Documentation for Half-Cheetah Control/Planning Using Mixed-Integer Optimization

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1 Optimization Problem Formulation

1.1 Variables Definitions

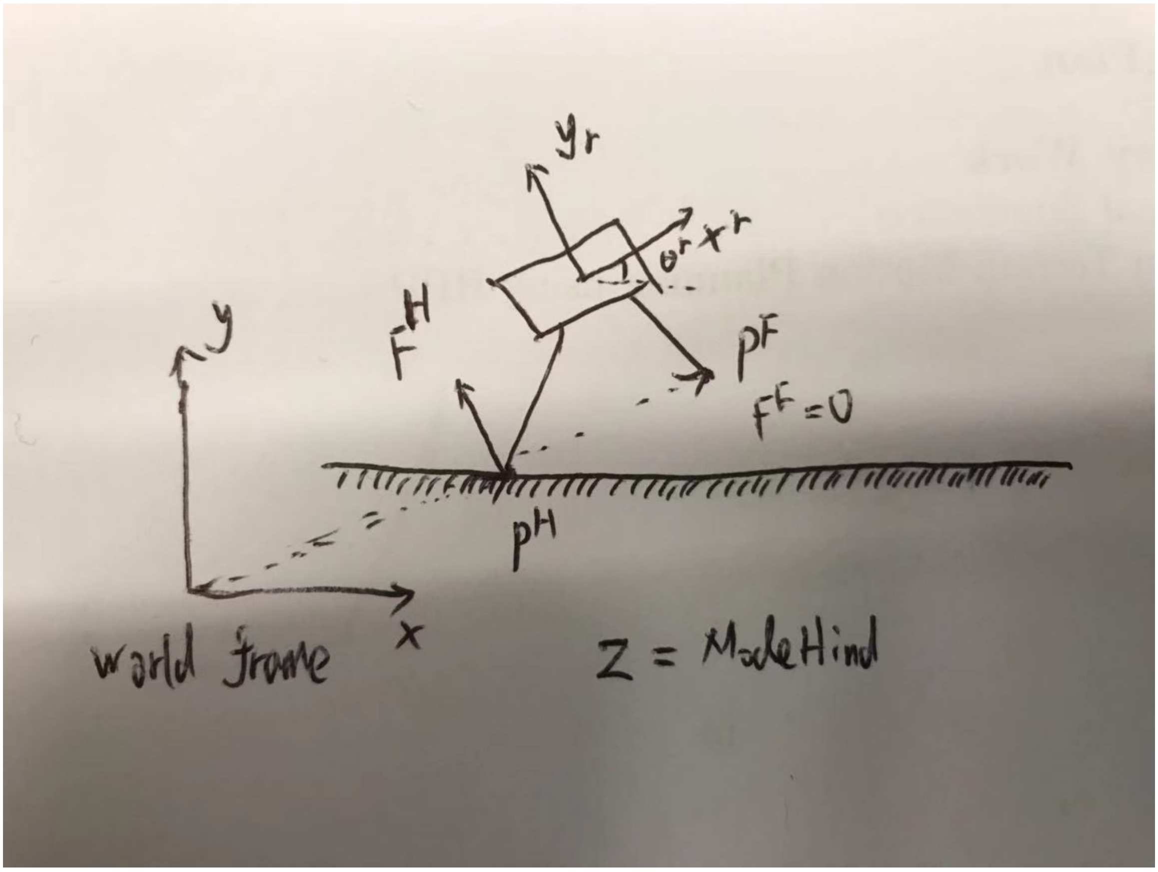

Figure 1: Variable Definition for the 2D Half Cheetah Robot

Variables:

Robot State: $[x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}]$
x: Horizontal position of the torso
y: Vertical position of the torso
θ: Torso orientation
$\dot{x}$: Horizontal Velocity of the torso
$\dot{y}$: Vertical Velocity of the torso
$\dot{\theta}$: Angular Velocity of the torso
Define $r = [x, y]$

