# Occasionally Misspecified

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  1. robust to small amounts of large contamination
  2. asymptotically unbiased
  3. not too hard to compute

# Motivating Example: Price Puzzle

- Estimand: effect of monetary shock on inflation
- Method: recursive VAR (OLS)
  - Variables: Interest Rates $R_t$, Inflation $\pi_t$, Unemployment: $u_t$
  - Specification: 4 lags
  - Data: US from 1960Q1 to 2000Q4
    (same as Stock and Watson, 2001)

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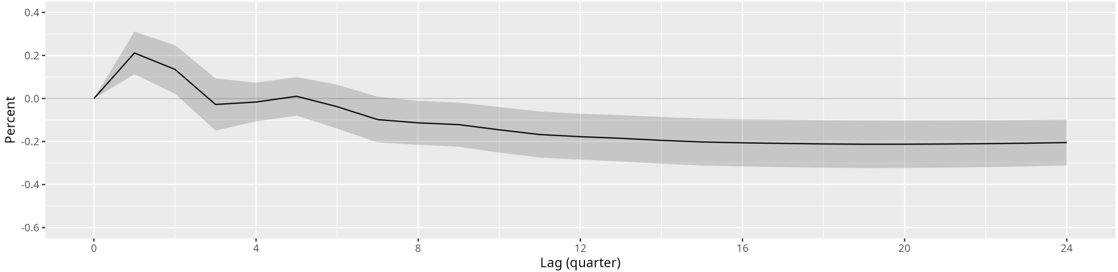
  - Data: US from 1960Q1 to 2000Q4
    (same as Stock and Watson, 2001)
- Results: when $R_t \nearrow$, $\pi_t \nearrow$



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- Let's look at some regression diagnostics

# Price Puzzle: Diagnostics

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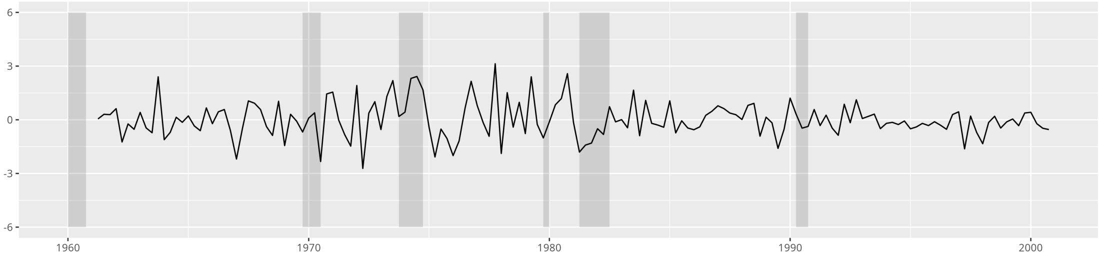
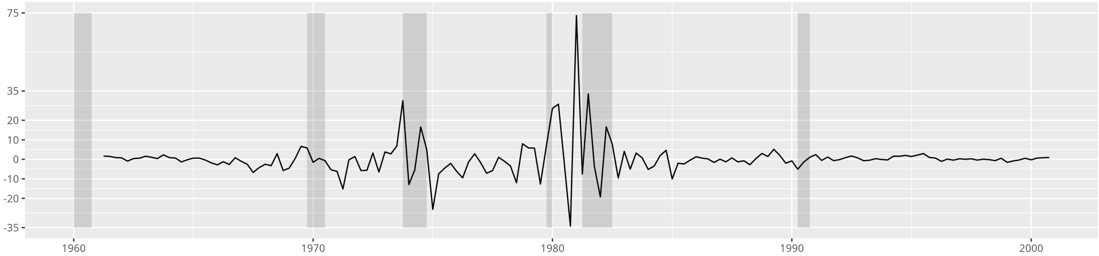
- Residuals $\hat{e}_{\pi,t}$ (standardized)




- Skewness: 0.36, Kurtosis: 3.78

# Price Puzzle: Diagnostics

- Contributions to $\hat{\beta}_{1n}$ – based on $(X'X/n)^{-1}x_t y_t$ (avg $= \hat{\beta}_n$)




- 1981Q1: $75/n \simeq 0.47$ vs. $\hat{\beta}_1 = 0.21$, 3.5 standard errors
- Skewness: 3.24, Kurtosis: 27.81

# Price Puzzle: Concerns

- 1979-1982: Federal Reserve no longer sets $R_t$ directly
  - Non-borrowed reserves targeting (misspecification)
  - Increased volatility in $R_t$ (leverage)

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  - + Reduce the influence of outliers
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- Robust estimation & inference desirable
  - $+$ Reduce the influence of outliers
  - $-$ Biased/Inconsistent for asymmetric data
  - $-$ Robust M-estimators **not robust to leverage** (Hamilton, 1992)

# Literature

- Robust M-estimation:
$$\min_{\theta} \sum_{t=1}^{n} \psi(y_t - x_t'\theta),$$
$\psi$ non-quadratic: LAD, Huber loss, trimming, etc.

