Explaining Implied Volatility Patterns for Options on Nikkei Futures

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December 1, 1994

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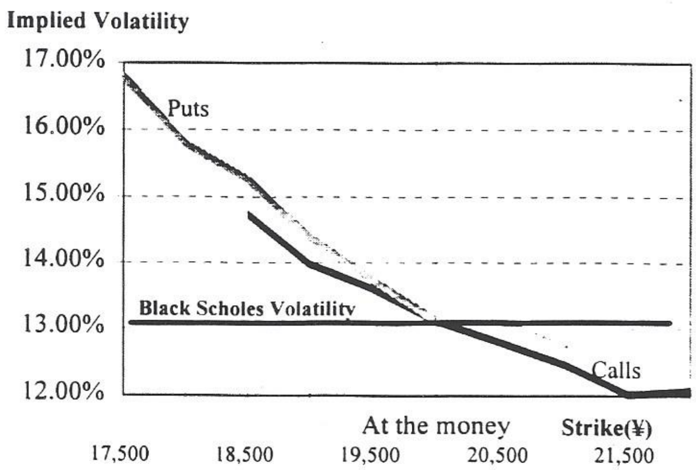
ABSTRACT

In all option pricing models, expected volatility of the underlying asset returns during the remaining life of the option is an important determinant in the price of the option. Such volatility reflects the potential of a particular option to move into the money. Expected volatility is not directly observable. Different methods have been developed to estimate or predict this volatility. One common method is to use the volatilities implied by the actual option prices observed in the market.

Implied price volatility is an estimate of expected volatility that can be derived from option prices. It is the volatility that the marketplace currently expects. In this case, the low strike prices of both put and call options on Nikkei-225 futures trade at high implied volatilities [1] relative to high strikes.

Implied volatility for options on a Nikkei Futures contract expiring in March 1995
Source: Bloomberg 10/31/94



The Black-Scholes options pricing model predicts constant implied volatilities across all strikes. The above observed pattern therefore either (i) violates the standard Black-Scholes model or (ii) represents a glaring arbitrage opportunity resulting from option mispricing by the market.

This paper explores the possibility of explaining the implied volatility pattern exhibited above by using alternative option pricing models including a fully specified recombining binomial tree recently developed by Mark Rubinstein.

Such models can infer that there are no such arbitrage opportunities but that dynamic trading strategies may be crafted that take this pattern into account.

[1] Numerical methods are used to estimate this implied volatility using the Black Scholes option pricing formula.

I. BACKGROUND

The Underlying Asset: March 1995 futures contract on the Nikkei -225.

The Nikkei-225 has long been considered the standard benchmark of the Japanese stock market. It is a price-weighted index of the top 225 Japanese companies that trade on the Tokyo Stock Exchange. The price of Komatsu, for example, represented 0.472% of the index on November 16, 1994. In 1994, the Nikkei has fluctuated between 17,370 and 21,552 and currently trades at the 19,400 level.

The options discussed in this paper are March 1995 options on the March 1995 Nikkei index futures, traded on the Singapore International Monetary Exchange (SMX). These futures are also traded on the Osaka Stock Exchange(OSE), the Chicago Mercantile Exchange(CME), and over the counter (OTC).

Trading of Nikkei-225 futures and options on these futures on the SMX tends to be dominated by sophisticated market professionals who reside or are legally chartered outside of Japan. As a general pattern, the volume of the futures tends to be greater on the Osaka exchange which is dominated by Japanese-based hedgers, arbitrageurs, and speculators. The options trading volume on the SMX for longer dated instruments is often greater than the OSE, and the OSE options volume is more likely to be greater for the most active futures contract (closest date - in this case December 1994).

The Nikkei futures price is related to the current Nikkei price by the relationship:

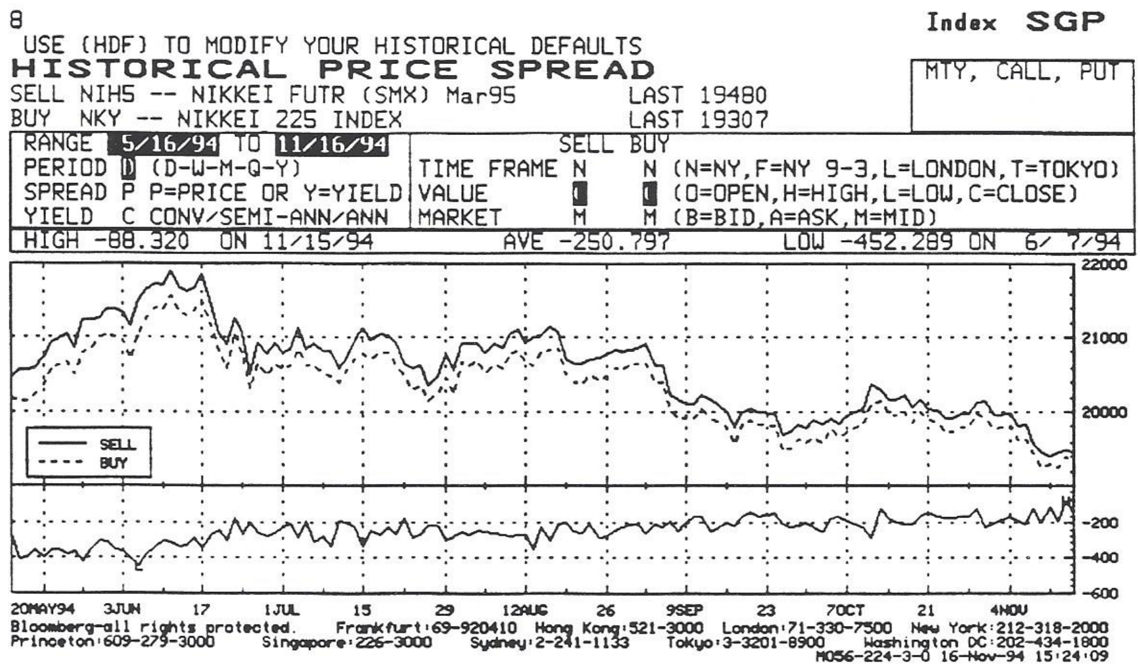
$F = S \times e^{(r-q)(T-t)}$, where:
F = Nikkei futures price.
S = Current Nikkei value.
r = risk free interest rate.
q = average annualized dividend yield expected to be paid during the remaining life of the futures contract.

T-t = time until the contract matures.

Therefore, any changes in the Nikkei will cause proportional changes in the value of the futures price. Thus, we need only look at fluctuations of the historical futures prices to estimate historical volatility of the Nikkei.

March 1995 Nikkei Index Futures Prices versus The Nikkei Index

Source: Bloomberg



Indices such as the Nikkei-225 can be thought of as securities that pay dividends. The security is the portfolio of stocks underlying the index, and the dividends are those paid to the holder of this portfolio. The value of a Nikkei futures contract is the index price times 500 yen. On October 28, 1994, the value of one Nikkei futures contract expiring in March 1995 was ¥500 × 19,970 = ¥9,985,000.

Stock index futures can be used to hedge the market risk in a well diversified portfolio of stocks. A hedge using index futures removes the risk arising from market moves and leaves the hedger exposed only to the performance of the portfolio relative to the market.

Option on a Nikkei futures contract.

When exercised, options on a futures contract require the delivery of the underlying futures contract. In our case, if a call futures option is exercised, the holder of the option receives a cash amount equal to the Nikkei futures price minus the strike price plus a long position in a Nikkei futures contract to buy (¥500 × Nikkei value) in March 1995. If a put futures option is exercised, the holder receives a cash amount equal to the exercise price minus the futures price plus a short position in the futures contract. If desired, the position in the futures contract can be closed out at no cost. This leaves the investor with the option cash payoff.

II. IMPLIED VOLATILITY PATTERNS

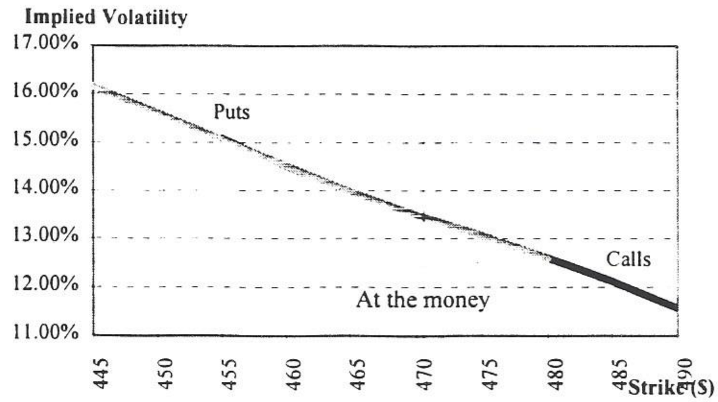
To determine whether declining volatility across strikes is unique to options on Nikkei futures, we examined the relationship between volatility and strike prices across other indices, security types and across various maturity dates and time periods for both calls and puts. The result was that the declining volatility against strike price relationship can be found among these other assets which suggests that the pattern is not exclusive to options on the Nikkei futures. There were also cases where a "smile" or jagged pattern was produced, as well. Appendix 1. shows a host of examples.

1. Across indices and security types

The following three graphs look at the trends of volatility for S&P 500 Futures Contracts, the US-Long Bond, and even Live Hog Futures on 11/16/94. As indicated, all three show the declining volatility as strike price increases.

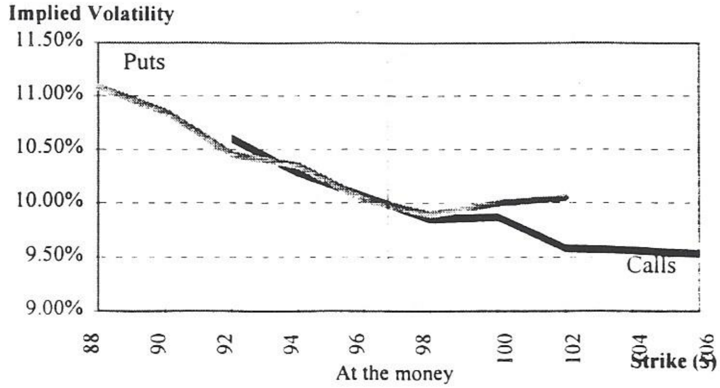
a) Implied volatility for options on S&P500 futures contract expiring March 1995

Source: Bloomberg 11/16/94



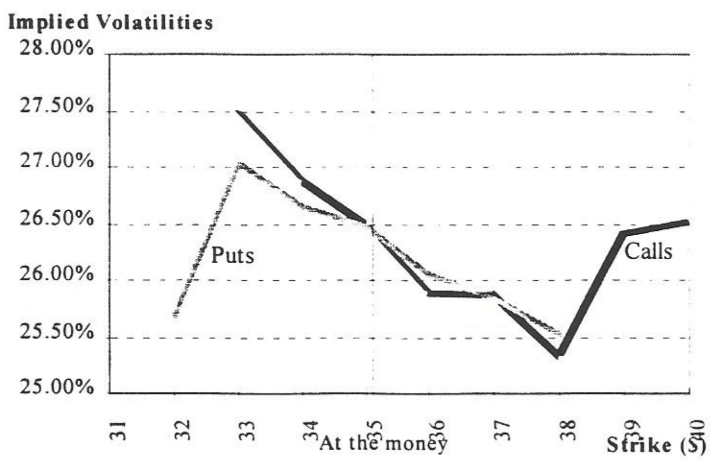
b) Implied volatility for options on the U.S. Long Bond

Source: Bloomberg 11/16/94



c) Implied volatility for options on Live Hog Futures

Source: Bloomberg 11/16/94

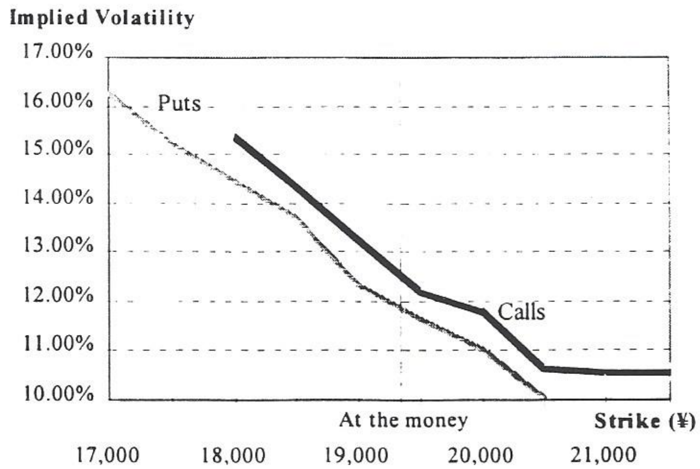


2. Across maturity dates and time periods

The following two graphs look at the trends of volatility on the same March 1995 Nikkei-225 Futures Contracts on 11/18. The volatility pattern is substantially the same as the October 31, 1994 pattern. Options on Nikkei Futures contracts expiring in June 1995 also show a similar pattern of declining volatility as strike prices increase.

