Mathematics Internal Assessment

How long does it take to make a grilled cheese sandwich?

word count: 1566

Contents

1 Introduction

This study aims to mathematically model **the least amount of time in which a grilled cheese sandwich can be made**. When making a grilled cheese sandwich, one is often faced with the challenge of balancing the temperature of the pan and the time it takes to finish melting the cheese. Originally, the topic was chosen in jest, though in attempting to research an answer, it became clear the problem was far more interesting and complex than I anticipated, which means that a few over-simplifications will be made. In mathematical terms, this is a clear study on one-dimensional heat conduction, which has been thoroughly studied – especially with so-called homogeneous boundaries, meaning that the temperature on the boundaries is always zero. In this study we will take case that does not have homogeneous boundaries, turn them homogeneous and attempt to solve them.

First let us tackle the practical definitions of our modelled grilled cheese: There are two parts to our sandwich, two slices of toast and one slice of cheese inbetween them, and it is prepared on a hot pan such that only one slice of toast can touch the pan at a time. The definition of how much time is acceptable is completely arbitrary, so we can opt a maximum of 10 minutes for discussion purposes, though matematically a variable will be used. The process that is being modelled is as follows:

1. the entire sandwich has a uniform initial temperature throughout it, $t_{\min}$;
2. Once the sandwich is placed on the the pan at $t = 0$, one side will touch the pan, which we define as the hot side ($x = 0$), which is at temperature $T_{\max}$. This is a temperature such that the sandwich would never burn, and so it can reach thermal equilibrium with the pan.
3. Conversely, there must be a cold side, at position $x = L$ and temperature $T_{\min}$, such that the difference between their positions is the thickness of the two toasts combined (L). The thickness of the cheese is negligible;

4. After $t_{\max}$ time elapses, the middle of the sandwich will reach the cheese's melting temperature, T_{melt}. The sandwich is then flipped, heated for the same amount of time $t_{\max}$, and the sandwich is said to be done;

5. Thus, the answer we search for is twice the time it takes for the temperature at $\frac{L}{2}$ (where the cheese is) to reach T_{melt}.

Theoretically, if enough time elapses we can expect that the cold side will reach T_{melt}, but from practical experience, expecting the top of the sandwich to be warm after heating it from below is completely impractical. This allows us to simplify our model even further by suggesting that the temperature on the cold side must be constant.

There are many challenges to modelling the sandwich, including density, specific heat capacity (the amount of energy needed to change a mass of 1 kg the temperature by 1 K), though these will lumped together under a constant. Instead we take the sandwich to consist of a simple cuboid with some physical properties which will be discussed below, and we will forego the use of physical units. A visualisation of most of what was so far can be seen in figure 1.

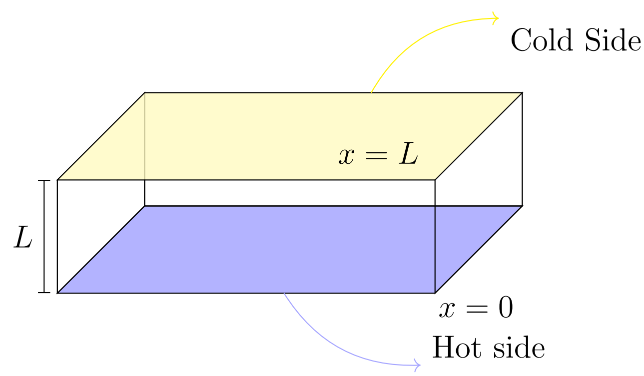

Figure 1: Section of the simplified infinitely long and wide toast with area A and thickness L.

From the above text, we can define a temperature function (u) dependent on both space

(x) and time (t), $u(x,t)$, with the following initial and boundary conditions:

$$\begin{aligned} \text{Initial: } & u(x,0) = T_{\min}, \forall x \neq 0 \qquad && u(0,0) = T_{\max} \\ \text{Boundary: } & u(L,t) = T_{\min} && u(0,t) = T_{\max} \end{aligned}$$

And with this function we can formalise the final answer we seek:

$$u(\frac{L}{2}, t_{\max}) = T_{melt},\ T_{\min} < T_{melt} < T_{\max}$$

from which we must then isolate $t_{\max}$, and multiply it by 2.

