Robust PCA

CS5240 Theoretical Foundations in Multimedia

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Previously...

not robust against outliers		robust against outliers
linear least squares	➡	trimmed least squares
PCA	➡	trimmed PCA

Various ways to robustify PCA:

- trimming: remove outliers
- covariance matrix with 0-1 weight [Xu95]: similar to trimming
- weighted SVD [Gabriel79]: weighting
- robust error function [Torre2001]: winsorizing

Strength: simple concepts
Weakness: no guarantee of optimality

Robust PCA

PCA can be formulated as follows:

> *Given a data matrix* $\mathbf{D}$, *recover a low-rank matrix* $\mathbf{A}$ *from* $\mathbf{D}$ *such that the error* $\mathbf{E} = \mathbf{D} - \mathbf{A}$ *is minimized:*

$$\min_{\mathbf{A},\mathbf{E}} \|\mathbf{E}\|_F, \text{ subject to } \operatorname{rank}(\mathbf{A}) \leq r, \; \mathbf{D} = \mathbf{A} + \mathbf{E}. \tag{1}$$

- $r \ll \min(m, n)$ is the target rank of $\mathbf{A}$.
- $\| \cdot \|_F$ is the Frobenius norm.

Notes:

- This definition of PCA includes dimensionality reduction.
- PCA is severely affected by large-amplitude noise; not robust.

[Wright2009] formulated the **Robust PCA** problem as follows:

> *Given a data matrix* $\mathbf{D} = \mathbf{A} + \mathbf{E}$ *where* $\mathbf{A}$ *and* $\mathbf{E}$ *are unknown but* $\mathbf{A}$ *is low-rank and* $\mathbf{E}$ *is sparse, recover* $\mathbf{A}$*.*

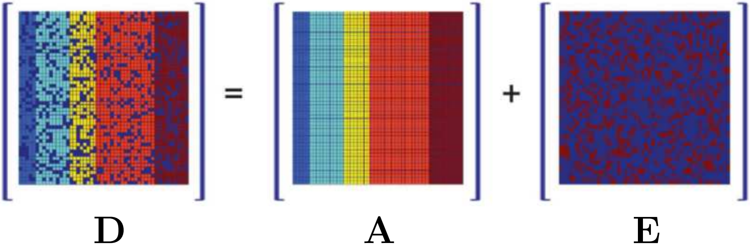
An obvious way to state the robust PCA problem in math is:

> *Given a data matrix* $\mathbf{D}$*, find* $\mathbf{A}$ *and* $\mathbf{E}$ *that solve the problem*

$$\min_{\mathbf{A},\mathbf{E}} \operatorname{rank}(\mathbf{A}) + \lambda \|\mathbf{E}\|_0, \text{ subject to } \mathbf{A} + \mathbf{E} = \mathbf{D}. \tag{2}$$

- λ is a Lagrange multiplier.
- $\|\mathbf{E}\|_0$: l_0-norm, number of non-zero elements in $\mathbf{E}$.
  $\mathbf{E}$ is sparse if $\|\mathbf{E}\|_0$ is small.

$$\min_{\mathbf{A},\mathbf{E}} \operatorname{rank}(\mathbf{A}) + \lambda \|\mathbf{E}\|_0, \text{ subject to } \mathbf{A} + \mathbf{E} = \mathbf{D}. \tag{2}$$



- Problem 2 is a **matrix recovery** problem.
- $\operatorname{rank}(\mathbf{A})$ and $\|\mathbf{E}\|_0$ are not continuous, not convex; very hard to solve; no efficient algorithm.

[Candès2011, Wright2009] reformulated Problem 2 as follows:

Given an $m \times n$ data matrix $\mathbf{D}$, find $\mathbf{A}$ and $\mathbf{E}$ that solve

$$\min_{\mathbf{A},\mathbf{E}} \|\mathbf{A}\|_* + \lambda\|\mathbf{E}\|_1, \text{ subject to } \mathbf{A} + \mathbf{E} = \mathbf{D}. \tag{3}$$

- $\|\mathbf{A}\|_*$: nuclear norm, sum of singular values of $\mathbf{A}$; surrogate for rank($\mathbf{A}$).
- $\|\mathbf{E}\|_1$: l_1-norm, sum of absolute values of elements of $\mathbf{E}$; surrogate for $\|\mathbf{E}\|_0$.

Solution of Problem 3 can be recovered exactly if

- $\mathbf{A}$ is sufficiently low-rank but not sparse, and
- $\mathbf{E}$ is sufficiently sparse but not low-rank,

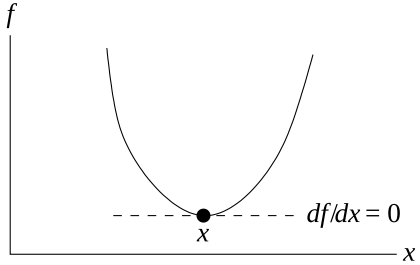
with optimal $\lambda = 1/\sqrt{\max(m, n)}$.

- $\|\mathbf{A}\|_*$ and $\|\mathbf{E}\|_1$ are convex; can apply convex optimization.

