as

$$y = X\beta + \sigma\epsilon \quad (33)$$

where the first 15 values of the coefficient vector β are ones and the rest are zeroes and $\epsilon \sim N(0, 1)$. σ was chosen such that the signal to noise ratio is 2.

Both gradient descents were initialized at $(0.01, 0.01)$ and $(10, 10)$. The grid for cross validation was 10×10 , starting from $1e-5$ to four times the largest eigenvalue of $X_T^T X_T$.

3.2 Sparse group lasso

We generated 30 datasets of 60 training and 15 validation observations with 1500 covariates. The predictors X were generated iid gaussian. The response y was constructed as

$$y = \sum_{l=1}^3 X^{(k)} \beta^{(k)} + \sigma \epsilon \quad (34)$$

where $\epsilon \sim N(0, 1)$, $\beta^{(k)} = (1, 2, 3, 4, 5, 0, \dots, 0)$ for $k = 1, 2, 3$. σ was chosen such that the signal to noise ratio was 2. The number of groups m in the criterion was set to 150.

Both gradient descents were initialized at three points, where all regularization parameters were set to 0.01, 1, and 100. We used a 10×10 grid for grid-based cross validation, starting from $1e-5$ to $\max(\{\|X^{(k)T} y\|_2\}_{l=1, \dots, m})$.

3.3 Additive partial linear models

We generated 30 datasets with 100 training and 25 validation observations and 21 predictors. The first 20 predictors were the inputs to the linear function and the last predictor was the input to the nonlinear function. Predictors for the linear function were generated such that the first two groups of three features were highly correlated and the rest of the features were generated iid Gaussian.

$$\begin{aligned} x_i &= Z_1 + \epsilon_i \text{ for } i = 1, 2, 3 \\ x_i &= Z_2 + \epsilon_i \text{ for } i = 4, 5, 6 \\ x_i &\sim N(0, 1) \text{ for } i = 7, \dots, 20 \end{aligned} \quad (35)$$

where $Z_1 \sim N(0, 1)$, $Z_2 \sim N(0, 1)$, and $\epsilon_i \sim N(0, \frac{1}{16})$. The predictors for the nonlinear component z were randomly drawn from a uniform distribution from 0 to 1. The response y was constructed as

$$y = X\beta + \kappa g(z) + \sigma \epsilon \quad (36)$$

The constants κ and σ were chosen such that the linear to nonlinear ratio $\frac{\|X\beta\|_2}{\|g(Z)\|_2}$ was 2 and the signal to noise ratio was 2, respectively. We set $\beta = (1, 1, 1, 1, 1, 1, 0, \dots, 0)$ and $g(z) = (2 - z) \sin(20z^4)$.

Both gradient descents were initialized at the four points $\lambda_1 = \lambda_2 = 10^i$ for $i = -2, -1, 0, 1$. We used a 10×10 grid for grid-based cross validation, with values ranging from $1e - 6$ to 10.

3.4 Discussion of results

As shown in Table 1, descent-based and grid-based joint optimization had the same performance in all three situations. The only major difference between the different implementations of joint optimization is in the runtimes. In every situation, gradient descent was over 40% faster than the grid-based approach and Nesterov's subgradient descent was over 50% faster.

Figure 2 shows example plots of how validation error decreases in the descent-based joint optimization procedures. The horizontal line in the plots are the lowest validation error achieved by grid-based joint optimization. In all three plots, Nesterov's gradient descent requires fewer iterations to minimize the validation loss.

