BLUEPRINT FOR THE ADJUNCTION FORMULA

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1. Auslander-Buchsbaum-Serre Theorem

We develop some homological and element-wise machinery from Commutative Algebra, culminating in the proof of the Auslander-Buchsbaum-Serre theorem, and the corollary that the condition of being a regular ring localizes. This corollary is crucial for our proof of adjunction.

We follow Bruns-Herzog almost exclusively.

The current version of this blueprint is incomplete–this reflects the fact that we have not yet chosen which proof of Auslander-Buchsbaum-Serre to formalize.

In particular, there are two detailed by Bruns-Herzog: one uses a fact about Cohen-Macaulay rings for the forward direction, and a theorem of Ferrand-Vasconcelos for the reverse direction. The other uses Koszul homology and some facts about Tor for the forward direction, and a statement of Serre on DG-algebras for the reverse direction. We can basically pick and choose which proof we want to use for each direction.

1.1. **Projective Resolutions.**

Definition 1.1.1. Let R be a ring. Let M be an R-module. A projective resolution of M over R Is an exact sequence

$$... \to P^2 \to P^1 \to P^0 \to M \to 0$$

. That is to say, a projective resolution is a quasi-isomorphism of complexes between the inclusion of M into chain complexes over R (i.e. the complex which has M in degree zero, trivial everywhere else, with trivial maps) and a bounded—below complex whose components are projective R-modules.

Definition 1.1.2. A free resolution of M over R is a projective resolution whose components P^i are free.

Definition 1.1.3. The length of a projective resolution $P_{\cdot}$ is the highest i such that P^i is nonzero. If there exists no such i, the length is infinity

Definition 1.1.4. Let M be an R-module. Then proj dim$M$ is the minimum length of a projective resolution of M. It lives in the set $\mathbb{N} \cap \infty$.

Lemma 1.1.5. *If there exists a projective resolution of M with finite length, then proj dim$M < \infty$.*

Proof. Expand definitions, use the definition of minimum. □

Lemma 1.1.6. *Any resolution of M over R has length at least that of the minimal resolution.*

Proof. Expand definitions, use definition of minimum □

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Proposition 1.1.7. *There exists a free resolution $F_.$ of M such that length of $F_.$ is equal to the projective dimension of M.*

Proof. One has the generators-relations explicit construction, which should eventually get its own definition. Then one has to show that this is indeed minimal among projective resolutions, which I don't remember how to do the top of my head. □

Definition 1.1.8. The global dimension of a ring R is the supremum over all R-modules of proj dim$M$.

Lemma 1.1.9. *Localization by a multiplicative set is an exact functor.*

Lemma 1.1.10. *Let M be an R-module. Let S be a multiplicative set. Let $\ldots \to A_2 \to A_1 \to A_0 \to M \to 0$ be a free resoltion of M. Then $\ldots S^{-1}(A_2) \to S^{-1}(A_1) \to S^{-1}(A_0) \to S^{-1}M \to 0$ is a free resolution of $S^{-1}M$.*

1.2. **Dimension Theory.**

Definition 1.2.1. The krull dimension of a module is the supremum of the lengths of chains of prime submodules.

Definition 1.2.2. Krull dim of ring (prime ideals are prime submodules, might need to show this)

Other dimension theory necessary? Probably?

1.3. **Regular elements and sequences.**

Definition 1.3.1. Let M be an R-module. An element $x \in R$ is M-regular if it does not annihilate any element in M. In colon notation, it is non a member of the ideal $(0 : M)$. If x is an R-regular element, we simply say it is a regular element.

Definition 1.3.2. Let M be an R-module. Then a weak M-regular sequence is a sequence $x_1, \ldots, x_n$ with $x_i \in R$ for all i, such that x_i is a $M/(x_1, \ldots, x_{i-1})M$-regular element for all i. If $M = R$, we say that $x_1, \ldots, x_n$ is a weak regular sequence.

Definition 1.3.3. Let M be an R-module. An M-regular sequence is a weak M-regular sequence such that $M/(x_1, \ldots, x_n)M \neq 0$. If $M = R$, we simply call this a regular sequence.

Theorem 1.3.4. *Any regular sequence is part of a system of parameters*

Proof. This depends on the notion of depth, a bunch of machinery from BH chapter 1, and the definition of associated primes. □

Definition 1.3.5. A local ring R is regular if the minimal number of generators of its maximal ideal is equal to the dimension of R.