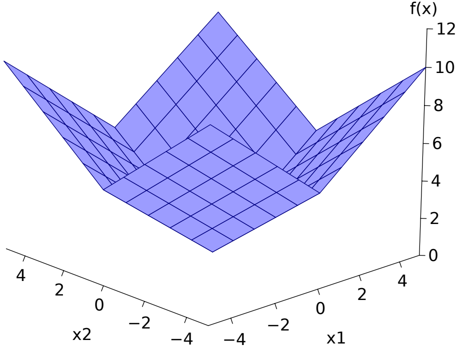
Introduction to Convex Optimization

For a differentiable function $f(\mathbf{x})$, its minimizer $\widehat{\mathbf{x}}$ is given by

$$\frac{df(\widehat{\mathbf{x}})}{d\mathbf{x}} = \left[\frac{\partial f(\widehat{\mathbf{x}})}{\partial x_1} \;\; \cdots \;\; \frac{\partial f(\widehat{\mathbf{x}})}{\partial x_m}\right]^\top = \mathbf{0}. \tag{4}$$



$\|\mathbf{E}\|_1$ is not differentiable when any of its element is zero!



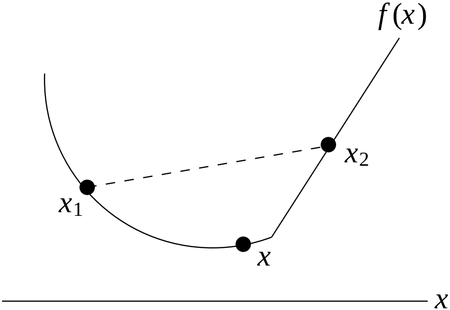
Cannot write the following because they don't exist:

$$\frac{d\|\mathbf{E}\|_1}{d\mathbf{E}}, \quad \frac{\partial\|\mathbf{E}\|_1}{\partial e_{ij}}, \quad \frac{d|e_{ij}|}{de_{ij}}. \quad \textbf{WRONG!} \tag{5}$$

Fortunately, $\|\mathbf{E}\|_1$ is convex.

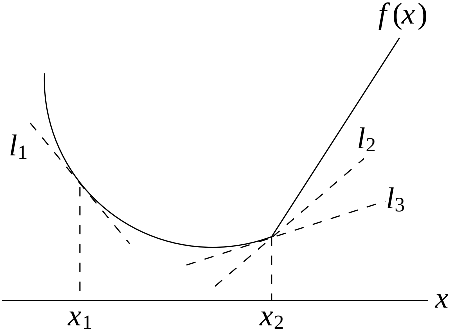
A function $f(\mathbf{x})$ is convex if and only if $\forall \mathbf{x}_1, \mathbf{x}_2$, $\forall \alpha \in [0, 1]$,

$$f(\alpha \mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1 - \alpha) f(\mathbf{x}_2). \tag{6}$$



A vector $\mathbf{g}(\mathbf{x})$ is a subgradient of convex function f at $\mathbf{x}$ if

$$f(\mathbf{y}) - f(\mathbf{x}) \geq \mathbf{g}(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}), \quad \forall \mathbf{y}. \tag{7}$$



- At differentiable point x_1: one unique subgradient = gradient.
- At non-differentiable point x_2: multiple subgradients.

The subdifferential $\partial f(\mathbf{x})$ is the **set** of all subgradients of f at $\mathbf{x}$:

$$\partial f(\mathbf{x}) = \left\{ \mathbf{g}(\mathbf{x}) \,|\, \mathbf{g}(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) \le f(\mathbf{y}) - f(\mathbf{x}), \ \forall \mathbf{y} \right\}. \tag{8}$$

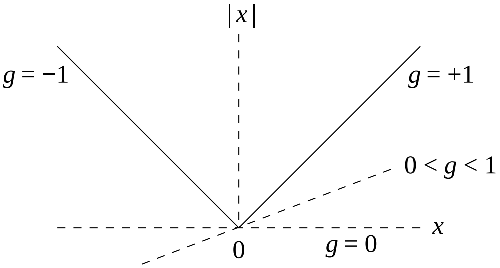
Caution:

- Subdifferential $\partial f(\mathbf{x})$ is **not** partial differentiation $\partial f(\mathbf{x})/\partial \mathbf{x}$. Don't mix up.
- Partial differentiation is defined for differentiable functions. It is a single vector.
- Subdifferential is defined for convex functions, which may be non-differentiable. It is a set of vectors.

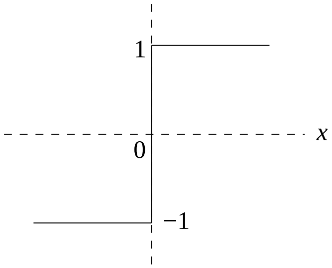
Example: Absolute value.

$|x|$ is not differentiable at $x = 0$ but subdifferentiable at $x = 0$:

$$|x| = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -x & \text{if } x < 0. \end{cases} \qquad \partial|x| = \begin{cases} \{+1\} & \text{if } x > 0, \\ [-1, +1] & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0. \end{cases} \tag{9}$$



absolute value



subdifferential

Example: l_1-norm of an m-D vector $\mathbf{x}$.