Foot/End-effector States (IN WORLD FRAME):
FrontLeg:

Position: $P^F = [P^F_x, P^F_y]$
Velocity: $\dot{P}^F = [\dot{P}^F_x, \dot{P}^F_y]$

HindLeg:
Position: $P^H = [P^H_x, P^H_y]$
Velocity: $\dot{P}^H = [\dot{P}^H_x, \dot{P}^H_y]$

where P^i_x and P^i_y are the positions of i^{th} foot/end-effector in **in world frame**, $i \in F, H$. $\dot{P}^i_x$ and $\dot{P}^i_y$ are the velocities of i^{th} foot/end-effector in **world frame**, $i \in F, H$.

Foot-Ground Reaction Forces (IN WORLD FRAME):
FrontLeg: $F^F = [F^F_x, F^F_y]$
HindLeg: $F^H = [F^H_x, F^H_y]$

Contact Modes (Discrete Variables; Two Tracks):

- **(Use only when necessary)** Mode Enumeration: $Mode = [Mode^{fly}, Mode^{double}, Mode^{front}, Mode^{hind}]$, where $Mode^{name} = 0, 1$
- **(Preferred)** Define Contact Configurations for Every Foot: $C = [C^F, C^H]$, where $C^i = 0, 1$

  Using this formulation, we can keep dynamic constraints invariant, and ask all foot to obey complementarity constraints, kinematics constraint and friction cone. So, the foot-ground reaction forces may flow into system dynamics equation with zero values when legs are set as swing legs, or they will flow with some values to drive the torso to move, when they are set as stance legs.

  In contrary, the Mode Enumeration is tedious because we need to write different system dynamic constraint, footstep location, foot-ground reaction forces constraints (big-M) based on mode indicator variables.

1.2 Objective/Cost Function

NOTE: For higher order integration scheme, we need to approximate the cost function using quadrature methods
Quadratic form: $obj = v^T Q v$
where v is the decision variable vector, it should be a column vector, Q is the selection matrix for the Quadratic Form.

$v^T Q v = v_1 Q_{11} v_1 + v_1 Q_{12} v_2 + ... + v_2 Q_{21} v_1 + v_2 Q_{22} v_2 + ...$
$Q_{i,j} \in R^{dim(v_i) \times dim(v_j)}$

1.3 Constraints

1.3.1 System Dynamics Constraint

System Dynamical Equations:
x-axis:
$m\ddot{x} = F^F_x + F^H_x \rightarrow$
$\ddot{x} = \frac{F^F_x}{m} + \frac{F^H_x}{m} \rightarrow$
$\ddot{x} = f_x()$

y-axis:
$m\ddot{y} = -mg + F^F_y + F^H_y \rightarrow$
$\ddot{y} = -g + \frac{F^F_y}{m} + \frac{F^H_y}{m} \rightarrow$
$\ddot{y} = f_y()$

rotation-axis:
$I\ddot{\theta} = (P^F - r) \times F^F + (P^H - r) \times F^H \rightarrow$
$I\ddot{\theta} - (P^F - r) \times F^F - (P^H - r) \times F^H = 0$

Quadrature Schemes:
Euler Integration (Currently in Use):
$x_{k+1} - x_k = h_k f_k \rightarrow$

$x_{k+1} - x_k - h_k f_k = 0$

Trapezoidal Collocation:
$x_{k+1} - x_k = \frac{1}{2}h_k(f_{k+1} + f_k) \rightarrow$
$x_{k+1} - x_k - \frac{1}{2}h_k f_{k+1} - \frac{1}{2}h_k f_k = 0$

