

Documentation for Half-Cheetah Control/Planning Using Mixed-Integer Optimization

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February 7, 2019

1 Optimization Problem Formulation

1.1 Variables Definitions

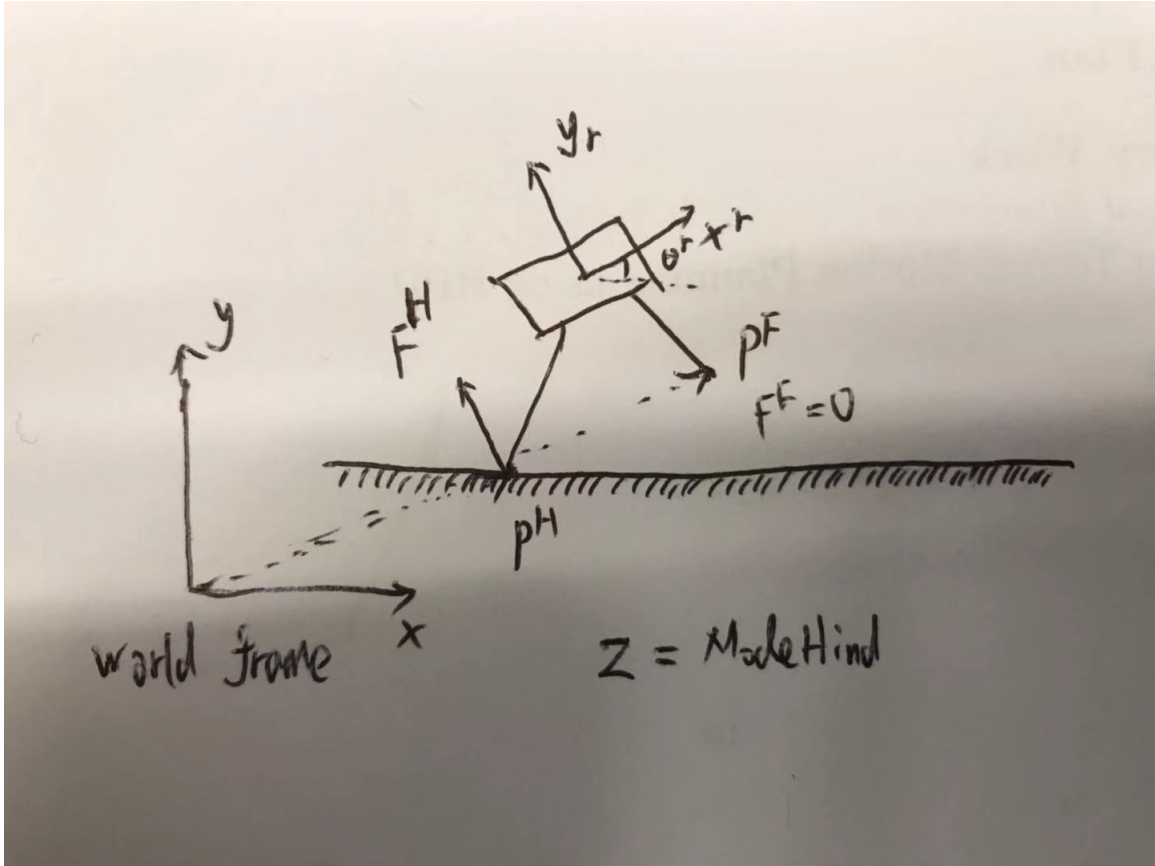


Figure 1: Variable Definition for the Half Cheetah Robot

Variables:

Robot State: $r = [x_r, y_r, \theta_r, \dot{x}_r, \dot{y}_r, \dot{\theta}_r]$

Foot/End-effector States (**IN WORLD FRAME**):

FrontLeg: $P^F = [P_x^F, P_y^F, \dot{P}_x^F, \dot{P}_y^F]$

HindLeg: $P^H = [P_x^H, P_y^H, \dot{P}_x^H, \dot{P}_y^H]$

where P_x^i and P_y^i are the positions of i^{th} foot/end-effector, $i \in F, H$. \dot{P}_x^i and \dot{P}_y^i are the velocities of i^{th} foot/end-effector, $i \in F, H$.