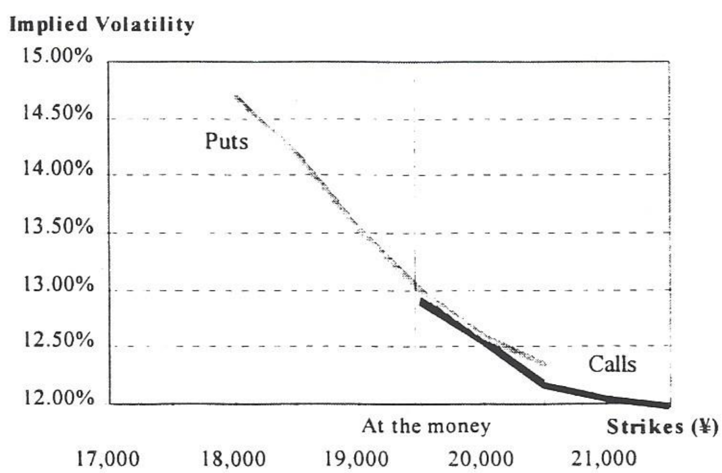
a) Implied volatility for options on a March 1995 Nikkei futures contract

Source: Bloomberg 11/18/94



b) Implied volatility for options on a June 1995 Nikkei futures contract

Source: Bloomberg 11/18/94



That differences in indices, markets, maturity dates, and time periods may not offer a different observed relationship between volatility and strike price indicate that this phenomenon is not exclusive to options on the March 1995 contract on Nikkei futures. This should allow us to (i) rule out the possibility of Japan Ministry of Finance or exchange regulatory constraints as a major causal factor of the observed implied volatility pattern and to (ii) concentrate on the need to modify standard options pricing models.

Ministry of Finance regulations do have an effect on who trades on the Osaka exchange and the SMX. Onerous disclosure, margin, tax, commission, and circuit breaker requirements in Japan (OSE) do tend to make the SMX a more attractive environment for offshore hedge funds and the like. Despite this, prices of futures and options between exchanges exhibit limited opportunity for arbitrage, although some traders and journalists have noted past instances of gains being achieved by exploiting very short-term regulatory driven phenomena.

III. OPTION PRICING MODELS

1. Black-Scholes formulation for a futures option

If the volatility of S, the Nikkei value, is constant, then the volatility of F is constant and equal to the volatility of S. Since it costs nothing to enter a futures contract, the expected growth rate in a futures price in a risk neutral world is 0. Therefore, the dividend yield paid out on the index must be equal to the repo rate (i.e. the risk free interest rate for futures contracts). Then, we can adjust the Black-Scholes formula to get values for calls and puts, assuming the options mature on the same date as the futures contract.

Assumptions underlying the Black-Scholes are:

- constant expected volatility during the remaining life of the option.
- constant level of risk free interest rate during the remaining life of the option.
- lognormal distribution of underlying asset (futures contract) returns. The futures price can either go up to u*F or down to d*F.

The formula used for the Black-Scholes for options on futures contracts is below.

Input parameters:[2]

r:	continuously compounded risk free interest rate = repo rate
t:	Time until the expiration date of the option = time until March 1995.
F:	futures price of the Nikkei 225.
N(x):	Cumulative normal distribution evaluated at x.
K:	Exercise price.
σ:	Expected volatility during the remaining life of the contract.

Output parameters:

$$\text{Call price} = C = FN(xe^{rt}) - K\, r^{-t}\, N(xe^{rt} - \sigma\sqrt{t})$$

[2] Source: Bloomberg, 1994.

Put price $= P = Kr^{-t} N(y + \sigma t^{1/2}) - FN(y)$

where

$$x = \frac{\log[F / Kr^{-t}]}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}$$

$$y = \frac{\log[Kr^{-t} / F]}{\sigma\sqrt{t}} - (1/2)\,\sigma\sqrt{t}$$

2. Constant elasticity of variance formulation

In general the price movement of the underlying asset (Nikkei futures contracts) that lead to the Black-Scholes formula has three important properties:

1. The possible percentage changes in the futures contracts over any period did not depend on the level of the futures contract price at the beginning of the period.

2. Over a very small interval of time, the size of the change in the futures contract was small; in other words, prices could not change all that much before we could do something about it.

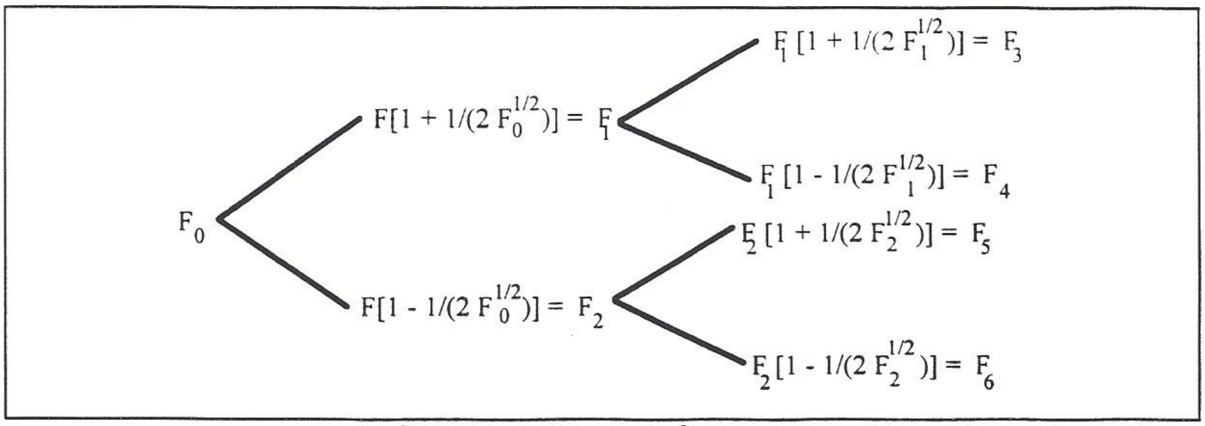
3. Over a single period, only two underlying futures contract prices were possible.

The Black-Scholes model assumes that the volatility is constant over the life of the option. However, if the volatility of the futures contracts on the Nikkei changes significantly, then it also must be true that the price of the futures contracts changes significantly.[3] The price movements of the futures contracts on the Nikkei with volatility depend on the level of the futures contract movements. Therefore, we could modify the binomial tree to account for changes in the price of the underlying asset if we allowed *u* and *d* in the binomial tree to differ predictably from period to period. In our case, we can allow the steps sizes to

[3]Cox and Rubinstein, *Options Markets*, pp. 360-364.

depend on the beginning-of-the-period futures contract prices. We could then provide step sizes and probabilities for every possible futures contract price for each day, as shown in figure 1. For figure 1, we assumed that the upward movement factor is represented by u(F,t), the downward movement factor by d(F,t), and the probability of an upward movement by q(F,t).

$$u(F,t) = 1+\frac{1}{2\sqrt{F}};\ d(F,t) = 1-\frac{1}{2\sqrt{F}};\ \text{and } q(F,t) = 1/2$$



Square root process of new tree.

To show this effect, we can reduce the new tree to the Constant Elasticity of Variance Option Pricing Formula (CEV) which is a modified version of the Black-Scholes formula that incorporates the variance of the return of the underlying asset (in our case the futures contracts) on the price of the underlying asset (futures contracts). An important illustration of this occurs when the instantaneous volatility σ(F,t) has the following form:

$$\sigma(F,t) * F = \hat{\sigma} * F^{\rho-1}$$

In our case for options on futures contracts, ρ should be less than one to indicate the inverse relationship between price and volatility. This type of process has the property that the elasticity of the variance is constant and it can be labeled in this way as a constant elasticity of variance diffusions. The Black-Scholes case corresponds to $\rho = 1$.[4] If ρ is less than 1 for these processes, the variance of the rate of return, $\sigma^2(F,t)$, will vary inversely with the price of the underlying asset, a feature which corresponds to be a characteristic of the actual underlying price movement. Therefore, in our case, we should see a negative correlation between volatility and the price of the futures contracts.

As we have already discussed, there is an economic rationale for this behavior based on the effect of financial and operating leverage for stocks in the Nikkei as well as their sensitivity to interest rate changes. When stock prices on the Nikkei decline, we can presume that the operating performance of firms in the index as a whole have declined and the fixed costs of the firms in the index have the effect of increasing volatility. When the stock prices of companies in this index increase, the volatility should then decrease. The same should be true for futures contracts on the Nikkei. In the general concept, the constant elasticity of variance model is therefore similar to the compound option model.[5]

The constant elasticity of variance option pricing formula is given below.[6,7] In addition, since we are dealing with options on futures, we have replaced x with x^*e^{rt}.[8]

$$C = F *[\sum_{n=1} g(n,xe^{rt}) * G(n+\lambda,y)] - K^*r^{-t} *[\sum_{n=1} g(n+\lambda,xe^{rt}) * G(n,y)]$$

where

$$\lambda = \frac{1}{2^*(1-\rho)}$$

$$x = \frac{2\lambda\log[r]}{\hat{\sigma}^2(r^{t/\lambda}-1)} F^{1/\lambda} r^{t/\lambda}$$

[4] Cox and Rubinstein (1985), pp. 360-364.

[5] Hull, *Options, Futures and Other Derivative Securities*, 1993, p. 442.

[6] Cox and Rubinstein (1985), pp. 360-364.

[7] Cox and Ross, "The valuation of Options for Alternative Stochastic Processes", pp. 145-166.

[8] Black, Fisher, "The Pricing of Commodity Contracts", pp. 145-166.

$$y = \frac{2\lambda \log[r]}{\hat{\sigma}^2(r^{t/\lambda} - 1)} K^{1/\lambda}$$

and the gamma density function is given by

$$g(n,w) = \frac{e^{-w} z^{n-1}}{(n-1)!} = \frac{e^{-w} x^{n-1}}{\text{Gamma}[n]}$$

where

$$(n-1)! = e^{-v} v^{n-1} dv = \text{Gamma}[n]$$

In addition, the Gamma distribution is given by

$$G(n,y) = \int_{y}^{\infty} g(n,y)\, dy$$

In the case of options on futures contracts on the Nikkei, we can describe the fluctuations in the futures price in terms of the constant elasticity of variance formula. In addition, it also explains the relation of volatility to the strike prices.[9,10] *As we can see, there exists an inverse relation between fluctuations in the futures price and the volatility. For a low strike prices out-of-the-money, the volatility must be high because the futures contract on the Nikkei must decreases to reach that strike price. In addition, for a high strike price in-the-money, the volatility must be low because the futures contract on the Nikkei must increases to reach that strike price.*

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[9] Cox and Rubinstein (1985), pp. 360-364.

[10] Hull (1993), p. 442.

Test Performed and Data Results

As shown in appendix 3, we performed our analysis by solving for the implied volatilities for at-the-money futures prices on the Nikkei using a modified Black-Scholes equation for options on futures contracts. We weighted the implied volatilities for the two closest strike prices that were near the futures price. Since we had a small data set of 50 observations, we then used the implied volatilities near-the-money regressed on the historical volatility multiplied by the futures prices raise and a function of the ρ.

$$\hat{\sigma}_{implied} = \sigma_{historical} * F^{\rho-1}$$

Finally, we estimated a 20 day historical volatility average using Hull's formula below[11]:

$$s = [\frac{1}{n-1} \sum_{n} \mu_i^2 - \frac{1}{n(n-1)} (\sum_{n} \mu_i)^2]^{1/2}$$

$$\sigma = s\sqrt{t}$$

In our case, t was 365 days to represent one year. Prior to our regression, we merely performed a correlation between the price of the futures contract and the 20 day historical volatility and received an correlation coefficient of -0.268, which indicates an inverse relationship between the price and volatility. Using a log regression, we found the following relationship:

$$\ln[\hat{\sigma}_{implied}] = \ln[\sigma_{historical} * F]^{\rho-1} = \beta_{his} * \ln[\sigma_{historical}] + (\rho-1)\ln[F] + \varepsilon$$

[11] Hull (1993), p. 215.

The results of our regression analysis are shown in **Table 1** below using a Micro TSP software package.

	Coefficient	**Standard Deviation**	**T-Statistic**
LN of Historical Volatility	0.9861361	0.2039426	4.8353612
LN of Futures Price	-0.0107297	0.0425506	-0.2521626

Table 1. Regression of implied volatility on historical volatility and price.

This creates the following equation relating ρ, volatility and the futures prices.

$$\hat{\sigma}_{implied} = 0.9861361 * \ln[\sigma_{historical}] - 0.0107297*\ln[F] + \varepsilon$$

where $\rho = 1 - 0.0107297 = 0.9892703$. In this case, r is extremely high and close to 1, which is close to making the CEV close to the Black-Scholes model for options on futures.

Using the Table 1, we can create volatilities for pricing the options using a simple example from our data on October 14th, for $\sigma_{historical} = 0.144547$.

K	**Volatility from Regression**
17500	0.13369898
18000	0.13365858
18500	0.13361929
19000	0.13358106
19500	0.13354384
20000	0.13350756
20500	0.13347220
21000	0.13343769

Table 2. Predicted volatilities from Regression and Black-Scholes for October 31st.

We can now use the relationship to replace F with the strike prices and show that the predicted volatility across different strikes has an inverse relationship as shown in Table 2.

Conclusions

We see only a mild effect of this relationship due to the small data sample that we used to prove this relationship. In addition, the t-statistic for the LN of Futures Price is not significant to the 5% level making the beta for LN of Futures Price not statistically significant.

