

Orthodox Gravity

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Abstract

A scalar field theory is investigated within the context of orthodox quantum gravity.

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1 Introduction

With quantum theory and general relativity being such good descriptions of the world, it is somewhat paradoxical that we have still not managed to wed the two theories [Isham, 1981]. Before embarking upon variations one might question the need to quantize gravity at all, since there is no direct experimental evidence demanding the quantization of the gravitational field [Feynman, 1963]. However, if gravity remains classical, since its fields are then not subject to the uncertainty principle of quantum theory, it might be employed to make an indirect measurement of a quantum field that would be more precise than that permitted. This argument for quantizing gravity is not watertight, as one might propose a gravitational coupling to the quantum expectation value, or some other alteration to quantum theory itself [Kibble, 1981]. However it does motivate one to begin by investigating the obstacles to naive quantization of the gravitational field.

The usual scheme of field quantization is plagued by divergences, but in some special cases those infinities can be consistently ploughed back into the theory to yield a finite end result with a small number of arbitrary constants remaining; these then being obtained from experiment [Ramond, 1990; Collins, 1984]. This is the renowned scheme of renormalization, disapproved of by some, but reasonably well defined and yielding results in excellent agreement with nature. The fact that only some theories are renormalizable has the beneficial effect of being selective, and so predictive. Unfortunately, in the usual sense, general relativity is *not* renormalizable [Veltman, 1976].

1.1 Traditional Formulation

Orthodox quantum gravity is a perturbatively unrenormalizable theory in the traditional sense, for starting from the example of a free scalar field in a gravitational field described by the Lagrangian:

$$\mathcal{L} = R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2$$

one discovers, upon quantizing both the matter and gravitational fields, that the counter terms do not fall back within the original Lagrangian. Already at one loop one observes the appearance of ϕ^4 and p^4 counter terms (most easily seen by power counting); where p^2 is shorthand for $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, and not the independent variable of Hamiltonian mechanics. At two loop one also has such divergences, along with the occurrence of additional counter terms of ϕ^6

and p^6 form. This continues indefinitely, and since the total number of counter terms is then infinite in number, their associated ambiguities destroy the predictive power of the theory. The presence of higher derivative counter terms further destroys the causal behaviour of the theory.

In summary, renormalizability requires that the number of independent counter terms be finite in number and that they do not spoil the physical behaviour of the original theory (modification is permitted). The fact that even after successful renormalization some factors, such as mass and charge are left undetermined should perhaps not be viewed as a predictive shortcoming, since the fundamental units of nature are relative; that is to say, the choice of reference unit (be it mass, length, time or charge) is always arbitrary, and then everything else can be stated in terms of these few units.

These observations motivate the consideration of the most general Lagrangian (in even powers of ϕ and p) permitted on the grounds of symmetry:

$$\begin{aligned}\mathcal{L} = & \Lambda + R + \frac{1}{2}p^2 + \frac{1}{2}m^2\phi^2 + p^2\phi^2\kappa(\phi^2) + \phi^4\lambda(\phi^2) \\ & + \underbrace{R\phi^2\gamma(\phi^2) + Rp^2e(p^2, \phi^2) + (\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \dots)}_{\text{higher derivative terms}} f(p^2, \phi^2)\end{aligned}$$

where κ , λ , γ , e , and f are arbitrary analytic functions.

Strictly this is formal in having neglected gauge fixing and the resulting presence of ghost particles. Symmetry now assures us that all counter terms must fall back within this Lagrangian, and it is this that motivated the construction. However the theory in this form has no predictive content, since there are an infinite number of arbitrary constants (in each arbitrary function: κ , λ , γ , e , f), and in this sense the theory is not renormalized. However, there remain physical criterion to pin down some of these arbitrary factors.

The cosmological constant is abandoned on the grounds of energy conservation [Shiekh, 1992], and since in general the higher derivative terms lead to acausal behaviour, their renormalised coefficient can also be put down to zero. This still leaves the three arbitrary functions κ , λ and γ , associated with the terms $p^2\phi^2\kappa(\phi^2)$, $\phi^4\lambda(\phi^2)$ and $R\phi^2\gamma(\phi^2)$. The last may be abandoned on the grounds of defying the equivalence principle. To see this, begin by considering the first term of the Taylor expansion, namely $R\phi^2$; this has the form of a mass term and so one would be able to make a local measurement of mass to determine the curvature, and so contradict the equivalence principle. The same line of reasoning applies to the remaining terms, $R\phi^4$, $R\phi^6$, ... etc.