Foot-Ground Reaction Forces (**IN WORLD FRAME**):

FrontLeg: $F^F = [F_x^F, F_y^F]$

HindLeg: $F^H = [F_x^H, F_y^H]$

Mode Selection (Two Tracks):

- Mode Enumeration: $Mode = [Mode^{fly}, Mode^{double}, Mode^{front}, Mode^{hind}]$, where $Mode^i = 0, 1$
- (Preferred) Define Contact Configuration for Every Foot: $C = [C^F, C^H]$, where $C^i = 0, 1$

Because we can just keep dynamic constraints invariant, and constrain different foot-ground reaction forces based on foot-ground contact indicators. However, the Mode Enumeration is tedious because we need to write different dynamics, footstep locations, foot-ground reaction forces constraints (big-M) based on mode indicator variables

1.2 Objective/Cost Function and Constraints

1.2.1 Objective/Cost Function

Quadratic form: $c = v^T Q v$

where v is the decision variable vector, it should be a column vector, Q is the selection matrix for the Quadratic Form.

$$v^T Q v = v_1 Q_{11} v_1 + v_1 Q_{12} v_2 + \dots + v_2 Q_{21} v_1 + v_2 Q_{22} v_2 + \dots$$

$$Q_{i,j} \in R^{dim(v_i) \times dim(v_j)}$$

1.2.2 Constraints

System Dynamics:

$$m\ddot{x} = F_x^F + F_x^H \rightarrow \ddot{x} = \frac{F_x^F}{m} + \frac{F_x^H}{m} \rightarrow \ddot{x} = f_x()$$

$$m\ddot{y} = -mg + F_y^F + F_y^H \rightarrow \ddot{y} = -g + \frac{F_y^F}{m} + \frac{F_y^H}{m} \rightarrow \ddot{y} = f_y()$$

Euler Integration:

$$x_{k+1} - x_k = h_k f_k \rightarrow x_{k+1} - x_k - h_k f_k = 0$$

Trapezoidal Collocation:

$$x_{k+1} - x_k = \frac{1}{2} h_k (f_{k+1} + f_k) \rightarrow x_{k+1} - x_k - \frac{1}{2} h_k f_{k+1} - \frac{1}{2} h_k f_k = 0$$

Apply Trapezoidal Collocation into translational dynamics:

$$\text{x-axis position: } x_{k+1} - x_k = \frac{1}{2} h_k (\dot{x}_{k+1} + \dot{x}_k) \rightarrow x_{k+1} - x_k - \frac{1}{2} h_k \dot{x}_{k+1} - \frac{1}{2} h_k \dot{x}_k = 0$$

x-axis velocity:

$$\dot{x}_{k+1} - \dot{x}_k - \frac{1}{2} h_k \left(\frac{F_{x,k+1}^F}{m} + \frac{F_{x,k+1}^H}{m} \right) - \frac{1}{2} h_k \left(\frac{F_{x,k}^F}{m} + \frac{F_{x,k}^H}{m} \right) = 0 \rightarrow$$

$$\dot{x}_{k+1} - \dot{x}_k - \frac{1}{2} h_k \frac{F_{x,k+1}^F}{m} - \frac{1}{2} h_k \frac{F_{x,k+1}^H}{m} - \frac{1}{2} h_k \frac{F_{x,k}^F}{m} - \frac{1}{2} h_k \frac{F_{x,k}^H}{m} = 0$$

$$\text{With on/off signals (Contact or not flag): } \dot{x}_{k+1} - \dot{x}_k - \frac{1}{2} h_k \frac{F_{x,k+1}^F}{m} C_{k+1}^F - \frac{1}{2} h_k \frac{F_{x,k+1}^H}{m} C_{k+1}^H - \frac{1}{2} h_k \frac{F_{x,k}^F}{m} C_k^F - \frac{1}{2} h_k \frac{F_{x,k}^H}{m} C_k^H = 0$$

