Contents

1 Calculus

1.1 a

Assuming the container is a cone, its volume V(t) is

$$V(t) = \frac{h(t) \cdot r(t)^2 \pi}{3} = \frac{\tan\left(\frac{\pi}{2} - \theta\right)\pi}{3} h(t)^3 \tag{1}$$

1.2 b

$$\begin{aligned} h(\tau) &= 0 \\ \implies h_0^{\frac{5}{2}} &= \frac{5a^2\sqrt{2g}}{2\tan^2\left(\frac{\pi}{2} - \theta\right)}\tau \\ \therefore \tau &= h_0^{\frac{5}{2}} \frac{2\tan^2\left(\frac{\pi}{2} - \theta\right)}{5a^2\sqrt{2g}} \end{aligned} \tag{2}$$

Assuming $\theta = \frac{\pi}{4}$; $g = 9.8\,\mathrm{m\,s^{-2}}$ and replacing the given values:

$$\tau = (0.30)^{\frac{5}{2}} \frac{2\tan^2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)}{5(0.01)^2\sqrt{2 \cdot 9.8}} = 44.5\,\mathrm{s} \tag{3}$$

As can be seen in Figure 1, the minimum height is 5 cm, which makes the equation not

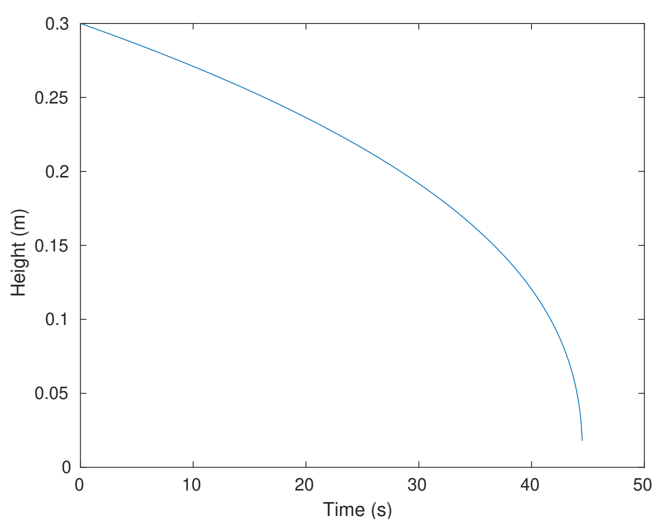

Figure 1: Height variation from $t = 0$ to $t = \tau = 44.5$

particularly valid – especially considering we start with a height of 30 cm. Changing the angle heavily affects how close we are able to get to a height of zero, but this will be discussed subsequent questions.

Listing 1: Topic 5. Question b

```
syms t
a=0.01; h0=0.30; theta=pi/4; g = 9.8;
tau = h0.^(5/2).*( 2.*tan(pi/2-theta).^2 )./( 5*a^2*sqrt(2*g) )
h=@(t) ( h0.^(5/2) - t.*( 5*a^2*sqrt(2*g) )./( 2.*tan(pi/2-theta).^2 ) ).^(2/5);
t = 0:0.1:tau;
plot(t, h(t));
xlabel('Time (s)'); ylabel('Height (m)');
print('eps/topic5_b.eps','-depsc')
```

1.3 c

Unsurprisingly, the closer to $\frac{\pi}{2}$, the less time it takes to empty the container. This means that we are approaching the shape of a cylinder, rather than a cone. Physically this requires a widening of the outlet, decreasing the speed of the liquid coming out. Furthermore, this points out an interesting feature of graph – at smaller angles the difference between both heights is far bigger than at larger angles.

