Detecting Identification Failure in Moment Condition Models

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Introduction: Identification in GMM

- Interested in moment condition models: GMM, MD, SMM,...

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$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \mathcal{N}(0, V)$$

where $\hat{\theta}_n = \text{argmin}_{\theta \in \Theta} \|\bar{g}_n(\theta)\|_W$, $W = W_n(\theta)$ weight matrix

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- Important assumption: global identification

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- However, typically non-Gaussian under:
  - i. **set/weak identification**: $\nexists \delta(\varepsilon) > 0$ or $\delta(\varepsilon) \asymp n^{-1/2}$
  - ii. **local identification failure**: $\partial_\theta g_n(\theta_0)$ (close to) singular

Dealing with Identification Failure

- Requires identification robust inference:

  e.g. Anderson and Rubin (1949); Stock and Wright (2000); Moreira (2003); Kleibergen (2005); Andrews and Cheng (2012); Andrews and Mikusheva (2016); Chen et al. (2018),...

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- Typically for the full vector θ; subvector inference:
  - i. *Projection* (Dufour and Taamouti, 2005)
  - ii. *Bonferroni* (McCloskey, 2017)
  - $\Rightarrow$ Conservative
  - $\nearrow$ power: concentrate out identified nuisance parameters

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  $\nearrow$ power: concentrate out identified nuisance parameters
- **This paper:** is about answering two questions:
  i. is θ strongly globally identified?
  ii. which components of θ are not identified?

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  ii. A two-step procedure for robust subvector inference

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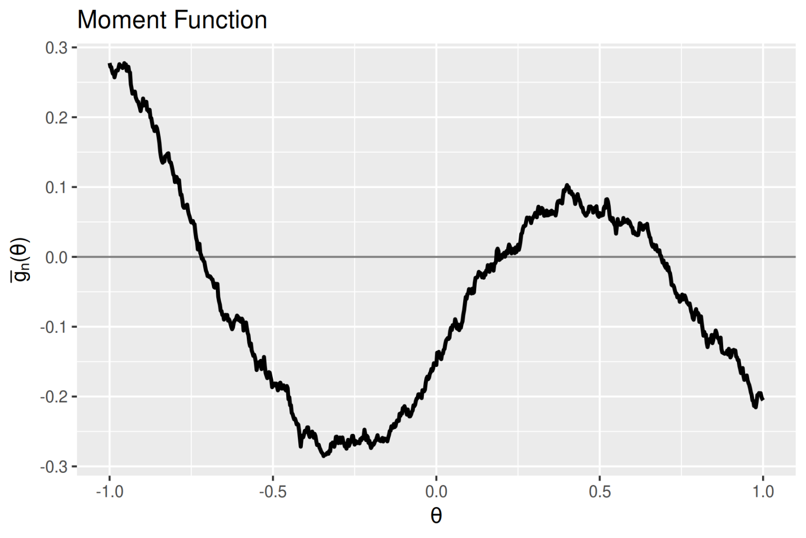
  i. A generic approach to detecting both weak/set identification and local identification failure
  ii. A two-step procedure for robust subvector inference
- Introduce a quasi-Jacobian matrix:
  - Jacobian $\simeq$ local linear approx. in a $\|\cdot\|$ neighborhood of θ_0
    - informative about *local identification only*
  - quasi-Jacobian local linear approx. in a $\|g_n(\cdot)\|_W$ neighborhood
    - informative about *local + global identification*

The quasi-Jacobian: Intuition



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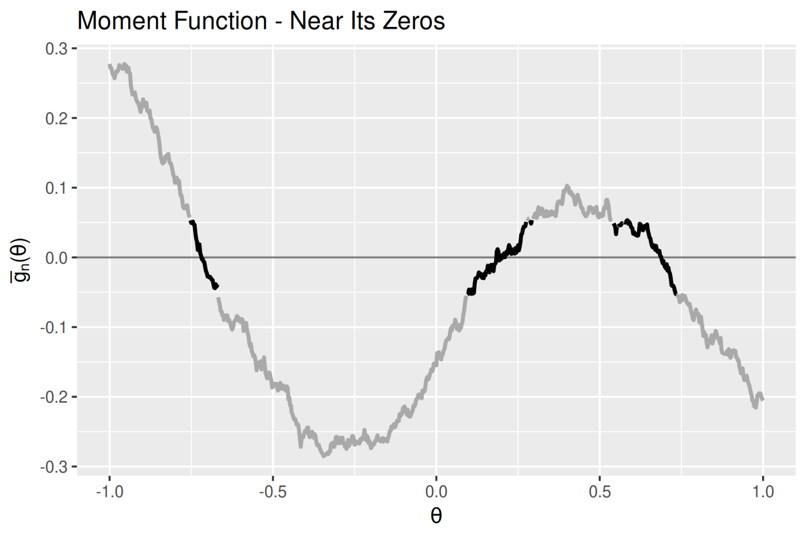

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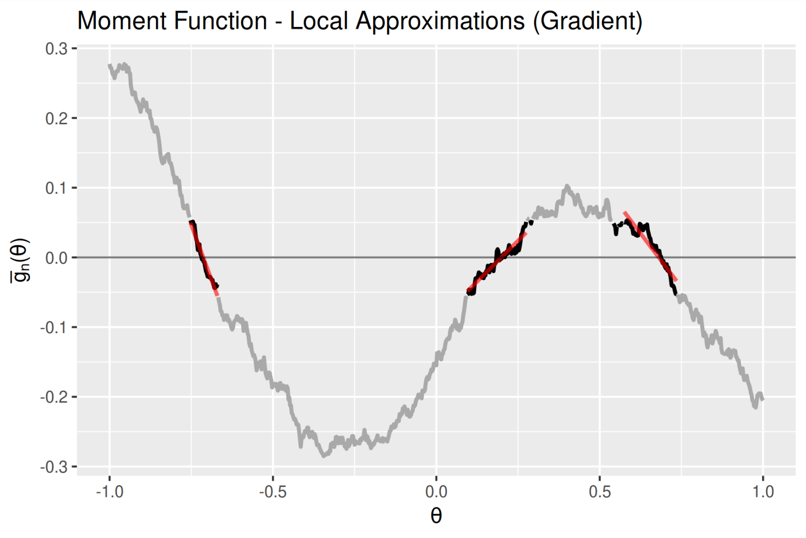

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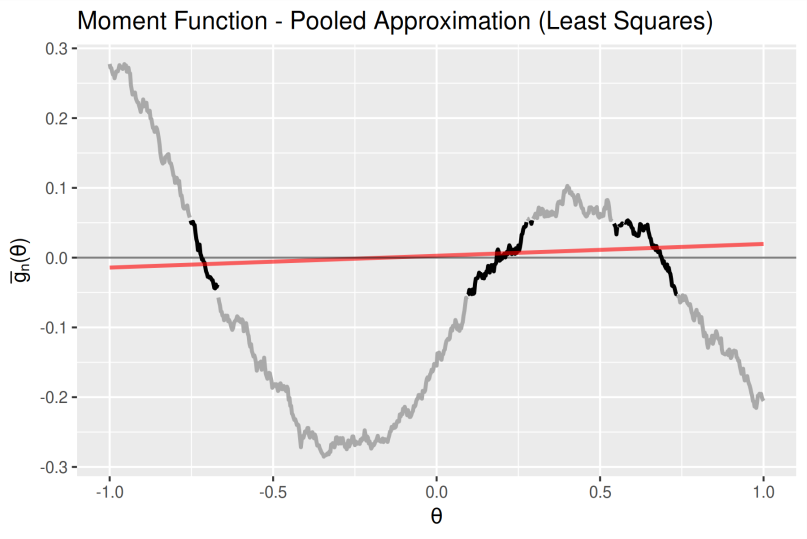


How to Build the Approximation in Practice

- Two estimators

  i. **Sup-norm (∞):** $(A_{n,\infty}, B_{n,\infty})$

$$\operatorname{argmin}_{A,B}\left\{\sup_{\theta\in\Theta}\left[\|\bar{g}_n(\theta)-A-B\theta\|\times K\left(\left\|\frac{\bar{g}_n(\theta)}{\kappa_n}\right\|_W\right)\right]\right\}$$

  ii. **Least-squares (LS):** $(A_{n,LS}, B_{n,LS})$

$$\operatorname{argmin}_{A,B}\int_\Theta\left[\|\bar{g}_n(\theta)-A-B\theta\|^2\times K\left(\left\|\frac{\bar{g}_n(\theta)}{\kappa_n}\right\|_W-\left\|\frac{\bar{g}_n(\hat{\theta}_n)}{\kappa_n}\right\|_W\right)\right]d\theta$$

- where

  i. K is a kernel, either:

     - Lipschitz-continuous, strictly positive on the support $[-1,1]$
     - Exponential $K(x)=C_1\exp(-C_2|x|^a)$, $C_1, C_2, a>0$ (LS only)

  ii. κ_n is a bandwidth

     - $\sqrt{n}\kappa_n\to\infty, \kappa_n^2=o(n^{-1/2})$ (e.g. $\kappa_n=\sqrt{2\log\log(n)/n}$)
     - $\tilde{\kappa}_n=\kappa_n\log(n)^{1/a}$ $\sqrt{n}\tilde{\kappa}_n\to\infty, \tilde{\kappa}_n^2=o(n^{-1/2})$ (exp. kernel)

Remarks

- **Sup-Norm** $(A_{n,\infty}, B_{n,\infty})$**:**
    i. strong theoretical predictions about $B_{n,\infty}$
    ii. convex optimization problem but more challenging to compute
- **Least-squares** $(A_{n,LS}, B_{n,LS})$**:**
    i. very easy to compute:
    $(A_{n,LS}, B'_{n,LS}) = \left(\int_\Theta X(\theta)X(\theta)'\hat{K}_n(\theta)d\theta\right)^{-1}\int_\Theta X(\theta)\bar{g}_n(\theta)'\hat{K}_n(\theta)d\theta$
    where $\hat{K}_n(\theta) = K\left(\|\bar{g}_n(\theta)\|_W/\kappa_n - \|\bar{g}_n(\hat{\theta}_n)\|_W/\kappa_n\right), X(\theta) = (1, \theta')$
    ii. theoretical predictions depend on the topology of Θ_0

What happens under weak or set identification?

$\Rightarrow$ $\text{plim}_{n\to\infty} B_n$ singular and

i. **eigenvalues** (rank) of B_n informative about identifiability of θ

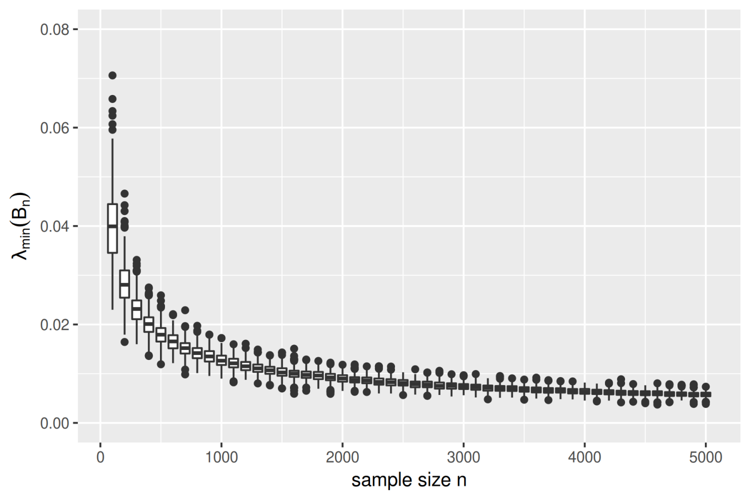
ii. **eigenvectors** informative about span of identification failure

- Remark: $g_n(\theta)$ linear $\Rightarrow$ approximation is exact - IV:

$$A_n = Z'y/n, B_n = -Z'X/n$$

- Stock and Yogo (2005) use Cragg-Donald rank test

Illustration: Smallest Eigenvalue and Sample Size



Boxplot: $\lambda_{\min}(B_n)$.

