Documentation for Half-Cheetah Control/Planning Using Mixed-Integer Optimization

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1 Optimization Problem Formulation

1.1 Variables Definitions

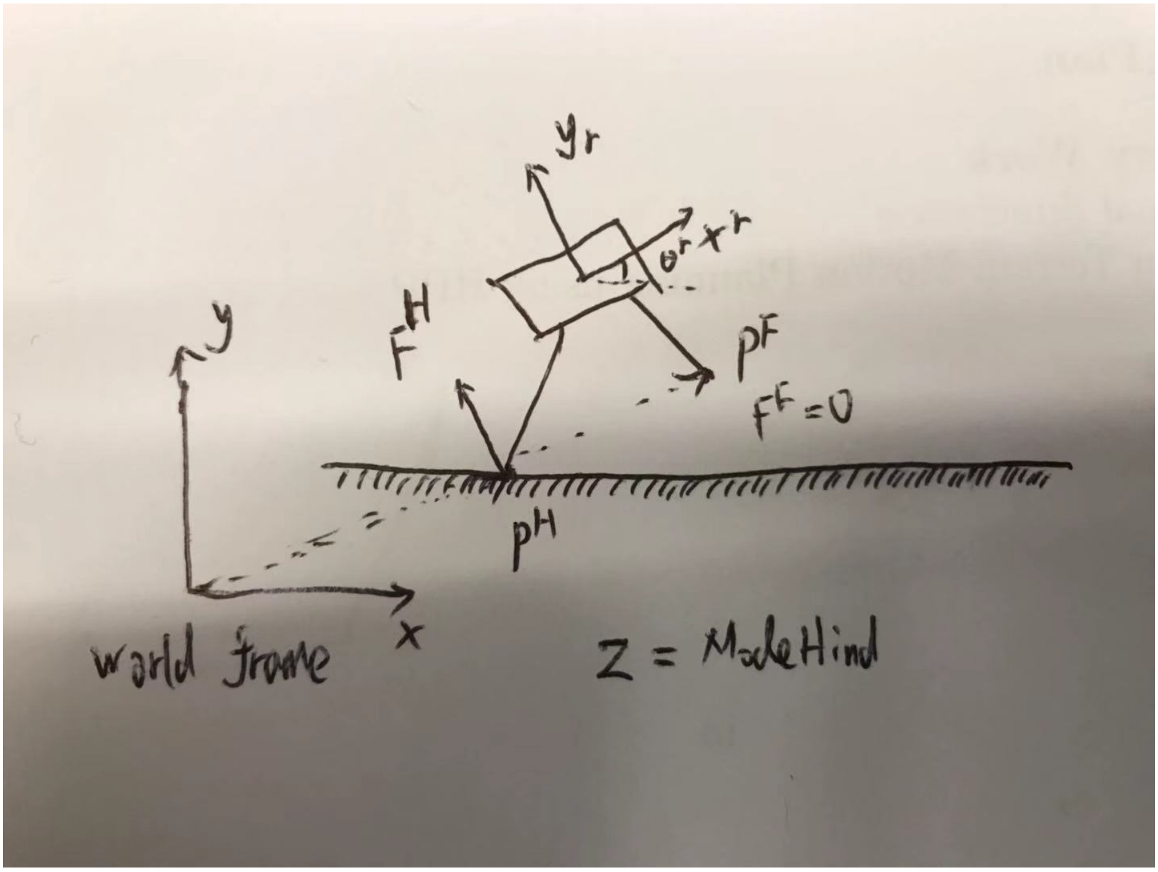

Figure 1: Variable Definition for the Half Cheetah Robot

Variables:

Robot State: $r = [x_r, y_r, \dot{x_r}, \dot{y_r}]$

Foot/End-effector States **(IN WORLD FRAME)**:
FrontLeg: $P^F = [P_x^F, P_y^F, \dot{P}_x^F, \dot{P}_y^F]$
HindLeg: $P^H = [P_x^H, P_y^H, \dot{P}_x^H, \dot{P}_y^H]$

where P_x^i and P_y^i are the positions of i^{th} foot/end-effector, $i \in F, H$. $\dot{P}_x^i$ and $\dot{P}_y^i$ are the velocities of i^{th} foot/end-effector, $i \in F, H$.

