Inference by Stochastic Optimization: A Free-Lunch Bootstrap

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Introduction: Challenging Inferences

- Extremum estimation: GMM, NLS, MLE, ...
  - where computing the asymptotic variance is not tractable
    e.g. rely on transformed/generated data, multi-step estimation, complicated moments/likelihood
  - or counterfactuals st. Δ-method is challenging to implement

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- Classical Bootstrap: run the optimizer many times
- This Paper: focused on a **Stochastic Newton-Raphson** algorithm, a single run produces
  - a consistent estimator by simple averaging
  - asymptotically valid Bootstrap draws (free-lunch)

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- Interested in the sample GMM/MLE/MD estimator:

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  for $b = 1, \ldots, B$, where $Q_m^{(b)}(\cdot)$ is an m out of n re-sampled objective (or re-weighted/multiplier)

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- **Goal: a 2-in-1 procedure for estimation and inference**

This Paper

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Bootstrap is '*free*' in the sense that we get both *an approximate minimizer* (iii) and *asym. valid standard errors* (iv) in one run

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$$\theta^b = \theta^{b-1} - \gamma \cdot [H_m^{(b)}(\theta^{b-1})]^{-1} G_m^{(b)}(\theta^{b-1})$$

$G_m^{(b)}, H_m^{(b)}$ re-sampled gradient, hessian; $m/n \to c \in (0, 1]$;
$\gamma \in (0, 1]$ fixed learning rate

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iii. $\frac{1}{B} \sum_{b=1}^{B} \theta^b \simeq \hat{\theta}_n$

iv. $var(\theta^b) \simeq \frac{\gamma^2}{1-[1-\gamma]^2} var(\hat{\theta}_n)$ (the asymptotic variance)

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Related Literatures

- **Computationally Attractive Bootstrap:**
  Davidson and MacKinnon (1999); Andrews (2002); Kline and Santos (2012); Armstrong et al. (2014); Honoré and Hu (2017),...
  k-step re-sampling at a converged estimate of $\hat{\theta}_n$
- **Stochastic Derivative-Based Optimization:**
  Robbins and Monro (1951); Dvoretzky (1956); Ruppert (1988); Polyak and Juditsky (1992),[...], Bach and Moulines (2011); Moritz et al. (2016); Mandt et al. (2017),...
  interested in optimization on very large or online data sets
- **Stochastic Optimization and Inference:**
  Chernozhukov and Hong (2003),...
  MCMC similar to Simulated Annealing with a fixed temperature; (quasi)-posterior distribution asymptotically valid for inference

Outline

An Overview of Derivative-Based Methods

Gradient Descent and Newton-Raphson Methods

Algorithm: Newton-Raphson

i. initialize: at θ^0, given

ii. for $b = 1, \ldots, B$ compute:

$$\theta^b = \theta^{b-1} - \underbrace{\gamma_b}_{\text{learning rate}} \cdot [H_n(\theta^{b-1})]^{-1} G_n(\theta^{b-1})$$

- **Gradient-Descent**: $\theta^b = \theta^{b-1} - \gamma_b \cdot \cancel{[H_n(\theta^{b-1})]^{-1}} G_n(\theta^{b-1})$

- less costly, slow convergence when $\lambda_{\max}(H_n)/\lambda_{\min}(H_n)$ large

Illustration OLS regression

- OLS regression: $y_i = x_i'\theta + u_i$; $\gamma_b = \gamma$ fixed
- **Newton-Raphson**:

$$\boxed{\theta^b - \hat{\theta}_n = (1-\gamma)^b[\theta^0 - \hat{\theta}_n]}$$

  - for $\gamma_b = 1$ convergence after one iteration

- **Gradient Descent**:

$$\boxed{\theta^b - \hat{\theta}_n = (I - 2\gamma[\sum_i x_i x_i'/n])^b[\theta^0 - \hat{\theta}_n]}$$

  - very slow convergence when $\lambda_{\max}(X'X)/\lambda_{\min}(X'X)$ large

Stochastic Gradient Descent

- Full sample G_n, H_n costly to compute for n very large
- **Solution**: use a *minibatch* (small) of subsamples $m \ll n$
- In practice: $m = 1$ is popular

Algorithm: Stochastic Gradient-Descent

i. initialize: at θ^0, given

ii. for $b = 1, \ldots, B$ compute:

$$\theta^b = \theta^{b-1} - \gamma_b \cdot G_m^{(b)}(\theta^{b-1})$$

Simple Illustration: OLS estimation ($m = 1$)

- Mini-batch with $m = 1$
- **Stochatic Gradient Descent**:

$$\theta^b - \hat{\theta}_n = (I - 2\gamma_b \underbrace{x_i^{(b)} x_i^{(b)\prime}}_{\text{noisy}})(\theta^{b-1} - \hat{\theta}_n) - 2\gamma_b \underbrace{x_i^{(b)} \hat{u}_i^{(b)}}_{\text{noisy}}$$

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- For $\theta^b \overset{p^\star}{\to} \hat{\theta}_n$ we need $\gamma_b \searrow 0$
  - fast enough so that $\mathbb{E}^\star \| 2\gamma_b [x_i^{(b)} x_i^{(b)\prime}] \theta^{(b-1)} \|^2 \to 0$
  - not too fast so that $\mathbb{E}^\star \| (1 - 2\gamma_b x_i^{(b)} x_i^{(b)\prime})(\theta^{b-1} - \hat{\theta}_n) \|^2 \to 0$

  $\Rightarrow$ convergence can be very slow
  - in practice: adaptive methods (adagrad, RMSprop,...)

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$$\theta^b - \hat{\theta}_n = (I - 2\gamma_b \underbrace{x_i^{(b)} x_i^{(b)\prime}}_{\text{noisy}})(\theta^{b-1} - \hat{\theta}_n) - 2\gamma_b \underbrace{x_i^{(b)} \hat{u}_i^{(b)}}_{\text{noisy}}$$

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  $\Rightarrow$ convergence can be very slow
  - in practice: adaptive methods (adagrad, RMSprop,...)
- **Stochastic Newton-Raphson**: $H_1^{(b)}$ often noisy/near singular
  - e.g. $x_i = (1, x_{i,1})$, $x_{i,1} \sim \textit{Bernoulli}(p)$

    $\Rightarrow x_i x_i'$ singular wp. 1 for any $p \in [0, 1]$

This Paper: S-NR with larger batches

- Three changes over S-GD:
  a. re-introduce the Hessian $H_m^{(b)}(\theta^{b-1})$ (NR)
  b. sample m out of n observations, $m/n \to c \in (0,1]$
  c. fixed learning rate $\gamma_b = \gamma \in (0,1]$

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$$\theta^b = \theta^{b-1} - \gamma \cdot [H_m^{(b)}(\theta^{b-1})]^{-1} G_m^{(b)}(\theta^{b-1})$$

Simple Illustration: OLS estimation ($m/n \to c \in (0,1]$)