Our results show that the closing prices for the Nikkei futures do not fluctuate by a large amount over the time period we studied. Therefore, the distribution of prices does not vary as much as we hoped for our data sample. However, we do see the inverse relationship between price and historical volatility. With more data samples, the relationship should be more pronounced and statistically significant.[12]

3. Rubinstein's implied binomial tree.

Overview

This approach to valuing options involves three steps. The first step is to derive the implied ending (i.e., at option expiration) risk-neutral probabilities of returns on the underlying asset (Nikkei futures). Armed with these ending risk-neutral probabilities and returns, we work backwards to generate a fully-specified binomial tree that models the local volatility of the underlying as a stochastic process that is a function of both the price of the underlying and time: $\sigma(S, t)$. We could then use this fully-specified binomial tree to

[12] Cox and Ross (January-March 1976), pp.145-166.

compute the value of both European and American options (and their associated hedging parameters) that mature on or before the end of the tree.

Derivation of Implied Ending Risk-Neutral Probabilities

First we must make a prior guess at the ending risk-neutral probabilities associated with the underlying returns. To do this, we use the now famous binomial method to value European options[13]. But what volatility do we select? As a result of the aforementioned skewness observed in the option volatilities, a sensible choice volatility would be to use the average (via linear interpolation) of the volatilities of the observed nearest-to-money options. In our case, the current value of the Mar95 Nikkei future (19970) was bracketed by the 19500 and 20000 strike calls. This gives:

$$\sigma^{imp}_{19970} = 13.60\% \times \frac{20000-19970}{20000-19500} + 13.10\% \times \frac{19970-19500}{20000-19500} = 13.13\%$$

[13] Cox and Rubinstein (1985) Chapter 5.

Supplying this average implied volatility to our n-step binomial tree allowed us to generate a vector of ending Mar95 Nikkei future prices which I will denote as S_j for j = 0,...,n ordered from lowest to highest. Associated with each of the S_j is a risk-neutral probability P'_j distributed lognormally as:

$$P'_j = \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}$$

Figure 1 in the next page illustrates how one would generate each of the S_j and P_j for an n-step binomial tree:

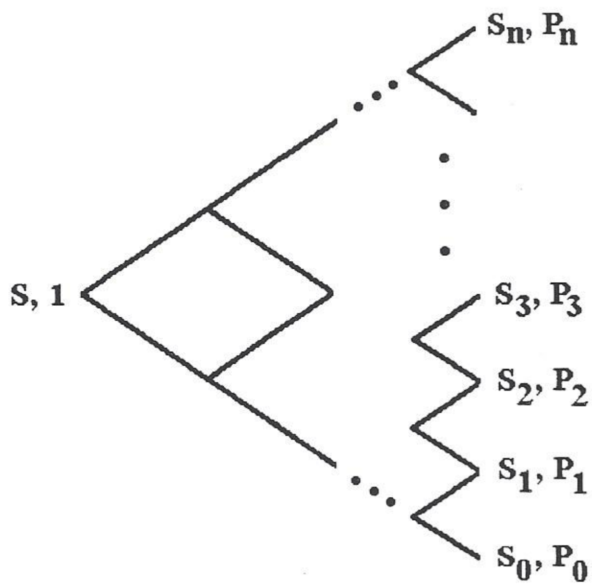


Figure 1.

As a result, our desired posterior risk-neutral probabilities, P_j, are the solution to the following linear program:

$$\min_{P_j} : \sum_{j=0}^{n} (P_j - P'_j)^2 \textbf{ subject to:}$$

$$\sum_{j=0}^{n} P_j =$$

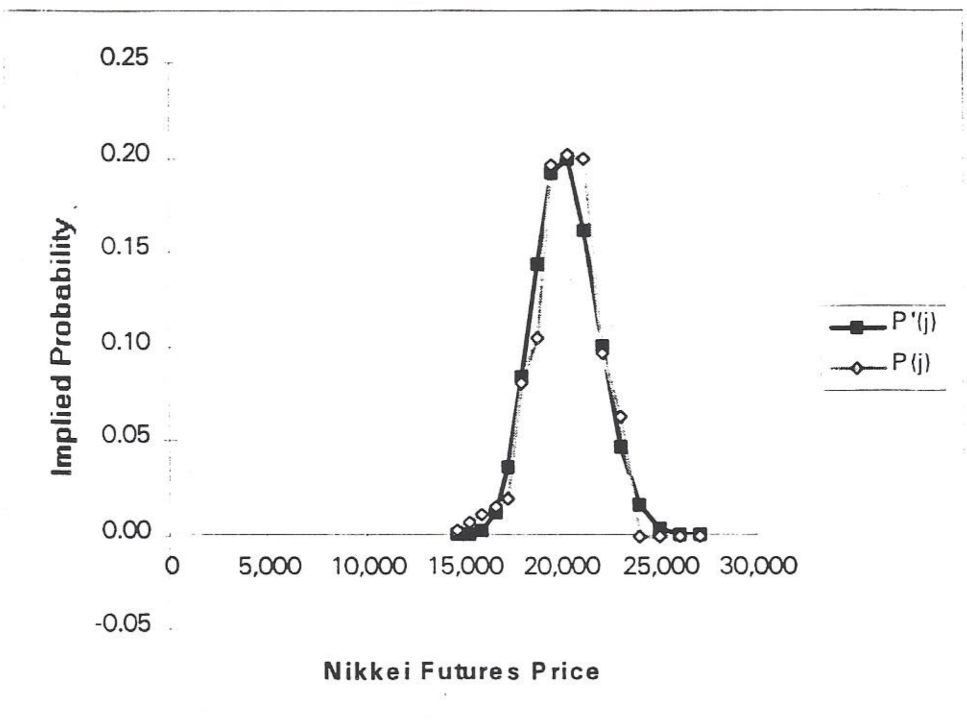
$$P_j \geq 0$$

$$S = \Big(\sum_{j=0}^{n} P_j S_j\Big) / r^n$$

$$C_i = \Big(\sum_{j=0}^{n} P_j \max[0, S_j - K_i]\Big) / r^n \quad \text{for } i = 1, \ldots, m$$

where r is the risk free rate associated with each binomial period and the C_i are observed prices of European calls (with associated strikes K_i) that mature at the end of the tree. As a result, the P_j's are the ending risk neutral probabilities that are, in an absolute difference sense, closest to lognormal that cause the value of the underlying asset to equate with its observed price and cause the call prices to fall between their observed bid ask spreads.

Using Excel solver to implement the linear program for a 15-step tree for the Mar95 Nikkei futures, we were able to obtain the following risk-neutral probability distributions:



Notice that compared to the smooth lognormal distribution of the risk-neutral probabilities of our prior (P_j), the distribution of the implied risk-neutral probabilities displays a fatter tail at asset prices and a much more attenuated tail at high asset prices.

We are now ready to proceed to the next stage which is to evaluate the stochastic process of the underlying asset's volatility implied by the S_j and P_j.

Implied Stochastic Process for Local Volatility of the Underlying Asset

In order to see how we use the S_j and P_j to flesh out our implied binomial tree, let's consider a simple n=2 step example:

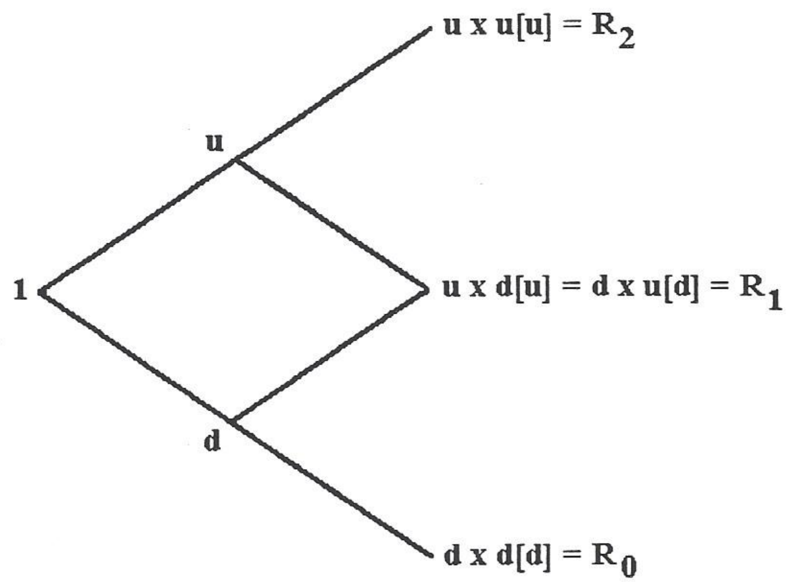


Figure 2.

Notice that we have only n+1=3 possible ending returns. The notation used in the figure above refers to the path dependence of the of each up and down move. For example, u[d] refers to an up move immediately following a down move.

We can draft an analogous diagram for the ending and interior node risk neutral probabilities. This is shown on figure 3 on the next page:

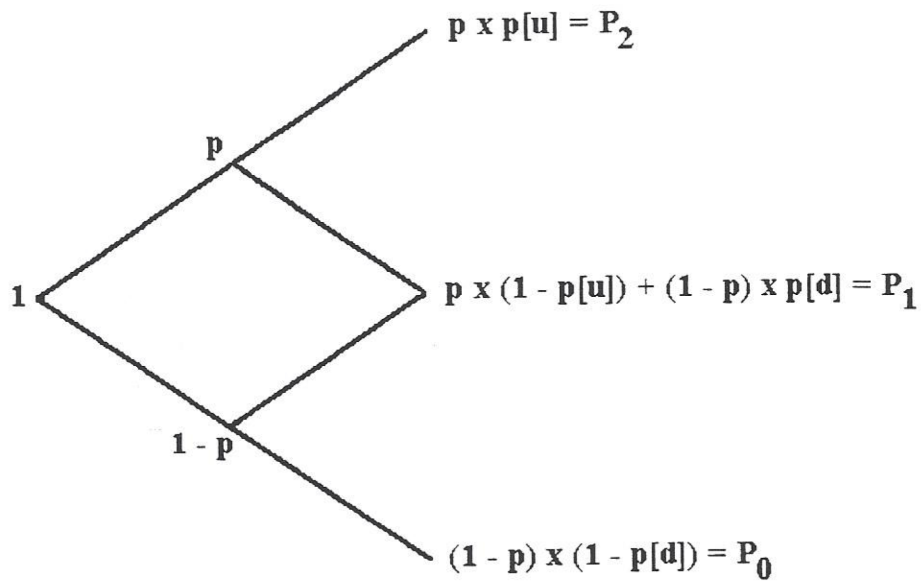


Figure 3.

Here again, the bracket notation indicates the path dependent nature of the risk neutral probabilities. For example, p(1-p[u]) is the probability of a down move in the second period immediately following an up move in the first period. Combining Figures 2 and 3 and are basic binomial option pricing assumption that the "move probabilities" are risk-neutral we must have:

$$r[\bullet] = ((1 - p[\bullet]) \times d[\bullet] + (p[\bullet] \times u[\bullet])$$

where $r[\bullet]$ is one plus the riskless rate of interest over the associated binomial step.

Before we actually proceed with fleshing-out our implied binomial tree, we must first clearly state the following economic and computational assumptions that underly the binomial process:

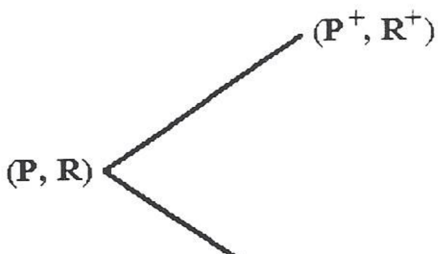
1. ***The underlying asset return follows a binomial process:*** This allows us to derive a completely general description of the volatility of the underlying asset as a function of the underlying asset return, the prior path the underlying asset return, and time.
2. ***The binomial tree is re-combining:*** This assumption imposes a computational constraint on our implied tree by forcing the asset returns to be *path-independent*. In other words, the asset return at any node of the implied tree is determined solely by the *number* of up and down moves required to reach the node and not on the *order* of the up and down moves. Of course, the imposition of this constraints restricts our description of the asset volatility to be a function of only the asset return and time (see assumption 1. above). The primary advantage of this assumption is that it significantly reduces the computational complexity of the fleshing out process. For example, the number of ending nodes for an n-step *non-recombining* binomial tree is 2^n compared to only n+1 for an n-step recombining tree.
3. ***The interest is constant:*** In our example, this means that r = r[u] = r[d]. As a result, we can drop the bracket notation when referring to r from now on. Note that to get an even more general description of the volatility of the underlying asset, we could relax this assumption. If, for instance, we developed a term structure model of interest rates, we could assign a different interest rate for every binomial period. Unfortunately, time did not permit us to investigate this possibility.

4. ***All paths that lead to the same ending node have the same risk-neutral probability:*** In the context of our 2-step binomial tree in Figure 2, this assumption allows us to derive the values of the variables P_{uu} , $P_{du} = P_{ud}$, P_{dd} which are the probabilities of a particular single path through the implied tree ("path probabilities"):

$$p \times p[u] = P_2 \equiv P_{uu}$$
$$(1 - p) \times p[d] = P_1 \div 2 \equiv P_{du}$$
$$p \times (1 - p[u]) = P_1 \div 2 \equiv P_{ud}$$
$$(1 - p) \times (1 - p[d]) = P_0 \equiv P_{dd}$$

Solution of the Implied Binomial Tree

The process of actually fleshing out the interior nodes of our implied binomial tree is an inherently recursive algorithm. Consider, for example, the solution for an arbitrary single interior node of our binomial tree in Figure 4 shown below:



1) $\mathbf{P = P^- + P^+}$
2) $\mathbf{p = P^+ / P}$
3) $\mathbf{R = [(1 - p)R^- + pR^+] / r}$

<u>**Figure 4.**</u>

Note that in this case, the unsubscripted P's represent the path probabilities and the unsubscripted R's represent the nodal asset return values.