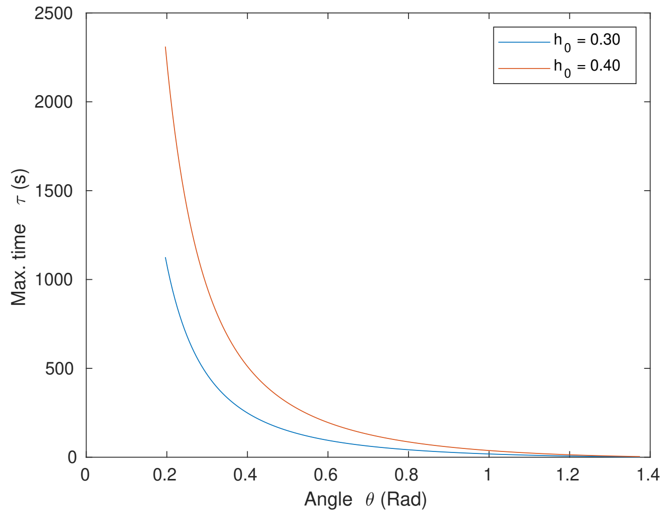

Figure 2: Variation of maximum time τ with the angle

Listing 2: Topic 5. Question c

```
theta = linspace(pi/16, 7*pi/16,200);
h0=[0.30; 0.40];
tau= h0.^(5/2).*( 2.*tan(pi/2-theta).^2 )./( 5*a^2*sqrt(2*g) )
plot(theta, tau(1,:)); hold on;
plot(theta, tau(2,:));
xlabel('Angle \theta (Rad)'); ylabel('Max. time \tau (s)');
legend('h_0 = 0.30', 'h_0 = 0.40')
print('eps/topic5_c.eps','-depsc')
```

1.4 d

Rearranging Equation (2) and replacing it into the given expression:

$$\frac{5a^2\sqrt{2g}}{2\tan^2\left(\frac{\pi}{2}-\theta\right)} = \frac{h_0^{\frac{5}{2}}}{\tau}$$

$$\begin{aligned} h(t) &= \left[h_0^{\frac{5}{2}} - t\cdot\frac{h_0^{\frac{5}{2}}}{\tau}\right]^{\frac{2}{5}} \\ \Longrightarrow \dot{h}(t) &= \frac{d}{dt}h(t) = \frac{2}{5}\left[h_0^{\frac{5}{2}} - t\cdot\frac{h_0^{\frac{5}{2}}}{\tau}\right]^{-\frac{3}{5}}\left(-\frac{h_0^{\frac{5}{2}}}{\tau}\right) \end{aligned} \tag{4}$$

From the definitions given

$$r(t) = h(t) \cdot \tan\left(\frac{\pi}{2} - \theta\right)$$
$$\therefore \dot{r}(t) = \dot{h} \tan\left(\frac{\pi}{2} - \theta\right)$$

Taking the derivative and replacing into Equation (1):

$$\begin{aligned}
V(t) &= \frac{h(t) \cdot r(t)^2 \pi}{3} \\
\dot{V}(t) &= \frac{\pi}{3}\left(\dot{h}(t) r(t)^2 + 2h(t) r(t) \dot{r}(t)\right) \\
\dot{V}(t) &= \frac{\pi}{3}\left(\dot{h}(t)\left[h(t)\tan\left(\frac{\pi}{2} - \theta\right)\right]^2 + 2[h(t)]^2 \tan\left(\frac{\pi}{2} - \theta\right) \dot{h}(t) \tan\left(\frac{\pi}{2} - \theta\right)\right) \\
\therefore \quad \dot{V}(t) &= \pi \cdot \dot{h}(t) \cdot [h(t)]^2 \cdot \tan^2\left(\frac{\pi}{2} - \theta\right)
\end{aligned} \tag{5}$$

1.5 f

Listing 3: Topic 5. Question f

```
syms t
a=0.01; h0=0.3; theta=pi/4; g=9.8;
tau= h0.^(5/2).*( 2.*tan(pi/2-theta).^2 )./( 5*a^2*sqrt(2*g) )
dhdt=@(t) 2/5*( h0.^(5/2)-t.*(h0^(5/2))./(tau) ).^(-3/5)*( (-h0.^(5/2))./(tau) );
dVdt=@(t) pi.*dhdt(t).*(h(t).*tan(pi/2-theta)).^2;
format long;
V=integral(dVdt,0,tau/2)
rsums(dVdt, [0,tau/2])
```

`rsums(dVdt, [0,tau/2])` gives us an interactive riemann sum where we can define the step sizes. The step sizes of 10 and 1 can be visualised on Figures 3 and 4. Finally, for step size of 0.01, it is expected that is small enough that given the number of significant figures, it is equivalent to matlab's `integral(dVdt, 0, tau/2) = -0.0159672`.