Elastic Net		
	Validation Error	Runtime (sec)
Grid search	0.34 (0.003)	10.74
Gradient Descent	0.34 (0.003)	4.43
Nesterov's Gradient Descent	0.34 (0.003)	2.28
Sparse Group Lasso		
	Validation Error	Runtime (sec)
Grid search	1.36 (0.09)	161.29
Gradient Descent	1.36 (0.09)	71.34
Nesterov's Gradient Descent	1.36 (0.10)	67.10
APLM		
	Validation Error	Runtime (sec)
Grid search	1.31 (0.05)	27.82
Gradient Descent	1.31 (0.05)	16.04
Nesterov's Gradient Descent	1.31 (0.05)	12.09

Table 1. Validation Error comparisons

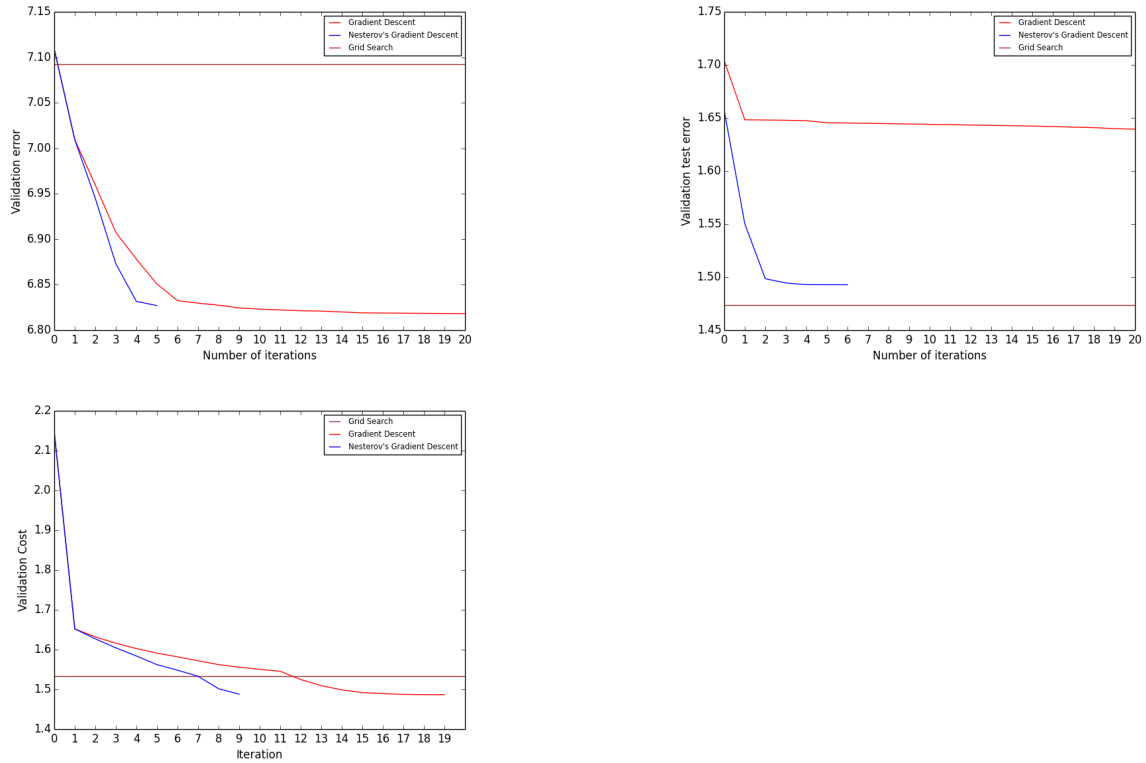


Figure 2. Validation error progression: Top Left - Elastic Net, Top Right - Grouped Lasso, Bottom Left - APLM

4 Results: regularizations with more than two penalties

Our second simulation study is to show that descent-based joint optimization can effectively and efficiently solve regressions with more than two regularization parameters.

For each experiment, given a regularization method, we compared tuning the original two-parameter version using a grid-based approach to tuning the generalized, many-parameter version using gradient descent. In the sparse group lasso experiment, we tested an “unpooled” version of sparse group lasso in which the number of regularization parameters is linear in the number of covariate groups. In the APLM experiment, we tested having an additional ridge penalty for the linear regression coefficients. To compare the regularization methods, we measured the loss on the test sets. As shown later, since gradient descent for the generalized regression achieved a lower test set loss on average, we have shown that a descent-based approach for many-penalty regressions can produce better models than those from 2-penalty regressions.