Definition 1.3.6. A ring R is regular if $R_{\mathfrak{p}}$ is regular for every $\mathfrak{p} \in R$.

Proposition 1.3.7. *The following are equivalent Firstly, R is regular. Secondly, the zariski cotangent space is a vector space of dimension* $\dim R$.

Proof. See in `adjunction_blob.txt` □

Lemma 1.3.8. *Every regular ring is an integral domain.*

Proposition 1.3.9 (BH 2.2.4)**.** *Let R be a regular local ring. Then R/I is regular local if and only if I is generated by a (regular) system of parameters (I.e. a generating set for $\mathfrak{m}$).*

Proof. This proof uses the following facts:
* a Nakayama corollary
* the fact that regular rings are integral domains
* the fact that you can't have a proper containment of integral domains with the same dimension

□

Proposition 1.3.10. *A local ring R is regular if and only if its maximal ideal is generated by a regular sequence.*

Proof. See Bruns-Herzog 2.2.5. This uses BH 2.2.4 for the forward direction, and for the reverse direction, we use BH 1.2.12 and the fact that the minimal number of generators for $\mathfrak{m}$ is at least $\dim R$ (need to account for the word "system of parameters". □

1.4. **Associated Primes.**

Definition 1.4.1. The support of a module is the set of prime ideals $\mathfrak{p} \subset R$ such that $M_{\mathfrak{p}} \neq 0$.

Proposition 1.4.2. *Let R a ring, and M an R-module. Let $\mathfrak{p}$ be a prime ideal, then the following are equivalent:*

(i) $\mathfrak{p} = Ann_R(m)$ *for some element* $m \in M$*.*
(ii) $R/\mathfrak{p}$ *embeds into* M*.*

Definition 1.4.3. The set $\mathrm{Assoc}_R(M)$ is the set of primes satisfying the preivious proposition

Proposition 1.4.4. *The associated primes of M are in the support of M.*

1.5. **Depth and Regular Sequences.** This mostly follows Bruns-Herzog 1.2

Definition 1.5.1. The depth of a module M is . . .

1.6. **Auslander-Buchsbaum Formula.**

Lemma 1.6.1. *BH 1.3.4*

Proof. Uses
* def of associated primes
* a fact about associated primes giving and embedding
* some facts about commutative squares (maybe already in lean?)
* tensor products of modules
* Nakayama (a corollary, like Atiyah-MacDonald 2.8 but uses maps, might just follow from AM 2.8) □

Lemma 1.6.2. *BH 1.3.5*

Proof. This uses
* BH 1.1.5
* The "tor" characterization of projective dimension

□

Theorem 1.6.3. *depth + projdim = depth*

Proof. □

1.7. **Proof of Auslander-Buchbaum-Serre.**

Lemma 1.7.1. *Let R be a regular local ring. Then R has finite global dimension. That is, any finitely generated module R has finite projective dimension.*

Proof. □

Theorem 1.7.2 (Ferrand-Vasconcelos, BH 2.2.8)**.** *Let $(R, \mathfrak{m}, k)$ be a local noetherian ring. Let I be a nonzero ideal with finite projective dimension. If I/I^2 is a free R-module, then I is generated by a regular sequence.*

Proof. Since I has finite projective dimension, it has a finite free resolution. Thus, by 1.4.6 it has must have an R-regular element x. . . . finish the proof . . . □

Theorem 1.7.3 (Auslander-Buchsbaum-Serre Criterion, BH 2.2.7)**.** *Let $(R, \mathfrak{m}, k$ be a noetherian local ring. The following are equivalent:*

(i) R is regular.
(ii) R has finite global dimension.
(iii) proj dim$k < \infty$

Proof. (i) $\implies$ (ii) is precisely Lemma **??**
(ii) $\implies$ (iii) follows by applying the definition of global dimension with $M = k$
(iii) $\implies$ (i) is a special case of Theorem **??**, using **??** to conclude regularity.
??. □

Theorem 1.7.4 (Regular Rings Localize, BH 2.2.9)**.** *Let R be a regular local ring, and let $\mathfrak{p}$ be a prime ideal in R. Then $R_\mathfrak{p}$ is a regular local ring.*

Proof. By Auslander-Buchsbaum-Serre, it is enough to show that $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ has finite projective dimension. Also by Auslander-Buchsbaum-Serre, we know that $k = R/\mathfrak{m}$ has finite projective dimension. Then k has a minimal free resolution of finite length by Proposition **??**. By the fact that the loclization of a resolution is a resolution, We get a finite resolution for $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, thus any minimal resolution is also finite, giving us what we want. □