The fleshing out process begins by generating an empty n-step implied binomial tree. We then take our output from the linear optimization model above, R_j and P_j, and assign them to the ending nodes of our empty implied tree[14]. To get the *path probability* for each of the ending nodes, we must take the ending node probability P_j and divide it by the number of paths to the node. In general, the probability is given by:

$$P_j \div \frac{n!}{j!(n-j)!}$$

In practice, however, most computer don't have enough bits to represents such large resulting factorials. Instead, it is easier to simply recursively traverse the tree and record, for each node, the number of times you visited the node. In our case, we implemented the following simple preorder traversal:

1. Visit the root of the binary tree.
2. Traverse the up subtree in preorder.
3. Traverse the down subtree in preorder.

Next, we work backwards from the end of the tree computing the path probabilities and asset returns for each parent of the ending nodes. Note that since we have a recombining tree, each ending node has *two* parents.

[14]Nota Bene: $R_j = S / S_j$ where S is the current price of our underlying asset.

Let's examine each of the steps in Figure 4. from an economic and probabilistic standpoint. Step one merely asserts that the path probability of any interior node must equal the sum of the probabilities of all paths that emanate from it to the end of the tree. Step 2 simply determines the probabilities associated with an up and down move from a given interior node. The third step uses the risk neutral probabilities computed in the second step to compute the nodal asset return by discounting the probability-weighted returns of the next period nodes. Note that in this context, r is defined as:

$$r^n = \sum_{j=0}^{n} P_j R_j$$

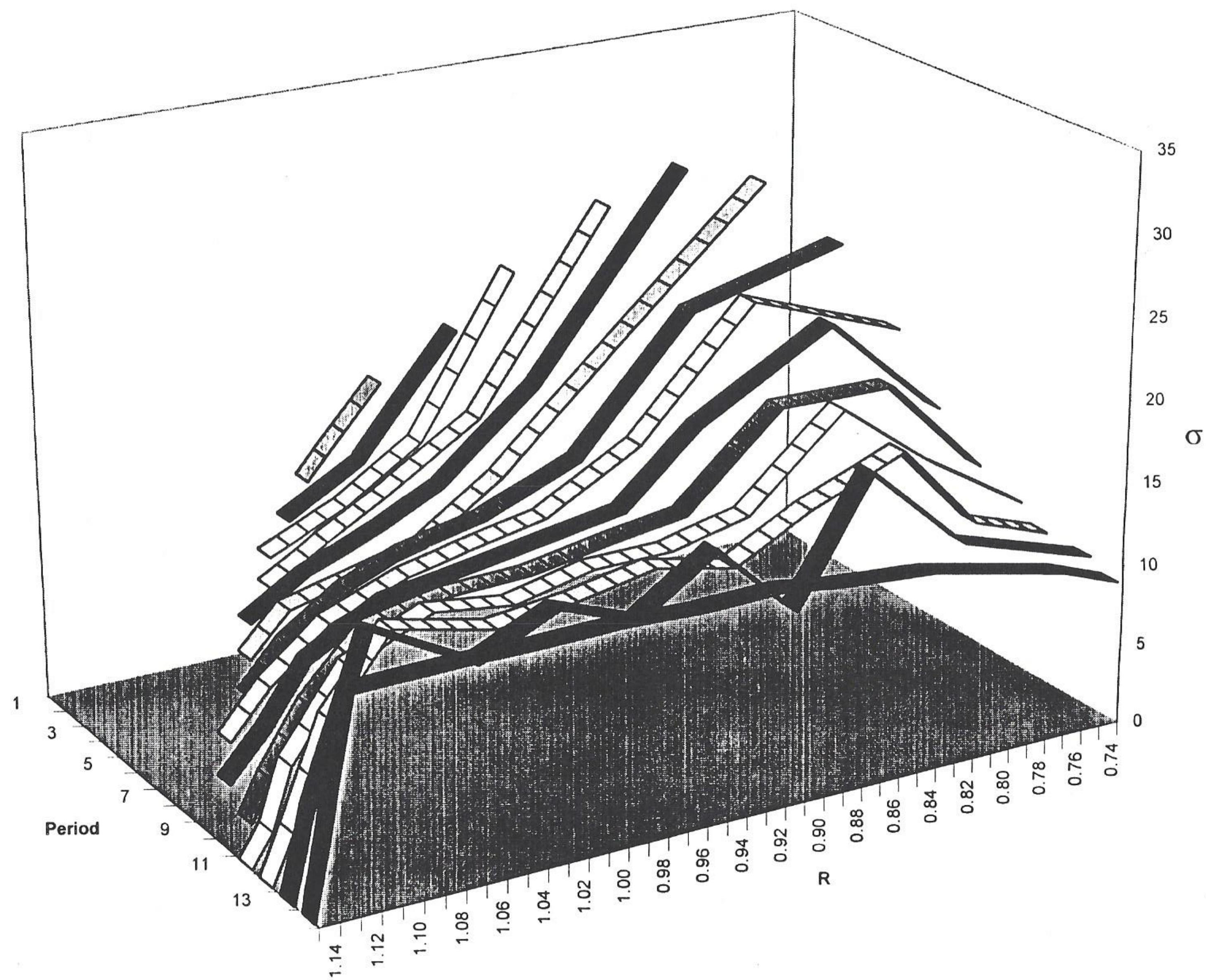
Computation of Local Volatilities

At the same time that we compute the risk neutral probabilities and nodal returns in the recursive manner described in the previous section, we can also compute the local underlying asset volatility for each node:

$$u[\bullet] \equiv R^+ / R$$
$$d[\bullet] \equiv R^- / R$$
$$\mu[\bullet] \equiv ((1 - p[\bullet]) \times \log d[\bullet]) + (p[\bullet] \times \log u[\bullet])$$
$$\sigma^2[\bullet] \equiv ((1 - p[\bullet]) \times [\log d[\bullet] - \mu[\bullet]]^2) + (p[\bullet] \times [\log u[\bullet] - \mu[\bullet]]^2)$$

The power of this technique is now clearly manifest. All existing closed-form European option pricing models assume that there exists a strict, parametric relationship between asset volatility and independent variables asset price and time. For instance, the Black Scholes model assumes that asset volatility is independent of both asset price and time and is assumed to be constant over the life of the option. The constant elasticity of variance model allows for asset volatility to change in a specific parametric fashion with asset price

VOLATILITIES FROM IMPLIED BINOMIAL TREE



VOLATILITIES FROM IMPLIED BINOMIAL TREE

	0		1		2		3		4		5		6		7		8		9	
	sigma	R	sigma	R	sigma	R	sigma	R	sigma	R	sigma	R	sigma	R	sigma	R	sigma	R	sigma	R
0	13.67	1.00	11.03	1.02	9.89	1.04	9.10	1.06	8.33	1.07	7.47	1.09	6.49	1.10	5.37	1.11	4.12	1.13	2.80	1.14
1			16.39	0.98	12.53	1.00	11.13	1.02	10.39	1.04	9.74	1.06	9.01	1.08	8.11	1.09	6.98	1.11	5.55	1.12
2					19.80	0.95	14.15	0.98	12.27	1.01	11.53	1.02	11.05	1.04	10.57	1.06	9.97	1.08	9.08	1.10
3							23.86	0.93	16.00	0.96	13.27	0.99	12.38	1.01	12.03	1.03	11.80	1.04	11.64	1.06
4									27.85	0.89	18.24	0.94	14.23	0.97	12.97	0.99	12.59	1.01	12.40	1.02
5											30.23	0.86	20.80	0.91	15.33	0.94	13.45	0.97	12.99	0.99
6													29.49	0.83	23.13	0.88	16.63	0.92	13.87	0.94
7															25.69	0.80	24.10	0.86	18.03	0.90
8																	20.59	0.78	22.75	0.83
9																			16.09	0.77

	10		11		12		13		14	
	sigma	R	sigma	R	sigma	R	sigma	R	sigma	R
0	1.46	1.14	0.05	1.15	7.65	1.35	6.95	1.35	6.36	1.35
1	3.86	1.13	2.00	1.14	0.01	1.15	9.48	1.29	8.68	1.30
2	7.69	1.11	5.68	1.13	2.97	1.14	0.00	1.15	10.25	1.24
3	11.42	1.08	10.72	1.10	9.04	1.12	5.15	1.14	0.00	1.15
4	12.43	1.04	12.90	1.06	13.63	1.09	14.98	1.11	12.92	1.13
5	12.64	1.00	12.26	1.02	12.26	1.04	11.65	1.06	13.06	1.08
6	13.43	0.96	13.34	0.98	12.83	1.00	13.79	1.02	13.04	1.04
7	14.02	0.92	13.36	0.94	13.76	0.96	11.96	0.98	13.13	1.00
8	19.32	0.87	14.26	0.90	12.72	0.92	15.11	0.94	12.90	0.96
9	19.21	0.81	19.53	0.85	15.07	0.88	10.40	0.90	13.04	0.92
10	13.05	0.76	15.03	0.80	17.31	0.83	17.62	0.86	12.55	0.89
11			11.33	0.75	11.99	0.79	12.15	0.81	12.04	0.84
12					10.37	0.75	11.03	0.78	11.25	0.81
13							9.54	0.74	10.18	0.77
14									8.83	0.74
15										

but not time. In contrast, the implied binomial tree method not only permits asset volatility to be a function of both asset price and time, it allows for the fact that the precise nature of this relationship can change with time. The implied tree, therefore, provides a discrete approximation to this complex relationship.

IV. CAUSES OF IMPLIED VOLATILITY PATTERNS.

1. Pricing biases

According to the Black Scholes formula, the volatility of the Nikkei futures contract should be constant across call and put option strike prices. As shown before, we observe an inverse relationship between implied volatility and exercise prices for puts and calls.

One important reason why these observed prices might not necessarily imply arbitrage opportunities is that the distribution of the terminal stock price is not lognormal in reality, as assumed by Black-Scholes The application of Black-Scholes will then cause pricing biases. In our case, if the left tail of the actual distribution of final stock prices is fatter and the right tail is thinner, then Black-Scholes will show the observed biases.

2. Skewness

The Black Scholes formula assumes that volatility changes are independent of changes in futures prices. One alternative assumption is that changes in volatility are inversely related with changes in futures prices. As the futures price rises, volatility decreases. This relationship would cause the observed pattern of implied volatilities for puts and calls.

3. Mean reversion.

As we look into strike prices that are much higher than at the money, we observe a stabilizing of the implied volatility. This could be due to a mean reversion effect for implied volatility.

4. Random causes.

Finally, there could be unexplained sources of variation that make the implied volatility curve stabilize and shift up for high strike prices. An alternative explanation for these random causes is an anticipation in the direction of the movement of the futures price.

Since July 1994, the futures price has been going down in general (from ¥21,370 to ¥19,480). If the market believes that the futures price will keep going down, then put options that are currently out of the money might be in the money in the near future, and therefore they trade at a higher price and higher implied volatility because demand for them increases and put writers want to be compensated for this extra risk. Puts that are currently in the money are more likely to remain so; therefore they will trade at lower implied volatilities and at lower prices with respect to Black Scholes valuation to adjust for the lower risk taken by put writers.

V. ESTIMATING VOLATILITY

Option pricing models, whether standard Black-Scholes or alternative methods, require several inputs concerning the exact specification of an option contract and the characteristics of the underlying security. While factors such as strike price and time to expiration are easily observed, the volatility of the underlying security σ must be estimated. The choice of estimation process by which σ is determined can lead to significantly different option values. The processes differ in several respects; the weights assigned to more recent and long-dated observations, whether or not past forecasts are included, and the presence of a long-term volatility trend.