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- Local asymptotics require $\hat{\theta}_n$ consistent, asymptotically normal

# Literature

- More recent, finite-sample: Median-of-Means
  - $K$-subsamples, $K$ means, return median
  - robust up to $n_o \leq K/2 - 1$ outliers
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- $n_o \to \infty$ requires $K \to \infty$
- bias of order: $\sqrt{K/n}$

# Sample

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- $n_o$ **outliers**: for $0 \leq A, \alpha < \infty$ and $t = n_P + 1, \ldots, n$

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- Sample mean $\overline{g}_n(\theta)$ can be
  - **asymptotically biased** if $n_o n^{\alpha}/\sqrt{n} \not\to 0$
  - **inconsistent** if $n_o n^{\alpha}/n \not\to 0$
  - **diverging** if $n_o n^{\alpha}/n \not\to 0$

# Estimator

1. Moments: find $\hat{\psi}_n(\theta;\nu) = (\hat{\mu}_n, \hat{\Sigma}_n)$ minimizing:

$$Q_n(\psi;\theta) = \frac{\nu+p}{n}\sum_{t=1}^{n}\log\left(1+\frac{\|g(z_t;\theta)-\mu\|^2_{\Sigma^{-1}}}{\nu}\right)+\log|\Sigma|+\frac{\kappa_1}{\nu}\|\mu\|^2_{\Sigma^{-1}}+\frac{\kappa_2}{\nu}\mathrm{trace}(\Sigma)$$

for $0 < \nu, \kappa_1, \kappa_2 < \infty$ over

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3. Estimates $\tilde{\theta}_n$:

$$\|\tilde{\mu}_n(\tilde{\theta}_n)\|^2_{W_n} \leq \inf_{\theta\in\Theta}\|\tilde{\mu}_n(\theta)\|^2_{W_n} + o_p(n^{-1})$$

# Tuning parameters, Properties

- $0 < \nu < \infty$: controls robustness,
  - $\nu = \infty$: $\hat{\mu}_n(\theta; \infty) = \overline{g}_n(\theta)$, $\hat{\Sigma}_n(\theta; \infty)$ sample covariance
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- Weighted average representation:

$$\hat{\mu}_n(\theta; \nu) = \sum_t \omega_t(\theta; \nu) g(z_t; \theta), \quad \tilde{\mu}_n(\theta; \nu) = \sum_t \tilde{\omega}_t(\theta; \nu) g(z_t; \theta)$$

where $0 \leq \omega_t$, $\sum_t \omega_t \leq 1$, $\tilde{\omega}_t = 2\omega_t(\nu) - \omega_t(\nu/2)$

$\Rightarrow$ Robust-LS is weighted-LS with weights $\tilde{w}_t$

# Some intuition

- Simplified estimator: $\theta_0 = \mathbb{E}_P(z_t)$

$$\hat{\mu}_n(\nu) = \frac{1}{n}\sum_{t=1}^{n} \frac{z_t}{1 + |z_t|^2/\nu}$$

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- Two biases:
  - outlier: $\sqrt{\nu} n_o/n$
  - asymmetry: $\mu(\nu) - \theta_0 = \mathbb{E}_P[\hat{\mu}_n(\nu)] - \theta_0$

# Some intuition

- Asymmetry bias is at most $\mathbb{E}_P(|z_t|^3)/\nu$:

$$\mu(\nu) = \theta_0 - \frac{1}{\nu}\mathbb{E}_P\left(\frac{z_t^3}{1 + z_t^2/\nu}\right)$$

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- Repeat bias correction $\tilde{\tilde{\mu}}(\nu) = 2\tilde{\mu}(\nu) - \tilde{\mu}(\nu/2)$:

$$\tilde{\mu}(\nu) = \theta_0 + \frac{2}{\nu^2}\mathbb{E}_P\left(\frac{z_t^5(1-z_t^4/\nu)}{(1+z_t^2/\nu)(1+2z_t^2/\nu)(1+2z_t^2/\nu)(1+4z_t^2/\nu)}\right)$$

numerator has 3 roots, better small sample properties (simulations)

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  - $n_o$ cannot increase too quickly...
  - $P$ symmetric, need: $n_o = o(n^{1/2})$
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- Undersmoothing (no bias correction): $n_o = o(n^{1/4})$