2. Adjunction Formula

2.1. **Nakayama's Lemma and Corollaries.**

Lemma 2.1.1 (Nakayama's Lemma)**.** *State Nakayama's Lemma here.*

Proof. The proof of this is already in Mathlib □

Corollary 2.1.2. *Let $(R, \mathfrak{m}, k)$ be a local ring. Let M be an R-module. If the elements $x_1, \dots, x_n$ are elements in M that form a basis in the projection $M/\mathfrak{m}M$, then $x_1, ldots, x_n$ generate M.*

Proof. See Atiyah-MacDonald Corollary 2.8 □

The following lemma is used by user6:

Lemma 2.1.3. *A finitely generated projective module over a regular local ring is free*

Proof. A proof of this theorem can be pieced together from this stack exchange answer: https://math.stackexchange.c
modules-over-local-rings-are-free-matsumuras-proof This proof needs

- Nakayama's lemma
- Equivalent definitions of projective modules

□

Lemma 2.1.4. *Let $f : M \to M$ be a surjection of modules (over a local ring?). Then f is an isomorphism.*

Proof. I think this uses the Corollary 2.8 version of Nakayama's Lemma □

2.2. **Regular Rings.** We summarize things which have been proved in the previous section.

Proposition 2.2.1. *Let $(R, \mathfrak{m}, k)$ be a noetherian ring. The following are equivalent:*

(a) Let n be the minimal number of generators of $\mathfrak{m}$. Then $n = \dim R$.
(b) $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim R$

Proof. It is enough to show that $\mu(\mathfrak{m}) = \dim \mathfrak{m}/\mathfrak{m}^2$, where $\mu(\mathfrak{m})$ is the minimal number of generators of $\mathfrak{m}$.

First, we show $\mu(\mathfrak{m}) \leq \dim \mathfrak{m}/\mathfrak{m}^2$. Let $\overline{x}_1, \dots, \overline{x}_n$ be a basis of $\mathfrak{m}/\mathfrak{m}^2$ over k. Then, by a corollary of Nakayama's lemma (Atiyah-MacDonald 2.8), the lifts of the $\overline{x}_i$ generate $\mathfrak{m}$, so $\mu(\mathfrak{m}) \leq n$ as desired.

Second, we show that $\dim \mathfrak{m}/\mathfrak{m}^2 \leq \mu(\mathfrak{m})$. Let $x_1, \dots, x_n$ be a generating set of $\mathfrak{m}$. Then, the residues of the x_i generate $\mathfrak{m}/\mathfrak{m}^2$ (quotient map is a homomorphism or something). Since it is a generating set, it contains a basis by linear algebra, and $\dim \mathfrak{m}/\mathfrak{m}^2 \leq n$ as desired. □

Lemma 2.2.2. *The localization of a regular local ring at (read: away from) a prime ideal is a regular local ring.*

2.3. **Misc Commutative Algebra.**

Lemma 2.3.1. *A surjection of finite free modules splits*

Lemma 2.3.2. *Rank is additive on short exact sequences*

Theorem 2.3.3. *Let R be a noetherian ring. Then $R[x]$ is noetherian.*

Proof. The proof of this is already in mathlib: □

2.4. **Kahler differentials.** We use the following strategy to define the Kahler differentials: first, we give the universal property, and then we give a few constructions that satisfy the universal property

Let A be an R-algebra.

Definition 2.4.1. An R-linear derivation of A into M is a map of R-modules $d\colon A \to M$

The set of derivations is denoted $\mathrm{Der}_R(A, M)$

Lemma 2.4.2. $\mathrm{Der}_R(A, M)$ *is an R-module.*

Proof. Derivations live in $\mathrm{Hom}_R(A, M)$, so all we need to check is that Leibnitz' rule still holds after addition, which we can do explicitly. □

Lemma 2.4.3. *The module of Kahler differentials $\Omega_{A/R}$ is the A-module that represents the functor $M \mapsto \mathrm{Der}_R(A, M)$ from A-modules to R-modules.*

Of course as we define by universal property via representaility, it is not clear that the module exists.

