Variance and Volatility Swaps

An Introduction

Volatility and Quantitative Research

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ERRATUM: This report includes revised versions of a few variance and volatility swap formulas, superseding our publication of March 19, 2004.

We provide a basic understanding of the mechanics behind variance and volatility swaps, as well as different valuation and hedging methods. We illustrate why variance swaps offer a viable alternative to hedging a portfolio of options.

- Basic Intuition Behind Variance Swaps. This section is designed for those traders or investors that have a limited understanding of derivatives and options. We explain the basic intuition behind the payout pattern of a variance swap, as well as the mechanics of pricing and hedging a variance swap with a limited amount of mathematical formulas. Finally, we show how the payout of variance swap can be used to hedge a portfolio using two sample market regimes (a volatile and a non-volatile regime).

- Detailed Formulation of a Variance Swap. This section is designed for those traders and investors that have a good overall understanding of derivatives and the pricing of options. We provide the formulas generally used to calculate, mark, hedge, and price variance swaps. In addition, we show how to calculate a daily mark for a variance swap. Finally, we illustrate three methods used to price and replicate a variance swap using a portfolio of options:

    - *Variance Swap Pricing Method 1*—The original academic variance swap pricing method, whereby the log or natural log contract is equal to a forward plus the sum of a portfolio of put and call options.

    - *Variance Swap Pricing Method 2*—Lehman Brothers' replication method that determines an upper and lower bound for variance swaps.

    - *Variance Swap Pricing Method 3*—Lehman Brothers' volatility surface method and the Gram-Charlier Expansion.

- Volatility Swap Contracts and Convexity. A volatility swap can be thought of as a derivative of a variance swap. While changes in volatility should have little effect on the variance swap strategy because it captures the total variance, a volatility swap is fundamentally a different instrument, which may require some re-hedging. Essentially, the payout of a volatility swap requires an adjustment for the convexity of a variance swap. This convexity can be estimated and used to adjust the fair strike of the volatility swap. This complication makes variance swaps the preferred quoted swap contract by most market makers.

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Basic Intuition Behind Variance and Volatility Swaps

ERRATUM: This report includes revised versions of a few variance and volatility swap formulas, superseding our publication of March 19, 2004.

Introduction to the Mechanics of a Variance Swap

The variance swap allows an investor to directly implement a view on the direction of future realized variance. Like an equity swap, in which two parties exchange cash flows based on the return of a specified equity, the variance swap is characterized by the exchange of cash flows tied to the "performance" of variance (volatility squared).

In a variance swap, an investor agrees to receive or pay the realized variance (typically measured as the annualized standard deviation squared) of an equity index or single stocks relative to an agreed-upon "strike level."

In a variance swap, an investor agrees to receive or pay the realized variance of an equity index or single stocks relative to an agreed-upon "strike level." Realized variance is typically measured as the annualized standard deviation squared of the daily natural log returns of the stock or index. Whereas, an equity swap is based on a specified number of shares, a variance swap is expressed in terms of the dollar value of each variance point (the number of "units"). The variance swap resembles a forward contract in that there is no initial exchange of cash flows between the two parties, only an agreement upon the strike level. In Figure 1, we show the exchange of cash flows at expiration for a short variance swap (where the investor sells the variance swap to Lehman Brothers).

Figure 1: Exchange of Cash Flows at Expiration for a Short Variance Swap

Source: Lehman Brothers

The payout of a variance swap is expressed below as a function of realized variance and fair variance (K_{Fair}), implied by the option market.

$$Payout = (\sigma_R^2 - K_{Fair}) * 100 * Units$$

$K_{Fair} = \sigma^2_I$ or the implied variance (implied volatility squared) represents the fair variance of the tradable options in the market, and σ^2_R is the realized variance of the underlying asset over that period. In a short variance swap, if σ^2_R is greater than strike level (K_{Fair}), then the investor pays Lehman Brothers, according to Figure 1. Conversely, if σ^2_R is less than strike level (K_{Fair}), then Lehman Brothers pays the investor, as shown in Figure 1.

We have presented in prior research the mechanics of a volatility swap, which differs from a variance swap in that the former is expressed in terms of the standard deviation, while the latter is the volatility squared.[1]

To illustrate the mechanics of a variance swap[2], suppose an investor has the following view:

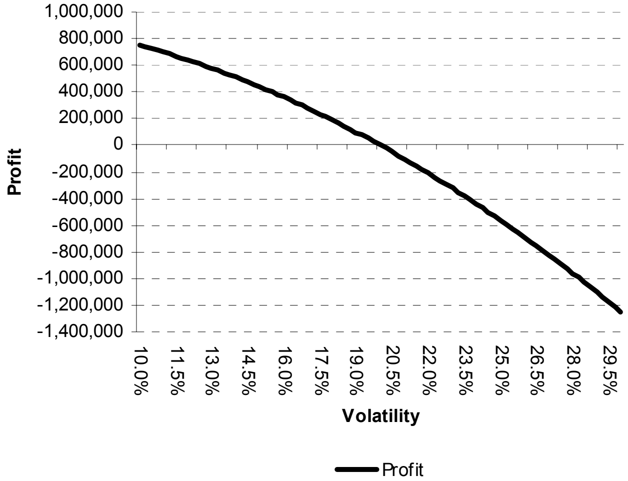
- Market conditions will be relatively stable over the next six months and realized variance (volatility squared) of the S&P 500 Index will be $(20\%)^2$ or lower. So the investor sells a variance swap.
- Assume that the investor can enter into a short variance swap with a strike level of 0.04 (or $20\%^2$) based on some implied volatility surface level for the S&P 500.
- The contract specifies that the investor will receive the difference between S&P 500 realized variance and the strike level, to the extent that realized variance is below the 0.04 variance strike level.
- If the investor's view is correct and realized variance drops below 0.04 to, say, 0.0225, he will earn a profit of 1.75 units [-(0.0225 - 0.04) * 100] scaled by the specified number of dollars per variance point units.
- If the transaction were on 250,000 units, the investor would profit by US$437,500.
- If, however, the realized variance rises to 0.0625 ($25\%^2$), the investor would pay out 2.25 units [–(0.0625 - 0.04) * 100] or lose US$562,500.

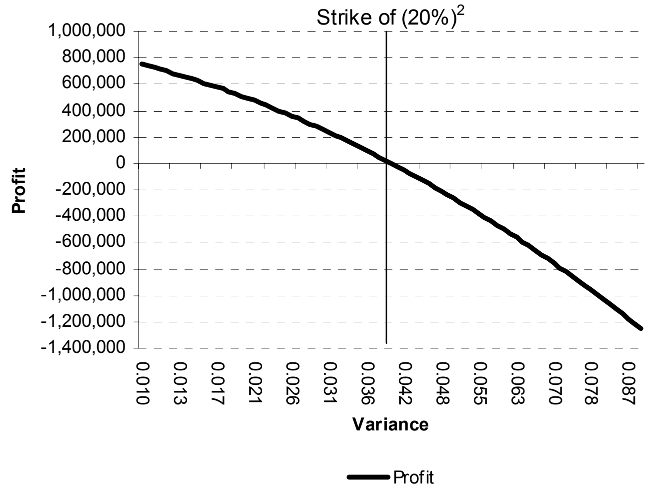
In Figure 2 below, we illustrate the payoff of our theoretical variance swap for both variance and the equivalent volatility moves.

[1] "Volatility Swaps," Lehman Brothers, April 1999.

[2] I would like to acknowledge the assistance of three individuals from our Quantitative Analytics Group at Lehman Brothers: Abdelkerim Karim in our London office, and Grigor Peradze and Peter B. Lee in our New York office.

Figure 2: Payoff of Short Variance Swap with Strike of $(20\%)^2$ on 250,000 Units per Variance and Volatility Points





Source: Lehman Brothers

We display indicative terms and conditions of such a contract in Figure 3.[3]

Figure 3: Short Variance Swap on 250,000 Units – Indicative Term Sheet

Party A:	Lehman Brothers
Party B:	Investor
Units:	250,000
Trade Date:	TBD
Effective Date:	Three Business Days after the Valuation Date
Underlying Index:	S&P 500 Index
Valuation Date:	6 Months from Trade Date
Strike Level:	$(20.0\%)^2$
Payoff at Expiration:	Units x (Variance – Strike Level) x 100 If such amount is a positive number then Party B shall pay such amount to Party A. If such amount is a negative number then Party B shall receive the absolute value of such amount from Party A.
Rate of Return:	On Expiration Date, $Units \cdot \left[\sigma_R^2 - \sigma_K^2\right] \cdot 100$ where, σ_R^2 is the Realized Variance as defined below σ_K^2 is the Fair Variance Strike
Volatility:	Realized Variance is defined as $\frac{252}{n-1} * \sum_{i=1}^{n} \left[R_i\right]^2$ where, $R_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$ Realized variance is calculated with a zero mean. S_i is the official closing price of the Underlying on the i[th] Exchange Business Day of the life of the Transaction (i = 1 to n) n is 126, the number of business day returns starting on the Trade Date and ending on the Expiration Date If a Market Disruption Event occurs on any Exchange Business Day, then for purposes of calculating R_i, the value of S_i shall equal the value of S_{i-1}.

Source: Lehman Brothers

Realized variance and volatility for variance and volatility swaps are generally calculated with a zero mean since the replication process is essentially a zero mean process. We

[3] Some contracts are expressed by normalizing the payout return by 2 * σ_K. This would scale the payout up or down by a constant factor and represent the percentage change from the strike level, as opposed to an absolute level. For our illustration, we decided not to show this type of variance swap return.

will examine this concept later in this report. However, variance swaps can be quoted with a mean as long as the contract reflects how it will be calculated.

Trading options, a complicated and demanding exercise, sometimes leads to unpredictable economic outcomes. The variance swap may be an attractive alternative for investors who have traditionally gained variance exposure through the option markets.

As described above, the variance swap allows an investor to directly express a view on future realized variance. In addition, academic research from journals widely documents how to calculate and develop variance swaps.[4] The variance swap may be an attractive alternative for investors who have traditionally gained volatility exposure through the option markets. In the following sections, we review how investors use options as a source of long or short volatility exposure and illustrate the inherent limitations of option-based volatility strategies. We show that trading options, although widely used to source or supply volatility, is a complicated and demanding exercise that sometimes leads to unpredictable economic outcomes. The variance swap is recommended as a method for effectively "outsourcing" the dynamic hedging process required in volatility-driven strategies that rely on options.

Brief Overview of Volatility Trading using a Single Option[5]

Volatility trading is focused on buying or selling options and then seeking to offset the option payoff through a dynamic hedging strategy.

Volatility trading is focused on buying or selling options and then seeking to offset the option payoffs through a dynamic hedging strategy. The volatility trader is not generally concerned with market direction, but rather with the price of an option relative to the cost of hedging it. If, for example, the volatility trader sells a call option, his goal is to create a portfolio that has the same payoff as the short call, but at a lower cost. If the position is hedged, the trader has not taken a view on whether the stock will rise or fall, but that the option was potentially overvalued.

Brief Review of the Importance of the Black/Scholes Model

By far, the most common method for valuing and hedging options is the Black/Scholes framework (or some close variant of it).

By far, the most common method for valuing and hedging options is the Black/Scholes[6] framework (or some close variant of it). Black/Scholes is a "no-arbitrage" model that determines the fair value of an option by identifying the portfolio (consisting of long and short positions in stocks and bonds) that, given some simplifying assumptions, perfectly reproduces the option payoff. The Black/Scholes option pricing formula is summarized by six variables, all of which affect the economics of constructing and maintaining the portfolio that reproduces the option payoff. These variables are the stock price, strike price, time to expiration, interest rates, dividends, and volatility (standard deviation). With the exception of the volatility input, each of these variables is hedgeable, or in the case of future dividends, dividends can be effectively made certain through contract specifications.

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[4] Chriss, Neil; Morokoff, William. "Market Risk for Volatility and Variance Swaps", *Risk Magazine*,1999. Lehman Brothers, "Volatility Swaps", April 12, 1999. Demeterfi, Kresimir; Derman, Emanuel; Kamal, Michael; and Zou, Joseph, "A Guide to Volatility and Variance Swaps", *The Journal of Derivatives*, Summer 1999.

[5] This and the following section are provided for readers who may not be familiar with volatility trading. Those experienced with the concepts of volatility trading may want to skip this section. For a more complete discussion see Hull, John, *Options, Futures and Other Derivatives*, 5th Edition, Prentice Hall, 1999.

[6] Black, F., and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81 (May-June 1973), 637-59.

Option Market Efficiency

If financial markets are efficient and there are no opportunities for arbitrage, the price of an option should equal the aggregate cost of reproducing its payoff. For example, in a perfectly efficient financial market, a call option that costs $10 will also cost $10 to reproduce via the Black/Scholes hedging strategy. This suggests that if an investor wrote this call and received $10 in premium, he would expend this same $10 in hedging (offsetting) the short call option position. If, on the other hand, the investor purchased the call for $10 and then sought to offset this payoff via hedging, the investor would gain $10. In each case, the investor has neither made nor lost money. The key point is that there are hedging gains to be derived from long option positions and hedging costs that arise from short option positions. We will see that these costs and gains are closely tied to the volatility component of an option for the index or equity.

Appendix A contains a review of option-based concepts including delta, gamma, theta, rho, and vega effects on a long European call position. If the reader is not familiar with these concepts, then we recommend first reading Appendix A for a further explanation and graphical examples of the concepts discussed briefly below.

The volatility trader's goal is to buy those options for which the premium paid is less than the benefit that can be potentially realized through delta hedging, and to sell those options for which the premium collected exceeds the cost of hedging the option.

- The Delta Hedging Strategy and the Benefits of Volatility: The Black/Scholes formula offers a "recipe" for pricing a put or call option by providing the elements of the hedge portfolio—the long or short stock position and the amount of money that should be borrowed or lent. The stock hedge, also known as the "delta," reflects the sensitivity of the option price to a change in the stock price. The delta measures the long or short stock position required to hedge the equity exposure in the option.[7] The long option position combined with the stock delta hedge allows the trader to profit from movements in the stock price in either direction. In fact, each time the stock price moves up or down, the long option holder will benefit, as the delta hedge becomes mismatched and the opportunity arises to rebalance the hedge. If the trader bought the option, then the greater the swing in the stock price, the greater the benefit to the trader and, potentially, the greater the profit. Therefore, traders may be willing to pay a higher premium for options on volatile stocks, provided the option premium is not prohibitive to the trader making a profit. The volatility trader's goal is to buy those options for which the premium paid is less than the benefit than can be potentially realized through delta hedging, and to sell those options for which the premium collected exceeds the cost of hedging the option.

[7] For example, two parties can contract to a "Total Return Option," which strips out dividend risk. Mathematically, the delta is the first derivative of the option price with respect to a change in the stock price.

Gamma indicates the extent to which the delta hedge becomes mismatched and for which there exists an economic benefit from large movements in the stock price. Theta indicates the economic consequence of these large movements not occurring (and the cost associated with waiting for a large movement).

- Option Gamma and the Gamma/Theta Trade-Off: The degree to which the delta hedge becomes mismatched when the stock price moves is reflected in the option's "gamma." Gamma measures the change in delta for a given change in the underlying asset or the acceleration (second momentum) of the option price movement. In general, gamma tends to be highest for options for which the strike price and underlying asset price are close together (at-the-money or ATM). Deep in-the-money or out-of-the-money options tend to have little gamma. Closely related to gamma is theta, which measures the extent to which an option's premium "decays" with the passage of time assuming that the stock price does not change. One can think of gamma and theta in terms of a trade-off: As described above, gamma indicates the extent to which the delta hedge becomes mismatched and for which there exists an economic benefit from large movements in the stock price. Theta indicates the economic consequence of these large movements not occurring (and the cost associated with waiting for a large movement). This concept is also important in the Gamma and Theta trade-off of a log contract (natural log contract) for a variance swap, which we will discuss further below.

- Option Rho, the Interest Rate Risk: Rho refers to the interest rate risk (sensitivity) in an option. This interest rate risk usually can be hedged out by buying or selling $US LIBOR contracts.

A European call option is more valuable at a higher implied volatility than at a lower implied volatility.

- Implied Volatility, Realized Volatility, and Vega: As noted above, an option price is affected by six variables, one of which is the expected volatility of the underlying equity or index. The term "implied volatility" refers to the volatility input that equates the option's observed market price with its theoretical (i.e., Black/Scholes) value given the five other option pricing inputs (stock price, strike price, time to expiration, interest rates, and dividend yield). A European call option is more valuable at a higher implied volatility than at a lower implied volatility. The change in option price due to a 1% change in implied volatility is measured by "vega."

The Limitations of Delta-Neutral Strategies for a Single Option

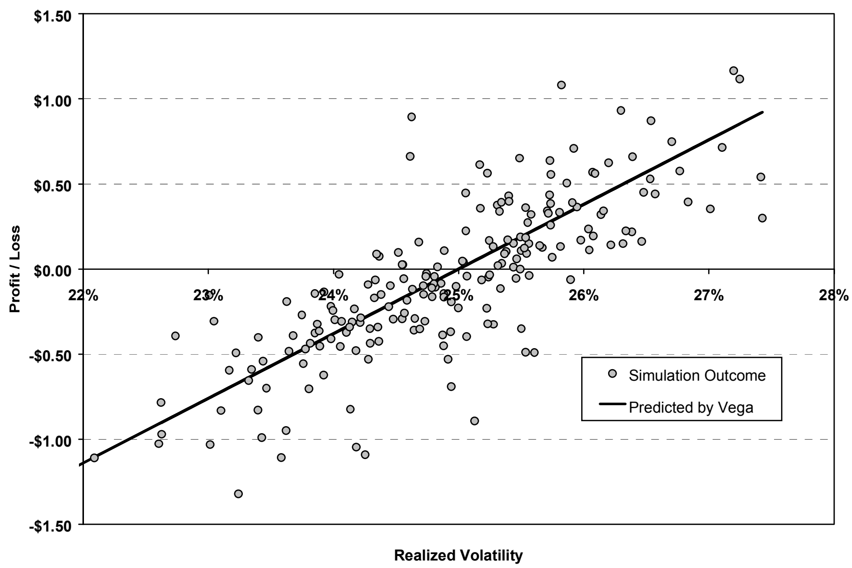
Delta-neutral option strategies have limitations in allowing an investor to capture the value of implied volatility in an option (e.g., an option's sensitivity to volatility changes due to movements in the stock price and the passage of time).

Although widely used, delta-neutral option strategies have limitations in allowing an investor to capture the value of implied volatility in an option. Among these limitations is an option's sensitivity to volatility changes due to movements in the stock price and the passage of time (among other factors). For example, a volatility trader may buy a one-year option on the S&P 500 Index at an implied volatility of 25%, with the belief that the aggregate benefit of hedging the option will exceed the premium paid for the option. Suppose that the trader's view is correct and that realized volatility is 27%. What is the profitability of the trade? As a rule of thumb, one can estimate the anticipated profit on the option as the spread between implied versus realized volatility (2% in this example) multiplied by the amount of vega in the option. However, as the simulation results[8] in

[8] The chart shows the results of a 200 trial simulation in which we draw from a normal distribution with a volatility of 25% and implement a delta-neutral strategy against a long call option. The pricing assumptions for the call option include an implied volatility of 25%, a dividend yield of 1.30%, and an interest rate of 5.38%. The call option is assumed to be re-hedged once daily, at the close of business.

Figure 4 show, although the profit is related to vega, there are cases in which significant deviations occur between the expected and realized profit.

Figure 4: Predicted vs. Actual P/L on 1-Yr, $100 Strike Long European Call



Source: Lehman Brothers

The profitability outcome is entirely path dependent—not only does the realized volatility (or variance) of the underlying equity matter (in relation to the implied volatility), but so too does the timing of this volatility (or variance).

The error in Figure 4 is introduced by the fact that an option's vega and gamma are sensitive to, among other factors, the positioning of the spot price relative to the strike price. The gamma and vega levels of the option that prevail, on average, during the life of the option have a significant impact on the overall profitability of a delta-neutral option trade. The key point is that the profitability outcome is entirely path dependent—not only does the realized volatility (or variance) of the underlying equity matter (in relation to the implied volatility), but so too does the timing of this volatility (or variance). For example, a short option position in which most of the volatility occurs close to expiration, and when the stock price is close to the strike price, may have a significantly different profit/loss outcome than a scenario in which the realized volatility is identical, but occurs earlier in the option's life, and when the stock and strike price are significantly apart. Because the gamma and vega of an option evolve uniquely during its life, the delta hedging strategy is inherently imperfect and the economics are subject to error. In large part, the variance (volatility) swap addresses this key shortcoming of the delta hedging strategy.

Basic Intuition Behind How a Variance Swap Works

In the prior section, we dealt with a single option. Here, we expand beyond the concept of just one option to create a portfolio of calls and puts, to gain a variance exposure while removing dependency on the stock price or index level movements. We provide the basic concepts behind a variance swap that are explained in the "Detailed Formulation of a Variance Swap" section of this report. There are four important concepts that illustrate how a variance swap works and how they can be replicated:

1. Gaining a Constant Exposure to Moves in Volatility Using a Portfolio of Options

A correctly weighted portfolio of just a few options will not provide full exposure to moderate movements in variance when downside moves of the index or stocks occur.

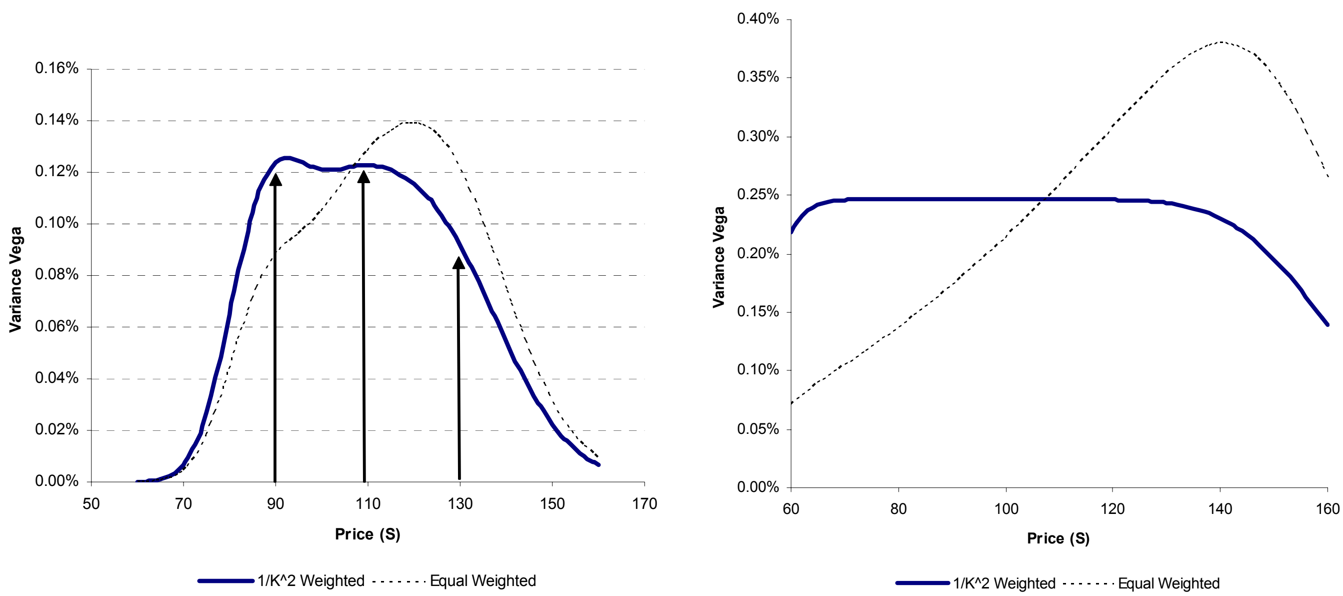
We know that one can buy straddles on a series of options to gain exposure to variance, but that will not replicate the payout of a variance swap. A correctly weighted portfolio of just a few options will not provide full exposure to moderate movements in variance when downside moves of the index or stocks occur. As the maturity of the variance swap increases beyond a few months, using a few straddles on ATM and slightly OTM calls and puts will not allow an investor to fully participate on the variance increases or decreases in the market. In Figure 5 below,[9] we illustrate variance sensitivity (know as the variance vega) using three options with three different strikes: 90, 110, and 130, and compare this to using 11 strikes from 60 to 130 (in increments of 10). If we compare the solid lines in the first graph using only three options to the solid line in the second graph using 11 options, the variance sensitivity using 11 options becomes flatter across a wider price range (weighted by $1/K^2$). In the case of 11 strikes, we have a constant participation in variance sensitivity at approximately 0.25% from a price level of 68 to almost 129, while in the case of only using three options, we have a constant participation in variance sensitivity at only 0.13% from a price level of only 90 to approximately 110. Therefore, if the underlying moves down by more than 10 for the case of using only three options, we would lose participation at a constant variance sensitivity below the price of 90. The difference between using three and 11 strikes amplifies as the maturity of the variance swap increases.

For example, if volatility moves from 20% to 30% as the market drops sharply, 11 options will provide a volatility participation approximately from 20% to 29%, while three options may only provide a volatility participation from approximately 20% to 25%. In the next section, we will compare the equal weighted option case to the case in which we weight the options by the $1/K^2$, where K is the strike of the option.