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{m} |x_i|, \quad \partial\|\mathbf{x}\|_1 = \sum_{i=1}^{m} \partial|x_i|. \tag{10}$$

Caution: Right-hand-side of Eq. 10 is **set addition**.
This gives a product of m sets, one for each element x_i:

$$\partial\|\mathbf{x}\|_1 = J_1 \times \cdots \times J_m, \quad J_i = \begin{cases} \{+1\} & \text{if } x_i > 0, \\ [-1, +1] & \text{if } x_i = 0, \\ \{-1\} & \text{if } x_i < 0. \end{cases} \tag{11}$$

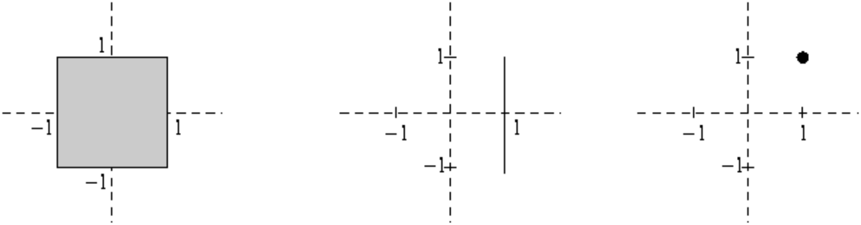
Alternatively, we can write $\partial\|\mathbf{x}\|_1$ as

$$\partial\|\mathbf{x}\|_1 = \{\,\mathbf{g}\,\} \text{ such that } g_i \begin{cases} = +1 & \text{if } x_i > 0, \\ \in [-1, +1] & \text{if } x_i = 0, \\ = -1 & \text{if } x_i < 0. \end{cases} \tag{12}$$

Another alternative:

$$\partial\|\mathbf{x}\|_1 = \{\,\mathbf{g}\,\} \;\text{ such that }\; \begin{cases} g_i = \operatorname{sgn}(g_i) & \text{if } |x_i| > 0, \\ |g_i| \le 1 & \text{if } x_i = 0. \end{cases} \tag{13}$$

Here are some examples of the subdifferentials of 2D l_1-norm.

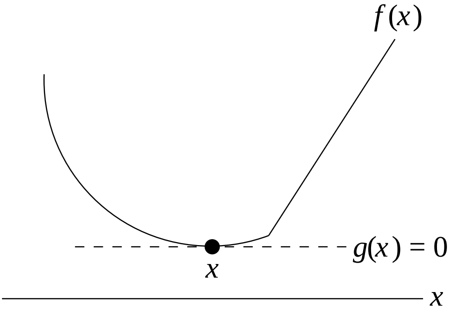


(a) (b) (c)

$\partial f(0,0) = [-1,1] \times [-1,1]$ $\quad\partial f(1,0) = \{1\} \times [-1,1]$ $\quad\partial f(1,1) = \{(1,1)\}$

Optimality Condition

$\mathbf{x}$ is a minimum of f if $\mathbf{0} \in \partial f(\mathbf{x})$, i.e., $\mathbf{0}$ is a subgradient of f.

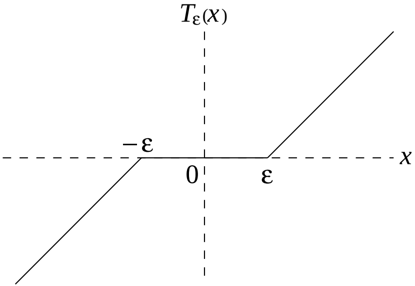


For more details on convex functions and convex optimization, refer to [Bertsekas2003, Boyd2004, Rockafellar70, Shor85].

Back to Robust PCA

Shrinkage or **soft threshold** operator:

$$T_\varepsilon(x) = \begin{cases} x - \varepsilon & \text{if } x > \varepsilon, \\ x + \varepsilon & \text{if } x < -\varepsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$



Using convex optimization, the following minimizers are shown [Cai2010, Hale2007]:

For an $m \times n$ matrix $\mathbf{M}$ with SVD $\mathbf{U}\mathbf{S}\mathbf{V}^\top$,

$$\mathbf{U}T_\varepsilon(\mathbf{S})\mathbf{V}^\top = \arg\min_{\mathbf{X}} \varepsilon\|\mathbf{X}\|_* + \frac{1}{2}\|\mathbf{X} - \mathbf{M}\|_F^2, \tag{15}$$

$$T_\varepsilon(\mathbf{M}) = \arg\min_{\mathbf{X}} \varepsilon\|\mathbf{X}\|_1 + \frac{1}{2}\|\mathbf{X} - \mathbf{M}\|_F^2. \tag{16}$$

Solving Robust PCA

There are several ways to solve robust PCA (Problem 3) [Lin2009,Wright2009]:

- principal component pursuit
- iterative thresholding
- accelerated proximal gradient
- augmented Lagrange multipliers

Augmented Lagrange Multipliers

Consider the problem

$$\min_{\mathbf{x}} f(\mathbf{x}) \quad \text{subject to } c_j(\mathbf{x}) = 0, j = 1, \ldots, m. \tag{17}$$

This is a constrained optimization problem.