Lemma 2.4.4. *The following module satisfies the universal property of $\Omega_{R/A}$: Take the free R-module on the symbols da for $a \in A$, and quotient out by the relations*

1. $dr = 0$ *for* $r \in R$
2. $d(a + a') = da + da'$
3. $d(aa') = ada' + a'da$.

We can state another version of the universal property:

Lemma 2.4.5. *The module of Kahler differentials has the following universal property: The map $d : A \to \Omega_{A/R}$ defined by $a \mapsto da$ is initial in the category whose objects are derivations $\delta : A \to M$ and morphisms are diagrams*

$$\begin{tikzcd} A \arrow[r, "\delta'"] \arrow[dr, "\delta"'] & M' \arrow[d] \\ & M \end{tikzcd}$$

Finally, there is a second construction:

Lemma 2.4.6. *Let I be the kernel of the multiplication map $A \otimes_R A \to A$. Then I/I^2 satisfies the universal property of $\Omega_{A/R}$*

Proof. This proof (at least in Vakil) is a bit long, uses a lot of properties of pure tensors, and I'm not sure if it's worth it. □

The following is quite important.

Lemma 2.4.7. *By ϕ, we mean the ring map $R \to A$ given by the algebra structure Let S a multiplicative subset of A, and let T be a multiplicative subset of R with $\phi(T) \subset S$. Assume the following diagram commutes*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \\ T^{-1}R \arrow[r] & S^{-1}A \end{tikzcd}$$

We have a (canonical) isomorphism

$$S^{-1}\Omega_{A/R} \cong \Omega_{S^{-1}A/T^{-1}R}$$

Proof. TODO □

Definition 2.4.8. Given a map of schemes $X \to S$, we have a sheaf $\Omega_{X/S}$ which globalizes the construction $\Omega_{A/R}$.

Proof. Use the fact that $\Omega_{A/R}$ commutes with localization plus general scheme machinery: if we have a sheaf on an affine cover that is compatible on the intersections, then we get a sheaf on the whole scheme. □

Lemma 2.4.9. $\Omega_{X/S}$ *is quasi-coherent*

Proof. Use the fact that it is defined locally as a module. This is mathematically trivial but is a good stress test of "quasicoherent sheaf machinery" in Lean. □

Proposition 2.4.10. *Let B be an A-algebra, and I some ideal of B. Let $C := B/I$. We have the following right-exact sequence:*

$$I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

Proof. □

Corollary 2.4.11 (Hartshorne II.8.5)**.** *If B is a finitely generated A-algebra or a localization thereof, then $\Omega_{B/A}$ is finitely generated as a B-module.*

Proof. Calculate $\Omega_{B/A}$ for the case of a polynomial ring. Then use the conormal sequence to pass to the quotient. Finally, Kahler differentials commute with localization. □

Corollary 2.4.12. *The sheaf of kahler differentials is coherent.*

Proof. Put together **??** and **??** □

Proposition 2.4.13. *There is a sheafy exact sequence globalizing* **??**

Proof. We will need ideal sheaves for this. □

Lemma 2.4.14. *An algebraically closed field is perfect.*

Proof. Is this already in lean? □

Definition 2.4.15. A field extension K/k is *separably generated* if there exists a trancendence basis $\{x_i\}$ for K/k such that K is a separable algebraic extension of $k(\{x_i\})$.

Lemma 2.4.16. *Let K/k be a perfect field extension. Then K is separably generated over k.*

Lemma 2.4.17 (Hartshorne II.8.6A)**.** *Let K be a finitely generated (as an algebra) field extension of k. Then $\operatorname{trdeg} K/k \leq \dim \Omega_{K/k}$, with equality if (and only if) K is separably generated over k.*

Theorem 2.4.18 (Hartshorne II.8.7)**.** *Let $(B, \mathfrak{m}, k)$ be a local ring which contains a field k isomorphic to its residue field Then the map $\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B k$ which is the first map in the conormal right-exact sequence is an isomorphism.*

Proof. □

Lemma 2.4.19 (Hartshorne II.8.9)**.** *Let A be a noetherian local integral domain, with residue field k and quotient field K. If M is a finitely generated A-module and $\dim_k M \otimes_A k = \dim_K M \otimes_A K = r$, then M is free of rank r.*

Proof. □

Theorem 2.4.20 (Hartshorne II.8.8)**.** *Let $(B, \mathfrak{m}, k)$ be a local ring of equal characteristic. In addition, assume that k is a perfect field, and that B is a localization of a finitely generatd k-algebra. Then $\Omega_{B/k}$ is a free B-module of rank equal to the dimension of B if and only if B is a regular local ring.*

Proof. This proof uses quite a few things. □

Theorem 2.4.21 (Hartshorne II.8.15)**.** *Let X be an irreducible separated scheme of finite type over an algebraically closed field k. Then $\Omega_{X/k}$ is a locally free sheaf of rank $\dim X$ if and only if X is regular.*

Proof. □

2.5. Coherent Sheaves and Stalks.

Definition 2.5.1. Let X be a scheme. We say X is *locally noetherian* if there exists an open affine cover $\{U_i = \operatorname{Spec} A_i\}$ such that all A_i are noetherian rings.