1. Historical Volatility

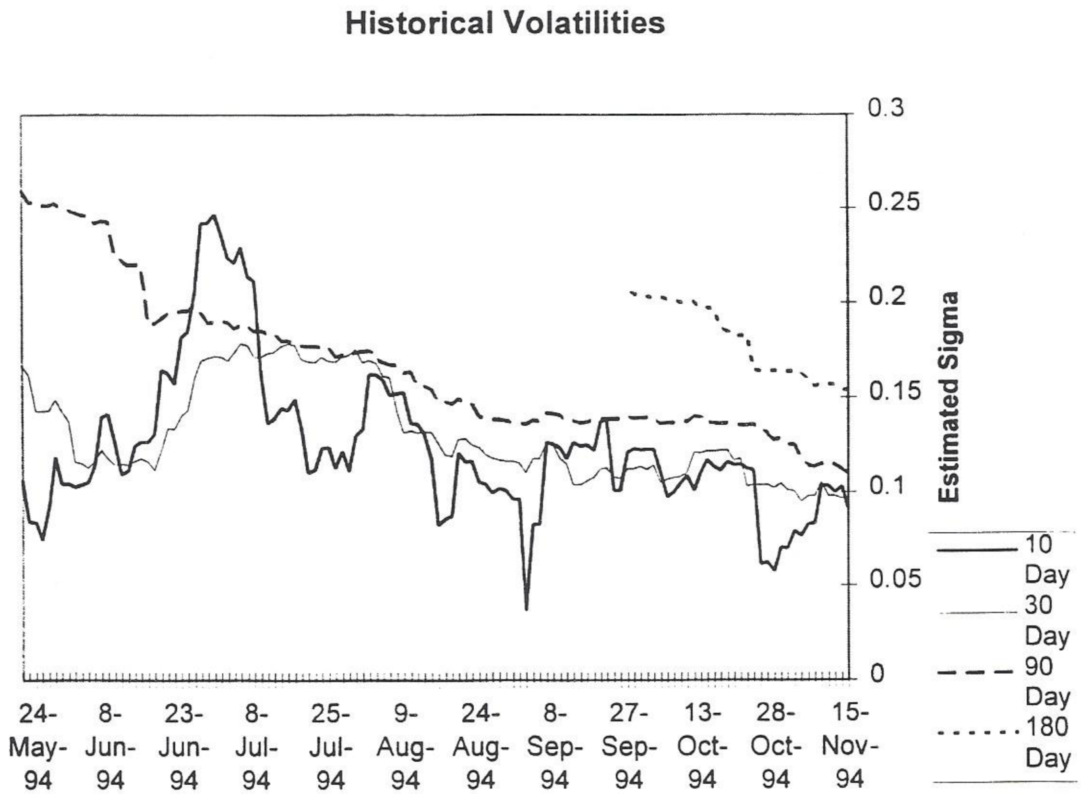
Calculating volatility from a specified number n days of observed prices of the underlying asset is perhaps the most simple and intuitive process for estimating σ. The sample estimate is simply:

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{p} u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} u_i\right)^2}$$

where $u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$ or the log return in the underlying asset.

Several researchers have found that using the last 90 to 180 days of observed security prices is generally a good approximation of the volatility implied by option prices. Shorter periods such as 10 to 20 days can occasionally differ significantly from observed put and call prices.

The following chart of historical volatilities presents σ estimates over a range of different n values, including 10, 30, 90, and 180 days. The Nikkei futures prices have definitely become more stable recently. The 10 day window, which most closely reflects "spot" volatility, has been close to 10%, far below its high of 25% in June. By November all of the volatility estimates except for the 180 day window had reached a relatively tight range between 9% and 13%, reflecting the stability of the Nikkei futures market over the last 90 days. The 180 day estimate remained above 15% as the higher volatility of the summer remained in the estimate. Current option pricing in the Nikkei futures market implies σ of about 13%, making a determination of which n day window is most appropriate for such estimation. Only the 180 day window can be said to clearly not estimate σ accurately.



2. Autoregressive Estimates

The simple estimate of σ tends to overlook the possibility that recent information is more relevant than observations at the start of the n day period. Procedures such as ARCH (Autoregressive Conditional Heteroskedasticity) and GARCH (Generalized ARCH) allow for such different weighting in computing an estimate h_t of conditional variance in the following manner[15]:

ARCH $$h_t = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-1}^2$$

where $\varepsilon_t = y_t - m_g$ or the difference between the observed return and the return conditional on the available information. Note that if the constant $\omega = 0$ and each $\alpha = 1/p$ we are calculating the simple estimate of σ. By examining the impact of past conditional estimates of volatility on the current estimate, we generalize the ARCH equation to the GARCH form as follows:

GARCH $$h_t = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-1}^2 + \sum_{i=1}^{q} \beta_i h_{t-1}$$

While GARCH models can help explain the "smile curve" observed in some options, they do not allow for positive and negative error terms to have different impacts on forecasts of future volatility (since the errors are squared.) Given the observed skewness pattern in Nikkei future options, this limitation in volatility forecasts could pose a problem. The EGARCH (exponential GARCH) allows for such different impacts as follows:

EGARCH $$\log h_t = \omega + \sum_{i=1}^{p} \beta_i \log h_{t-1} + \sum_{i=1}^{p} \alpha_i |\varepsilon_{t-1}| / h_{t-i}^{1/2} + \sum_{i=1}^{p} \gamma_i \varepsilon_{t-i} / h_{t-i}^{1/2}$$

The EGARCH model also has the advantage of never forecasting negative volatility due to the use of logarithms.

[15] Engle, pp. 73-75.

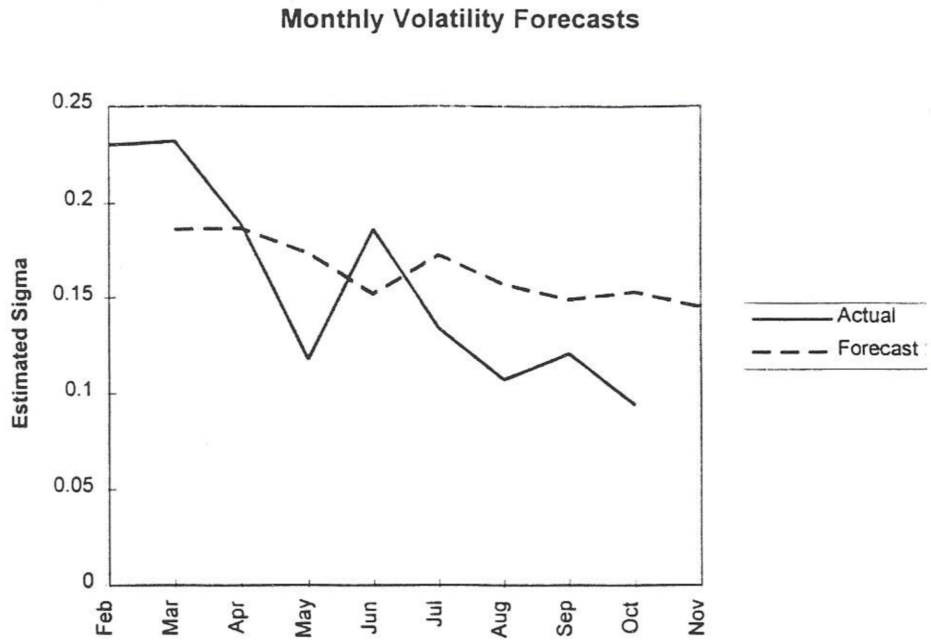
3. Mean Reversion Model

Volatility in equity markets has been shown by several researchers to revert towards a long-term level after a positive or negative shock, such as the stock market crash in October 1987. The speed by which observed volatility returns to the long-run average is the "half-life" of the process. The model can be expressed as follows: $\delta = \alpha + \beta s$ where δ is the drift parameter of the volatility process and β is the reversion rate.[16] The expected volatility at any given time is defined as follows:

$$E\left[s_t\right] = e^{\beta T}\left[s_{t-1} - \omega\right] + \omega$$

where ω is the mean reversion level. From the last year of Nikkei futures data, we find the long-term average to be 0.1671 or 16.71% annual deviation. Picking a value of β corresponding to Randolph's estimate of -0.06 implies a volatility half-life of -ln2/β or 11.6 trading days (over two weeks.) The following graph presents monthly volatility forecasts for February through November using the preceding month's observations:

[16] Randolph, pp. 22-25.



The mean reversion model assumes that volatility will revert to a long-term trend relatively quickly. In the Nikkei futures market however, there have been prolonged departures from the trend.

VI. TRADING STRATEGIES

If investors believed that implied volatility is a good predictor of average future volatility, and they calculated a weighted average of the implied volatilities at the different strike prices, the composite implied volatility would usually be between the low and high strikes implied volatility. Therefore, one risky strategy would be to sell low strike options trading at high implied volatility (and therefore apparently overpriced) and buy high strike options trading at low volatility.

This strategy would be risky because it would not guarantee that the traded options would finish in the money. Therefore, we would have to hedge this portfolio dynamically to protect it from possible future losses.

In order to test the profitability of strategy, we could use simulation in a two pronged approach. First, develop a profitable dynamic hedging scenario using one year of data starting two years ago. Then, test that profitable strategy using the most recent year of data. In the following simple example of a dynamic trading strategy, we assume that interest rates will remain relatively stable during the life of the contract.

1. Delta hedging.

Delta measures the sensitivity of the option value to changes in the futures price. The idea of delta hedging is to offset possible changes in the value of the option position with an opposite position in the futures contract. When the hedge is perfect, the portfolio of options and futures contract is delta neutral. For our hedge to be perfect, the maturity of the futures contract must match the expiration date on the option. Also, we must hold delta futures contracts for every call option that we write. Finally, because delta changes as time passes and market conditions change, we have to rebalance our portfolio continuously to always have a delta neutral portfolio. This involves transaction costs and thus might be costly to do every day.

In our case, if we sell a call futures option at a strike price of 18,500, the delta of this position is -0.81 on October 28, 1994. For every dollar increase in the futures price, the call option will increase by $0.81. To offset this delta, we will buy 0.00162 (0.81/500) Nikkei futures contracts at the same strike for a delta on the futures contracts of 0.81 (since the Nikkei futures has a delta of 1). Overall, our portfolio delta is zero (-0.81+0.81) and our portfolio is delta neutral.

When the futures price drops, delta drops. If the option delta drops from 0.81 to 0.75, then we must sell 0.00012 (0.06/500) futures contracts to keep our portfolio delta neutral. Delta hedging can be expensive when the futures price drops by a large amount. In effect delta hedging buys high and sells low. To prevent this loss and reduce the frequency of the rebalancing, we make our portfolio gamma neutral.

2. Simultaneous Gamma and Vega hedging.

Gamma measures the sensitivity of delta to changes in the futures price. If the gamma of a portfolio of futures options is positive, it can be shown that the value of the portfolio declines if there is no change in the futures price, and increases in value when there is a large positive or negative change in the futures price.

If the gamma of portfolio is negative, the portfolio increases in value when there is no change in the futures price and decreases in value when there is a large positive or negative change in the futures price. A position in the futures contract has zero gamma. Therefore, we must trade in options to make our portfolio gamma neutral.

Vega measures the sensitivity of the call value to a change in the value of the volatility. The vega of the futures contract is 0. Therefore, we need to establish a position with options to have a vega-neutral portfolio.

In our previous example, our portfolio is delta neutral and has a gamma of -1.5 and a vega of -32. Call options at 21,500 have a gamma of 1.5 and a vega of 29. Call options at 22,000 have a gamma of 1 and a vega of 21. Let w1 and w2 be the amounts of the option to be included in the portfolio. We want:

$-1.5 + 1.5w1 + w2 = 0$

$-32 + 29w1 + 21w2 = 0$

Solving for w1 and w2, we get: w1=-0.6, w2=2.4.

Therefore we want to write 0.6 call futures options at 21,500 and buy 2.4 call futures options at 22,000 to have a gamma and vega neutral portfolio.

3. Rebalancing delta.

This will make the delta of our new portfolio to be -0.26 (=-0.16-0.1). Therefore, to finally rebalance our portfolio we will need to buy 0.00052 (0.26/500) futures contracts. Our final position is therefore:

Position	Profit
long 0.00214 Futures contracts	0
sell 1 call option at 18,500	1,640
sell 0.6 call options at 21,500	72
buy 2.4 call options at 22,000	-156
Total profit assuming perfect dynamic hedge:	**1,556**

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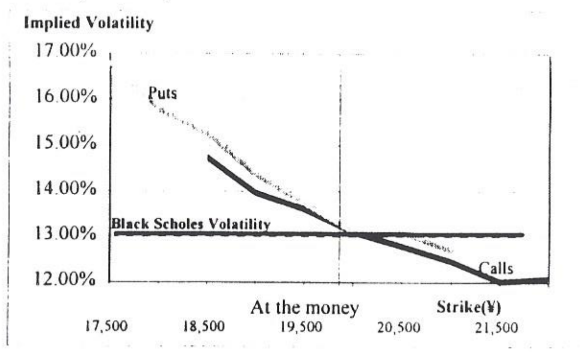
APPENDIX 1.

