Math 7580 Project

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Douglas G. Bonett -- Confidence Intervals for Mean Absolute Deviations.

Introduction:

Norma-theory tests and confidence intervals for variances are known to be hypersensitive to minor violations of the normality assumption. The mean absolute deviation an alternative way to give us informative measure of variability. Approximate confidence intervals for mean absolute deviations in one-group and two-group designs are derived are shown to have excellent small-sample properties under moderate non-normality.

Methodology:

Let Y_{ij} *(i = 1, 2, ...*n_j *; j = 1, 2)* be continuous, independent and identically distributed random variables within group j with 0 < var(Y_{ij}) = σ_j^2 < ∞ and E(Y_{ij}) = μ_{ij}. The population median of $Y_{ij}$is denoted as η_j.

A consistent estimator of the mean absolute deviation τ is:

(1) $\hat{\tau} = \sum\left|Y_{ij} - \hat{\eta}\right|/n$, where $\hat{\eta}$ is the sample mean

The estimator of var $\ln(\hat{\tau})$ is:

(2) $\overline{var}\ln(\hat{\tau}) = (\hat{\delta} - \hat{\gamma} - 1)/n$, where $\hat{\delta}$= ($\hat{\mu} - \hat{\eta}$)/ $\hat{\tau}$ and $\hat{\gamma} = \hat{\sigma}^2/\hat{\tau}^2$

In a one-group design the following $100(1-\alpha)\%$ confidence interval τ is proposed

(3) $\exp\left[\ln\left(c\hat{\tau} \pm Z_{\frac{\alpha}{2}}\{\overline{var}\ln(\hat{\tau})\}^{\frac{1}{2}}\right)\right]$

In a two-group design the following $100(1-\alpha)\%$ confidence interval τ_1/τ_2 is proposed

$$(4)\ \exp\left[ln\left(\frac{c_1\widehat{\tau_1}}{c_2\widehat{\tau_2}} \pm Z_{\frac{\alpha}{2}}\{\overline{var}\ ln(\hat{\tau}_1)+\overline{var}\ ln(\hat{\tau}_2)\}^{\frac{1}{2}}\right)\right] \quad \text{, where } c_j = n_j/(n_j-1)$$

We want to find out the coverage probability for (3) and (4) when $\alpha = 0.1, 0.05, and\ 0.01$ under different size(n) by utilizing Monte Carol method. Beside, we want to see if we can improve accuracy when we employ some techniques of variance reduction.

Table 1. Shape Parameters for Distributions Used in Simulations

Distribution	δ	γ	α_3	α_4
		Set A		
Beta(2,2)	0	1.23	0	1.5
Uniform	0	1.33	0	1.8
Tukey(3)	0	1.42	0	2.1
		Set B		
Normal	0	1.57	0	3
Logistic	0	1.71	0	4.2
Laplace	0	2.00	0	6
		Set C		
ScConN(.05,9)	0	1.82	0	7.6
Student's *t*(5)	0	2.00	0	9
ScConN(.1,25)	0	2.73	0	16.5
		Set D		
Gamma(1,7)	0.158	1.63	0.76	3.9
Half-Normal	0.261	1.62	0.99	3.9
Gumbel(0,1)	0.217	1.76	1.14	5.4
		Set E		
Chi-square(4)	0.305	1.81	1.41	6
Chi-square(3)	0.356	1.89	1.63	7
Exponential	0.443	2.08	2.00	9

(Table1. Is used in simulation.)

*** Gamma(1=rate, 7=shape)*

***Beta (2=1/shape, 2=1/rate)*

Table 2. Coverage Probabilities (× 100) for (3)

Distributions	n	$1-\alpha$ .90	.95	.99
Set A	10	91-95	95-97	99
	25	91-95	95-97	99
	50	91-93	95-97	99
	200	90-91	95-96	99
Set B	10	88-90	93-95	98-99
	25	89-90	94-95	98-99
	50	90-90	94-95	99
	200	90	95	99
Set C	10	82-89	89-95	96-99
	25	82-90	89-95	96-99
	50	85-90	90-95	96-99
	200	89-90	95-95	98-99
Set D	10	89-90	94-95	98-99
	25	89-90	94-95	98-99
	50	89-90	94-95	98-99
	200	90	95	99
Set E	10	83-87	89-92	95-97
	25	86-89	92-93	97-98
	50	88-89	93-94	98
	200	89	95	99