Lagrange multipliers method reformulates Problem 17 as:

$$\min_{\mathbf{x}} f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j c_j(\mathbf{x}) \tag{18}$$

with some constants λ_j.

Penalty method reformulates Problem 17 as:

$$\min_{\mathbf{x}} f(\mathbf{x}) + \mu \sum_{j=1}^{m} c_j^2(\mathbf{x}). \tag{19}$$

- Parameter μ increases over iteration, e.g., by factor of 10.
- Need $\mu \to \infty$ to get good solution.

Augmented Lagrange multipliers method combines Lagrange multipliers and penalty:

$$\min_{\mathbf{x}} f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j c_j(\mathbf{x}) + \frac{\mu}{2} \sum_{j=1}^{m} c_j^2(\mathbf{x}). \tag{20}$$

Denote $\boldsymbol{\lambda} = [\lambda_1 \ \cdots \ \lambda_m]^\top$, $\mathbf{c}(\mathbf{x}) = [c_1(\mathbf{x}) \ \cdots \ c_m(\mathbf{x})]^\top$.
Then, Problem 20 becomes

$$\min_{\mathbf{x}} f(\mathbf{x}) + \boldsymbol{\lambda}^\top \mathbf{c}(\mathbf{x}) + \frac{\mu}{2} \|\mathbf{c}(\mathbf{x})\|^2. \tag{21}$$

If the constraints form a matrix $\mathbf{C} = [c_{jk}]$, then define $\boldsymbol{\Lambda} = [\lambda_{jk}]$.

Then Problem 20 becomes

$$\min_{\mathbf{x}} f(\mathbf{x}) + \langle \boldsymbol{\Lambda}, \mathbf{C}(\mathbf{x}) \rangle + \frac{\mu}{2} \|\mathbf{C}(\mathbf{x})\|_F^2. \tag{22}$$

- $\langle \boldsymbol{\Lambda}, \mathbf{C} \rangle$ is sum of product of corresponding elements:

$$\langle \boldsymbol{\Lambda}, \mathbf{C} \rangle = \sum_j \sum_k \lambda_{jk} c_{jk}.$$

- $\|\mathbf{C}\|_F$ is the Frobenius norm:

$$\|\mathbf{C}\|_F^2 = \sum_j \sum_k c_{jk}^2.$$

Augmented Lagrange Multipliers (ALM) Method

1. Initialize $\mathbf{\Lambda}$, $\mu > 0$, $\rho \geq 1$.
2. Repeat until convergence:
    2.1 Compute $\mathbf{x} = \arg\min_{\mathbf{x}} L(\mathbf{x})$ where
    $$L(\mathbf{x}) = f(\mathbf{x}) + \langle \mathbf{\Lambda}, \mathbf{C}(\mathbf{x}) \rangle + \frac{\mu}{2} \|\mathbf{C}(\mathbf{x})\|_F^2.$$
    2.2 Update $\mathbf{\Lambda} = \mathbf{\Lambda} + \mu \mathbf{C}(\mathbf{x})$.
    2.3 Update $\mu = \rho\mu$.

What kind of optimization algorithm is this?

ALM does not need $\mu \to \infty$ to get good solution.
That means, can converge faster.

With ALM, robust PCA (Problem 3) is reformulated as

$$\min_{\mathbf{A},\mathbf{E}} \|\mathbf{A}\|_* + \lambda\|\mathbf{E}\|_1 + \langle \mathbf{Y}, \mathbf{D} - \mathbf{A} - \mathbf{E}\rangle + \frac{\mu}{2}\|\mathbf{D} - \mathbf{A} - \mathbf{E}\|_F^2, \tag{23}$$

- $\mathbf{Y}$ are the Lagrange multipliers.
- Constraints $\mathbf{C} = \mathbf{D} - \mathbf{A} - \mathbf{E}$.

To implement Step 2.1 of ALM, need to find minima for $\mathbf{A}$ and $\mathbf{E}$.
Adopt alternating optimization.

Trace of Matrix

The trace of a matrix $\mathbf{A}$ is the sum of its diagonal elements a_{ii}:

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}. \tag{24}$$

Strictly speaking, scalar c and a 1×1 matrix $[\,c\,]$ are not the same thing. Nevertheless, since $\operatorname{tr}([\,c\,]) = c$, we often write, for simplicity $\operatorname{tr}(c) = c$.