Foot-Ground Reaction Forces **(IN WORLD FRAME)**:
FrontLeg: $F^F = [F_x^F, F_y^F]$
HindLeg: $F^H = [F_x^H, F_y^H]$

Mode Selection (Two Tracks):

- Mode Enumeration: $Mode = [Mode^{fly}, Mode^{double}, Mode^{front}, Mode^{hind}]$, where $Mode^i = 0, 1$
- (Preferred) Define Contact Configuration for Every Foot: $C = [C^F, C^H]$, where $C^i = 0, 1$
  Because we can just keep dynamic constraints invariant, and constrain different foot-ground reaction forces based on foot-ground contact indicators. However, the Mode Enumeration is tedious because we need to write different dynamics, footstep locations, foot-ground reaction forces constraints (big-M) based on mode indicator variables

1.2 Objective/Cost Function and Constraints

1.2.1 Objective/Cost Function

Quadratic form: $c = v^T Q v$

where v is the decision variable vector, it should be a column vector, Q is the selection matrix for the Quadratic Form.

$v^T Q v = v_1 Q_{11} v_1 + v_1 Q_{12} v_2 + ... + v_2 Q_{21} v_1 + v_2 Q_{22} v_2 + ...$
$Q_{i,j} \in R^{dim(v_i) x dim(v_j)}$

1.2.2 Constraints

System Dynamics:
$m\ddot{x} = F_x^F + F_x^H \rightarrow \ddot{x} = \frac{F_x^F}{m} + \frac{F_x^H}{m} \rightarrow \ddot{x} = f_x()$

$m\ddot{y} = -my + F_y^F + F_y^H \rightarrow \ddot{y} = -g + \frac{F_y^F}{m} + \frac{F_y^H}{m} \rightarrow \ddot{y} = f_y()$

Trapezoidal Collocation:
$x_{k+1} - x_k = \frac{1}{2} h_k (f_{k+1} + f_k) \rightarrow x_{k+1} - x_k - \frac{1}{2} h_k f_{k+1} - \frac{1}{2} h_k f_k = 0$

Apply Trapezoidal Collocation into translational dynamics:
x-axis position: $x_{k+1} - x_k = \frac{1}{2} h_k (\dot{x}_{k+1} + \dot{x}_k) \rightarrow x_{k+1} - x_k - \frac{1}{2} h_k \dot{x}_{k+1} - \frac{1}{2} h_k \dot{x}_k = 0$
x-axis velocity:
$\dot{x}_{k+1} - \dot{x}_k - \frac{1}{2} h_k (\frac{F_{x,k+1}^F}{m} + \frac{F_{x,k+1}^H}{m}) - \frac{1}{2} h_k (\frac{F_{x,k}^F}{m} + \frac{F_{x,k}^H}{m}) = 0 \rightarrow$
$\dot{x}_{k+1} - \dot{x}_k - \frac{1}{2} h_k \frac{F_{x,k+1}^F}{m} - \frac{1}{2} h_k \frac{F_{x,k+1}^H}{m} - \frac{1}{2} h_k \frac{F_{x,k}^F}{m} - \frac{1}{2} h_k \frac{F_{x,k}^H}{m} = 0$
With on/off signals (Contact or not flag): $\dot{x}_{k+1} - \dot{x}_k - \frac{1}{2} h_k \frac{F_{x,k+1}^F}{m} C_{k+1}^F - \frac{1}{2} h_k \frac{F_{x,k+1}^H}{m} C_{k+1}^H - \frac{1}{2} h_k \frac{F_{x,k}^F}{m} C_k^F - \frac{1}{2} h_k \frac{F_{x,k}^H}{m} C_k^H = 0$
They are linear constraints