Setting: $y_i = \theta_1 x_{1,i} + \theta_1\theta_2 x_{2,i} + e_i$, $\theta_1 = 2/\sqrt{n}$, 500 replications

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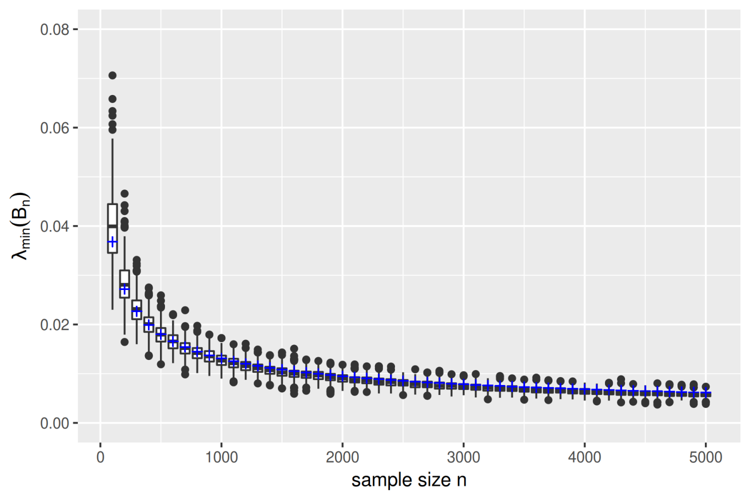


Boxplot: $\lambda_{\min}(B_n)$. Blue Cross: fitted predicted rate

Setting: $y_i = \theta_1 x_{1,i} + \theta_1\theta_2 x_{2,i} + e_i$, $\theta_1 = 2/\sqrt{n}$, 500 replications

Related Literatures (non-exhaustive)

- For an identification safari: Lewbel (2018)
- **Local/global identification in the population**:
  - Koopmans and Reiersol (1950); Fisher (1967); Rothenberg (1971); Brown (1983); Komunjer (2012), ...
  - hard to check for many models (e.g. SMM/Indirect Inference)
- **Detecting identification failure in finite samples**:
  - Stock and Yogo (2005); Olea and Pflueger (2013); Wright (2003); Inoue and Rossi (2011); Arellano et al. (2012); Bravo et al. (2012); Antoine and Renault (2017), ...
- **Distribution under weak identification**:
  - Nelson and Startz (1990); Choi and Phillips (1992); Dufour (1997); Staiger and Stock (1997); Stock and Wright (2000)...
- **Identification robust inference**: (prev. slide)

Definitions and Main Assumptions

Identification Regimes

Definition (Identification Regimes - $g_n(\theta) = \mathbb{E}(\bar{g}_n(\theta))$)

1. **strong identification**
   i. $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 : \inf_{\|\theta-\theta_0\|>\varepsilon} \|g_n(\theta)\|_W \geq \delta(\varepsilon)$
   ii. $\exists \varepsilon > 0$ and $\underline{c} > 0$, $\|\theta - \theta_0\| \leq \varepsilon \Rightarrow \|g_n(\theta)\|_W \geq \underline{c}\|\theta - \theta_0\|$
2. **semi-strong identification**
   i. 1.i. holds;
   ii. $n \times \lambda_{\min}(\partial_\theta g_n(\theta_0)'\partial_\theta g_n(\theta_0)) \to \infty$; cond. on hod $+ \exists \varepsilon > 0$, $\|\theta - \theta_0\| \leq \varepsilon \Rightarrow \|g_n(\theta)\|_W \asymp \|\partial_\theta g_n(\theta_0)[\theta - \theta_0]\|$
3. **higher-order local identification**
   i. 1.i. holds
   ii. $\exists \varepsilon > 0$, $P_1, \dots, P_r$ projection matrices with $P_r \neq 0$, $\|\theta - \theta_0\| \leq \varepsilon \Rightarrow \|g_n(\theta)\|_W \asymp \sum_{j=1}^r \|P_j(\theta - \theta_0)\|^j$
4. **weak/set identification**:
   i. $\exists \theta_0 \neq \theta_1 \in \Theta_0 = \{\theta, \lim_{n\to\infty} \sqrt{n}\|g_n(\theta)\|_W < \infty\}$

Identification Regimes and Asymptotic Properties of $\hat{\theta}_n$

Identification Regime	$\hat{\theta}_n$ consistent?	Rate of convergence	Limiting distribution
Strong	Yes	$\sqrt{n}$	Gaussian
Semi-Strong	Yes	slower than $\sqrt{n}$	Gaussian
Higher-Order	Yes	$n^{1/4}$ or slower	non-Gaussian
Weak or Set	No	-	non-Gaussian

- Goal: characterize the behaviour of $B_{n,\infty}$, $B_{n,LS}$ in each regime
- assume $\exists \theta_0$ st. $\|g_n(\theta_0)\|_W = 0$
- for (semi)-strong, assume $\theta_0 \in int(\Theta)$ - boundary not covered

References

Simple Examples

1. (Cheng, 2015) **Nonlinear Least-Squares:**

$$y_t = \theta_{1,n} x_{1,t} + \theta_{1,n}\theta_2 x_{2,t} + e_t$$

   - semi/strong $\sqrt{n} \times |\theta_{1,n}| \to +\infty$
   - weak/set $\sqrt{n} \times |\theta_{1,n}| = O(1)$

Some Details

2. (Gospodinov and Ng, 2015) **Possibly non-invertible MA(1):**

$$y_t = \sigma[e_t - \vartheta e_{t-1}], \quad e_t \sim (0, 1, \tau_n) \quad \tau_n = \text{skewness}$$

   - strong $\sqrt{n} \times |\tau_n| \to \infty$; $\vartheta \in \mathbb{R}/\{-1, 0, 1\}$
   - weak/set $\sqrt{n} \times |\tau_n| = O(1)$; $\vartheta \in \mathbb{R}/\{-1, 0, 1\}$
   - second-order $\vartheta \in \{-1, 1\}; \tau \in \mathbb{R}$

Assumptions on $\bar{g}_n$ (Common)

Assumption (Common)

1. **Uniform CLT**
2. **Stochastic Equicontinuity**

+ conditions on weighting matrix $W_n(\theta)$ (invertible, ...)

allows for:

1. non-smooth, discontinuous moments
   e.g. quantile IV, SMM with indicator function, etc.
2. profile moments $\bar{g}_n\big[\theta, \hat{\gamma}(\theta)\big]$ - $\hat{\gamma}(\theta)$ (semi)-strongly identified

Preview: Asymptotic Properties of $\hat{\theta}_n$ and $B_{n,LS/\infty}$

Identification Regime	Asymptotics for $\hat{\theta}_n$	Asymptotics for $B_{n,LS/\infty}$
Strong	Gaussian	$B_{n,LS/\infty} \simeq$ Jacobian
Semi-Strong	Gaussian	$B_{n,LS/\infty} \simeq$ Jacobian
Higher-Order	Non-Gaussian	$B_{n,LS/\infty} v_j \asymp \text{bandwidth}^{1-1/j}$
Weak or Set	Non-Gaussian	$B_{n,LS/\infty} v \asymp$ bandwidth

- for all directions:
    - v_j in which $\|g_n(\theta_0 + v_j)\| \asymp \|v_j\|^j$
    - $v = \theta_0 - \theta_1$ with $\theta_0, \theta_1 \in \Theta_0$, weakly identified set

Asymptotic Behaviour of $B_{n,LS/\infty}$

$A_{n,LS/\infty}, B_{n,LS/\infty}$ - (Semi)-Strong Identification

Theorem (Semi-Strong Identification)

Suppose the model is (semi)-strongly identified, compact kernel conditions + assumptions above hold and

$$\kappa_n^2 = o\Big[\lambda_{\min}\Big(\partial_\theta g_n(\theta_0)'\partial_\theta g_n(\theta_0)\Big)\Big]$$

then:

i. $A_{n,LS/\infty} = \bar{g}_n(\theta_0) - B_{n,LS/\infty}\theta_0 + o_p(n^{-1/2})$

ii. $B_{n,LS/\infty}H_n = \partial_\theta g_n(\theta_0)H_n + o_p(n^{-1/2}\kappa_n^{-1})$

where $H_n = [\partial_\theta g_n(\theta_0)'\partial_\theta g_n(\theta_0)]^{-1/2}$

Implication: the estimator based on $A_{n,LS/\infty}, B_{n,LS/\infty}$

$$H_n^{-1}[\hat{\theta}_{n,LS/\infty} - \hat{\theta}_{n,GMM}] = o_p(n^{-1/2})$$

$B_{n,LS/\infty}$ is a smoothed estimate of the Jacobian $\partial_\theta g_n(\theta_0)$

Theorem (Weak and Set Identification)

Suppose the model is weakly and/or set identified, compact kernel conditions + assumptions above hold, then:

i. $|\lambda_{\min}(B_{n,\infty})| = O_p(\kappa_n)$

ii. $\forall v \in V = Span(\{\theta_0 - \theta_1, \theta_0, \theta_1 \in \Theta_0\})$: $B_{n,\infty} v = O_p(\kappa_n)$

V is the span of the identification failure

Theorem 2: sketch of the proof

- By construction, $\|\bar{g}_n(\theta)\|\hat{K}_n(\theta) \leq \|K\|_\infty \kappa_n / \lambda_{\min}(W)$

$\Rightarrow$ By minimization, we have

$$\sup_{\theta\in\Theta}\left[\|A_{n,\infty} + B_{n,\infty}\theta - \bar{g}_n(\theta)\|_W \hat{K}_n(\theta)\right] \leq \sup_{\theta\in\Theta}\left[\|\bar{g}_n(\theta)\|_W \hat{K}_n(\theta)\right] \leq O(\kappa_n)$$

$+$ reverse triangle inequality:

$$O(\kappa_n) \geq \|A_{n,\infty} + B_{n,\infty}\theta - \bar{g}_n(\theta)\|\hat{K}_n(\theta) \geq \|A_{n,\infty} + B_{n,\infty}\theta\|\hat{K}_n(\theta) - O(\kappa_n)$$
$$\Rightarrow O(\kappa_n) \geq \|A_{n,\infty} + B_{n,\infty}\theta\|\hat{K}_n(\theta) \geq 0$$

- Also, wp $\nearrow 1$, both $\hat{K}_n(\theta_0)$ and $\hat{K}_n(\theta_1) \geq \underline{K} > 0$
- Apply inequality above for θ_0, then for θ_1 and we get:

$$0 \leq \|B_{n,\infty}(\theta_1 - \theta_0)\| \leq O_p(\kappa_n)$$

- For any pair $(\theta_0, \theta_1) \in \Theta_0$, the results follow

$A_{n,\infty}, B_{n,\infty}$ - Higher-Order Local Identification

Theorem (Higher-Order Local Identification)

Suppose the model is locally higher-order identified at order $r \geq 2$, compact kernel conditions + assumptions above hold, then:

i. $|\lambda_{\min}(B_{n,\infty})| = O_p(\kappa_n^{1-1/r})$

ii. $\forall v_j \in Span(P_j)$: $B_{n,\infty} v_j = O_p(\kappa_n^{1-1/j})$

Recall: P_j is the direction in which $\|g_n\|_W$ goes to 0 no faster than a polynomial of order j