Properties

- $\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$, where λ_i are the eigenvalues of $\mathbf{A}$.
- $\operatorname{tr}(\mathbf{A}+\mathbf{B}) = \operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B})$
- $\operatorname{tr}(c\mathbf{A}) = c\operatorname{tr}(\mathbf{A})$
- $\operatorname{tr}(\mathbf{A}) = \operatorname{tr}(\mathbf{A}^\top)$

- $\text{tr}(\mathbf{ABCD}) = \text{tr}(\mathbf{BCDA}) = \text{tr}(\mathbf{CDAB}) = \text{tr}(\mathbf{DABC})$
- $\text{tr}(\mathbf{X}^\top\mathbf{Y}) = \text{tr}(\mathbf{X}\mathbf{Y}^\top) = \text{tr}(\mathbf{Y}^\top\mathbf{X}) = \text{tr}(\mathbf{Y}\mathbf{X}^\top) = \displaystyle\sum_i \sum_j x_{ij} y_{ij}$

From Problem 23, the minimal $\mathbf{E}$ with other variables fixed is given by

$$
\begin{aligned}
& \min_{\mathbf{E}} \lambda \|\mathbf{E}\|_1 + \langle \mathbf{Y}, \mathbf{D} - \mathbf{A} - \mathbf{E} \rangle + \frac{\mu}{2} \|\mathbf{D} - \mathbf{A} - \mathbf{E}\|_F^2 \\
\Rightarrow \quad & \min_{\mathbf{E}} \lambda \|\mathbf{E}\|_1 + \mathrm{tr}(\mathbf{Y}^\top (\mathbf{D} - \mathbf{A} - \mathbf{E})) + \frac{\mu}{2} \|\mathbf{D} - \mathbf{A} - \mathbf{E}\|_F^2 \\
\Rightarrow \quad & \min_{\mathbf{E}} \lambda \|\mathbf{E}\|_1 + \mathrm{tr}(-\mathbf{Y}^\top \mathbf{E}) + \frac{\mu}{2} \mathrm{tr}((\mathbf{D} - \mathbf{A} - \mathbf{E})^\top (\mathbf{D} - \mathbf{A} - \mathbf{E})) \\
\vdots \quad & \text{(Homework)} \\
\Rightarrow \quad & \min_{\mathbf{E}} \lambda \|\mathbf{E}\|_1 + \frac{\mu}{2} \mathrm{tr}((\mathbf{E} - (\mathbf{D} - \mathbf{A} + \mathbf{Y}/\mu))^\top (\mathbf{E} - (\mathbf{D} - \mathbf{A} + \mathbf{Y}/\mu))) \\
\Rightarrow \quad & \min_{\mathbf{E}} \lambda \|\mathbf{E}\|_1 + \frac{\mu}{2} \|\mathbf{E} - (\mathbf{D} - \mathbf{A} + \mathbf{Y}/\mu)\|_F^2 \\
\Rightarrow \quad & \min_{\mathbf{E}} \frac{\lambda}{\mu} \|\mathbf{E}\|_1 + \frac{1}{2} \|\mathbf{E} - (\mathbf{D} - \mathbf{A} + \mathbf{Y}/\mu)\|_F^2. \qquad (25)
\end{aligned}
$$

Compare Eq. 16

$$T_\varepsilon(\mathbf{M}) = \arg\min_{\mathbf{X}} \varepsilon \|\mathbf{X}\|_1 + \frac{1}{2}\|\mathbf{X} - \mathbf{M}\|_F^2,$$

and Problem 25

$$\min_{\mathbf{E}} \frac{\lambda}{\mu}\|\mathbf{E}\|_1 + \frac{1}{2}\|\mathbf{E} - (\mathbf{D} - \mathbf{A} + \mathbf{Y}/\mu)\|_F^2.$$

Set $\mathbf{X} = \mathbf{E}$, $\mathbf{M} = \mathbf{D} - \mathbf{A} + \mathbf{Y}/\mu$. Then,

$$\mathbf{E} = T_{\lambda/\mu}(\mathbf{D} - \mathbf{A} + \mathbf{Y}/\mu). \tag{26}$$

From Problem 23, the minimal $\mathbf{A}$ with other variables fixed is given by

$$\begin{aligned} & \min_{\mathbf{A}} \|\mathbf{A}\|_* + \langle \mathbf{Y}, \mathbf{D} - \mathbf{A} - \mathbf{E} \rangle + \frac{\mu}{2} \|\mathbf{D} - \mathbf{A} - \mathbf{E}\|_F^2 \\ \Rightarrow \quad & \min_{\mathbf{A}} + \operatorname{tr}(\mathbf{Y}^\top (\mathbf{D} - \mathbf{A} - \mathbf{E})) + \frac{\mu}{2} \|\mathbf{D} - \mathbf{A} - \mathbf{E}\|_F^2 \\ \vdots \quad & \text{(Homework)} \\ \Rightarrow \quad & \min_{\mathbf{A}} \frac{1}{\mu} \|\mathbf{A}\|_* + \frac{1}{2} \|\mathbf{A} - (\mathbf{D} - \mathbf{E} + \mathbf{Y}/\mu)\|_F^2 \end{aligned} \tag{27}$$

Compare Problem 27 and Eq. 15

$$\mathbf{U} T_\varepsilon(\mathbf{S}) \mathbf{V}^\top = \arg\min_{\mathbf{X}} \varepsilon \|\mathbf{X}\|_* + \frac{1}{2} \|\mathbf{X} - \mathbf{M}\|_F^2,$$

Set $\mathbf{X} = \mathbf{A}$, $\mathbf{M} = \mathbf{D} - \mathbf{E} + \mathbf{Y}/\mu$. Then,

$$\mathbf{A} = \mathbf{U}\, T_{1/\mu}(\mathbf{S})\, \mathbf{V}^\top. \tag{28}$$

Robust PCA for Matrix Recovery via ALM

Inputs: $\mathbf{D}$.