[9] We make the following assumptions to calculate our variance sensitivity exposure (vega): implied volatility is constant at 10% for all strikes, time to expiration is one year, strikes range from 60 to 160, the current spot level is 100, and interest rates are 2%.

Figure 5: Variance Exposure using Three European Options with Strikes of 90, 110, and 130, and using 11 European Options with Strikes of 60 to 130 for Both the Equal Weighted Case and $1/K^2$ Weighted Cases

Variance Sensitivity Using Three Options | Variance Sensitivity Using 11 Options



Source: Lehman Brothers

2. A Variance Swap Is Similar to a Log or Natural Log Contract

A variance swap can be represented by a log contract (natural log contract), but log contracts exist only in theory.

A variance swap can be represented by what is known as a log contract (natural log contract), but log contracts exist only in theory. The log or natural log contract is used to provide a constant exposure to variance vega or sensitivity independent of the underlying price. We can attempt to replicate a natural log contract using an infinite number of strikes across both calls and puts. However, in the options market, an infinite number of calls and puts do not exist; therefore, we are limited to strikes that are both listed and actually traded. In the following section of this report, "Detailed Formulation of a Variance Swap," we derive a relationship where the puts and calls are not equally weighted by the same number of option contracts but weighted by $1/K^2$ contracts.

Essentially we are vega weighting each option by $1/K^2$ (to determine the number of contracts to buy/sell), which will provide us the constant variance sensitivity required so that at every price within the strike range, we gain the same exposure to variance whether it increases or decreases. The OTM puts are weighted by more contracts because vega is directly proportional to the stock or index price (S) multiplied by the square root of time ($\sqrt{T}$), according to equations A12 and A13 in Appendix A. From these equations, we could observe that a put option with an $80 strike at a spot level of $80 would have a lower vega than a call option with a $120 strike at a spot level of $120 (all other inputs being equal). Therefore, we would need more OTM puts than OTM calls to achieve a constant level of vega across a range of strikes. This also falls in line with our intuition that if volatility generally increases when the market sells off, then the investor would want to have increased exposure to variance or volatility to the downside represented by more contracts on the further OTM puts. Conversely, if the market moves higher, then the investor would require a smaller number of contracts on the OTM calls. A variance swap can be represented as a log contract (natural log contract) payout expressed as follows:

Payout Log Contract = Forward Contract + Sum All Put Options + Sum All Call Options

The log contract (natural log contract) basically has three components. The first is a forward contract equal to the difference between the current and the future spot levels over some maturity (time = T) normalized by the future spot level at time T. The second component is a weighted portfolio of puts across different strikes with a maturity T, ranging from the at-the-money (ATM) to the out-of-the-money (OTM) put strikes. The third component is a weighted portfolio of calls across different strikes with maturity T, ranging from the ATM to the OTM call strikes. Once the option portfolio is selected, the option hedge against a variance swap would be static until maturity, but we would still have to delta hedge the option positions, potentially adding to our costs.

The concept of a natural log contract has two important implications for a variance swap:

- It replicates the payout pattern that we desire in a variance swap.
- The payout pattern is independent of moves in the underlying stock price or index level, providing exposure only to the change in the level of volatility.

If we weight the option contracts inversely to $1/K^2$ and calculate the variance sensitivity, we find that within the strike range, variance sensitivity becomes more independent of the price level and changes at a constant rate.

As shown in Figure 5 above, we compare the equal weighted option contract case to the case of weighting the option contracts by $1/K^2$ (where K is the strike of the option) for the two scenarios, using three and 11 options. For the dashed lines in Figure 5, we take a set of three and 11 options and equally weight them by the same number of contracts. The case of using an equal number of contracts across all strikes creates an imperfect exposure to variance sensitivity to the downside, because it overweights the calls to the upside and underweights the puts to the downside, as shown by the dashed lines in Figure 5. For the solid lines in Figure 5, we weight the option contracts inversely to $1/K^2$ and calculate the variance sensitivity, and find that within the strike range, the variance sensitivity becomes more independent of the price level and changes at a constant rate. When sharp moves to the downside occur, the volatility sensitivity will increase at a constant rate. Using an equal weighted number of contracts, the investor would have less of an increase in variance when the market sells off, which is the exact time when a constant or increasing variance sensitivity is required.

As shown in Figure 6, we essentially replicate the log or natural log payout using a discrete set of call and put options weighted by $1/K^2$.

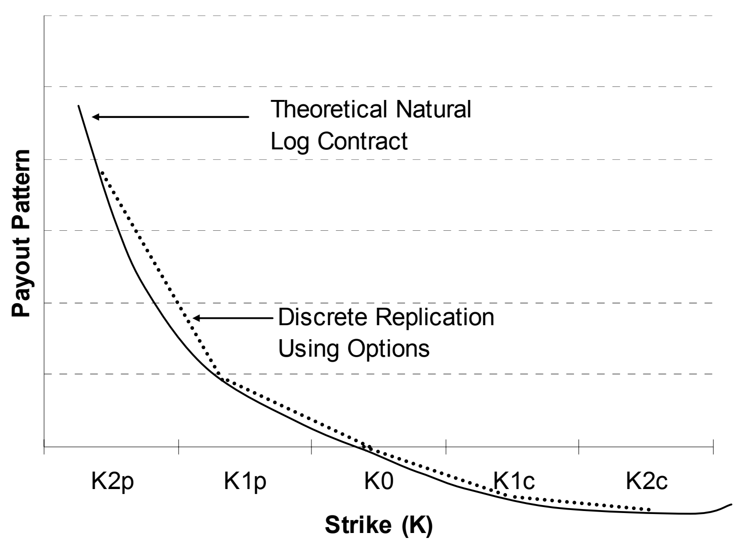
As shown in Figure 6 below, we essentially replicate the log or natural log payout using a discrete set of call and put options weighted by $1/K^2$. The calculations of the contract weighting scheme are explained in more detail in the "Detailed Formulation of a Variance Swap" section of this report. Here, we present only the hypothetical payout of a perfect natural log contract. The forward spot price marks the demarcation line between where puts end and calls begin. The weight of the options will depend on the slope of the line between the discrete strikes available for replication. Below, we separate all the puts options from 0 to F and all the call options from F to ∞.

$$K_o = F < K_{1c} < K_{2c} < K_{3c} < .....$$ for the ATM to OTM calls

$$K_o = F > K_{1p} > K_{2p} > K_{3p} > .....$$ for the ATM to OTM puts

where K_{1c} is the first call strike and K_{1p} is the first put strike.

Figure 6: Illustration of the Replication of a Natural Log Contract using Options (Time = $T_{Expiration}$)



Source: Lehman Brothers

3. Problems with Replicating a Variance Swap with a Portfolio of Options

A variance swap has a form of a log contract (natural log contract) that can be replicated with options where the effects of varying the underlying price of the stock price or level of the index are removed. The two biggest problems with this replication are the following:

There is a limited range of strikes available to replicate the log or natural log payoff.

- There is a limited range of strikes available to replicate the log or natural log payoff. Log contracts (natural log contracts) do not trade, and using a limited number of strikes may fail to capture the true variance. In fact, the discrete replication of a continuous payout using options can results in points above, below, or on the log (natural log) contract curve, as shown in Figure 6. To adjust for the lack of strikes that are available, traders may increase the number of contracts on the last few OTM puts and calls that are being used, to ensure they get the proper volatility exposure for large downside moves.

Stock prices and index levels can change in large, discrete jumps.

- Stock prices and index levels can change in large, discrete jumps, which often occurs during incidents such as the market crash of 1987, the Persian Gulf war in 1990, the Mexican devaluation in 1994, the Asian debt crisis of 1998, the end of the Tech bubble run-up in 2000, the terrorist attack in 2001, the U.S. accounting crisis of 2002, and the Iraq war in 2003. This scenario can also occur for single stocks around stock-specific events. Again, buying more OTM puts and calls can assist in hedging out this effect.

Both of these problems are dealt with in more detail in the "Detailed Formulation of a Variance Swap" section of this report. The important point is to be aware that we are creating a discrete payout pattern for a continuous function.

4. The Gamma versus Theta Trade-Off

Again, we are showing negative theta (till expiration) is offset by the potential benefit of positive gamma (non-linear payout).

We discussed the trade-off of Gamma and Theta for a single option. As described in Appendix A (for a single option) and in the prior section, gamma indicates the extent to which the delta hedge becomes mismatched and for which there exists an economic benefit from large movements in the stock price. Again, we are showing negative theta (till expiration) is offset by the potential benefit of positive gamma (non-linear payout). Similarly, in a long variance swap, the investor wants a higher future variance above the strike level to benefit from gamma. In addition, if the market moves down and since we have purchased more contracts on OTM puts, our gamma exposure will likely increase. However, if the market is not volatile, then the investor will not benefit from gamma; instead, volatility will be lower than the initial strike value, which is similar to paying your theta in a long variance swap contract. Later in this report, we will derive the relationship between the gamma and theta of a log contract (natural log contract).

Three Pricing Models for Calculating Fair Variance

Here, we review the advantages and disadvantages of the three pricing model for a variance swap in this document.

Academic Model

The disadvantage of the Academic Model is that it may overestimate the fair variance (K_{Fair}), because the discrete replication will result in a point on the log contract (natural log contract) curve that could be at or above the log contract (natural log contract) curve.

The Academic Model, as outlined above, consists of a forward contact plus a portfolio of puts and calls. The academic model can be expanded to include bid and offer option prices to get a bid and offer on a variance swap strike. The disadvantage of the Academic Model is that it may overestimate or underestimate the fair variance (K_{Fair}), because the discrete replication will result in a points that could be at, above or below the log contract (natural log contract) curve.

Lehman Brothers Model with Upper and Lower Bound

The second model is the Lehman Brothers Upper and Lower Bound Model. The advantages to the first model are the following:

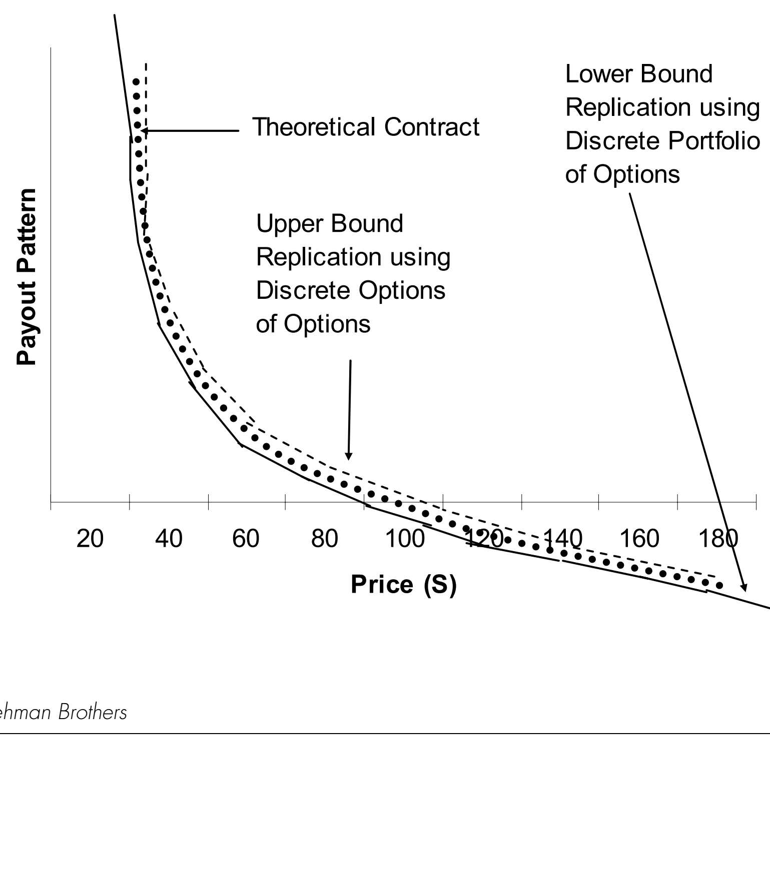
For our Upper and Lower Bound Model, the upper bound replication is always at or above the theoretical contract, and the lower bound is always at or below the theoretical log contract (natural log contract).

- It calculates an upper and lower bound of variance by expanding on the replication methodology of the Academic Model. As shown in Figure 7 below, the upper bound replication is always at or above the theoretical contract value, and the lower bound is always at or below the theoretical log contract (natural log contract). This methodology establishes a lower bound that never gives a value greater than the theoretical contract value, unlike the academic model, which can result in points above, below, or on the log contract (natural log contract) curve. In addition, the upper bound never provides a value below the log (natural log contract) curve; therefore, the variance value from the upper bound model tends to be near or slightly higher than the academic model.

- It can be expanded to create a variance swap bid price, using the lower bound method with the option bid prices, and an offer price, using the upper bound method with option offer prices.

- This method establishes a unique value associated with the points that are near or at the strikes for the lower and upper bound.

Figure 7: Lehman Brothers Upper and Lower Bound Replication Methodology



Source: Lehman Brothers

Lehman Brothers Volatility Surface Model and the Gram-Charlier Expansion

The Lehman Brothers Volatility Surface Model, combined with the Gram-Charlier expansion, can accurately represent the volatility surface.

The third model is obtained by solving three equations and three unknowns, using a more complicated concept of the volatility surface and the Gram-Charlier expansion to solve for the overall volatility, skew, and smile. We can observe and calculate the skew, kurtosis, and smile from any volatility surface across any maturity of options (after solving for ξ, η, $\sigma_{ATMsolve}$). We then can use three equations and three unknowns to directly solve for σ^2 (variance) as well as S (skew) and K (kurtosis). We can use σ as the fair variance (K_{Fair}) calculation. In addition, we can also calculate the fair skew and fair kurtosis of the volatility surface. The main advantage is that the Lehman Volatility Surface Model, combined with the Gram-Charlier expansion, can accurately represent the volatility surface. Finally, we can create a bid and an offer variance using the bid and offer of option prices. However, the main disadvantage is that many of the options used to create the volatility surface may have very low liquidity and may not be tradable.

The advantages of the volatility surface and Gram-Charlier expansion method, compared to the Lehman Brothers method and the academic method, are as follows:

- This method provides a comparison actual fair variance (implied by the volatility surface in the market) to any closed form or estimated replication of the variance log contract (natural log contract).
- Although individual option weights are not calculated in this method, one can use this volatility surface model to calculate the option contract weights, to better reflect the volatility surface in the marketplace.
- With this method, we can easily use many more option strikes (deep OTM strikes for calls and puts). We avoid the effect of truncating the strikes in calculating the fair variance across a limited set of options (which is found in the academic model and the Lehman Brothers upper and lower bound model).
- This method reflects the fair variance of the volatility surface. This comparison is valuable in shorter-term maturities that may overestimate implied volatility, based on the tradable strikes, rather than what is implied by the volatility surface.

Comparison of All Three Pricing Methods

Figure 8 below shows the calculation differences between all three pricing methods. Note that the Lehman Brothers Volatility Surface and Gram-Charlier method has a low one-month volatility calculation, compared to the academic model and the Lehman Brothers upper and lower bound models. This may be due to the use of options that are less liquid in the creation of the volatility surface.[10]

[10] For this table we are using the S_r = Forward price for each maturity.

Figure 8: Comparison of Three Replication Methods to Calculate the Fair Strike Variance

Expiration	Method 1: Academic Model $K_{variance}$ using Strikes 50% to 145%	Method 2: Lehman Upper Bound $K_{variance}$ using Strikes 50% to 145%	Method 2: Lehman Lower Bound $K_{variance}$ using Strikes 50% to 145%	Method 3: Lehman Volatility Surface and Gram-Charlier Expansion $K_{variance}$ using Strikes 50% to 145%
1-mth	$(16.58)^2$	$(16.61)^2$	$(14.03)^2$	$(15.67)^2$
3-mth	$(15.61)^2$	$(15.70)^2$	$(14.84)^2$	$(15.75)^2$
6-mth	$(15.53)^2$	$(15.74)^2$	$(15.28)^2$	$(15.91)^2$
1-yr	$(16.15)^2$	$(16.21)^2$	$(15.96)^2$	$(16.44)^2$

Source: Lehman Brothers

A Case Study Using Variance Swaps over Two Periods

Over the last 14 years since 1990, there have been several incidents in which variance swaps[11] could help the performance of a portfolio (including the Latin American debt crisis of 1994, the Asian debt crisis of 1998, the end of the Tech bubble run-up in 2000, the terrorist attack in 2001, the U.S. accounting crisis of 2002, and the Iraq war in 2003). For brevity, we examine two periods over the last few years to illustrate the use of realized and implied variance in a variance swap on the S&P 500. We have chosen a period of high implied and realized variance around August 2002 (during the U.S. accounting crisis of that year) and a period of falling implied and realized volatility from June–December 2003. We utilize the mark-to-market variance swap formula outlined in the "Detailed Formulation of a Variance Swap" section of this report.

Hedging Cost of a Strip of Options versus a Variance Swap

The two periods that we examine are July 1–December 18, 2002, and July 1–December 18, 2003. We assume that we are long variance swaps in both cases, and examine our profits and losses.

We compare the option portfolio replication (hedge) to a variance swap contract, to get an understanding of what would be involved in creating a variance swap using only equity or index options. In addition, we show the difference in performance of a variance swap versus a portfolio replication using options that would require delta hedging. We then compare the profits of a variance swap in two different realized and implied variance (volatility) regimes (period 1 and period 2) for a six-month variance swap. The two extreme cases that we examine are July 1–December 18, 2002 (period 1), during the U.S. accounting crisis and before the Iraq War, when volatility increased significantly; and July 1–December 18, 2003 (period 2), when the S&P 500 rallied, fundamentals improved, and variance decreased significantly. We assume that we are long variance swaps in both cases, and examine our profits and losses.

We note the following about all the theoretical P/L graphs to follow in this report: Although we delta hedge the option replication, it is based only on historical data in and simulations from our databases, and does not include transaction costs. In addition, we calculated realized variance with a zero mean.

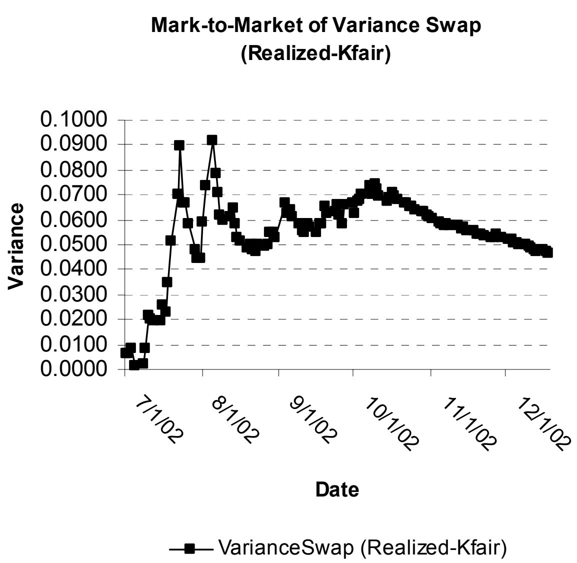
[11] We assume zero mean variance swaps for this analysis.

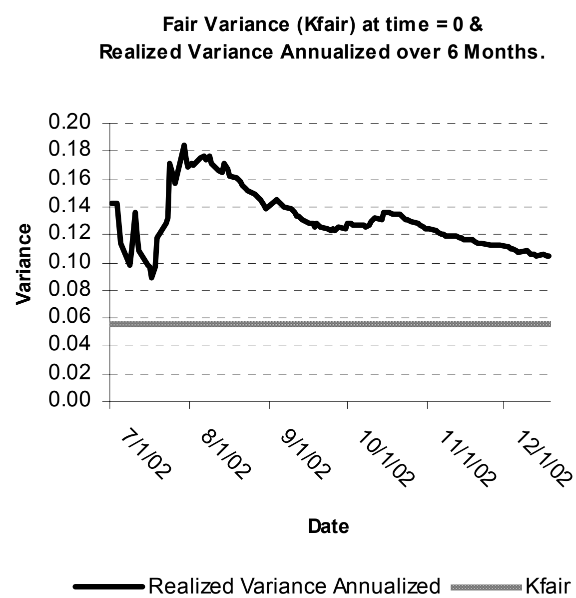
Period 1: July 1–December 18, 2002 (Accounting Scandals, Iraq War, and Increasing Variance)

At time t= 0, K_{Fair} was $(25.3\%)^2$ but at maturity, $\sigma^2_{Realized}$ was $(32.4\%)^2$. Therefore, variance increased significantly during this six-month period and the variance swap had a high positive payout, during a regime where the S&P 500 index level had very large movements.

The daily mark-to-market variance, the fair strike variance, and the realized variance are shown in Figure 9 for period 1, which incorporates the high-volatility period during the U.S. accounting crisis of 2002 and the beginnings of a potential war with Iraq. Figure 9 shows the original fair variance calculated at the start (t= 0) as well as the daily annualized realized variance. At time t= 0, K_{Fair} was $(23.8\%)^2$ but at maturity, $\sigma^2_{Realized}$ was $(32.3\%)^2$. Therefore, variance increased significantly during this six-month period and the variance swap had a high positive payout, during a regime where the S&P 500 index level had very large movements. As the index level moved down significantly, we benefited from the additional gamma exposure represented in the increased number of OTM puts contracts. The gamma of our options hedge for period 1 ranged from 0.1104 to 0.18612 (near expiration) with a mean level of 0.1348. The large jumps at the beginning of the period in Figure 9 represent the beginning of the accounting crisis in August 2002.

Figure 9: Mark-to-Market of a Variance Swap on the S&P 500 as well as Fair Variance versus Actual Realized Variance on the S&P 500 from July 1 – December 18, 2002



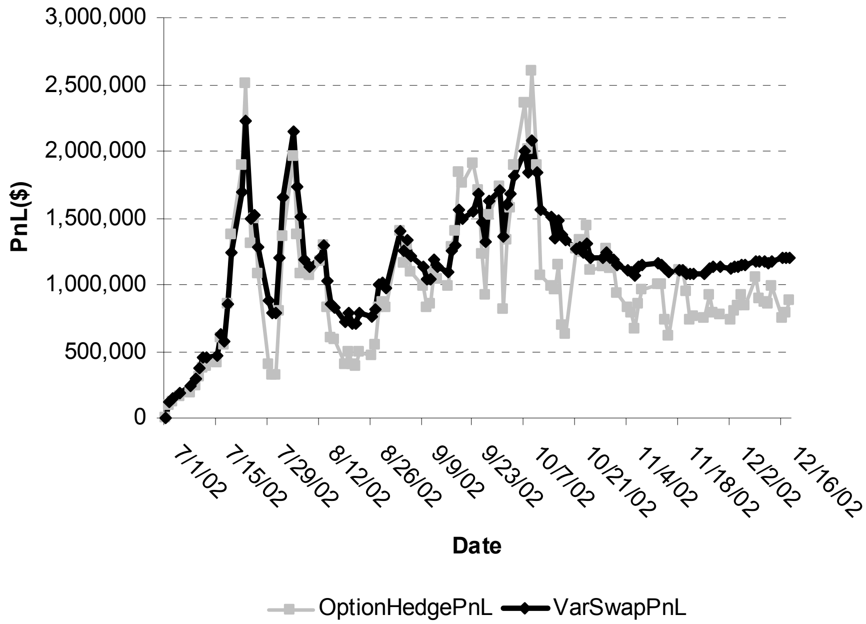


Source: Lehman Brothers

Even if one had the ability to replicate the payout of a variance swap using options and delta hedge the option position, we would still do better owning the variance swap in period 1 in our simulated scenario.

Figure 10 below shows the variance swap P/L and the option portfolio replication P/L. The P/L of the 11 option portfolio (four calls and seven puts ranging from approximately 25% OTM Puts to 25% OTM Calls) and the actual payout of a variance swap to expiration for 250,000 units is shown in Figure 10, where one unit represents $[\sigma^2_R - \sigma^2_K]$ * 100 *$1 according to our initial contract. The 11 options used to replicate the variance swap actually underperformed by about 27%, making approximately +$884,000 versus a profit of +$1.201 million using the variance swap. This shows that large jumps in the index can create differences using options, which can be significant if one does not own additional puts for the downside event. Even if one had the ability to replicate the payout of a variance swap using options and delta hedge the option position, we would still do better owning the variance swap in period 1 in our simulated scenario. In addition, we assumed daily re-hedging of any residual delta to capture some of the large jumps in variance in our example. Finally, the difference in performance would only be amplified if we used only three options instead of 11.