Least-Squares Approximation: Notation

- Let $\hat{\pi}_n$ be the density implied by $\hat{K}_n$:

$$\hat{\pi}_n(\theta) = \frac{K(\|\bar{g}_n(\theta)\|_W/\kappa_n - \|\bar{g}_n(\hat{\theta}_n)\|_W/\kappa_n)}{\int_\Theta K(\|\bar{g}_n(\theta)\|_W/\kappa_n - \|\bar{g}_n(\hat{\theta}_n)\|_W/\kappa_n)d\theta}$$

- K = Gaussian density, $\kappa_n = n^{-1/2}$: quasi-Bayesian estimation (Chernozhukov and Hong, 2003; Creel et al., 2015)
- Quasi-posterior mean, variance:

$$\bar{\theta}_n = \int_\Theta \theta \hat{\pi}_n(\theta) d\theta, \quad \Sigma_n = \int_\Theta (\theta - \bar{\theta}_n)(\theta - \bar{\theta}_n)' \hat{\pi}_n(\theta) d\theta$$

- Moon and Schorfheide (2012); Chen et al. (2018): posterior concentrates on the identified set

$B_{n,LS}$ and the Quasi-Posterior Variance

Lemma (Relation between $B_{n,LS}$ and Σ_n)

Under any identification regime, compact/exponential kernel conditions + assumptions above hold + technical cond. for exponential kernel, then:

i. $\mathit{trace}\left(B_{n,LS}\Sigma_n B'_{n,LS}\right) = O_p(\tilde{\kappa}_n^2)$

ii. $\lambda_{\min}\left(B'_{n,LS}B_{n,LS}\right)\lambda_{\max}(\Sigma_n) = O_p(\tilde{\kappa}_n^2)$

where $\tilde{\kappa}_n = \kappa_n$ for compact kernel and $\kappa_n \log(n)^{1/a}$ for exp. kernel

Lemma 1: sketch of the proof

- Least-squares formula: $A_{n,LS} = \int \bar{g}_n(\theta)\hat{\pi}_n(\theta)d\theta - B_{n,LS}\bar{\theta}_n$
- Objective becomes

$$\int \|B_{n,LS}(\theta - \bar{\theta}_n) - [\bar{g}_n(\theta) - \int \bar{g}_n(\tilde{\theta})\hat{\pi}_n(\tilde{\theta})d\tilde{\theta}]\|^2 \hat{\pi}_n(\theta)d\theta \leq O_p(\tilde{\kappa}_n^2)$$

- Similar strategy as before implies:

$$\int \|B_{n,LS}(\theta - \bar{\theta}_n)\|^2 \hat{\pi}_n(\theta)d\theta \leq O_p(\tilde{\kappa}_n^2)$$

- By definition of the Frobenius norm, it implies:

$$\underbrace{\int \text{trace}\left(B_{n,LS}(\theta - \bar{\theta}_n)(\theta - \bar{\theta}_n)' B'_{n,LS}\right) \hat{\pi}_n(\theta)d\theta}_{=\text{trace}(B_{n,LS}\Sigma_n B'_{n,LS})} \leq O_p(\tilde{\kappa}_n^2)$$

- Which implies the results

Proposition (Weak and Set Identification)

Suppose the model is weakly and/or set identified, compact/exponential kernel conditions + assumptions above hold and $\exists \theta_0 \neq \theta_1 \in \Theta_0$ *with:*

a. $0 < \varepsilon \leq \|\theta_0 - \theta_1\|$
b. $\exists \eta > 0$, *for* $j \in \{0, 1\}$*:*

$$\hat{\pi}_n \left(\mathcal{B}_{\varepsilon/3}\left(\theta_j\right)\right) \geq \eta + o_p(1)$$

then:

i. $\lambda_{\max}(\Sigma_n) \geq \eta\varepsilon^2/[36 d_\theta] + o_p(1)$
ii. $\lambda_{\min}(B_{n,LS}) \leq O_p(\tilde{\kappa}_n)$

$A_{n,LS}, B_{n,LS}$ - Anatomy of an Identification Failure

Theorem (Topology of the Weakly Identified Set)

Suppose one the following holds

a. $int(\Theta_0) \neq \emptyset$ *(omni-directional failure)*
b. $\Theta_0 = \cup_{j=1}^{k}\{\theta_j\}$ + *same local behaviour*
c. $\Theta_0 = \cup_{j=1}^{k} S_j$, k_j *dimensional manifolds* + *local behaviour*

then the previous Theorem holds

Remark: sets S_j with largest k_j dominate the (quasi)-posterior

Corollary (Global Re-Parameterization)

Suppose that $\theta = \varphi(\alpha, \gamma)$; $int(\mathcal{A}_0) \neq \emptyset$ *($\cup$ manifolds)*, $\Gamma_0 = \{\gamma_0\}$ + *conds. on* φ, *local behaviour,... then Proposition c. above holds.*

Counter-Example

Detecting Identification Failure and Two-Step Subvector Inference

Subvector Inference: General Idea

- Focus on weak/set vs. (semi)-strong identification
- Linear hypothesis:

$$H_0 : \; R\theta_0 = c \text{ vs. } R\theta_0 \neq c$$

- Main idea $\theta = (\theta_1, \theta_2)$
  - θ_1 weak/set/higher-order identified: needs to be fixed
  - θ_2 (semi)-strongly identified, estimable for θ_1 fixed
- Simple case: $\text{span}(R) = \text{span}(P_{\theta_1})$, i.e.

$$\Theta_0 \cap \{\theta \in \Theta, \; R\theta = c\} = \{\theta_{0,c}\} \text{ singleton}$$

  or empty depending on c
- If not, can add restrictions $\tilde{R}$ until θ is point identified
- We'll use this to do **two-step inference**

Two-Step Subvector Inference: Second Step

- Suppose θ weak/set identified on Θ, (semi)-strongly identified on $\Theta \cap \{\theta \in \Theta, R\theta = c, \tilde{R}\theta = \tilde{c}\}$ for each $\tilde{c}$
- Projection Inference:
  i. construct $\tilde{CS}_{1-\alpha}$ for $(R', \tilde{R}')'\theta$ assuming the remaining coefficients are (semi)-strongly identified
  ii. the confidence set for $R\theta$ collects all values of $R\theta$ in $\tilde{CS}_{1-\alpha}$
- Remarks:
  - a lower rank for $\tilde{R} \Rightarrow$ less conservative, more power
  - full projection when $\text{rank}(R', \tilde{R}') = d_\theta$

First Step: Collapsing the Identified Set into a Singleton

- Consider a deterministic sequence of constraint matrices

$$R_1 = R, R_2 = (R_1', \tilde{R}_2')', \dots, R_{\mathcal{L}} = (R_{\mathcal{L}-1}', \tilde{R}_{\mathcal{L}}')'$$
$$1 \leq \text{rank}(R_1) < \cdots < \text{rank}(R_{\mathcal{L}}) = d_\theta$$

- By construction $\exists \ell^\star \leq \mathcal{L}$ (smallest) such that $\forall \ell \geq \ell^\star$:

$$\Theta_0 \cap \{\theta \in \Theta,\ R_\ell \theta = c_\ell\}$$

  is either a singleton or the empty set depending only on c_ℓ
- Assume remaining parameters are (semi)-strongly identified
  - we could re-compute $B_{n,LS}$ with the restrictions to check
- Want an algorithm that finds $\hat{\ell}_n \geq \ell^\star$ wp $\nearrow 1$

Which parameters to fix?

Lemma (Collapsing the Weakly Identified Set)

Let $\underline{\lambda}_n > 0$ st. $\kappa_n = o(\underline{\lambda}_n)$; suppose B_n is a $O_p(\kappa_n)$ on V. If we use $\underline{\lambda}_n$ as a cutoff to pick $\hat{\ell}_n$ st:

i. $\hat{d}_V = \#\{j \leq d_\theta, \lambda_j(B_n) \leq \underline{\lambda}_n\}$

ii. $rank(R_{\hat{\ell}_n}) \geq \hat{d}_V$, $\#\{j \leq d_\theta - rank(R_{\hat{\ell}_n}), \lambda_j(B_n P^{\perp}_{R_{\hat{\ell}_n}}) \leq \underline{\lambda}_n\} = 0$

then, wp $\nearrow 1$, $rank(P_{R_{\hat{\ell}_n}} P_V) = rank(P_V)$, i.e. $\hat{\ell}_n \geq \ell^\star$

Remarks:

- rule-of-thumb for $\underline{\lambda}_n$ in a few slides
- prev. results $\Rightarrow \#\{j \leq d_\theta - \text{rank}(R_{\hat{\ell}_n}), \lambda_j(B_n P^{\perp}_{R_{\hat{\ell}_n}}) \leq \underline{\lambda}_n\} \geq 1$ wp $\nearrow 1$ for each $1 \leq \ell < \ell^\star$
- use a family-wise error rate argument for the group $1 \leq \ell < \ell^\star$

Implications for Subvector Inference

Theorem (Two-Step Subvector Inference)

Under weak or set identification: suppose test statistics $S_{\ell,n}$ for $R_\ell\theta = c_\ell$ satisfies

$$\inf_{\ell^\star \leq \ell \leq \mathcal{L}} \mathbb{P}(S_{\ell,n} \leq c_{1-\alpha,\ell}) \geq 1 - \alpha + o(1)$$

then $\hat{\ell}_n \geq \ell^\star$ wp $\nearrow$ 1 implies

$$\mathbb{P}(S_{\hat{\ell}_n,n} \leq c_{1-\alpha,\hat{\ell}_n}) \geq 1 - \alpha + o(1)$$

Under (semi)-strong identification: conditions on eigenvalues & $\underline{\lambda}_n$ imply $\hat{\ell}_n = 1$ wp $\nearrow 1$ and

$$\mathbb{P}(S_{\hat{\ell}_n,n} \leq c_{1-\alpha,\hat{\ell}_n}) = \mathbb{P}(S_{1,n} \leq c_{1-\alpha,1}) + o(1)$$

Remarks

Designing a Cutoff $\underline{\lambda}_n$ - 1/2

- Use a simple asymptotic framework
  - (semi)-strong local asymptotics
  - look at simple linear t-test over all directions
  - bound worst-case size distortion in terms of $\lambda_{\min}(B_n)$
  - function of n, the signal - i.e. $\lambda_{\min}(B_n)$ - and the noise
- A given level of size distortion requires:
  - $\lambda_{\min}(B_n) \leq$ quantities(n,co-variances)
  - use this as a cutoff $\underline{\lambda}_n$ to detect identification failure

Designing a Cutoff $\underline{\lambda}_n$ - 2/2

- Non-local asymptotics (MA model)
  - partition the parameter space into clusters
  - within each cluster use rule of thumb above
  - distance between clusters also implies size distortion
- Higher-order asymptotics
  - check residual curvature
  - non-linearities $\Rightarrow$ size distortion

Details

Monte-Carlo Illustrations

Example 1: NLS

- Simple example:

$$y_t = \theta_1 x_{1,t} + \theta_1 \theta_2 x_{2,t} + e_t$$

- Identification failure $\theta_{1,0} = 0$, weak identification $\theta_{1,0} \simeq 0$
- Two cutoffs $\underline{\lambda_n}$: $\sqrt{\log(n)/n}$, rule-of-thumb
- Null hypothesis:

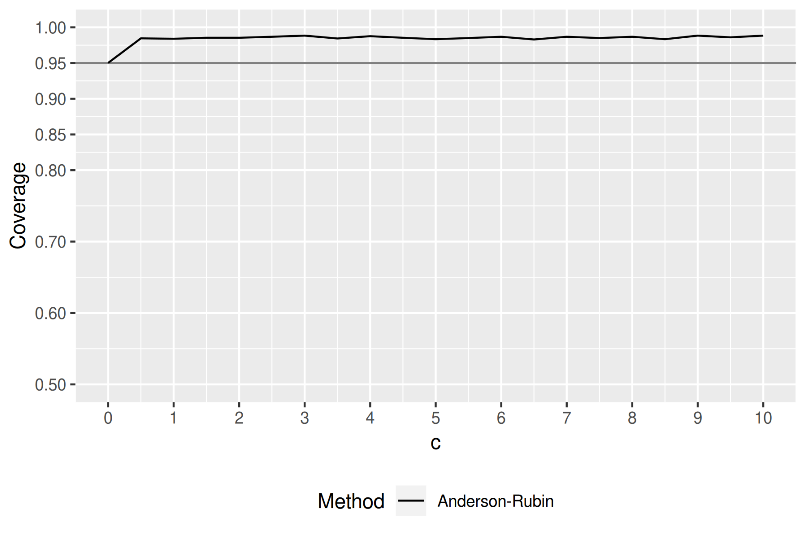
$$H_0 : \theta_1 = \theta_{1,0} = c/\sqrt{n}$$

- Pretend like we don't know the identification structure
- $\lambda_{\min}(B_{n,LS/\infty}) \leq \lambda_n$ suggests weak identification

Example 1: NLS

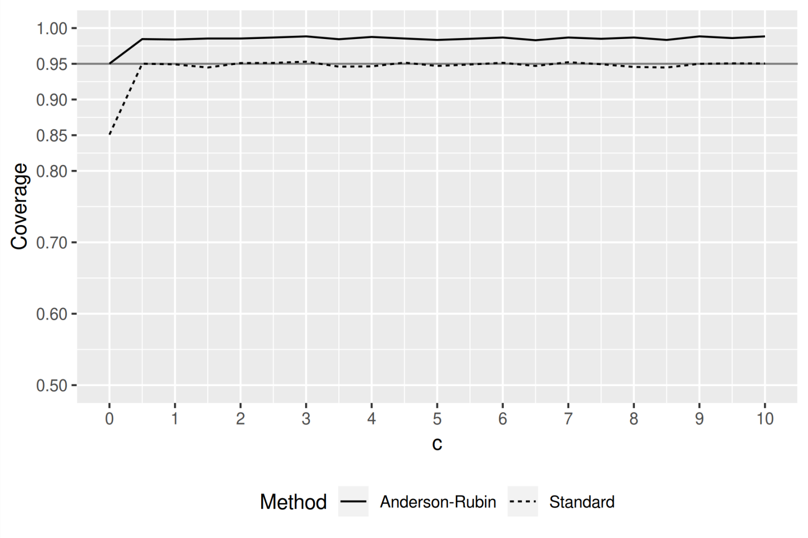
- Use projected S-statistic (two-step/Anderson-Rubin) and Wald/QLR test (standard)
- Main difference between two-step & Anderson-Rubin is critical value: data-driven (χ^2_1 or χ^2_2) vs. fixed (χ^2_2)
- For $H_0 : \theta_1 = 0$, projection inference is not conservative; it has exact asymptotic coverage
  - AR/S-statistic does not depend on $\theta_2 \Rightarrow \chi^2_2$ distribution

Identification Robust Projection Inference



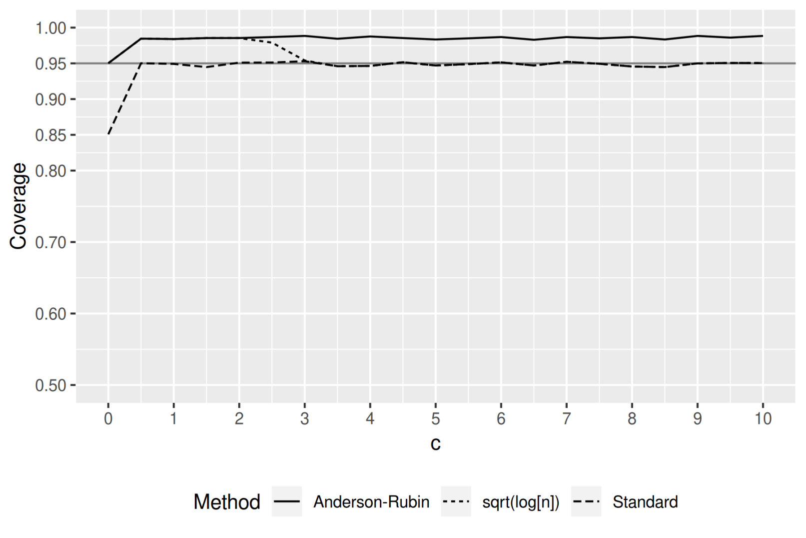
Note: $n = 1,000$, $\theta_1 = c/\sqrt{n}$, $\theta_2 = 5$, $x_1, x_2, e \sim \mathcal{N}(0,1)$

Standard QLR Inference



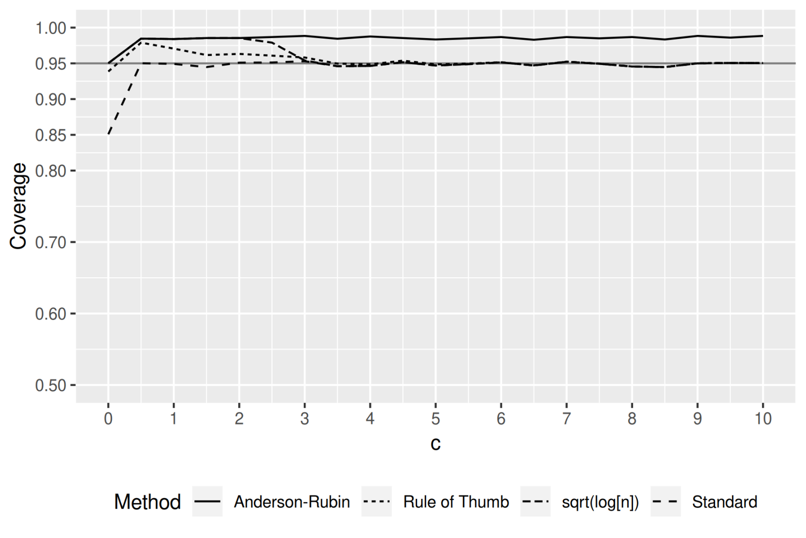
Note: $n = 1,000$, $\theta_1 = c/\sqrt{n}$, $\theta_2 = 5$, $x_1, x_2, e \sim \mathcal{N}(0,1)$

Two-Step Approach $\lambda_n = \sqrt{\log n / n}$



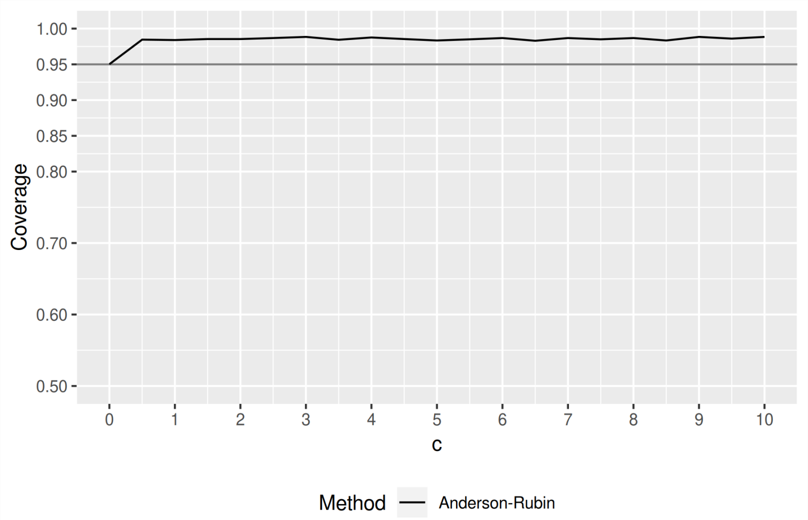
Note: $n = 1,000$, $\theta_1 = c/\sqrt{n}$, $\theta_2 = 5$, $x_1, x_2, e \sim \mathcal{N}(0,1)$

Two-Step Approach λ_n = Rule-of-Thumb



Note: $n = 1,000$, $\theta_1 = c/\sqrt{n}$, $\theta_2 = 5$, $x_1, x_2, e \sim \mathcal{N}(0,1)$

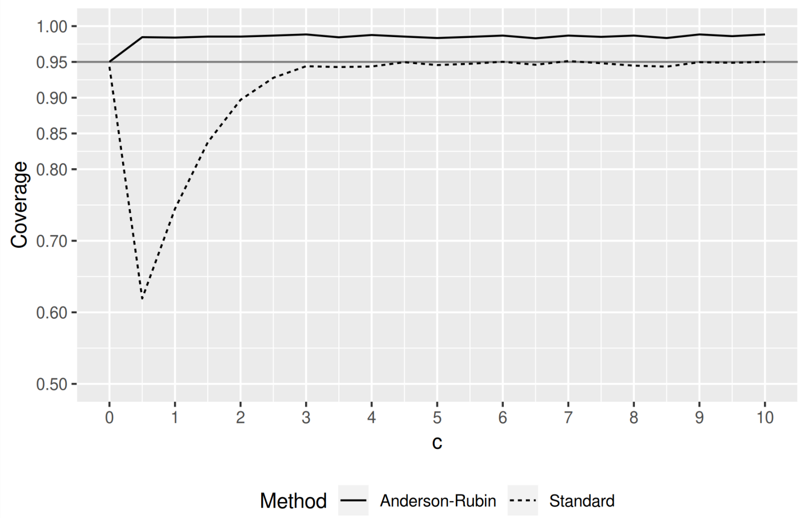
Identification Robust Projection Inference



Note: $n = 1{,}000$, $\theta_1 = c/\sqrt{n}$, $\theta_2 = 5$, $x_1, x_2, e \sim \mathcal{N}(0, 1)$

Example 2 | Example 3 | Empirical Application

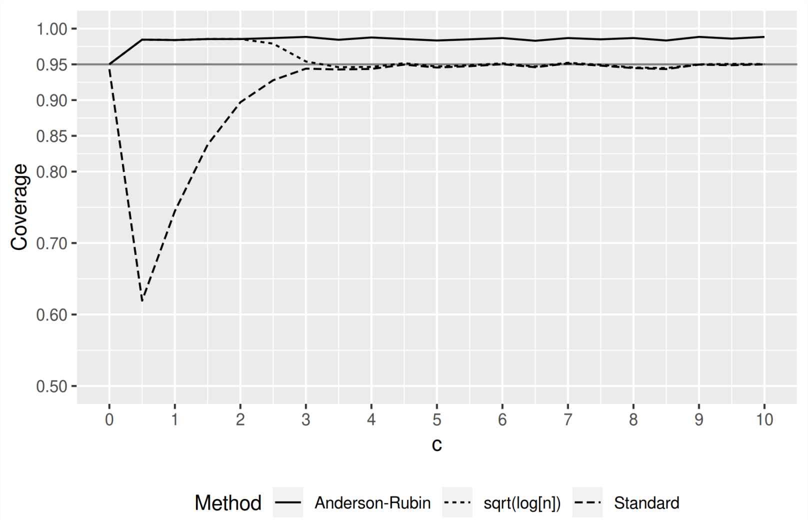
Standard Wald Inference



Note: $n = 1,000$, $\theta_1 = c/\sqrt{n}$, $\theta_2 = 5$, $x_1, x_2, e \sim \mathcal{N}(0, 1)$