# Finite Samples

$$Q_\nu(\psi;\theta) = \mathbb{E}_P\left[(\nu+p)\log\left(1+\|g(z_t;\theta)-\mu\|^2_{\Sigma^{-1}}/\nu\right)\right] + \log|\Sigma| + \frac{\kappa_1}{\nu}\|\mu\|^2_{\Sigma^{-1}} + \frac{\kappa_2}{\nu}\mathrm{trace}(\Sigma).$$

- $Q_\nu$: population analog of $Q_n$ with $n_o = 0$, let $\psi(\theta;\nu)$ minimize $Q_\nu$

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- $Q_\nu$: population analog of $Q_n$ with $n_o = 0$, let $\psi(\theta;\nu)$ minimize $Q_\nu$
- If i. $\mathbb{E}_P[\|g(z_t;\theta)\|^2] < \infty$, $\forall\theta$,
  ii. $\|g(z_t;\theta_1) - g(z_t;\theta_2)\| \leq G_t\|\theta_1-\theta_2\|$, $\mathbb{E}_P(\|G_t\|^2) < \infty$, then, for iid data:

$$\mathbb{P}\Bigg( \sup_{\theta\in\Theta} \sup_{z_t\in\mathcal{O}_n, t>n_P} \left\{ Q_\nu(\hat\psi_n(\theta;\nu);\theta) - Q_\nu(\psi(\theta;\nu);\theta) \right\} \geq C_{\mathcal{O}} \frac{n_o(\nu+p)}{n}[1+\log(n)]$$

$$+ L\frac{n_P}{n}(\nu+p)\log(1+\nu p)\left[\sqrt{\frac{x}{n_P}} + \frac{x}{n_P} + \sqrt{\frac{C_n}{n_P}} + \frac{C_n}{n_P}\right]\Bigg) \leq 4\exp(-x),$$

where $C_n = 1 + (k+2p^2)[\log(p) + \log(\nu) + \log(n_P)]$

# Large Samples: Oracle Equivalence

- Further assume that $\sup_{\theta\in\Theta}\mathbb{E}_P[\|g(z_t;\theta)\|^4]<\infty$, and

$$n_o = o\left(\frac{n}{\nu\log(n)}\right),\ \nu\log(\nu) = o\left(\sqrt{\frac{n}{\log(n)}}\right).$$

- Let $M_{r,\delta} = \max\left(\mathbb{E}_P(\|g(z_t;\theta_0)\|^{r+\delta}), \mathbb{E}_P(\|G_t\|^{r+\delta})\right)$ and $\overline{g}_{n_P}(\theta) = \frac{1}{n_P}\sum_{t=1}^{n_P} g(z_t;\theta)$

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- If $M_{3,\delta} < \infty$ for some $\delta > 0$:

$$\sup_{\theta\in\Theta}\left(\sup_{z_t\in\mathcal{O}_n, t>n_P} \|\hat{\mu}_n(\theta;\nu) - \overline{g}_{n_P}(\theta)\|\right) = O_p\left(\max\left[\frac{1}{\nu}, \frac{\sqrt{\nu} n_o}{n}\right]\right)$$

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- If $M_{5,\delta} < \infty$ for some $\delta > 0$:

$$\sup_{\theta\in\Theta}\left(\sup_{z_t\in\mathcal{O}_n, t>n_P} \|\tilde{\mu}_n(\theta;\nu) - \bar{g}_{n_P}(\theta)\|\right) = O_p\left(\max\left[\frac{1}{\nu^2}, \frac{\sqrt{\nu}n_o}{n}\right]\right).$$

# Large Samples: Oracle Equivalence (estimator)

- Assume standard regularity conditions for $1 \leq t \leq n_P$, and

$$\frac{\sqrt{n}}{\nu^2} = o(1), \text{ and } \sqrt{\frac{\nu}{n}} n_o = o(1).$$

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$$\frac{\sqrt{n}}{\nu^2} = o(1), \text{ and } \sqrt{\frac{\nu}{n}} n_o = o(1).$$

- Let $\hat{\theta}_{n_P}$ minimize $\|\overline{g}_{n_P}(\theta)\|_{W_n}$; $\tilde{\theta}_n$ minimize $\|\tilde{\mu}_n(\theta;\nu)\|_{W_n}$ then:

$$\sup_{z_t \in \mathcal{O}_n, t > n_P} \|\sqrt{n_P}(\tilde{\theta}_n - \hat{\theta}_{n_P})\| = o_p(1), \quad \text{and} \quad \sqrt{n_P}(\tilde{\theta}_n - \theta_0) \xrightarrow{d} \mathcal{N}(0, V)$$

for any sequence $(z_t)_{t>n_P} \in \mathcal{O}_n$, $V = \text{avar}(\hat{\theta}_{n_P})$