1. Initialize $\mathbf{A} = \mathbf{0}$, $\mathbf{E} = \mathbf{0}$.
2. Initialize $\mathbf{Y}$, $\mu > 0$, $\rho > 1$.
3. Repeat until convergence:
4. Repeat until convergence:
5. $\mathbf{U}, \mathbf{S}, \mathbf{V} = \text{SVD}(\mathbf{D} - \mathbf{E} + \mathbf{Y}/\mu)$.
6. $\mathbf{A} = \mathbf{U}\, T_{1/\mu}(\mathbf{S})\, \mathbf{V}^\top$.
7. $\mathbf{E} = T_{\lambda/\mu}(\mathbf{D} - \mathbf{A} + \mathbf{Y}/\mu)$.
8. Update $\mathbf{Y} = \mathbf{Y} + \mu(\mathbf{D} - \mathbf{A} - \mathbf{E})$.
9. Update $\mu = \rho\mu$.

Outputs: $\mathbf{A}$, $\mathbf{E}$.

Typical initialization [Lin2009]:

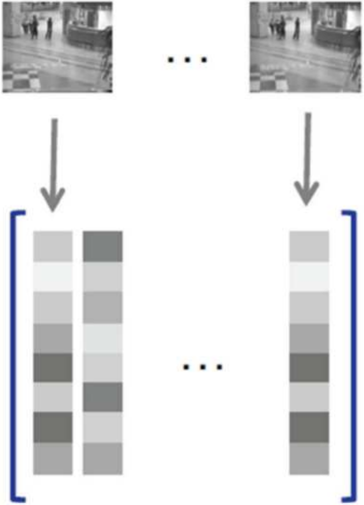
- $\mathbf{Y} = \text{sgn}(\mathbf{D})/J(\text{sgn}(\mathbf{D}))$.
- $\text{sgn}(\mathbf{D})$ gives sign of each matrix element of $\mathbf{D}$.
- $J(\cdot)$ gives scaling factors:

$$J(\mathbf{X}) = \max(\|\mathbf{X}\|_2, \lambda^{-1}\|\mathbf{X}\|_\infty).$$

- $\|\mathbf{X}\|_2$ is spectral norm, largest singular value of $\mathbf{X}$.
- $\|\mathbf{X}\|_\infty$ is largest absolute value of elements of $\mathbf{X}$.
- $\mu = 1.25\,\|\mathbf{D}\|_2$.
- $\rho = 1.5$.
- $\lambda = 1/\sqrt{\max(m, n)}$ for $m \times n$ matrix $\mathbf{D}$.

Example: Recovery of video background.

video frames

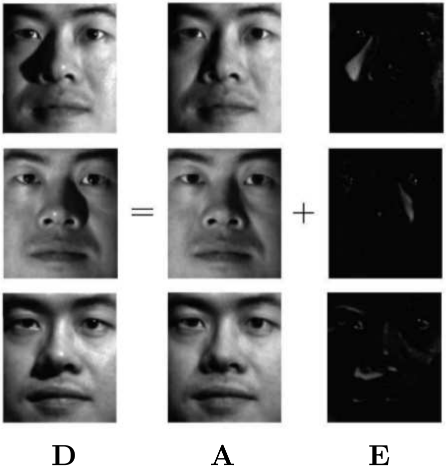


data matrix $\mathbf{D}$

Sample results:

data matrix low-rank error low-rank error

Example: Removal of specular reflection and shadow.



Fixed-Rank Robust PCA

In reflection removal, reflection may be global.

ground-truth

input

Then, $\mathbf{E}$ is not sparse: violate RPCA condition!
But, rank of $\mathbf{A} = 1$.

Fix the rank of $\mathbf{A}$ to deal with non-sparse $\mathbf{E}$ [Leow2013].

Fixed-Rank Robust PCA via ALM

Inputs: $\mathbf{D}$.

1. Initialize $\mathbf{A} = \mathbf{0}$, $\mathbf{E} = \mathbf{0}$.
2. Initialize $\mathbf{Y}$, $\mu > 0$, $\rho > 1$.
3. Repeat until convergence:
4. Repeat until convergence:
5. $\mathbf{U}, \mathbf{S}, \mathbf{V} = \text{SVD}(\mathbf{D} - \mathbf{E} + \mathbf{Y}/\mu)$.
6. If $\text{rank}(T_{1/\mu}(\mathbf{S})) < r$, $\mathbf{A} = \mathbf{U}\, T_{1/\mu}(\mathbf{S})\, \mathbf{V}^\top$; else $\mathbf{A} = \mathbf{U}\, \mathbf{S}_r \mathbf{V}^\top$.
7. $\mathbf{E} = T_{\lambda/\mu}(\mathbf{D} - \mathbf{A} + \mathbf{Y}/\mu)$.
8. Update $\mathbf{Y} = \mathbf{Y} + \mu(\mathbf{D} - \mathbf{A} - \mathbf{E})$.
9. Update $\mu = \rho\mu$.