Figure 10: Variance Swap P/L versus 9 Option Hedge P/L (7 puts and 4 calls)

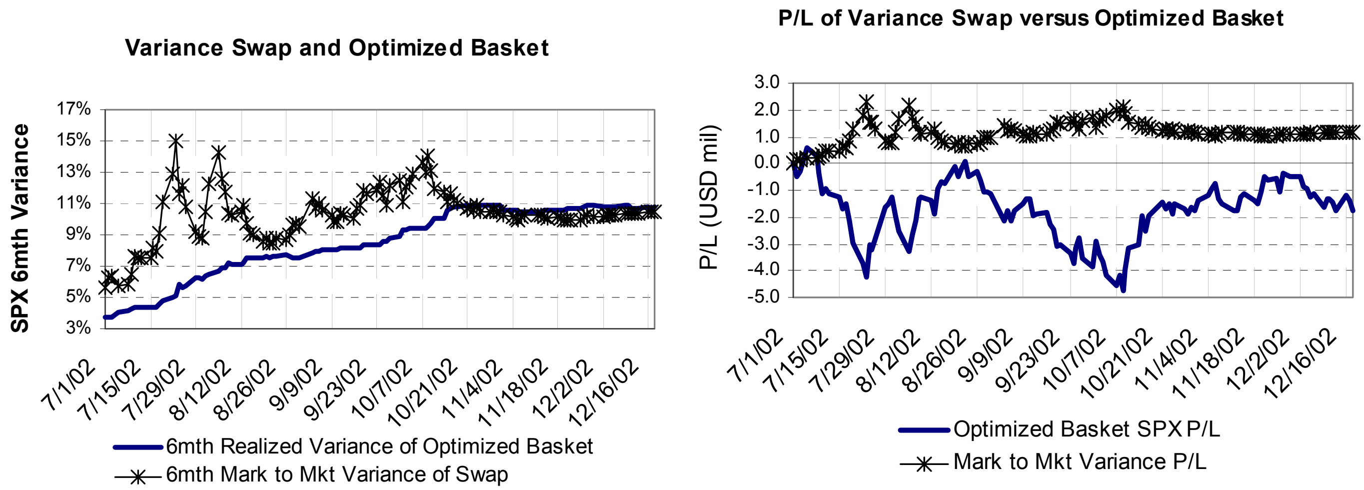


Source: Lehman Brothers

In Figure 11 below, we show how a six-month variance swap could be used to hedge a portfolio during this volatile period. We took a $25 million notional portfolio and optimized a basket of 100 stocks on the S&P 500, which had a tracking error of 100 basis points (bps). The final realized variance of our variance swap was 10.48% and the final variance of the 100-stock portfolio was 10.74%. Using a $25 million long variance swap to hedge the 100-stock optimized portfolio would have resulted in a return of –2.2%, which is higher than the –6.92% unhedged return of only the 100-stock optimized portfolio on the S&P 500.[12] . Moreover, the variance of the 100-stock portfolio was $(32.77\%)^2$, while the variance swap reduced the overall variance of the combined portfolio to $(21.4\%)^2$ (due to the negatively correlated returns of the variance swap and the 100-stock portfolio).

[12] On a risk-adjusted return basis, hedging the 100-stock portfolio with a long variance swap outperforms only the 100-stock portfolio, with an annualized risk adjusted return of –0.08 and –0.23, respectively.

Figure 11: Variance and P/L of Optimized 100-Stock Portfolio and 6-Month Variance Swap on S&P 500 for Period 1



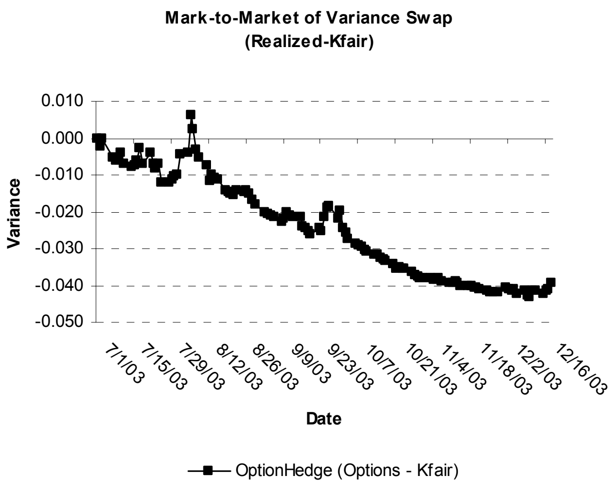
Source: Lehman Brothers

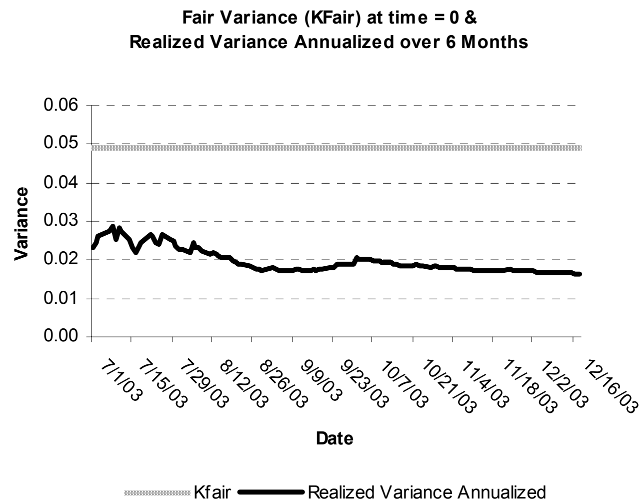
Period 2: July 1–December 18, 2003 (Market Rally and Decreasing Variance)

At time t= 0, K_{Fair} was $(22.2\%)^2$ but at maturity $\sigma^2_{Realized}$ was only $(12.9\%)^2$. Therefore, variance decreased significantly during this six-month period.

From July 1–December 18, 2003, after the U.S. accounting crisis and the Iraq war, the variance of the S&P 500 decreased substantially. The daily mark-to-market variance, the fair strike variance, and the realized variance for period 2 are shown in Figure 12 below. We compare the payout of a variance swap to the alternative of buying a strip of puts and calls, but we limit the strike range to −25 OTM% puts to +25% OTM calls on the S&P 500 (seven puts and four calls). At time t= 0, K_{Fair} was $(22.2\%)^2$ but at maturity $\sigma^2_{Realized}$ was only $(12.9\%)^2$. Therefore, variance decreased significantly during this six-month period. The variance swap resulted in a high negative payout during a low volatility regime. The gamma of our options hedge for period 2 ranged from 0.0996 to 0.1268 (near expiration) with a mean level of 0.1171.

Figure 12: Mark-to-Market of SPX Variance Swap and Fair Variance vs. Actual Realized SPX Variance (2H03)

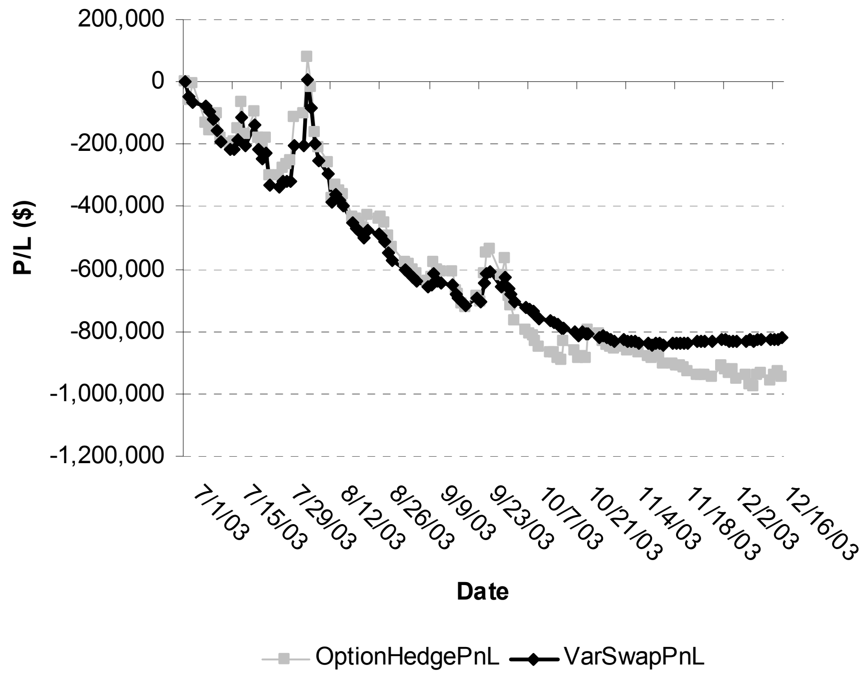




Source: Lehman Brothers

Figure 13 below shows the P/L of the 11 option portfolio (four calls and seven puts) versus the actual payout of a variance swap to expiration for 250,000 units, where, again, one unit represents $[\sigma^2_R - \sigma^2_K]$ * 100 *$1 according to our initial contract. In this declining volatility environment, implied volatility for index options remained high until the end of the contract. However, the variance swap payout was very close to the payout of the option until expiration was reached. The option hedge underperformed by about 15%, losing approximately $945,000 vs. a loss of $820,000 for the variance swap.

Figure 13: Daily Mark-to-Market P/L of Variance Swap versus 11 Option Replication Basket

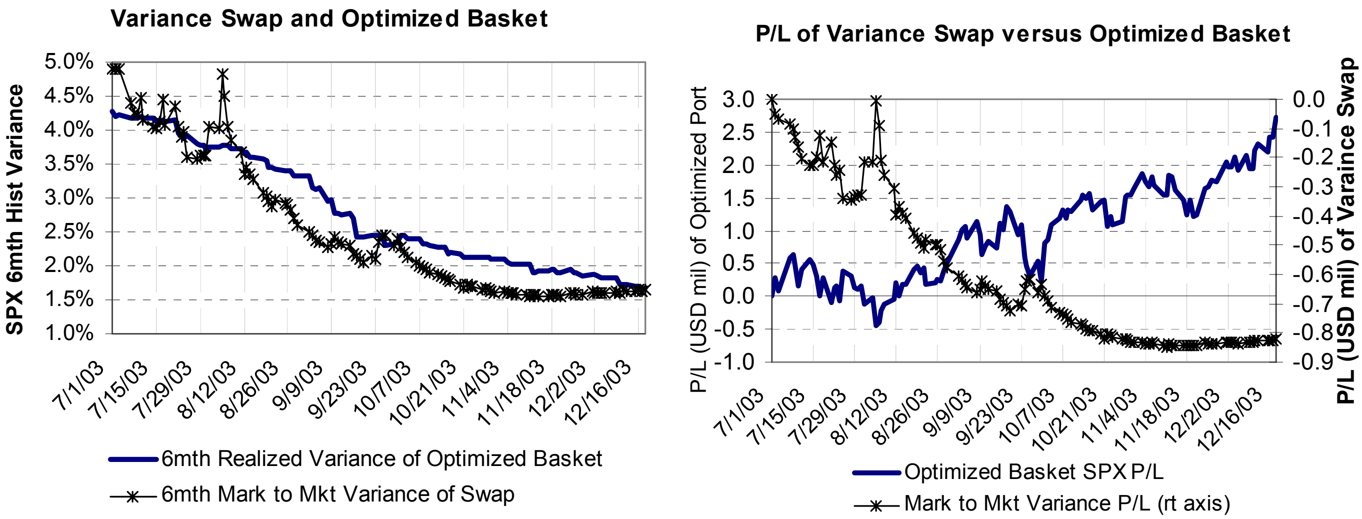


Source: Lehman Brothers

In Figure 14 below, we show how a six-month variance swap could be used to hedge a portfolio during a non-volatile period. We took a \$25 million notional portfolio and optimized a basket of 100 stocks on the S&P 500, which had a tracking error of 104 bps. The final realized variance of our variance swap was 1.66% and the final variance of the optimized portfolio was 1.68%. Using a \$25 million long variance swap to hedge the 100-stock optimized portfolio would have resulted in a return of +7.6%, which is lower than the 10.93% unhedged return of only the 100-stock optimized portfolio on the S&P 500.[13] However, the volatility of the 100-stock portfolio was 12.96%, while the variance swap reduced the overall volatility of the combined portfolio to 12.1%, but also reduced the returns of the combined portfolio due to the negative return correlation.

[13] On a risk-adjusted return basis, hedging the 100-stock portfolio with a long variance swap underperforms only the 100-stock portfolio, with an annualized risk-adjusted return of 0.25 and 0.8, respectively.

Figure 14: Variance and P/L of Optimized 100-Stock Portfolio and 6-Month Variance Swap on S&P 500 for Period 2



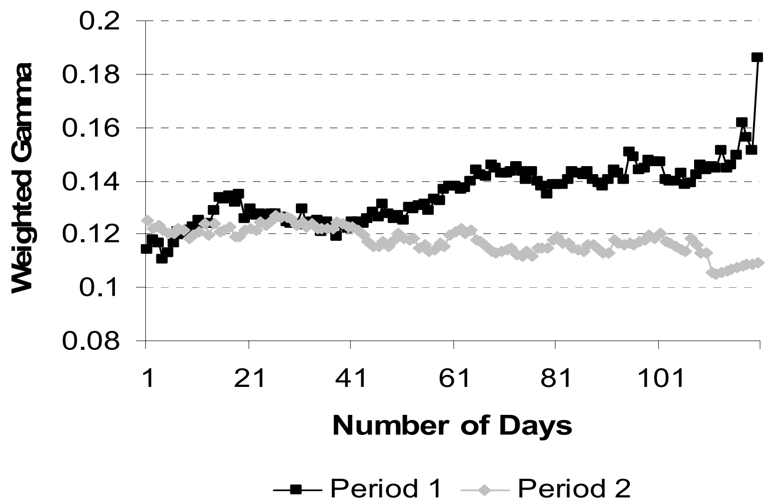
Source: Lehman Brothers

Comparison of Gamma in Period 1 and Period 2

The weighted gamma across the strip of options in period 1, in a more volatile regime, was relatively higher on a daily basis than in period 2.

Although gamma is not necessarily higher in a more volatile regime, Figure 15 below shows that weighted gamma across the strip of options was relatively higher in period 1 (high volatility regime) than in period 2, the less volatile regime. Long variance swap payout was positive in period 1 and negative in period 2, which reinforces the idea that to benefit from paying theta for all the options, one must take advantage of gamma in a higher-volatility regime. Period 1 has a higher gamma because the index level moved down significantly, and, as part of our replication process, we had bought a larger number of OTM put contracts, which created an increased exposure to gamma. Conversely, in period 2, gamma decreased as the market rallied because we had bought fewer OTM call contracts.

Figure 15: Weighted Gamma Profiles of Option Hedges for Period 1 and Period 2



Source: Lehman Brothers

The Benefits of Using Variance Swaps

Trading the volatility or variance component of an option is complicated by the fact that the option's gamma and vega are influenced by movements in the underlying equity. With an exact payoff based on realized variance, the variance swap avoids these complications.

We have illustrated the inherent limitations of option-based volatility strategies and have shown that replicating a variance swap using a portfolio of options is more complex, and costly, and may also require delta hedging. Additionally, trading the volatility or variance component of an option is complicated by the fact that the option's gamma and vega are influenced by movements in the underlying equity. With an exact payoff based on realized variance, the variance swap avoids these complications. The overall simplicity of the variance swap may also provide a significant advantage for investors who lack the necessary modeling infrastructure, risk management systems, and market access to engage in delta-neutral strategies. In order to hedge a long or short S&P 500 index option, an investor would likely need to satisfy the following infrastructure requirements:

- Options pricing software and risk management systems would be needed to price and mark options, and to determine the appropriate delta hedge for a portfolio of options.
- Futures trading agreements would be needed to establish the initial delta hedge and periodically rebalance the hedge as the delta changes.
- Eurodollar trading agreements would be needed to hedge the interest rate risk inherent in options.
- Dedicated operations professionals would be needed to manage the cash flows that result from delta-hedging activity, as well as managing the movements of margin.

By effectively packaging the option position and the delta hedge, the variance swap allows the investor to express a view on variance without many of the requirements necessary for delta hedging.

Finally, delta hedging is costly. Each time a delta hedge is rebalanced, commissions are incurred and some portion of the bid/offer spread is likely absorbed. Moreover, delta-neutral strategies may at times be capital intensive, requiring the investor to post significant margin or collateral. For options that move deep in-the-money and require a large delta hedge (i.e., a short call option for which the investor must be long a substantial hedge in stock or futures), these collateral requirements may be especially significant. By effectively packaging the option position and the delta hedge, the variance swap allows the investor to express a view on variance without many of the requirements necessary for delta hedging.

Users of Variance Swaps

Variance swaps can be used actively to assume long or short variance exposure, or defensively, to hedge existing variance risk contained in a portfolio.

The variance swap has the most obvious appeal for investors seeking to express views on market variance without implementing the delta hedging activity required in options-based variance strategies. Variance swaps can be used actively to assume long or short variance exposure, or defensively, to hedge existing variance risk contained in a portfolio. A variance swap is beneficial because it is:

- An asset that is generally uncorrelated with market returns or direction. The variance swap may appeal to investors seeking to increase portfolio diversification by adding an asset class with a negative (or low) correlation to stock market or equity returns.

- A method for expressing views on the future realized variance in one equity index versus another. A hedge fund manager might believe that a variance swap can be sold on one index and purchased on another index at favorable levels (relative to historical levels).

- A general method for controlling Value-at-Risk (VAR). The increased focus on value-at-risk suggests that the variance swap may be a useful addition to the tools available for controlling overall portfolio risk.

- An overlay against a naturally short variance portfolio. For example, a firm that issues guaranteed products linked to stock market performance may find the variance swap a useful method for offsetting short option exposure.

- An overlay against a naturally long variance portfolio. For instance, a convertible bond arbitrageur may use the variance swap as a way to capture the value of the embedded long options in a portfolio of convertibles.

- A means of minimizing tracking error in a benchmarked portfolio. The variance swap may be a useful product for a portfolio manager seeking to address the increased tracking error that typically occurs in more volatile markets. This is especially true for benchmarked portfolios that would require more frequent rebalancing in more volatile times, generating higher transaction costs for the portfolio manager.

- A hedge for investors that are likely short variance because they are long equities. Variance and returns of the equity or index generally move in opposite directions.

Therefore, diversifying across countries can become less of an effective hedge, as shocks are felt universally in a time of global crisis or terrorist event. Variance and volatility swaps are among the few entities that tend to be negatively correlated with global index and equity returns.

- A hedging mechanism for mutual funds, hedge funds, and broker-dealers, including using variance swaps for dispersion trading.

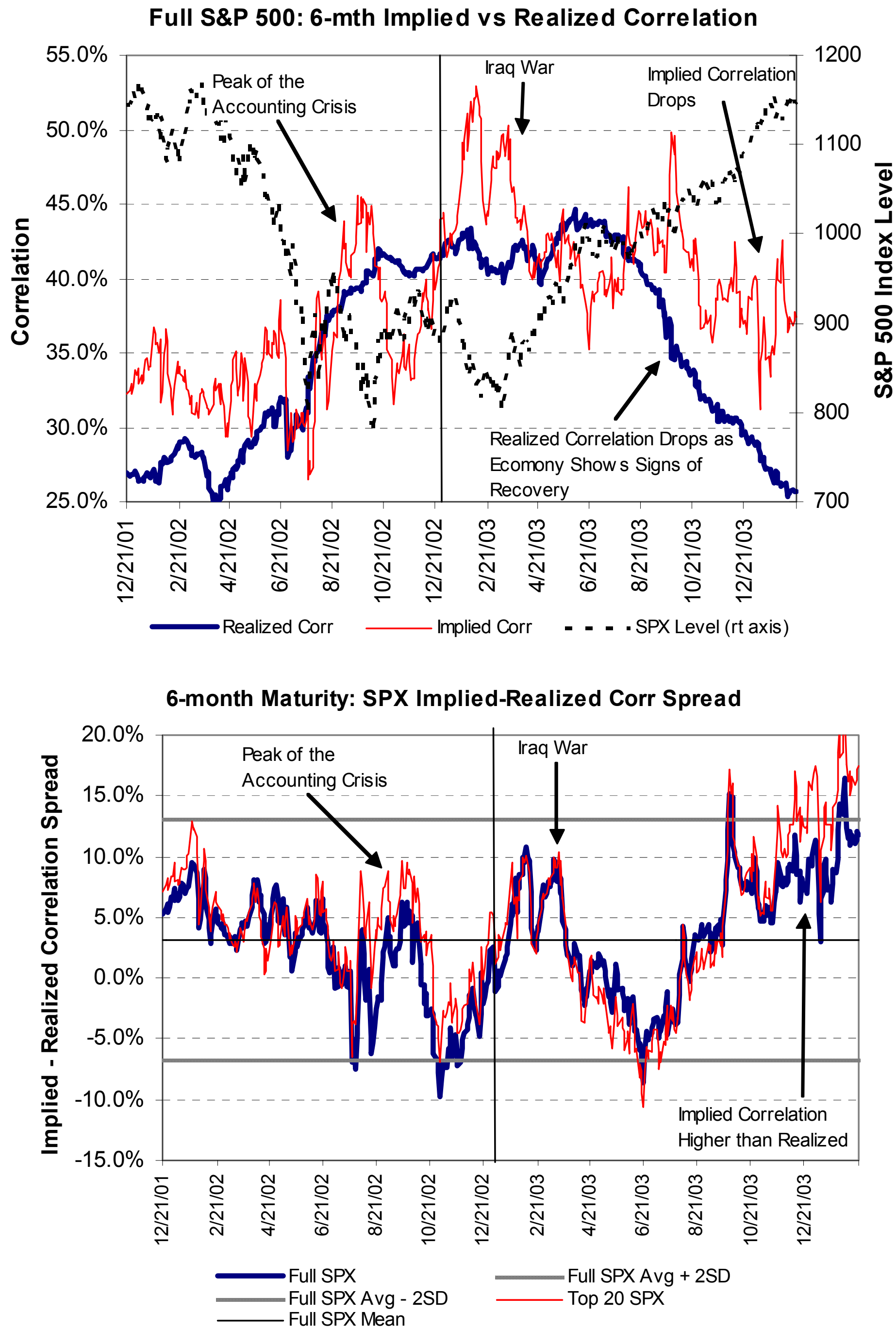
Finding Buy and Sell Signals Using Implied and Realized Correlation for the S&P 500 over a Six-Month Maturity

An important overlay to variance swaps is to monitor realized and implied correlation levels.

An important overlay to variance swaps is to monitor realized and implied correlation levels. As implied correlation becomes greater than realized correlation, then typically, there is a dispersion trade that can be implemented by investors in buying single stock variance (or implied volatility of single stock options) and selling index variance (or implied volatility of index options). Dispersion trading is common when implied correlation is higher than realized correlation, and sharp changes in correlation generally create the profitable trading opportunities. A positive implied less realized correlation spread suggests that single stock implied volatility is a better buy, and that index implied volatility is a better sell, relative to one another. Typically, the maturity of such a trade is at least six months or more to capture the effect. This becomes an arbitrage opportunity that allows the investor to buy single stock implied volatility or variance, versus selling index implied volatility or variance. Variance swaps can be used to play these arbitrage opportunities. Alternatively, a negative implied less realized correlation spread indicates that single stock implied volatility is a better sell, and index implied volatility is a better buy, relative to one another, but this is a riskier trade as it involves shorting single stock options (or variance contracts on single stocks).[14]

[14] See our "Weekly Index Dispersion Monitor" publication, February 23, 2004, or any date, for a further explanation of the calculations used for implied less realized correlation.

Figure 16: S&P 500 Correlation Levels (using top 20 mkt cap constituents) & Implied less Realized Correlation Spread



Source: Lehman Brothers

Figure 16 above shows the six-month implied less realized correlation graphs for the S&P 500 over the last few years, calculated for the ATM implied volatility levels. Clearly, buying single stock volatility and selling S&P 500 variance swaps with a maturity of six months could have been a viable strategy during the U.S. accounting crisis of 2002, the Iraq war in 2003, and most recently, from year-end 2003–early 2004. From December 2003–February 2004, realized correlation dropped, while index implied volatility remained high compared to correlation-adjusted implied volatility levels of index constituents. The drop in realized correlation caused the implied less realized correlation spread to widen to more than two standard deviations above historical average, creating a strong dispersion signal. Using variance swaps to express a view on single stock versus index volatility levels is a viable option for investors.

Near the end of June 2003, the six-month implied less realized correlation spread fell to more than two standard deviations below its long-term mean. Under these conditions, buying index implied volatility and selling single stock implied volatility looked attractive, but again, this is a more difficult and risky trade. In this lower correlated environment, generally, statistical arbitrage and pairs trading strategies perform better.

Conclusions

We have shown that variance swaps offer a viable alternative to buying and hedging a strip of index or equity options. In addition, we have provided the basic intuition behind variance swaps and discussed three pricing methodologies. We have shown that, in more volatile regimes, variance swaps offer a way to control risk. Finally, Lehman Brothers has an active variance swap book and makes two-sided swap markets on the S&P 500 index, on other U.S. indices and ETFs, and on single stocks, as well as for major European and Asian indices and single stocks.

Detailed Formulation of a Variance Swap

The Formulation Behind a Variance Swap

Since the late 1990s, there has been much periodic discussion on the formulation of both volatility and variance swaps.[15] In this section, we attempt to summarize the final formulation and factors that go into setting and pricing a variance swap. To replicate a variance swap, we start by building on the Black/Scholes (BS) model (and our knowledge from Appendix A). First, we define realized and implied volatility.

Realized (Historical) Volatility is based on the past price (historical) volatility of the stock price.

Realized (Historical) Volatility is based on the past price (historical) volatility of the stock price. It is backward-looking volatility, explained by the historical price movements of the stocks from t-1 to t.