# Monte Carlo

- Simulation design – for $1 \leq t \leq n_P$

$$y_t = \theta_0 + \theta_1 x_{1t} + \theta_2 x_{2t} + \theta_3 x_{3t} + e_t,$$

- $n = 150$, $x_{1t}, x_{2t}, x_{3t}, e_t = (\chi_5^2 - 5)/\sqrt{10}$. $\theta_0 = (0, 1, 1, 1)$

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- Outliers: $y_t = x_t' \theta_\dagger$, $\theta_\dagger = (0, 1/2, 1/2, 1/2)$. $x_{1t} = x_{2t} = x_{3t} = \sqrt{n}$
- $n_o = 0, 1, 5, 10, \ldots$

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- $n = 150$, $x_{1t}, x_{2t}, x_{3t}, e_t = (\chi^2_5 - 5)/\sqrt{10}$. $\theta_0 = (0, 1, 1, 1)$
- Outliers: $y_t = x_t' \theta_\dagger$, $\theta_\dagger = (0, 1/2, 1/2, 1/2)$. $x_{1t} = x_{2t} = x_{3t} = \sqrt{n}$
- $n_o = 0, 1, 5, 10, \ldots$
- Several estimators:
    - OLS,
    - oracle estimator,
    - Robust M-estimator with Huber loss,
    - This paper: $\hat{\theta}_n$, $\tilde{\theta}_n$, $\tilde{\tilde{\theta}}_n$
    - Undersmoothing: no bias correction, $\nu^2$

# Monte Carlo: $n = 150$, $n_o = 0$

| | 100 × RMSE | | | | | | | Rejection Rate | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $n_o = 0$ | | | | | | | | | | | | | |
| | $\hat{\theta}_n^{ols}$ | $\hat{\theta}_{n_P}^{ols}$ | $\hat{\theta}_n^{rlm}$ | $\hat{\theta}_n$ | $\tilde{\theta}_n$ | $\tilde{\tilde{\theta}}_n$ | $\hat{\theta}_n^{un}$ | $\hat{\theta}_n^{ols}$ | $\hat{\theta}_{n_P}^{ols}$ | $\hat{\theta}_n^{rlm}$ | $\hat{\theta}_n$ | $\tilde{\theta}_n$ | $\tilde{\tilde{\theta}}_n$ | $\hat{\theta}_n^{un}$ |
| $\theta_0$ | 8.05 | 8.05 | 12.00 | 11.84 | 9.31 | 8.11 | 7.94 | 0.04 | 0.04 | 0.24 | 0.29 | 0.14 | 0.05 | 0.06 |
| $\theta_1$ | 8.00 | 8.00 | 7.15 | 7.97 | 7.79 | 7.78 | 7.92 | 0.06 | 0.06 | 0.06 | 0.11 | 0.08 | 0.07 | 0.06 |
| $\theta_2$ | 8.10 | 8.10 | 7.46 | 8.45 | 8.21 | 8.11 | 8.06 | 0.04 | 0.04 | 0.05 | 0.10 | 0.06 | 0.05 | 0.05 |
| $\theta_3$ | 8.19 | 8.19 | 7.43 | 8.55 | 8.30 | 8.16 | 8.14 | 0.06 | 0.06 | 0.06 | 0.10 | 0.07 | 0.06 | 0.06 |

Average $\hat{\nu}_n$: 35.85, 16.00, 11.00, 10.71 for $n_0 = 0, 1, 5, 10$ respectively. Each $\hat{\nu}_n$ is selected on a grid $[\nu_0, \dots, \nu_J]$ where $\nu_0 = 8.77$, $\nu_J = 584.69$.

# Monte Carlo: $n = 150$, $n_o = 1$

| | 100 × RMSE | | | | | | | Rejection Rate | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $n_o = 1$ | | | | | | | | | | | | | |
| | $\hat{\theta}_n^{ols}$ | $\hat{\theta}_{n_P}^{ols}$ | $\hat{\theta}_n^{rlm}$ | $\hat{\theta}_n$ | $\tilde{\theta}_n$ | $\tilde{\tilde{\theta}}_n$ | $\hat{\theta}_n^{un}$ | $\hat{\theta}_n^{ols}$ | $\hat{\theta}_{n_P}^{ols}$ | $\hat{\theta}_n^{rlm}$ | $\hat{\theta}_n$ | $\tilde{\theta}_n$ | $\tilde{\tilde{\theta}}_n$ | $\hat{\theta}_n^{un}$ |
| $\theta_0$ | 10.71 | 8.04 | 13.01 | 14.18 | 10.97 | 8.52 | 10.32 | 0.03 | 0.04 | 0.20 | 0.46 | 0.23 | 0.08 | 0.08 |
| $\theta_1$ | 38.57 | 8.07 | 15.23 | 8.27 | 7.97 | 7.87 | 32.24 | 0.00 | 0.06 | 0.01 | 0.14 | 0.10 | 0.07 | 0.40 |
| $\theta_2$ | 38.39 | 8.11 | 15.09 | 8.73 | 8.36 | 8.14 | 32.08 | 0.01 | 0.04 | 0.01 | 0.12 | 0.06 | 0.06 | 0.38 |
| $\theta_3$ | 39.94 | 8.20 | 15.75 | 8.83 | 8.49 | 8.27 | 33.47 | 0.00 | 0.06 | 0.00 | 0.12 | 0.09 | 0.07 | 0.39 |