Table 3. Coverage Probabilities (× 100) for (4)

Distributions				$1-\alpha$		
Sample 1	Sample 2	n_1	n_2	.90	.95	.99
Set A	Set B	10	10	90-94	95-97	99
		25	10	92-95	96-97	99
		10	25	88-92	93-96	99
		25	25	91-93	95-97	99
		100	100	90-91	95-96	99
Set B	Set C	10	10	87-91	92-95	98-99
		25	10	89-91	94-95	98-99
		10	25	85-90	91-95	97-99
		25	25	86-90	92-95	98-99
		100	100	88-90	93-95	98-99
Set B	Set D	10	10	89-92	94-96	98-99
		25	10	88-91	94-96	98-99
		10	25	89-91	94-95	98-99
		25	25	90-91	95	99
		100	100	90	95	99
Set D	Set E	10	10	87-90	93-94	98
		25	10	89-90	94-95	98-99
		10	25	86-89	91-93	97-98
		25	25	89-90	94-95	98-99
		100	100	90	95	99

Two tables above are author's results. Left table is one-group probability, and right table is two-sample group probability. I chose a distribution from in each set for one-group design and two-group design.

Set A	Set B	Set C	Set D	Set E
Uniform(0,1)	Normal(0,1)	Student's(5)	Gamma(7,1)	Chi-square(4)

(Table 4)

<u>Monte Carlo experiment:</u>

(I) One-group design:

This simulation study examined the small-sample performance of (3)

First, we need to know the true τ. Using $\gamma = \sigma^2/\tau^2$, we can calculate the true τ. The Table1. helps us to obtain true τ , which is used in the simulation. Then, we sampled size n from a distribution to get the estimator of τ. For the estimator of var $\ln(\hat{\tau})$, we used the formula (1) and (2) to obtain. We finally got the value of estimators of τ and var $\ln(\hat{\tau})$. With the information from samples, we can construct the confidence interval. Repeat the procedure m times. We will have m sets of confidence intervals. The standard error is $\sqrt{\hat{p}(1-\hat{p})/m}$. $\hat{p}$ is the rate that the number of confidence intervals really covering true τ. For example, the confidence interval covers 46,500 times under 50,000 iterations. The $\hat{p}$ is 0.93, and the standard error is 0.0011.

The author used 50,000 Monte Carlo random samples of a given sample size(n= 10, 25, 50) from a wide variety of distribution and 200. I also used Monte Carlo study of 50,000 iterations. It's time-consuming if I run all distributions in each set. I decided to run a distribution in each set.

I checked the γ value in Table1. From (1), we can estimator τ from the formula $\hat{\gamma} = \hat{\sigma}^2/\hat{\tau}^2$. I got a large samples randomly from a distribution to get $\hat{\sigma}^2$ and $\hat{\tau}^2$, so we can get $\hat{\gamma}$. Therefore, we can compare the $\hat{\gamma}$ and γ value in Table1. I noticed that the Gamma(1, 7) on the article is Gamma(1=rate, 7=shape) and Beta(2,2) is Beta (0.5=shape. 0.5=rate). I think the author should have labeled the shape and rate because it's very confusing.

I wrote a function to get the confidence interval of (3). In the function, we will need to obtain n samples from a distribution for m times. From the samples, I could calculate $\overline{var}\ln(\hat{\tau})$ for m times. Then, I wrote another function to see how many confidence intervals can cover the true τ. In the function, I calculated true τ and checked if those confidence intervals from previous function cover the τ. If it covers, it counted 1. The probability would be total times over m(=50,000), and the standard error would be $\sqrt{\hat{p}(1-\hat{p})/m}$. In set A, I chose Uniform(0, 1) distribution to represent set A. The variance is 1/12 and γ is 1.33. According the Formula, true τ is .2503. When α is 0.1 and n=10, the 46,786 sets of confidence interval covers τ. Coverage probability is .93572, and S.E is .0011.