Outputs: $\mathbf{A}$, $\mathbf{E}$.

$\mathbf{S}_r$ is $\mathbf{S}$ with last $m - r$ singular values set to 0.

Example: Removal of local reflections.

ground-truth

input

FRPCA

RPCA

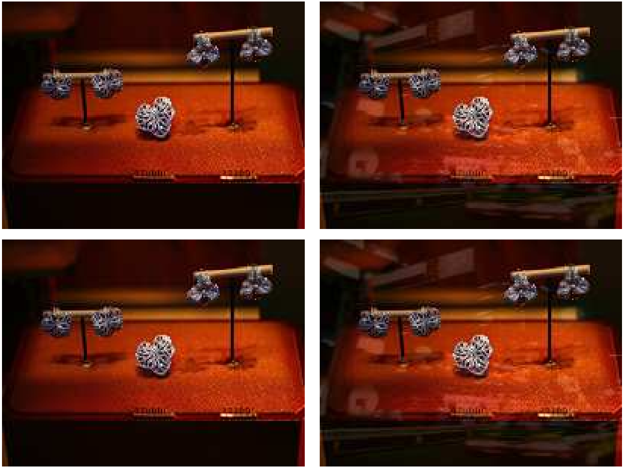
Example: Removal of global reflections.

ground-truth input

FRPCA RPCA

Example: Removal of global reflections.

ground-truth input



FRPCA RPCA

Example: Background recovery for traffic video: fast moving vehicles.

input

FRPCA

RPCA

Example: Background recovery for traffic video: slow moving vehicles.

input

FRPCA

RPCA

Example: Background recovery for traffic video: temporary stop.

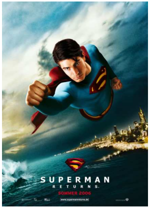
input

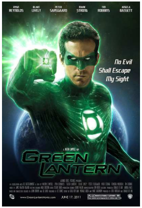
FRPCA

RPCA

Matrix Completion

Customers are asked to rate the movies from 1 (poor) to 5 (excellent).





A	5	5	3	2		
B		3	4			3
C	3			4	5	4
D	4		5	5	3	

⋮

Customers rate only some movies ⇒ some data are missing.
How to estimate the missing data? matrix completion.

Let $\mathbf{D}$ denote data matrix with missing elements set to 0, and $M = \{(i, j)\}$ denote the indices of missing elements in $\mathbf{D}$.

Then, the **matrix completion** problem can be formulated as

Given $\mathbf{D}$ *and* M*, find matrix* $\mathbf{A}$ *that solves the problem*

$$\min_{\mathbf{A}} \|\mathbf{A}\|_* \quad \text{subject to } \mathbf{A} + \mathbf{E} = \mathbf{D}, E_{ij} = 0 \; \forall \, (i, j) \notin M. \tag{29}$$

- For $(i, j) \notin M$, constrain $E_{ij} = 0$ so that $A_{ij} = D_{ij}$; no change.
- For $(i, j) \in M$, $D_{ij} = 0$, i.e., $A_{ij} = E_{ij}$; recovered value.

Reformulating Problem 29 using augmented Lagrange multipliers gives

$$\min_{\mathbf{A}} \|\mathbf{A}\|_* + \langle \mathbf{Y}, \mathbf{D} - \mathbf{A} - \mathbf{E} \rangle + \frac{\mu}{2} \|\mathbf{D} - \mathbf{A} - \mathbf{E}\|_F^2 \tag{30}$$

such that $E_{ij} = 0 \; \forall \, (i, j) \notin M$.

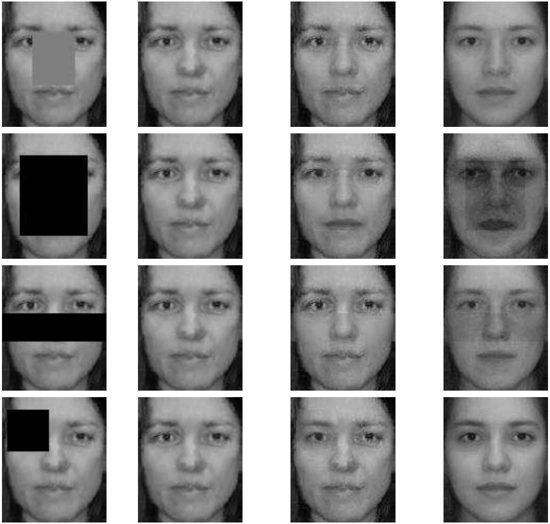
Robust PCA for Matrix Completion

Inputs: $\mathbf{D}$.