Average $\hat{\nu}_n$: 35.85, 16.00, 11.00, 10.71 for $n_0 = 0, 1, 5, 10$ respectively. Each $\hat{\nu}_n$ is selected on a grid $[\nu_0, \ldots, \nu_J]$ where $\nu_0 = 8.77$, $\nu_J = 584.69$.

# Monte Carlo: $n = 150$, $n_o = 5$

| | 100 × RMSE | | | | | | | Rejection Rate | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $n_o = 5$ | | | | | | | | | | | | | |
| | $\hat{\theta}_n^{ols}$ | $\hat{\theta}_{nP}^{ols}$ | $\hat{\theta}_n^{rlm}$ | $\hat{\theta}_n$ | $\tilde{\theta}_n$ | $\tilde{\tilde{\theta}}_n$ | $\hat{\theta}_n^{un}$ | $\hat{\theta}_n^{ols}$ | $\hat{\theta}_{nP}^{ols}$ | $\hat{\theta}_n^{rlm}$ | $\hat{\theta}_n$ | $\tilde{\theta}_n$ | $\tilde{\tilde{\theta}}_n$ | $\hat{\theta}_n^{un}$ |
| $\theta_0$ | 11.98 | 8.14 | 16.57 | 16.98 | 13.38 | 9.82 | 13.45 | 0.10 | 0.04 | 0.24 | 0.59 | 0.38 | 0.13 | 0.16 |
| $\theta_1$ | 47.57 | 8.40 | 47.17 | 9.02 | 8.62 | 8.40 | 46.72 | 0.99 | 0.06 | 0.99 | 0.12 | 0.08 | 0.06 | 0.99 |
| $\theta_2$ | 47.48 | 8.26 | 48.25 | 9.28 | 8.80 | 8.53 | 47.14 | 0.99 | 0.04 | 1.00 | 0.12 | 0.05 | 0.03 | 1.00 |
| $\theta_3$ | 49.17 | 8.28 | 49.48 | 9.33 | 8.94 | 8.72 | 48.65 | 0.98 | 0.06 | 0.98 | 0.10 | 0.08 | 0.04 | 0.98 |

Average $\hat{\nu}_n$: 35.85, 16.00, 11.00, 10.71 for $n_0 = 0, 1, 5, 10$ respectively. Each $\hat{\nu}_n$ is selected on a grid $[\nu_0, \ldots, \nu_J]$ where $\nu_0 = 8.77$, $\nu_J = 584.69$.