- Based on historical prices: $x = \ln\left[\dfrac{S_t}{S_{t-1}}\right]$

- $\sigma_{ann} = \sqrt{\sum_{t=1}^{n} \dfrac{(x_t - \mu)^2}{(n-1)}} * \sqrt{252}$ (unbiased)

$$= \sqrt{\frac{n * \sum_{t=1}^{n} x_t^2 - \left(\sum_{t=1}^{n} x_t\right)^2}{n*(n-1)}} * \sqrt{252}$$

where μ = mean of x, n is the total number of daily return samples over time, t is a specific sample, S_t is the t[th] period price, S_{t-1} is the t-1 period price, and σ_{ann} = the annualized realized volatility. Variance swaps can be quoted with a mean as long as the contract reflects how it will be calculated. However, for a variance or volatility swap, we assume a zero mean process for the variance calculation because the replication of the variance swap using options assumes a zero mean volatility or variance. This has become the standard in the Over-The-Counter (OTC) market. Therefore, we will calculate variance for our contract using equation 1 below:

$$\sigma_{ann}^2 = \sum_{t=1}^{n} \frac{(x_t)^2}{(n-1)} * 252 \qquad \text{(equation 1)}$$

[15] Carr, Peter and Madan, Dilip, 1998, "Towards a theory of volatility trading", in Robert A. Jarrow, ed: *Volatility: New Estimation Techniques for Pricing Derivatives*, Chapter 29, pp.417-427, *Risk* Books , London. Chris, Neil and Morokoff, William, 1999, "Market Risk for Variance Swaps", *Risk*, 12, pp. 55-59. Demeterfi, Kresimir; Derman, Emanuel; Kamal , Michael; and Zou, Joseph, 1999, "A guide to volatility and variance swaps", *Journal of Derivatives*, 6, pp. 9-32. Breeden, D. and Litzenberger, R., 1978,"Price of state-contingent claims implicit in option prices", *Journal of Business*, 51, pp. 621-651. Cox, John, C.; Ingersoll, Johnathan, E.; and Ross, Steven, A., 1985, "A theory of the term structure of interest rates", *Econometrica*, 53, 385-407.

Implied Volatility explains the future expected uncertainty of the stock or index, and is forward looking.

Implied Volatility explains the future expected uncertainty of a stock or index, and is forward looking. Listed options on indices or stocks can be used to calculate implied volatility levels, assuming other factors in the Black/Scholes model are constant. Implied volatility from the Black/Scholes model for a European call[16] assuming no dividends is calculated from the following equations:

- $C = S * N(d_1) - K * e^{-rt} * N(d_2)$ (equation 2)
- $P = K * e^{-rt} * N(-d_2) - S * N(-d_1)$ (equation 3)
- $d_1 = \dfrac{\ln\left(\dfrac{S}{K}\right) + \left(r + \dfrac{\sigma^2}{2}\right) * t}{(\sigma * \sqrt{t})}$ (equation 4)
- $d_2 = d_1 - \sigma * \sqrt{t}$ (equation 5)
- C = call price and P = put price from the option market, K= option strike, S = stock price, N = cumulative normal distribution, r = continuously compounded risk-free rate, t = time to maturity, and σ = option implied volatility.

We assume that the risk-free rate and dividends are zero.

Since rho (r) is likely to be hedged out in our variance swap example, we develop our initial analysis around the assumption that the risk-free rate is zero and dividends are zero. Therefore, by building on a familiarity with the standard Black/Scholes model, we can replicate the variance path for a set of options to accurately represent a variance swap. If τ is the time to expiration, then the total variance of the stock to expiration:

$$v = \sigma^2 * \tau \qquad \text{(equation 6)}$$

where σ is the volatility of the underlying and τ is the time till expiration.

[16] Hull, John, *Options, Futures and Other Derivatives*, 5th Edition, Prentice Hall, 1999, pp. 237-272.

We will call the exposure to an option's stock's variance (V), which measures the change in the value of the position, resulting in a change in variance.

We will call the exposure to an option's stock's variance (V), which measures the change in the value of the position resulting in a change in variance. V is called the variance vega (the rate of change of the volatility or the variance sensitivity).

$$V = \frac{\partial C}{\partial \sigma^2} = \frac{S * \sqrt{\tau}}{2\sigma} * \frac{\exp(-d_1^2 / 2)}{2\pi} \qquad \text{(equation 7)}$$

where from equation 4, $d_1 = \dfrac{\ln\left(\dfrac{S}{K}\right) + \dfrac{(\sigma^2)}{2} * \tau}{(\sigma * \sqrt{\tau})}$ since t = τ and the risk free rate is zero. Note that as S decreases and moves far away from K, V decreases rapidly. This illustrates the sensitivity of the change in variance to moves in the stock price.

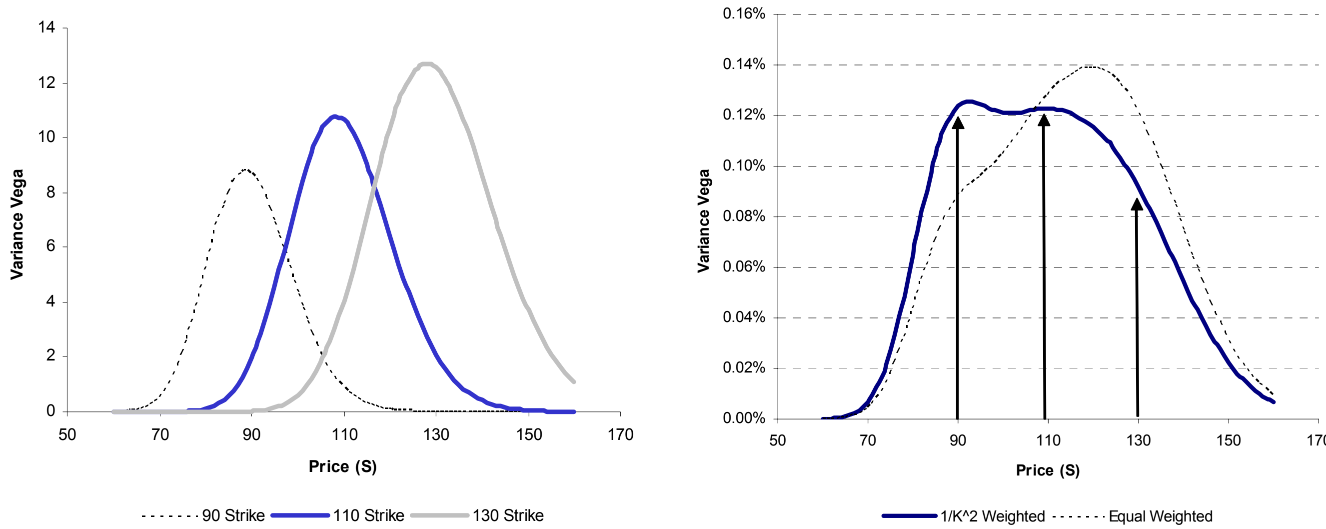
To achieve a portfolio that only moves with variance (or volatility) independent of other variables, one would need to combine options across many strikes to fix an initial variance level, and then allow variance to move from that fixed level.

As we can see from equation 7 above, a single option position or even an option straddle would provide imperfect exposure, since a move in the price of the stock or index would alter one's sensitivity to the changes in variance. To achieve a portfolio that moves only with variance (or volatility) independent of other variables (such as movements in the stock or index price), one would need to combine options across many strikes to fix an initial variance (or volatility) level, and then allow variance to move from that fixed level to gain an exposure to only a move in variance.

In Figure 17 below,[17] we illustrate the variance sensitivity (variance vega) using three options with three different strikes: 90, 110, and 130. We then compare the equal weighted option contract case to the case where we weight the option contracts by $1/K^2$, where K is the strike of the option. In the first case (the dashed line), we take a set of three options and equally weight each strike with the same number of contracts, but this creates an imperfect exposure to variance vega. In the second case (the solid line), we weight the option contracts inversely to $1/K^2$ and calculate the variance vega, and find that within the strike range, the variance vega becomes more independent of the price level. So, from the price level of 90 to 130, we notice that the variance vega profile of the $1/K^2$ weighted case is more within a constant range of 0.13%–0.10% when compared to the equal weighted case. In addition, with only three options, we are not properly hedging our exposure to the variance vega for large underlying moves.

[17] We make the following assumptions to calculate our variance sensitivity exposure (vega): implied volatility is constant at 10% for all strikes, time to expiration is one year, strikes range from 60 to 160, the current spot level is 100, and interest rates are 2%.

Figure 17: Variance Exposure using Three European Call Options, with Strikes of 90, 110, and 130, for Equal Weighted and $1/K^2$ Weighted Replications



Source: Lehman Brothers

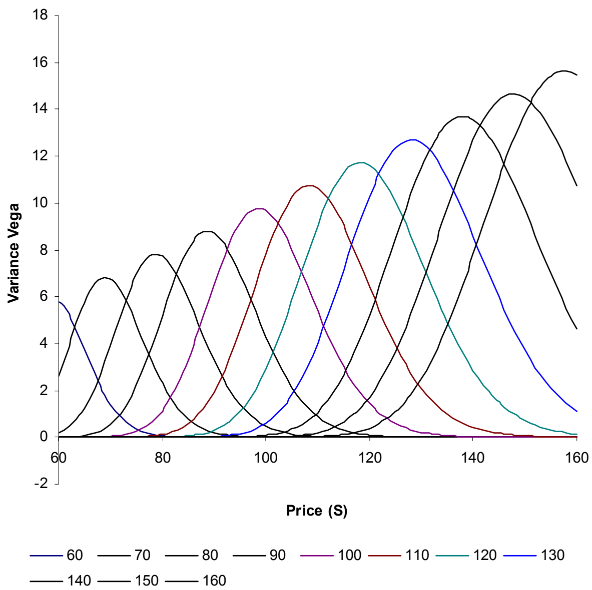
Inherently, by weighting the number of contracts by $1/K^2$, we are increasing the contract amounts (weights) for the lower strike options (puts).

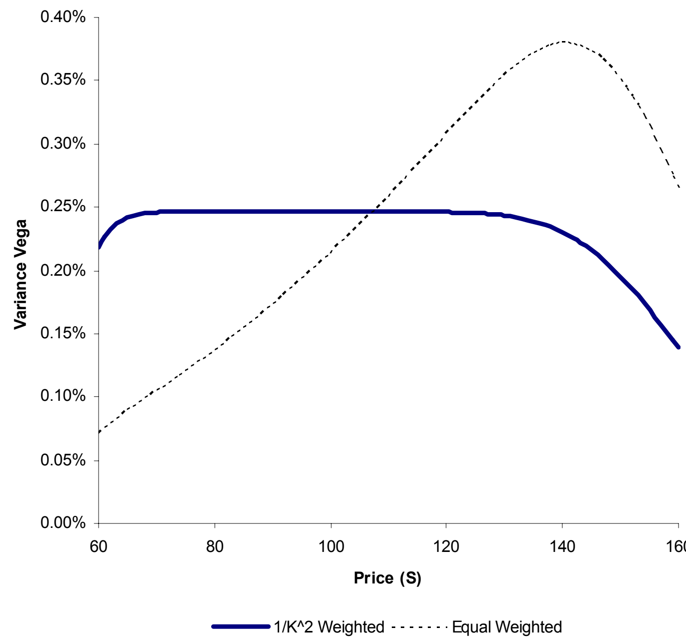
Therefore, as shown in Figure 18 below, we now use 11 options and find that our variance sensitivity (variance vega) becomes even flatter across a wider range of price variations when weighting the number of contracts by $1/K^2$. Now, from a price level of 68 to almost 129, the variance vega remains nearly constant at 0.25%, while the equal weighted contract case overweights the variance vega at higher price levels and underweights the variance vega at the lower price levels (to the downside), which is not the desired outcome. When the index or stock moves down and the variance (volatility) increases, the investor would have a reduced sensitivity to variance. Therefore, the investor would have less of an increase in variance when the market sells off, just at a time when the investor wants a constant or increasing variance vega.

Essentially we are vega weighting each option by $1/K^2$ contracts, which will provide us the constant variance sensitivity required so that at every price within the strike range, we gain the same exposure to variance whether it increases or decreases. The OTM puts are weighted by more contracts because vega is directly proportional to the stock or index price (S) multiplied by the square root of time($\sqrt{T}$), according to equations A12 and A13 in Appendix A. From these equations, we could observe that a put option with an \$80 strike at a spot level of \$80 would have a lower vega than a call option with a \$120 strike at a spot level of \$120 (all other inputs being equal). Therefore, we would need more OTM puts than OTM calls to achieve a constant level of vega across a range of strikes.

This also falls in line with our intuition that if volatility generally increases as the market sells off, then the investor would want to have increased exposure to variance or volatility to the downside represented by more contracts on the further OTM puts. Conversely, if the market moves higher, then the investor would require a smaller number of contracts on the OTM calls. Note that by weighting the number of contracts by $1/K^2$, inherently we are increasing the contract amounts (weights) for the lower strike options (puts), because we know that (in general) variance (volatility) increases large downside price moves more than upside price moves. We also note that one option or a few straddles equally weighted will not replicate the desired variance exposure that we wish to achieve. As the maturity of the variance swap increases beyond a few months, using a few straddles on ATM and slightly OTM calls and puts will not allow an investor to fully participate on the variance increases or decreases in the market. Once the option and the number of contracts weighted by $1/K^2$ are selected, the hedge would be static until maturity; however, one would still have to delta hedge the residuals from the option positions.

Figure 18: Variance Exposure using 11 European Call Options with Strikes of 60 to 130 (Increments of 10) for Equal Weighted and $1/K^2$ Weighted Replications





Source: Lehman Brothers

Weighting a Portfolio of Options by $1/K^2$ and the Log Contract

We assume that we want to replicate the aggregate payout pattern of a series of strikes across many options.

Kresimir Demeterfi, Emanuel Derman, Michael Kamal and Joseph Zou developed the first model that we will discuss, which we term the academic model for a fair variance calculations.[18] In Appendix B, we derive a relationship incorporating the requirement that options are weighted by $1/K^2$ to reach a constant V. In addition, we show how to replicate the payout of a log contract (here, we use the natural log contract) using a discrete number of options. We assume that we want to replicate the aggregate payout pattern of a series of strikes across many options. S_r is the forward reference price or level that will mark the demarcation between calls and puts. If S_r is a forward reference point that marks the boundary between liquid calls and puts, we can establish a continuously changing strike from 0 to S_r for puts and from S_r to ∞ for calls. We express the $1/K^2$ relationship from Appendix B as follows, where BS_c is the Black/Scholes value of the calls and BS_p is the Black/Scholes value of the puts using only S, K, and $\sigma\sqrt{\tau}$ as input variables (assuming other BS variables are zero or constant).

$$F(S,\sigma\sqrt{\tau}) = \sum_{K>S_r} \frac{1}{K^2} * BS_c(S,K,\sigma\sqrt{\tau}) + \sum_{K<S_r} \frac{1}{K^2} * BS_p(S,K,\sigma\sqrt{\tau})$$

(equation 8)

[18] Demeterfi, Kresimir; Derman, Emanuel; Kamal , Michael; and Zou, Joseph, 1999, "A guide to volatility and variance swaps", *Journal of Derivatives*, 6, pp. 9-32.

From Appendix B and equation 8, we can further derive the relationship for a portfolio of calls and puts with a varying S, K, and $\sigma\sqrt{\tau}$, where F is the portfolio of options. At expiration, when time= T, the present value payoff of the portfolio of options (F) can be shown to be the following assuming zero dividends:

$$F(S_T,0)=\left[\frac{S_T-S_r}{S_r}-\ln\left(\frac{S_T}{S_r}\right)\right] \qquad \text{(equation 9)}$$

where S_T is the final stock price at expiration. Here, we use the natural log function (ln) for the log contract. For stock prices prior to expiration at time t, where S_t is the price at time t, the total portfolio value can be shown to be:

$$F(S_t,\sigma\sqrt{\tau})=\left[\frac{S_t-S_r}{S_r}-\ln\left(\frac{S_t}{S_r}\right)\right]+\frac{\sigma^2\tau}{2} \qquad \text{(equation 10)}$$

There is a small difference between the value of the portfolio at expiration (T) and any time prior to expiration (t). The difference is only $\frac{\sigma^2\tau}{2}$ or half of the total variance multiplied by the time remaining to expiration (τ) where the variance exposure of F is $V=\frac{\tau}{2}$. Therefore, to get initial exposure to $1 per volatility point squared, one must hold 2/T units of the total portfolio F. Now, using this relationship, if we multiply by $\frac{2}{T}$ * $F(S_t,\sigma\sqrt{\tau})$, we can show the following:

$$F(S_t,\sigma\sqrt{\tau})=\frac{2}{T}*\left[\frac{S_t-S_r}{S_r}-\ln\left(\frac{S_t}{S_r}\right)\right]+\frac{\sigma^2\tau}{T} \qquad \text{(equation 11)}$$

In the creation of a weighted portfolio of options, we have created the desired log contract (natural log contract) that has the desired exposure to the variance (or volatility) sensitivity or vega from the options.

In equation 11, the $\left[\frac{S_t-S_r}{S_r}\right]$ term is essentially $1/S_r$ forward contract on S_t which is equivalent to a long position at time t in the stock (with a current value of S_t) and a short position in a bond represented by S_r. In addition, the term $\left[-\ln\left(\frac{S_t}{S_r}\right)\right]$ is a short log contract (ln) that starts at the price of S_r . Neuberger and Dupire discussed the creation of a log contract whose payoff is proportional to the log of the stock price at expiration, and for which the hedging of the contract relies on the volatility (variance) of the stock.[19] In the creation of a weighted portfolio of options, we have created the desired log

[19] The log contracts were discussed by Dupire and Neuberger. Dupire, 1993, "Model Art", *Risk Magazine*, Sept 1993, pp. 118-120. The log contract was also discussed by Neuberger, Neuberger,, A.; *Volatility Trading*, 1990, London Business School working paper. Neuberger, A; "Variance Swap Volatility and Option Strategies," *Derivatives Week*, Volume 9, No. 46 , pp. 1-4.

contract (natural log contract) that has the desired exposure to the variance (or volatility) sensitivity or vega from the options.

Log Contracts Used to Trade Realized and Implied Volatility

Why is a log contract (natural log contract) important to understand?

- It replicates the payout pattern that we desire in a variance swap.
- The payout pattern is independent of moves in the underlying stock price or index level.

For our example, let's assume implied volatility (σ_I) is the estimate of future realized volatility. If one buys a portfolio options at time zero (t=0) with a starting stock price of S_o, then the payout (F) can be constructed below using equations 10 and 11:

$$F_o = \frac{2}{T} * \left[\frac{S_o - S_r}{S_r} - \ln\left(\frac{S_o}{S_r} \right) \right] + \sigma_I^2 \qquad \text{(equation 12)}$$

We calculate the realized volatility at expiration to be σ_R, then the position value captured would be:

$$F_o = \frac{2}{T} * \left[\frac{S_o - S_r}{S_r} - \ln\left(\frac{S_o}{S_r} \right) \right] + \sigma_R^2 \qquad \text{(equation 13)}$$

After hedging to expiration (F_{exp}), we therefore can create a natural log contract payout of:

$$F_{\exp} = (\sigma_R^2 - \sigma_I^2) \qquad \text{(equation 14)}$$

If the investor buys the variance swap (long position), then the investor makes money if realized variance is above the strike of implied variance and the investor loses money if realized variance is below the strike of implied variance.

If you re-hedge the delta and rho of the log-contract (natural log contract) position, then essentially you have created a variance swap with an initial strike of σ_I^2. If the investor buys the variance swap (long position), then the investor makes money if realized variance is above the strike of implied variance and the investor loses money if realized variance is below the strike of implied variance.

Vega, Gamma, and Theta of a Log Contract

As we show in Appendix A of this report for individual options, here, we show the calculation of vega, gamma, and theta, assuming zero interest rates and zero dividends. We have shown early that the variance vega must be independent of the stock price, and equation 11 provides us such an exposure to vega that is independent:

$$F(S_t,\sigma,t,T)=\frac{2}{T}*\left[\frac{S_t-S_r}{S_r}-\ln\left(\frac{S_t}{S_r}\right)\right]+\frac{\sigma^2*(T-t)}{T}$$

We can hedge the term $S_t - S_r$, which is a forward with a long position in the stock and a short position in a bond, with no effect on the variance or volatility. The value of the natural log contract, which inherently has a logarithmic payout at expiration (T), is below:

$$LogPayout(S_t,\sigma,t,T)=\frac{2}{T}*\left[-\ln\left(\frac{S_t}{S_r}\right)\right]+\frac{\sigma^2*(T-t)}{T} \qquad \text{(equation 15)}$$

Sensitivity to vega in equation 15 is:

Vega: $$V=\frac{T-t}{T} \qquad \text{(equation 16)}$$

We can see the exposure to variance equal to one at t = 0 and then decreasing as we get closer to expiration, which makes sense intuitively.

The time decay of the log (natural log) contract is given by:

Theta $$\theta=-\frac{\sigma^2}{T} \qquad \text{(equation 17)}$$

The log (natural log) contract value decreases directly proportional to its variance and inversely proportional to its time to expiration at a constant rate.

The log (natural log) contract's exposure to price is given shares of stock below:

Delta: $$\Delta=-\frac{2}{T}*\frac{1}{S_t} \qquad \text{(equation 18)}$$

The investor needs a constant long position of 2/T worth of stock to be hedged, since each share of stock is worth S_t.

The rate of change in the exposure to price is given by gamma:

Gamma: $$\Gamma=\frac{2}{T}*\frac{1}{S_t^2} \qquad \text{(equation 19)}$$

For hedging an option position, gamma is a constant concern. However, in a portfolio of many calls and puts, the natural log contract gamma has a wider distribution and is, therefore, smoother than the gamma of a single option. This can be seen in the term $1/S_t^2$.

The Gamma/Theta Log Contract Trade-Off

In Appendix A, we discuss the trade-off of Gamma and Theta for a single option. As described in Appendix A for a single option, gamma indicates the extent to which the delta hedge becomes mismatched and for which there exists an economic benefit for large movements in the stock price. Theta indicates the economic consequence of these large movements not occurring (and the cost associated with waiting for a large movement). We can combine equations 17 and 19 to derive the gamma and theta trade-off for a natural log contract (assuming $\Delta = 0$):

$$0 = \theta + \frac{1}{2} * \Gamma * S_t^2 * \sigma^2 \qquad \text{(equation 20)}$$

Again, we are showing negative theta (until expiration) is offset by the potential benefit of positive gamma (non-linear payout).

Again, we are showing negative theta (until expiration) is offset by the potential benefit of positive gamma (non-linear payout). Gamma also increases for more contracts that have been purchased for the OTM puts, if the index or stock price moves down rapidly.

Trading an Imperfect Log Contract and Our Imperfect Assumptions

As long as the stock or index remains within the strike range, trading an imperfectly replicated log contract will tend to increase or decrease at a constant rate in line with the increase or decrease in the variance of the stock or index.

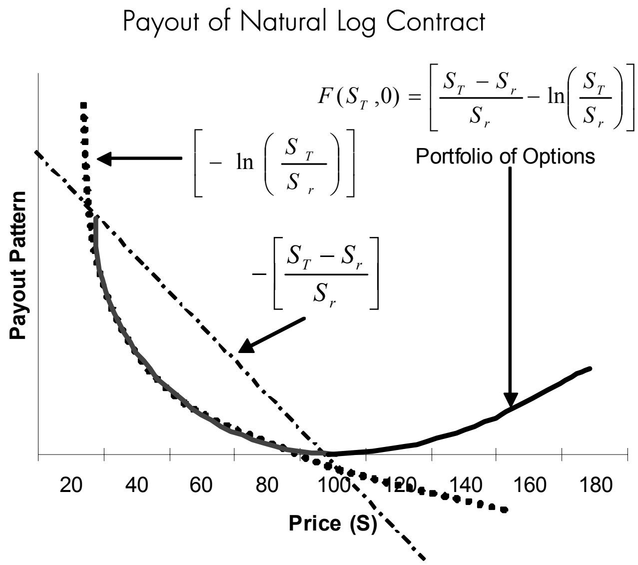
Log contracts (natural log contracts) exist only in theory. We attempt to replicate a natural log contract using an infinite number of strikes across both calls and puts. As shown in Figures 17 and 18 above as long as the stock or index remains within the strike range, then trading an imperfectly replicated log contract (natural log contract) will tend to increase or decrease at a constant rate in line with the increase or decrease in the variance of the stock or index. This is because the variance vega remains approximately constant within this strike range. We have shown that dynamic hedging of a natural log contract produces a payout whose value is proportional to future realized variance to a particular expiration, and that this natural log contract can be replicated (imperfectly) using calls and puts weighted by $1/K^2$. In Figure 19 below, we show a hypothetical payout of our imperfect log contract (natural log contract) using equation 9 where we assume $S_r = 100/(1+r)^{-T}$, assuming no dividends. In addition, from Appendix B, we show how options may not perfectly replicate the log or natural log payout, per Figure 19. As shown in Figure 19, the weight of the options will depend on the slope of the line between the discrete strikes available for replication. Below, we separate all the put options form 0 to S_r and all the call options from S_r to ∞.