(II) Two-group design:

This simulation study examined the small-sample performance of (4)

We want to see if the constructed confidence interval can covers the ratio of τ from two distributions. Those two distributions represent two sets in Table 3. Two distribution are not required to be identical. By $\gamma = \sigma^2/\tau^2$, we can calculate the ratio of $\tau_1$and τ_2.The Table1. helps us to get true τ_1/τ_2 which is used in the simulation. If two distributions have similar shape such that τ_1/τ_2=σ_1/σ_2, (4) will generates confidence interval for σ_1/σ_2.

We sampled size n_1 and n_2 from two distributions to get the $\widehat{\tau_1}/\widehat{\tau_2}$ .To estimate var ln($\widehat{\tau_1}$)and var ln($\widehat{\tau_2}$), we used the formula (1) and (2) to get them. With the information from samples, we can construct the confidence interval to see if a confidence interval covers τ_1/τ_2. Repeat the procedure for m times. We can generate m sets of confidence intervals.

The standard error is $\sqrt{\hat{p}(1-\hat{p})/m}$. $\hat{p}$ is the coverage probability that confidence interval can cover τ_1/τ_2. This simulation is to see how many constructed confidence intervals cover the τ_1/τ_2.

The procedure is similar to one-group design. The τ_1/τ_2 can be calculated from the table. I wrote a function to find (4). In the function, we need to find out the estimators for var ln($\widehat{\tau_1}$), var ln($\widehat{\tau_2}$), τ_1 *and* τ_2. I sampled n1 and n2 for distribution 1 and distribution 2 for m times. With these samples, we can construct (4). Then, I wrote another function to see how many confidence intervals can cover τ_1/τ_2 under 50,000 iterations, and the standard error would be $\sqrt{\hat{p}(1-\hat{p})/m}$. I chose Uniform(0,1) for Set A and N(0,1) for Set B. τ_1/τ_2 is 0.3136. When α is 0.1 and n=10, there are 46,520 confidence intervals covering τ_1/τ_2. Coverage probability is .9304, and S.E is .0011. I still use one distribution to represent a set as I concluded in Table 4.

Result:

Result for (I) one- group design *(Corresponded to Table 2)*

Set A:

	n	alpha_0.1	alpha_0.05	alpha_0.01
Coverage.Prob	10	93.57200	96.41800	98.80000
Standard.Error	*10*	*0.00110*	*0.00083*	*0.00049*
Coverage.Prob1	25	93.20400	96.76800	99.24800
Standard.Error1	*25*	*0.00113*	*0.00079*	*0.00039*
Coverage.Prob2	50	91.60400	95.92600	99.04800
Standard.Error2	*50*	*0.00124*	*0.00088*	*0.00043*
Coverage.Prob3	200	90.59600	95.38800	99.04800
Standard.Error3	*200*	*0.00131*	*0.00094*	*0.00043*

Set B:

	n	alpha_0.1	alpha_0.05	alpha_0.01
Coverage.Prob	10	91.25400	95.21400	98.51800
Standard.Error	*10*	*0.00126*	*0.00095*	*0.00054*

```
Coverage.Prob1   25  91.22600  95.40800   98.84400
Standard.Error1  25   0.00127   0.00094    0.00048
Coverage.Prob2   50  90.46600  95.14800   98.85600
Standard.Error2  50   0.00131   0.00096    0.00048
Coverage.Prob3  200  90.01400  95.05600   99.03000
Standard.Error3 200   0.00134   0.00097    0.00044
```

Set C:

```
                  n alpha_0.1 alpha_0.05 alpha_0.01
Coverage.Prob    10  87.92000  93.10000   97.75800
Standard.Error   10   0.00146   0.00113    0.00066
Coverage.Prob1   25  88.22000  93.49800   98.19600
Standard.Error1  25   0.00144   0.00110    0.00060
Coverage.Prob2   50  87.32800  92.84800   98.10200
Standard.Error2  50   0.00149   0.00115    0.00061
Coverage.Prob3  200  83.94600  90.62000   97.28000
Standard.Error3 200   0.00164   0.00130    0.00073
```