# Monte Carlo: $n = 150$, $n_o = 10$

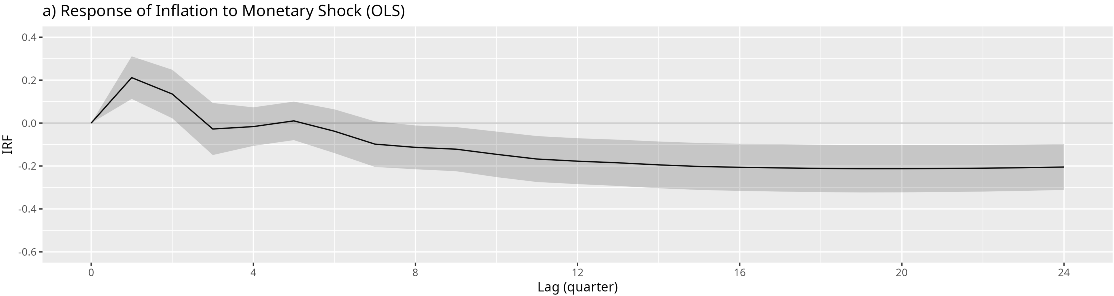
| | 100 × RMSE | | | | | | | Rejection Rate | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $n_o = 10$ | | | | | | | | | | | | | |
| | $\hat{\theta}_n^{ols}$ | $\hat{\theta}_{n_P}^{ols}$ | $\hat{\theta}_n^{rlm}$ | $\hat{\theta}_n$ | $\tilde{\theta}_n$ | $\tilde{\tilde{\theta}}_n$ | $\hat{\theta}_n^{un}$ | $\hat{\theta}_n^{ols}$ | $\hat{\theta}_{n_P}^{ols}$ | $\hat{\theta}_n^{rlm}$ | $\hat{\theta}_n$ | $\tilde{\theta}_n$ | $\tilde{\tilde{\theta}}_n$ | $\hat{\theta}_n^{un}$ |
| $\theta_0$ | 12.21 | 8.21 | 17.33 | 16.78 | 13.27 | 10.35 | 14.13 | 0.09 | 0.04 | 0.23 | 0.47 | 0.22 | 0.07 | 0.17 |
| $\theta_1$ | 49.14 | 8.54 | 48.38 | 10.22 | 11.68 | 19.76 | 48.65 | 0.99 | 0.04 | 0.99 | 0.01 | 0.01 | 0.09 | 1.00 |
| $\theta_2$ | 49.05 | 8.31 | 49.67 | 10.76 | 12.40 | 20.28 | 48.92 | 0.99 | 0.04 | 0.99 | 0.01 | 0.01 | 0.09 | 1.00 |
| $\theta_3$ | 50.52 | 8.51 | 50.70 | 11.04 | 13.00 | 20.96 | 50.19 | 0.98 | 0.06 | 0.98 | 0.00 | 0.01 | 0.09 | 0.99 |

Average $\hat{\nu}_n$: 35.85, 16.00, 11.00, 10.71 for $n_0 = 0, 1, 5, 10$ respectively. Each $\hat{\nu}_n$ is selected on a grid $[\nu_0, \ldots, \nu_J]$ where $\nu_0 = 8.77$, $\nu_J = 584.69$.

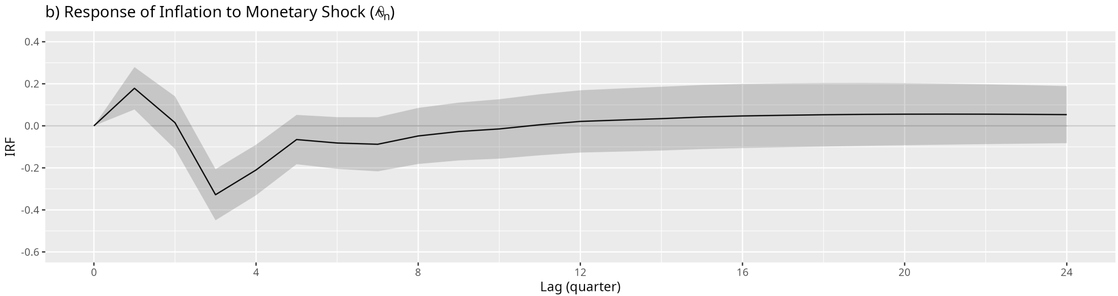
# Back to the Price Puzzle

- Re-estimate the model using OLS, $\hat{\theta}_n$, $\tilde{\theta}_n$, $\tilde{\tilde{\theta}}_n$
- Same data
- Same model