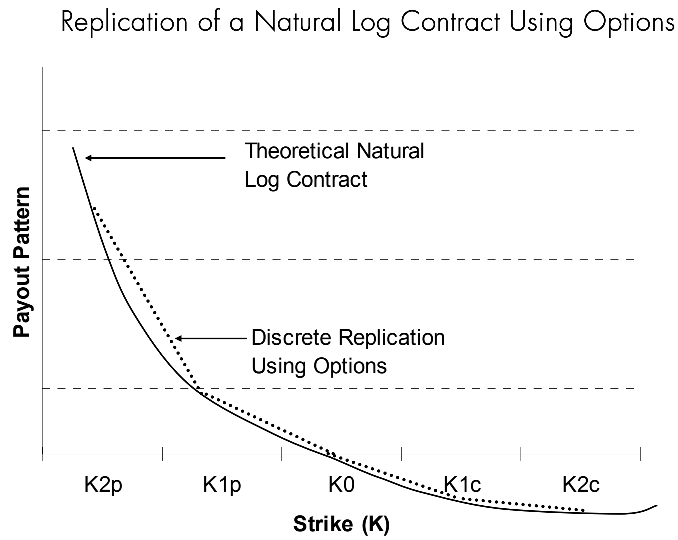
$$K_o = S_r < K_{1c} < K_{2c} < K_{3c} < .....$$

$$K_o = S_r > K_{1p} > K_{2p} > K_{3p} > .....$$

where K_{1c} is the first call strike and K_{1p} is the first put strike. Once the options are selected, the option hedge against a variance swap would be static until maturity, but we would still have to delta hedge the option positions, potentially adding to our costs.

Figure 19: Payout of Natural Log Contract and Illustration of the Replication of a Natural Log Contract using Options at Time = $T_{Expiration}$





Source: Lehman Brothers

In addition, we made some imperfect assumptions when we started with zero interest rates, zero dividend rate, no jumps, and constant implied volatility across strikes (without skew). We assumed that stock price evolution was derived from Geometric Brownian motion[20] (without a jump process), which is given below:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dz \qquad \text{(equation 21)}$$

where t represents each point in time with different values of μ and σ at time t. Since we replicate variance swap with options assuming a zero mean process, σ in equation 21 will also be calculated with a zero mean. Valuing a forward variance contract can be expressed as follows:

$$F_V = E[e^{-rT} * (V - K)] \qquad \text{(equation 22)}$$

With zero present value (r = 0 and F_v = 0), we can solve for the fair variance below.

$$K_{\text{variance}} = E[V]$$

[20] Hull, John, *Options, Futures and Other Derivatives*, 4th Edition, Prentice Hall, pp. 220–226.

Since log contracts (using the natural log function) do not trade, we can derive a relationship to duplicate the log or natural log payout with liquid options.

$$\ln\frac{S_T}{S_o} = \ln\left(\frac{S_T}{S_r}\right) + \ln\left(\frac{S_r}{S_o}\right) \qquad \text{(equation 23)}$$

where S_o is the original stock price, S_T is the stock price at expiration and S_r is the forward price boundary between liquid call and put strikes. Since $\ln(S_r/S_o)$ is a constant independent of the final stock price at time (T), we need only replicate $\ln(S_T/S_r)$ for the present value payout below:

Payout Log Contract = Forward Contract + Sum All Put Options + Sum All Call Options

$$-\ln\frac{S_T}{S_r} = -\left(\frac{S_T - S_r}{S_r}\right)$$

$$+\int_o^{S_r} \frac{1}{K^2} * Max(K - S_T, 0)dK$$

$$+\int_{S_r}^{\infty} \frac{1}{K^2} * Max(S_T - K, 0)dK \qquad \text{(equation 24)}$$

To replicate a log contract (natural log contract), we will need the following:

- Forward Contract: Short $(1/S_r)$ forward contract with initial price of S_r.
- Put Options: Long position in $1/K^2$ put options struck at K for all strikes from 0 to S_r.
- Call Options: Long position in $1/K^2$ call options struck at K for all strikes from S_r to ∞.

As shown in Figure 19 above, this replicates the log contract payout.

Calculating a Fair Value of a Variance Contract

Applying our analysis in equation 11, the fair value of future variance at any point in the life of the contract can be computed by the equation below:

$$K_{\text{var}iance} = -\frac{2}{T}\left(rT - \left(\frac{S_o}{S_r} e^{rT} - 1 \right) - \ln\left(\frac{S_r}{S_o} \right) \right.$$

$$+ e^{rT} \int_o^{S_r} \frac{1}{K^2} * BS_p(K)dK$$

$$\left. + e^{rT} \int_{S_r}^{\infty} \frac{1}{K^2} * BS_c(K)dK \right) \qquad \text{(equation 25)}$$

If we use the current market prices of the put and call options, we can obtain an estimate of the current market price of future variance.

where $BS_p(K)$ and $BS_c(K)$ are the value of calls and puts at strike K, respectively. If we use the current market prices of the put and call options (the bid or ask, depending on whether one is a buyer or seller of variance), we can obtain an estimate of the current market price of future variance. We inherently are using implied volatilities from the option market as the expectation of future realized volatilities. It makes a connection between the cost of option prices (using the bid/ask spreads) and how to capture future realized volatility, even if there is an implied volatility skew.[21]

Variance Swap Pricing Method 1: Academic Model of Variance Swap Pricing

We use a set of options to price forward variance for any index or stock with the skew inherently included in the option prices. This model uses equation 24 to calculate the forward plus the portfolio of puts and calls, to replicate the natural log contract payout. For this example, we compute fair variance from the expectation of equation below:

$$K_{\text{var}} = K_{\text{var}aince} = \frac{2}{T} E\left[\int_0^T \frac{dS_t}{S_t} - \ln\left(\frac{S_T}{S_o} \right) \right] \qquad \text{(equation 26)}$$

If $S_r = S_0$, we can then take the expected value and get:

$$K_{\text{var}} = \frac{2}{T} E\left[rT - \left(\frac{S_o}{S_r} * e^{rT} - 1 \right) - \ln\left(\frac{S_r}{S_o} \right) \right] + e^{rT} * F_{Calls\&Puts} \qquad \text{(equation 27)}$$

[21] The CBOE uses a similar methodology for the VIX and VXN indices.

From Appendix B, Equation B4 shows how to calculate the variance payout in the future below:

$$F(S_T) = \frac{2}{T} * \left[\frac{S_T - S_r}{S_r} - \ln\left(\frac{S_T}{S_r}\right) \right] \qquad \text{(equation 28)}$$

Therefore, from Appendix B using equations B5 and B6, we can derive below the calculations of the weights in the number of options for calls and puts using equation 28 above. We need to evaluate variance according to its future payout when we calculate it by multiplying by e^{rT}.

$$W_c(K_n) = \left[\frac{F(K_{n+1,c}) - F(K_{n,c})}{K_{n+1,c} - K_{n,c}} \right] * e^{rT} - \sum_{i=0}^{n-1} W_c(K_{i,c}) \qquad \text{(equation 29)}$$

$$W_p(K_n) = \left[\frac{F(K_{n+1,p}) - F(K_{n,p})}{K_{n,p} - K_{n+1,p}} \right] * e^{rT} - \sum_{i=0}^{n-1} W_p(K_{i,p}) \qquad \text{(equation 30)}$$

We assume that the forward spot at time T is equivalent to the strike to determine the number of contracts. The weight for calls $W_c(K_n)$ is the number of contracts required for a strike near or at S_r. For $W_c(K_{1c})$, we calculate the weight for K_{1c} and then subtract the weight of all former discrete weight calculation for each option from S_r to the highest call strike (until ∞) for the calls. Similarly, for the puts, we use the same methodology with a few adjustments and subtract the weight of all former discrete weight calculations for each option from the lowest puts strike (until 0) to S_r for the puts.

Figure 20 shows a theoretical payout profile of our variance swap.

Figure 20 shows a theoretical payout profile of our variance swap. We assume European calls and puts using a European Black/Scholes formula to value the options with an interest rate of 3.5% over expiration of six months and zero dividends. We assume a spot level of 100, and use the forward demarcation of liquid calls and puts as 101.7345 [where $S_r = 100/(1+r)^{-T} = 100/(1+3.5\%)^{-0.5}$]. We choose the lower strike below the forward level and, therefore, we choose the put and call options with a strike of 100 as our demarcation point. As noted above, when the index moves lower, the deep OTM put strikes are weighted by higher number contracts using the $1/K^2$ methodology, which allows us to capture variance to the downside at a constant rate.

Figure 20: Theoretical Variance Value of Weight Profile for Puts and Calls

Call or Put	Spot	Strike	Implied Volatility	Weight in Number of Contracts (W)	Value per Option (V)	Contribution (W*V)
Put	100	50	30.0%	81.80	0.0015624	0.128
Put	100	55	27.0%	67.54	0.0023893	0.161
Put	100	60	26.0%	56.72	0.0083829	0.475
Put	100	65	25.0%	48.30	0.0253740	1.226
Put	100	70	23.0%	41.63	0.0494248	2.058
Put	100	75	22.0%	36.25	0.1259065	4.564
Put	100	80	19.0%	31.85	0.1745721	5.561
Put	100	85	18.0%	28.21	0.4148454	11.703
Put	100	90	17.0%	25.16	0.9063452	22.802
Put	100	95	16.0%	22.58	1.8263325	41.232
Put	100	100	15.0%	17.47	3.3956530	59.307
Call	100	100	15.0%	2.91	5.1010156	14.828
Call	100	105	14.00%	18.48	2.6086921	48.199
Call	100	110	13.50%	16.83	1.1545118	19.434
Call	100	115	13.25%	15.40	0.4453905	6.859
Call	100	120	13.00%	14.14	0.1437481	2.033
Call	100	125	12.80%	13.03	0.0394392	0.514
Call	100	130	12.50%	12.05	0.0082941	0.100
Call	100	135	12.25%	11.17	0.0014416	0.016
Call	100	140	12.10%	10.39	0.0002274	0.002
Call	100	145	11.90%	9.68	0.0000274	0.000
					Total Variance Value	241.20

Source: Lehman Brothers

We include the following example of the calculation for the number of contracts on the 90 put strikes using our assumptions. For the 90 put strike, we can calculate the weight in the number of contracts below. Let us also assume that the forward S_r = $100/(1+3.5\%)^{-0.5}$ = 101.7345 and that there just happen to be strikes for puts and calls near the forward price at 100. We first calculate the payoff of F at the 90 and 85 strike levels, and then calculate the W_p at 90.

$$F_{3p}(90) = \frac{2}{0.5}\left[\frac{90-101.73}{101.73} - \ln\left(\frac{90}{101.73}\right)\right] = 0.0288519$$

$$F_{4p}(85) = \frac{2}{0.5}\left[\frac{85-101.73}{101.73} - \ln\left(\frac{85}{101.73}\right)\right] = 0.0608962$$

$$W_p(90) = \left[\frac{0.0608962 - 0.0288519}{90-85}\right] * \frac{1}{(1+0.035)^{-0.5}} - 0.00400421_{\sum W_p} = 0.00251586$$

where $\sum_{i=0}^{1} W_p(K_{i,p}) = W_p(100_{0,p}) + W_p(95_{1,p}) = 0.00400421_{\sum W_p}$.

Finally, we multiply by $(100)^2$ because the weights are in percentages and variance is squared in order to represent the number of contracts.

$$W_p(90) = 0.00251586 * (100)^2 = 25.1586 \text{ contracts}$$

One can clearly see that cost of capturing variance properly changes with different sets of discrete strikes, due to the inherent skew in the implied volatility levels with more options.

In the example in Figure 20 above, we used five point strike increments and calculated a total variance of $K_{var} = 241.20 = (15.53)^2$. However, if we used 2.5 point increments, total variance would have been reduced to $K_{var} = 235.17 = (15.33)^2$, Assuming linear skew, the use of smaller strike increments (especially for the puts) will generally lower calculated fair variance, and the difference will become greater for longer swap maturities. Thus, the cost of properly capturing variance is dependent on the set of discrete available strikes, which are determined by inherent option skew. While the use of smaller strike increments will produce a more accurate fair variance estimate, the larger number of total option contracts traded (to achieve smaller strike increments) may also result in higher transaction costs. The swap investor must therefore choose an optimal tradeoff between accurate fair variance replication and cost of implementation. Finally, we suggest using this methodology to create and bid and offer for a variance contract, using the bid and offer prices for the actual options shown in Figure 20.

Effect of Skew on Variance Swap

We assume that implied volatility varies linearly with the strike, but in reality, Figure 20 above shows that it is likely not the case. The formula for skew that varies linearly with strike is shown below. We typically use such a formula to interpolate skew.

$$\sigma_{imp}(K) = \sigma_{Fo} - c * \frac{K - S_F}{S_F} \qquad \text{(equation 31)}$$

where σ_{Fo} is the implied volatility at a strike going forward and S_F is the forward price level of the index or stock. The variable *c* is the slope or the change in skew. However, at some point, the implied volatility would become negative within the formula above. Therefore, we can derive a non-linear relationship where the fair variance is:

$$K_{\text{var}} \approx (\sigma_{Fo})^2 * (1 + 3Tc^2 + ...) \qquad \text{(equation 32)}$$

As time (T) increases, the fair variance above the ATM forward level, $(\sigma_{Fo})^2$, increases, because skew has more effect in the calculation of fair variance over longer maturities than shorter maturities.

An added complication is that skew varies linearly with the Black/Scholes delta of an option. Therefore, as implied volatility increases for a put, the delta of the option decreases because the likelihood of the option being in- or out-of-the-money on expiration is more uncertain and, therefore, requires a lower delta to hedge. Similarly, for a calculation of fair variance, we can compute the relationship below, assuming a non-linear skew.

$$K_{\text{var}} \approx (\sigma_{Fo})^2 * \left(1 + \frac{1}{\sqrt{\pi}} c\sqrt{T} + \frac{1}{12}\frac{c^2}{\sigma_{Fo}^2} + ...\right) \qquad \text{(equation 33)}$$

The first-order correction has a magnitude of $c\sqrt{T}$. The variance of the ATM forward, $(\sigma_{Fo})^2$, is no longer equivalent to a variation linear in strike.

Imperfect Replication Due to Strike Range and Stock or Index Jumps

We have shown that a variance swap has the form of a log contract (natural log contract) that can be replicated with options, so that the effects of varying the underlying price of the stock or level of the index are removed. The two biggest problems with this replication are the following:

There is a limited range of strikes available to replicate the log or natural log payoff.

- There is a limited range of strikes available to replicate the log or natural log payoff. Log contracts (natural log contracts) do not trade, and using a limited number of strikes may fail to capture the true variance. In fact, the discrete replication of a continuous payout using options can results in points above, below, or on the log contract curve, as shown in Figure 19 above. To adjust for the lack of strikes that are available, traders may increase the number of contracts on the last few OTM puts and calls that are being used, to ensure they get the proper volatility exposure for large downside moves.

Stock prices and index levels can jump in one large move.

- Stock prices and index levels can jump in one large move, which often occurs during broad market crises. Disaster scenarios can also occur for single stocks based on stock-specific news or events. Again, buying more OTM puts and calls can assist in hedging out this effect.

Imperfect Replication Due to Limited Strike Range (for listed options)

The limited range of listed option strikes will affect an investor's ability to accurately estimate future realized variance using implied volatility. The difference between estimated and actual future realized volatility is magnified as swap maturity increases.

The limited range of available listed option strikes will affect an investor's ability to accurately estimate future realized variance using implied volatility. Additionally, the difference between estimated and actual future realized variance is magnified as swap maturity increases. Let us return to our example in Figure 20: If we limit ourselves to 80%–120% strikes for replication, and compare the estimate to another obtained using 50%–145% strikes, we find that the difference in variance is not that great for short-term contracts with a maturity of one month or three months. However, as shown in Figure 21, the effects of using fewer strikes to estimate future realized variance are clearer for six-month and one-year expirations. The approximation produced with a smaller 80%–120% strike range will tend to underestimate realized variance compared to an estimate produced with a broader 50%–145% strike range (assuming interest rates, spot levels, strikes, and implied volatilities are fixed across maturities, and dividends are zero). The limitation of using a smaller strike range is magnified over longer maturities because the stock price or index level is more likely to move in a price path outside of the strike range over a longer period. Therefore, a variance contract priced from fewer options would probably result in a lower strike variance level, due to the underestimation of realized variance that can occur.

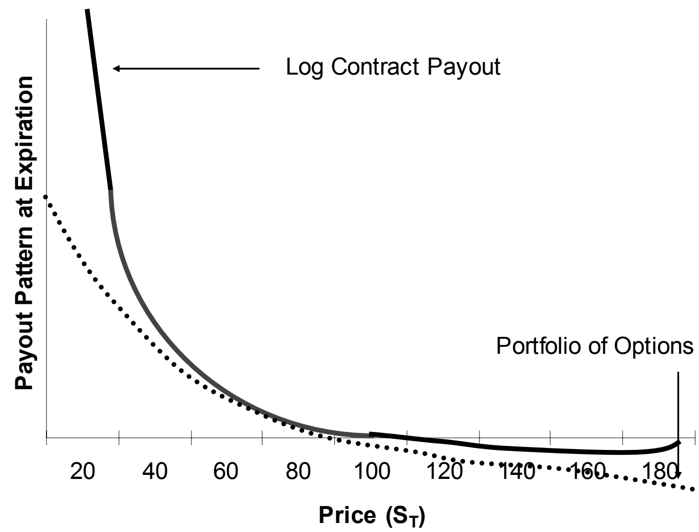
Figure 21: Fair Variance Estimates with Different Strike Range

Expiration	$K_{variance}$ using Strikes 50% to 145%	$K_{variance}$ using Strikes 80% to 120%
1-mth	$(16.58025)^2$	$(16.58023)^2$
3-mth	$(15.6071)^2$	$(15.5757)^2$
6-mth	$(15.5307)^2$	$(15.2302)^2$
1-yr	$(16.1535)^2$	$(14.9698)^2$

Source: Lehman Brothers

However, if the stock price path moves in a range outside of the strike region (where the curvature of the portfolio is not large enough), then the investor will capture less of the realized variance. Capturing variance is a continuous process; therefore, one must maintain the curvature of the log contract (natural log contract) at each stock price at any value. Again, per equation 9 and Figure 19, we are using options to construct an expiration payoff that does not perfectly match the log contract (natural log contract) payoff. Such a payout pattern at expiration is shown in Figure 22 below, using our example of 80%–120% options for our six-month case.

Figure 22: Example of Perfect Natural Log Contract Payout Pattern versus Payout of Imperfect Portfolio of Options (80%–120%)



Source: Lehman Brothers

Effects of Stock Price or Index Level Jumps

When a stock price or index level jumps, the log (natural log) contract may not accurately capture realized volatility for two major reasons:

- Since the log (natural log) contract is replicated using a finite number of strikes, a large movement may take the stock or index into a range where the variance vega does not move at a constant rate.
- If we assume perfect strike-based replication, a large continuous stock price or index level jump will not perfectly capture an index payout pattern.

Appendix C contains the derivation of the jump process. For equation 34, using a Taylor series expansion, we can reduce the jump process to a cubic term and higher order terms, of which the cubic generates the most P/L movement. The contribution to error from the jump process is mostly from the cubic term and higher order terms:

$$P\&L_{jump} = \frac{2}{3}\frac{J^3}{T} + ... \qquad \text{(equation 34)}$$

For a short variance swap, a large move down leads to a profit because J > 0 but a large move up leads to a loss because J < 0. A large move up one day and a large move down the next day will offset each other.

The cubic term in the jump process has different signs for upward and downward jumps. For a short variance swap, a large move down leads to a profit because J > 0 but a large move up leads to a loss because J < 0. A large move up one day and a large move down the next day will offset each other. However, if the skew shifts, then the P/L effect may not always be zero, given the limited number of strikes and options used to replicate the log contract (natural log contract).

Variance Swap Pricing Method 2: The Lehman Brothers Variance Swap Replication Method for Upper and Lower Bound

Alternatively, Lehman Brothers had developed a slightly different methodology than the academic model described above.[22] We begin with a review of the basic principles behind variance swaps. If volatility is diffused or continuous (assuming no jumps), stock price evolution is the following using Brownian Motion:

$$\frac{dS_t}{S_t} = u(t, S, ...)dt + \sigma(t, S, ...)dZ_t \qquad \text{(equation 35)}$$

We assume that the drift factor (μ) and continuously sampled volatility (σ) are arbitrary across time, stock prices, and other variables. Again, σ is a determined assuming a zero mean process. Implied volatility tree models using time and other factors can be used to develop the future path of volatility or variance. The realized variance is path dependent based on several variables and, therefore, can be expressed below:

$$V = \frac{1}{T} * \int_0^T \sigma^2 (t, S, ...) dt$$

Fair variance can be expressed by the risk-neutral expectation of future variance:

$$K_{\text{var}iance} = \frac{1}{T} E\left[\int_0^T \sigma^2 (t, S, ...) * dt \right]$$

Replicating a variance contract is therefore difficult because one needs to express the expectation of future variance. To obtain a payout that is incremental to the variance over time for the stock, we use Ito's Lemma to explain S_t and then can show the following:

$$d(\ln(S_t)) = \left(u(t, S, ...) - \frac{1}{2} * \sigma^2 (t, S, ...) \right) dt + \sigma(t, S, ...) * dZ_t \qquad \text{(equation 36)}$$

[22] Lehman Brothers' methodology was developed by Abdelkerim Karim (Ph.D.) of our Quantitative Analytics Team in London.

We can express the difference between equation 35 and 36 as the following:

$$\frac{dS_t}{S_t} - d\left(\ln(S_t)\right) = -\frac{1}{2} * \sigma^2\left(t, S, ...\right)dt \quad \text{(equation 37)}$$

The drift term (μ) is removed in equation 37, along with the noise. If we sum the discrete time intervals from 0 to T, we can create a continuous form for the variance of equation 38 expressed below:

$$V = \frac{1}{T} * \int_0^T \sigma^2\left(t, S, ...\right)dt = \frac{2}{T} * E\left[\int_0^T \frac{dS_t}{S_t} - \ln\left(\frac{S_T}{S_0}\right)\right] \quad \text{(equation 38)}$$

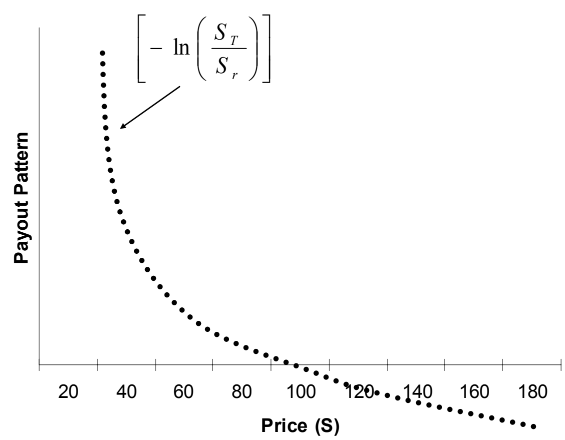
The $E\left[\int_0^T \frac{dS_t}{S_t}\right]$ term reduces to a risk-neutral constant identified as r*T. Therefore, from equation 38 and equation 8, we can express the expected value of the variance as:

$$V = -\frac{2}{T} * E\left[\ln\left(\frac{S_T}{S_r}\right)\right] \quad \text{(equation 39)}$$

where S_r is the forward price of the index or stock at time T. .

Rather than solving for this equation, we suggest a replication strategy for the log contract (natural log contract) itself using a different weighting scheme. Unlike the academic method outlined above, the Lehman Brothers upper and lower bound model directly replicates the payout of the forward, along with the calls and the puts, by creating a minimum point estimation of the log contract (natural log contract).

Figure 23: Payout Pattern of Log Contract

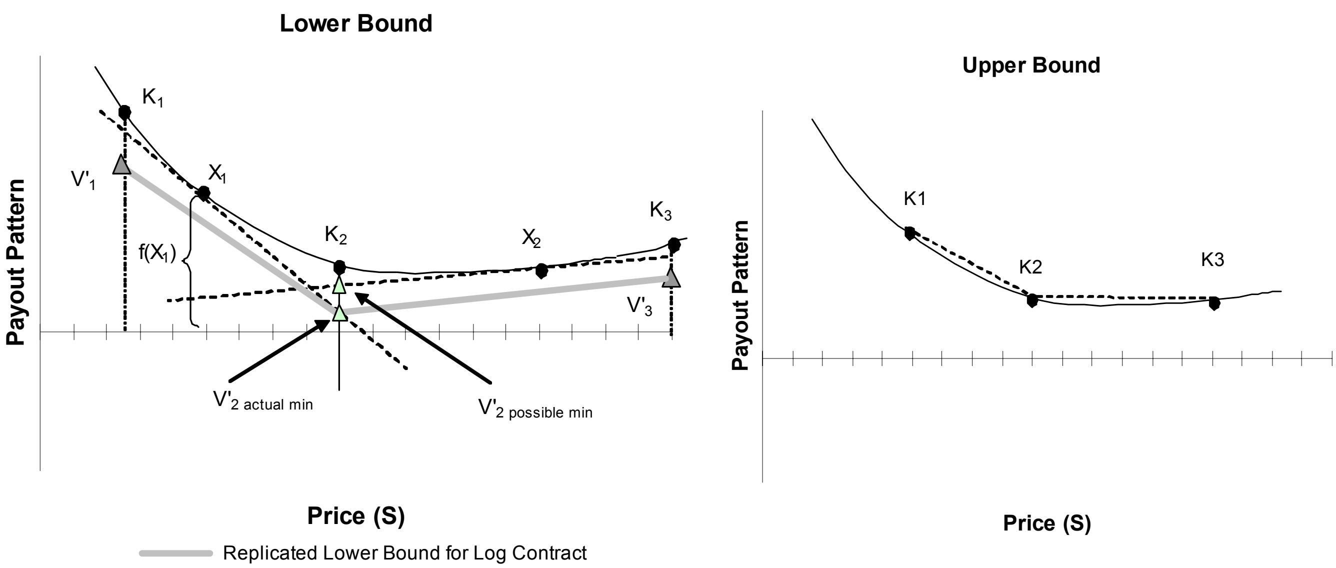


Source: Lehman Brothers

Figure 24 below shows how we would replicate the contract and always arrive at a value equal to or lower than the log contract (natural log contract) for a lower bound. We calculate the upper bound using a method similar to the academic method (Method 1).