- **Stochatic Newton-Raphson**:

$$\theta^b - \hat{\theta}_n = \underbrace{(1-\gamma)(\theta^{b-1} - \hat{\theta}_n)}_{\text{deterministic cv.}} + \underbrace{\gamma(\hat{\theta}_m^{(b)} - \hat{\theta}_n)}_{\text{noise}}$$

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- For $\gamma = 1$, $\theta^b = \hat{\theta}_m^{(b)}$ the bootstrapped estimate
- $\theta^b \overset{p}{\not\to} \hat{\theta}_n$ with γ fixed but
    - $\mathbb{E}^\star(\theta^b) \simeq \hat{\theta}_n$ and $var^\star(\theta^b) \simeq \frac{\gamma^2}{1-[1-\gamma]^2} var^\star(\hat{\theta}_m^{(b)})$

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  - $\mathbb{E}^\star(\theta^b) \simeq \hat{\theta}_n$ and $var^\star(\theta^b) \simeq \frac{\gamma^2}{1-[1-\gamma]^2} var^\star(\hat{\theta}_m^{(b)})$
- Now: extend this result to a class of non-linear models

Asymptotic Results

Assumptions: Q_n

Assumption (Sample Objective Function)

i. $\|H_n(\theta)^{-1}G_n(\theta)\|_2 \leq \overline{C}\|\theta - \hat{\theta}_n\|_2$,

ii. $\underline{C}\|\theta - \hat{\theta}_n\|_2^2 \leq \langle \theta - \hat{\theta}_n, H_n(\theta)^{-1}G_n(\theta)\rangle$,

iii. $\underline{c}_H \leq \lambda_{\min}(H_n(\theta)^{-1}) \leq \lambda_{\max}(H_n(\theta)^{-1}) \leq \overline{C}_H$,

iv. $\|H_n(\theta) - H_n(\hat{\theta}_n)\|_2 \leq C_{n,1} \times \|\theta - \hat{\theta}_n\|_2$,

v. $\|\sup_{\theta\in\Theta} G_n(\theta)\|_2 \leq \overline{C}_n$

Remark: conditions i-iii. imply strong convexity

Newton-Raphson

Lemma (Newton-Raphson)

Suppose Assumption 1 holds, then for $\gamma \in (0,1)$ small enough, $\exists \bar{\gamma} \in (0,1)$ such that for any θ^0:

$$\|\theta^b_{NR} - \hat{\theta}_n\|_2 \leq (1-\bar{\gamma})\|\theta^{b-1} - \hat{\theta}_n\|_2 \leq (1-\bar{\gamma})^b\|\theta^0 - \hat{\theta}_n\|_2$$

Assumptions: $Q_m^{(b)}$

Assumption (Re-Sampled Objective Function)

Suppose the following holds uniformly over $\theta \in \Theta$:

i. $\|[H_m^{(b)}(\theta)]^{-1}[G_m^{(b)}(\theta) - G_m^{(b)}(\hat{\theta}_n) - H_m^{(b)}(\theta)(\theta - \hat{\theta}_n)]\|_2 \leq C_{m,1} \times \|\theta - \hat{\theta}_n\|_2^2$,

ii. $\mathbb{E}^\star \left(\sup_{\theta\in\Theta} \|[H_n(\theta)]^{-1} - [H_m^{(b)}(\theta)]^{-1}\|_2^2\right)^{1/2} \leq C_{m,2} \times m^{-1/2}$,

iii. $\left[\mathbb{E}^\star \left(\sup_{\theta\in\Theta} \|H_n(\theta) - H_m^{(b)}(\theta)\|_2^2\right)\right]^{1/2} \leq C_{m,3} \times m^{-1/2}$,

iv. $\left[\mathbb{E}^\star \left(\sup_{\theta\in\Theta} \|\mathbb{G}_m^{(b)}(\theta)\|_2^2\right)\right]^{1/2} \leq \overline{C}$, *for* $\mathbb{G}_m^{(b)}(\theta) \stackrel{def}{=} \sqrt{m}[G_m^{(b)}(\theta) - G_n(\theta)]$.

where $C_{m,1/2/3}$ and $(C_n)_{n\geq 1}$ are bounded above, $\overline{C} < +\infty$.

Stochastic Newton-Raphson: Asymptotic Linearization

Lemma (Linearization of the S-NR Markov-Chain)

Suppose Assumptions 1-3 hold, then for $\gamma \in (0,1)$ *small enough,* $\exists \bar{\gamma} \in (0,1)$ *such that* $\forall \theta^0$*, uniformly in* $b \geq 1$*:*

$$\mathbb{E}^\star\Big(\|\theta^b_{NR} - \hat{\theta}_n + \gamma \sum_{j=0}^{b-1} (1-\gamma)^j \mathbb{Z}_m^{b-j}\|_2^2\Big)^{1/2} \lesssim m^{-1} + b\rho^b[d_{0,n} + d_{0,n}^2]$$

where $\rho = \max(1-\gamma, 1-\bar{\gamma}) \in [0,1)$*;* $d_{0,n} = \mathbb{E}^\star\left(\|\theta^0 - \hat{\theta}_n\|_2^2\right)^{1/2}$ *and* $\mathbb{Z}_m^{b-j} = [H_n(\hat{\theta}_n)]^{-1} G_m^{(b-j)}(\hat{\theta}_n)$

Theorem (Convergence in Distribution)

Suppose Assumptions 1-3 hold, let $\mathbb{Z}_m^b = [H_n(\hat{\theta}_n)]^{-1}\mathbb{G}_m^{(b)}(\hat{\theta}_n)$ *and* $\Sigma_m = var^\star(\mathbb{Z}_m^b)$*. Suppose* $0 < \underline{\lambda} \leq \lambda_{\min}(\Sigma_m) \leq \overline{\lambda} \leq \lambda_{\max}(\Sigma_m) < +\infty$*, and conditions on the characteristic function of* $\mathbb{Z}_m^b$ *hold then:*

$$\sqrt{m}\Sigma_m^{-1/2}(\theta^b - \hat{\theta}_n) \xrightarrow{d^\star} \mathcal{N}\left(0, \frac{\gamma^2}{1-[1-\gamma]^2} I\right),$$

as $m, b \to \infty$*; if* $\log(m)/b \to 0$ *and* $d_{0,n} = O(1)$*,* $n/m = O(1)$*.*

Empirical Illustration

Simple Example: Mroz (1987) Probit model

- Probit model: $\mathbb{P}(y_i = 1|x_i) = \Phi(x_i'\theta)$
- Sample of $n = 753$ observations, $m = n$, $\gamma = 0.3$
- $SNR_{np/m}$: iid re-sampling and multiplier Bootstrap (5k draws)
- Compare $\hat{\theta}_{n,MLE}$, asym. & boot. standard errors with SNR

	nwifeinc	educ	exper	exper2	age	kidslt6	kidsge6	constant
$\hat{\theta}_{n,MLE}$	-0.012	0.131	0.123	-0.002	-0.053	-0.868	0.036	0.270
Asym.	(0.005)	(0.025)	(0.019)	(0.001)	(0.008)	(0.119)	(0.043)	(0.509)
$\bar{\theta}_{n,boot}$	-0.012	0.134	0.124	-0.002	-0.054	-0.883	0.036	0.275
Boot.	(0.005)	(0.027)	(0.020)	(0.001)	(0.009)	(0.122)	(0.046)	(0.524)
SNR_{np}	-0.012	0.133	0.123	-0.002	-0.053	-0.873	0.038	0.263
	(0.005)	(0.026)	(0.019)	(0.001)	(0.010)	(0.119)	(0.046)	(0.510)
SNR_m	-0.012	0.132	0.124	-0.002	-0.053	-0.873	0.037	0.267
	(0.005)	(0.025)	(0.019)	(0.001)	(0.008)	(0.115)	(0.044)	(0.500)