They are linear constraints

$$\text{y-axis position: } y_{k+1} - y_k = \frac{1}{2} h_k (\dot{y}_{k+1} + \dot{y}_k) \rightarrow y_{k+1} - y_k - \frac{1}{2} h_k \dot{y}_{k+1} - \frac{1}{2} h_k \dot{y}_k = 0$$

y-axis velocity:

$$\dot{y}_{k+1} - \dot{y}_k - \frac{1}{2} h_k \left(-g + \frac{F_{y,k+1}^F}{m} + \frac{F_{y,k+1}^H}{m} \right) - \frac{1}{2} h_k \left(-g + \frac{F_{y,k}^F}{m} + \frac{F_{y,k}^H}{m} \right) = 0 \rightarrow$$

$$\dot{y}_{k+1} - \dot{y}_k + \frac{1}{2} h_k g - \frac{1}{2} h_k \frac{F_{y,k+1}^F}{m} - \frac{1}{2} h_k \frac{F_{y,k+1}^H}{m} + \frac{1}{2} h_k g - \frac{1}{2} h_k \frac{F_{y,k}^F}{m} - \frac{1}{2} h_k \frac{F_{y,k}^H}{m} = 0 \rightarrow$$

$$\dot{y}_{k+1} - \dot{y}_k - \frac{1}{2} h_k \frac{F_{y,k+1}^F}{m} - \frac{1}{2} h_k \frac{F_{y,k+1}^H}{m} - \frac{1}{2} h_k \frac{F_{y,k}^F}{m} - \frac{1}{2} h_k \frac{F_{y,k}^H}{m} = -h_k g$$

$$\text{With on/off signals (Contact or not flag): } \dot{y}_{k+1} - \dot{y}_k - \frac{1}{2} h_k \frac{F_{y,k+1}^F}{m} C_{k+1}^F - \frac{1}{2} h_k \frac{F_{y,k+1}^H}{m} C_{k+1}^H - \frac{1}{2} h_k \frac{F_{y,k}^F}{m} C_k^F - \frac{1}{2} h_k \frac{F_{y,k}^H}{m} C_k^H = -h_k g$$

Foot/End-effector Dynamics:

Front Leg:

FrontLeg x-axis:

$$P_{x,k+1}^F - P_{x,k}^F = \frac{1}{2} h_k (\dot{P}_{x,k+1}^F + \dot{P}_{x,k}^F) \rightarrow P_{x,k+1}^F - P_{x,k}^F - \frac{1}{2} h_k \dot{P}_{x,k+1}^F - \frac{1}{2} h_k \dot{P}_{x,k}^F = 0$$

FrontLeg y-axis:

$$P_{y,k+1}^F - P_{y,k}^F = \frac{1}{2} h_k (\dot{P}_{y,k+1}^F + \dot{P}_{y,k}^F) \rightarrow P_{y,k+1}^F - P_{y,k}^F - \frac{1}{2} h_k \dot{P}_{y,k+1}^F - \frac{1}{2} h_k \dot{P}_{y,k}^F = 0$$

Hind Leg:

HindLeg x-axis:

$$P_{x,k+1}^H - P_{x,k}^H = \frac{1}{2} h_k (\dot{P}_{x,k+1}^H + \dot{P}_{x,k}^H) \rightarrow P_{x,k+1}^H - P_{x,k}^H - \frac{1}{2} h_k \dot{P}_{x,k+1}^H - \frac{1}{2} h_k \dot{P}_{x,k}^H = 0$$

HindLeg y-axis:

$$P_{y,k+1}^H - P_{y,k}^H = \frac{1}{2}h_k(\dot{P}_{y,k+1}^H + \dot{P}_{y,k}^H) \rightarrow P_{y,k+1}^H - P_{y,k}^H - \frac{1}{2}h_k\dot{P}_{y,k+1}^H - \frac{1}{2}h_k\dot{P}_{y,k}^H = 0$$