Figure 23 above shows our replication objective. Figure 24 shows how we would replicate the contract and always arrive at a value equal to or lower than the log contract (natural log contract) for a lower bound. We calculate the upper bound using the same method. The value of the upper bound is generally closer to the academic method, but unlike the academic method, it will only allow points at or above the contract curve. In addition, Figure 25 below shows how we actually replicate the upper and lower bound for the log contract (natural log contract) in order to get an upper and lower bound of variance. We then use these upper and lower bounds to construct put and call spreads to estimate the payout of the combined log contract (natural log contract), consisting of the forward plus calls and puts.

Figure 24: Lehman Brothers Contract Replication Methodology for the Lower and Upper Bounds

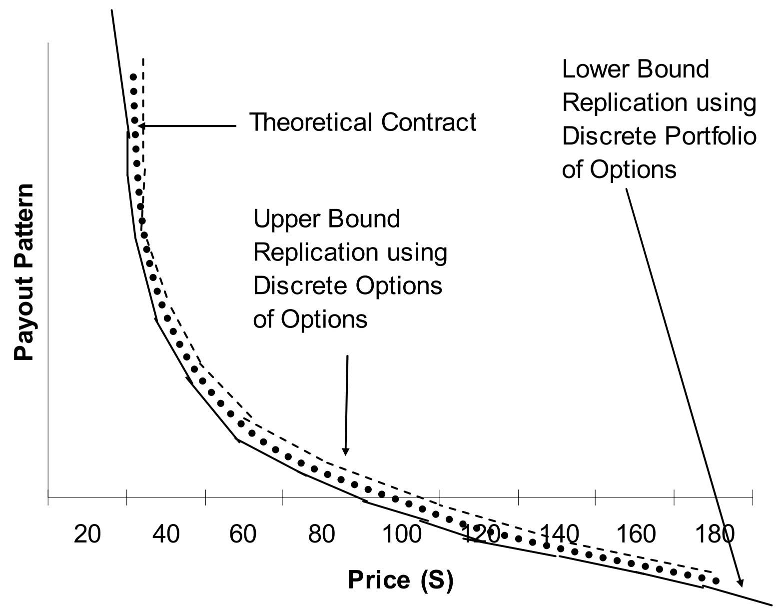


Source: Lehman Brothers

This methodology establishes a lower bound that never gives a value greater than the theoretical log contract (natural log contract) at any discrete strike, unlike the academic model.

This methodology can ideally be used to calculate a lower bound using the bid price and an upper bound using offer prices to get a range of volatility represented by the options that trade in the market. As shown in Figure 24, we can derive the value of $V'_{2possiblemin}$ and $V'_{2actualmin}$ and then choose the actual minimum between value of either $V'_{2possiblemin}$ and $V'_{2actualmin}$ as our new log contract (natural log contract) point. This process will establish a unique value associated with the points that are near the strikes and create a new natural log contract that is close to the old using the options listed in the market. In addition, this methodology establishes a lower bound that never gives a value greater than the theoretical log contract (natural log contract) at any discrete strike, unlike the academic model, which can result in points above, below, or at the log contract (natural log contract) curve. We then connect each minimum point for the lower bound and each discrete point for the upper bound, for a new estimated log contract (natural log contract), as shown in Figure 25 below. For the last put and call strikes, we must choose an OTM contract to create a linear payout beyond a selected +/− percentage from the spot as the disaster scenario, which will affect our calculation of fair variance in some way.

Figure 25: Lehman Brothers Replication Methodology Using an Upper and Lower Bound



Source: Lehman Brothers

Lower Bound Method

We use the following formulas to replicate the log contract (natural log contract) curve for all strikes (i) except the one near the outer bound:

$$x_i = \frac{(K_{i+1} - K_i)}{\ln\left(\frac{K_{i+1}}{K_i}\right)} \qquad \text{(equation 40)}$$

$$f(x_i) = -\ln\left(\frac{x_i}{S_r}\right) \qquad \text{(equation 41)}$$

The slope at $f(x_i)$ is derived below:

$$\frac{\partial f(x_i)}{\partial x_i} = \frac{\partial}{\partial x_i}\left[-\ln\left(\frac{x_i}{S_r}\right)\right] = \frac{1}{x_i}$$

The slope at $f(x_i)$ can be used to calculate the intercept points of the tangent line at that point with any K_i.

$$V_{i_1} = f(x_i) - \left[\frac{K_i}{x_i} - 1 \right] \quad \text{(equation 42)}$$

$$V_{i_2} = f(x_{i-1}) - \left[\frac{K_i}{x_{i-1}} - 1 \right] \quad \text{(equation 43)}$$

$$V_i^{'} = Min(V_{i_1}, V_{i_2}) \quad \text{(equation 44)}$$

where V_{i_1} and V_{i_2} are either the possible minimum or the actual minimum. There are three modifications to the formulas above for specific cases:

- Adjustments will have to be made to the lowest put and the highest call strike by selecting a deep OTM put or call beyond the strike required range. We utilize the same equation for x_i above, but modify the calculation of V' to calculate the starting minimum points based on the selection of a deep OTM strike for one call and put. If the strike range is chosen from the lowest selected OTM put strike (NP) to the highest OTM call strike (NC), then the strike range is NP < K_i < NC. For the two furthest OTM options (one for the call and one for the put), the V' from a selected way OTM strike can be calculated directly from the formulas below for the call and put.

$$V_{NP}^{'} = f(NP) - \left[\frac{K_{NP}}{x_{NP}} - 1 \right] \quad \text{(equation 45)}$$

$$V_{NC}^{'} = f(NC) - \left[\frac{K_{NC}}{x_{NC}} - 1 \right] \quad \text{(equation 46)}$$

- Finally, we can calculate the weights and implicitly determine the forward when calculating the quantity of options at each strike below, using put and call spreads. Since we are calculating the forward variance, we need to divide by $(1+r)^{-T}$, which is (assuming no dividends) the discrete forward of variance (instead of multiplying by the continuous form using e^{rT}). Again, we need to calculate the values for the furthest OTM options using a modification to the method below, but what we are calculating is the proper quantity of put and call spreads to replicate the payout at and between each strike:

$$QP_i = \left[\frac{\left(\frac{(V_{i-1} - V_i)}{K_i - K_{i-1}} \right)}{(1+r)^{-T}} - QP_{i-1} \right] * \frac{2}{T} \quad \text{for i = NP to Call/Put Start Strike (equation 47)}$$

$$QC_i = \left[\frac{\left(\frac{(V_{i-1} - V_i)}{K_i - K_{i-1}} \right)}{(1+r)^{-T}} - QC_{i-1} \right] * \frac{2}{T} \text{ for i = Call/Put Start Strike to NC (equation 48)}$$

- For the put and call strike closest to the ATM forward level, we do not need to subtract the prior put or call QP_{i-1} or QC_{i-1}.

We then multiply the weights by the option price and 10,000 (variance) to get the contribution of each option to the variance (similar to the academic model) and then sum that contribution across all strikes. We then add back in the constant of the forward to the variance summation, which is the following (assuming zero dividends):

$$V_o^{adjust} = \frac{2}{T} * V_0^{'} * 10{,}000 \qquad \text{(equation 49)}$$

Upper Bound Method

For the upper bound, we calculate the log (natural log) contract curve using a similar method to the academic model, as shown in equation 50. In addition, we formulate the quantities for QP_i and QC_i. This upper bound result should be close or slightly more than the academic model output.

$$V_i^{'} = -\ln\left(\frac{K_i}{S_r} \right) \qquad \text{(equation 50)}$$

Again, we must add back the constant of the forward to the variance summation.

Comparison of Methods

Similar to Figure 20 for our academic model, Figure 26 below shows a theoretical payout profile of a variance swap using the Lehman Brothers method to calculate an upper and lower bound. We value European calls and puts using a Black/Scholes formula with an interest rate of 3.5%, expiration of six months and zero dividends. We assume a spot level of 100 and use the forward spot level of S_r = 100*exp(3.5%*T) = 100*/$(1+r)^{-T}$ without dividends. The $1/K^2$ strike weighting methodology will effectively overweight OTM puts (by increasing the number of contracts), which ensures roughly constant variance exposure for steep downside index moves Finally, we can calculate the lower bound as only K_{var} = 233.69 = $(15.28)^2$ for our six-month variance swap example.

Figure 26: Theoretical Variance Value of Put/Call Weight Profile Using Lehman Brothers Lower Bound Methodology

Call or Put	Spot	Strike	Implied Volatility	Weight in Number of Contracts (W)	Value per Option (V)	Contribution (W*V)
Put	100	50	30.0%	83.85	0.0015624	0.13
Put	100	55	27.0%	67.03	0.0023893	0.16
Put	100	60	26.0%	56.36	0.0083829	0.47
Put	100	65	25.0%	48.05	0.0253740	1.22
Put	100	70	23.0%	41.44	0.0494248	2.05
Put	100	75	22.0%	36.11	0.1259065	4.55
Put	100	80	19.0%	31.75	0.1745721	5.54
Put	100	85	18.0%	28.13	0.4148454	11.67
Put	100	90	17.0%	25.09	0.9063452	22.74
Put	100	95	16.0%	22.52	1.8263325	41.14
Put	100	100	15.0%	417.17	3.3956530	1,416.56
Call	100	100	15.0%	-396.84	5.1010156	(2,024.28)
Call	100	105	14.00%	18.44	2.6086921	48.11
Call	100	110	13.50%	16.80	1.1545118	19.40
Call	100	115	13.25%	15.38	0.4453905	6.85
Call	100	120	13.00%	14.12	0.1437481	2.03
Call	100	125	12.80%	13.02	0.0394392	0.51
Call	100	130	12.50%	12.03	0.0082941	0.10
Call	100	135	12.25%	11.16	0.0014416	0.02
Call	100	140	12.10%	10.17	0.0002274	0.00
Call	100	145	11.90%	9.33	0.0000274	0.00
					Forward Adustment	674.72
					Total Variance Value	233.69

Source: Lehman Brothers

As one can see, our method more closely represents the variance based on the options that can be used to replicate a log contract.

The synthetic forward position appearing at strike 100 is offset by the sensitivity of the *Forward Adjustment* to the spot price. To hedge the *Forward Adjustment* we would net the put and call positions and only buy the residual puts. The weight for 100 strike puts can be adjusted to buying (414.17- 396.84) or approximately 20.33 puts and no calls.

Why do the ATM calls have a negative contract value? First, we should review some basic concepts around the premium associated with a piece-wise linear function. If the function is positive (or negative), we will have to add (or subtract) a put spread or a call spread. But when the function crosses the zero axis (at 100 in this example), we must be careful.

Let us illustrate using the example above in Figure 26: The minimum value (V') that we use changed from positive at 100 to negative at 105. Therefore, when we use equation 48 to calculate the value spread, we will get a negative quantity. If, at a price of 100, the function V'_0 is worth 0.01716, but at 105, it is worth -0.03244, this interval will have a spread of:

$$-\frac{(0.0176-(-0.03244))}{(105-100)}*\frac{2}{0.5}*10{,}000=-396.84$$

This interval's contribution will be the following:

$$0.0176-\frac{(0.0176-(-0.03244))}{(105-100)}*\frac{2}{0.5}*\left(5.101_{BS_{call}^{at100}}-2.609_{BS_{call}^{at105}}\right)=-0.08174$$

Figure 27 shows a comparison of all the variance swap pricing methods presented thus far in this report.

Figure 27: Comparison of Two Methods to Calculate Fair Variance Across Different Maturities

Expiration	Method 1: Academic Model $K_{variance}$ using Strikes 50% to 145%	Method 2: Lehman Upper Bound $K_{variance}$ using Strikes 50% to 145%	Method 2: Lehman Lower Bound $K_{variance}$ using Strikes 50% to 145%
1-mth	$(16.58)^2$	$(16.61)^2$	$(14.03)^2$
3-mth	$(15.61)^2$	$(15.70)^2$	$(14.84)^2$
6-mth	$(15.53)^2$	$(15.74)^2$	$(15.28)^2$
1-yr	$(16.15)^2$	$(16.21)^2$	$(15.96)^2$

Source: Lehman Brothers

The advantages of the Lehman Brothers Upper and Lower Bound Model (compared to the academic model discussed previously). are the following:

For the Lehman Upper and Lower Bound Model, the upper bound replication is always at or above the theoretical contract, and the lower bound is always at or below the theoretical log contract (natural log contract).

- It calculates an upper and lower bound of variance by expanding on the replication methodology of the academic model. As shown in Figure 25 above, upper bound replication is always at or above the theoretical contract value, and the lower bound is always at or below the theoretical log contract (natural log contract). This methodology establishes a lower bound that never gives a value greater than the theoretical contract value, unlike the academic model, which can result in points above, below, or on the log contract (natural log contract) curve. In addition, the upper bound never provides a value below the log contract (natural log contract) curve; therefore, the variance value from the upper bound model tends to be near or slightly higher than the academic model.

- It can be expanded to create a variance swap bid price using the lower bound method with the option bid prices, and an offer price using the upper bound method with option offer prices.

- This method establishes a unique value associated with the points that are near or at the strikes for the lower and upper bound.

Variance Swap Pricing Method 3: The Lehman Brothers Volatility Surface Method and the Gram-Charlier Expansion

This method is popular with the trading desk because it uses the current volatility surface across strikes for a specific maturity to solve for the fair variance ($K_{variance}$).

We examine a third method to calculate the fair variances using the Lehman Brothers volatility surface method combined with the Gram-Charlier expansion (also known as the Edgeworth expansion introduced by others).[23] Gram-Charlier is essentially used to solve a non-linear set of equations with a quadratic variable. This method is popular with the trading desk because it uses the current volatility surface across strikes for a specific maturity to solve for the fair variance ($K_{variance}$). We devised this methodology as part of a method to calculate the volatility surface and compare it to the volatility surface of a basket of options correlation adjusted at Lehman Brothers.[24] If S_t is the stock price at the time to expiration, then the following can be the potential price of the asset:

$$z = \frac{\ln\left(\frac{S_T}{S_t}\right) - (\mu - \frac{\sigma^2}{2}) * (T - t)}{\sigma * \sqrt{T - t}} \qquad \text{(equation 51)}$$

The risk-neutral probability density function for z is expressed as:

$$f(z) = \left(1 + \frac{s}{3!}\sigma^3 + \frac{k}{4!}\sigma^4\right) * n(z) \qquad \text{(equation 52)}$$

where d = the partial derivative with respect to z, u = the carry rate, σ = the annualized standard deviation of the return, s = skewness, k = excess kurtosis, and n(z) is the standard normal. We define the standard normal, skew, and kurtosis below[25]:

$$n(z) = \frac{1}{\sqrt{2\pi}} * e^{-z^2/2}$$

We can then derive the following relationship where volatility (or variance) is a function of x to calculate the volatility across the entire volatility surface:

$$\sigma(x) = \sigma_{ATM} + Skew_{solve} * x + Smile_{solve} * x^2 \qquad \text{(equation 53)}$$

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[23] Lee, Peter; Wang, Limin; and Karim, Abdelkerim. "Index Volatility Surface via Moment Matching Techniques", *Risk Magazine*, December 2003. Jarrow, R. and Rudd, A., "Approximate Option Valuations for Arbitrary Stochastic Processes", *Journal of Financial Economics*, 1982, vol 10, pp. 342–369.

[24] This method was published in *Risk Magazine* in December 2003. Abdelkerim Karim (Ph.D.), Peter Lee (Ph.D.), and Limin Wang (Ph.D.) of Lehman Brothers developed this methodology.

[25] Press, William; Flannery, Brian; Teukolsky, Saul; and Vetterling, William, *Numerical Recipes in C: The Art of Scientific Computing*, Cambridge University Press, 1991, pp.474–475. Skew is normally defined as $s = skew = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{x_i - \mu_x}{\sigma}\right]^3$ and kurtosis is defined as $k = kurtosis = \left\{\frac{1}{N}\sum_{i=1}^{N}\left[\frac{x_i - \mu}{\sigma}\right]^4\right\} - 3$ .for each x where the mean is $u = \frac{1}{N}\sum_{i=1}^{N}\frac{x_i}{N}$.

and then show the following:

$$x_i = \ln\left[\frac{S_r}{K_i}\right] \text{ for all i, representing different strikes} \quad \text{(equation 54)}$$

$$\sigma_{ATMSolve} = \sigma * \left[1 + \frac{s}{3!} * \frac{3}{2} * \sigma * \sqrt{T-t} + \frac{k}{4!} * \left(\frac{7}{4} * \sigma^2 * (T-t) - 1\right)\right]$$

(equation 55)

$$\xi = Skew_{solve} = \frac{1}{\sqrt{T-t}} * \left(\frac{s}{3!} + \frac{2k\sigma\sqrt{T-t}}{4!}\right) \quad \text{(equation 56)}$$

$$\eta = Smile_{solve} = \frac{1}{T-t} * \left(\frac{k}{4! * \sigma}\right) \quad \text{(equation 57)}$$

where S_r is the forward price level, T = time to expiration, and the t = current time days into the total days of maturity T.

We can observe and calculate skew, kurtosis, and smile from any volatility surface (ξ, η, $\sigma_{ATMsolve}$) across any maturity, and then use these three equations (equations 55–57) and three unknowns above in order to solve for σ as well as S and K. We can use these to calculate the fair variance, skew, and kurtosis below:

$$FairVariance_{volsurface} = K_{variance} = \sigma^2 \quad \text{(equation 58)}$$

$$FairSkew_{volsurface} = s \quad \text{(equation 59)}$$

$$FairKurtosis_{volsurface} = k \quad \text{(equation 60)}$$

However, for each strike, we need to calculate the skew, smile, and implied volatility. To infer and solve for $skew_{solve}$, $\sigma_{ATMsolve}$ and $smile_{solve}$, we need to minimize the system of equations to calculate the following for each strike (i):

$$x_i = \ln\left[\frac{S_r}{K_i}\right] \text{ for all i, representing different strikes}$$

$$\sigma_i(x_i) = \sigma_{ATMsolve} + \xi * x_i + \eta * x_i^2 \quad \text{across each strike} \quad \text{(equation 61)}$$

where for the entire volatility surface, ξ is the $skew_{solve}$, η is the $smile_{solve}$ and $\sigma_{ATMsolve}$ is the ATM implied volatility. We then can use these variables in equations 55–57 for the Gram-Charlier expansion to solve for implied volatility, skew, and smile (s, k, and σ) of the volatility surface. First, we need to solve the system of equations below to get the ξ, η, and $\sigma_{ATMsolve}$ using all the existing options at different strikes:

$$\chi^2 = \sum_{i=1}^{N}\left(\sigma_{ATMsolve} + \xi * x_i + \eta * x_i^2 - \sigma_i^2\right)^2 \quad \text{(equation 62)}$$

$$\frac{\partial \chi^2}{\partial \sigma_{ATMsaolve}} => N * \sigma_{ATMsolve} + \xi * \sum_{i=1}^{N} x_i + \eta * \sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} \sigma_i^2 = 0 \quad \text{(equation 63)}$$

$$\frac{\partial \chi^2}{\partial \xi} => \sigma_{ATMsolve} * \sum_{i=1}^{N} x_i + \xi * \sum_{i=1}^{N} x_i^2 + \eta * \sum_{i=1}^{N} x_i^3 - \sum_{i=1}^{N}\left(x_i * \sigma_i\right) = 0$$

(equation 64)

$$\frac{\partial \chi^2}{\partial \eta} => \sigma_{ATMsolve} * \sum_{i=1}^{N} x_i^2 + \xi * \sum_{i=1}^{N} x_i^2 + \eta * \sum_{i=1}^{N} x_i^4 - \sum_{i=1}^{N}\left(x_i^2 * \sigma_i\right) = 0$$

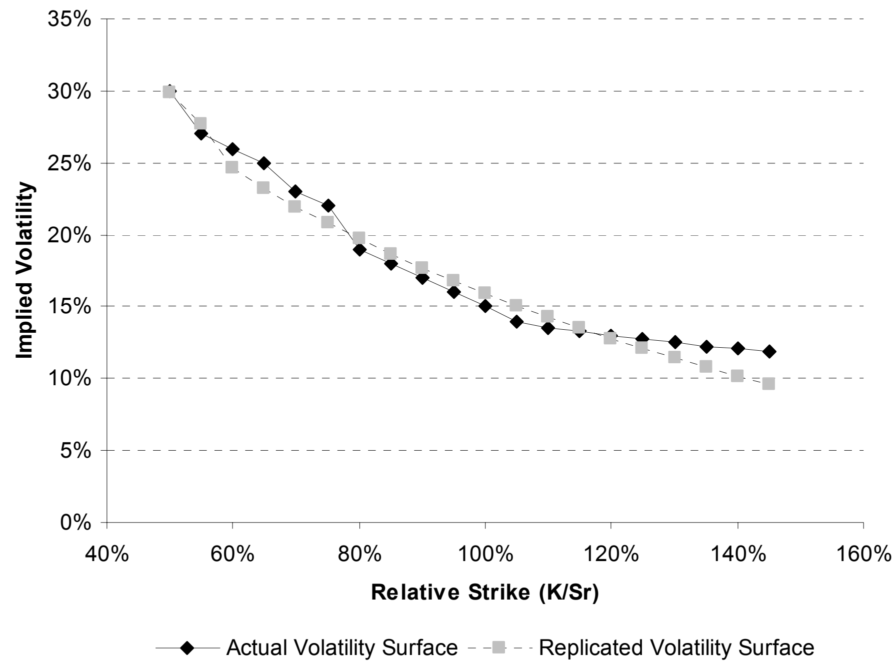
(equation 65)

We then can solve the following simultaneous set of equations using linear algebra where σ_i is the implied volatility obtained for each option's strike, using the Black/Scholes model:

$$\begin{pmatrix} N.......\sum_{i=1}^{N} x_i ... \sum_{i=1}^{N} x_i^2 \\ \sum_{i=1}^{N} x_i ... \sum_{i=1}^{N} x_i^2 ... \sum_{i=1}^{N} x_i^3 \\ \sum_{i=1}^{N} x_i^2 ... \sum_{i=1}^{N} x_i^3 ... \sum_{i=1}^{N} x_i^4 \end{pmatrix} * \begin{pmatrix} \sigma_{ATMsolve} \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N} \sigma_i \\ \sum_{i=1}^{N}\left(x_i * \sigma_i\right) \\ \sum_{i=1}^{N}\left(x_i^2 * \sigma_i\right) \end{pmatrix} \quad \text{(equation 66)}$$

We can then calculate ξ which is the skew_{solve}, η which is the smile_{solve} and $\sigma_{ATMsolve}$. We then can use these variables to solve for the Fair Variance = $K_{variance}$ using equations 55–57. Figure 28 shows how a parabolic fit can be generated to replicate an estimated volatility surface, VS(ξ, η and $\sigma_{ATMsolve}$), and compare it to the actual volatility surface used in our six-month example, assuming the same inputs.

Figure 28: Volatility Surface Replication versus the Actual Volatility Surface Using the Options from Our Six-Month Variance Swap Example



Source: Lehman Brothers

Figure 29 shows a comparison of the fair variance calculations using the three different methods, assuming interest rates, spot levels, strikes, and implied volatilities are fixed across maturities (and zero dividends).

Figure 29 shows a comparison of the fair variance calculations using the three different methods, assuming interest rates, spot levels, strikes, and implied volatilities are fixed across maturities (and zero dividends). We adjust the academic model by using a forward for the calculation of the log contract (natural log contract) payout, and we use the forward for the Lehman Brothers upper and lower bound method so that $S_r = 100/(1+3.5\%)^{-T}$. As we can see in Figure 29, the differences begin to widen out as the maturity increases. For the one-month case, the differences are greater, again because the one-month maturity overweights the near-term strikes in Method 1. However, Gram-Charlier reflects the volatility surface level of fair variance, rather than choosing strikes to overweight or underweight, as with the academic or Lehman Brothers upper and lower bound models.

Figure 29: Fair Variance Estimate Comparison

Expiration	Method 1: Academic Model $K_{variance}$ using Strikes 50% to 145%	Method 2: Lehman Upper Bound $K_{variance}$ using Strikes 50% to 145%	Method 2: Lehman Lower Bound $K_{variance}$ using Strikes 50% to 145%	Method 3: Lehman Volatility Surface and Gram-Charlier Expansion $K_{variance}$ using Strikes 50% to 145%
1-mth	$(16.58)^2$	$(16.61)^2$	$(14.03)^2$	$(15.67)^2$
3-mth	$(15.61)^2$	$(15.70)^2$	$(14.84)^2$	$(15.75)^2$
6-mth	$(15.53)^2$	$(15.74)^2$	$(15.28)^2$	$(15.91)^2$
1-yr	$(16.15)^2$	$(16.21)^2$	$(15.96)^2$	$(16.44)^2$

Source: Lehman Brothers

The main disadvantage is that many of the options it uses to create the volatility surface may have very low liquidity and may not be tradable.