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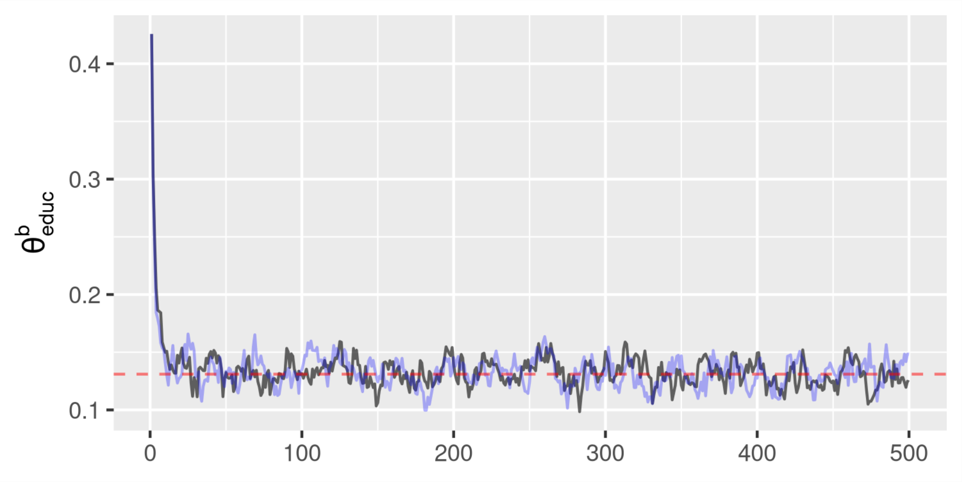

red: $\hat{\theta}_n$; black/blue: SNR with iid/multiplier Bootstrap

Simple Example: $m = n$

	nwifeinc	educ	exper	exper2	age	kidslt6	kidsge6	constant
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Simple Example: $m = 500$, ($n = 753$)

	nwifeinc	educ	exper	exper2	age	kidslt6	kidsge6	constant
$\hat{\theta}_{n,MLE}$	-0.012	0.131	0.123	-0.002	-0.053	-0.868	0.036	0.270
Asym.	(0.005)	(0.025)	(0.019)	(0.001)	(0.008)	(0.119)	(0.043)	(0.509)
$\bar{\theta}_{n,boot}$	-0.012	0.134	0.124	-0.002	-0.054	-0.883	0.036	0.275
Boot.	(0.005)	(0.027)	(0.020)	(0.001)	(0.009)	(0.122)	(0.046)	(0.524)
SNR_{np}	-0.012	0.133	0.123	-0.002	-0.053	-0.876	0.037	0.277
	(0.005)	(0.026)	(0.020)	(0.001)	(0.008)	(0.120)	(0.045)	(0.505)
SNR_m	-0.012	0.134	0.123	-0.002	-0.053	-0.876	0.036	0.259
	(0.006)	(0.029)	(0.022)	(0.001)	(0.010)	(0.133)	(0.053)	(0.585)

Simple Example: $m = 200$, $(n = 753)$

	nwifeinc	educ	exper	exper2	age	kidslt6	kidsge6	constant
$\hat{\theta}_{n,MLE}$	-0.012	0.131	0.123	-0.002	-0.053	-0.868	0.036	0.270
Asym.	(0.005)	(0.025)	(0.019)	(0.001)	(0.008)	(0.119)	(0.043)	(0.509)
$\bar{\theta}_{n,boot}$	-0.012	0.134	0.124	-0.002	-0.054	-0.883	0.036	0.275
Boot.	(0.005)	(0.027)	(0.020)	(0.001)	(0.009)	(0.122)	(0.046)	(0.524)
SNR_{np}	-0.013	0.135	0.124	-0.002	-0.054	-0.884	0.039	0.267
	(0.005)	(0.026)	(0.020)	(0.001)	(0.009)	(0.121)	(0.046)	(0.512)
SNR_m	-0.014	0.140	0.122	-0.002	-0.054	-0.898	0.036	0.248
	(0.007)	(0.033)	(0.025)	(0.001)	(0.011)	(0.158)	(0.058)	(0.655)

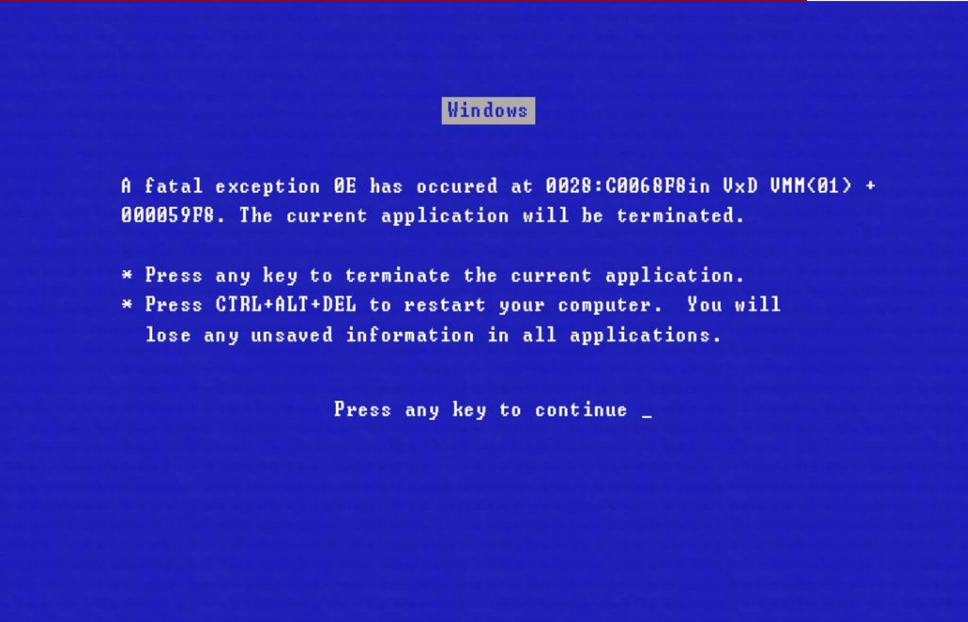
Simple Example: $m = 100$, $(n = 753)$

	nwifeinc	educ	exper	exper2	age	kidslt6	kidsge6	constant
$\hat{\theta}_{n,MLE}$	-0.012	0.131	0.123	-0.002	-0.053	-0.868	0.036	0.270
Asym.	(0.005)	(0.025)	(0.019)	(0.001)	(0.008)	(0.119)	(0.043)	(0.509)
$\bar{\theta}_{n,boot}$	-0.012	0.134	0.124	-0.002	-0.054	-0.883	0.036	0.275
Boot.	(0.005)	(0.027)	(0.020)	(0.001)	(0.009)	(0.122)	(0.046)	(0.524)
SNR_{np}	-0.014	0.139	0.125	-0.002	-0.055	-0.906	0.038	0.281
	(0.006)	(0.027)	(0.021)	(0.001)	(0.009)	(0.128)	(0.047)	(0.525)
SNR_m	-0.015	0.144	0.127	-0.002	-0.056	-0.930	0.039	0.263
	(0.007)	(0.035)	(0.027)	(0.001)	(0.012)	(0.164)	(0.061)	(0.681)