1. Initialize $\mathbf{A} = \mathbf{0}$, $\mathbf{E} = \mathbf{0}$.
2. Initialize $\mathbf{Y}$, $\mu > 0$, $\rho > 1$.
3. Repeat until convergence:
4. Repeat until convergence:
5. $\mathbf{U}, \mathbf{S}, \mathbf{V} = \mathrm{SVD}(\mathbf{D} - \mathbf{E} + \mathbf{Y}/\mu)$.
6. $\mathbf{A} = \mathbf{U}\, T_{1/\mu}(\mathbf{S})\, \mathbf{V}^\top$.
7. $\mathbf{E} = \Gamma_M(\mathbf{D} - \mathbf{A} + \mathbf{Y}/\mu)$, where
$$\Gamma_M(\mathbf{X}) = \begin{cases} X_{ij}, & \text{for } (i,j) \in M. \\ 0, & \text{for } (i,j) \notin M, \end{cases}$$
8. Update $\mathbf{Y} = \mathbf{Y} + \mu(\mathbf{D} - \mathbf{A} - \mathbf{E})$.
9. Update $\mu = \rho\mu$.

Outputs: $\mathbf{A}$, $\mathbf{E}$.

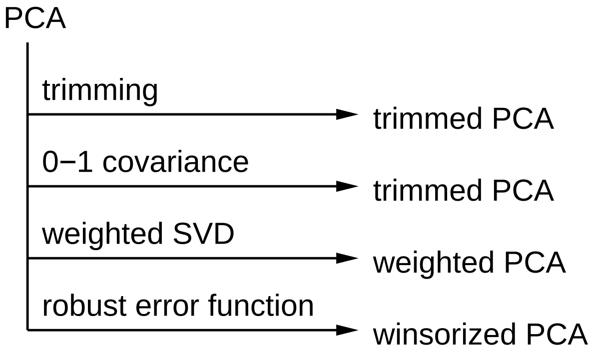
Example: Recovery of occluded parts in face images.



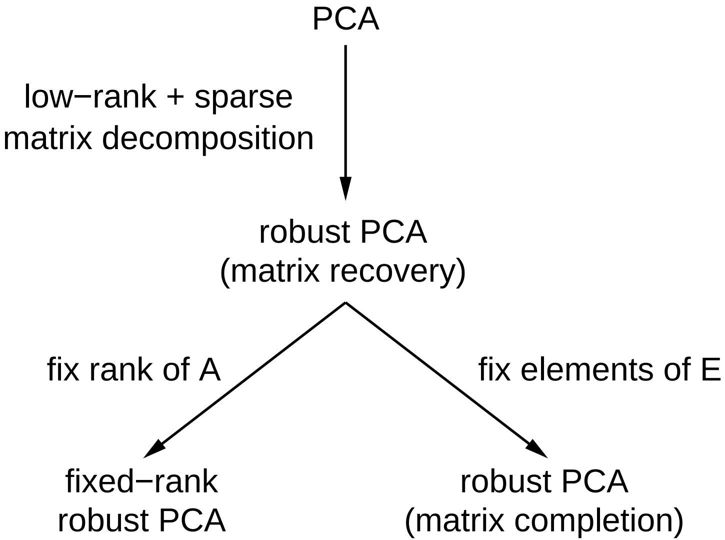
input | ground truth | matrix completion | matrix recovery

Summary

Robustification of PCA



Robust PCA


PCA
low−rank + sparse
matrix decomposition
robust PCA
(matrix recovery)
fix rank of A
fix elements of E
fixed−rank
robust PCA
robust PCA
(matrix completion)

Probing Questions

- If the data matrix of a problem is composed of a low-rank matrix, a sparse matrix and something else, can you still use robust PCA methods? If yes, how? If no, why?
- In application of robust PCA to high-resolution colour image processing, the data matrix contains three times as many rows as the number of pixels in the images, which can lead to a very large data matrix that takes a long time to compute. Suggest a way to overcome this problem.
- In application of robust PCA to video processing, the data matrix contains as many columns as the number of video frames, which can lead to a very large data matrix that is more than the available memory required to store the matrix. Suggest a way to overcome this problem.

Homework

1. Show that
$$\mathrm{tr}(\mathbf{X}^\top \mathbf{Y}) = \sum_i \sum_j x_{ij} y_{ij}.$$

2. Show that the following two optimization problems are equivalent:
$$\min_{\mathbf{E}} \lambda \|\mathbf{E}\|_1 + \mathrm{tr}(-\mathbf{Y}^\top \mathbf{E}) + \frac{\mu}{2} \mathrm{tr}((\mathbf{D} - \mathbf{A} - \mathbf{E})^\top (\mathbf{D} - \mathbf{A} - \mathbf{E}))$$
$$\min_{\mathbf{E}} \lambda \|\mathbf{E}\|_1 + \frac{\mu}{2} \mathrm{tr}((\mathbf{E} - (\mathbf{D} - \mathbf{A} + \mathbf{Y}/\mu))^\top (\mathbf{E} - (\mathbf{D} - \mathbf{A} + \mathbf{Y}/\mu)))$$

3. Show that the minimal $\mathbf{A}$ of Problem 23 with other variables fixed is given by
$$\min_{\mathbf{A}} \frac{1}{\mu} \|\mathbf{A}\|_* + \frac{1}{2} \|\mathbf{A} - (\mathbf{D} - \mathbf{E} + \mathbf{Y}/\mu)\|_F^2.$$

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