Nikkei-225 March 1995 Futures Contract (10/31/94)

Asset Level: 19,970

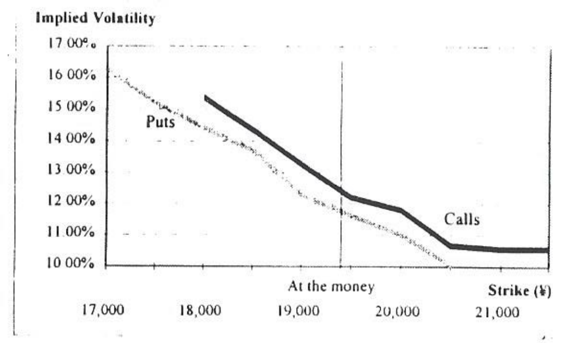
Strike	Call Volatility	Put Volatility
17,500		16.79%
18,000		15.78%
18,500	14.74%	15.26%
19,000	13.97%	14.38%
19,500	13.60%	13.73%
20,000	13.10%	13.11%
20,500	12.79%	13.11%
21,000	12.46%	12.70%
21,500	12.02%	
22,000	12.08%	



Nikkei-225 March 1995 Futures Contract (11/18/94)

Asset Level: 19,425

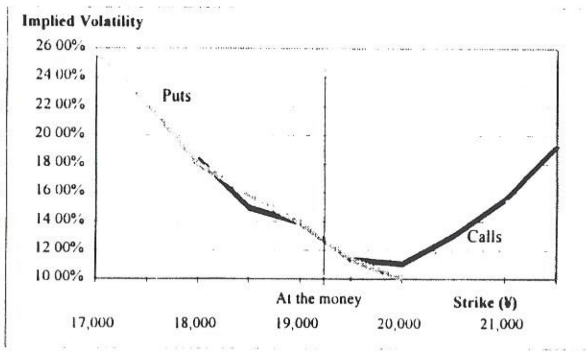
Strike	Call Volatility	Put Volatility
17,000		16.28%
17,500		15.22%
18,000	15.39%	14.43%
18,500	14.36%	13.72%
19,000	13.27%	12.31%
19,500	12.21%	11.64%
20,000	11.81%	11.02%
20,500	10.67%	10.06%
21,000	10.57%	
21,500	10.57%	



Nikkei-225 December 1994 Futures Contract (11/16/94)

Asset Level: 19,340

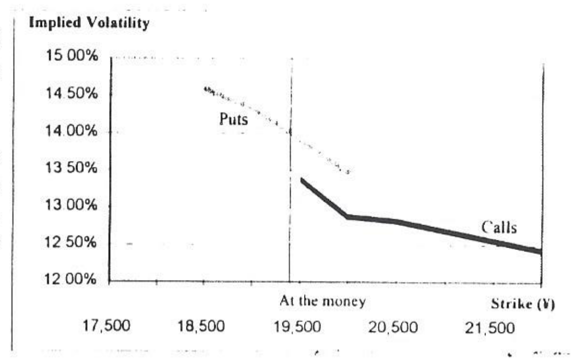
Strike	Call Volatility	Put Volatility
17,000		25.72%
17,500		22.25%
18,000	18.35%	17.99%
18,500	14.96%	15.86%
19,000	13.91%	13.88%
19,500	11.37%	11.38%
20,000	11.06%	10.03%
20,500	13.09%	
21,000	15.57%	
21,500	19.29%	



Nikkei-225 December 1994 Futures Contract (11/18/94)

Asset Level: 19,305

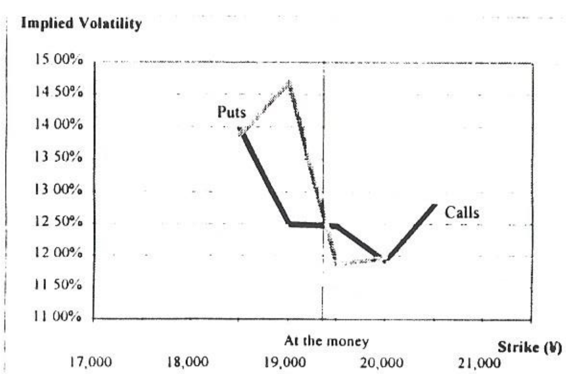
Strike	Call Volatility	Put Volatility
17,500		
18,000		
18,500		14.60%
19,000		14.34%
19,500	13.38%	13.93%
20,000	12.88%	13.48%
20,500	12.82%	
21,000	12.69%	
21,500	12.56%	
22,000	12.43%	



Nikkei-225 Index spot (11/16/94)

Asset Level: 19,307

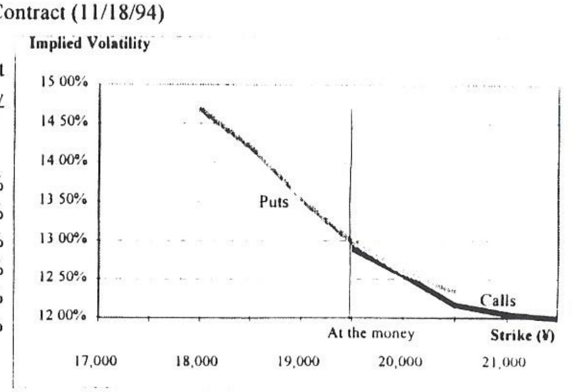
Strike	Call Volatility	Put Volatility
17,000		
17,500		
18,000		
18,500	13.98%	13.88%
19,000	12.50%	14.70%
19,500	12.47%	11.87%
20,000	11.92%	11.97%
20,500	12.82%	
21,000		
21,500		



Nikkei-225 June 1995 Futures Contract (11/18/94)

Asset Level: 19,495

Strike	Call Volatility	Put Volatility
17,000		
17,500		
18,000		14.68%
18,500		14.20%
19,000		13.54%
19,500	12.90%	12.99%
20,000	12.56%	12.59%
20,500	12.17%	12.35%
21,000	12.05%	
21,500	11.99%	

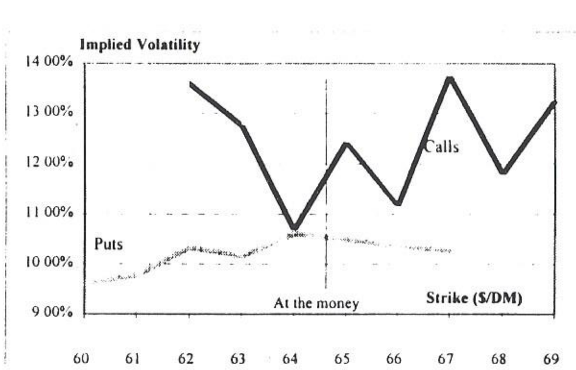


Source: Bloomberg

APPENDIX 1.

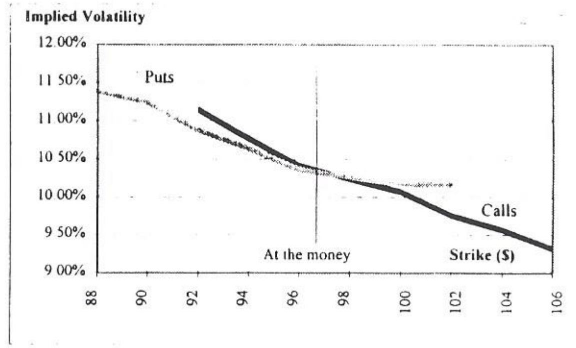
D-Mark spot (11/16/94)
Asset Level: 64.6

Strike	Call Volatility	Put Volatility
60		9.62%
61		9.78%
62	13.60%	10.33%
63	12.77%	10.17%
64	10.68%	10.61%
65	12.43%	10.50%
66	11.16%	10.37%
67	13.74%	10.28%
68	11.80%	
69	13.24%	



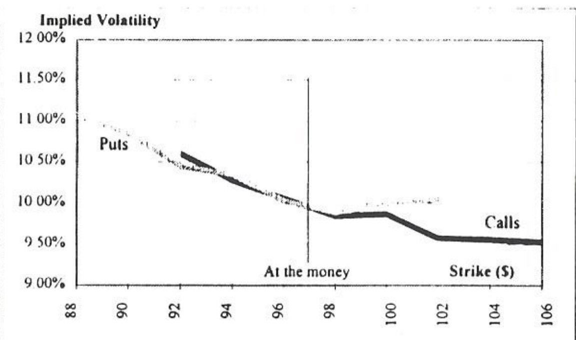
Long Bond Futures (11/14/94)
Asset Level: 96.6

Strike	Call Volatility	Put Volatility
88		11.39%
90		11.24%
92	11.15%	10.88%
94	10.77%	10.64%
96	10.42%	10.37%
98	10.24%	10.26%
100	10.07%	10.18%
102	9.76%	10.17%
104	9.57%	
106	9.32%	



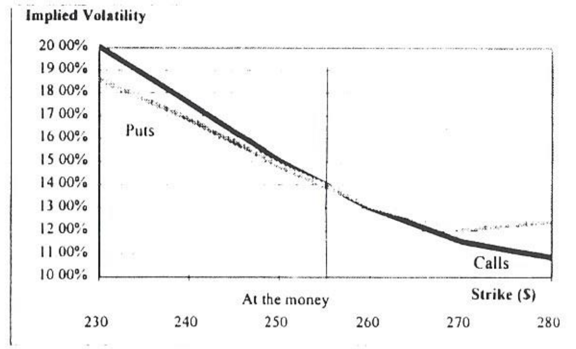
US-Long Bond (11/16/94)
Asset Level: 96.5

Strike	Call Volatility	Put Volatility
88		11.08%
90		10.85%
92	10.60%	10.45%
94	10.28%	10.35%
96	10.06%	10.03%
98	9.85%	9.89%
100	9.87%	10.00%
102	9.58%	10.05%
104	9.56%	
106	9.53%	



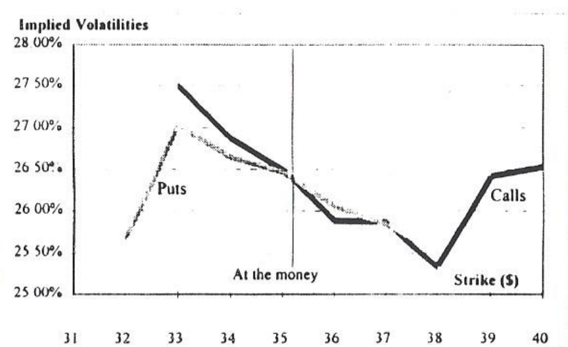
NYSE Index Futures (11/14/94)
Asset Level: 256.9

Strike	Call Volatility	Put Volatility
230	20.05%	18.59%
240	17.57%	16.87%
250	15.07%	14.84%
260	13.02%	13.12%
270	11.56%	12.10%
280	10.86%	12.45%



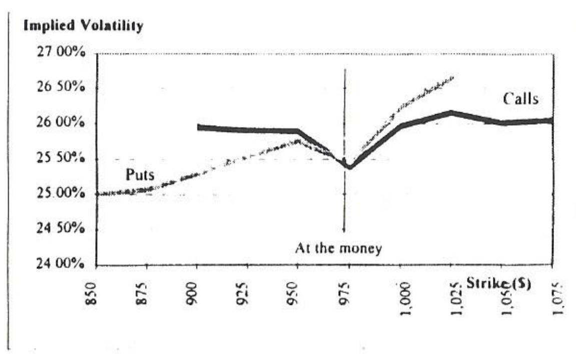
Live Hog Futures (11/16/94)
Asset Level: 35.1

Strike	Call Volatility	Put Volatility
31		
32		25.69%
33	27.50%	27.03%
34	26.88%	26.65%
35	26.50%	26.47%
36	25.89%	26.06%
37	25.87%	25.84%
38	25.34%	25.53%
39	26.42%	
40	26.53%	



Cocoa Futures (11/14/94)
Asset Level: 967

Strike	Call Volatility	Put Volatility
850		25.01%
875		25.07%
900	25.95%	25.30%
925	25.91%	25.54%
950	25.90%	25.76%
975	25.39%	25.45%
1,000	25.97%	26.24%
1,025	26.16%	26.66%
1,050	26.02%	
1,075	26.05%	



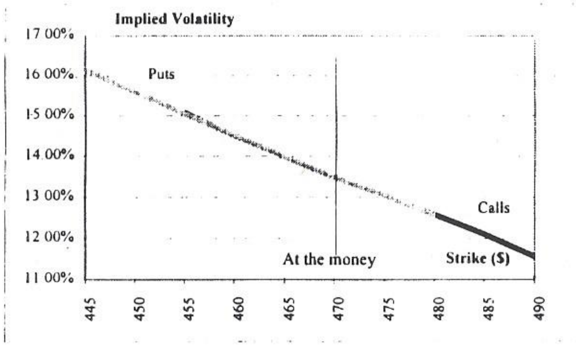
Source: Bloomberg

APPENDIX 1.