Apply Euler Integration into Dynamical Equations:
x-axis position (linear constraints; first-order):
$x_{k+1} - x_k = h\dot{x}_k \rightarrow$
$x_{k+1} - x_k - h\dot{x}_k = 0$
x-axis velocity (linear constraints; second-order):
$\dot{x}_{k+1} - \dot{x}_k = \frac{h}{m}F_x^F + \frac{h}{m}F_x^H \rightarrow$
$\dot{x}_{k+1} - \dot{x}_k - \frac{h}{m}F_x^F - \frac{h}{m}F_x^H = 0$
y-axis position (linear constraints; first-order):
$y_{k+1} - y_k = h\dot{y}_k \rightarrow$
$y_{k+1} - y_k - h\dot{y}_k = 0$
y-axis velocity (linear constraints; second-order):
$\dot{y}_{k+1} - \dot{y}_k = -hg + \frac{h}{m}F_y^F + \frac{h}{m}F_y^H \rightarrow$
$\dot{y}_{k+1} - \dot{y}_k - h\frac{F_y^F}{m} - h\frac{F_y^H}{m} = -hg$
rotation-axis position (linear constraints; first-order):
$\theta_{k+1} - \theta_k = h\dot{\theta} \rightarrow$
$\theta_{k+1} - \theta_k - h\dot{\theta} = 0$

rotation-axis velocity Vector Form (Nonlinear Constraints; second order):
$I\dot{\theta}_{k+1} - I\dot{\theta}_k = h[(P_k^F - r_k) \times F_k^F + (P_k^H - r_k) \times F_k^H] \rightarrow$
$I\dot{\theta}_{k+1} - I\dot{\theta}_k - h[(P_k^F - r_k) \times F_k^F - (P_k^H - r_k) \times F_k^H = 0$
Cross Product Note:
$a = [a_x, a_y], b = [b_x, b_y]$
$a \times b = a_x b_y - a_y b_x$
Cross Produce of Contact Forces and Foot Position Vectors:
$\tau_k^{sum} = (P_{x,k}^F - x_k)F_{y,k}^F - (P_{y,k}^F - y_k)F_{x,k}^F + (P_{x,k}^H - x_k)F_{y,k}^H - (P_{y,k}^H - y_k)F_{x,k}^H$
Then rotation axis velocity contraints becomes (Nonlinear Constraints; second order):
$I\dot{\theta}_{k+1} - I\dot{\theta}_k - h\tau_k^{sum} = 0 \rightarrow$
$I\dot{\theta}_{k+1} - I\dot{\theta}_k - h[(P_{x,k}^F - x_k)F_{y,k}^F - (P_{y,k}^F - y_k)F_{x,k}^F + (P_{x,k}^H - x_k)F_{y,k}^H - (P_{y,k}^H - y_k)F_{x,k}^H] = 0$