Example 2 Example 3 Empirical Application

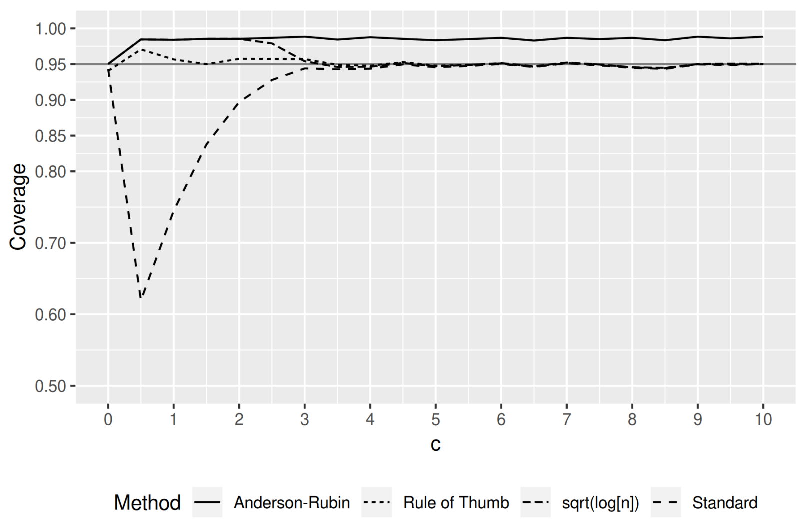
Two-Step Approach $\lambda_n = \sqrt{\log n / n}$



Note: $n = 1,000$, $\theta_1 = c/\sqrt{n}$, $\theta_2 = 5$, $x_1, x_2, e \sim \mathcal{N}(0,1)$

Example 2 Example 3 Empirical Application

Two-Step Approach $\lambda_n =$ Rule-of-Thumb



Note: $n = 1,000,\ \theta_1 = c/\sqrt{n},\ \theta_2 = 5,\ x_1, x_2, e \sim \mathcal{N}(0,1)$

Example 2 Example 3 Empirical Application

Conclusion

Conclusion: A Simple Solution to a Complex Problem

- Covers a wide range of moments and identification failures
- Computationally attractive: massively parallel
- Open questions
    i. Beyond GMM: general M-estimation problems
    ii. From type I to uniform type II inferences?
    iii. Identification failure in semi-nonparametric models?

THANK YOU!

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Illustration: NLS

$$y_t = \theta_{1,n} x_{1,t} + \theta_{1,n}\theta_2 x_{2,t} + e_t$$

- Suppose $x_{1,t}, x_{2,t} \sim \mathcal{N}(0,1)$ uncorrelated
- Moments:

$$\bar{g}_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} y_t (x_{1,t}, x_{2,t})' - \theta_1 (1, \theta_2)'$$

- Suppose $\theta_{1,n} = c_0/\sqrt{n}$ then for any c, θ_2

$$\sqrt{n} \times \bar{g}_n(c/\sqrt{n}, \theta_2) \xrightarrow{d} \mathcal{N}\left(\underbrace{c_0(1,\theta_{2,0})' - c(1,\theta_2)'}_{\text{Information}}, \underbrace{V}_{\text{Noise}}\right)$$

Main

$A_{n,LS}, B_{n,LS}$ - (Semi)-Strong Identification

Theorem (Semi-Strong Identification)

Suppose the model is (semi)-strongly identified, compact or exponential kernel conditions + assumptions above hold and

$$\kappa_n^2 = o\left[\lambda_{\min}(\partial_\theta g_n(\theta_0)'\partial_\theta g_n(\theta_0))\right]$$

then:

i. $A_{n,LS} = \bar{g}_n(\hat{\theta}_{n,GMM}) - B_{n,LS}\hat{\theta}_{n,GMM} + o_p(n^{-1/2})$

ii. $B_{n,LS}H_n = \partial_\theta g_n(\hat{\theta}_{n,GMM})H_n + o_p(1)$ *(full rank)*

iii. $H_n^{-1}[\hat{\theta}_{n,LS} - \hat{\theta}_{n,GMM}] = o_p(n^{-1/2})$

iv. $H_n^{-1}\Sigma_n H_n^{-1} = O_p(\tilde{\kappa}_n^2)$

$A_{n,LS}, B_{n,LS}$ - Higher-Order Local Identification

Theorem (Higher-Order Local Identification)

Suppose the model is higher-order locally identified at an order $r \geq 2$, compact/exponential kernel conditions + assumptions above hold then:

$$\Sigma_n = \sum_{j=1}^{r} P_j O_p(\tilde{\kappa}_n^{2/j}) P_j'$$

using the Lemma, this implies that:

a. $v_j \in Span(P_j) \Rightarrow B_{n,LS} v_j = O_p(\tilde{\kappa}_n^{1-1/j})$
b. $|\lambda_{\min}(B_{n,LS})| = O_p(\tilde{\kappa}_n^{1-1/r})$

Summary of the Results for $B_{n,LS/\infty}$ Main

References for Indentification Regimes

- Each regime has asymptotic implications for $\hat{\theta}_{n,GMM}$:
  1. $\Rightarrow$ consistent + $\sqrt{n}$ asymptotically normal (1.ii. $\Rightarrow \partial_\theta g_n$ full rank) (Newey and McFadden, 1994; van der Vaart, 1998)
  2. $\Rightarrow$ consistent + slower than $\sqrt{n}$ asymptotically normal (Antoine and Renault, 2012; Andrews and Cheng, 2012)
  3. $\Rightarrow$ consistent + slower than $\sqrt{n}$ convergent, not asymptotically normal (Rotnitzky et al., 2000; Dovonon and Hall, 2018)
  4. $\Rightarrow$ not consistent, not asymptotically normal (Staiger and Stock, 1997; Stock and Wright, 2000)

Main

Illustration

- We know that $B_{n,LS/\infty}$ is $O_p(\kappa_n)$ on the span of the identification failure V
- Example: $\theta = (\theta_1, \theta_2)$; $\theta_1 - \theta_2$ point identified, $\theta_1 + \theta_2$ set identified, the model is linear and:

$$B_n = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

- We could use $B_n(1,1) = (0,0)$ and fix $\theta_1 + \theta_2$ (eigenvector)...
- if we **want to fix something interpretable** i.e. θ_1 or θ_2:

$$B_{n,LS/\infty}(1,0) = (1,-1); \quad B_{n,LS/\infty}(0,1) = (-1,1)$$

- fixing either θ_1/θ_2 makes the other point identified as seen by:

$$B_n P^{\perp}_{(1,0)} \text{ and } B_n P^{\perp}_{(0,1)} \text{ have rank} = 1, B_n \text{ has rank } 1$$

Remarks on the assumptions

- The critical assumption is that the free parameters are (semi)-strongly identified so that the $T_{j,n}$s (e.g. S, K stat) have asymptotically correct size
- To be sure, we could compute $\tilde{B}_{n,LS/\infty}$ for $\bar{g}_n(\theta)$ on $\{R_{\hat{j}_n}\theta - c_{\hat{j}_n}\}$ and compare its eigenvalues with $\underline{\lambda}_n$
- Can also check for first-order identification failure after collapsing the identified set

Main

Empirical Applications

Assumptions a., b. Counter-Example

- Let
$$g_n(\theta) = \theta^4 \sin(1/\theta)$$
- g is smooth - but not analytical - has infinitely many zeros
- $\nexists \underline{c}, k$ such that $\underline{c} d(\theta, \Theta_0)^k \leq |g(\theta)|$
- For any $\varepsilon > 0$, $\hat{\pi}_n \left(\mathcal{B}_\varepsilon(0)\right) \to 1$
- Nb: if g_n were analytic with ∞-many zeros around $\theta_0 \Rightarrow g_n$ identically zero around θ_0

Main

Further Asymptotic Results

Proposition (Quasi-Central Limit Theorem)

Suppose that $\bar{g}_n$ is smooth, $\partial_\theta g_n$ satisfies stoch. equicont., CLT then under (semi)-strong identification:

$$\sqrt{n}\begin{pmatrix} A_{n,LS} - B_{n,LS}\theta_0 \\ vec(B_{n,LS} - \bar{B}_{nLS}) \end{pmatrix} = \sqrt{n}\begin{pmatrix} \bar{g}_n(\theta_0) \\ vec(\partial_\theta \bar{g}_n(\theta_0) - g_n(\theta_0)) \end{pmatrix} + o_p(1)$$

$$\xrightarrow{d} \mathcal{N}(0, V)$$

where

$$\bar{B}_{n,LS} = \Sigma_n^{-1}\int_\Theta (\theta - \bar{\theta}_n)\int_\Theta \{g_n(\theta) - g_n(\tilde{\theta})\}'\hat{\pi}_n(\tilde{\theta})d\tilde{\theta}\hat{\pi}_n(\theta)d\theta$$

Remark on (Quasi)-CLT for $A_{n,LS}, B_{n,LS}$

- After some re-centering, we always have

$$\begin{aligned}&\sqrt{n}[B_{n,LS} - \bar{B}_{n,LS}]\\&= \Sigma_n^{-1} \int_\Theta (\theta - \bar{\theta}_n) \int_\Theta [\mathbb{G}_n(\theta) - \mathbb{G}_n(\tilde{\theta})]' \hat{\pi}_n(\tilde{\theta}) d\tilde{\theta} \hat{\pi}_n(\theta) d\theta\end{aligned}$$

- $\text{int}(\Theta_0) \neq \emptyset$ implies it is a sequence of bounded linear operators applied to an empirical process; which can be used to prove a CLT
- Higher-order local identification and manifold valued identified set are more difficult...

Main

Practical Implications: which parameters to fix?

- We can re-write each $\theta \in \Theta_0$ as:

$$\theta = \theta_0 + v,\; v \in V = \mathrm{Span}(\{\theta_1 - \theta_0,\, \theta_0, \theta_1 \in \Theta_0\})$$

- For the projection matrix P_V and the orthogonal $P_V^{\perp}$:

$$P_V\theta = P_V\theta_0 + v, \quad P_V^{\perp}\theta = P_V^{\perp}\theta_0 + 0$$

- The first one is not unique: v can vary
- The second one is unique $\Rightarrow$ identified

Practical Implications: which parameters to fix?

- Suppose $(u, v^\star)$ forms a basis with $\text{rank}(P_V^\perp P_{v^*}^\perp) = \text{rank}(P_{v^*}^\perp)$
- Pick $\theta_1 \in \Theta_0$ with $P_{v^\star}\theta_1 = c$ fixed

$$P_V^\perp \theta_1 = P_V^\perp (P_{v^\star}^\perp \theta_1 + P_{v^\star}\theta_1) = P_V^\perp P_{v^\star}^\perp \theta_1 + P_V^\perp \underbrace{P_{v^\star}\theta_1}_{=c \text{ fixed}}$$

- Since $P_V^\perp \theta_1 = P_V^\perp \theta_0$, we have the system:

$$P_V^\perp P_{v^\star}^\perp \theta_1 = P_V^\perp P_{v^\star}^\perp \theta_0 - P_V^\perp c$$
$$P_{v^\star}\theta_1 = c$$

- Rk: $\text{rank}(P_V^\perp P_{v^*}^\perp) = \text{rank}(P_{v^*}^\perp) \Rightarrow$ the system has full rank

$\Rightarrow$ The solution is unique: θ is identified up to $P_{v^\star}\theta$ (fixed)

Main

Remarks

- Weak/set: if free (nuisance) parameters (semi)-strongly identified when $\ell \geq \ell^\star$ and $S_{n,\ell} =$ S/K/cQLR statistic:

$$\mathbb{P}(S_{\hat{\ell}_n,n} \leq c_{1-\alpha,\hat{\ell}_n}) \geq 1 - \alpha + o(1)$$

- Semi-strong: $\lambda_{\min}[\partial_\theta g_n(\theta_0)]$ slightly larger than $O(n^{-1/2}) \Rightarrow$ false positives... better than false negatives?