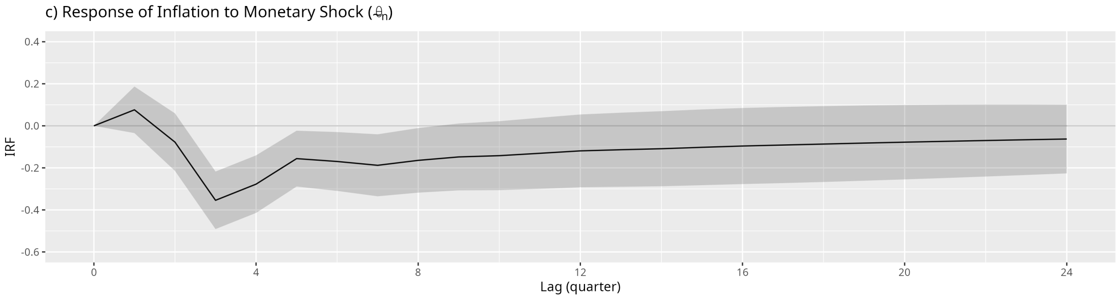
# Price Puzzle: OLS



# Price Puzzle: Robust, No Bias Correction

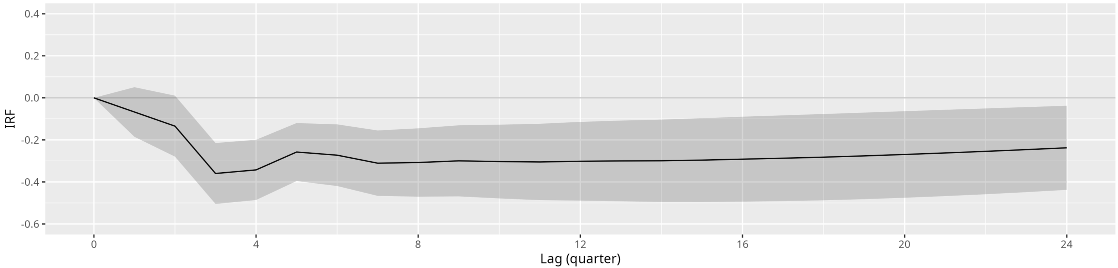


# Price Puzzle: Robust, Bias Correction

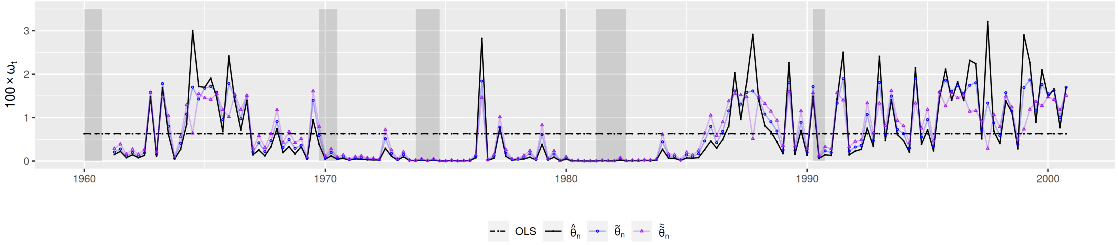


# Price Puzzle: Robust, Repeat Bias Correction

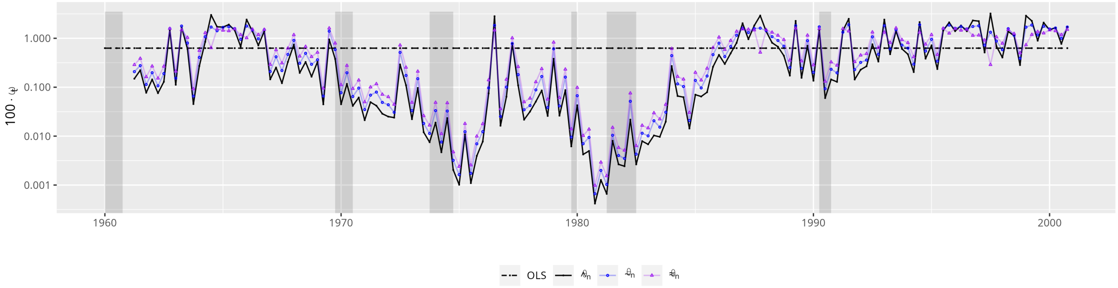
d) Response of Inflation to Monetary Shock ($\vartheta_n$)



# Price Puzzle: Estimation Weights



# Price Puzzle: Estimation Weights (log scale)



# Conclusion

- Misspecification can occur
  - don't always know which $t$ are involved
  - diagnostics useful but not definitive
- Robust estimation:
  - more resilient
  - potentially biased/inconsistent
- This paper: simple estimates, bias correction
  - some finite and large sample results
  - robustness to leveraged outliers
  - weights: make the results transparent

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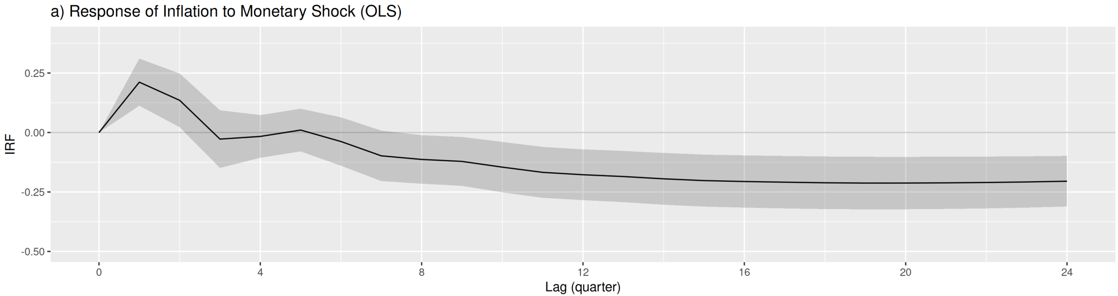
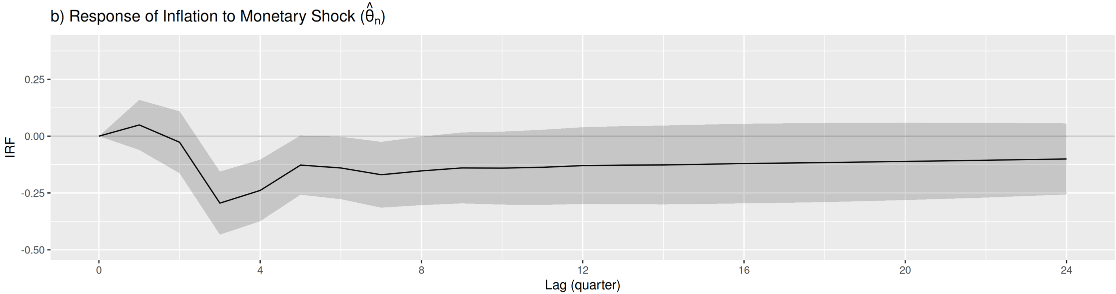
# Back to the Price Puzzle