Set D:

```
                  n alpha_0.1 alpha_0.05 alpha_0.01
Coverage.Prob    10  89.92400   94.29000   98.16400
Standard.Error   10   0.00135    0.00104    0.00060
Coverage.Prob1   25  90.00600   94.62200   98.59400
Standard.Error1  25   0.00134    0.00101    0.00053
Coverage.Prob2   50  89.52000   94.45200   98.68400
Standard.Error2  50   0.00137    0.00102    0.00051
Coverage.Prob3  200  89.85200   94.87200   98.90200
Standard.Error3 200   0.00135    0.00099    0.00047
```

Set E:

```
                  n alpha_0.1 alpha_0.05 alpha_0.01
Coverage.Prob    10  86.82000  91.97800   97.12600
Standard.Error   10   0.00151   0.00121    0.00075
Coverage.Prob1   25  88.34000  93.37200   97.98800
Standard.Error1  25   0.00144   0.00111    0.00063
Coverage.Prob2   50  88.72800  93.82400   98.33000
Standard.Error2  50   0.00141   0.00108    0.00057
Coverage.Prob3  200  89.59600  94.72400   98.84800
Standard.Error3 200   0.00137   0.00100    0.00048
```

We can see the coverage probabilities will approach $1 - \alpha$ as the sample size increases. The S.E is small because the number of iterations is relatively big. When α decreases, S.E also decreases. It makes sense that confidence covers more when α decrease, since the range of lower and upper bound becomes wider.

I only ran a distribution to represent in each set. Most of coverage probabilities align with author's results. Few coverage probabilities are higher than author's result.

Result for (II) two- group design *(Corresponded to Table 3)*

Set A and Set B:

	n1	n2	alpha_0.1	alpha_0.5	alpha_0.01
Coverage.Prob	10	10	93.04200	96.45800	99.16000
Standard.Error	*10*	*10*	*0.00114*	*0.00083*	*0.00041*
Coverage.Prob1	25	10	91.91400	95.74400	98.93400
Standard.Error1	*25*	*10*	*0.00122*	*0.00090*	*0.00046*
Coverage.Prob2	10	25	93.39800	96.74400	99.22400
Standard.Error2	*10*	*25*	*0.00111*	*0.00079*	*0.00039*
Coverage.Prob3	25	25	92.30800	96.27000	99.26800
Standard.Error3	25	25	0.00119	0.00085	0.00038
Coverage.Prob4	100	100	90.67200	95.33400	99.09600
Standard.Error4	*100*	*100*	*0.00130*	*0.00094*	*0.00042*

Set B and Set C:

	n1	n2	alpha_0.1	alpha_0.5	alpha_0.01
Coverage.Prob	10	10	90.52200	94.85000	98.61200
Standard.Error	*10*	*10*	*0.00131*	*0.00099*	*0.00052*
Coverage.Prob1	25	10	89.07400	93.97400	98.25800
Standard.Error1	*25*	*10*	*0.00140*	*0.00106*	*0.00059*
Coverage.Prob2	10	25	90.86800	95.14000	98.63200
Standard.Error2	*10*	*25*	*0.00129*	*0.00096*	*0.00052*
Coverage.Prob3	25	25	90.03600	94.73400	98.76800
Standard.Error3	*25*	*25*	*0.00134*	*0.00100*	*0.00049*
Coverage.Prob4	100	100	88.46600	93.71000	98.47400
Standard.Error4	*100*	*100*	*0.00143*	*0.00109*	*0.00055*

Set B and Set D:

	n1	n2	alpha_0.1	alpha_0.5	alpha_0.01
Coverage.Prob	10	10	91.24600	95.31600	98.79200
Standard.Error	*10*	*10*	*0.00126*	*0.00094*	*0.00049*
Coverage.Prob1	25	10	90.83400	95.00200	98.62800
Standard.Error1	*25*	*10*	*0.00129*	*0.00097*	*0.00052*
Coverage.Prob2	10	25	91.42800	95.55200	98.81200
Standard.Error2	*10*	*25*	*0.00125*	*0.00092*	*0.00048*
Coverage.Prob3	25	25	91.12200	95.47600	99.06400
Standard.Error3	*25*	*25*	*0.00127*	*0.00093*	*0.00043*
Coverage.Prob4	100	100	90.28800	95.27600	99.03400
Standard.Error4	*100*	*100*	*0.00132*	*0.00095*	*0.00044*