This leaves us the two remaining infinite families of ambiguities within the terms $\phi^4\lambda(\phi^2)$ and $p^2\phi^2\kappa(\phi^2)$. In the limit of flat space in 3+1 dimensions this will reduce to a renormalized theory in the traditional sense if $\lambda(\phi^2) = \text{constant}$, and $\kappa(\phi^2) = 0$. So one is lead to proposing that the renormalised theory of quantum gravity for a scalar field should have the form:

$$\begin{aligned}\mathcal{L}_{ren} = & \Lambda_0 + R_0 + \frac{1}{2}p_0^2 + \frac{1}{2}m_0^2\phi_0^2 + p_0^2\phi_0^2\kappa_0(\phi_0^2) + \phi_0^4\lambda_0(\phi_0^2) \\ & + R_0\phi_0^2\gamma_0(\phi_0^2) + R_0p_0^2e_0(p_0^2, \phi_0^2) + (\alpha_0R_0^2 + \beta_0R_{0\mu\nu}R_0^{\mu\nu} + \dots) f_0(p_0^2, \phi_0^2)\end{aligned}$$

where the physical parameters:

$$\begin{aligned}\Lambda &= \kappa(\phi^2) = \gamma(\phi^2) = e(p^2, \phi^2) = f(p^2, \phi^2) = 0 \\ \lambda(\phi^2) &= \lambda = \text{scalar particle selfcoupling constant} \\ m &= \text{mass of the scalar particle}\end{aligned}$$

One might wonder about the renormalization group parameter. Although this would only be one additional free parameter, there are hints that this might be fixed [Culumovic et al., 1990; Leblanc et al., 1991], and one might there anticipate the appearance of the Plank mass.

We are left with a finite theory that has few arbitrary constants. Despite the patch work line of reasoning invoked to arrive at this hypothesis, one might alter perspective and simply be interested in investigating the consequences of such a scheme for its own sake, where many of the arbitrary factors are set to zero, for whatever reason. At this stage any well behaved, finite theory is worth investigating; and it is unfortunate that we don't have the guiding hand of mother nature to assist us in the guessing game.

1.2 Viable Formulation

Having discussed this approach within the context of traditional renormalization; it is intriguing to note that the use of analytic continuation [Bollini et al., 1964; Speer, 1968; Salam and Strathdee, 1975; Hawking, 1975; Dowker and Critchley, 1976] and the more recent method of operator regularization [McKeon and Sherry, 1987; McKeon et al., 1987; McKeon et al., 1988; Mann, 1988; Mann et al., 1989; Culumovic et al., 1990; Shiekh, 1990], implements the above scheme in a much cleaner way.

In operator regularization one removes divergences using the analytical continuation:

$$H^{-m} = \lim_{\varepsilon \rightarrow 0} \frac{d^n}{d\varepsilon^n} \left(\frac{\varepsilon^n}{n!} H^{-\varepsilon-m} \right)$$

where n is chosen sufficiently large that one is without infinities. This is explicitly illustrated through an example later.

The method of operator regularization has the strength of explicitly maintaining invariances, as well as being applicable to all loop levels; unlike the original Zeta function technique [Salam and Strathdee, 1975; Dowker and Critchley, 1976; Hawking, 1975] that only applied to one loop.

To see this method in action, we will walk through a simple example of a divergent one loop diagram of a massive scalar particle in quantum gravity. So begin with an investigation of a massive scalar theory in its own induced gravitational field, described by the action:

$$\mathcal{S} = \int_{-\infty}^{\infty} \sqrt{g} d^4x \left(R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 + \lambda \phi^4 \right)$$

The Feynman rules (of which there are an infinite number) we explicitly list; the gauged graviton propagator being derived from the gravitational, R , Lagrangian [M. Veltman, Les Houches XXVIII, 1976] (see FIG. 1):

$$\frac{\delta_{\mu\alpha} \delta_{\nu\beta} + \delta_{\mu\beta} \delta_{\nu\alpha} - \delta_{\mu\nu} \delta_{\alpha\beta}}{p^2}$$

The scalar propagator (see FIG. 2):

$$\frac{1}{p^2 + m^2}$$

First interaction vertex (see FIG. 3):

$$\kappa_{21} \left[\frac{1}{2} \delta_{\mu\nu} (p \cdot q - m^2) - p_\mu q_\nu \right]$$

etc.

Although there are an infinite number of Feynman diagrams, only a finite number are used to any finite loop order.

1.2.1 Divergent One loop diagram example:

Set about a one loop investigation with matter particles on the external legs (see FIG. 4).

$$= \int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4} \left(\frac{1}{l^2 + m^2} \right) \left(\frac{\delta_{\mu\alpha} \delta_{\nu\beta} + \delta_{\mu\beta} \delta_{\nu\alpha} - \delta_{\mu\nu} \delta_{\alpha\beta}}{(l+p)^2} \right) \kappa_{21} \left[\frac{1}{2} \delta_{\mu\nu} (p \cdot l - m^2) - p_\mu l_\nu \right] \kappa_{21} \left[\frac{1}{2} \delta_{\alpha\beta} (p \cdot l - m^2) - p_\alpha l_\beta \right]$$

expand out the indices to yield:

$$= \kappa_{21}^2 \int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 + m^2} \frac{1}{(l+p)^2} (p^2 l^2 + 2m^2 p \cdot l - 2m^4)$$