1.1 Physics

Consider the physics of the sandwich in a pan: The cheese may melt due to the conduction of the heat going through the bread, the convection of the warm air inside the pan, the radiation of heat from the pan itself, or – more appropriately – a combination of all of them, plus many more nuances on the edges of the sandwich. This leads us to the following simplifications for our model which tackle all of the above. Our overall model can be seen in figure 1.

- The sandwich is infinitely long and wide, and has thickness L;
- Since it's infinitely long, all of concepts discussed henceforth are implicitely "per unit area";
- The sandwich is in space, hence there is only conduction;
- The heat transfer is purely one-dimensional.

Finally, we must also introduce Fourier's heat conduction equation, relating the rate of temperature change to the gradient, such that k is a constant of proportionality related to the material's density and specific heat capacity (Lienhard IV and Lienhard V, 2001, pp.17-19).

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

What is important to understand about partial derivatives (denoted by ∂) is that we treat the other function's variables as constants, so that $\frac{\partial u}{\partial t}$ evaluates the change in u due to t, while x is kept constant. This means that the partial derivative of a single-variable function is the same as an ordinary derivative.

2 Solution

Consider if enough time has elapsed, far more than the minimum we seek, such that the temperature in the entire toast is in equilibrium (meaning it no longer changes, rather than the same temperature throughout). If we define this as a function, then this equilibrium temperature, $u_e(x)$, would have to be such that it satisfies our established boundary conditions as well as Fourier's equation (1.1). Below we opt for $t \to \infty$, as for practical purposes, whether the sandwich takes 30 minutes or 30 hours to make is irrelevant, as either is far beyond what we decided to be acceptable.

$$\lim_{t\to\infty} u(x,t) = u_e(x)$$

$$\frac{d^2u_e}{dx^2} = 0 \qquad u_e(0) = T_{\max} \qquad u_e(L) = T_{\min} \tag{2.1}$$

Solving the differential equation:

$$\int \frac{d^2u_e}{dx^2}\, dx = \int 0\, dx \to \frac{du_e}{dx} + c_1 = 0 \to \int \frac{du_e}{dx} + c_1\, dx = 0$$

$$\therefore u_e(x) = c_1 x + c_2$$

applying boundary conditions to find the constants:

$$u_e(0) = c_2 = T_{\max}$$

$$\therefore u_e(L) = T_{\min} = c_1 L + T_{\max}$$

$$\therefore c_1 = \frac{T_{melt} - T_{\max}}{L}$$

$$\therefore u_e(x) = T_{\max} + x\frac{T_{\min} - T_{\max}}{L} \tag{2.2}$$

If we define a function $\psi(x,t)$ such that it represents the difference between a local temperature in time $u(x,t)$ and the equilibrium temperature (u_e)

$$u(x,t) = \psi(x,t) + u_e(x) \tag{2.3}$$

and take the partial derivatives with respect to time and with respect to space, it can be seen that $\psi(x,t)$ and $u(x,t)$ must satisfy the same partial differential equations

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= \frac{\partial}{\partial t}\psi(x,t) + \frac{\partial}{\partial t}u_e(x) \\ \therefore \frac{\partial}{\partial t}u(x,t) &= \frac{\partial}{\partial t}\psi(x,t) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial x^2}u(x,t) &= \frac{\partial^2}{\partial x^2}\psi(x,t) + \cancelto{0}{\frac{\partial}{\partial x^2}u_e(x)} \qquad \text{from (2.1)} \\ \therefore \frac{\partial^2}{\partial x^2}u(x,t) &= \frac{\partial^2}{\partial x^2}\psi(x,t) \end{aligned}$$

Hence, we may establish the initial and boundary conditions for $\psi(x,t)$.

$$\begin{aligned} \text{Initial: } & \psi(x,0) = u(x,0) - u_e(x) \\ \text{Boundary: } & \psi(0,t) = u(0,t) - u_e(0) = T_{\max} - T_{\max} = 0 \\ & \psi(L,t) = u(L,t) - u_e(L) = T_{\min} - T_{\min} = 0 \end{aligned}$$