On the Cutoff $\underline{\lambda}_n$ for the Eigenvalues (Just-Identified)

- Which cutoff $\lambda_{min}(B_{n,LS}) \leq \underline{\lambda}_n$ to detect identification failure?
- Similar to (Stock and Yogo, 2005): just-identified + gaussian

$$\bar{g}_n(\theta) = A_n + B_n(\theta - \theta_0)$$

$$A_n = \bar{g}_n(\theta_0) - B_n\theta_0,\ B_n - \overline{B}_n = \Delta_n = O_p(1/\sqrt{n})$$

quasi-CLT for $A_{n,LS}, B_{n,LS}$

- Using the Woodbury identity recursively:

$$\hat{\theta}_n - \theta_0 = \underbrace{-\overline{B}_n^{-1}\bar{g}_n(\theta_0)}_{\text{CLT term}} + \underbrace{\overline{B}_n^{-2}\Delta_n\bar{g}_n(\theta_0)}_{\text{Non-Standard Term}} - \overline{B}_n^{-3}\Delta_n^2 \dots$$

$$\Rightarrow \text{bias} \simeq \overline{B}_n^{-2}\underbrace{\mathbb{E}[\Delta_n\bar{g}_n(\theta_0)]}_{=O(1/n)} \quad \text{variance} \simeq \overline{B}_n^{-1}\underbrace{\mathbb{V}[\bar{g}_n(\theta_0)]}_{=O(1/n)}\overline{B}_n^{-1\prime}$$

A Rule-of-Thumb for $\underline{\lambda}_n$

- Rate of convergence depends on $\lambda(\overline{B}_n^{-1})$
- Pick $v_{j,n}$ (complex) left-eigenvector of $\bar{B}_{n,LS}$:

$$v_{j,n}(\hat{\theta}_n - \theta_0) = \lambda_j^{-1} v_{j,n} \bar{g}_n(\theta_0) + \lambda_j^{-2} v_{j,n} \Delta_n \bar{g}_n(\theta_0) + \dots$$

- Size distortion in that direction depends on (*bias*^{2}/*variance*):

$$\frac{1}{n|\lambda_j|^2} \frac{v_{j,n}^\star V_{12} V_{21} v_{j,n}}{v_{j,n}^\star V_1 v_{j,n}} \leq \frac{1}{n|\lambda_{\min}(\bar{B}_{n,LS})|^2} \frac{|V_{12} V_{21}|}{\lambda_{\min}(V_1)}$$

- Design cutoff $\underline{\lambda}_n$ based on a sequence of size distortions $\searrow 0$
- Over-identified: involves W as well
- Higher-Order: residual curvature matters

Additional Remarks

Remarks

- Rule-of-thumb designed for problems with $\bar{g}_n$ flat around θ_0
  - Counter-example: MA(1) locally identified but not globally
- Alternative Representation:
  - Think of $\Theta_0 = \cup_{j=1}^{k} S_j$ - disjoint sets S_j - then $\hat{\theta}_n \in \mathcal{N}(S_j)$ for some $j \in \{1, \ldots, k\}$ wp $\nearrow 1$
  - Compute a Wald statistic for $H_0 : \theta = \theta_{0,j^\star} \in S_{j^\star}$
  - Size distortions: within ($j = j^\star$) and between sets ($j \neq j^\star$)
- Simple idea: partition $\hat{\Theta}_{0,n} = \{\theta, \|\bar{g}_n(\theta)\|_W \leq \kappa_n\}$ using cluster algorithm (e.g. k-means), then
  - Compute rule-of-thumb within cluster (as prev. slides)
  - Compute rule-of-thumb between clusters (distance)

Example 2: MA(1)

- Simple example:

$$y_t = \sigma[e_t + \vartheta e_{t-1}], e_t \sim (0, 1, \tau)$$

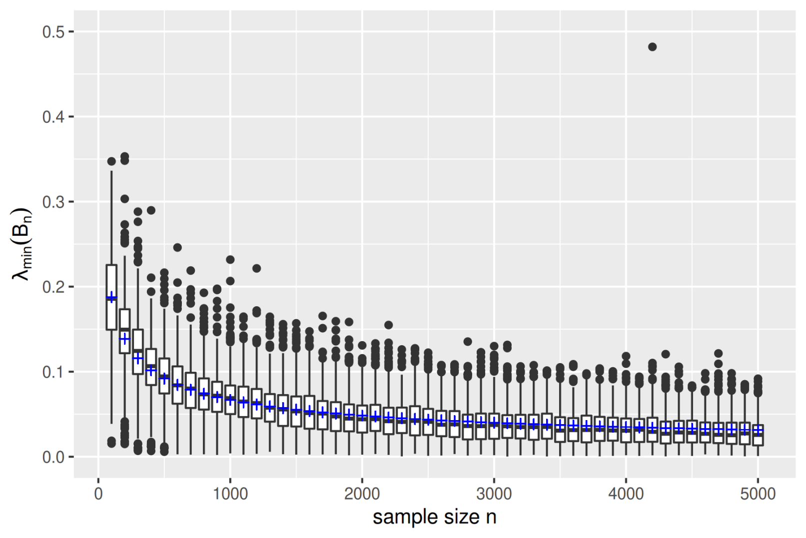
- Identification failure $\tau = 0$, weak identification $\tau \simeq 0$
- Two cutoffs $\underline{\lambda_n}$: $\sqrt{\log(n)/n}$, rule-of-thumb
- Null hypothesis:

$$H_0 : \theta = \theta_0 = 2$$

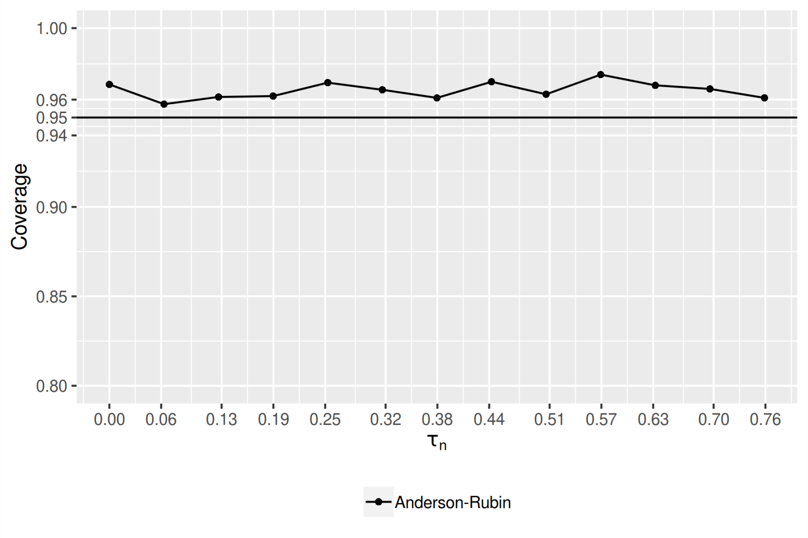
- With 4 estimating moments and
  - $\tau_n = 2 \times n^{-1/2}$, $e_t \sim GEV(0, 1, \tau_n)$
  - $\kappa_n = \max(q_{0.99}(\chi^2_4), \sqrt{2\log(\log[n])/n})$
- Compare AR (χ^2_2 critical value: oracle), Wald/QLR and Two-Step

Main

Example 2: MA(1) - Distribution of $\lambda_{\min}(B_{n,LS})$

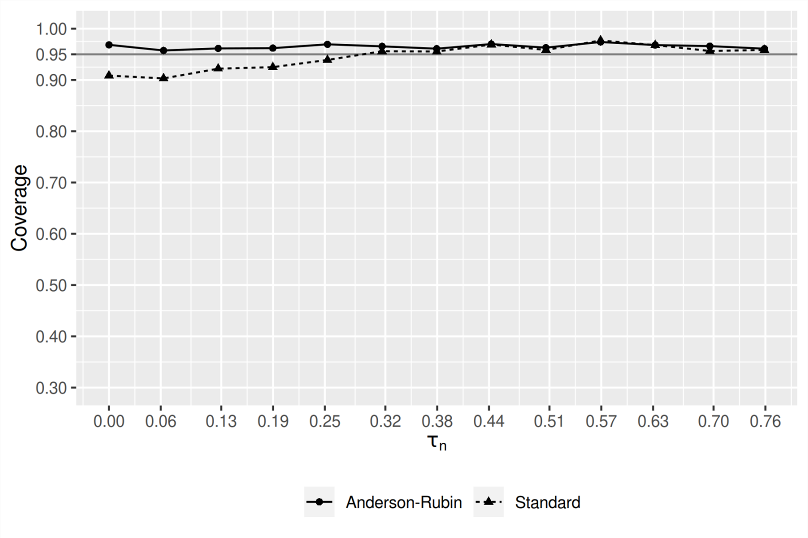


Identification Robust Projection Inference



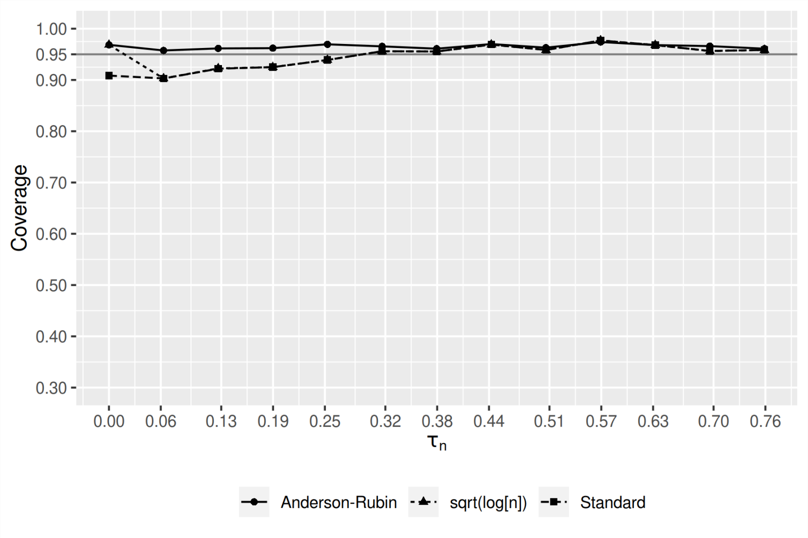
Note: $n = 1,000$, $\tau = c/\sqrt{n}$, $\vartheta_0 = 2$, $e_t \sim \mathcal{N}(0,1)$

Standard QLR Inference



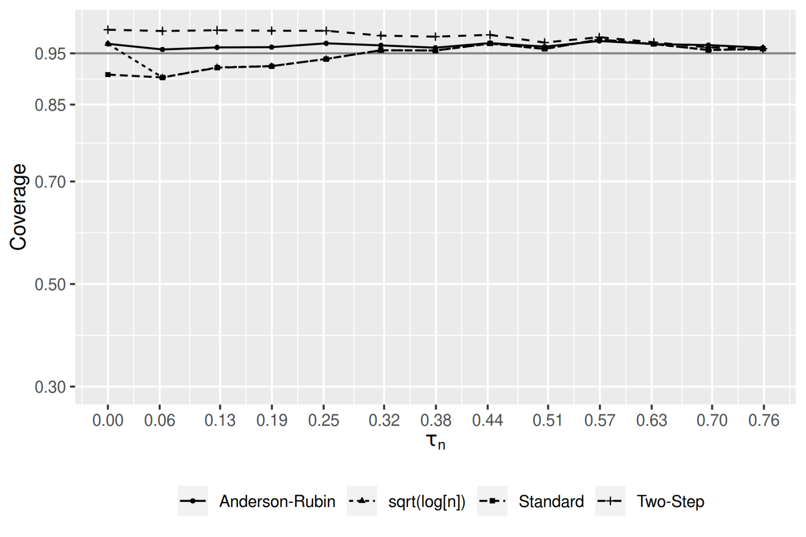
Note: $n = 1,000$, $\tau = c/\sqrt{n}$, $\vartheta_0 = 2$, $e_t \sim \mathcal{N}(0,1)$