Then introduce the standard Feynman parameter 'trick':

$$\frac{1}{D_1^{a_1} D_2^{a_2} \dots D_k^{a_k}} = \frac{\Gamma(a_1 + a_2 + \dots a_k)}{\Gamma(a_1) \Gamma(a_2) \dots \Gamma(a_k)} \int_0^1 \dots \int_0^1 dx_1 \dots dx_k \frac{\delta(1 - x_1 - \dots x_k) x_1^{a_1-1} \dots x_k^{a_k-1}}{(D_1 x_1 + \dots D_k x_k)^{a_1 + \dots a_k}}$$

to yield:

$$= \kappa_{21}^2 \int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4} \int_0^1 dx \frac{p^2 l^2 + 2m^2 p \cdot l - 2m^4}{[l^2 + m^2 x + p^2 (1-x) + 2l \cdot p (1-x)]^2}$$

Remove divergences using the analytic continuation:

$$H^{-m} = \lim_{\varepsilon \rightarrow 0} \frac{d^n}{d\varepsilon^n} \left(\frac{\varepsilon^n}{n!} H^{-\varepsilon-m} \right)$$

n being chosen sufficiently large to cancel the infinities. For the case in hand $n = 1$ is adequate.

$$H^{-2} = \lim_{\varepsilon \rightarrow 0} \frac{d}{d\varepsilon} (\varepsilon H^{-\varepsilon-2})$$

This yields:

$$= \kappa_{21}^2 \int_0^1 dx \lim_{\varepsilon \rightarrow 0} \frac{d}{d\varepsilon} \int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4} \left(\varepsilon \frac{p^2 l^2 + 2m^2 p \cdot l - 2m^4}{[l^2 + m^2 x + p^2 (1-x) + 2l \cdot p (1-x)]^{\varepsilon+2}} \right)$$

Then performing the momentum integrations using [Ramond, 1990]:

$$\int_{-\infty}^{\infty} \frac{d^{2\omega} l}{(2\pi)^{2\omega}} \frac{1}{(l^2 + M^2 + 2l \cdot p)^A} = \frac{1}{(4\pi)^\omega \Gamma(A)} \frac{\Gamma(A-\omega)}{(M^2 - p^2)^{A-\omega}}$$

$$\int_{-\infty}^{\infty} \frac{d^{2\omega} l}{(2\pi)^{2\omega}} \frac{l_\mu}{(l^2 + M^2 + 2l \cdot p)^A} = -\frac{1}{(4\pi)^\omega \Gamma(A)} p^\mu \frac{\Gamma(A-\omega)}{(M^2 - p^2)^{A-\omega}}$$

$$\int_{-\infty}^{\infty} \frac{d^{2\omega} l}{(2\pi)^{2\omega}} \frac{l_\mu l_\nu}{(l^2 + M^2 + 2l \cdot p)^A} = \frac{1}{(4\pi)^\omega \Gamma(A)} \left[p_\mu p_\nu \frac{\Gamma(A-\omega)}{(M^2 - p^2)^{A-\omega}} + \frac{\delta_{\mu\nu}}{2} \frac{\Gamma(A-\omega-1)}{(M^2 - p^2)^{A-\omega-1}} \right]$$

yields the finite expression:

$$= \frac{\kappa_{21}^2}{(4\pi)^2} \int_0^1 dx \lim_{\varepsilon \rightarrow 0} \frac{d}{d\varepsilon} \left(\frac{p^4 (1-x)^2 \Gamma(\varepsilon)}{[m^2 x + p^2 x (1-x)]^\varepsilon} + 2 \frac{p^2 \Gamma(\varepsilon-1)}{[m^2 x + p^2 x (1-x)]^{\varepsilon-1}} \right) \frac{\varepsilon}{\Gamma(\varepsilon+2)}$$

Doing the ε differential using:

$$\lim_{\varepsilon \rightarrow 0} \frac{d}{d\varepsilon} \left(\frac{\varepsilon}{\Gamma(\varepsilon + 2)} \left(a \frac{\Gamma(\varepsilon)}{\chi^\varepsilon} + b \frac{\Gamma(\varepsilon - 1)}{\chi^{\varepsilon-1}} \right) \right) = -a \ln(\chi) + b \chi (\ln(\chi) - 1)$$

yields:

$$= \frac{\kappa_{21}^2}{(4\pi)^2} \int_0^1 dx \left(\frac{((2m^4 + 2m^2p^2 - p^4) + p^4x(4 - 3x)) \ln(m^2x + p^2x(1 - x))}{-2p^2(m^2x + p^2x(1 - x))} \right)$$

and finally performing the x integration gives rise to the final result:

$$= \frac{\kappa_{21}^2}{(4\pi)^2} m^4 \left(\left(3 + 2 \frac{p^2}{m^2} + \frac{m^2}{p^2} \right) \ln \left(1 + \frac{p^2}{m^2} \right) - 3 - \frac{9}{2} \frac{p^2}{m^2} - \frac{1}{6} \frac{p^4}{m^4} + 2 \left(1 + \frac{p^2}{m^2} \right) \ln \left(\frac{m^2}{\mu^2} \right) \right)$$

where there is no actual divergence at $p = 0$, and it should be commented that the use of a computer mathematics package can in general greatly reduced 'calculator' fatigue. The factor μ appears on dimensional grounds.

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3 References

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