y-axis position: $y_{k+1} - y_k = \frac{1}{2} h_k (\dot{y}_{k+1} + \dot{y}_k) \rightarrow y_{k+1} - y_k - \frac{1}{2} h_k \dot{y}_{k+1} - \frac{1}{2} h_k \dot{y}_k = 0$
y-axis velocity:
$\dot{y}_{k+1} - \dot{y}_k - \frac{1}{2} h_k (-g + \frac{F_{y,k+1}^F}{m} + \frac{F_{y,k+1}^H}{m}) - \frac{1}{2} h_k (-g + \frac{F_{y,k}^F}{m} + \frac{F_{y,k}^H}{m}) = 0 \rightarrow$
$\dot{y}_{k+1} - \dot{y}_k + \frac{1}{2} h_k g - \frac{1}{2} h_k \frac{F_{y,k+1}^F}{m} - \frac{1}{2} h_k \frac{F_{y,k+1}^H}{m} + \frac{1}{2} h_k g - \frac{1}{2} h_k \frac{F_{y,k}^F}{m} - \frac{1}{2} h_k \frac{F_{y,k}^H}{m} = 0 \rightarrow$
$\dot{y}_{k+1} - \dot{y}_k - \frac{1}{2} h_k \frac{F_{y,k+1}^F}{m} - \frac{1}{2} h_k \frac{F_{y,k+1}^H}{m} - \frac{1}{2} h_k \frac{F_{y,k}^F}{m} - \frac{1}{2} h_k \frac{F_{y,k}^H}{m} = -h_k g$
With on/off signals (Contact or not flag): $\dot{y}_{k+1} - \dot{y}_k - \frac{1}{2} h_k \frac{F_{y,k+1}^F}{m} C_{k+1}^F - \frac{1}{2} h_k \frac{F_{y,k+1}^H}{m} C_{k+1}^H - \frac{1}{2} h_k \frac{F_{y,k}^F}{m} C_k^F - \frac{1}{2} h_k \frac{F_{y,k}^H}{m} C_k^H = -h_k g$

Foot/End-effector Dynamics:
Front Leg:
FrontLeg x-axis:
$P_{x,k+1}^F - P_{x,k}^F = \frac{1}{2} h_k (\dot{P}_{x,k+1}^F + \dot{P}_{x,k}^F) \rightarrow P_{x,k+1}^F - P_{x,k}^F - \frac{1}{2} h_k \dot{P}_{x,k+1}^F - \frac{1}{2} h_k \dot{P}_{x,k}^F = 0$

FrontLeg y-axis:
$P_{y,k+1}^F - P_{y,k}^F = \frac{1}{2} h_k (\dot{P}_{y,k+1}^F + \dot{P}_{y,k}^F) \rightarrow P_{y,k+1}^F - P_{y,k}^F - \frac{1}{2} h_k \dot{P}_{y,k+1}^F - \frac{1}{2} h_k \dot{P}_{y,k}^F = 0$

Hind Leg:
HindLeg x-axis:
$P_{x,k+1}^H - P_{x,k}^H = \frac{1}{2} h_k (\dot{P}_{x,k+1}^H + \dot{P}_{x,k}^H) \rightarrow P_{x,k+1}^H - P_{x,k}^H - \frac{1}{2} h_k \dot{P}_{x,k+1}^H - \frac{1}{2} h_k \dot{P}_{x,k}^H = 0$

HindLeg y-axis:

$P^H_{y,k+1} - P^H_{y,k} = \frac{1}{2}h_k(\dot{P}^H_{y,k+1} + \dot{P}^H_{y,k}) \rightarrow P^H_{y,k+1} - P^H_{y,k} - \frac{1}{2}h_k\dot{P}^H_{y,k+1} - \frac{1}{2}h_k\dot{P}^H_{y,k} = 0$

Complementarity Constraint:

General Idea:

- if $C = 1 \rightarrow$ foot in contact, then
  - Foot stay on the ground: $P_y = 0$ or $P_y = height(P_x, Py)$ (terrain height map)
  - Vertical force pointing upwards: $F_y \geq 0$
  - Non-slippage: $\dot{P}_x = 0, \dot{P}_y = 0$ (may remove if we want slippage dynamics in the future) (SEE if there are other better way to define end-effector velocity)
    May use logical variables, z_{k+1} and $z_k = 0/1$? to identify phase boundaries and knots inside a phase.
- if $C = 0 \rightarrow$ foot in the air, then
  - No forces: $F_x = 0, F_y = 0$
  - Foot above the terrain: $P_y \geq 0$

In Optimizer Constraint Form:

- Foot/End-effector position:
  $P_y \leq height + M(1 - C)$
  $P_y \geq height$
- Foot/End-Effector Velocity:
  x-axis:
  $\dot{P}_x \leq 0 + M(1 - C)$
  $\dot{P}_x \geq 0 - M(1 - C)$
  y-axis:
  $\dot{P}_y \leq 0 + M(1 - C)$
  $\dot{P}_y \geq 0 - M(1 - C)$
- Foot-ground reaction forces:
  x-axis:
  $F_x \leq 0 + MC$
  $F_x \geq 0 - MC$
  y-axis:
  $F_y \leq 0 + MC$
  $F_y \geq 0$

Complete form for Programming:
Front Leg:

- Foot/End-Effector Position:
  $P^F_y \leq height + M^{pos}_y(1 - C^F) \rightarrow P^F_y + M^{pos}_y C^F \leq height + M^{pos}_y$
  $P^F_y \geq height$
- Foot/End-effector Velocity:
  x-axis:
  $\dot{P}^F_x \leq 0 + M_{vel}(1 - C^F) \rightarrow \dot{P}^F_x + M_{vel}C^F \leq 0 + M_{vel}$
  $\dot{P}^F_x \geq 0 - M_{vel}(1 - C^F) \rightarrow \dot{P}^F_x - M_{vel}C^F \geq 0 - M_{vel}$
  y-axis:
  $\dot{P}^F_y \leq 0 + M_{vel}(1 - C^F) \rightarrow \dot{P}^F_y + M_{vel}C^F \leq 0 + M_{vel}$
  $\dot{P}^F_y \geq 0 - M_{vel}(1 - C^F) \rightarrow \dot{P}^F_y - M_{vel}C^F \geq 0 - M_{vel}$

- Foot-Ground Reaction Forces:

  x-axis:

  $F_x^F \leq 0 + M_{fx}C^F \rightarrow F_x^F - M_{fx}C^F \leq 0$

  $F_x^F \geq 0 - M_{fx}C^F \rightarrow F_x^F + M_{fx}C^F \geq 0$

  y-axis:

  $F_y^F \leq 0 + M_{fy}C^F \rightarrow F_y^F - M_{fy}C^F \leq 0$

  $F_y^F \geq 0$

Hind Leg:

- Foot/End-Effector Position:

  $P_y^H \leq height + M_y^{pos}(1 - C^H) \rightarrow P_y^H + M_y^{pos}C^H \leq height + M_y^{pos}$

  $P_y^H \geq height$

- Foot/End-effector Velocity:

  x-axis:

  $\dot{P}_x^H \leq 0 + M_{vel}(1 - C^H) \rightarrow \dot{P}_x^H + M_{vel}C^H \leq 0 + M_{vel}$

  $\dot{P}_x^H \geq 0 - M_{vel}(1 - C^H) \rightarrow \dot{P}_x^H - M_{vel}C^H \geq 0 - M_{vel}$

  y-axis:

  $\dot{P}_y^H \leq 0 + M_{vel}(1 - C^H) \rightarrow \dot{P}_y^H + M_{vel}C^H \leq 0 + M_{vel}$