Apply Trapezoidal Collocation into Dynamical Equations (Currently Not in Use; Missing Rotation Dynamics):
x-axis position (linear constraints; first-order):
$x_{k+1} - x_k = \frac{1}{2}h_k(\dot{x}_{k+1} + \dot{x}_k) \rightarrow$
$x_{k+1} - x_k - \frac{1}{2}h_k\dot{x}_{k+1} - \frac{1}{2}h_k\dot{x}_k = 0$
x-axis velocity (linear constraints; second-order):
$\dot{x}_{k+1} - \dot{x}_k - \frac{1}{2}h_k(\frac{F_{x,k+1}^F}{m} + \frac{F_{x,k+1}^H}{m}) - \frac{1}{2}h_k(\frac{F_{x,k}^F}{m} + \frac{F_{x,k}^H}{m}) = 0 \rightarrow$
$\dot{x}_{k+1} - \dot{x}_k - \frac{1}{2}h_k\frac{F_{x,k+1}^F}{m} - \frac{1}{2}h_k\frac{F_{x,k+1}^H}{m} - \frac{1}{2}h_k\frac{F_{x,k}^F}{m} - \frac{1}{2}h_k\frac{F_{x,k}^H}{m} = 0$
y-axis position (linear constraints; first-order):
$y_{k+1} - y_k = \frac{1}{2}h_k(\dot{y}_{k+1} + \dot{y}_k) \rightarrow$
$y_{k+1} - y_k - \frac{1}{2}h_k\dot{y}_{k+1} - \frac{1}{2}h_k\dot{y}_k = 0$
y-axis velocity (linear constraints; second-order):
$\dot{y}_{k+1} - \dot{y}_k - \frac{1}{2}h_k(-g + \frac{F_{y,k+1}^F}{m} + \frac{F_{y,k+1}^H}{m}) - \frac{1}{2}h_k(-g + \frac{F_{y,k}^F}{m} + \frac{F_{y,k}^H}{m}) = 0 \rightarrow$
$\dot{y}_{k+1} - \dot{y}_k + \frac{1}{2}h_k g - \frac{1}{2}h_k\frac{F_{y,k+1}^F}{m} - \frac{1}{2}h_k\frac{F_{y,k+1}^H}{m} + \frac{1}{2}h_k g - \frac{1}{2}h_k\frac{F_{y,k}^F}{m} - \frac{1}{2}h_k\frac{F_{y,k}^H}{m} = 0 \rightarrow$
$\dot{y}_{k+1} - \dot{y}_k - \frac{1}{2}h_k\frac{F_{y,k+1}^F}{m} - \frac{1}{2}h_k\frac{F_{y,k+1}^H}{m} - \frac{1}{2}h_k\frac{F_{y,k}^F}{m} - \frac{1}{2}h_k\frac{F_{y,k}^H}{m} = -h_k g$

Foot/End-effector Dynamics:

Euler Integration Formulation:
Front Leg x-axis position (linear constraint; first-order):

$P^F_{x,k+1} - P^F_{x,k} = h\dot{P}^F_{x,k} \rightarrow$
$P^F_{x,k+1} - P^F_{x,k} - h\dot{P}^F_{x,k} = 0$
Front Leg y-aixs position (linear constraint; first-order):
$P^F_{y,k+1} - P^F_{y,k} = h\dot{P}^F y,k \rightarrow$
$P^F_{y,k+1} - P^F_{y,k} - h\dot{P}^F_{y,k} = 0$
Hind Leg x-axis position (linear constraint; first-order):
$P^H_{x,k+1} - P^H_{x,k} = h\dot{P}^H_{x,k} \rightarrow$
$P^H_{x,k+1} - P^H_{x,k} - h\dot{P}^H_{x,k} = 0$
Hind Leg y-axis position (linear constraint; first order):
$P^H_{y,k+1} - P^H_{y,k} = h\dot{P}^H_{y,k}$
$P^H_{y,k+1} - P^H_{y,k} - h\dot{P}^H_{y,k} = 0$

Trapezoidal Quadrature for Foot Dynamics (Not in Use, because it cannot handle the discontinuity of control signals, introduced by contact mode switches)]
Front Leg:
FrontLeg x-axis:
$P^F_{x,k+1} - P^F_{x,k} = \frac{1}{2}h_k(\dot{P}^F_{x,k+1} + \dot{P}^F_{x,k}) \rightarrow P^F_{x,k+1} - P^F_{x,k} - \frac{1}{2}h_k\dot{P}^F_{x,k+1} - \frac{1}{2}h_k\dot{P}^F_{x,k} = 0$
FrontLeg y-axis:
$P^F_{y,k+1} - P^F_{y,k} = \frac{1}{2}h_k(\dot{P}^F_{y,k+1} + \dot{P}^F_{y,k}) \rightarrow P^F_{y,k+1} - P^F_{y,k} - \frac{1}{2}h_k\dot{P}^F_{y,k+1} - \frac{1}{2}h_k\dot{P}^F_{y,k} = 0$