Lemma 2.5.2. *The functor on sheaves of abelian groups (and in particular, quasi-coherent sheaves) on a scheme X which takes a sheaf $\mathcal{F}$ to its stalk at the point x, $\mathcal{F}_x$, is a functor that preserves colimits.*

Proof. Taking stalks is itself a colimit, and colimits commute with colimits. $\square$

Theorem 2.5.3. *Consider an exact sequence of abelian (coherent) sheaves on a scheme. Can be left-exact, right-exact, exact in the middle, short exact, longer, anything. The seqeunce is exact iff it exact on all stalks.*

Corollary 2.5.4. *A stalk of a quotient of ideal sheaves is isomoprhic to the quotient of the stalks of the ideal sheaves.*

Lemma 2.5.5. *Let A be a ring, and M an A-module. Suppose that, for some prime ideal $\mathfrak{p}$, $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module. Then there exists an element $f \in A \setminus \mathfrak{p}$ such that M_f is a free A_f-module.*

Proof. This is exactly the same proof as the next lemma, minus the reduction to being an affine scheme. $\square$

Theorem 2.5.6 (Hartshorne Exercise II.5.7a)**.** *Let X be a locally noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf. If the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for some point $x \in X$, then there exists a neighborhood U containing x such that $\mathcal{F}|_U$ is free.*

Proof. WLOG, we can assume that X is affine, so $X = \operatorname{Spec} A$, and furthermore we can assume A is noetherian. Indeed, take an open affine noetherian cover as guarenteed by local noetherianity. Then x is contained in some neighborhood $U = \operatorname{Spec} A$ in the cover. If we show the theorem for $\operatorname{Spec} A$, then we have shown it for X.

Now, as $\mathcal{F}$ is a coherent sheaf on A, $\mathcal{F} \cong \tilde{M}$ for some finitely generated module M on A. x, being a point of $\operatorname{Spec} A$, is/corresponds to a prime ideal $\mathfrak{p}$ in A. Now, our assumption about freeness of the stalk says that $M_{\mathfrak{p}} \cong A_{\mathfrak{p}}^{\oplus n}$ for some n. Indeed, $M_{\mathfrak{p}}$ is the stalk of $\tilde{M}$ and $A_{\mathfrak{p}}$ is the local ring of $\operatorname{Spec} A$ at $\mathfrak{p}$. Let $m_1, \dots, m_n$ the a basis/free generating set for $M_{\mathfrak{p}}$. In other words, m_i is the image (along some isomorphism $A_{\mathfrak{p}}^{\oplus n}$) of $(0, \dots, 1, \dots, 0) \in A_{\mathfrak{p}}^{\oplus n}$ where the 1 is in the i-th place. Since the m_i are elements of the stalk $\mathcal{F}_x \cong M_{\mathfrak{p}}$, we can choose a neighborhood U' of x with representatives $m_1', \dots, m_n'$, whose images in the stalk are the m_i. Indeed, we can do this individually for each m_i, and since there are finitely many, we can choose U' a neighborhood which dominates each of them in the diagram (i.e. a neighborhood which is contained in all of them, for example the intersection). Moreover, we can choose U' to be affine by taking a smaller neighborhood (X is a scheme). If we prove the theorem for U', then we have proven it for U, so we can assume that $U = U'$. Aside: we can see that the m_i' here aren't zero, because there is a ring map from the sections over U' to the stalk, and the images are nontrivial in the stalk.

Let $x_1, \dots, x_k$ be a finite generating set for M. Now, we have the following equations in $M_{\mathfrak{p}}$:

$$\frac{x_i}{1} = \sum_{j=1}^{n} \frac{a_{ij}}{b_{ij}} m_j.$$

Aside: if $\frac{x_i}{1} = 0$, then all the a_{ij} are zero and the b_{ij} are 1 and the rest of the proof is not affected.