S&P500 March 1995 Futures Contract (11/16/94)

Asset Level: 470

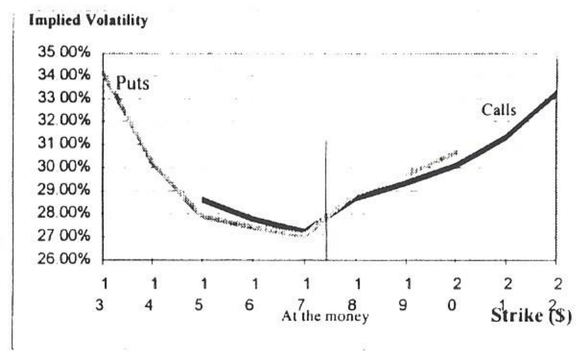
Strike	Call Volatility	Put Volatility
445		16.15%
450		15.61%
455	15.09%	15.06%
460	14.51%	14.52%
465	13.98%	13.98%
470	13.50%	13.50%
475	13.06%	13.06%
480	12.59%	12.58%
485	12.12%	
490	11.57%	



Crude Oil Futures (11/14/94)

Asset Level: 17.44

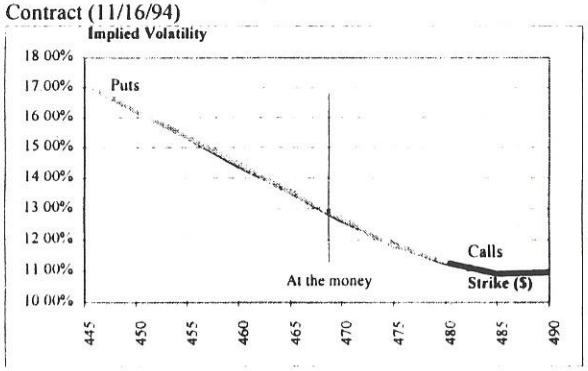
Strike	Call Volatility	Put Volatility
13		34.15%
14		30.23%
15	28.63%	27.91%
16	27.78%	27.42%
17	27.25%	27.08%
18	28.69%	28.91%
19	29.35%	29.71%
20	30.10%	30.69%
21	31.32%	
22	33.27%	



S&P500 December 1994 Futures Contract (11/16/94)

Asset Level: 467

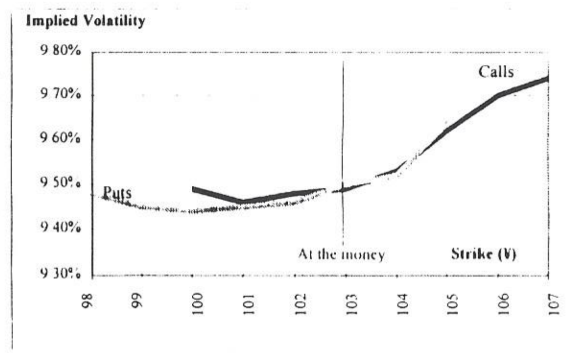
Strike	Call Volatility	Put Volatility
445		17.15%
450		16.24%
455	15.37%	15.37%
460	14.42%	14.46%
465	13.56%	13.54%
470	12.66%	12.69%
475	11.84%	11.81%
480	11.28%	11.31%
485	10.92%	
490	10.96%	



Yen Futures (11/14/94)

Asset Level: 102.91

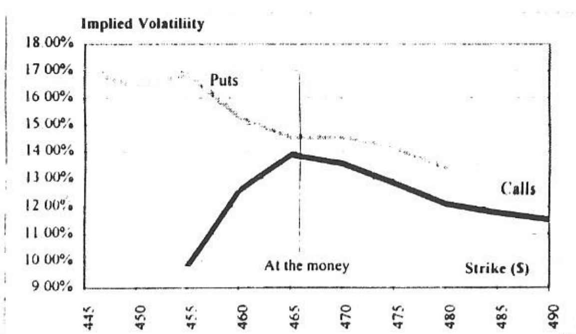
Strike	Call Volatility	Put Volatility
98		9.48%
99		9.45%
100	9.49%	9.44%
101	9.46%	9.45%
102	9.48%	9.46%
103	9.49%	9.50%
104	9.53%	9.52%
105	9.62%	9.64%
106	9.70%	
107	9.74%	



S&P500 spot (11/16/94)

Asset Level: 466

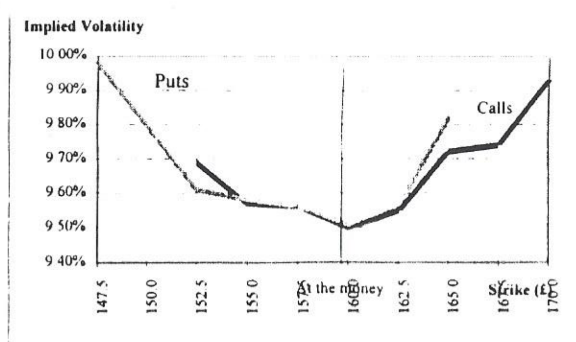
Strike	Call Volatility	Put Volatility
445		17.08%
450		16.44%
455	9.81%	16.92%
460	12.57%	15.33%
465	13.90%	14.56%
470	13.56%	14.59%
475	12.85%	14.16%
480	12.10%	13.41%
485	11.78%	



British Pound Futures (11/14/94)

Asset Level: 158.56

Strike	Call Volatility	Put Volatility
147.5		9.98%
150.0		9.80%
152.5	9.69%	9.61%
155.0	9.57%	9.58%
157.5	9.56%	9.56%
160.0	9.50%	9.51%
162.5	9.55%	9.57%
165.0	9.72%	9.82%
167.5	9.74%	
170.0	9.93%	



Source: Bloomberg

Appendix 2. Implied Binomial Tree Generator

```
#include <windows.h>
#include "foo.h"
#include "platform.h"
#include <stdlib.h>
#include <stdio.h>
#include <search.h>
#include <time.h>

#define TOL                     0.000000001
#define MAX(x,y)    ((x)>(y)?(x):(y))
#define EQDBL(x,y)  (-TOL < (x-y) && (x-y) < TOL)

#define   UP          0
#define DOWN     1
#define ROOT      -1

typedef struct {
   int rows;
   int columns;
   double vals[1];
} XlParms;

typedef struct enode *EndNodePtr;

typedef struct enode {
        double value;
        double prob;
} EndNode;

typedef struct tnode *TreePtr;

typedef struct tnode {
        double value;
        double optionValue;
        double prob;
        double u;
        double d;
        double mu;
        double sigma;
        int period;
        int type;
        int visited;
        int numVisits;

        TreePtr up;
        TreePtr parent;
        TreePtr down;
} TreeNode;
```

```
/* global variables and arrays */
static EndNode BT_FAR endNodes[MAX_END_NODES];
static XlEndNodeVals BT_FAR retEndNodes;
static XlImpVols BT_FAR retImpVols;
static XlResVals BT_FAR foo_result_vals;
static nodeIndex = 0;

TreePtr talloc(void)
{
        return (TreePtr) malloc(sizeof(TreeNode));
}


void tfree(TreePtr p)
{
        if (p == NULL)
                return;

        tfree(p->up);
        tfree(p->down);
        free(p);
}

void setEndNodes(TreePtr p, int lastPeriod)
{
        if (p->period != lastPeriod) {
                setEndNodes(p->up, lastPeriod);
                setEndNodes(p->down, lastPeriod);
        }
        else {
                endNodes[nodeIndex].value = p-
>value;
                endNodes[nodeIndex].prob = p->prob;
                nodeIndex++;
        }
}

void freeLattice(TreeNode **lattice, int n)
{
        int i;

        for (i=0; i < n+1; i++) {
                free(lattice[i]);
        }
        free(lattice);
}
```

```
void buildLattice(TreeNode ***treePtr, int n)
{
        int i,j,k;
        TreeNode **local_tree, *tree;


        local_tree = (TreeNode
**)malloc((n+1)*sizeof(TreeNode *));
        for (i=0; i < n+1; i++) {
                tree = (TreeNode
*)malloc((i+1)*sizeof(TreeNode));
                local_tree[i] = tree;
        }

        for (i=0; i < n; i++) {
                for (j=0; j < i+1; j++) {
                        local_tree[i][j].up =
&(local_tree[i+1][j]);
                        local_tree[i][j].down =
&(local_tree[i+1][j+1]);
                        local_tree[i][j].visited =
FALSE;
                        local_tree[i][j].numVisits =
0;
                }
        }

        for (i=0; i < n+1; i++) {
                local_tree[n][i].up = NULL;
                local_tree[n][i].down = NULL;
                local_tree[n][i].visited = FALSE;
                local_tree[n][i].numVisits = 0;

        }

        *treePtr = local_tree;
}

void initLattice(TreePtr p, TreePtr parent, int type,
        double value, double nodeProb, double prob,
double u, double d,
        int currPeriod, int lastPeriod)
{
        p->numVisits++;
        if (!p->visited) {
                p->value = value;
                p->optionValue = 0.0;
                p->type = -1;
```

```
		p->prob = nodeProb;
		p->u = 0.0;
		p->d = 0.0;
		p->mu = 0.0;
		p->sigma = 0.0;
		p->period = currPeriod;
		p->parent = NULL;
		p->visited = TRUE;
	}

	if (currPeriod != lastPeriod) {
		initLattice(p->up, p, UP, u*value,
prob*nodeProb, prob, u, d,
				currPeriod+1, lastPeriod);
		initLattice(p->down, p, DOWN,
d*value, (1.0-prob)*nodeProb, prob, u, d,
				currPeriod+1, lastPeriod);
	}

	return;
}

TreePtr addTree(TreePtr p, TreePtr parent, int type,
	double value, double prob, double u, double d,
	int currPeriod, int lastPeriod)
{
	p = talloc();

	p->value = value;
	p->optionValue = 0.0;
	p->type = type;
	switch (type) {
	case ROOT:
		p->prob = 1.0;
		break;
	case UP:
		p->prob = prob * parent->prob;
		break;
	case DOWN:
		p->prob = (1.0-prob) * parent->prob;
		break;
	default:
		;
	}
	p->period = currPeriod;
	p->parent = parent;
	p->up = p->down = NULL;

	if (currPeriod != lastPeriod) {
		p->up = addTree(p->up, p, UP,
u*value, prob, u, d,
				currPeriod+1, lastPeriod);
		p->down = addTree(p->down, p,
DOWN, d*value, prob, u, d,
				currPeriod+1, lastPeriod);
	}

	return p;
}

double valueTree(TreePtr p, double prob, double strike,
double r)
{
	double upValue, downValue;

	if (p->up == NULL) {
		p->optionValue = MAX(0, p->value -
strike);
		return p->optionValue;
	}

	upValue = valueTree(p->up, prob, strike, r);
	downValue = valueTree(p->down, prob, strike, r);

	p->optionValue = (prob*upValue + (1.0-
prob)*downValue)/r;

	return (p->optionValue);
}

void swap(EndNode v[], int i, int j)
{
	EndNode temp;

	temp.value = v[i].value;
	temp.prob = v[i].prob;

	v[i].value = v[j].value;
	v[i].prob = v[j].prob;

	v[j].value = temp.value;
	v[j].prob = temp.prob;
}

void quicksort(EndNode v[], int left, int right)
{
	int i, last;
	void swap(EndNode v[], int i, int j);

	if (left >= right)
		return;
	swap(v, left, (left + right)/2);
	last = left;
	for (i= left+1; i <= right; i++)
		if(v[i].value > v[left].value)
			swap(v, ++last, i);
	swap(v, left, last);
	quicksort(v, left, last-1);
	quicksort(v, last+1, right);
}

EndNode calcImpBinomLattice(TreePtr p, double r)
{
	EndNode retVal, upVal, downVal;

	if (p->up == NULL) {
		/* optionValue is used to hold the path
probabilities! */
		p->optionValue = p->prob/(double)p-
>numVisits;
		retVal.prob = p->optionValue;
		retVal.value = p->value;
		return retVal;
	}

	upVal = calcImpBinomLattice(p->up, r);
	downVal = calcImpBinomLattice(p->down, r);
	/* step ONE */
	p->optionValue = upVal.prob + downVal.prob;
	/* step TWO */
	p->prob = upVal.prob / p->optionValue;
	/* step THREE */
	p->value = (p->prob*upVal.value + (1.0 - p-
>prob)*downVal.value)/r;
	/* compute move sizes u, d */
	p->u = upVal.value/p->value;
	p->d = downVal.value/p->value;

	/* calc local volatility */
	p->mu = ( (1 - p->prob)*log(p->d) ) + ( p-
>prob*log(p->u) );
	p->sigma = sqrt(
```

```
                    ( (1 - p->prob)*pow(log(p->d) - p->mu,
2.0) ) +
                    ( p->prob*pow(log(p->u) - p->mu, 2.0)
)
                    );

          /* set return values */
          retVal.prob = p->optionValue;
          retVal.value = p->value;

          return (retVal);
}


void initImpBinomLattice(TreePtr p, int currPeriod, int
lastPeriod)
{
          p->numVisits++;
          if (!p->visited) {
                    p->value = 0.0;
                    p->optionValue = 0.0;
                    p->type = -1;
                    p->prob = 0.0;
                    p->u = 0.0;
                    p->d = 0.0;
                    p->mu = 0.0;
                    p->sigma = 0.0;
                    p->period = currPeriod;
                    p->parent = NULL;
                    p->visited = TRUE;
          }

          if (currPeriod != lastPeriod) {
                    initImpBinomLattice(p->up,
currPeriod+1, lastPeriod);
                    initImpBinomLattice(p->down,
currPeriod+1, lastPeriod);
          }

          return;
}

XlImpVols BT_FAR * BT_GLOBAL_FUNC
xl_calc_imp_binom_tree(long int periods, XlParms BT_FAR
*inputs,
 XlParms BT_FAR *inP, XlParms BT_FAR *inR)
{
          int n;
          double rhat, sumPR;
          long int status;
          int i, j, k, l;
          double t, h;
          int numCols;
          TreePtr root;
          TreeNode **lattice;
          int currPeriod, lastPeriod;


          for (i=0; i<MAX_IMP_VOLS; i++) {
                    retImpVols.vals[i] = 0.0;
          }


          n = periods;
          t = inputs->vals[0];
          h = t/(double)n;

          buildLattice(&lattice, n);
          root = lattice[0];
          currPeriod = 0;
          lastPeriod = n;
          initImpBinomLattice(root, currPeriod, lastPeriod);

          for (i=0; i < n+1; i++) {
                    lattice[n][i].prob = inP->vals[i];
                    lattice[n][i].value = inR->vals[i];
          }

          sumPR = 0.0;
          for (i=0; i < n+1; i++) {
                    sumPR += lattice[n][i].prob *
lattice[n][i].value;
          }
          rhat = pow(sumPR, 1.0/(double)n);

          (void) calcImpBinomLattice(root, rhat);


          retImpVols.rows = n+1;
          retImpVols.columns = (n+1)*2;


          k=0;
          numCols = (n+1)*2;
          for (i=0; i < n+1; i++) {
                    for (j=0; j < i+1; j++) {

          retImpVols.vals[(numCols*j)+k] =

          lattice[i][j].sigma*100.0/sqrt(h);
                    }
                    for (l=j; l < n+1; l++) {

          retImpVols.vals[(numCols*l)+k] = 0.0;
                    }
                    k++;

                    for (j=0; j < i+1; j++) {

          retImpVols.vals[(numCols*j)+k] =
lattice[i][j].value;
                    }
                    for (l=j; l < n+1; l++) {

          retImpVols.vals[(numCols*l)+k] = 0.0;
                    }
                    k++;
          }

          freeLattice(lattice, n);

          return (&retImpVols);
}



XlEndNodeVals BT_FAR * BT_GLOBAL_FUNC
xl_opt_binom_eur(long int resType, long int periods,
XlParms BT_FAR *inputs)
{
          double s, r, sigma, x, t;
          int n;
          double h, p, u, d, rhat;
          long int status;
          int i, j, k;
          long int l;
          int numCols;
          TreePtr root, parent;
          EndNode currEndNode;
          TreeNode **lattice;
          double value, prob, optionValue, strike;
          int currPeriod, lastPeriod, type;
```

```
	for (i=0; i<MAX_END_NODES; i++) {
		endNodes[i].value = 0.0;
		endNodes[i].prob = 0.0;
	}

	for (i=0; i<MAX_OPT_VALS; i++) {
		retEndNodes.vals[i] = 0.0;
	}

	nodeIndex = 0;
	n = periods;
	s = inputs->vals[0];
	r = inputs->vals[1];
	sigma = inputs->vals[2];
	x = inputs->vals[3];
	t = inputs->vals[4];

	h = t/(double)n;
	u = exp(sigma*sqrt(h));
	d = 1.0/u;
	rhat = pow(r, h);
	p = (rhat-d)/(u-d);

	root = NULL;
	value = s;
	prob = p;
	currPeriod = 0;
	lastPeriod = n;
	parent = NULL;
	type = ROOT;

	buildLattice(&lattice, n);
	root = lattice[0];
	initLattice(root, parent, type, value, 1.0, prob, u, d,
		currPeriod, lastPeriod);

	for (i=0; i < n+1; i++) {
		lattice[n][i].prob =
			lattice[n][i].prob *
(double)lattice[n][i].numVisits;
	}

	strike = x;
	optionValue = valueTree(root, prob, strike, rhat);

	if (resType == RES_NODES) {
		for (i=0; i < n+1; i++) {
					endNodes[nodeIndex].value
= lattice[n][i].value;
					endNodes[nodeIndex].prob
= lattice[n][i].prob;
					nodeIndex++;
				}

				k=0;
				numCols = 2;
				currEndNode.value =
endNodes[0].value;
				retEndNodes.vals[0] =
endNodes[0].value;
				retEndNodes.vals[1] =
endNodes[0].prob;
				j = 0;
				for (i=1; i < nodeIndex; i++) {
					if
(EQDBL(currEndNode.value, endNodes[i].value)) {

			retEndNodes.vals[(numCols*j)+0] =
endNodes[i].value;

			retEndNodes.vals[(numCols*j)+1] +=
endNodes[i].prob;
					}
					else {
						j++;

			currEndNode.value = endNodes[i].value;

			retEndNodes.vals[(numCols*j)+0] =
endNodes[i].value;

			retEndNodes.vals[(numCols*j)+1] =
endNodes[i].prob;
					}
				}


				retEndNodes.rows =
MAX_OPT_VALS / 2;
				retEndNodes.columns = numCols;
			}
			else {
				numCols = 2;
				retEndNodes.vals[0] = optionValue;
				retEndNodes.rows =
MAX_OPT_VALS / 2;
				retEndNodes.columns = numCols;
			}

			freeLattice(lattice, n);
			return (&retEndNodes);
}
```