Hind Leg:
HindLeg x-axis:
$P^H_{x,k+1} - P^H_{x,k} = \frac{1}{2}h_k(\dot{P}^H_{x,k+1} + \dot{P}^H_{x,k}) \rightarrow P^H_{x,k+1} - P^H_{x,k} - \frac{1}{2}h_k\dot{P}^H_{x,k+1} - \frac{1}{2}h_k\dot{P}^H_{x,k} = 0$
HindLeg y-axis:
$P^H_{y,k+1} - P^H_{y,k} = \frac{1}{2}h_k(\dot{P}^H_{y,k+1} + \dot{P}^H_{y,k}) \rightarrow P^H_{y,k+1} - P^H_{y,k} - \frac{1}{2}h_k\dot{P}^H_{y,k+1} - \frac{1}{2}h_k\dot{P}^H_{y,k} = 0$

1.3.2 Complementarity Constraint:

General Idea:

- if $C^i = 1 \rightarrow$ foot in contact, then
  - Foot stay on the ground: $P_y = 0$ or $P_y = height(P_x, Py)$ (terrain height map)
  - Vertical force pointing upwards: $F_y \geq 0$
  - Non-slippage: $\dot{P}_x = 0, \dot{P}_y = 0$ (may remove if we want slippage dynamics in the future) (SEE if there are other better way to define end-effector velocity, using footstep locations only)
    May use logical variables, z_{k+1} and $z_k = 0/1$? to identify phase boundaries and knots inside a phase.
- if $C^i = 0 \rightarrow$ foot in the air, then
  - No forces: $F_x = 0, F_y = 0$
  - Foot above the terrain: $P_y \geq 0$

In Optimizer Constraint Form:

- Foot/End-effector position:
  $P_y \leq height + M^{pos}(1 - C)$
  $P_y \geq height$
- Foot/End-Effector Velocity:
  x-axis:
  $\dot{P}_x \leq 0 + M^{vel}(1 - C)$
  $\dot{P}_x \geq 0 - M^{vel}(1 - C)$
  y-axis:
  $\dot{P}_y \leq 0 + M^{vel}(1 - C)$
  $\dot{P}_y \geq 0 - M^{vel}(1 - C)$

- Foot-ground reaction forces:
  x-axis:
  $F_x \leq 0 + M^f C$
  $F_x \geq 0 - M^f C$
  y-axis:
  $F_y \leq 0 + M^f C$
  $F_y \geq 0$

Complete form for Programming:
Front Leg:

- Foot/End-Effector Position (y-axis only):
  $P_y^F \leq height + M_y^{pos}(1 - C^F) \rightarrow P_y^F + M_y^{pos} C^F \leq height + M_y^{pos}$
  $P_y^F \geq height$
- Foot/End-effector Velocity:
  x-axis:
  $\dot{P}_x^F \leq 0 + M^{vel}(1 - C^F) \rightarrow \dot{P}_x^F + M^{vel} C^F \leq 0 + M^{vel}$
  $\dot{P}_x^F \geq 0 - M^{vel}(1 - C^F) \rightarrow \dot{P}_x^F - M^{vel} C^F \geq 0 - M^{vel}$
  y-axis:
  $\dot{P}_y^F \leq 0 + M^{vel}(1 - C^F) \rightarrow \dot{P}_y^F + M^{vel} C^F \leq 0 + M^{vel}$
  $\dot{P}_y^F \geq 0 - M^{vel}(1 - C^F) \rightarrow \dot{P}_y^F - M^{vel} C^F \geq 0 - M^{vel}$
- Foot-Ground Reaction Forces:
  x-axis:
  $F_x^F \leq 0 + M_{fx} C^F \rightarrow F_x^F - M^f C^F \leq 0$
  $F_x^F \geq 0 - M_{fx} C^F \rightarrow F_x^F + M^f C^F \geq 0$
  y-axis:
  $F_y^F \leq 0 + M_{fy} C^F \rightarrow F_y^F - M^f C^F \leq 0$
  $F_y^F \geq 0$

Hind Leg:

- Foot/End-Effector Position:
  $P_y^H \leq height + M_y^{pos}(1 - C^H) \rightarrow P_y^H + M_y^{pos} C^H \leq height + M_y^{pos}$
  $P_y^H \geq height$
- Foot/End-effector Velocity:
  x-axis:
  $\dot{P}_x^H \leq 0 + M^{vel}(1 - C^H) \rightarrow \dot{P}_x^H + M^{vel} C^H \leq 0 + M^{vel}$
  $\dot{P}_x^H \geq 0 - M^{vel}(1 - C^H) \rightarrow \dot{P}_x^H - M^{vel} C^H \geq 0 - M^{vel}$
  y-axis:
  $\dot{P}_y^H \leq 0 + M^{vel}(1 - C^H) \rightarrow \dot{P}_y^H + M^{vel} C^H \leq 0 + M^{vel}$
  $\dot{P}_y^H \geq 0 - M^{vel}(1 - C^H) \rightarrow \dot{P}_y^H - M^{vel} C^H \geq 0 - M^{vel}$
- Foot-Ground Reaction Forces:
  x-axis:
  $F_x^H \leq 0 + M^f C^H \rightarrow F_x^H - M^f C^H \leq 0$
  $F_x^H \geq 0 - M^f C^H \rightarrow F_x^H + M^f C^H \geq 0$
  y-axis:
  $F_y^H \leq 0 + M^f C^H \rightarrow F_y^H - M^f C^H \leq 0$
  $F_y^H \geq 0$