These boundary conditions are so-called homogenous, that is, $\psi(0,t) = \psi(L,t) = 0$ for any t, and as such we are able utilise a well-established solutions for them (Dawkins, 2018) which stem from Fourier's incredible work in the field. That is:

$$\psi(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(\frac{n\pi}{L})^2 t} \tag{2.4}$$

where B_n are the Fourier sine coefficients given by

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L (\psi(x,0)) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, 3, \ldots \\ &= \frac{2}{L}\int_0^L (u(x,0) - u_e(x)) \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned}$$

Replacing (2.2) and (2.4) into (2.3):

$$u(x,t) = T_{\max} + x\frac{T_{\min} - T_{\max}}{L} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \tag{2.5}$$

As we attempt to integrate B_n by parts, we are faced with the integration of $\psi(x,0)$. We can attempt to simplify this calculation by graphing $\psi(x,0)$ within its domain of $0 \leq x \leq L$ with the help of $u_e(x)$ and $u(x,0)$. Of note is that, since $u(x,0)$ is a step function, we can look at it as a graph transformation, especifically a vertical shift, aiding in finding the graph for $\psi(x,0)$. This can be seen in figure 2

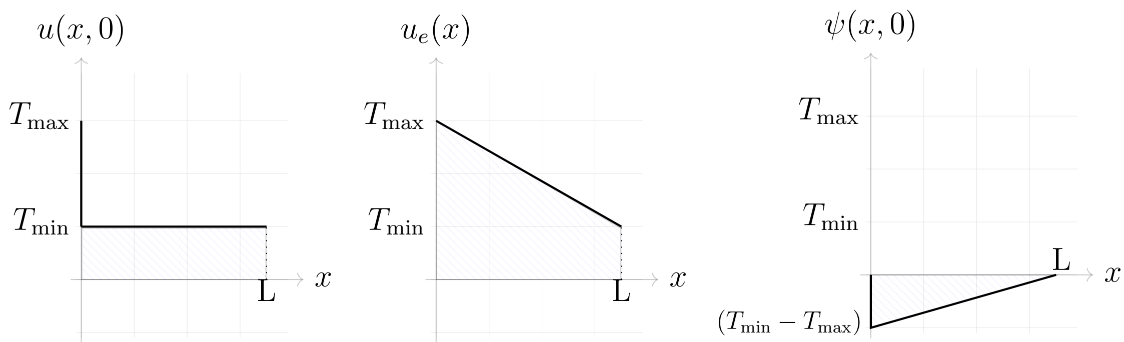

Figure 2: Graphical representation of $u(x,0)$ and $u_e(x)$, from which we able to deduce the shape of $\psi(x,0)$, and the three functions' respective integrals shaded in blue

If we define $\Psi(x)$ such that

$$\Psi(x) = \int \psi(x,0)dx \tag{2.6}$$

we can solve the integrals geometrically with the aid of figure 2, though disregarding the one point which is a step (at $x = 0$). This is done for simplicity sake.

$$\Psi(x) = \int u(x,0)dx - \int u_e(x)dx$$
$$\therefore \Psi(x) = T_{\min}x - \frac{T_{\min} + T_{\max}}{2}x$$
$$\therefore \Psi(x) = x\left(T_{\min} - \frac{T_{\min} + T_{\max}}{2}\right) = x\left(\frac{T_{\min} - T_{\max}}{2}\right)$$

Which we can verify is true from the graphical representation of $\psi(x, 0)$. Then B_n can be integrated by parts

$$B_n = \frac{2}{L}\sin\left(\frac{n\pi x}{L}\right)\Psi(x) - \frac{n\pi}{L}\int_0^L \Psi(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad \text{(Replacing } \Psi(x)\text{)}$$
$$\therefore B_n = \frac{T_{\min} - T_{\max}}{L}\sin\left(\frac{n\pi x}{L}\right)x - \frac{n\pi}{L}\frac{T_{\min} - T_{\max}}{2}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$$