Data on Volatility

r annualized	2.4600%
r compounded	2.4905%

Date	K for Implied Low Price	K for Implied High Price	Call Price at K Low	Call Price at K High	Implied Volatility Low	Implied Volatility High	r Annual Compounded	Time	Price of Futures (S)	High Implied X for B/S	High Implied B/S Call Price	Low Implied X for B/S	Low Implied B/S Call Price	Weighted Implied Volatility	Near-the-Money Implied Volatility	20 Day Historical Ave Volatility
15-Nov-94	19,000	19,500	N/A	415	N/A	0.07125	1.02491	0.36438	19,480	0.10166	415	N/A	N/A	0.07125	0.07125	0.09994
14-Nov-94	19,000	19,500	N/A	415	N/A	0.07389	1.02491	0.36712	19,455	0.08758	415	N/A	N/A	0.07389	0.07389	0.09954
11-Nov-94	19,000	19,500	N/A	380	N/A	0.07189	1.02491	0.36986	19,405	0.06373	380	N/A	N/A	0.07189	0.07189	0.09905
10-Nov-94	19,000	19,500	N/A	445	N/A	0.07825	1.02491	0.37260	19,465	0.09089	445	N/A	N/A	0.07825	0.07825	0.09995
9-Nov-94	19,500	20,000	485	375	0.08014	0.10326	1.02491	0.37534	19,555	(0.05945)	375	0.13114	485	0.10071	0.08014	0.12702
8-Nov-94	19,500	20,000	630	520	0.07739	0.10593	1.02491	0.37808	19,825	0.03598	520	0.25954	630	0.08738	0.10593	0.11720
7-Nov-94	19,500	20,000	620	510	0.07678	0.10495	1.02491	0.38082	19,810	0.03120	510	0.25412	620	0.08749	0.10495	0.11977
4-Nov-94	19,500	20,000	770	600	0.08505	0.10416	1.02491	0.38356	19,980	0.08904	600	0.30462	770	0.08582	0.10416	0.11608
1-Nov-94	19,500	20,000	740	630	0.08201	0.11263	1.02491	0.38904	19,950	0.07882	630	0.30055	740	0.08507	0.08201	0.11848
31-Oct-94	20,000	20,500	700	450	0.10850	0.10423	1.02491	0.39178	20,145	(0.01951)	450	0.14178	700	0.10547	0.10850	0.11093
28-Oct-94	20,000	20,500	700	455	0.11127	0.10756	1.02491	0.39452	20,115	(0.02571)	455	0.13088	700	0.10841	0.10756	0.11328
27-Oct-94	19,500	20,000	770	610	0.08403	0.10477	1.02491	0.39726	19,970	0.08742	610	0.30192	770	0.08527	0.10477	0.11116
26-Oct-94	19,500	20,000	830	650	0.09532	0.11173	1.02491	0.40000	19,975	0.08812	650	0.27442	830	0.09614	0.11173	0.11313
25-Oct-94	19,500	20,000	800	650	0.09464	0.11619	1.02491	0.40274	19,930	0.07457	650	0.25939	800	0.09765	0.11619	0.11321
24-Oct-94	19,500	20,000	780	650	0.09200	0.11729	1.02491	0.40548	19,915	0.07058	650	0.25934	780	0.09630	0.09200	0.13006
21-Oct-94	20,000	20,500	675	440	0.11343	0.11033	1.02491	0.40822	20,025	(0.04732)	440	0.10390	675	0.11048	0.11343	0.12812
20-Oct-94	20,000	20,500	700	480	0.11546	0.11593	1.02491	0.41096	20,045	(0.03492)	480	0.10951	700	0.11589	0.11546	0.12817
19-Oct-94	20,000	20,500	775	525	0.11766	0.11409	1.02491	0.41370	20,155	(0.00353)	525	0.14054	775	0.11520	0.11766	0.12586
18-Oct-94	20,000	20,500	715	475	0.11377	0.11081	1.02491	0.41644	20,080	(0.02775)	475	0.12092	715	0.11128	0.11377	0.13367
17-Oct-94	20,000	20,500	800	545	0.11529	0.11176	1.02491	0.41918	20,210	0.01255	545	0.15809	800	0.11324	0.11529	0.13936
14-Oct-94	20,000	20,500	800	550	0.11973	0.11648	1.02491	0.42192	20,165	0.00283	550	0.14273	800	0.11755	0.11973	0.14455
13-Oct-94	20,000	20,500	850	580	0.12986	0.12265	1.02491	0.42466	20,155	0.00450	580	0.13554	850	0.12489	0.12265	0.14477
12-Oct-94	20,000	20,500	890	630	0.12346	0.12017	1.02491	0.42740	20,275	0.03639	630	0.17041	890	0.12198	0.12017	0.14435
11-Oct-94	20,000	20,500	965	685	0.12440	0.11907	1.02491	0.43014	20,385	0.06661	685	0.19861	965	0.12318	0.12440	0.14203
7-Oct-94	20,000	20,500	750	520	0.11985	0.11882	1.02491	0.43288	20,055	(0.02368)	520	0.11320	750	0.11893	0.11882	0.12413
5-Oct-94	19,500	20,000	905	780	0.10786	0.13348	1.02491	0.43836	19,940	0.08242	780	0.23699	905	0.11093	0.13348	0.12132
4-Oct-94	19,500	20,000	845	750	0.10887	0.13857	1.02491	0.44110	19,830	0.05694	750	0.20212	845	0.11897	0.13857	0.14343
3-Oct-94	19,500	20,000	915	780	0.10942	0.13291	1.02491	0.44384	19,935	0.08186	780	0.23293	915	0.11248	0.13291	0.14286
30-Sep-94	19,500	20,000	845	710	0.10917	0.13088	1.02491	0.44658	19,820	0.05339	710	0.19878	845	0.11698	0.13088	0.14815
29-Sep-94	19,500	20,000	885	750	0.10868	0.13134	1.02491	0.44932	19,885	0.07010	750	0.21887	885	0.11389	0.13134	0.15018
28-Sep-94	19,500	20,000	815	710	0.10869	0.13528	1.02491	0.45205	19,765	0.04214	710	0.18284	815	0.12119	0.10869	0.14599
27-Sep-94	19,500	20,000	825	750	0.10144	0.13485	1.02491	0.45479	19,840	0.06054	750	0.21497	825	0.11213	0.10144	0.14681
26-Sep-94	19,500	20,000	N/A	710	10.7[illegible]/4[illegible]8	0.13714	1.02491	0.45753	19,735	0.03663	710	N/A	N/A	0.13714	0.13714	0.14281
22-Sep-94	19,500	20,000	N/A	685	N/A	0.13670	1.02491	0.46027	19,685	0.02505	685	N/A	N/A	0.13670	0.13670	0.14428
21-Sep-94	19,500	20,000	N/A	820	N/A	0.13305	1.02491	0.46301	19,970	0.09270	820	N/A	N/A	0.13305	0.13305	0.13500
19-Sep-94	20,000	20,500	N/A	595	N/A	0.13045	1.02491	0.46849	20,010	(0.01697)	595	N/A	N/A	0.13045	0.13045	0.13537
16-Sep-94	20,000	20,500	N/A	605	N/A	0.12963	1.02491	0.47123	20,035	(0.01091)	605	N/A	N/A	0.12963	0.12963	0.13730
14-Sep-94	20,000	20,500	N/A	635	N/A	0.13723	1.02491	0.47397	20,005	(0.01152)	635	N/A	N/A	0.13723	0.13723	0.13677
13-Sep-94	19,500	20,000	N/A	755	N/A	0.13751	1.02491	0.47671	19,780	0.05052	755	N/A	N/A	0.13751	0.13751	0.12396
12-Sep-94	19,500	20,000	N/A	875	N/A	0.13748	1.02491	0.47945	19,995	0.10027	875	N/A	N/A	0.13748	0.13748	0.13494
8-Sep-94	20,000	20,500	N/A	715	N/A	0.13510	1.02491	0.48493	20,170	0.02719	715	N/A	N/A	0.13510	0.13510	0.13215
7-Sep-94	20,000	20,500	N/A	745	N/A	0.13632	1.02491	0.48767	20,210	0.03733	745	N/A	N/A	0.13632	0.13632	0.13603
6-Sep-94	20,000	20,500	N/A	705	N/A	0.13901	1.02491	0.49041	20,095	0.01347	705	N/A	N/A	0.13901	0.13901	0.13271
5-Sep-94	20,000	20,500	N/A	715	N/A	0.13941	1.02491	0.49315	20,105	0.01646	715	N/A	N/A	0.13941	0.13941	0.13397
2-Sep-94	20,000	20,500	N/A	825	N/A	0.15260	1.02491	0.49589	20,170	0.03744	825	N/A	N/A	0.15260	0.15260	0.13602
1-Sep-94	20,000	20,500	N/A	860	N/A	0.15415	1.02491	0.49863	20,215	0.04751	860	N/A	N/A	0.15415	0.15415	0.13717
31-Aug-94	20,500	21,000	N/A	830	N/A	0.15217	1.02491	0.50137	20,625	0.03096	830	N/A	N/A	0.15217	0.15217	0.11917
30-Aug-94	20,500	21,000	N/A	790	N/A	0.14479	1.02491	0.50411	20,625	0.02767	790	N/A	N/A	0.14479	0.14479	0.11924
29-Aug-94	20,500	21,000	N/A	915	N/A	0.13897	1.02491	0.50685	20,920	0.08745	915	N/A	N/A	0.13897	0.13897	0.10558
26-Aug-94	20,500	21,000	N/A	900	N/A	0.14199	1.02491	0.50959	20,855	0.07470	900	N/A	N/A	0.14199	0.14199	0.10953