Simple Example: $m = 50$, $(n = 753)$

	nwifeinc	educ	exper	exper2	age	kidslt6	kidsge6	constant
$\hat{\theta}_{n,MLE}$	-0.012	0.131	0.123	-0.002	-0.053	-0.868	0.036	0.270
Asym.	(0.005)	(0.025)	(0.019)	(0.001)	(0.008)	(0.119)	(0.043)	(0.509)
$\bar{\theta}_{n,boot}$	-0.012	0.134	0.124	-0.002	-0.054	-0.883	0.036	0.275
Boot.	(0.005)	(0.027)	(0.020)	(0.001)	(0.009)	(0.122)	(0.046)	(0.524)
SNR_{np}	-0.015	0.149	0.128	-0.002	-0.057	-0.948	0.037	0.270
	(0.006)	(0.029)	(0.024)	(0.001)	(0.009)	(0.137)	(0.050)	(0.566)
SNR_m	-0.016	0.155	0.133	-0.002	-0.059	-0.974	0.045	0.243
	(0.008)	(0.037)	(0.030)	(0.001)	(0.012)	(0.180)	(0.065)	(0.735)

Simple Example: $m \leq 30$ $(n = 753)$



Going Forwad: Beyond SNR?

- OLS regression: run SGD with $m \to \infty$

$$\theta^b = \theta^{b-1} - \gamma \cdot G_m^{(b)}(\theta^{b-1})$$

- As $m, b \to \infty$, we have a VAR(1) representation:

$$\mathbb{E}^\star(\|\theta^b - \hat{\theta}_n - \gamma \sum_{j=0}^{b-1}(I - 2\gamma X_n' X_n/n)^j \mathbb{Z}_m^{(b-j)}\|) \lesssim m^{-1} + \rho^b d_{0,n}$$

where $\rho = \lambda_{\max}(I - 2\gamma X_n' X_n/n)$ (assuming $\lambda_{\min} \geq 0$); $d_{0,n} = \mathbb{E}^\star\left(\|\theta^0 - \hat{\theta}_n\|\right)$ and $\mathbb{Z}_m^{b-j} = -2X_m^{(b)\prime}\hat{e}_m^{(b)}/m$

Going Forwad: Beyond SNR?

- As $m, b \to \infty$, we have a VAR(1) representation:

$$\theta^b - \hat{\theta}_n \overset{approx.}{=} (I - 2\gamma X_n' X_n / n)(\theta^{b-1} - \hat{\theta}_n) + \gamma \mathbb{Z}_m^b$$

- Run a VAR(1) regression:

$$\theta^b = A_0 + A_1 \theta^{b-1} + \varepsilon^b$$

Going Forwad: Beyond SNR?

- As $m, b \to \infty$, we have a VAR(1) representation:

$$\theta^b - \hat{\theta}_n \overset{approx.}{=} (I - 2\gamma X_n' X_n / n)(\theta^{b-1} - \hat{\theta}_n) + \gamma \mathbb{Z}_m^b$$

- Run a VAR(1) regression:

$$\theta^b = A_0 + A_1 \theta^{b-1} + \varepsilon^b$$

- Heuristically we should have:

$$\begin{aligned} X_n' X_n / n &\overset{approx.}{=} 0.5\gamma^{-1}(I - A_1) \\ \hat{\theta}_n &\overset{approx.}{=} (I - A_1)^{-1} A_0 \end{aligned}$$

without explicitly computing these quantities

Going Forwad: Beyond SNR?

- As $m, b \to \infty$, we have a VAR(1) representation:

$$\theta^b - \hat{\theta}_n \overset{approx.}{=} (I - 2\gamma X_n' X_n / n)(\theta^{b-1} - \hat{\theta}_n) + \gamma \mathbb{Z}_m^b$$

- Run a VAR(1) regression:

$$\theta^b = A_0 + A_1 \theta^{b-1} + \varepsilon^b$$

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$$X_n' X_n / n \overset{approx.}{=} 0.5\gamma^{-1}(I - A_1)$$
$$\hat{\theta}_n \overset{approx.}{=} (I - A_1)^{-1} A_0$$

without explicitly computing these quantities

- QUESTION: can we transform θ^b into asymptotically valid Bootstrap draws without additional information?

Conclusion

Conclusion

- SNR: simultaneous estimation and Bootstrapping
- Appealing for two-step estimators with complicated variance
- Potential avenues of research:
    - Stochastic quasi-Newton Methods (S-BFGS)

      computationally very attractive
    - Alternative sampling schemes

      look for theoretical guarantees in non-convex settings

THANK YOU!

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Application:
Sensitivity Analyses

Influential Observations

- **Influential observations:**
  a subset $\mathcal{I} \subset \{1, \ldots, n\}$ of the data which impacts the conclusions significantly
- e.g. leads to very different point estimates or standard errors
  - outliers
  - leverage points
  - ...
- Idea: under H_0 (no influential observations) removing $\mathcal{I}$ during the iterations should not significantly affect the Markov-Chain
- Under H_1 (influential observations) should lead to a structural break in the levels/variance

Main Example: PSID Income Dynamics

- Panel Study of Income Dynamics (PSID)
- Moffitt and Zhang (2018) earnings volatility
- $3,508$ males ($36,403$ person-year obs.)
- Model permanent and transitory components:

$$y_{iat} = \alpha_t \mu_{ia} + \beta_t \nu_{ia}$$

$$\mu_{ia} = \mu_{i0} + \sum_{s=1}^{a} \omega_{is}$$

$$\nu_{ia} = \varepsilon_{ia} + \sum_{s=1}^{a-1} \psi_{a,a-s} \varepsilon_{is}, \quad a \geq 2$$

