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1 Building Mathematical Models

1.1 a

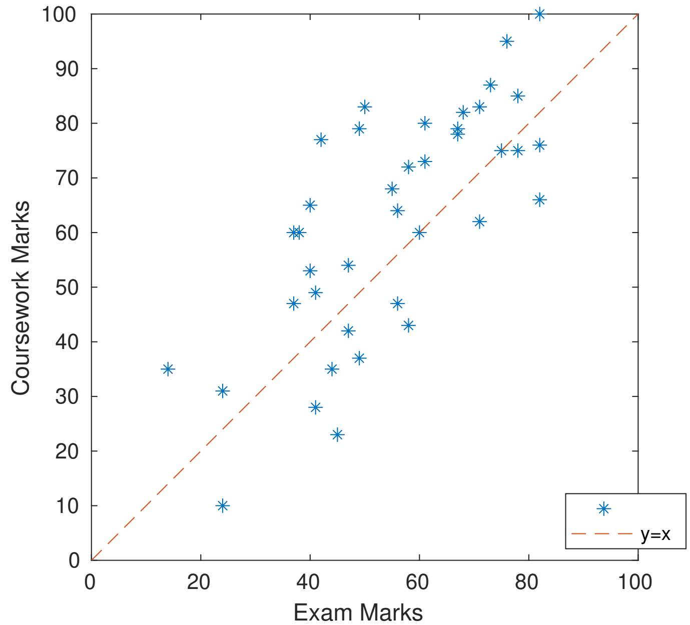

Figure 1: Coursework marks plotted against exam marks. Note the orange dashed line of symmetry

In Figure 1 the line of symmetry, $x = y$, was plotted. Because the data are reasonably spread around the line of symmetry, there is evidence for some proportionality between the two data sets.

Listing 1: Topic 1. Question a. Note all code per topic belongs to same file

```
figure,plot(examMarks,courseworkMarks, '*');
xlabel('Exam Marks'); ylabel('Coursework Marks')
axis square; axis ([0 100 0 100])
hold on; plot(0:100,0:100, '--'); hold off;
legend('', 'y=x', 'Location', 'best');
print('eps/topic1_a.eps','-depsc')
```

1.2 b

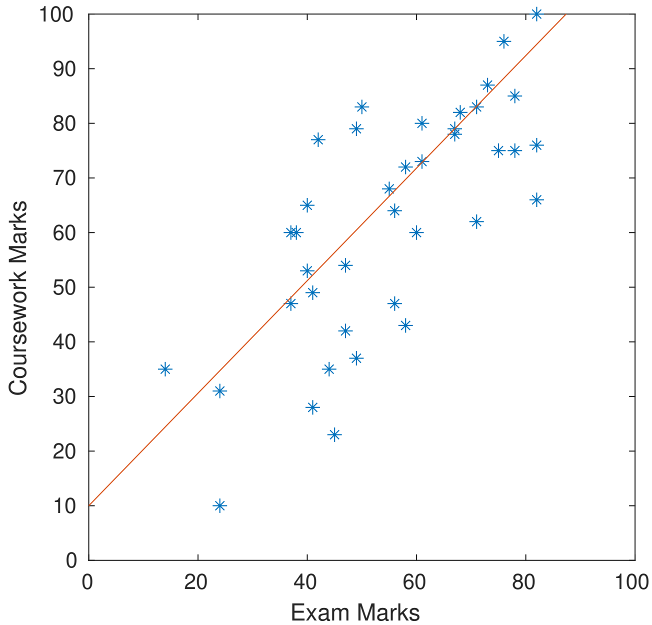

Figure 2: Coursework marks plotted against exam marks. Line of best fit estimated by eye

Given a line $y = ax + b$, the parameters were estimated as per below. This is represented in Figure 2.

$$
\begin{aligned}
a &= \frac{90 - 10}{78 - 0} = 1.03 \\
b &= 10
\end{aligned}
\tag{1}
$$