Set D and Set E:

	n1	n2	alpha_0.1	alpha_0.05	alpha_0.01
Coverage.Prob	10	10	89.38000	94.23800	98.41200
Standard.Error	*10*	*10*	*0.00138*	*0.00104*	*0.00056*
Coverage.Prob1	25	10	88.29600	93.30400	97.94200
Standard.Error1	*25*	*10*	*0.00144*	*0.00112*	*0.00063*
Coverage.Prob2	10	25	90.28600	94.68800	98.59200
Standard.Error2	*10*	*25*	*0.00132*	*0.00100*	*0.00053*
Coverage.Prob3	25	25	89.59200	94.59000	98.70200
Standard.Error3	*25*	*25*	*0.00137*	*0.00101*	*0.00051*
Coverage.Prob4	100	100	89.89800	94.69000	98.85800
Standard.Error4	*100*	*100*	*0.00135*	*0.00100*	*0.00048*

We can see the coverage probabilities will approach $1-\alpha$ as the sample size increases in balanced and imbalanced designs. The standard errors are small because the number of iterations is relatively big.

Variance Reduction:

I tried three method for variance reduction for Monte Carol experiment. They are Antithetic Sampling, Importance Sampling and Control Variates. Unfortunately, these three method cannot work well. I didn't apply variance reduction for two-group design because all three variance reduction techniques can't work well for one-group design.

(I)Antithetic Sampling:

We find two identical distributed unbiased estimator. They are $\hat{\mu}_1$ and $\hat{\mu}_2$, and they are negative correlated. Averaging two estimators will be superior to using either alone with double the sample size. The estimator will be$\hat{\mu}_{AS} = (\hat{\mu}_1 + \hat{\mu}_2)/2$. The variance of var($\hat{\mu}_{AS}$)=$\frac{(1+\rho)\sigma^2}{2n}$, where ρ is the correlation between $\hat{\mu}_1$ and $\hat{\mu}_2$, and $\sigma^2\ /n$ is the variance of either estimator using sample size n.

I tried uniform distribution from set A first. I sampled size n randomly from uniform distribution for m/2 times, say u_1. I set u_2=1-u_1. Two sample paths are negative correlated under the setup. Since I have the 2 sets of samples, I can construct two confidence intervals. We want to see if two confidence interval can cover true τ. If the confidence interval covers true τ, we count 1. After we check (m/2) sets of confidence interval for, we sum all the counts for two confidence intervals. Averaging two numbers will be $\hat{\mu}_{AS}$. If we divide $\hat{\mu}_{AS}$ by (m/2), it will be the coverage probability. The standard error of the coverage probability is the standard deviation of averaging count divided by square root of m. ($sd(count)/\sqrt{m}$). It changed a little based on one- group design. The point is to get two sample paths and see how many sets of confidence intervals which are constructed from two samples(u_1 and u_2) cover true τ. Then, we collect the counts and average two numbers.

I set m =1000, since it takes so much time to run.

Naïve Monte Carol (Uniform (0,1) distribution Set A, m=1,000):

	n	alpha_0.1	alpha_0.05	alpha_0.01
Coverage.Prob	10	94.00000	96.70000	99.30000
Standard.Error	10	0.00751	0.00565	0.00264
Coverage.Prob1	25	94.90000	96.80000	99.30000
Standard.Error1	25	0.00696	0.00557	0.00264
Coverage.Prob2	50	91.30000	95.50000	99.20000
Standard.Error2	50	0.00891	0.00656	0.00282
Coverage.Prob3	200	91.40000	95.50000	98.90000
Standard.Error3	200	0.00887	0.00656	0.00330