- $a = \text{age} \in [24, 54]$

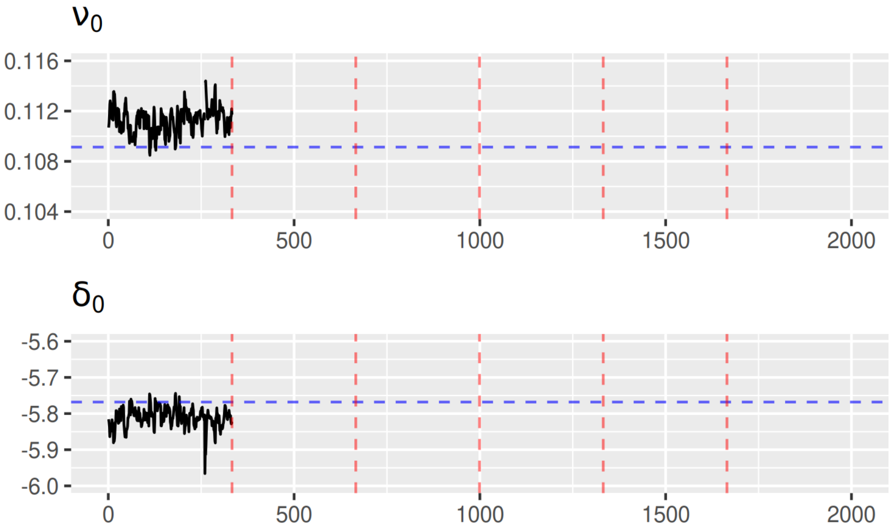
Empirics

- De-trend the data using OLS with polynomial regressors
- Aggregate residual autocovariances by age-group
- Match sample with model-based autocovariance matrix
- Warning:
  - original paper estimates 11 variance parameters
  - we only estimate 4 because of identification issues

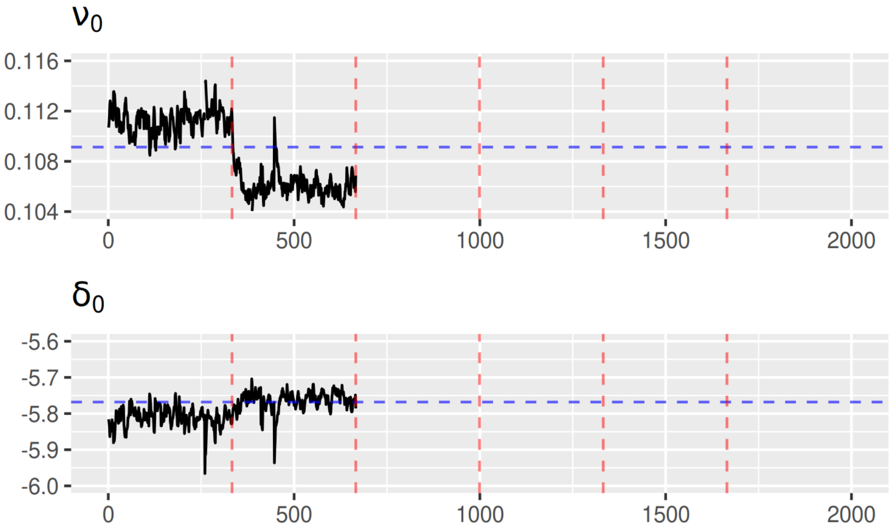
$$\begin{aligned} var(\mu_{i,0}) &: \nu_0 \\ var(\omega_{ir}) &: \delta_0, \cancel{\delta_1} \\ var(\varepsilon_{ir}) &: \gamma_0, \gamma_1, \cancel{k} \\ \psi_{a,a-r} &: \cancel{\pi}, \cancel{\lambda_1}, \cancel{\eta_1}, \cancel{\eta_2}, \cancel{\eta_3} \end{aligned}$$

- **Goal: Are the results sensitive to particular age groups?**

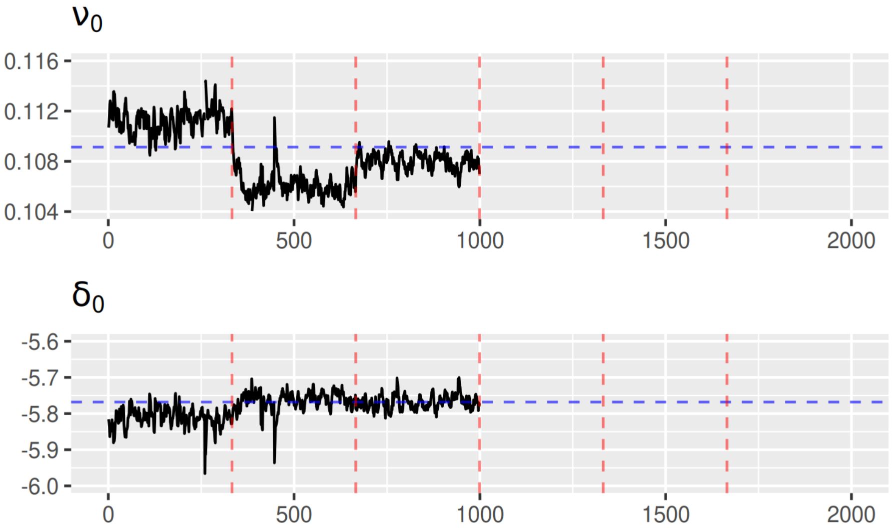
Influence: estimates without 24-28 age group



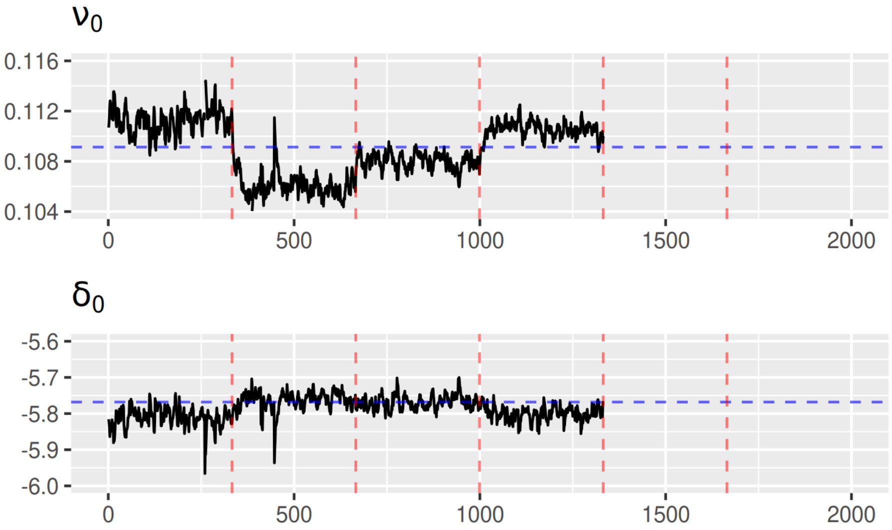
Influence: estimates without 29-33 age group