1.3 c

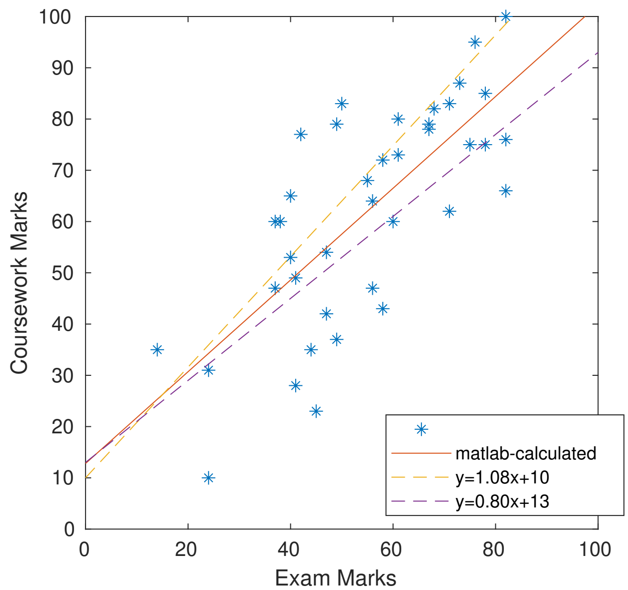

Figure 3: Coursework marks plotted against exam marks.

By comparing the maximum and minimum fits, we find the uncertainties on slope (Δa) and y-intercept(Δb) to be

$$
\begin{gathered}
Y_{\max} = 1.08x + 10; \qquad Y_{\min} = 0.80x + 13 \\
\therefore \Delta a = \frac{1.08 - 0.80}{2} = 0.14; \qquad \Delta b = 1.5
\end{gathered} \tag{2}
$$

Listing 2: Topic 1. Question c. Note all code per topic belongs to same file

```
figure,plot(examMarks,courseworkMarks, '*');
xlabel('Exam Marks'); ylabel('Coursework Marks')
hold on;
axis square; axis ([0 100 0 100]);
p=polyfit(examMarks, courseworkMarks, 1); %Generate polynomial of best fit
xfit=0:100; yfit=polyval(p,xfit); %Line of best fit
plot(xfit,yfit)
plot(i,10+i.*1.08, '--'); %max slope
plot(i,13+i.*0.80, '--'); %min slope
legend('', 'matlab-calculated', 'y=1.08x+10', 'y=0.80x+13', 'Location', 'best')
hold off;
```

1.4 d

The general trend is similar, but Matlab calculates a higher y-intercept (12.7990) and a lower slope (0.8950). This suggests that I overvalue the density of points towards the extremity of the graph, while the software appropriately looks at all points equally.

1.5 e

$$\begin{aligned} f(x) &= 0.8950x + 12.7990 \\ f(60) &= 0.8950 \cdot 60 + 12.7990 = 66.499 \end{aligned} \tag{3}$$

Predicted grade would be 66.

1.6 f

We could present a more quantitative marker of reliability by measuring the correlation coefficient, but it can also be done qualitatively. There is a general correlation between exam marks and coursework marks, but the scattering suggests this will be imperfect (have an R-coefficient less than 1). From the data we have students with coursework marks around 60% getting exam marks ranging from approximately 40% to 80%. This is a really large spread that covers failing and getting first-class, so while the correlation is there, and it is accurate, it is not precise enough.

2 Employ assumptions to simplify systems

2.1 a

Length = L, Mass = M, Time = T

- $[x] = L$
- $[m] = M$
- $[\frac{d^2x}{dt^2}] = \frac{L}{T^2}$
- $[k] = \frac{M}{T^2}$

2.2 b

$$
\begin{gathered}
A \approx 10^{-3}; k \approx 10^{3}; m = 1; t = 10^{-3} \\
x(10^{-3}) \approx 10^{-3} \cdot \cos \cancelto{1}{\left(10^{-3} \cdot \sqrt{10^{3}}\right)} \approx 10^{-3}
\end{gathered} \tag{4}
$$

3 Matrices and vectors

3.1 a

Listing 3: Topic 3. Question a. Note all code per topic belongs to same file

```
t=0:0.4:40
p=[20*sin(t); 20*cos(t); 10-(t./4).^2]
```