Using the characterization of when elements of the localization are zero (i.e. that they are s-torsion for some $s \in A \setminus \mathfrak{p}$), we have the equations

$$t_i \prod b_{ij} \left(x_i - \sum_j \frac{a_{ij}}{b_{ij}} m_i' \right) = 0$$

for some $t_i \in A \setminus \mathfrak{p}$, where the sum takes place in in M (which is a module over A). Note that we must have the factor of $\prod b_{ij}$ as one multiply "top and bottom" by this term to put the element $x_i - \sum \frac{a_{ij}}{b_{ij}} m_i$ into the form $\frac{p}{q}$ with $p, g \in M$.

Let $b := \prod_i t_i \prod_{i,j} b_{ij}$. We know that the $\frac{x_i}{1}$ generate M_b as an A_b-module by the characterization of elements in the localization as fractions (given an element of Mb, one has an equation in M for its numerator, and then one over its denominator is in A_b). Thus, the equations above and the fact that $A \setminus \mathfrak{p}$-torsion is the kernel of the localization map show that m_i' generate M_b.

Thus, we have that the following map

$$\begin{aligned} A_b^{\oplus n} &\longrightarrow M_b \\ (0, \dots, 0, 1, 0, \dots, 0) &\longmapsto m_i' \end{aligned}$$

is surjective (where the 1 is in the i-th place). Let the Kernel of the above map be denoted K. We want to consider the exact sequence

$$0 \longrightarrow K \longrightarrow A_b^{\oplus n} \longrightarrow M_b \longrightarrow 0 \ .$$

As A is noetherian, A_b is noetherian, and thus so is $A_b^{\oplus n}$. Thus, K, which is a submodule of $a_b^{\oplus n}$, is finitely generated, say by $k_1, \dots, k_\ell$.

Now, we apply the localization functor to the above exact sequence. However, we note that the second (nontrivial) map is indeed the same map as the isomorphism we gave in the first place. Thus, we conclude that $K_\mathfrak{p} = 0$. This means, by the characterization of the kernel of a localization map, that there are $s_i \in A \setminus \mathfrak{p}$ such that $s_i k_i = 0$ in A for all i. (the same is true if we replace A with A_b, one can use whichever ring is more convenient). Now, if we let $b' = b \prod_i s_i$, then we see that by the equations above and the characterization of the kernel of the localization as torsion, that $K_{b'} = 0$. This means, by the exact sequence above and/or the fact that kernels commute with localization, that $A_{b'}^{\oplus n} \cong M_{b'}$, and we have what we want.

□

Corollary 2.5.7. *Let X be a variety over k. Same comclusion as above.*

Proof. X has an affine cover of finite type k-algebras. Thus, by hilbert basis, it is (locally) noetherian. Apply the previous theorem. □

2.6. **Irreducible Schemes.** Let $|X|$ be a topological space.

Definition 2.6.1. $|X|$ is irreducible if it cannot be written as a union of two closed subsets $Z_1 \cup \mathbf{Z}_2$.

Lemma 2.6.2. *Let X be a suitable (noetherian? irred? zariski?) topological space. Then the intersection of an open with an irreducible is irreducible.*

Now, let X be a scheme with $|X|$ its associated topological space.

Definition 2.6.3. X is irreducible if $|X|$ is.

Lemma 2.6.4. *Let X be a scheme, U an affine open neighborhood and Y a closed (irreducible?) subscheme. Then $U \cap Y$ is an affine open neighborhood of Y*

Proposition 2.6.5. *Let $X = \operatorname{Spec} R$ be an affine scheme. Then the irreducible subsets of the topological space $|X|$ are in one-to-one correspondence with the prime ideals of R on the association $\mathfrak{p} \mapsto V(\mathfrak{p})$.*

Proof. This statement looks a lot like the nullstellensatz, but it actually just follows straight from the definitions. □

2.7. **Left-exactness of conormal bundle.**

Lemma 2.7.1. *Let X be a regular variety, and let Y be a regular subvariety. Let $\mathcal{I}$ be the ideal sheaf of Y. Then $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf of rank $\dim X - \dim Y$.*

Proof. Let $U = \operatorname{Spec} R$ be an affine open neighborhood of y in X.