  $\dot{P}_y^H \geq 0 - M_{vel}(1 - C^H) \rightarrow \dot{P}_y^H - M_{vel}C^H \geq 0 - M_{vel}$

- Foot-Ground Reaction Forces:

  x-axis:

  $F_x^H \leq 0 + M_{fx}C^H \rightarrow F_x^H - M_{fx}C^H \leq 0$

  $F_x^H \geq 0 - M_{fx}C^H \rightarrow F_x^H + M_{fx}C^H \geq 0$

  y-axis:

  $F_y^H \leq 0 + M_{fy}C^H \rightarrow F_y^H - M_{fy}C^H \leq 0$

  $F_y^H \geq 0$

Kinematics Constraint:

Boundary Constraints:
Initial Condition:
$x_0 = x(0)$
$y_0 = y(0)$
$\dot{x}_0 = 0$
$\dot{y}_0 = 0$

Terminal Condition:
$x_T = x(T)$
$y_T = y(T)$
$\dot{x}_T = \dot{x}(T)$
$\dot{y}_T = \dot{y}(T)$

2 To-Do List

1. Add testing functions, checking by verifying matrix dimensionality.
2. Foot/End-effector velocity constraint, is there any better to define it, to remove end-effector velocity state, using logical operations.

3 Example Codes

3.1 Motion Planning Problem for Hybrid Dynamical Systems

Given initial state x_{init}, goal state x_{goal} and environment model Ω, generate a robot motion plan:

$$\pi(t) = f_{plan}(x_{init}, x_{goal}, \Omega), t \in [0, T] \tag{1}$$

For a hybrid dynamical system like legged robots, in addition to continuous variables such as joint torque and/or foot-ground reaction forces, CoM trajectory, etc., the motion plan $\pi(t)$ also needs to include the mode switching sequence $M(t)$:

$$(2)$$

$$\pi(t) = [x(t), P_{foot}(t), \lambda(t), u(t), M(t)] \tag{3}$$

where $x(t)$ is the robot state, $P_{foot}(t)$ is the footstep locations, $\lambda(t)$ is the foot-ground reaction forces, $u(t)$ is the control input and $M(t)$ **is the mode (dynamics) that the robot undergoes at any time** t:

$$(4)$$

$$M(t) \in \{m_1, m_2, m_3, ...m_N\}, \text{Time-Indexed} \tag{5}$$

where the number of modes N is determined by the number of different contact configurations of the robot.

Note: the mode switching sequence can be also represented as a series (not time-indexed):

$$(6)$$

$$M(i) \in \{m_1, m_2, m_3, ...m_N\} \tag{7}$$

In this case scenario, after determining mode switching sequence (usually a pre-defined), it will be further augmented toward complete whole-body motion plan includes time-indexed mode switching sequences along with continuous variables (i.e. contact forces, CoM trajectory, etc.)

[**?**] introduces complementarity constraints to keep the trajectory optimization formulation being consistent with respect to contact changes:

$$\underset{\{x(t),u(t),\lambda(t)\}}{\text{minimize}} \; J(x(t), u(t), \lambda(t))$$

$$\text{subject to:}$$

$$\phi(\mathbf{x(t)}) \geq \mathbf{0} \qquad \textbf{Complementarity Constraint 1} \tag{8}$$

$$\lambda(\mathbf{t}) \geq \mathbf{0} \qquad \textbf{Complementarity Constraint 2} \tag{9}$$

$$\phi(\mathbf{x(t)})^{\mathbf{T}}\lambda(\mathbf{t}) = \mathbf{0} \qquad \textbf{Complementarity Constraint 3} \tag{10}$$

$$g(x(t), u(t), \lambda(t)) \leq 0 \qquad \text{(Other inequility constraints)}$$

$$h(x(t), u(t), \lambda(t)) = 0 \qquad \text{(Other equality Constraints)} \tag{11}$$

4 Plan

(1) Reduce the search branches
(2) One step look ahead

References