In addition, we can create a bid and an offer variance using the bid and offer of option prices to calculate separate implied volatility surfaces. However, the main disadvantage is that many of the options it uses to create the volatility surface may have very low liquidity and may not be tradable. The advantages of the volatility surface and Gram-Charlier expansion method, compared to the Lehman Brothers method and the academic method, are as follows:

This method provides a comparison actual fair variance (implied by the volatility surface in the market) to any closed form or estimated replication of the variance log contract (natural log contract).

- This method provides a comparison actual fair variance (implied by the volatility surface in the market) to any closed form or estimated replication of the variance log contract (natural log contract).
- Although individual option weights are not calculated in this method, one can use this volatility surface model to calculate the option contract weights, to better reflect the volatility surface in the marketplace.
- With this method, we can easily use many more option strikes (deep OTM strikes for calls and puts). We avoid the effect of truncating the strikes in calculating the fair variance across a limited set of options (which is found in the academic model and the Lehman Brothers upper and lower bound model).
- This method reflects the fair variance of the volatility surface. This comparison is valuable in shorter-term maturities that may overestimate implied volatility based on the tradable strikes, rather than what is implied by the volatility surface. We see this for the one-month maturity in Figure 29, where the differences are greater because the one-month case overweights the near-term ATM strikes compared to the academic model.

Marking and Unwinding a Variance Swap

Prior to the expiration of a variance swap, how does one mark the position?

Prior to the expiration of a variance swap, how does one mark the position? Let us go back to equation 1 to calculate the realized variance (σ^2_{ann}), and to equations 28, 29, and 30 to calculate the fair variance remaining to expiration (σ^2_{Fair}).

Again, variance swaps can be quoted with a mean as long as the contract reflects how it will be calculated[26]. However, for a variance or volatility swap, we assume a zero mean process for the variance calculation because the replication of the variance swap using options assumes a zero mean volatility or variance. This has become the standard in the Over-The-Counter (OTC) market. Therefore, we will calculate variance for our contract using equation 1 below:

$$\sigma^2_{ann} = \sum_{t=1}^{n} \frac{(x_t)^2}{(n-1)} * 252 \qquad \text{(equation 1)}$$

[26] With a mean realized volatility would be calculated as $\sigma_{ann} = \sqrt{\frac{n * \sum_{t=1}^{n} x_t^2 - \left(\sum_{t=1}^{n} x_t\right)^2}{n * (n-1)}} * \sqrt{252}$.

where $x_t = \ln\left[\frac{S_t}{S_{t-1}}\right]$. We can now formulate a methodology to mark the value of the current variance swap. For a long variance swap that has an expiration of T and a current time (t), the fair value of the variance swap can be calculated as follows:

$$\sigma^2_{R\&F} = \sigma^2_{ann,t} * \frac{t}{T} + \sigma^2_{Fair,T-t} * \frac{(T-t)}{T} \qquad \text{(equation 67)}$$

where $\sigma^2_{R\&F}$ is the combination of what has been already realized into the life of the contract and what we can estimate using the fair variance calculation. The current value ($F_{val;ue}$) of the variance swap can then be estimated to be:

$$F_{value} = (\sigma^2_{R\&F} - K_{\text{var}}) \qquad \text{(equation 68)}$$

$$PayoutToday = \frac{F_{value} * Notional * Multiplier}{e^{r(T-t)}} \qquad \text{(equation 69)}$$

where $K_{\text{var}} = \sigma^2_{Fair,T}$, which is the original fair variance calculated on the first day with expiration T. Finally, we have to determine the present value of the payout as of today; therefore, we divide by the remaining discount rate.

Lehman Brothers makes two-way markets in equity index and some single stock variance swaps, giving clients the opportunity to buy or sell realized variance.

Lehman Brothers makes two-way markets in equity index and some single stock variance swaps, giving clients the opportunity to buy or sell realized variance. The strike level for an index variance swap is largely determined by the level of implied variance in the relevant index options. For example, perhaps implied volatility in EuroStoxx 50 index options is higher than implied volatility in S&P 500 index options; thus, a variance swap on the EuroStoxx 50 will be struck at a higher level than one on the S&P 500. Other factors that influence pricing of variance swaps include:

- *The stance of the broker-dealer's variance book*—Pricing will depend in part on whether the broker-dealer is better positioned to buy or to sell variance at the time the variance swap is priced.

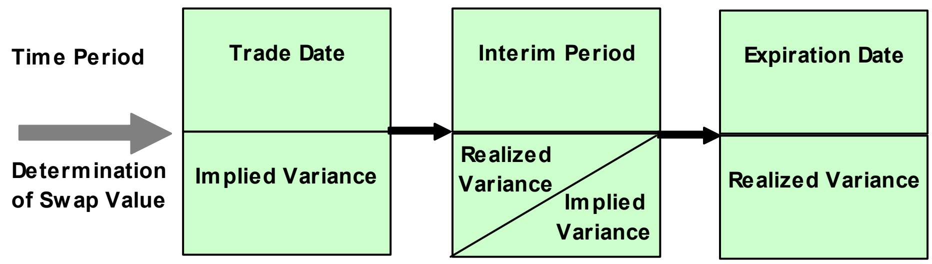
- *The size of the transaction*—Bid/offer spreads may be wider for transactions of larger size.

- *Overall market conditions*—Highly volatile market conditions generally make variance swaps increasingly difficult to hedge, and may contribute to wider bid/offer spreads.

Since the variance swap is a specialized instrument, liquidity can be limited. However, this will change when the CBOE lists variance swaps against the VIX index for a one-month period in 2004. In March 2004, the CBOE is projected to list futures on the VIX. However, we will leave these topics for future reports.

If an investor implements a variance swap, initial and mark-to-market collateral is required similar to a future. The mark-to-market value of a variance swap is determined by both realized and implied variance. In Figure 30, we illustrate how the value of a variance swap is first determined primarily by implied volatility squared (variance), but as index returns are realized, the value is determined by a weighted combination of realized and implied variance, as described in equations 67–69. At expiration, the value of a variance swap is determined entirely by realized variance.

Figure 30: Determination of Variance Swap Value



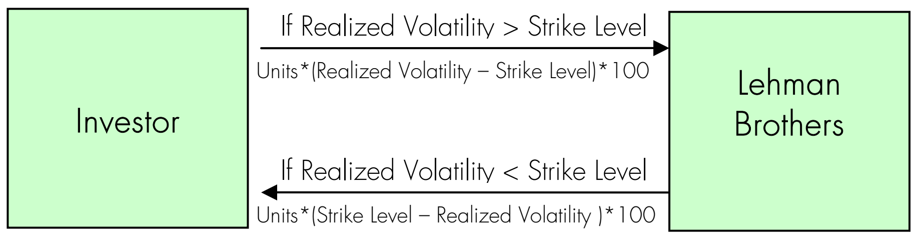
Source: Lehman Brothers

Volatility Swap Contracts and Convexity

Mechanics of a Volatility Swap

In a volatility swap, an investor agrees to receive or pay the realized volatility[27] of an equity index or single stocks relative to an agreed-upon "strike level." Whereas, an equity swap is based on a specified number of shares, a volatility swap is expressed in terms of the dollar value of each volatility point (the number of "units"). The volatility swap resembles a forward contract in that there is no initial exchange of cash flows between the two parties, only an agreement upon the strike level. In Figure 31, we show the exchange of cash flows at expiration for a short volatility swap.

Figure 31: Exchange of Cash Flows at Expiration for a Short Volatility Swap



Source: Lehman Brothers

We have presented in prior research the mechanics of volatility swaps. The payout structure of a volatility swap is provided below. In addition, we display indicative terms and conditions of such a volatility contract in Figure 32 below.[28]

After hedging to expiration (F_{exp}), we can create a natural log contract payout of:

$$F_{\exp} = (\sigma_R - \sigma_K)$$

where σ_K is the fair volatility (σ_I) calculated from the fair variance with a convexity adjustment, as detailed below.

[27] Typically measured as the annualized standard deviation of the daily natural log returns of the stock or index.

[28] Some contracts are expressed by normalizing the payout return by 2 * σ_K. This would scale the payout up or down by a constant factor and represent the percentage change from the strike level, as opposed to an absolute level. For our illustration, we decided not to show this type of variance swap return.

Figure 32: Short Volatility Swap on 250,000 Units – Indicative Term Sheet

Party A:	Lehman Brothers
Party B:	Investor
Units:	250,000
Trade Date:	TBD
Effective Date:	Three Business Days after the Valuation Date
Underlying Index:	S&P 500 Index
Valuation Date:	6 Months from Trade Date
Strike Level:	20.0%
Payoff at Expiration:	Units x (Variance – Strike Level) x 100 If such amount is a positive number then Party B shall pay such amount to Party A. If such amount is a negative number then Party B shall receive the absolute value of such amount from Party A.
Rate of Return:	On Expiration Date, $Units \cdot [\sigma_R - \sigma_K] \cdot 100$ where, σ_R is the Realized Volatility as defined below σ_K is the Volatility Strike
Volatility:	Realized Volatility is defined as $\sqrt{\frac{252}{n-1} * \sum_{i=1}^{n} [R_i]^2}$ where, $R_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$ Realized volatility is calculated with a zero mean. S_i is the official closing price of the Underlying on the i^{th} Exchange Business Day of the life of the Transaction (i = 1 to n) n is 126, the number of business day returns starting on the Trade Date and ending on the Expiration Date If a Market Disruption Event occurs on any Exchange Business Day, then for purposes of calculating R_i, the value of S_i shall equal the value of S_{i-1}.

Source: Lehman Brothers

The volatility swap may be an alternative for investors who have traditionally gained volatility exposure through the option markets.

As noted above, the volatility swap allows an investor to directly express a view on future realized variance. The volatility swap may be an attractive alternative for investors who have traditionally gained volatility exposure through the option markets, as well as an additional way to effectively "outsource" the dynamic trading process (delta hedging) required in volatility-driven strategies that rely on options.

Realized volatility for a volatility swaps is generally calculated with a zero mean since the replication process is essentially a zero mean process.

Formulation of a Volatility Swap and the Convexity Adjustment

Volatility swaps are difficult to replicate because the instrument to be hedged is no longer a quadratic contract; it is a variance contract that uses volatility as its measure.

Volatility swaps are difficult to replicate because the instrument to be hedged is no longer a quadratic contract; it is a variance contract that uses volatility as its measure. A variance swap makes no volatility assumptions other than assuming that stock prices evolve continuously without jumps. Changes in volatility should have little effect on the variance swap strategy because it captures the total variance. However, a volatility swap is fundamentally a different instrument: It is affected by changes in volatility, and its value does depend on the volatility path of future realized volatility. A volatility swap can be thought of as a derivative of variance swaps. Volatility is a linear function since it is the square root of variance, which makes it more difficult to hedge and replicate. Variance swaps are the preferred quoted swap contracts by many broker-dealers.

We can approximate a volatility contract by holding a variance contract. Using variance, we can calculate only an approximate strike of a volatility swap contract struck at $K_{volatility}$, which has a payout of $(\sigma_R - K_{volatility})$:

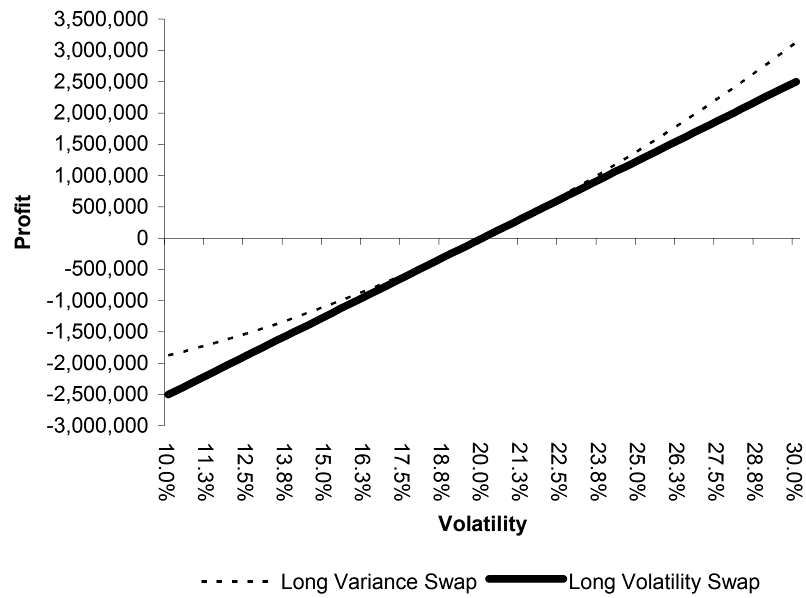
$$(\sigma_R - K_{volatility}) \approx \frac{1}{2K_{volatility}} * \left(\sigma_R^2 - K_{volatility}^2\right) \qquad \text{(equation 70)}$$

This means that $1/(2^* K_{volatility})$ variance contracts with a strike of $\sqrt{K_{volatility}^2}$ can be used to approximate the payoff of a volatility swap with a notional of \$1/(volatility point) for each realized volatility point from $K_{volatility}$. As it is only an estimate, this would mean that

$$K_{volatility} \approx \sqrt{K_{\text{var}iance}} \qquad \text{(equation 71)}$$

If we return to the variance swap payoff profile shown in Figure 1 and compare it to a volatility swap payoff profile using equation 72, it is evident that a variance swap shows a convex payoff compared to a volatility swap, as shown in Figure 33 below

Figure 33: Long Volatility and Variance Swap Payoff (Strike Volatility = 20%)



Source: Lehman Brothers

The difference between a quadratic variance swap payoff and linear volatility swap payoff increases as volatility deviates from the initial strike level.

The difference between a quadratic variance swap payoff profile and linear volatility swap payoff profile increases as volatility deviates from the initial strike level. The difference between the two payoffs is measured by the convexity of the volatility swap.

$$Convexity \approx \frac{1}{2K_{volatility}} * \left(\sigma_R - K_{volatility}\right)^2 \qquad \text{(equation 72)}$$

Since the square of $\left(\sigma_R - K_{volatility}\right)^2$ is always positive, the variance swap will always outperform the volatility swap the further one gets from the strike. We can adjust for this convexity by making the fair strike of the volatility swap lower than the fair strike of the variance swap, and using the convexity adjustment to shift the volatility swap strike level appropriately.

To adjust for this convexity difference, we could dynamically trade variance swaps during the life of the volatility swap.

To adjust for this convexity difference, we could dynamically trade variance swaps during the life of the volatility swap, which would produce a payoff independent of future volatility moves. Such a method would be similar to delta hedging an option to produce a curved payoff pattern using the linear stock price movements.

Dynamic replication of the option hedge ratio would require a model for the volatility of volatility. We would have to hold a variance delta of the total variance contracts to hedge the volatility of volatility, and make some assumptions about changes in the volatility surface of the stock or index going forward.

Alternatively, we can take a view on the direction and volatility of volatility, so that we can pick the strike and size of variance delta contracts that, on average, would potentially replicate the payoff pattern of a volatility swap.

Alternatively, we can take a view on the direction and volatility of volatility, so that we can pick the strike and size of variance delta contracts that, on average, would potentially replicate the payoff pattern of a volatility swap. The tracking error of the replication will depend on how closely realized volatility moves follow our initial expectations. This difference between the option hedge and the volatility swap contract is smaller as the maturity of the contract decreases.

If we choose a hedge realized variance (σ_{TR}) and assume the replication target realized volatility is (σ_T), then we can adjust for the convexity (the volatility contract replication error) using equation 73. See Appendix D for the derivation of equation 73.

$$Minimize\left(E\left[(\sigma_T - x * \sigma_{TR}^2 - y)^2\right]\right) = \frac{\sigma_{\sigma_{TR}}^2}{1 + \frac{2 * \overline{\sigma_{TR}}^2}{\sigma_{\sigma_{TR}}^2}} \qquad \text{(equation 73)}$$

Equation 73 can be used to adjust the initial strike, whether a buyer or seller of a volatility swap, to compensate for trading the convexity during the life of the volatility contract.

Appendix A: Delta, Gamma, Theta, Rho, and Vega

Review of the Black/Scholes Model

To replicate a variance swap, we start by building on the Black/Scholes model. First, we define realized and implied volatility.

Realized (historical) volatility is based on the past price (historical) volatility of the stock price.

Realized (historical) volatility is based on the past price (historical) volatility of the stock price. It is backward-looking volatility, explained by the historical price movements of the stocks from i-1 to i.

- Based on historical prices: $x = \ln\left[\frac{S_i}{S_{i-1}}\right]$

- $$\sigma_{ann} = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{(n-1)}} * \sqrt{252} \qquad \text{(unbiased)}$$

$$= \sqrt{\frac{n * \sum_{t=1}^{n} x_t^2 - \left(\sum_{t=1}^{n} x_t\right)^2}{n*(n-1)}} * \sqrt{252}$$

where μ = mean of x, n is the total number of sample over time, i is a specific sample, S_i is the i[th] period price, S_{i-1} is the i-1 period price, and σ_{ann} = the annualized realized volatility. Variance swaps can be quoted with a mean as long as the contract reflects how it will be calculated. However, for a variance or volatility swap, we assume a zero mean process for the variance calculation because the replication of the variance swap using options assumes a zero mean volatility or variance. This has become the standard in the Over-The-Counter (OTC) market. Therefore, we will calculate variance for our contract using equation A1 below:

$$\sigma_{ann}^2 = \sum_{t=1}^{n} \frac{(x_t)^2}{(n-1)} * 252 \qquad \text{(equation A1)}$$

Implied volatility explains the expected future uncertainty of the stock or index, and is forward looking.

Implied volatility explains the expected future uncertainty of the stock or index, and is forward looking. Listed options on indices or stocks can be used to calculate implied volatility levels, assuming other factors in the Black/Scholes model are constant. Implied volatility from the Black/Scholes model for a European call[29] (assuming no dividends) is given by the following equations:

- $C = S * N(d_1) - K * e^{-rt} * N(d_2)$ (equation 2)

- $P = K * e^{-rt} * N(-d_2) - S * N(-d_1)$ (equation 3)

[29] Hull, John, "Options, Futures and Other Derivatives", 5[th] Edition, Prentice Hall, 1999, pp. 237–329.

- $$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) * t}{(\sigma * \sqrt{t})}$$ (equation 4)

- $$d_2 = d_1 - \sigma * \sqrt{t}$$ (equation 5)

- C = call price and P = put price form options market, K= option strike, S = stock price, N = cumulative normal distribution, r = continuously compounded risk-free rate, t = time to maturity, and σ = option implied volatility.

The Delta Hedging Strategy and the Benefits of Volatility

The stock hedge, also known as the "delta," reflects the sensitivity of the option price to a change in the stock price.

The Black/Scholes formula offers a "recipe" for pricing a put or call option by providing the elements of the hedge portfolio—the long or short stock position and the amount of money that should be borrowed or lent. The stock hedge, also known as the "delta," reflects the sensitivity of the option price to a change in the stock price. The delta measures the long or short stock position required to hedge the equity exposure in the option.[30] Equation A1 shows the expression of delta for a long European call, and equation A2 shows the expression of delta for a long European put. For both the European call and put, we assume dividends are zero.

$$\Delta_c = \frac{\partial C}{\partial S} = N(d_1)$$ (equation A1)

$$\Delta_p = \frac{\partial P}{\partial S} = N(d_1) - 1$$ (equation A2)

- For example, a call option with a delta of 0.70 suggests that the investor should short 70 shares of stock for each long option contract (on 100 shares).

- If the stock price goes up $1, the option should rise by approximately $0.70, so that the gain on the option ($0.70 * 100) is equal to the loss on the short stock position ($1 * -70 shares), assuming no change in volatility and the gamma of the option).

An important consideration is the fact that an option's delta changes as the stock price changes. This feature of options requires that the delta hedge be periodically rebalanced.

In Figure 34 below, we illustrate how the delta of a six-month, $100 strike European call option changes as the price of the underlying stock changes, with a starting price of $100.[31] At one end of the spectrum, when the stock price is significantly below the

[30] For example, two parties can contract to a "total return option," which strips out dividend risk. Mathematically, the delta is the first derivative of the option price with respect to a change in the stock price.

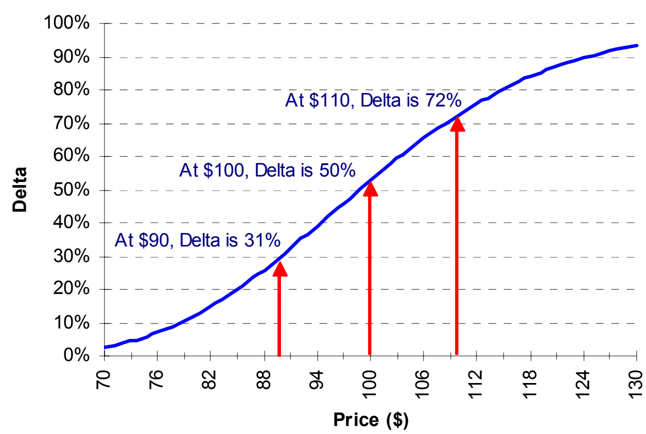
[31] The assumptions that go into pricing the option include a constant dividend yield of 1.50%, volatility of 25%, a maturity of six months, and LIBOR interest rate of 1.2%.

strike price, the call option delta is almost zero; at the other end, when the option is deep in-the-money, the call option delta is almost one.

Although the delta hedging strategy offsets the directional risk in the stock, its success hinges on the ability to adjust the required hedge as the delta changes.

Although the delta hedging strategy offsets the directional risk in the stock, its success hinges on the ability to adjust the required hedge as the delta changes. Large, discrete jumps in the stock price will leave the trader either over-hedged or under-hedged. For example, in Figure 34, the delta of the option when the stock price is \$100 is 0.50; this suggests that the trader who is long this call should be short 50 shares of stock. If the stock were to suddenly jump to \$110, the appropriate delta hedge would increase to a short position of 72 shares. The stock price movement leaves the investor under-hedged (short only 50 shares instead of 72 shares).

Figure 34: Delta of Six-Month \$100 Strike Long European Call Option



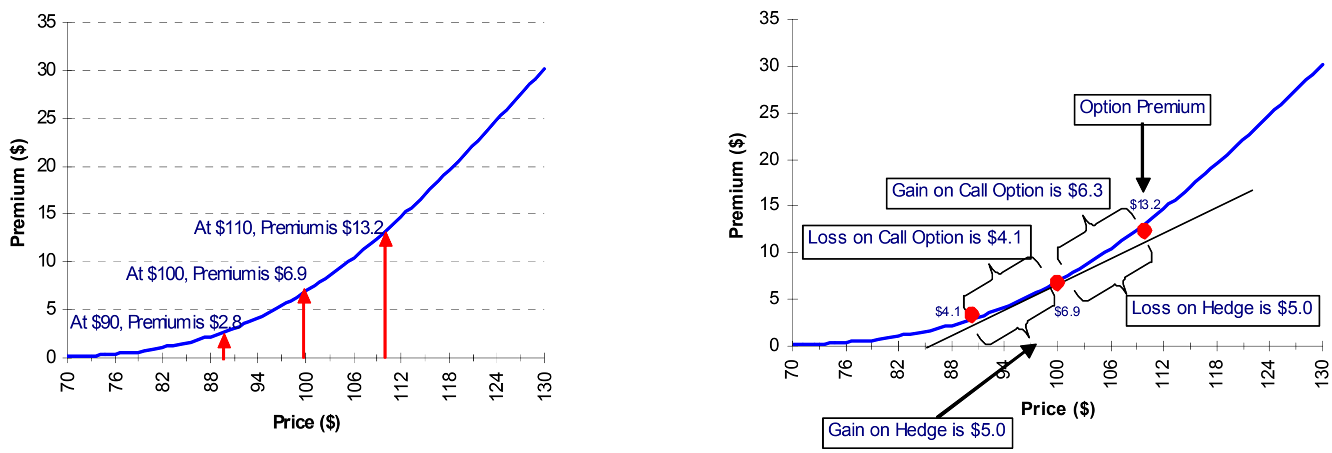
Source: Lehman Brothers

In Figure 35 below, we illustrate what has happened in this scenario:

- With the stock price moving to \$110 from \$100, the option is initially worth \$13.2. This is a gain of \$6.3 from the value of the option when the stock was at \$100 (\$13.2 - \$6.9). However, the trader has lost only \$5 on his short stock position (\$10 change in stock price multiplied by the 0.50 delta). This results in a net gain of \$1.3.

- Suppose the stock were to instead suddenly fall from \$100 to \$90. At a stock price of \$90, the option value is \$2.8, resulting in a loss of \$4.1 (\$6.9 – \$2.8). However, the trader was short stock as a delta hedge on this position and has profited by \$5.0 from the fall in stock price. The trader has a net gain of \$0.9.

Figure 35: Changes in Option Value and Delta Hedge for a Long European Call Option



Source: Lehman Brothers

The volatility trader's goal is to buy those options for which the premium paid is less than the benefit to potentially be realized through delta hedging, and to sell those options for which the premium collected exceeds the cost of hedging the option.