The joint optimization problem formulations and gradient derivations for these generalized regularizations are given in the Appendix.

4.1 Unpooled sparse group lasso

In this experiment, we compared the effectiveness of applying gradient descent on an “unpooled” variant of sparse group lasso to grid-based joint optimization on the regular sparse group lasso. In unpooled sparse group lasso, the group lasso penalty parameter λ_1 is replaced with an individual parameter for each covariate group:

$$\sum_{l=1}^m \lambda_1^{(k)} \|\beta^{(k)}\|_2 + \lambda_2 \|\beta\|_1 \quad (37)$$

Hence, the number of regularization parameters increased from two to one plus the number of coefficient groups. This unpooled variant allows different thresholds to be used for setting coefficients and coefficient groups to zero, as opposed to the original sparse group lasso where the thresholds are shared. Potentially, this increased flexibility can produce more accurate models, especially when the covariate groups have very different distributions.

We ran three experiments, each with a different number of covariate groups m and total covariates p . The simulation settings were similar to the first simulation study. Grid-based joint optimization for sparse group lasso was initialized with the same grid. Gradient descent for unpooled sparse group lasso was initialized with a lower set of values 1e-4, 1e-3, and 1e-2 to compensate for the fact that there are now $m + 1$ regularization parameters.

We measured three metrics to assess model performance: test error, β error, which is defined as $\|\beta - \hat{\beta}\|_2^2$, and the percentage of nonzero coefficients correctly identified among all the true nonzero coefficients. As shown in Table 2, unpooled sparse group lasso performed better according to all three metrics in all the experiments. Furthermore, gradient descent for unpooled sparse group lasso is faster than grid-based joint optimization for two-parameter sparse group lasso, even in the case with 151 regularization parameters.

4.2 Additive partial linear models with three regularization parameters

In this example of joint optimization for APLMs, we experimented with a generalized version of the APLM regularization by using the elastic net instead of the lasso for the linear regression coefficients:

$$\lambda_1 \|\beta\|_1 + \frac{1}{2} \lambda_2 \|\beta\|_2^2 + \frac{1}{2} \lambda_3 \|D^{(2)}\theta\|_2^2 \quad (38)$$

We compared model fits from this three-parameter regularization tuned using gradient descent to those from the original two-parameter regularization tuned using a grid-based approach. Our hypothesis was

n=60, p=300, g=3, m=30				
	β Error	% correct nonzero β	Test Error	Runtime (sec)
Gridsearch (baseline)	1.13	10.70	0.04	15.81
Gradient Descent	0.18	23.79	0.01	5.62
n=60, p=1500, g=3, m=50				
	β Error	% correct nonzero β	Test Error	Runtime (sec)
Gridsearch (baseline)	7.79	9.63	0.28	148.64
Gradient Descent	4.00	17.79	0.14	88.78
n=60, p=1500, g=3, m=150				
	β Error	% correct nonzero β	Test Error	Runtime (sec)
Gridsearch (baseline)	2.20	10.69	0.080	162.14
Gradient Descent	0.06	15.34	0.002	48.63

Table 2. Unpooled sparse group lasso

$g(z) = 4z^3 - z^2 + 2z$				
	β Error	θ error	Test Error	Runtime (sec)
Gridsearch	0.59	3.35	3.78	35.48
Gradient Descent	0.38	2.96	3.73	43.44
$g(z) = \sin(5z) + \sin(15(z-3))$				
	β Error	θ error	Test Error	Runtime (sec)
Gridsearch	0.51	3.76	3.90	37.04
Gradient Descent	0.34	3.73	3.79	45.95
$g(z) = (2-z)\sin(20z^4)$				
	β Error	θ error	Test Error	Runtime (sec)
Gridsearch	0.58	4.91	4.13	40.75
Gradient Descent	0.41	4.85	4.08	54.63

Table 3. Additive Partial Linear Model

that the elastic net penalty in this generalized regularization would produce more accurate models since it encourages both sparsity of representation and grouping among strongly correlated predictors.