Once again solving with integration by parts with the aid of Mathematica:

$$\therefore B_n = \frac{T_{\min} - T_{\max}}{L}\sin\left(\frac{n\pi x}{L}\right)x - \left(\frac{n\pi}{L}\frac{T_{\min} - T_{\max}}{2}\right)\left(\frac{L^2(n\pi\sin(n\pi) + \cos(n\pi) - 1)}{n^2\pi^2}\right)$$

Factoring out $\frac{T_{\min} - T_{\max}}{L}$

$$B_n = \left(\frac{T_{\min} - T_{\max}}{L}\right)\left(\sin\left(\frac{n\pi x}{L}\right)x - \frac{L^2(n\pi\sin(n\pi) + \cos(n\pi) - 1)}{2n\pi}\right)$$

Given that n can only be positive integers, $\sin(n\pi)$ will always equal to zero, hence we can simplify B_n further:

$$B_n = \left(\frac{T_{\min} - T_{\max}}{L}\right)\left(\sin\left(\frac{n\pi x}{L}\right)x - \frac{L^2(\cos(n\pi) - 1)}{2n\pi}\right) \tag{2.7}$$

Finally, we can start solving for our specific conditions, which we defined would answer our question:

$$
\begin{aligned}
&u(\frac{L}{2}, t_{\max}) = T_{melt} \\
&T_{melt} = T_{\max} + \frac{L}{2}\frac{T_{\min} - T_{\max}}{L} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi L}{2L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t_{\max}} \\
&T_{melt} - \frac{T_{\max} + T_{\min}}{2} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{2}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t_{\max}}
\end{aligned}
$$

At this point we can make two important observations which will greatly simplify our calculations: First, for every **even** n, the sine function $\sin\left(\frac{n\pi}{2}\right) = 0$, and consequently so will that term of the infinite sum. Second, if we look back at (2.7), for any odd n, $\cos(n\pi) = -1$, and so

$$
\frac{L^2(\cos(n\pi) - 1)}{2n\pi} = \frac{L^2(-1-1)}{2n\pi} = \frac{-L^2}{n\pi}
$$

Hence we can simplify the expression to the following, where $n \in \{1, 3, 5, \ldots\}$

$$
T_{melt} - \frac{T_{\max} + T_{\min}}{2} = \sum_{n=1}^{\infty} \left(\frac{T_{\min} - T_{\max}}{L}\right) \left(\sin\left(\frac{n\pi}{2}\right)\frac{L}{2} + \frac{L^2}{n\pi}\right) \sin\left(\frac{n\pi}{2}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t_{\max}}
$$

And remove the term independent from n from the infinite sum

$$
\begin{aligned}
&T_{melt} - \frac{T_{\max} + T_{\min}}{2} = \left(\frac{T_{\min} - T_{\max}}{L}\right) \sum_{n=1}^{\infty} \left(\sin\left(\frac{n\pi}{2}\right)\frac{L}{2} + \frac{L^2}{n\pi}\right) \sin\left(\frac{n\pi}{2}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t_{\max}} \\
&\therefore \frac{L(2T_{melt} - T_{\max} - T_{\min})}{2[T_{\min} - T_{\max}]} = \sum_{n=1}^{\infty} \left(\sin\left(\frac{n\pi}{2}\right)\frac{L}{2} + \frac{L^2}{n\pi}\right) \sin\left(\frac{n\pi}{2}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t_{\max}}
\end{aligned}
$$

For $n = n_1 \in \{1, 5, 9, 13, \ldots\}$ we will have all sine functions in the sum with the value of 1, while for $n = n_3 \in \{3, 7, 11, \ldots\}$ they will equal to -1. So in order to understand the sum, we will analyse its behaviour for one element at a time and analyse their individual weights to the overall sum. This means that we have to replace $t_{\max}$ for a fraction of it, $t_{n\,\max}$, such that the sum of all fractions will be $t_{\max}$:

$$\sum_{n=1}^{\infty} t_{n\max} = t_{\max}$$

First let us look at the expression for n_1:

$$\begin{aligned}
\frac{L(2T_{melt} - [T_{\max} + T_{\min}])}{2[T_{\min} - T_{\max}]} &= \left(\cancelto{1}{\sin\left(\frac{n_1\pi}{2}\right)} \frac{L}{2} + \frac{L^2}{n_1\pi} \right) \cancelto{1}{\sin\left(\frac{n_1\pi}{2}\right)} e^{-k\left(\frac{n_1\pi}{L}\right)^2 t_{n\max}} \\
\therefore \frac{2T_{melt} - [T_{\max} + T_{\min}]}{2(T_{\min} - T_{\max})} &= \left(\frac{n_1\pi + 2L}{2n_1\pi} \right) e^{-k(\frac{n_1\pi}{L})^2 t_{n\max}} \qquad (2.8) \\
\therefore \frac{(2T_{melt} - [T_{\max} + T_{\min}])n_1\pi}{(T_{\min} - T_{\max})(n_1\pi + 2L)} &= e^{-k(\frac{n_1\pi}{L})^2 t_{n\max}}
\end{aligned}$$

At this stage we would take ln on both sides, however that can only be done if the left hand side (l.h.s) is positive, and since $T_{\min} - T_{\max}$ is definitely negative, we need to make sure that on $2T_{melt} - T_{\max} + T_{\min}$ is also negative. Since we did not establish a relationship between them, other than $T_{\min} < T_{melt} < T_{\max}$, we are only able to reach a conclusion by formulating a system of equations. This can be done by looking at the values for n_3:

$$\begin{aligned}
\frac{2T_{melt} - [T_{\max} + T_{\min}]}{2[T_{\min} - T_{\max}]} &= -\left(\frac{2L - n_3\pi}{2n_3\pi} \right) e^{-k(\frac{n_3\pi}{L})^2 t_{n\max}} \\
-\frac{(2T_{melt} - [T_{\max} + T_{\min}])n_3\pi}{(T_{\min} - T_{\max})(2L - n_3\pi)} &= e^{-k(\frac{n_3\pi}{L})^2 t_{n\max}}
\end{aligned}$$

L is finite as established in the introduction, so for all values of $n_3 > \frac{2L}{\pi}$, the expression $2L - n_3\pi$ will be negative. This tells us that $2T_{melt} - [T_{\max} + T_{\min}]$ must also be negative for the whole term to be positive, and consequently for there to always exist a solution. Hence we proceed to solve n_1 by continuing with (2.8), taking ln on both sides:

$$\begin{aligned}
-k\left(\frac{n_1\pi}{L}\right)^2 t_{n\max} &= \ln\left(\frac{(2T_{melt} - [T_{\max} + T_{\min}])n_1\pi}{(T_{\min} - T_{\max})(n_1\pi + 2L)} \right) \\
\therefore t_{n\max} &= \left(\frac{L}{kn_1\pi} \right)^2 \ln\left(\frac{(2T_{melt} - [T_{\max} + T_{\min}])n_1\pi}{(T_{\min} - T_{\max})(n_1\pi + 2L)} \right)
\end{aligned}$$

Correspondingly for n_3 (multiplying $[2L - n_3\pi]$ by the -1):

$$t_{n\,\max} = \left(\frac{L}{kn_3\pi}\right)^2 \ln\left(\frac{(2T_{melt} - [T_{\max} + T_{\min}])n_3\pi}{(T_{\min} - T_{\max})(n_3\pi - 2L)}\right)$$

The same process would have to be repeated infinitely, so the final thing left to do is check the weight of each term. By disregarding all constants, we can quickly evaluate it for both subsets of n:

$$t_{n\,\max} \propto \frac{1}{n^2}$$
$$\therefore t_{\max} \propto \sum_{n=1}^{\infty} \frac{1}{n^2}$$

From the p-series test, we know that this is a convergent sum, which of course means we do not have to wait infinitely for the sandwich to be ready. The relative weight of each term can be visualised in figure 3.