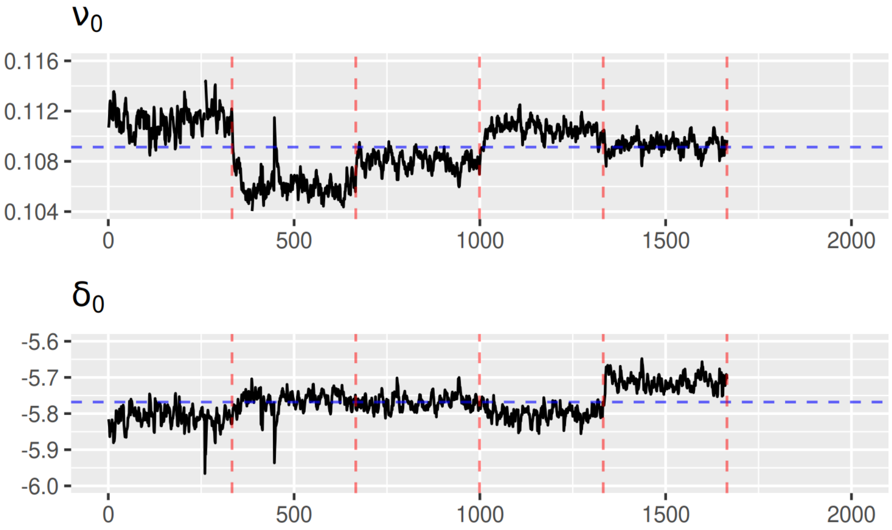
Influence: estimates without 34-38 age group



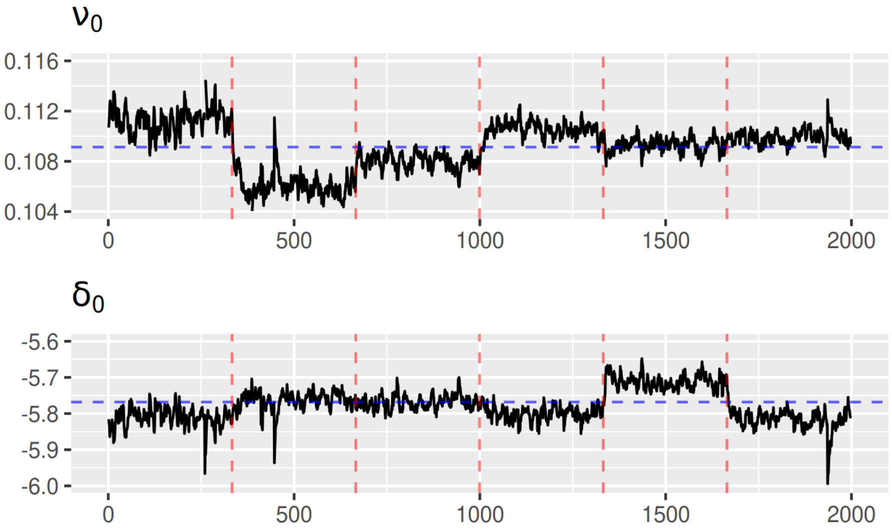
Influence: estimates without 39-43 age group



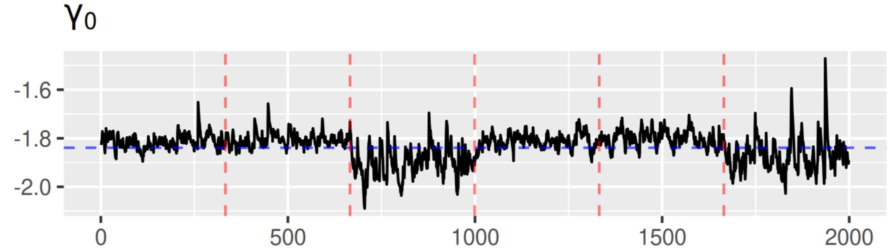
Influence: estimates without 44-48 age group

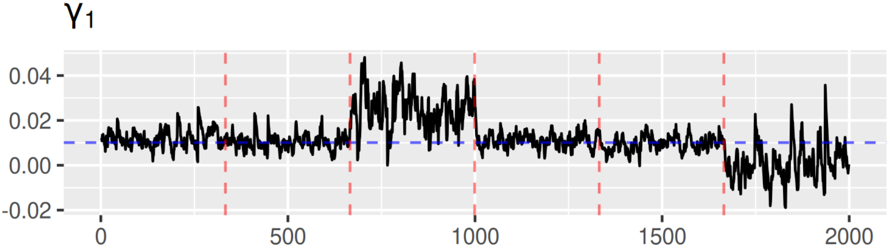


Influence: estimates without 49-54 age group



Influence of individual age-groups



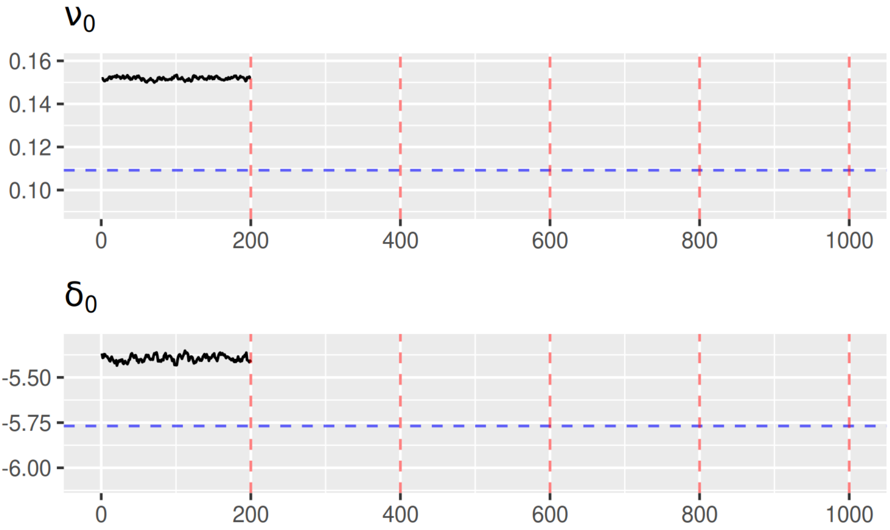


Trimming

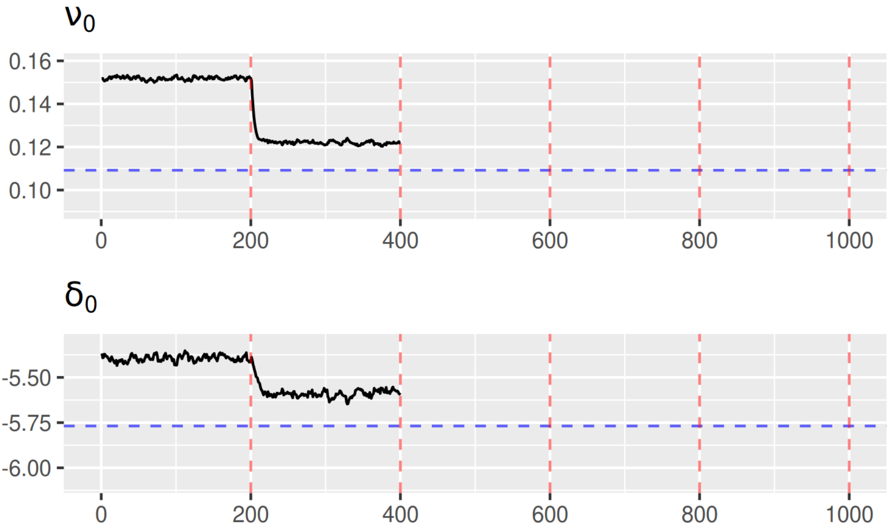
PSID Income Dynamics and Outlier Trimming