Two-Step Approach $\lambda_n = \sqrt{\log n / n}$



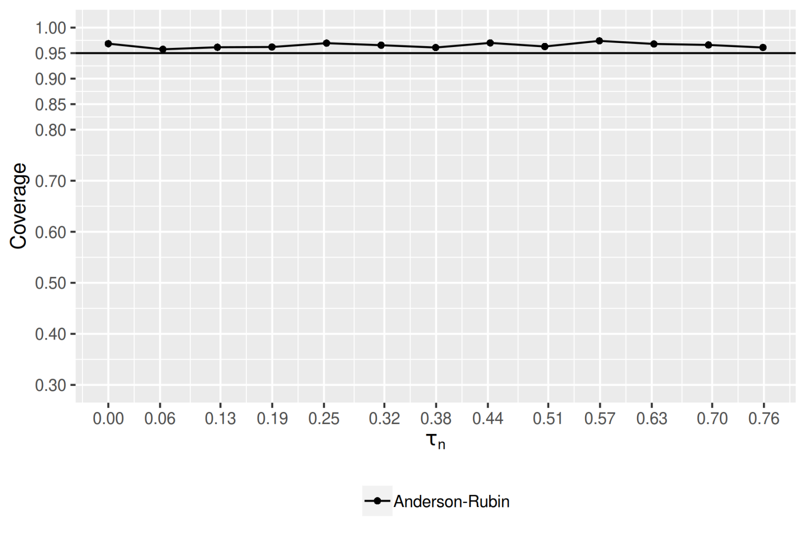
Note: $n = 1,000$, $\tau = c/\sqrt{n}$, $\vartheta_0 = 2$, $e_t \sim \mathcal{N}(0,1)$

Two-Step Approach λ_n = Rule-of-Thumb



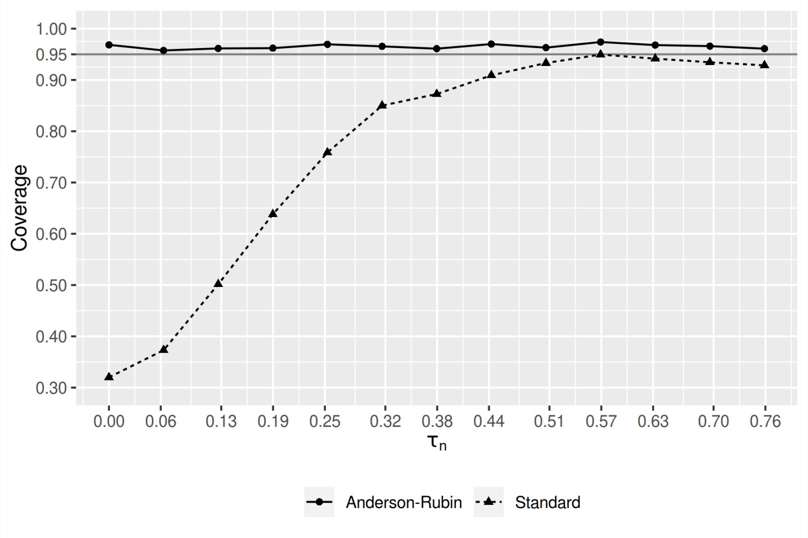
Note: $n = 1,000$, $\tau = c/\sqrt{n}$, $\vartheta_0 = 2$, $e_t \sim \mathcal{N}(0, 1)$

Identification Robust Projection Inference



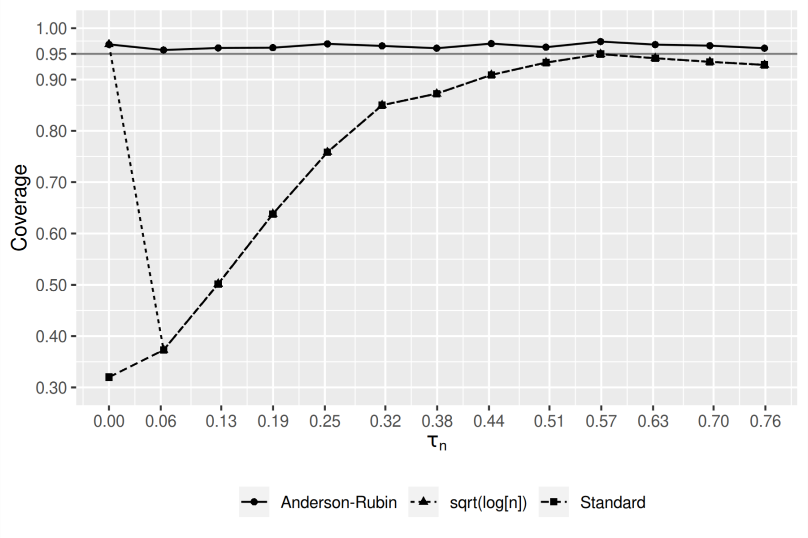
Note: $n = 1,000$, $\tau = c/\sqrt{n}$, $\vartheta_0 = 2$, $e_t \sim \mathcal{N}(0,1)$ Main

Standard Wald Inference



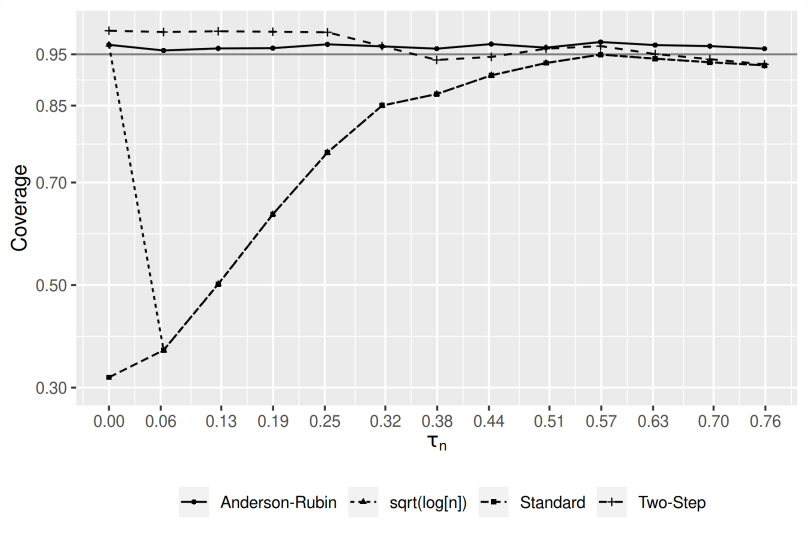
Note: $n = 1,000$, $\tau = c/\sqrt{n}$, $\vartheta_0 = 2$, $e_t \sim \mathcal{N}(0,1)$ Main

Two-Step Approach $\lambda_n = \sqrt{\log n / n}$



Note: $n = 1,000,\ \tau = c/\sqrt{n},\ \vartheta_0 = 2,\ e_t \sim \mathcal{N}(0,1)$ Main

Two-Step Approach λ_n = Rule-of-Thumb



Note: $n = 1,000$, $\tau = c/\sqrt{n}$, $\vartheta_0 = 2$, $e_t \sim \mathcal{N}(0,1)$ Main

Example 3: Higher-Order Identified NLS

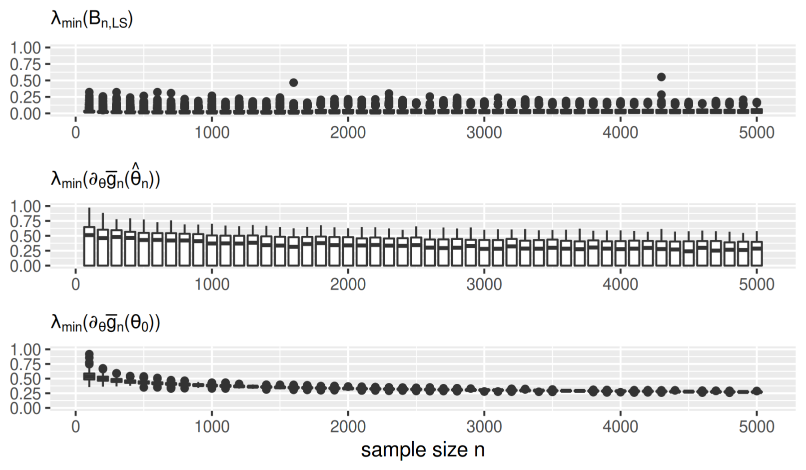
- Simple example:

$$y_t = \theta_1 x_{i,1} + \theta_{2,n}(\theta_{2,n} - \theta_1)^2 x_{2,i} + e_i, (x_{i,1}, x_{i,2}, e_i) \sim \mathcal{N}(0, I_3)$$

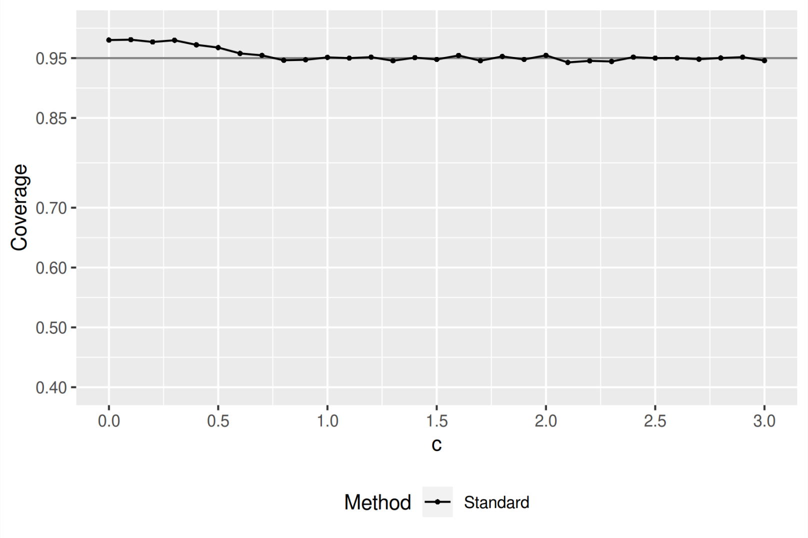
- Higher-order identification $\theta_{2,n} - \theta_1 = O(n^{-1/4})$
- Cutoff $\underline{\lambda_n}$: based on rule-of-thumb
- Estimating moments $\mathbb{E}(y_i(x_{i,1}, x_{i,2})) - (\theta_1, \theta_2(\theta_2 - \theta_1)^2)$
- Compare AR (χ^2_2 critical value: oracle), Wald/QLR and Two-Step