We ran experiments on three nonlinear functions of varying levels of smoothness. The simulation settings were similar to before. The only difference was that gradient descent was initialized at the six points $\lambda_1 = \lambda_2 = \lambda_3 = 10^i$ for $i = -4, \dots, 1$.

Models were judged by their predictive accuracy. In addition, we used β error, defined as $\|\beta - \hat{\beta}\|_2^2$, and θ error, defined as $\|g(z) - \theta\|_2^2$, to measure the error in the estimated linear and nonlinear components, respectively.

The results in Table 3 show that using three penalty terms in the criterion improved the models by all metrics. The biggest improvement was in the estimation of the linear regression components, which is expected since we added an penalty term for the linear coefficients. Somewhat surprisingly, the estimation of the nonlinear components also improved slightly, even though no penalty term was added for the nonlinear component. The runtime for tuning the three-parameter regularization was slightly longer than the grid-based version, but remained reasonable nonetheless. Most importantly, the increase in runtime is not exponential in the number of penalty parameters.

Figure 3 shows example model fits from gradient descent and grid-based joint optimization. The plots confirm the hypothesis that the highly-correlated covariates are estimated better using the three-parameter regularization as opposed to the two-parameter one.

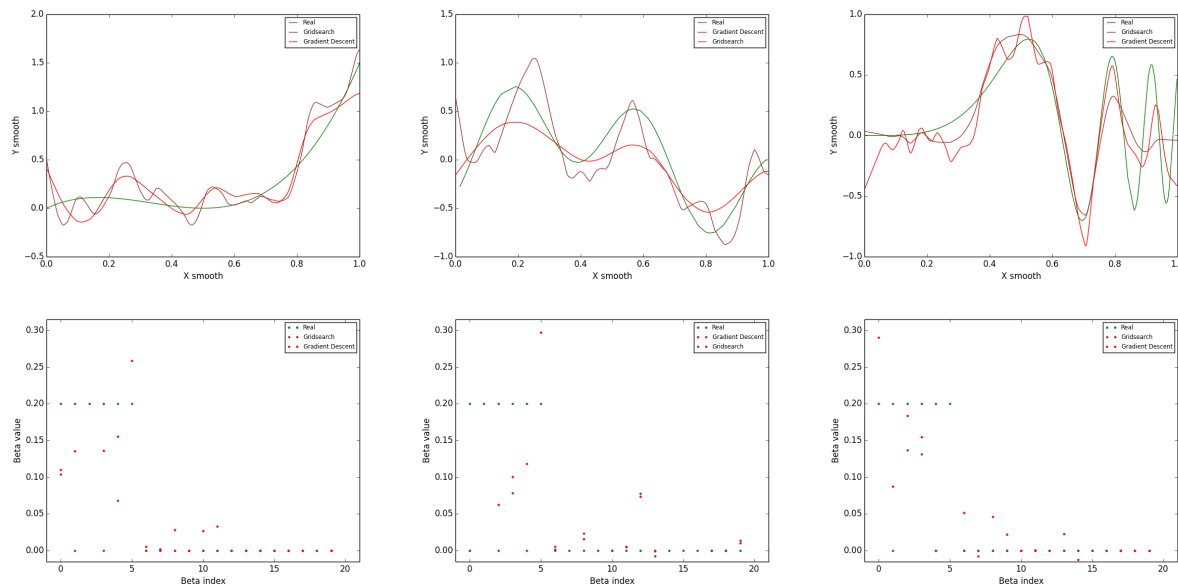


Figure 3. Top: Example fits to the three smooth functions used in the experiments. The green line is the true function. Bottom: Example fits for the linear regression coefficients. The green is the true beta.

5 Application to Real Data

In this section, we discuss the performance and efficiency of joint optimization when applied to real data. We use un-pooled sparse group lasso as our example, since it can have the most number of penalty parameters among the regularization methods we previously mentioned. We compare this to sparse group lasso tuned using grid search.