The difference between the extremes is $0.000\,013\,\mathrm{m}^3$. There is largely no computational difference between the results at this scale, so using 0.01 over 1 will yield more accurate numbers, but it's arguably not necessary. Ultimately I would choose 0.01.

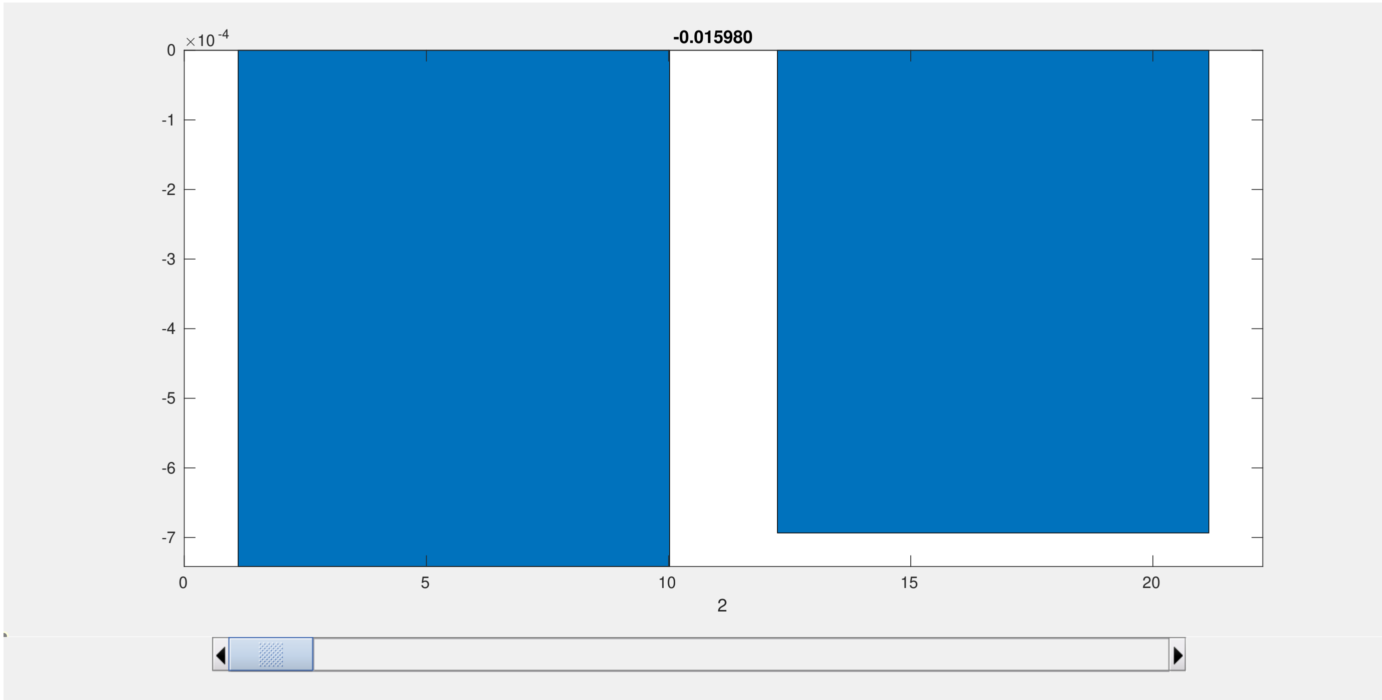

Figure 3: Step size of approximately 10

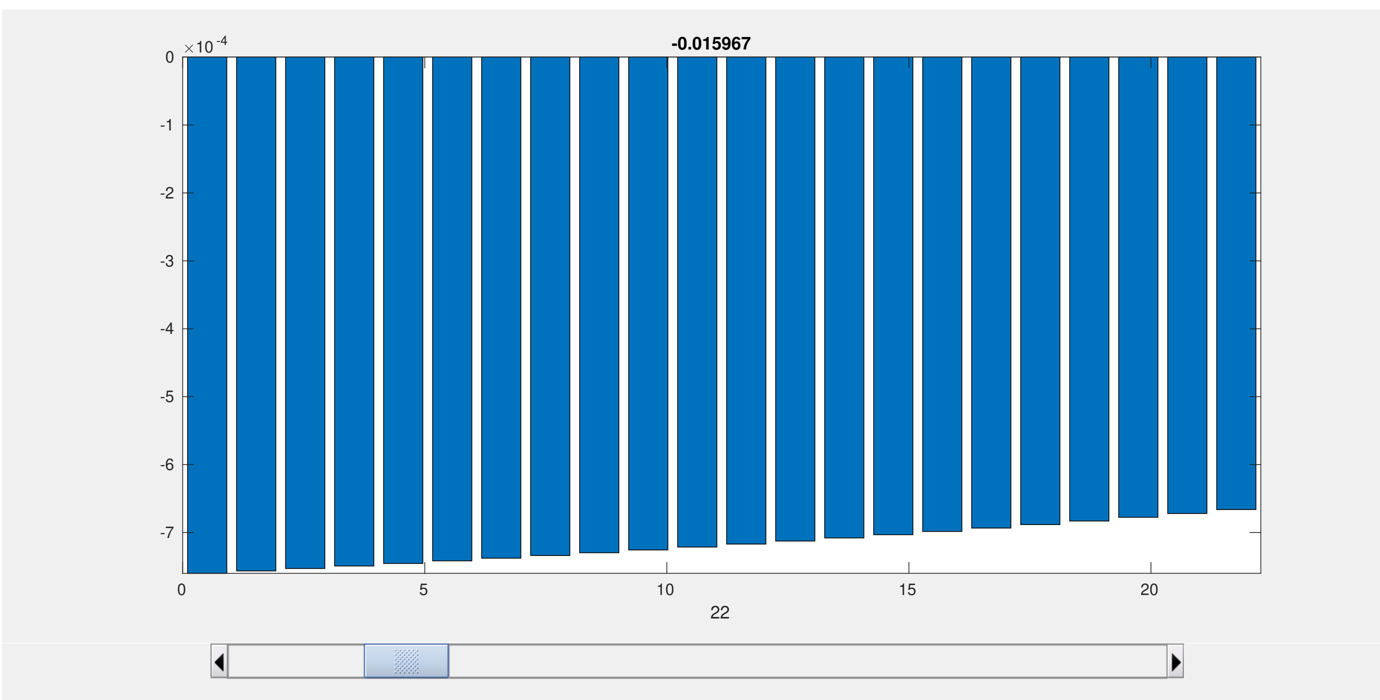

Figure 4: Step size of approximately 1

2 Ordinary Differential Equations

2.1 a

Rearranging the equation and finding its characteristic equation:

$$
\begin{aligned}
&M\frac{d^2h}{dt^2} + c\frac{dh}{dt} + kh = 0 \implies \frac{d^2h}{dt^2} + \frac{c}{M}\frac{dh}{dt} + \frac{k}{M}h = 0 \\
&\frac{d^2h}{dt^2} + 6\frac{dh}{dt} + 25h = 0 \\
&m^2 + 6m + 25 = 0 \implies m = -3 \pm 4i \\
\therefore\ &h(t) = e^{-3t}\left(Ae^{4it} + Be^{-4it}\right)
\end{aligned}
\tag{6}
$$

We are given two initial conditions: $h(0) = 0.15$ and $h'(0) = -0.05$. Using matlab to solve it:

$$h(t) = e^{-3t}\,\frac{(3\cos(4\,t) + 2\sin(4\,t))}{20}$$

Listing 4: Topic 6. Question a

```
syms h(t)
Dh = diff(h);
ode = diff(h,t,2) + 6*diff(h,t) + 25*h == 0;
conds = [Dh(0)==-0.05, h(0)==0.15];
hSol(t) = dsolve(ode,conds);
hSol=simplify(hSol)
```

As can be seen in Figure 6, between $t = 0$ and approximately $t = 0.6$, the suspension is being compressed. Ultimately it reaches -0.02m, which means that vehicle body is under its normal resting position (conceptually similar to it being above the resting position in the beginning of the movement) The oscillation is heavily dampened, which means the car does not move up-and-down continuously following the first bump. This is shown in the graph between about $t = 0.7$ and about $t = 1.5$.