Antithetic Sampling (Unifrom(0,1) distribution Set A, m=1,000):

	n	alpha_0.1	alpha_0.05	alpha_0.01
Coverage.Prob	10	94.60000	97.00000	99.20000
Standard.Error	10	0.00715	0.00540	0.00282
Coverage.Prob1	25	95.00000	98.20000	98.80000
Standard.Error1	25	0.00690	0.00421	0.00345
Coverage.Prob2	50	92.40000	95.80000	98.60000
Standard.Error2	50	0.00839	0.00635	0.00372
Coverage.Prob3	200	89.20000	94.40000	99.00000
Standard.Error3	200	0.00982	0.00728	0.00315

Naïve Monte Carol (Chi-square (4) distribution Set E, m=1,000):

	n	alpha_0.1	alpha_0.5	alpha_0.01
Coverage.Prob	10	85.50000	91.90000	97.40000
Standard.Error	10	0.01113	0.00863	0.00503
Coverage.Prob1	25	87.90000	93.80000	98.40000
Standard.Error1	25	0.01031	0.00763	0.00397
Coverage.Prob2	50	87.50000	93.20000	98.70000
Standard.Error2	50	0.01046	0.00796	0.00358
Coverage.Prob3	200	89.80000	94.60000	99.20000
Standard.Error3	200	0.00957	0.00715	0.00282

Antithetic Sampling (Chi-square(4) distribution Set E, m=1,000):

	n	alpha_0.1	alpha_0.05	alpha_0.01
Coverage.Prob	10	86.40000	93.00000	96.30000
Standard.Error	10	0.00822	0.00601	0.00438
Coverage.Prob1	25	89.30000	91.30000	95.90000
Standard.Error1	25	0.00708	0.00686	0.00468
Coverage.Prob2	50	88.40000	91.90000	97.20000
Standard.Error2	50	0.00732	0.00633	0.00377
Coverage.Prob3	200	89.40000	94.90000	99.10000
Standard.Error3	200	0.00741	0.00519	0.00210

The standard error doesn’t change a lot for Antithetic Sampling method in Set A. We can see some Standard error s are smaller than Monte Carlo but some standard errors are even bigger. It doesn’t improve variance reduction. I think this it's related to the distribution. When I calculated the correlation between the counts from two confidence intervals. The correlation is 1. It means two confidence intervals, which is created from two sets of sample paths in Antithetic Sampling, are identical for uniform distribution. When I went back to test two confidence interval, I realized that two confidence intervals are the same because twos sets of samples are negative correlated and $\overline{var}\ln(\hat{\tau})$ and $\hat{\tau}$ from two sets sample are identical, too. This situation also happens in set B(Normal distribution) and Set C (t distribution). The correlation from two counts of confidence intervals is 1 for Normal distribution and are t distribution because they are symmetric. However, for set D(Gamma) and set E(Ch-square), the results changed. Since G(7,1) and Chi-square(4) are not symmetric, the $\overline{var}\ln(\hat{\tau})$ and $\hat{\tau}$ from two sets samples, which is created in Antithetic Sampling, are not identical. This two distributions can really improve the accuracy. From the Chiq-square table, when α is 0.1 and n=10, the variance can be decreased by 88%. When α is 0.01 and n=10, the antithetic sampling can reduce 31.8% of variance compared to Monte carol's variance. I don't think the result is desirable. I checked the correlation between two counts of two confidence intervals, and the correlation between two counts is 0.21. It's not closed to -1. That's the reason why Antithetic sampling cannot improve accuracy a lot for Chi-square distribution.

Since I found out that the Antithetic sampling method doesn't work well, I decided to try Importance sampling.

(II)Importance Sampling

The idea behind importance sampling is that certain values of the input random variables in a simulation have more impact on the parameter being estimated than others. If these "important" values are emphasized by sampling more frequently, then the estimator variance can be reduced.

In other words, the improved accuracy is achieved by causing the event of interest to occur more frequently than it would in the naïve Monte carol sampling framework.

Hence, the basic methodology in importance sampling is to choose a distribution which "encourages" the important values. The importance sampling method is based on the principal that the expectation of h(X) w.r.t its density f can be written in the form

$\mu = \int h(X)f(X)dx = \int \frac{h(X)f(X)}{g(x)} * g(x)dx$, where g is another density function, called envelope.