3.2 b

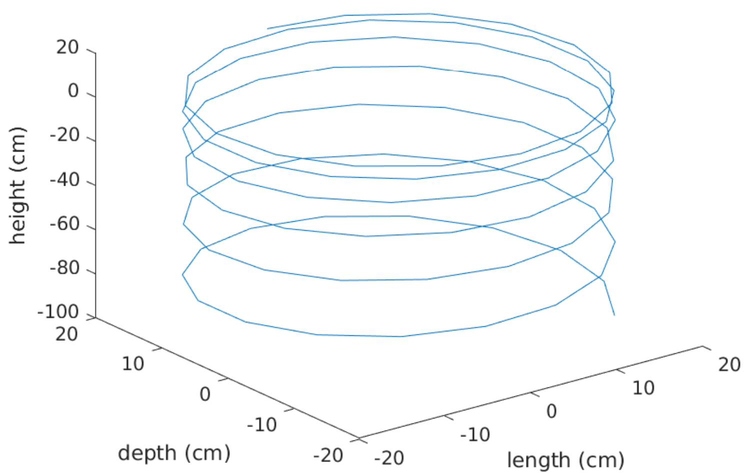

Figure 4: Particle position from $t = 0$s to $t = 40$s with 0.4 second intervals.

Listing 4: Code for Topic 3. Question b. Note all code per topic belongs to same file

```
plot3(p(1,:),p(2,:),p(3,:))
xlabel('length (cm)'),ylabel('depth (cm)'), zlabel('height (cm)')
```

3.3 c

Given that the speed is the modulus of velocity,

$$|\overrightarrow{v}(15)| = \left|\begin{bmatrix} 20\cos(15) \\ -20\sin(15) \\ -\frac{2}{16}15 \end{bmatrix}\right| = \left|\begin{bmatrix} -15.1938 \\ -13.0058 \\ -1.8750 \end{bmatrix}\right| = 20.0877 \text{ cms}^{-1} \tag{5}$$

3.4 d

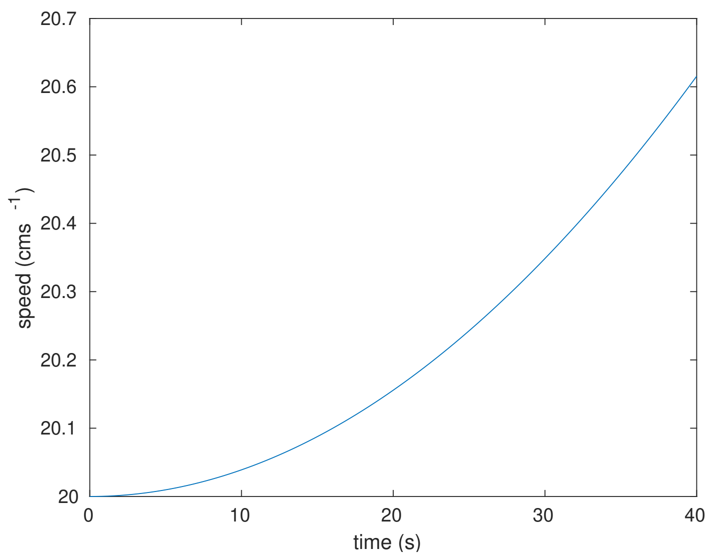

Figure 5: Particle speed varying with time.

Listing 5: Topic 3. Question d. Note all code per topic belongs to same file

```
v=[20*cos(t); -20*sin(t); -2.*t./16]
speed=sqrt(sum(v.^2))
plot(t,speed)
xlabel('time (s)'),ylabel('speed (cms^{-1})')
```

4 Complex numbers

4.1 a

Given that j^α is a positive imaginary number, we can conclude that its argument is $\frac{\pi}{2}$, that is

$$\begin{aligned}
&\left(re^{jx}\right)^\alpha = r^\alpha\left(\cos(x\alpha) + j\sin(x\alpha)\right) \quad \text{(De Moivre's theorem)} \\
&j^\alpha = r^\alpha e^{jx\alpha} \implies x = \frac{\pi}{2}; r = 1 \\
&\therefore j^\alpha = e^{j\frac{\pi\alpha}{2}} = \cos\left(\frac{\alpha\pi}{2}\right) + j\sin\left(\frac{\alpha\pi}{2}\right)
\end{aligned} \tag{6}$$