2.2 b

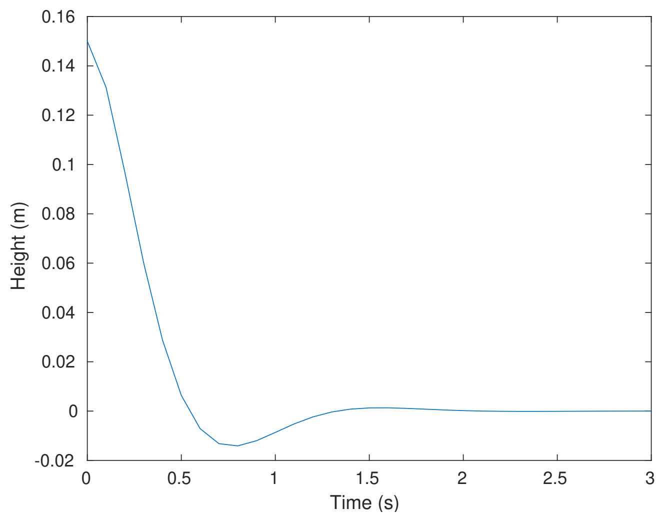

Figure 5: Suspension height during impact

Listing 5: Topic 6. Question b. Note this is a continuation of the code for a)

```
t=0:0.1:3
hVector = eval(vectorize(hSol))
plot(t, hVector)
xlabel('Time (s)'); ylabel('Height (m)')
print('eps/topic6_b.eps','-depsc')
```

2.3 c

$$\begin{aligned} \frac{d^2h}{dt^2} + \frac{c}{M}\frac{dh}{dt} + \frac{k}{M}h &= \frac{F}{M}e^{i\omega t} \\ \frac{d^2h}{dt^2} + 6\frac{dh}{dt} + 25h &= 2.5e^{i\omega t} \end{aligned} \tag{7}$$

Equation (6) gave us the complimentary function. We could find the particular integral manually, however matlab is utilised

Listing 6: Topic 6. Question c (i)

```
syms h(t) w
Dh = diff(h);
ode = diff(h,t,2) + 6*diff(h,t) + 25*h == 2.5*exp(i*w*t);
conds = [Dh(0)==-0.05 h(0)==0.15];
General=simplify(dsolve(ode))
Particular=simplify(dsolve(ode,conds))
```

General solution

$$C_1 \cos(4\,t)\ \mathrm{e}^{-3\,t} - C_2 \sin(4\,t)\ \mathrm{e}^{-3\,t} + \frac{5\cos(4\,t)^2\,\mathrm{e}^{t\,\omega\,\mathrm{i}}}{\sigma_1} + \frac{5\sin(4\,t)^2\,\mathrm{e}^{t\,\omega\,\mathrm{i}}}{\sigma_1}$$
$$\text{where} \tag{8}$$
$$\sigma_1 = 2\,(-\omega^2 + 6\,\omega\,\mathrm{i} + 25)$$

Particular solution

$$\frac{\mathrm{e}^{-3\,t}\left(50\cos(4\,t) + 25\sin(4\,t) + 100\,\mathrm{e}^{t\,(3+\omega\,\mathrm{i})} - 6\,\omega^2\cos(4\,t) - 4\,\omega^2\sin(4\,t) + 36\,\omega\cos(4\,t)\ \mathrm{i} - \omega\sin(4\,t)\ \mathrm{i}\right)}{40\,(-\omega^2 + 6\,\omega\,\mathrm{i} + 25)} \tag{9}$$

Listing 7: Topic 6. Question c (ii). Note it is the continuation of the code for c (i)

```
w=[0:1:100]'; t=0:0.1:2
plot(w, abs(eval(vectorize(Particular))));
xlabel('Angular displacement \omega'), ylabel('Height')
title('Each line is a distinct moment in time, t=0 to 3')
```

For ω around 10 to 20, there is heavy oscillation which gets reduced with an increase in the angular frequency. This suggests that a very large number of bumps on the road "dillutes" the impact – presumably imitating the conditions of a poorly maintained road. If the objective is to punish those who choose to go over the grid at higher speeds, then the graph suggests that the grid will achieve that. Ultimately, there are alternative speed control systems, such as the use of cameras and fines, though their compared effectiveness is up for contention.