4. $U \cap Y$ is also affine and irreducible (Y is irred) By Lemmas **??**

5. As $U \cap Y$ is irreducibe, it corresponds to a prime ideal $\mathfrak{p}$. By Lemma **??**.

6. We work in the local ring of X at $\mathfrak{p}$, which is $R_{\mathfrak{p}}$. Let $\mathfrak{m}$ be the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. Then $\mathfrak{m}/\mathfrak{m}^2$ is a free $R_{\mathfrak{p}}/\mathfrak{m}$ module of rank $\dim R_{\mathfrak{p}} = \text{ht}P = n - s$, where $n = \dim X$. By **??** and **??**

7. Thus, by Hartshorne II.5.7 (**??**) we have some f not in $\mathfrak{p}$ such that $(\mathfrak{p}/\mathfrak{p}^2)_f$ is free of rank r. □

Theorem 2.7.2. *The conormal sequence is exact on the left.*

Proof. 1. Y is regular, therefore $\Omega_{Y/k}$ is free of rank $s = \dim Y$. Likewise, X is regular and $\Omega_{X/k}$ is free of rank $n = \dim X$. By Theorem **??**

2. We have the conormal exact sequence

3. Take the stalk of this sequence at an arbitrary closed point $y \in Y$. By Theorem **??**

3a. The stalk-taking/localization pulls into the $\mathcal{I}/\mathcal{I}^2$. By Corollary **??**

3b. The stalk-taking/localization pulls into the restriction/tensor product in the middle of the sequence By Lemma **??** (tensor is a colimit, in fact a pushout).

Steps 4-7: Apply the previous lemma to conclude that $\mathcal{I}/\mathcal{I}^2$ is locally free.

Now, we have the conormal exact sequence localized at y:

$$\mathcal{I}_y/\mathcal{I}_y^2 \to \Omega_{X/k,y} \otimes_{\mathcal{O}_{X,y}} \mathcal{O}_{Y,y} \to \Omega_{Y/k,y} \to 0$$

8. The module in the middle of the (localized) exact sequence is free as it is the localization of a free module.

8a. First, we see that $\Omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is a free module, first by using part 1 to see that Ω_X is free, then using that tensor products commute with direct sums to reduce to the case of $\mathcal{O}_X$, and finally using the identity property of the tensor product. (tensor product identity property will need to be proved, but it's not currently written down)

8b. Then, we use the fact that localization pulls in/out of the construction (3b).

Let u be the surjective map in the (localized) conormal exact sequence.

9. Then $\operatorname{Ker} u$ is a projective module over $R_{\mathfrak{p}}$, as the exact sequence

$$0 \to \operatorname{Ker} u \to \Omega_{X/k,y} \otimes_{\mathcal{O}_{X,y}} \mathcal{O}_{Y,y} \to \Omega_{Y/k,y} \to 0$$

splits (both of the second components are free). By **??**

10. Ker u is a (finitely generated) projective module over a local ring, it is free. By **??**
11. By the additivity of rank on exact sequences, Ker u has rank $n - s$. By **??**
12. The first map of the localized conormal exact sequence is a surjection from $\mathcal{I}_y/\mathcal{I}_y^2$ and Ker u. Both of these are free modules of rank $n - s$. By Nakayama's lemma (the endomorphism from free to free corollary), this is an isomoprhism. By **??**
13. Thus, on stalks the conormal sequence is exact on the left, so it is exact globally. By Theorem **??**. □

2.8. Main Proof.

Theorem 2.8.1. *Theorem statement here*

Proof. The proof of this is detailed in a nice amount of detail in stack overflow, see code comment □

Theorem 2.8.2. *Theorem statement here.*

Theorem 2.8.3 (Adjunction Formula)**.** *Let X be a smooth variety and D a (smooth?) divisor. Then*

$$(\omega_X \otimes \mathcal{O}_X(D))|_D \cong \omega_D$$

Proof. We have an exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_X|_D \to \Omega_D \to 0$$

from the conormal exact sequence, and it is exact on the left by Theorem **??**. We apply fact that the determinant is "multiplicative" on short exact sequences„ concluding that

$$\omega_D \otimes \mathcal{I}/\mathcal{I}^2 \cong \omega_X|_D.$$

Note that $\mathcal{I}/\mathcal{I}^2$ has rank one, so it's determinant bundle is itself. Finally, by the alternate description of the conormal sheaf (reference), we tensor both sides by $\mathcal{O}_X(D)$ which is the inverse of the conormal sheaf, and we conclude the thoerem.

□