The long option position, combined with the stock delta hedge, allows the trader to profit from movements in the stock price in either direction. In fact, each time the stock price moves up or down, the long option holder will benefit as the delta hedge becomes mismatched and the opportunity arises to rebalance the hedge. If the trader has purchased the option, the greater the swing in the stock price, the greater the benefit to the trader and, potentially, the greater the profit. This can be seen in Figure 35 via the straight line, which indicates the magnitude of the delta hedge mismatch resulting from a movement in the stock price. Since larger price swings result in greater profitability for delta-neutral long option positions, traders are willing to pay higher premium for options on volatile stocks. Looked at from the perspective of an option seller, more premium is demanded for an option on a volatile stock, since the anticipated wide price fluctuations increase the cost of maintaining an ongoing delta hedge. The volatility trader's goal is to buy those options for which the premium paid is less than the benefit to potentially be realized through delta hedging, and to sell those options for which the premium collected exceeds the cost of hedging the option.

Option Gamma and the Gamma/Theta Trade-Off

Gamma

The degree to which the delta hedge becomes mismatched when the stock price moves is reflected in the option's "gamma."

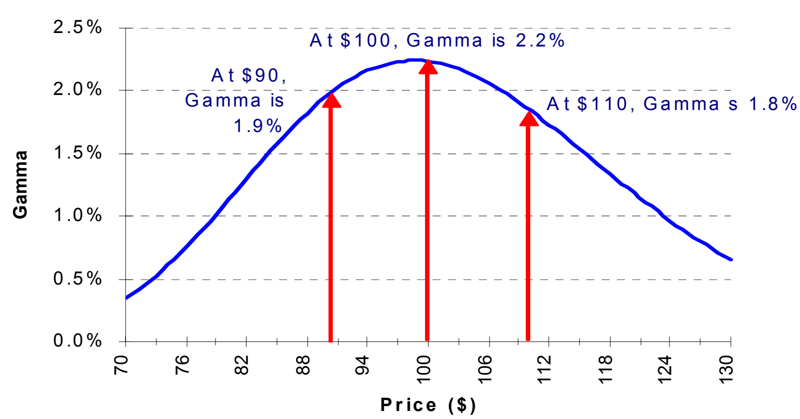
The degree to which the delta hedge becomes mismatched when the stock price moves is reflected in the option's "gamma." Gamma measures the change in delta for a given change in the underlying asset or the acceleration (second momentum) of the option price movement. Equation A3 shows the gamma of a long European call, and equation A4 shows the gamma of a long European put. For both the European call and put, we assume dividends are zero and T = time to maturity.

$$\Gamma_c = \frac{\partial^2 C}{\partial S^2} = \frac{N'(d_1)}{S_o \sigma \sqrt{T}} \qquad \text{(equation A3)}$$

$$\Gamma_p = \frac{\partial^2 P}{\partial S^2} = \frac{N'(d_1)}{S_o \sigma \sqrt{T}} \qquad \text{(equation A4)}$$

For example, in Figure 35 above, when the stock price moves from \$90 to \$100, the option's delta increases from 0.31 to 0.50. Thus, a \$10 increase in the stock price results in a change in delta of 0.19. The gamma per \$1 movement in stock price is then 0.019. When the stock moves from \$100 to \$110, the delta increase is 0.22. The gamma per \$1 movement in stock price is then 0.022. Thus, the gamma is higher when the stock is at \$100 than it is when the stock is at \$110, and higher at \$100 than at \$90, in this case. In Figure 36 below, we show how the gamma of the option changes based on different levels of the underlying equity. In general, gamma tends to be highest for options for which the strike price and underlying asset price are close together. Deep in-the-money or out-of-the-money options tend to have little gamma.

Figure 36: Gamma of a Six-Month \$100 Strike Long European Call



Source: Lehman Brothers

Gamma/Theta Trade-Off

Theta measures the extent to which an option's premium "decays" with the passage of time, assuming that the stock price does not change.

Closely related to gamma is theta. Theta measures the extent to which an option's premium "decays" with the passage of time, assuming that the stock price does not change.[32] Equation A5 shows the theta of a long European call, and equation A6 shows the theta of a long European put. For both the European call and put, we assume dividends are zero and T = time to maturity.

$$\theta_c = \frac{\partial C}{\partial t} = -\frac{S_o * N^{'}(d_1) * \sigma}{2 * \sqrt{T}} - r * K * e^{-rt} * N(d_2) \qquad \text{(equation A5)}$$

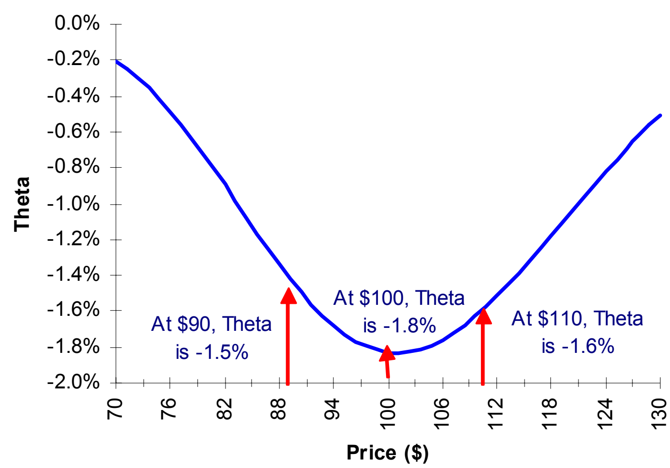
$$\theta_p = \frac{\partial P}{\partial t} = -\frac{S_o * N^{'}(d_1) * \sigma}{2 * \sqrt{T}} + r * K * e^{-rt} * N(-d_2) \qquad \text{(equation A6)}$$

where $N^{'}(x)$ is a Gaussian process with the form below:

$$N^{'}(x) = \frac{1}{2\pi} * e^{-\frac{x^2}{2}} \qquad \text{(equation A7)}$$

For example, the premium of the six-month $100 strike call option will fall from $6.92 to $6.91 when one day passes (assuming no movement in the stock price and that all the other variables are constant except time to expiration). Theta can accelerate depending on the spot, strike level and time left till expiration of the option. In Figure 37, we show the theta of the call option.

Figure 37: Theta of Six-Month $100 Strike of Long European Call Option



Source: Lehman Brothers

[32] Mathematically, theta is the first derivative of option price with respect to passage of time. Theta is most extreme for an ATM option with little time to expiration.

Note that theta, like gamma, is influenced in part by where the stock price is relative to the strike price. For deep in-the-money or deep out-of-the-money options, theta is small. The idea is that when the stock price is well away from the strike price, there is little uncertainty as to the outcome of the option and the passage of time makes little difference. However, when the stock price is close to the strike price, an option tends to be composed of more "time value" (the excess value of the option over its intrinsic value) which depreciates with the passage of time.

The trade-off: Gamma indicates the extent to which the delta hedge becomes mismatched and the economic benefit of large movements in the stock price. Theta indicates the economic consequence of these large movements not occurring (and the cost associated with waiting for a large movement).

One can think of gamma and theta in terms of a trade-off: As described above, gamma indicates the extent to which the delta hedge becomes mismatched and the economic benefit of large movements in the stock price. Theta indicates the economic consequence of these large movements not occurring (and the cost associated with waiting for a large movement). The relationship trade-off between gamma and theta can be expressed in equation A8 and A9 below for a delta-neutral portfolio $(\Delta = 0)$, for either European calls or puts with zero dividends.

$$\theta_c + \frac{1}{2} * \sigma^2 * S^2 * \Gamma_c = r * C = 0 \qquad \text{(equation A8)}$$

$$\theta_p + \frac{1}{2} * \sigma^2 * S^2 * \Gamma_p = r * P = 0 \qquad \text{(equation A9)}$$

Rho: A Measure of Interest Rate Risk

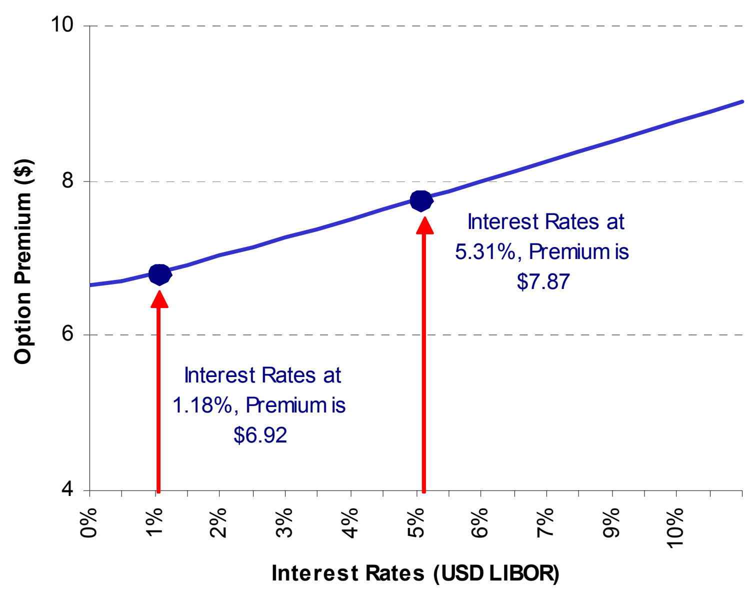
Option Rho refers to the rate of change of the portfolio value with respect to the interest rate inherent in the option. Equation A10 shows the rho of a long European call, and equation A11 shows the rho of a long European put. For both the European call and put, we assume dividends are zero and T = time to maturity.

$$Rho_c = \frac{\partial C}{\partial r} = K * T * e^{-rt} * N(d_2) \qquad \text{(equation A10)}$$

$$Rho_p = \frac{\partial P}{\partial r} = -K * T * e^{-rT} * N(-d_2) \qquad \text{(equation A11)}$$

This interest rate risk usually can be hedged out by buying or selling $US LIBOR contracts. If one buys a call, one would sell LIBOR futures bootstrapped out to six months to hedge the interest rate risk. Otherwise, as interest rates move, option premium moves as well. Figure 38 shows how interest rate risk can affect option prices. For example, if interest rates out six months for $US LIBOR move from 1.18% to 5.31%, then the option premium would move from $6.92 to $7.87.

Figure 38: Interest Rate Risk of Six-Month $100 Strike of Long European Call Option



Source: Lehman Brothers

Implied Volatility, Realized Volatility, and Vega

The term "implied volatility" refers to the volatility input that equates the option's observed market price with its theoretical (i.e., Black/Scholes) value.

As noted above, an option price is affected by six variables, one of which is the expected volatility of the underlying equity. Vega is the rate of change of the value of the option with respect to the volatility of the index or stock. The term "implied volatility" refers to the volatility input that equates the option's observed market price with its theoretical (i.e., Black/Scholes) value, given the five other option pricing inputs (stock price, strike price, time to expiration, interest rates, and dividend yields). Equation A12 shows the vega of a long European call, and Equation A13 shows the vega of a long European put. For both the European call and put, we assume dividends are zero, T = time to maturity, and $N^{'}(x)$ is given by equation A7.

$$V_c = \frac{\partial C}{\partial \sigma} = S_o * \sqrt{T} * N^{'}(d_1) \qquad \text{(equation A12)}$$

$$V_p = \frac{\partial P}{\partial \sigma} = S_o * \sqrt{T} * N^{'}(d_1) \qquad \text{(equation A13)}$$

For example, suppose the six-month-European call option that was discussed in the previous section traded in the market for $6.92. The other option inputs are shown in Figure 39, where only option price changes in relation to volatility.

Figure 39: Option Pricing Inputs for Our Example

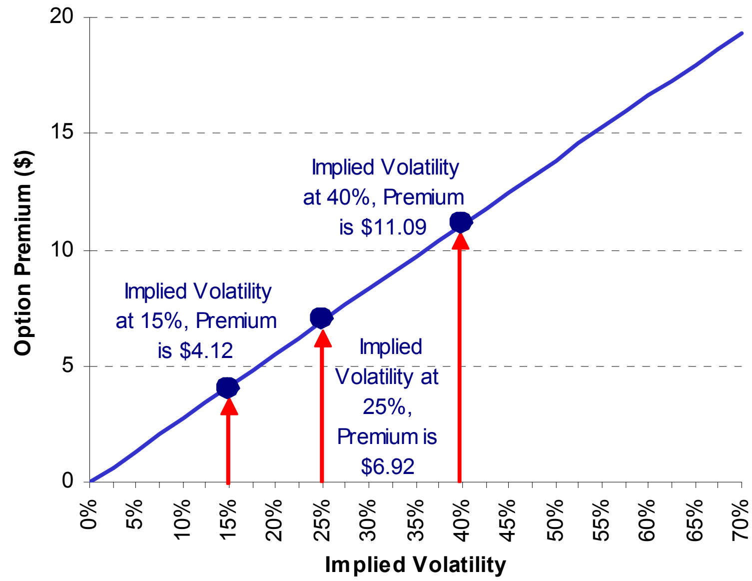
Input Variable	Values
Option Price	?
Maturity	6-Mth
Stock Price	$100
Strike	$100
Div Yield	1.50%
Interest Rate	1.18%
Volatility	?

Source: Lehman Brothers

Given the price inputs, what volatility level produces an option price of $6.92?[33] The level, 25%, is known as the option's implied volatility. If the price of the option were instead $8.31, the option's implied volatility would be 30%. In Figure 40 below, we show the price of the six-month- call option with different volatility inputs.

[33] The Black/Scholes model is typically used to compute the implied volatility of a European option.

Figure 40: Value of Six-Month 100 Strike European Call Option at Different Implied Volatilities



Source: Lehman Brothers

As is evident in Figure 40, the call option is more valuable at a higher implied volatility than at a lower implied volatility. The effect of a change on the option price due to a 1% change in implied volatility is measured by "vega." For example, in Figure 40, the option price changes by $1.89, from $6.92 to $11.09, when the implied volatility moves from 25% to 40%. Thus, the vega of the option is roughly $0.28 per a 1% move in implied volatility ($4.17/15 volatility points).[34]

[34] Mathematically, the vega of an option is the first derivative of option price with respect to a very small change in implied volatility. Since vega itself is not constant, larger changes in implied volatility in the calculation of vega will tend to produce more error in the approximation.

Appendix B: Derivations of 1/K² Relationship and the Log Contract

Derivation of the 1/K² Relationship

Using a similar analysis found in prior research, we can show the $1/K^2$ relationship.[35] If we take a portfolio of standard options, we can represent the payout below:

$$F(S) = \int_0^\infty P(K) * BS(S, K, v) * dK \quad \text{(Equation B1)}$$

Where BS is the Black/Scholes Model formulation of one option at Price (S), Strike (K), and variance ($v = \sigma^2 * \tau$). We represent the sensitivity of an individual option above (BS) by[36]:

$$V_o = \tau * S * W\left(\frac{K}{S}, v\right)$$

where

$$W\left(\frac{S}{K}, v\right) = \frac{1}{2\sqrt{v}} * \frac{\exp(-d_1^2/2)}{\sqrt{2\pi}}$$

and where r =0:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \frac{(v)}{2}}{\sqrt{v}}$$

The variance sensitivity of the portfolio is then,

$$F(S) = \tau \int_0^\infty P(K) * S * W(\frac{K}{S}, v) * dK$$

[35] Demeterfi, Kresimir; Derman, Emanuel; Kamal , Michael; and Zou, Joseph, 1999, "A guide to volatility and variance swaps", *Journal of Derivatives*, 6, pp. 9-32.

[36] We assume that r = 0 and that dividends are also zero.

The sensitivity of vega to S is:

$$\frac{\partial V_{F(s)}}{\partial S} = \tau \int_0^\infty \frac{\partial}{\partial S} * [S^2 P(xS)] * W(x,v) * dx \qquad \text{(Equation B2)}$$

$$= \tau \int_0^\infty S[2P(xS) + xSP^{'}(xS)] * W(x,v) * dx$$

where x = K/S or the percentage of how ITM or OTM the option is in relation to the current price level (S). We want $\frac{\partial V}{\partial S}$ = 0, so we can solve for P below:

$$2P + K\frac{\partial P}{\partial K} = 0$$

Therefore, we have the relationship below (where C = constant) showing that the weight for each option is inversely proportional to K2.

Therefore, we have the relationship below (where C = constant) showing that the weight for each option is inversely proportional to K^2.

$$P = \frac{C}{K^2} \qquad \text{(Equation B3)}$$

The Log Contract

Realized variance is similar to trading a log contract (using the natural log function) of a price series for a stock or index.

Realized variance is similar to trading a log contract (using the natural log function) of a price series for a stock or index. Since no log contract (natural log contract) is traded, we need to replicate the payout in terms of options in equation B4, where S_r is the forward reference price and S_T is the current price at time T:

$$F(S_T) = \frac{2}{T} * \left[\frac{S_T - S_r}{S_r} - \ln\left(\frac{S_T}{S_r}\right) \right] \qquad \text{(Equation B4)}$$

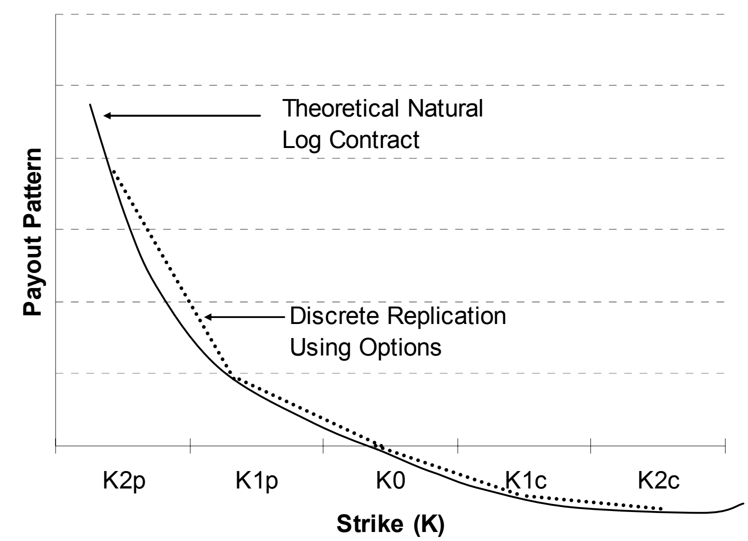
However, there is only a discrete set of option price available for replication of the variance swap, and we need to determine the number of options for each strike to trade.

$$K_o = S_r < K_{1c} < K_{2c} < K_{3c} < .....$$

$$K_o = S_r > K_{1p} > K_{2p} > K_{3p} > .....$$

We need to replicate the log-payoff and option portfolio at maturity T. Figure 41 below shows the log payout path we need to replicate with a discrete set of options.

Figure 41: Natural Log-Payout and Option Duplication at Maturity (time = T)



Source: Lehman Brothers

The number of options is determined by the slope between the strikes of the options:

$$W_c(K_0) = \left[\frac{F(K_{1c}) - F(K_0)}{K_{1c} - K_0}\right] * e^{rT}$$

and

$$W_c(K_1) = \left[\frac{F(K_{2c}) - F(K_{1c})}{K_{2c} - K_{1c}}\right] * e^{rT} - W_c(K_o)$$

where W_c is the weight of the calls. We need to evaluate variance according to its future payout when we calculate it by multiplying by e^{rT}. Therefore, for any call, the weight of the calls can be represented by the following:

$$W_c(K_n) = \left[\frac{F(K_{n+1,c}) - F(K_{n,c})}{K_{n+1,c} - K_{n,c}}\right] * e^{rT} - \sum_{i=0}^{n-1} W_c(K_{i,c}) \qquad \text{(equation B5)}$$

and for the puts, W_p is:

$$W_p(K_n) = \left[\frac{F(K_{n+1,p}) - F(K_{n,p})}{K_{n,p} - K_{n+1,p}}\right] * e^{rT} - \sum_{i=0}^{n-1} W_p(K_{i,p}) \qquad \text{(equation B6)}$$

Appendix C: Derivations of the P/L Adjustment Using a Jump Process

Using the analysis by Kresimir Demeterfi, Emanuel Derman, Michael Kamal and Joseph Zou, we can show the adjustment for a jump process.[37] A normal price path has the properties of adding the daily natural log returns, as seen in equation C1:

$$V = \frac{2}{T}\sum_{t=1}^{N}\left[\frac{\Delta S_t}{S_{t-1}} - \ln\left(\frac{\Delta S_t}{S_{t-1}}\right)\right] \qquad \text{(equation C1)}$$

where $\Delta S_t = S_t - S_{t-1}$.

Since the contribution of one jump to the variance is additive, the total unrealized variance is given in equation C2, assuming a zero-mean process:

$$V = \frac{1}{T}\sum_{t=1}^{N}\left[\frac{\Delta S_t}{S_{t-1}}\right]^2 = \frac{1}{T}\sum_{nojumps}\left[\frac{\Delta S_t}{S_{t-1}}\right]^2 + \frac{1}{T}\left[\frac{\Delta S_t}{S_{t-1}}\right]^2_{jump} \qquad \text{(equation C2)}$$

The jump contribution is broken out from equation C2 and reduced to the equation below.

$$\frac{1}{T}\left[\frac{\Delta S_t}{S_{t-1}}\right]^2_{jump} = \frac{J^2}{T}$$

P/L due to a jump can be determined to be:

$$P\,\&\,L_{jump} = \frac{2}{T}\left[-J - \ln(1-J)\right]^2 - \frac{J^2}{T} \qquad \text{(equation C3)}$$

Using a Taylor series expansion, the log function can be expressed by:

$$-\ln(1-J) = J + \frac{J^2}{2} + \frac{J^3}{2} + \ldots \qquad \text{(equation C4)}$$

[37] Demeterfi, Kresimir; Derman, Emanuel; Kamal , Michael; and Zou, Joseph, 1999, "A guide to volatility and variance swaps", *Journal of Derivatives*, 6, pp. 9-32.

The first constant scalar J in the expansion to equation C4 cancels out with the J in equation C3. The second quadratic term $\frac{J^2}{2}$ can be absorbed by the variance contract, since variance by nature is quadratic. In a hedged long position, and because the second term is a quadratic, large moves in either direction benefit the position and can be captured in the variance. In contrast, the variance replication strategy is long quadratic, cubic and higher-order terms due to the jump, while a short variance swap is short the quadratic terms. Therefore, the quadratic terms can be removed (long for the jump and short for the short variance swap). This leaves only the cubic term and higher order terms, of which the cubic generates the most movement in P/L. The contribution to error from the jump process is really the cubic term and higher order terms:

$$P\,\&\,L_{jump} = \frac{2}{3}\frac{J^3}{T} + ... \qquad \text{(equation C5)}$$

If we are short a variance swap, then a large move down leads to a profit because J > 0 but a large move up leads to a loss because J < 0. A large move up one day and a large move down the next day will offset each other.

The cubic term in the jump process has different signs for upward and downward jumps. If we are short a variance swap, then a large move down leads to a profit because J > 0, but a large move up leads to a loss because J < 0. A large move up one day and a large move down the next day will offset each other. However, if the skew shifts, then the P/L effect may not always be zero, given the limited number of strikes and options used to replicate the log contract.

Appendix D: Derivations of the Convexity Relationship for a Volatility Swap

As shown in prior research, we can derived the convexity relationship for a volatility swap.[38] We choose a hedge-realized variance (σ^2_{TR}) and assume the replication target realized volatility is (σ_T). Volatility as a function of variance is then:

$$\sigma_T \approx x * \sigma^2_{TR} + y \qquad \text{(equation D1)}$$

The variables x and y are chosen to minimize the expected squared deviation.

$$Minimize\left(E\left[(\sigma_T - x * \sigma^2_{TR} - y)^2\right]\right) \qquad \text{(equation D2)}$$

We can take the first and second derivate of equation D2 to produce the following equation, where future volatility is assumed to be normally distributed below:

$$E[\sigma_T] = x * E\left[\sigma^2_{TR}\right] + y$$

$$E[\sigma^3_T] = x * E\left[\sigma^4_{TR}\right] + y * E\left[\sigma^2_{TR}\right]$$

where $\sigma_T \approx N(\mu = \overline{\sigma_{TR}}, \sigma_{\sigma_{TR}})$.

If negative probabilities are negligible and $G_\rho[\sigma_T, \sigma^2_{TR}]$ is the correlation of σ_T and σ^2_{TR}, then

$$Minimize\left(E\left[(\sigma_T - x * \sigma^2_{TR} - y)^2\right]\right) = Var(\sigma_T) * \left[1 - (G_\rho[\sigma_T, \sigma^2_{TR}])^2\right]$$

Solving to minimize the difference, the hedging coefficients are then:

$$x = \frac{1}{2\overline{\sigma_{TR}} + \dfrac{\sigma^2_{\sigma_{TR}}}{\overline{\sigma_{TR}}}} \qquad \text{(equation D3)}$$

$$y = \frac{\overline{\sigma_{TR}}}{2 + \dfrac{\sigma^2_{\sigma_{TR}}}{\overline{\sigma_{TR}}^2}} \qquad \text{(equation D4)}$$

[38] Demeterfi, Kresimir; Derman, Emanuel; Kamal , Michael; and Zou, Joseph, 1999, "A guide to volatility and variance swaps", *Journal of Derivatives*, 6, pp. 9-32.

The replication error is then expressed as:

$$Minimize\left(E\left[(\sigma_T - x * \sigma_{TR}^2 - y)^2\right]\right) = \frac{\sigma_{\sigma_{TR}}^2}{1 + \frac{2 * \overline{\sigma_{TR}}^2}{\sigma_{\sigma_{TR}}^2}} \qquad \text{(equation D5)}$$

Equation D5 can be used to adjust the initial strike, whether a buyer or seller of a volatility swap, to compensate for trading the convexity during the life of the volatility contract.

Equation D5 can be used to adjust the initial strike, whether a buyer or seller of a volatility swap, to compensate for trading the convexity during the life of the volatility contract.

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