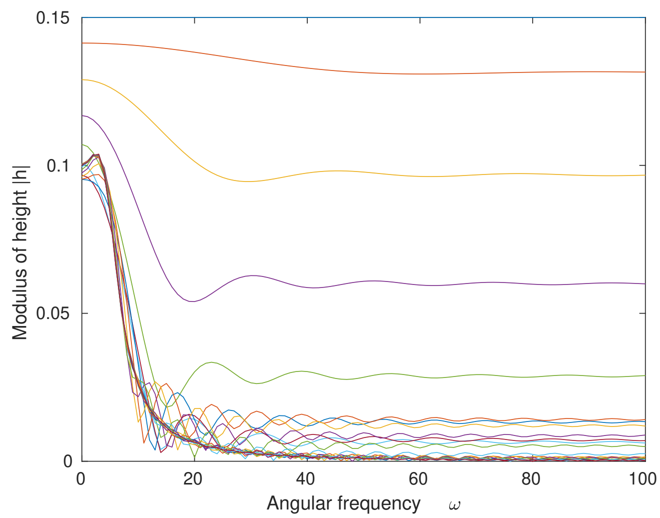

Figure 6: Note that each line represents a distinct moment in time from t=0 to t=2

3 Numerical Ordinary Differential Equations

3.1 a

Given that we want 20 steps of size 0.1, we center the polynomial around $20 \cdot 0.1 = 2$. We also calculate the integral of the expression, as seen in the matlab code.

Listing 8: Topic 8. Question a

```
syms t
y=@(t)-1/4*cos(4*t); %Define function
T=vectorize(taylor(y(t),t,2)); %Create taylor polynomial as a matlab expression
t=[0:0.1:2];
p=eval(T);
p(end)
```

`p(end)` evaluates to 0.0364

3.2 b

Listing 9: Topic 8. Question b

```
h=0.05; N=40; y=zeros(1,41); y(1)=-0.25;
t=0:h:2;
dy=@(t)sin(4*t);
for n=1:N
    y(n+1)=y(n)+h*dy(t(n));
end
y(end);
```

`yb(end)` evaluates to 0.0107

3.3 c

By inspection we find that $y(t) = -\frac{1}{4}\cos(4t)$, and therefore $y(2) = 0.0364$. As expected, the taylor polynomial gave us a similar answer; however if we look at Figure 7, a period cannot be deduced. On the other hand, Euler's method, Figure 8, gives us an inaccurate value (0.0107), but a very good shape for the graph – from which we can estimate the period to be 1.5. From the equation we know the period to be $\frac{2\pi}{4} \approx 1.57$.

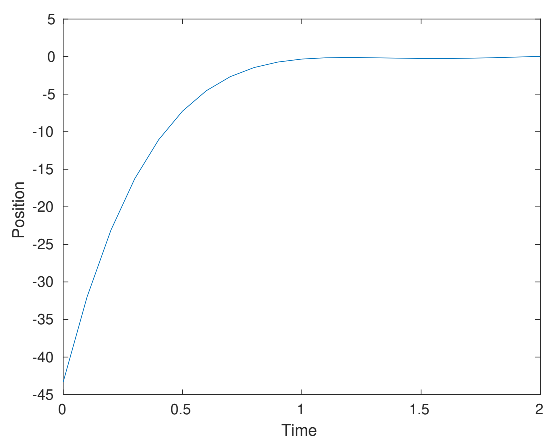

Figure 7: Taylor polynomial centered around 2.