We run regression to predict disease phenotype from gene expression data, as presented by Noah et. al. (2007). In this problem, the genes are grouped by gene pathways and we are interested in the pathways of interest and, from them, the driving genes.

Our dataset is from a colitis study of 127 total patients, 85 with colitis (59 crohn’s patients + 26 ulcerative colitis patients) and 42 health controls (Burczynski 2006). Expression data was measured for 22,283 genes on affymetrix U133A microarrays. We grouped the genes according to the 326 C1 positional gene sets from MSigDb v5.0 (Subramanian 2005) and discarded the 2358 genes not found in the gene set. For each experiment, we randomly shuffled the data and used the first 50 observations for the training set and the remaining 77 for the test set. The experiment was repeated 10 times.

5-fold cross-validation was used to fit models. The penalty parameters in un-pooled sparse group lasso were all initialized at 0.5. For sparse group lasso, models were fit over a 5×5 grid of parameter values from $1e-4$ to 5.

Perhaps the most important result is that joint optimization makes it possible to compare un-pooled sparse group lasso to sparse group lasso. Un-pooled sparse group lasso requires 327 regularization parameters for this problem, which is impossible to tune using grid search. Treating the problem as a continuous optimization problem not only makes it possible to tune many penalty parameters, but also makes the runtime comparable to grid search over two parameters, as seen in Table 4.

Un-pooled sparse group lasso and sparse group lasso achieve similar classification rates around 83%. The primary difference is that un-pooled sparse group lasso finds solutions that are significantly more sparse than sparse group lasso – on average, 9 genesets were identified, as opposed to 38. In addition, the number of genesets identified by un-pooled sparse group lasso has significantly lower variance. The number of genesets identified by sparse group lasso ranged from 2 to 73. The number of genesets identified by un-pooled sparse group lasso ranged from 8 to 12. Consistency is certainly an useful quality in regularization methods. These results suggest that un-pooling the penalty parameters in sparse group

	% correct	Num. Genesets	Num. Genes	Runtime (sec)
SGL	82.47 (0.7)	38.4 (671.2)	207.0 (22206.2)	2722.4
Unpooled SGL	84.29 (0.3)	8.9 (1.9)	83.9 (664.5)	2298.5

Table 4. Ulcerative Colitis Data: SGL = sparse group lasso, variance in parentheses

lasso can improve the regularization method. Further research is needed to determine if un-pooled sparse group lasso is a useful regularization method.

In this example, we have demonstrated that joint optimization is highly efficient at tuning parameters and can produce meaningful solutions to regression problems for real data.

6 Discussion

We proposed using joint optimization for tuning regularization parameters in both linear and nonlinear regressions. By treating the regression as an optimization problem over the regularization parameter space, joint optimization is highly scalable in the number of parameters. Through experiments on simulated data, we demonstrated that joint optimization is able to fit models with validation errors as low as grid search. We then increased the number of regularization parameters in traditional regularization methods and used joint optimization to fit even better models. During this exploration, we discovered that "unpooled" sparse group lasso has significantly better performance than the traditional sparse group lasso. Further research should be done to explore this new regularization method. Finally, we demonstrated that this method works well on real data.

This paper applied joint optimization to a number of criterions, but this method should be tested on more criterions to analyze its robustness. We also expect that by implementing joint optimization with more sophisticated optimization methods, we can expect significant gains in efficiency. Finally, it would be worthwhile to understand what types of situations are ill-suited for joint optimization and what is the maximum number of regularization parameters joint optimization can tune with reasonable computation time.

Appendix

6.1 Proof for Theorem 1

Proof. It is sufficient to show that the theorem holds for a given λ_0 . For a given λ_0 , let B' be the basis vectors that span the differentiable space $\Omega^{L_T}(\hat{\theta}(\lambda_0), \lambda_0)$. Let B be the subset of the basis vectors that span the model parameter space.