4.2 b

$$\begin{aligned}
Z &= R_\infty + \frac{R_0 - R_\infty}{1 + (j\omega\tau)^\alpha} \qquad = R_\infty + \frac{R_0 - R_\infty}{1 + (\omega\tau)^\alpha \cdot j^\alpha} \\
&= R_\infty + \frac{R_0 - R_\infty}{1 + \theta^\alpha\left(c + js\right)} = R_\infty + \frac{R_0 - R_\infty}{1 + \theta^\alpha c + \theta^\alpha js} \\
&= R_\infty + \frac{R_0 - R_\infty}{1 + \theta^\alpha c + \theta^\alpha js} \cdot \frac{1 + c\theta^\alpha - js\theta^\alpha}{1 + c\theta^\alpha - js\theta^\alpha} \\
&= R_\infty + \frac{\left(R_0 - R_\infty\right)\left(1 + c\theta^\alpha - js\theta^\alpha\right)}{1 + c\theta^\alpha + \cancel{js\theta^\alpha} + c\theta^\alpha + c^2\theta^{2\alpha} + \cancel{cjs\theta^{2\alpha}} - \cancel{js\theta^\alpha} - \cancel{jsc\theta^{2\alpha}} - (js)^2\theta^{2\alpha}} \\
&= R_\infty + \frac{\left(R_0 - R_\infty\right)\left(1 + c\theta^\alpha - js\theta^\alpha\right)}{1 + 2c\theta^\alpha + \theta^{2\alpha}\left(c^2 - j^2s^2\right)} \\
&= R_\infty + \frac{\left(R_0 - R_\infty\right)\left(1 + c\theta^\alpha - js\theta^\alpha\right)}{1 + 2c\theta^\alpha + \theta^{2\alpha}\cancelto{1}{\left(c^2 + s^2\right)}} \qquad \left(\text{from } \cos^2(x) + \sin^2(x) = 1\right) \\
\therefore Z &= R_\infty + \frac{\left(R_0 - R_\infty\right)\left(1 + c\theta^\alpha - js\theta^\alpha\right)}{1 + 2c\theta^\alpha + \theta^{2\alpha}} \qquad \text{as required}
\end{aligned} \tag{7}$$

4.3 c

Given that $\Delta = R_0 - R_\infty$, and $q = 1 + 2c\theta^\alpha + \theta^{2\alpha}$,

$$
\begin{aligned}
&Z = R_\infty + \Delta \frac{1 + c\theta^\alpha - js\theta^\alpha}{q} \\
&Z = R_\infty + \frac{\Delta + \Delta c\theta^\alpha}{q} - j\frac{s\Delta\theta^\alpha}{q} \\
&\implies |Z| = \sqrt{\left(R_\infty + \frac{\Delta + \Delta c\theta^\alpha}{q}\right)^2 + \left(\frac{s\Delta\theta^\alpha}{q}\right)^2} \\
&|Z| = \sqrt{R_\infty^2 + \frac{2R_\infty\Delta + 2R_\infty\Delta c\theta^\alpha}{q} + \frac{\Delta^2 + 2\Delta^2 c\theta^\alpha + \Delta^2 c^2\theta^{2\alpha}}{q^2} + \frac{(s\Delta\theta^\alpha)^2}{q^2}} \\
&\left[\text{note that } \theta^{2\alpha}\Delta^2\left(c^2 + s^2\right) = \theta^{2\alpha}\Delta^2\right] \\
&|Z| = \sqrt{R_\infty^2 + \frac{2R_\infty\Delta + 2R_\infty\Delta c\theta^\alpha}{q} + \Delta^2\left(\cancelto{\frac{1}{q}}{\frac{1 + 2c\theta^\alpha + \theta^{2\alpha}}{q^2}}\right)} \\
&|Z| = \sqrt{\frac{qR_\infty^2 + 2R_\infty\Delta + 2R_\infty\Delta c\theta^\alpha + \Delta^2}{q}} \\
&|Z| = \sqrt{\frac{(1 + 2c\theta^\alpha + \theta^{2\alpha})R_\infty^2 + 2R_\infty(R_0 - R_\infty) + 2R_\infty(R_0 - R_\infty)c\theta^\alpha + (R_0 - R_\infty)^2}{1 + 2c\theta^\alpha + \theta^{2\alpha}}} \\
&|Z| = \sqrt{\frac{\cancel{R_\infty^2} + \cancel{2c\theta^\alpha R_\infty^2} + R_\infty^2\theta^{2\alpha} + \cancel{2R_\infty R_0} - \cancel{2R_\infty^2} + 2R_\infty R_0 c\theta^\alpha - \cancel{2R_\infty^2 c\theta^\alpha} + R_0^2 + \cancel{R_\infty^2} - \cancel{2R_0 R_\infty}}{1 + 2c\theta^\alpha + \theta^{2\alpha}}} \\
&\therefore |Z| = \sqrt{\frac{R_\infty^2\theta^{2\alpha} + 2cR_\infty R_0\theta^\alpha + R_0^2}{1 + 2c\theta^\alpha + \theta^{2\alpha}}} \quad \text{as required}
\end{aligned}
\tag{8}
$$