Complementarity Constraint:

General Idea:

- if $C = 1 \rightarrow$ foot in contact, then
 - Foot stay on the ground: $P_y = 0$ or $P_y = \text{height}(P_x, P_y)$ (terrain height map)
 - Vertical force pointing upwards: $F_y \geq 0$
 - Non-slippage: $\dot{P}_x = 0, \dot{P}_y = 0$ (may remove if we want slippage dynamics in the future) (SEE if there are other better way to define end-effector velocity)
May use logical variables, z_{k+1} and $z_k = 0/1?$ to identify phase boundaries and knots inside a phase.
- if $C = 0 \rightarrow$ foot in the air, then
 - No forces: $F_x = 0, F_y = 0$
 - Foot above the terrain: $P_y \geq 0$

In Optimizer Constraint Form:

- Foot/End-effector position:

$$P_y \leq \text{height} + M(1 - C)$$

$$P_y \geq \text{height}$$

- Foot/End-Effector Velocity:

x-axis:

$$\dot{P}_x \leq 0 + M(1 - C)$$

$$\dot{P}_x \geq 0 - M(1 - C)$$

y-axis:

$$\dot{P}_y \leq 0 + M(1 - C)$$

$$\dot{P}_y \geq 0 - M(1 - C)$$

- Foot-ground reaction forces:

x-axis:

$$F_x \leq 0 + MC$$

$$F_x \geq 0 - MC$$

y-axis:

$$F_y \leq 0 + MC$$

$$F_y \geq 0$$

Complete form for Programming:

Front Leg:

- Foot/End-Effector Position:

$$P_y^F \leq \text{height} + M_y^{\text{pos}}(1 - C^F) \rightarrow P_y^F + M_y^{\text{pos}}C^F \leq \text{height} + M_y^{\text{pos}}$$

$$P_y^F \geq \text{height}$$

- Foot/End-effector Velocity:

x-axis:

$$\dot{P}_x^F \leq 0 + M_{vel}(1 - C^F) \rightarrow \dot{P}_x^F + M_{vel}C^F \leq 0 + M_{vel}$$

$$\dot{P}_x^F \geq 0 - M_{vel}(1 - C^F) \rightarrow \dot{P}_x^F - M_{vel}C^F \geq 0 - M_{vel}$$

y-axis:

$$\dot{P}_y^F \leq 0 + M_{vel}(1 - C^F) \rightarrow \dot{P}_y^F + M_{vel}C^F \leq 0 + M_{vel}$$

$$\dot{P}_y^F \geq 0 - M_{vel}(1 - C^F) \rightarrow \dot{P}_y^F - M_{vel}C^F \geq 0 - M_{vel}$$

- Foot-Ground Reaction Forces:

x-axis:

$$F_x^F \leq 0 + M_{fx}C^F \rightarrow F_x^F - M_{fx}C^F \leq 0$$

$$F_x^F \geq 0 - M_{fx}C^F \rightarrow F_x^F + M_{fx}C^F \geq 0$$

y-axis:

$$F_y^F \leq 0 + M_{fy}C^F \rightarrow F_y^F - M_{fy}C^F \leq 0$$

$$F_y^F \geq 0$$

Hind Leg:

- Foot/End-Effector Position:

$$P_y^H \leq height + M_y^{pos}(1 - C^H) \rightarrow P_y^H + M_y^{pos}C^H \leq height + M_y^{pos}$$

$$P_y^H \geq height$$

- Foot/End-effector Velocity:

x-axis:

$$\dot{P}_x^H \leq 0 + M_{vel}(1 - C^H) \rightarrow \dot{P}_x^H + M_{vel}C^H \leq 0 + M_{vel}$$

$$\dot{P}_x^H \geq 0 - M_{vel}(1 - C^H) \rightarrow \dot{P}_x^H - M_{vel}C^H \geq 0 - M_{vel}$$

y-axis:

$$\dot{P}_y^H \leq 0 + M_{vel}(1 - C^H) \rightarrow \dot{P}_y^H + M_{vel}C^H \leq 0 + M_{vel}$$

$$\dot{P}_y^H \geq 0 - M_{vel}(1 - C^H) \rightarrow \dot{P}_y^H - M_{vel}C^H \geq 0 - M_{vel}$$