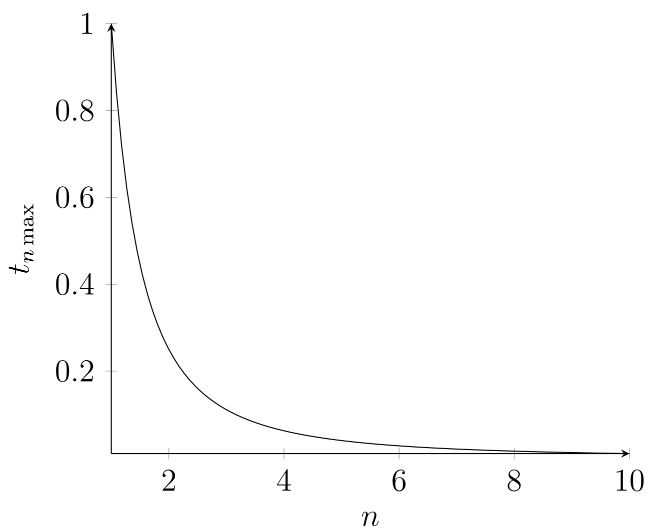

Figure 3: Demonstration of $\frac{1}{n^2}$, showing how quickly the weight of each $t_{n\,\max}$ decreases.

We originally stated that the acceptable time would have to be less than 10 minutes. This means that $t_{3\,\max}$, which contributes 11% to the total time, would be a little over a minute, and by $n = 9$, $t_{9\,\max}$'s contribution would be a little over seven seconds! This means that we can easily be satisfied by having only the first term of the infinite sum. As such, the time it takes to make a sandwich is, approximately,

$$2t_{\max} = 2\left(\frac{L}{k\pi}\right)^2 \ln\left(\frac{(2T_{melt} - [T_{\max} + T_{\min}])\pi}{(T_{\min} - T_{\max})(\pi + 2L)}\right) \tag{2.9}$$

3 Conclusion

Given our model, we have found a simple expression that greatly approximates the time needed to make the sandwich.

$$2t_{\max} = 2\left(\frac{L}{k\pi}\right)^2 \ln\left(\frac{(2T_{melt} - [T_{\max} + T_{\min}])\pi}{(T_{\min} - T_{\max})(\pi + 2L)}\right)$$

Although we have reached a simple expression, our model is not a good reflection of the real world. Foremost due to the simplifications that had to be made in order to be able to tackle the problem at all – most importantly that the sandwich's hot side (side facing the pan) and a hot pan reach thermal equilibrium. This is often not the case, although perhaps as we strive to balance temperature and time in order to not burn the toast before the cheese has melted, it makes this model a little more realistic.

It was argued that $2T_{melt} < (T_{\max}+T_{\min})$ in order for there to be a solution to our problem, and this is definitely congruent with reality. A grilled cheese being done usually implies that the toast has browned, a process called the Maillard reaction, thought to only occur above 140 °C. Thus, this must be the lowest acceptable value for $T_{\max}$. Furthermore, I noted during my Physics internal assessment that cheese (of that specific variety) melted at 50 °C to 60 °C. Finally, $T_{\min}$ is somewhere between fridge temperature and room temperature, 2 °C to 20 °C. Plugging in the maximum value for T_{melt} and lowest of $T_{\min}$ gives us: $120 < 142$, which is, of course, true.

Overall, the study could be greatly improved in two ways: By simplyfing the number of variables that were being used; and by having the pan be an infinite source of heat. The former could be done with non-dimensionalisation of the variables: Although we did not express any of the physical units, this is the process of tying multiple of our dimensional variables into non-dimensional numbers. Indeed, this is often done in solutions of similar problems. Having the pan as an infinite source of heat would present an even greater challenge – so while it would provide a much more realistic reflection of the real world, the complexity of unsteady state conduction, as these problems are called, is far above what is reasonable for a study of this format.

4 Bibliography

References

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5 Appendix

5.1 Glossary

Specific heat capacity (c): Amount of thermal energy needed to raise the temperature of one kilogram of mass by 1 kelvin. $c = \frac{\Delta H}{m\Delta T}$

Latent heat (C): Amount of thermal energy needed to change phase of one kilogram of mass while the temperature is constant. $C = \frac{\Delta H}{m}$