- Common empirical practice: remove extreme observations
- Here: the authors of the original paper trimmed the top and bottom 1% observations in each age-time group
  - are the results sensitive to the level of trimming?
- We look at a range of trimming levels: 0, 0.5, 1, 1.5 and 2%

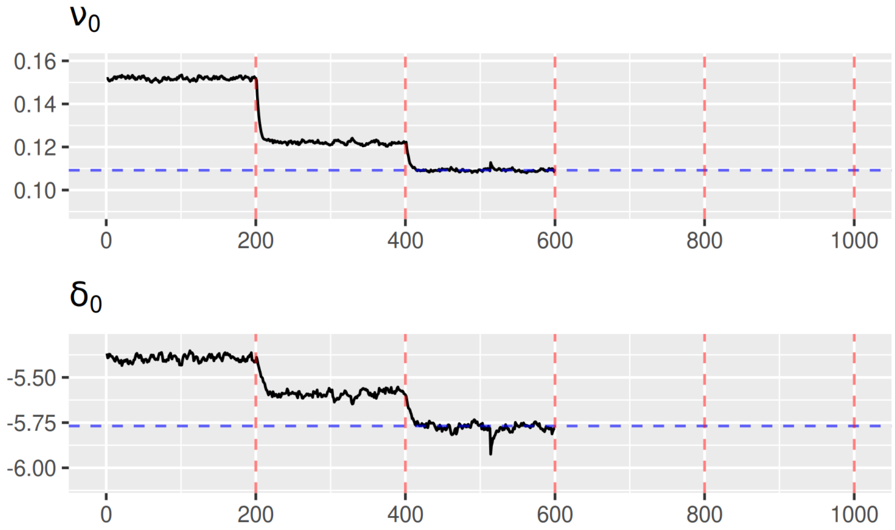
Influence of trimming: no trimming



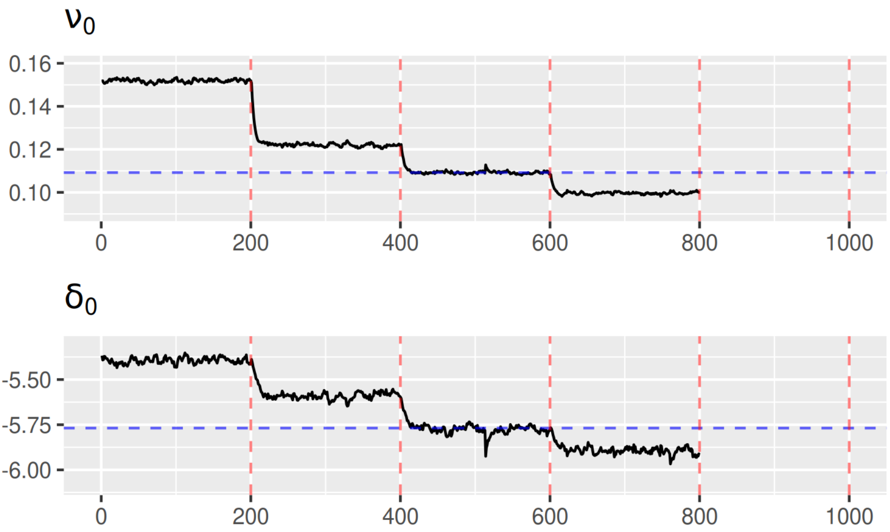
Influence of trimming: trim 0.5%



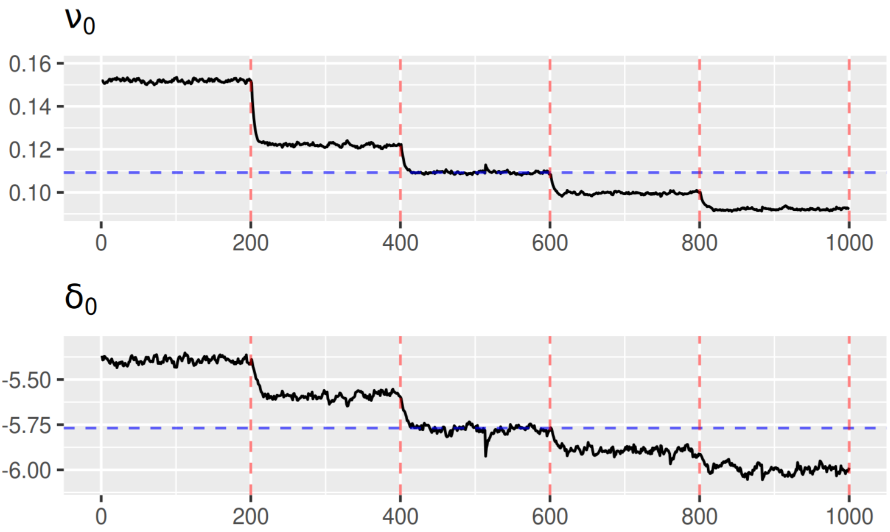
Influence of trimming: trim 1% (baseline)