4.4 d

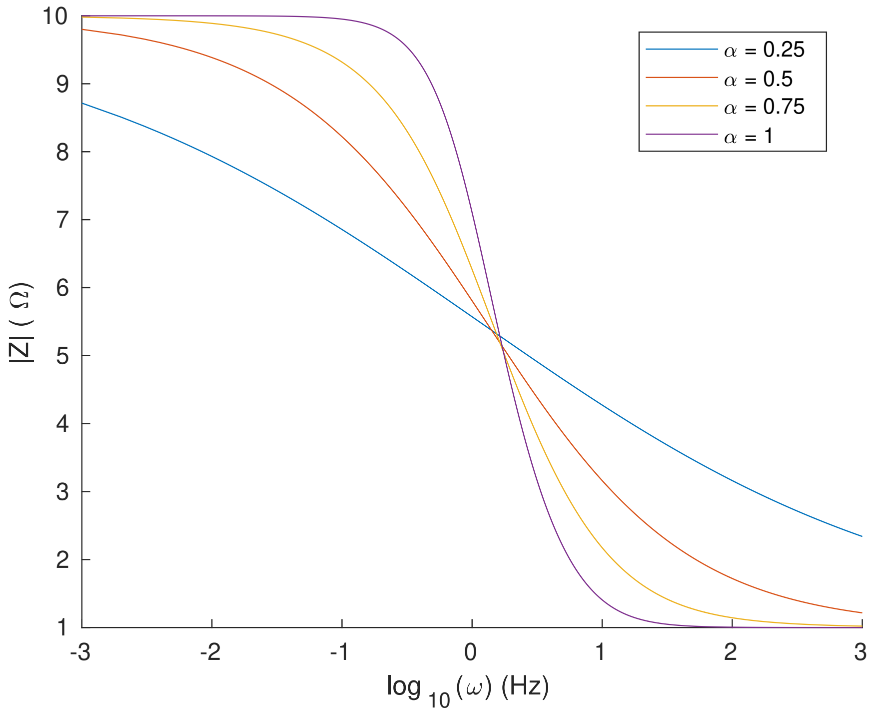

Figure 6: Magnitude of impedance plotted against angular frequency

Listing 6: Code for Topic 4. Question d.

```
R_infty = 1
R_0 = 10
tau = 1

omega=0.001:0.001:10^3
theta = omega*tau
figure, hold on;
for alpha = [0.25, 0.5, 0.75, 1]
    c=cos(alpha*pi/2)
    modZ = sqrt( ((R_infty)^2.*theta.^(2*alpha) + 2*c*R_0*R_infty.*theta.^alpha + (R_0)^2)./
         (1+2*c*theta.^alpha + theta.^(2.*alpha)) )
    plot(log10(omega), modZ);
end
xlabel('log_{10}(\omega) (Hz)'); ylabel('|Z| (\Omega)')
legend('\alpha = 0.25','\alpha = 0.5','\alpha = 0.75','\alpha = 1', 'Location','best')
print('eps/topic4_d.eps','-depsc')
```

4.5 e

We see variations of logistic curves, which is surprising, given that the equation for $|Z|$ is quite different from the standard $f(x)$, such that

$$f(x) = \frac{L}{1 + e^{-k(x-x_0)}} \tag{9}$$

Nevertheless, for $\alpha = 1$, we are left with $|Z| = \sqrt{\frac{\log(\omega)^2 + 10^2}{1 + \log(\omega)^2}}$