I tried chi-square(4) first. I used Chi-square(3.3) as envelope. (P*100). Then chi-square(4)/chi-square(3.3) is the weight function($\frac{f(X)}{g(x)}$ is weight function).I used the opposite way to calculate the coverage probability. If true τ is larger than upper bound of confidence. U is 1- weight; If true τ isn't larger than upper bound of confidence, U =1. If true τ is smaller than lower bound of confidence. L is 1- weight; If true τ isn't smaller than upper bound of confidence, L =1. U*L is like the count is previous method. Finally, we sum (U*L)/m. This is the coverage probability. The standard error is standard deviation of U*L divided by square root of m ($sd(U * L)/\sqrt{m}$)

I also set m=1,000 here.

Naïve Monte Carlo:

Chi-square(4)	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
n=10	85.5	91.9	97.4
n=10	0.01113	0.00863	0.00503

Importance Sampling:

Chi-square(4)	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
n=10	86.8680	89.351	97.84
n=10	0.0175	0.01925	0.0044

From the tables, we can see that the method doesn't achieve variance reduction at all. This is two-sided problem. Here we want to find the coverage probability for two-sided confidence interval for τ. Also, for other distributions, it's hard to find an appropriate distribution as an envelope to make it bounded.

(III) Control Variates:

The control variates strategy improves estimation of an unknown integral by relating to correlated estimator of an integral whose value is known.

Control variates estimator: $\widehat{\mu_{CV}} = \widehat{\mu_{MC}} + \lambda(\widehat{\theta_{MC}} - \theta)$, *λ is a parameter to be chosen by the user.*

$\widehat{\mu_{CV}} = \widehat{\mu_{MC}} + \lambda(\widehat{\theta_{MC}} - \theta) = \widehat{\beta_0} + \widehat{\beta_1}\theta$

I still used chi-square(4) as the first sample. The procedure is similar to Monte carol design. I construct the confidence intervals to see how many times the confidence interval can cover τ over m iterations. But I need to record the variance(ss) from n samples for m times. After I get the counts and variances, I can create the regression model. With the model, I can get the $\widehat{\beta_1}$. Then $\widehat{\mu_{CV}} = \widehat{\mu_{MC}} + \lambda(\widehat{\theta_{MC}} - \theta)$= counts +$(\widehat{-\beta_1})$ (ss-8), and 8 is the variance for Chi-square(4). Coverage probability is the sum of $\widehat{\mu_{CV}}$ divided by m.

I set m=1,000 here.

Naïve Monte Carlo:

Chi-square(4)	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
n=10	85.5	91.9	97.4
n=10	0.01113	0.00863	0.00503

Control Variates:

Chi-square(4)	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
n=10	87.42	92.79	97.386
n=10	0.0105	0.00818	0.00503
R-squared	0.0005265	0.00010145	0.00170

We can see the Standard errors don't change a lot. The S.Es from naïve Monte carol and Control variates are very close. The technique doesn't improve accuracy. I noticed that the R^2 for regression model is extremely low when $\alpha = 0.1, 0.05\ and\ 0.01$. I think that is the reason why it cannot reduce the variance. Theoretically, the control variates estimator is the fitted value on the regression model at the mean value of the predictor, and S.E of the control variates estimator is the S.E for the fitted value from the regression. In this case, if R^2 is small, the data are not close to fitted regression line. Lower R^2 tell us that the correlation between counts, which are the numbers when confidence covers τ, and variance of sample n is weak. This causes the failure in achieving variance reduction.

<u>Conclusions:</u>

I think the confidence intervals for τ_1/τ_2 and τ are not hard to compute. They are good alternatives for the classic confidence intervals for variances. We only need the Table2. to get the τ. The author uses the connection between τ and variance to design a new way to substitute the classic confidence interval for variances.

When I proceeded the techniques of variance reduction, I got very close estimates (The coverage probability) and Standard errors. In the article, there are many distributions. That's why it's harder to find a good method to improve accuracy. Furthermore, this is two-sided confidence interval problem. In class, we usually consider one-sided problems. It definitely increases the difficulty of using most of variance reduction techniques for the Monte Carol in the article.

Appendices:

Please see the attached file. In the file, the number of iteration is 1,000.