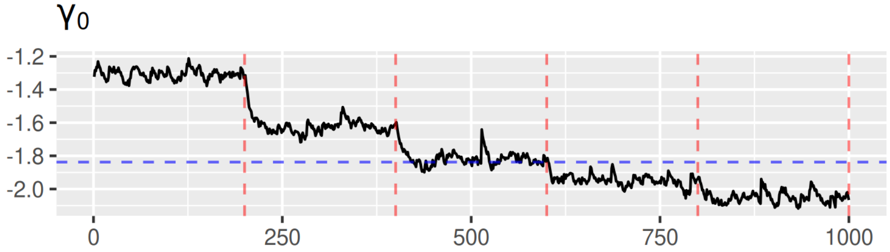
Influence of trimming: trim 1.5%

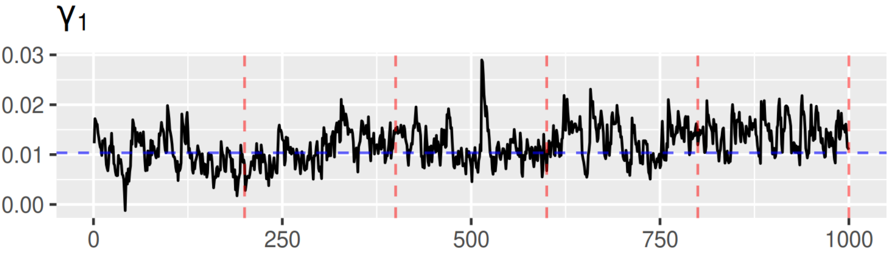


Influence of trimming: trim 2%



Influence of trimming





Age Groups

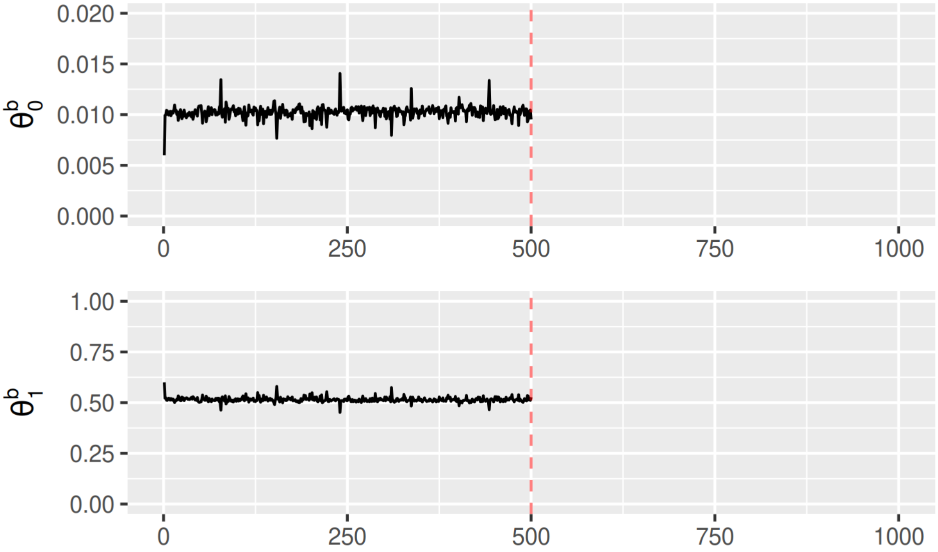
Simple Example: IBES

- Institutional Brokers' Estimate System (IBES)
- Large database of analyst earnings estimates vs. realized
- Predictive regression:

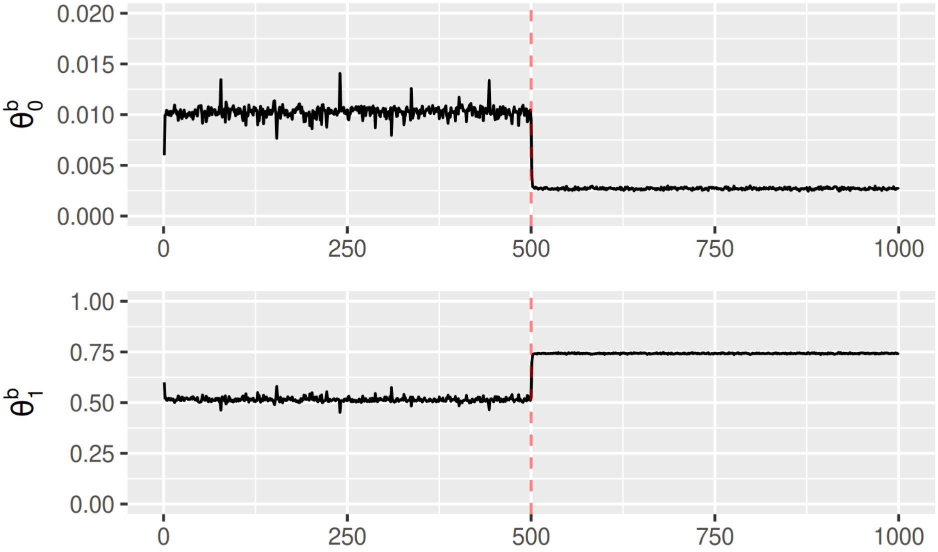
$$y_i^{\text{actual}} = \theta_0 + \theta_1 y_i^{\text{medest}} + e_i$$

- medest = median estimate
- Rational expectations: $\theta_0 \simeq 0, \theta_1 \simeq 1$
- Experiment: remove the 2% most influential obs. & compare
- $n = 9,278$ firms, $t = 01/1985 - 12/2017$
- $\gamma = 0.8$; $m = 6,000$ (re-sample firms)

Simple Example: IBES - Full Sample



Simple Example: IBES - Without Outliers



Simple Illustration: OLS estimation by Newton-Raphson

- OLS regression: $y_i = x_i'\theta + u_i$
- Sample objective: $Q_n(\theta) = \sum_{i=1}^n (y_i - x_i'\theta)^2/n$, $G_n(\theta) = -2\sum_i x_i(y_i - x_i'\theta)/n$, $H_n(\theta) = -2\sum_i x_i x_i'/n$
- **Newton-Raphson Iterations**:

$$\begin{aligned}\theta^b &= \theta^{b-1} + \gamma_b(\sum_i x_i x_i')^{-1}[\sum_i (x_i y_i - \theta^{b-1} x_i x_i')] \\ &= (1-\gamma_b)\theta^{b-1} + \gamma_b \hat{\theta}_n\end{aligned}$$

- For $\gamma_b = 1$ convergence after one iteration
- For $\gamma_b = \gamma \in (0,1]$ fixed, the error $\theta^b - \hat{\theta}_n$ is:

$$\theta^b - \hat{\theta}_n = (1-\gamma)^b[\theta^0 - \hat{\theta}_n]$$

Simple Illustration: OLS estimation by Gradient-Descent

- **Gradient Descent Iterations**:

$$\begin{aligned}\theta^b &= \theta^{b-1} + 2\gamma_b \sum_i x_i[y_i - x_i'\theta^{b-1}]/n \\ &= (I - 2\gamma_b \sum_i x_i x_i'/n)\theta^{b-1} + 2\gamma_b[\sum_i x_i x_i'/n]\hat{\theta}_n\end{aligned}$$

- Re-write the error $\theta^b - \hat{\theta}_n$ as:

$$\theta^b - \hat{\theta}_n = (I - 2\gamma_b[\sum_i x_i x_i'/n])(\theta^{b-1} - \hat{\theta}_n)$$

- For $\gamma_b = \gamma \le \lambda_{\max}(\sum_i x_i x_i'/n)/2$ fixed, the error $\theta^b - \hat{\theta}_n$ is:

$$\theta^b - \hat{\theta}_n = (I - 2\gamma[\sum_i x_i x_i'/n])^b[\theta^0 - \hat{\theta}_n]$$

- Convergence after one iteration in one direction if $\gamma = \lambda_{\max}(\sum_i x_i x_i'/n)/2$

Issues with Mini-Batch Stochastic Newton-Raphson

- Deterministic case: $\theta^b_{NR} \to \hat{\theta}_n$ faster than $\theta^b_{GD} \to \hat{\theta}_n$
- Why is S-GD more popular than S-NR?
  - need to compute $[H_1^{(b)}(\theta)]^{-1}$ often (near)-singular
  - e.g. $x_i = (1, x_{i,1})$, $x_{i,1} \sim Bernoulli(p)$
    $\Rightarrow x_i x_i'$ singular wp. 1 for any $p \in [0, 1]$

$\Rightarrow$ mini-batch S-NR can be infeasible/unstable

- some solutions:
  - use more observations for H (Byrd et al., 2016; Li et al., 2018)
  - use accumulated gradient for scaling: adagrad, RMSprop,...