Main

Example 3: NLS - Distribution of $\lambda_{\min}(B_{n,LS})$

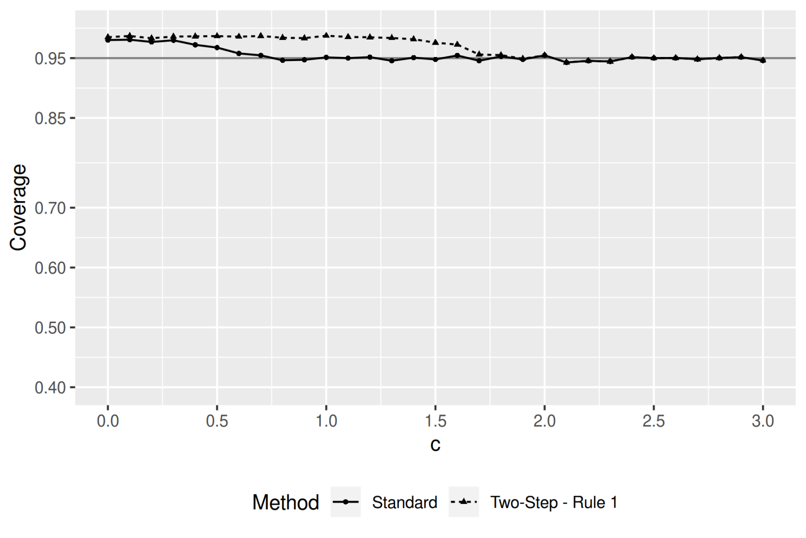


Standard QLR Inference



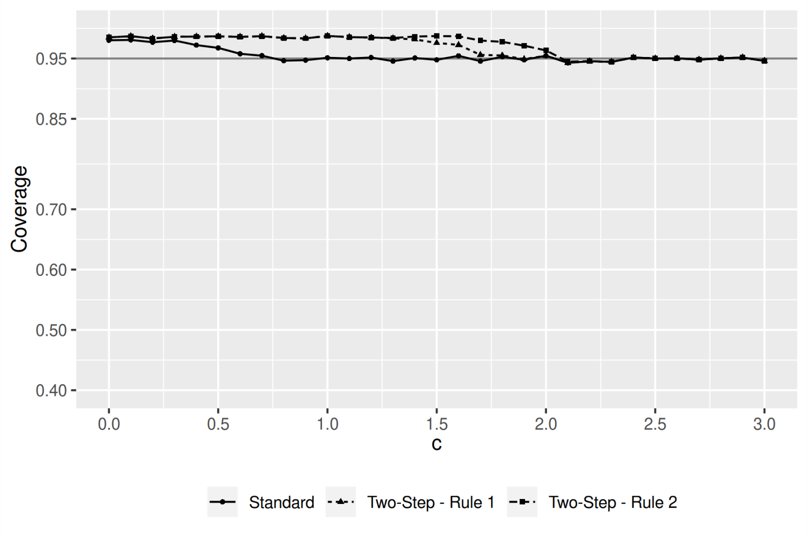
Note: $n = 1,000$

Two-Step: Rule-of-Thumb 1



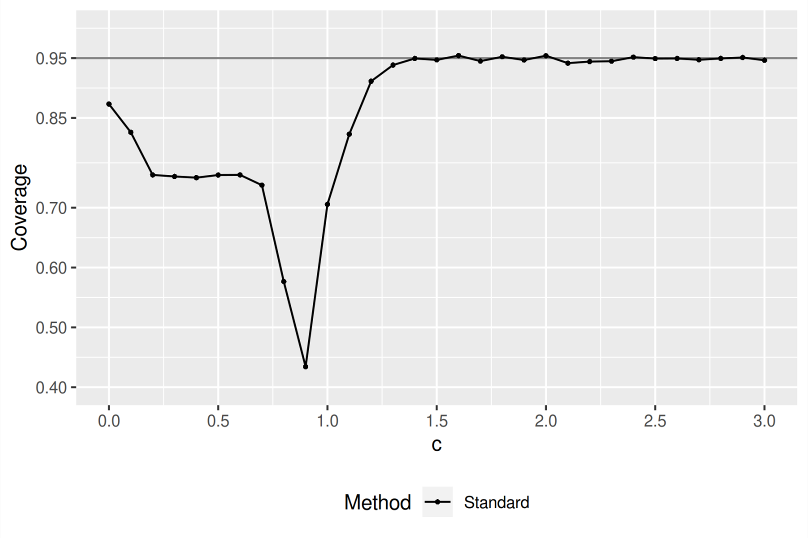
Note: $n = 1,000$

Two-Step: Rule-of-Thumb 2



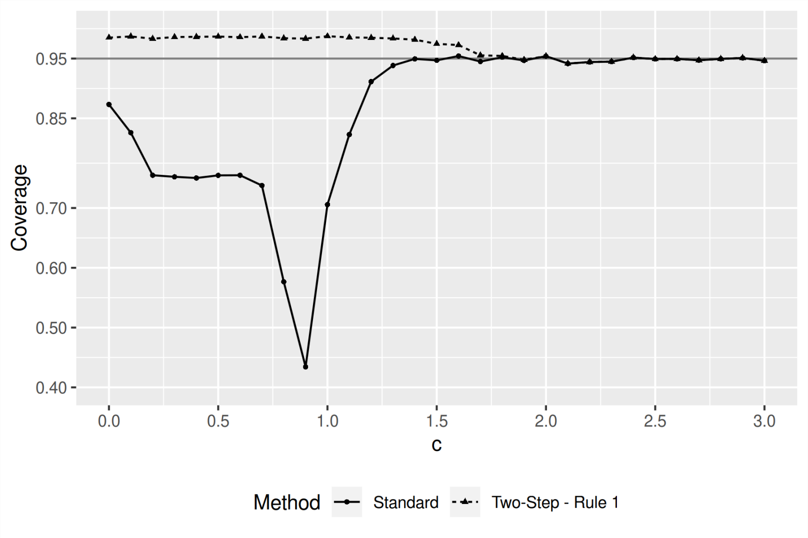
Note: $n = 1,000$

Standard Wald Inference



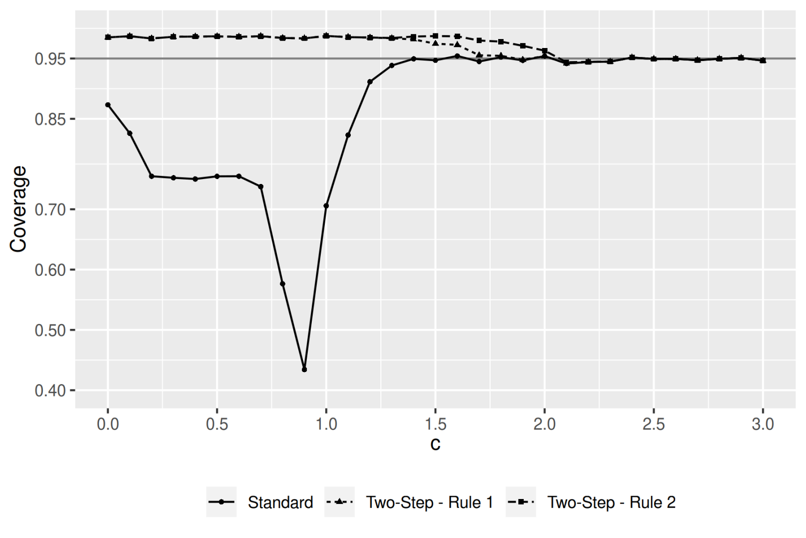
Note: $n = 1,000$ Main

Two-Step: Rule-of-Thumb 1



Note: $n = 1,000$ Main

Two-Step: Rule-of-Thumb 2



Note: $n = 1,000$ Main

Empirical Illustration

Illustration: Euler Equation

- Data: Stock and Wright (2000)
- Model:

$$\mathbb{E}\left(\left[\delta\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma} R_t - 1\right] Z_t\right) = 0$$

- $Z_t = (1, C_{t-1}, R_{t-1})$, $n = 103$ after taking lags
- W = Continuously-Updated Newey-West
- Bounds: $(\delta, \gamma) \in [0.7, 1.2] \times [0, 20]$
- Grid: 10^4 points from the Sobol sequence (quasi Monte-Carlo, see e.g. Owen, 2003, for an introduction)
- **Compute a quasi-Jacobian matrix $B_{n,LS}$ that summarizes the identifiability of (δ, γ)**

Illustration: $\hat{\Theta}_n = \left\{ \theta \in \Theta, \|\bar{g}_n(\theta)\|_W - \|\bar{g}_n(\hat{\theta}_n)\|_W \leq \kappa_n \right\}$

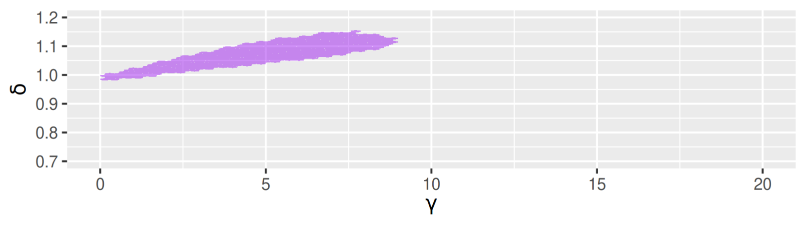


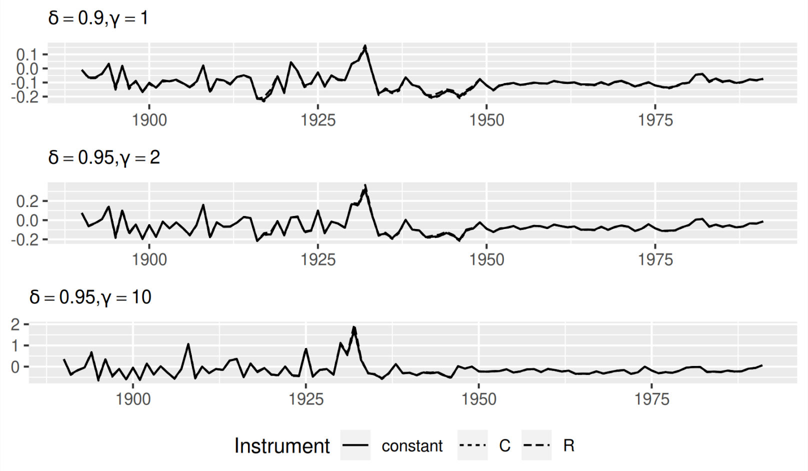
Illustration: Euler Equation - Linear Approximation

- Results $\theta = (\delta, \gamma)$

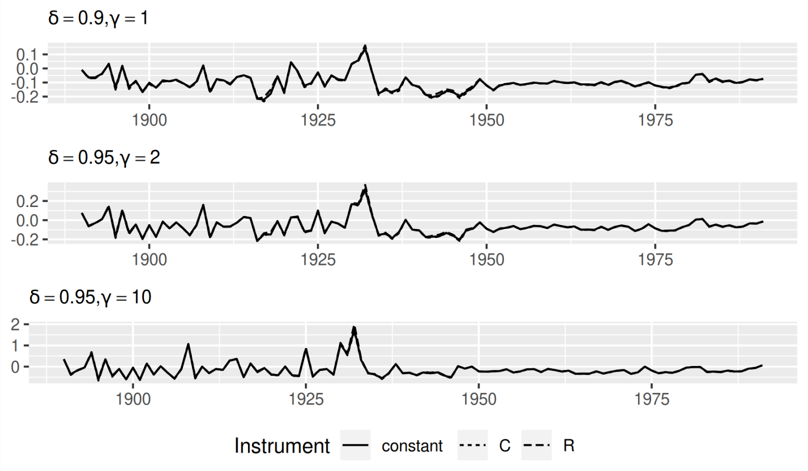
$$B_{n,LS} = \begin{pmatrix} 0.669 & -0.001 \\ 0.685 & -0.001 \\ 0.682 & 0.000 \end{pmatrix}$$

- Note that $\lambda(B_{n,LS}) \times \sqrt{n} = (11.929, 0.006)$

The Identification Problem in the Euler Equation



The Identification Problem in the Euler Equation



Moments are singular: amount to a single moment condition

Identification Robust Inference

- Require Singularity and Identification Robust Inference (Andrews and Guggenberger, 2019)
- Drop 2 moments, keep $Z_t = 1$; invert an AR test with χ^2_1 critical value:

$$CI_{95\%}(\delta) = [0.98, 1.17]; CI_{95\%}(\gamma) = [0.03, 20]$$

Main