- Re-estimate with a larger $\nu = 15$ vs. $\nu = 8.99$ in the main results

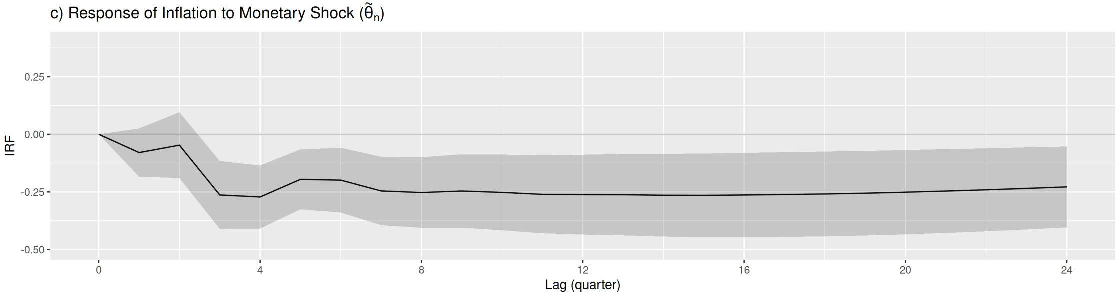
# Price Puzzle: OLS



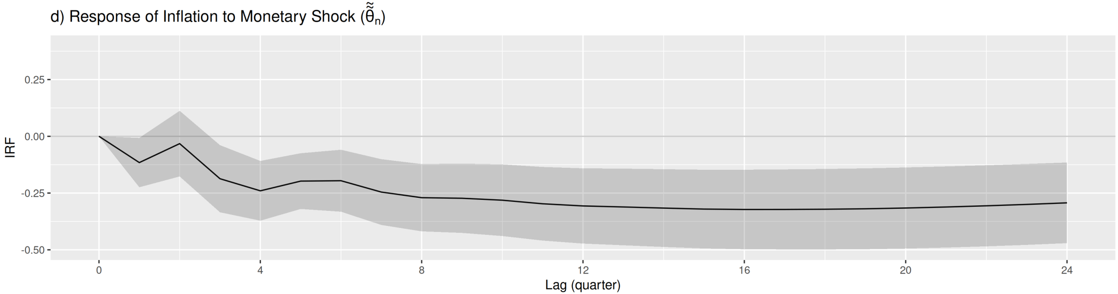
# Price Puzzle: Robust, No Bias Correction



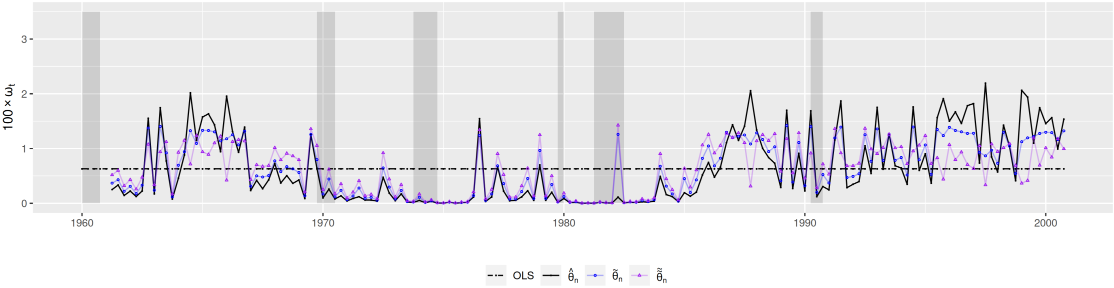
# Price Puzzle: Robust, Bias Correction



# Price Puzzle: Robust, Repeat Bias Correction



# Price Puzzle: Estimation Weights



# Price Puzzle: Estimation Weights (log scale)