Let $\tilde{L}_T(\theta, \lambda)$ be the gradient of $L_T(\cdot, \lambda)$ at θ with respect to the basis B :

$$\tilde{L}(\theta, \lambda) = \nabla_B L_T(\cdot, \lambda)|_{\theta} \quad (39)$$

Since $\hat{\theta}(\lambda_0)$ is the minimizer of the training loss, the gradient of $L_T(\cdot, \lambda_0)$ with respect to the basis B must be zero at $\hat{\theta}(\lambda_0)$:

$$\nabla_B L_T(\cdot, \lambda_0)|_{\hat{\theta}(\lambda_0)} = \tilde{L}_T(\hat{\theta}(\lambda_0), \lambda_0) = 0 \quad (40)$$

From our assumptions, we know that there exists a neighborhood W such that \tilde{L}_T is continuously differentiable along directions in the differentiable space $\Omega^{L_T}(\hat{\theta}(\lambda_0), \lambda_0)$. Also, the Jacobian matrix $D\tilde{L}_T(\cdot, \lambda_0)|_{\hat{\theta}(\lambda_0)}$ with respect to basis B is nonsingular. Therefore, by the implicit function theorem, there exist open sets U containing λ_0 and V containing $\hat{\theta}(\lambda_0)$ and a continuously differentiable function $\gamma : U \rightarrow V$ such that for every $\lambda \in U$, we have that

$$\tilde{L}_T(\gamma(\lambda), \lambda) = \nabla_B L_T(\cdot, \lambda)|_{\gamma(\lambda)} = 0 \quad (41)$$

That is, we know that $\gamma(\lambda)$ is a continuously differentiable function that minimizes $L_T(\cdot, \lambda)$ in the differentiable space $\Omega^{L_T}(\hat{\theta}(\lambda_0), \lambda_0)$. Since we assumed that the differentiable space is a local optimality space of $L_T(\cdot, \lambda)$ in the neighborhood W , then there must exist some neighborhood W' containing λ_0 such that for every $\lambda \in W'$,

$$\hat{\theta}(\lambda) = \arg \min_{\theta} L_T(\theta, \lambda) = \arg \min_{\theta \in \Omega^{L_T}(\hat{\theta}(\lambda_0), \lambda_0)} L_T(\theta, \lambda) = \gamma(\lambda) \quad (42)$$

Therefore, we have shown that if λ_0 satisfies the assumptions given in the theorem, the fitted model parameters $\hat{\theta}(\lambda)$ is a continuously differentiable function within a neighborhood of λ_0 . If the assumptions in the theorem hold true for almost every set of regularization parameters, then $\hat{\theta}(\lambda)$ is continuously differentiable almost everywhere. Furthermore, the Jacobian matrix of the model parameters, where it is defined, is

$$\frac{\partial}{\partial \lambda} \hat{\theta}(\lambda) = \frac{\partial}{\partial \lambda} \dot{\theta}(\lambda) \quad (43)$$

where

$$\dot{\theta}(\lambda) = \arg \min_{\theta \in \Omega^{L_T}(\hat{\theta}(\lambda_0), \lambda_0)} L_T(\theta, \lambda) \quad (44)$$

We can then apply the chain rule to get the gradient of the validation loss. \square

6.2 Elastic Net Gradient Derivations

MATHS

6.3 Sparse Group Lasso Derivations

6.3.1 Unpooled Sparse Group Lasso Derivations

The gradient of the validation error with respect to the regularization parameters is:

$$\begin{aligned} \nabla_{\lambda_1^{(k)}} L(\mathbf{y}_V, X_V \hat{\beta}) &= \frac{1}{n} \left(X_V \left(\frac{1}{n} X^T X + M_1 + M_2 \right)^{-1} \begin{bmatrix} 0 \\ \vdots \\ \frac{\hat{\beta}^{(k)}}{\|\hat{\beta}^{(k)}\|_2} \\ \vdots \\ 0 \end{bmatrix} \right)^T (\mathbf{y}_V - X_V \hat{\beta}) \\ \nabla_{\lambda_2} L(\mathbf{y}_V, X_V \hat{\beta}) &= \frac{1}{n} \left(\left(\frac{1}{n} X^T X + M_1 + M_2 \right)^{-1} \text{sign}(\hat{\beta}) \right)^T (\mathbf{y}_V - X_V \hat{\beta}) \end{aligned} \quad (45)$$