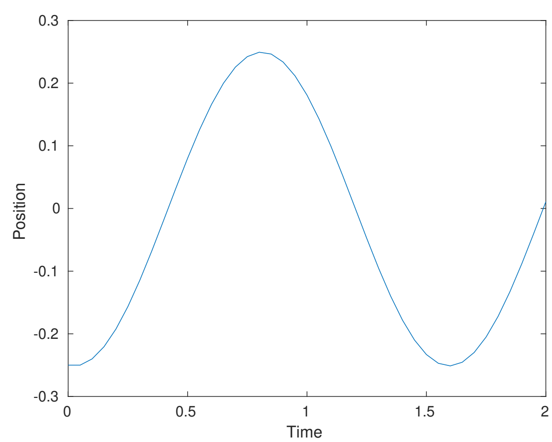

Figure 8: Euler method.

By increasing the order of the Taylor polynomial, we are able to reduce this difference significantly. Below is the same expression with order 100, which can be seen to have a similar shape to Euler's method and more appropriate values.

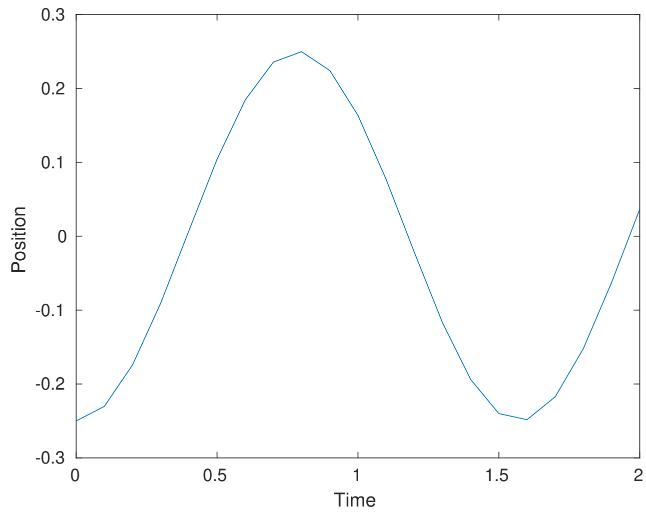

Figure 9: Taylor polynomial with high order.

Listing 10: Taylor polynomial of order 100

```
syms t
y=@(t)-1/4*cos(4*t);
T=vectorize(taylor(y(t),t,2, 'order', 100));
t=0:0.1:2;
p=eval(T);
plot(t, p)
xlabel('Time'); ylabel('Position')
print('eps/topic8_c.eps','-depsc')
```

4 Uncertainty and data

4.1 a

Listing 11: Topic 9/10. Question 4. a

```
mu=370
sigma=47
ai=1-normcdf(500,370,47)
aii=normcdf(200,370,47)
```

(i) $2.8 \cdot 10^{-3}$

(ii) $1.4901 \cdot 10^{-4}$

(iii) Assuming one bit is $8 \cdot 10^6$ megabytes, then

$$\text{time} = \frac{1.345 \cdot 8 \cdot 10^6}{370 \cdot 60 \cdot 60} = 2.9081 \cdot 10^4 \text{s}$$

Which is approximately 8h 47min

4.2 b

(i)

$$\mu(1-10) = 1160; \quad \sigma(1-10) = 288.27$$

$$\mu(11-20) = 721.8; \quad \sigma(11-20) = 302.37$$

(ii) Years 1-10: 953.8 to 1366.2; Years 11-20: 505.5 to 938.1

Listing 12: Topic 9/10. Question 4. b

```
%(i)

flow=[1460 1240 1230 1270 861 1355 612 822 1370 1380 ...
    810 735 259 1290 1125 528 622 468 664 717]
pd01_10=fitdist(flow(1:10)', 'Normal')
pd11_20=fitdist(flow(11:20)', 'Normal')
%%
%(ii)
paramci(pd01_10)
paramci(pd11_20)
```

(iii) Given the results from (ii), the upper band for rainfall in years 11-20 fall strictly under the minimum for years 1-10. Alternatively, for the average rainfall of years 11-20 to be larger than the 1-10 average, it must be 2 standard deviations away or above – meaning there's about 97% chance its value is lower than mean for years 1-10, 1160.