- Foot-Ground Reaction Forces:

x-axis:

$$F_x^H \leq 0 + M_{fx}C^H \rightarrow F_x^H - M_{fx}C^H \leq 0$$

$$F_x^H \geq 0 - M_{fx}C^H \rightarrow F_x^H + M_{fx}C^H \geq 0$$

y-axis:

$$F_y^H \leq 0 + M_{fy}C^H \rightarrow F_y^H - M_{fy}C^H \leq 0$$

$$F_y^H \geq 0$$

Kinematics Constraint:

Boundary Constraints:

Initial Condition:

$$x_0 = x(0)$$

$$y_0 = y(0)$$

$$\dot{x}_0 = 0$$

$$\dot{y}_0 = 0$$

Terminal Condition:

$$x_T = x(T)$$

$$y_T = y(T)$$

$$\dot{x}_T = \dot{x}(T)$$

$$\dot{y}_T = \dot{y}(T)$$

2 To-Do List

1. Add testing functions, checking by verifying matrix dimensionality.
2. Foot/End-effector velocity constraint, is there any better to define it, to remove end-effector velocity state, using logical operations.

3 Example Codes

3.1 Motion Planning Problem for Hybrid Dynamical Systems

Given initial state x_{init} , goal state x_{goal} and environment model Ω , generate a robot motion plan:

$$\pi(t) = f_{plan}(x_{init}, x_{goal}, \Omega), t \in [0, T] \quad (1)$$

For a hybrid dynamical system like legged robots, in addition to continuous variables such as joint torque and/or foot-ground reaction forces, CoM trajectory, etc., the motion plan $\pi(t)$ also needs to include the mode switching sequence $M(t)$:

$$\pi(t) = [x(t), P_{foot}(t), \lambda(t), u(t), M(t)] \quad (2)$$

where $x(t)$ is the robot state, $P_{foot}(t)$ is the footstep locations, $\lambda(t)$ is the foot-ground reaction forces, $u(t)$ is the control input and $M(t)$ **is the mode (dynamics) that the robot undergoes at any time t :**

$$M(t) \in \{m_1, m_2, m_3, \dots, m_N\}, \text{Time-Indexed} \quad (3)$$

where the number of modes N is determined by the number of different contact configurations of the robot.

Note: the mode switching sequence can be also represented as a series (not time-indexed):

$$M(i) \in \{m_1, m_2, m_3, \dots, m_N\} \quad (4)$$

In this case scenario, after determining mode switching sequence (usually a pre-defined), it will be further augmented toward complete whole-body motion plan includes time-indexed mode switching sequences along with continuous variables (i.e. contact forces, CoM trajectory, etc.)

[?] introduces complementarity constraints to keep the trajectory optimization formulation being consistent with respect to contact changes:

$$\underset{\{x(t), u(t), \lambda(t)\}}{\text{minimize}} \quad J(x(t), u(t), \lambda(t))$$

subject to:

$$\phi(\mathbf{x}(t)) \geq \mathbf{0} \quad \text{Complementarity Constraint 1} \quad (5)$$

$$\lambda(t) \geq \mathbf{0} \quad \text{Complementarity Constraint 2} \quad (6)$$

$$\phi(\mathbf{x}(t))^T \lambda(t) = \mathbf{0} \quad \text{Complementarity Constraint 3} \quad (7)$$

$$g(x(t), u(t), \lambda(t)) \leq 0 \quad (\text{Other inequality constraints})$$

$$h(x(t), u(t), \lambda(t)) = 0 \quad (\text{Other equality Constraints}) \quad (8)$$

4 Plan

- (1) Reduce the search branches
- (2) One step look ahead

References