1.3.3 Kinematics Constraint (Nonlinear Constraints)

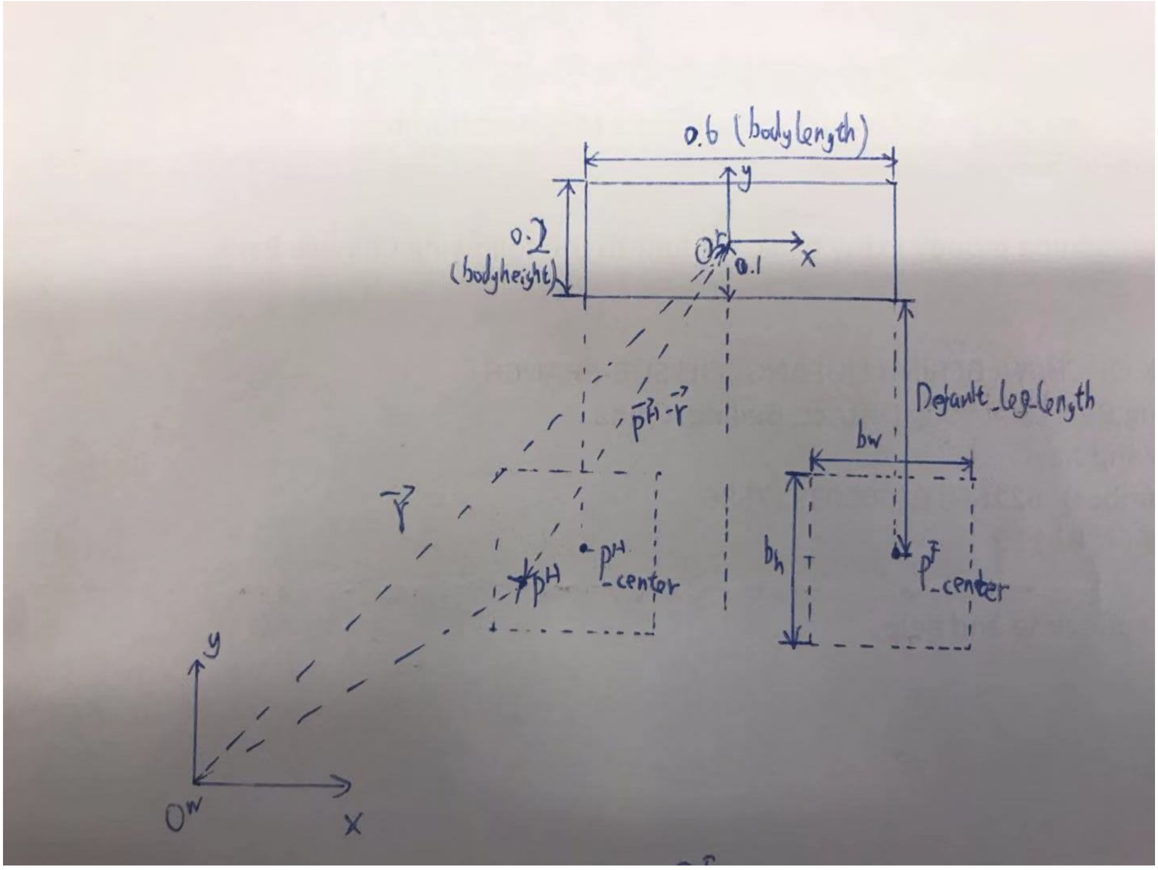

Figure 2: Kinematics Constraint of the 2D Half Cheetah Robot

Variable Definitions:
Body Length: L_{Body}
Body Height: H_{Body}
Default Leg Length: $\bar{L}_{leg}$
Default (Center) Position of Front Leg (**in robot local frame**):
$P^F_{center} = [\frac{1}{2}L_{Body}, -(\frac{1}{2}H_{Body} + \bar{L}_{leg})]$
Default (Center) Position of Hind Leg (**in robot local frame**):
$P^H_{center} = [-\frac{1}{2}L_{Body}, -\frac{1}{2}H_{Body} + \bar{L}_{leg}]$
Width of the bounding box: b_w
Height of the bounding box: b_h
bounding box vector: $b = [b_w, b_h]$

Kinematics Constraint (Augment to be consistent with program):
$-b/2 \leq R(\theta_k)(P_k - r_k) - P_{center} \leq b/2$

1.3.4 Friction Cone Constraint (Nonlinear Constraint for Uneven Terrain)

Variable Definition:
Terrain Norm: $N = [N_x, N_y]$
Terrain Norm will become a function of $r = [x, y]$ for uneven terrain cases
Friction Coefficient: μ

Friction Cone Constraint:
In 2D:
$F^F_{x,k} \leq \mu(N_x(r_k)F_{x,k} + N_y(r_k)F_{y,k})$
In 3D **(Nonlinear Constraint):**
$\sqrt{F_x^2 + F_y^2} \leq \mu f_n$

$f_n = [F^F_x, F^F_y]' * [N_x, N_y]$

1.3.5 Boundary Constraints

Initial Condition:
$x_0 = x(0)$
$y_0 = y(0)$
$\theta_0 = \theta(0)$
$\dot{x}_0 = \dot{x}(0)$
$\dot{y}_0 = \dot{y}(0)$
$\dot{\theta}_0 = \dot{\theta}(0)$

Terminal Condition:
$x_T = x(T)$
$y_T = y(T)$
$\theta_T = \theta(T)$
$\dot{x}_T = \dot{x}(T)$
$\dot{y}_T = \dot{y}(T)$
$\dot{\theta}_T = \dot{\theta}(T)$

1.3.6 Switching Time

Introduce a variable $\tau \in [0, 1]$, the upper bound of τ can be larger but better take it normalised.
Cut τ into N phases.
Switching Time Vector (Terminal Time of Each Phase) $Ts = [Ts_1, Ts_2, ..., Tend]$
Integration Step for each Phase:
$h = N * (Ts_{i+1} - Ts_i) * \tau_h$
Switching Time Follow the Constraint:
(1) $0 \leq Ts_1 \leq Ts_2... \leq Tend$
where $\tau_h = 1/\text{Number of Knots}$

2 To-Do List

1. Add testing functions, checking by verifying matrix dimensionality.
2. Foot/End-effector velocity constraint, is there any better to define it, to remove end-effector velocity state, using logical operations.

References