where M_1 is the block diagonal matrix with l components

$$\frac{\lambda_1^{(k)}}{\|\beta^{(k)}\|_2^3} \begin{bmatrix} \beta_1^{(k)} & 0 & & \\ 0 & \beta_2^{(k)} & & \\ & & \ddots & \\ & & & \beta_l^{(k)} \end{bmatrix} \begin{bmatrix} - & \beta^{(k)T} & - \\ - & \beta^{(k)T} & - \\ \vdots & & \end{bmatrix} \text{ from top left to bottom right and } M_2 \text{ is the diagonal matrix}$$

with $\frac{\lambda_1^{(k)}}{\|\beta^{(k)}\|_2}$ from top left to bottom right.

6.4 Additive Partial Linear Model Derivations

6.4.1 Additive Partial Linear Model Derivations: 3-penalties

To perform joint optimization, we formulate the problem as follows:

$$\min_{\lambda_1, \lambda_2, \lambda_3} \frac{1}{2} \|\mathbf{y}_V - X_V \hat{\beta} - M_V \hat{\theta}\|_2^2$$

$$\text{where } \hat{\beta}, \hat{\theta} = \arg \min_{\beta, \theta} \frac{1}{2} \|y - X\beta - M_T \theta\|_2^2 + \lambda_1 \|\beta\|_1 + \frac{1}{2} \lambda_2 \|\beta\|_2^2 + \frac{1}{2} \lambda_3 \|D^{(2)} \theta\|_2^2 \quad (46)$$

where $M_V \in \mathbb{R}^{|V| \times n}$ is the matrix such that $M_V X = X_V$ and $M_T \in \mathbb{R}^{|T| \times n}$ is the matrix such that $M_T X = X_T$.

The gradients of the validation set error with respect to the regularization parameters are:

$$\nabla_{\lambda_i} L(\mathbf{y}_V, X_V \hat{\beta} + \hat{\theta}) = (X_V \frac{\partial \beta}{\partial \lambda_i} + M_V \frac{\partial \theta}{\partial \lambda_i})^T (\mathbf{y}_V - X_V \hat{\beta} - \hat{\theta}_V) \quad (47)$$

where

$$\begin{aligned} \frac{\partial \beta}{\partial \lambda_1} &= -(X_T^T X_T - X_T^T M_T (M_T^T M_T + \lambda_3 D^T D)^{-1} M_T^T X_T + \lambda_2 I)^{-1} \text{sign}(\hat{\beta}) \\ \frac{\partial \theta}{\partial \lambda_1} &= -(M_T^T M_T + \lambda_3 D^T D)^{-1} M_T^T X_T \frac{\partial \beta}{\partial \lambda_1} \\ \frac{\partial \beta}{\partial \lambda_2} &= -(X_T^T X_T - X_T^T M_T (M_T^T M_T + \lambda_3 D^T D)^{-1} M_T^T X_T + \lambda_2 I)^{-1} \hat{\beta} \\ \frac{\partial \theta}{\partial \lambda_2} &= -(M_T^T M_T + \lambda_3 D^T D)^{-1} M_T^T X_T \frac{\partial \beta}{\partial \lambda_2} \\ \frac{\partial \theta}{\partial \lambda_3} &= -(M_T^T M_T - M_T^T X_T (X_T^T X_T + \lambda_2 I)^{-1} X_T^T M_T + \lambda_3 D^T D)^{-1} D^T D \theta \\ \frac{\partial \beta}{\partial \lambda_3} &= -(X_T^T X_T + \lambda_2 I)^{-1} X_T^T M_T \frac{\partial \theta}{\partial \lambda_3} \end{aligned} \quad (48)$$

Acknowledgments

BLAH

References

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