Adaptive Output Regulation for Uncertain Nonlinear Systems: an Additive Decomposition-based Method

Qinxiao Ma, Yizhou Gong, Yang Wang

***Abstract*— This paper investigates the robust output regulation problem for a class of uncertain nonlinear multiple-input and multiple-output (MIMO) systems affected by disturbances from known exosystems. The considered system is assumed to be minimum-phase but subject to large parameter uncertainties. To this end, a novel *internal model-based* control protocol is proposed, under the framework of *additive state decomposition*. The approach decomposes the original output regulation problem into a more tractable disturbance rejection problem for a *linear* system, and a tracking problem for a nominal system with the *nonlinear* term. By employing this decomposition, the proposed method effectively mitigates the design difficulty associated with the output regulation problem for nonlinear MIMO systems. Finally, a numerical experiment is conducted to demonstrate the effectiveness of the presented method.**

I. Introduction

The robust output regulation problem, namely having the output of a system asymptotically tracking prescribed trajectories and rejecting unwanted disturbances simultaneously, in the presence of parameter uncertainties, has received everlasting attention in the control community. Several conceptually different approaches to this problem have been proposed in literature [1]–[5]. One notable approach is based on the celebrated internal model principle [6], rendering a systematic control protocol to tackle with the tracking trajectories or disturbances to be canceled.

The problem of robust output regulation applied to nonlinear systems with parameter uncertainties has been extensively investigated in seminal works [7], [8]. Based on the non-equilibrium theory, the work by [7] established a clear link between the design of internal model-based regulators and nonlinear high-gain observers. Additionally, in the work [8], it demonstrates that there always exists a general and fixed structure of the controller such that the nonlinear output regulation problem can be solved. Although theoretically appealing, the result of [8] is not constructive, in the sense that no trivial design procedures of the controller are given even for plants with a simple nonlinearity. It is worth noting that the above works are essentially restrictive to minimum phase single-input and single-output (SISO) systems with a well-defined normal form between the control input and regulated error. By means of the method of [7] for the design of the internal model and the approach of [9] for dynamic feedback-linearization, asymptotic output regulation is achieved for a broad class of invertible MIMO nonlinear systems in [10]. However, the design of the internal model in [7], [10] requires an analytic solution to the partial differential equation, which becomes a hard task and consequently, makes the construction of the controller almost unfeasible. To circumvent the difficulty of solving the partial differential equation directly, Marconi et. al. [11] introduced an adaptive unit that tunes the regulator online with system identification algorithms to nonlinear minimum phase systems. This result extends the content of [8], but the systems consider here are SISO and relative-degree-unity.

Recently, the implementation of the internal model principle has been successfully achieved for a more general class of multivariable systems to solve the robust output regulation problem [12]–[17]. A particular case of output regulation problem for MIMO nonlinear non-minimum phase systems is considered affected by constant perturbations in [15]. The extension of [15] to the case in the presence of periodic perturbations is presented in [12]. However, the asymptotic regulation can only be retained under sufficiently small parameter variations of the plant's dynamics. From a practical point of view, it is essential to acknowledge that the parameters of the plants could be large or even completely unknown, which makes the application of aforementioned methods impractical.

In this paper, a new solution is proposed for a class of uncertain MIMO nonlinear systems in the presence of periodic perturbations and parameter uncertainties under the framework of *additive state decomposition* [18]. Thanks to the decomposition, the original output regulation problem is transformed into two subproblems, i.e., designing an output-feedback regulator for the linear primary system with unknown parameters, and developing a state-feedback regulator for the nominal secondary system with the nonlinear term. Both regulators are designed based on the internal model principle. Note that, an adaptive law is developed to tackle the parameter uncertainties of the plant. Instead of directing solving the partial differential equation of the original system, as done in the aforementioned works, we solve such a partial differential equation of a nominal system instead, which significantly alleviates the computational burden and widens the application of this method. As a result, this decomposition aids the tracking control scheme in dealing with a regulation equation easier to solve and enables it to tackle the rejection problem involving any external periodic signal. The main contributions of this paper lie in the following:

1) A new control framework to streamline the system analysis and control design is presented, which solves

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the robust output regulation problem for a class of nonlinear MIMO systems;
2) Based on the additive state decomposition technique, the robustness of embedding two internal models are improved in the presence of large parameter uncertainties, which is illustrated in Section IV;
3) Thanks to the additive state decomposition, we avoid computing the partial differential equation of the original system directly, thus significantly simplifying the computation and increasing the practical application of nonlinear output regulation problem.

The remainder of the paper is organized as follows. Section II formulates the output regulation problem of uncertain nonlinear MIMO systems and introduces the additive state decomposition. In Section III, a controller is designed, and a stability analysis is presented. Numerical examples are provided to illustrate the performance of the proposed controller under the additive state decomposition in Section IV. The paper is wrapped up with conclusions in Section V.

II. Problem Formulation

This paper addresses the output regulation problem for a class of nonlinear MIMO uncertain systems described by

$$\begin{aligned} &\dot{x} = A(\mu)x + B(\mu)u + P(\mu)w + \Phi(y), \quad x(0) = x_0 \in \mathcal{X} \\ &y = C_o^\top x \\ &e = C_o^\top x - Qw \end{aligned} \tag{1}$$

where[1] $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $e \in \mathbb{R}^m$, $u \in \mathbb{R}^m$ represent the state, the measurable output signal, the output to be regulated, and the input signal to be designed, respectively. $w \in \mathbb{R}^{n_w}$ accounting for the external disturbances and desired trajectories is thought to be generated by the following exosystem

$$\begin{aligned} &\dot{w} = Sw, \quad w(0) \in \mathcal{W} \in \mathcal{R}^{2r+1} \\ &y_d = Qw, \end{aligned} \tag{2}$$

which is assumed to be known in advance. The uncertainty of the external disturbance and the plant model is collected by the parameter vector $\mu \in \mathcal{P}$, ranging over a given compact set $\mathcal{P} \subset \mathbb{R}^p$. Note that, only the coefficient matrices $A(\mu), B(\mu), P(\mu)$ depend on μ, whereas S, Q, $C_o^\top$ are assumed to be known as priori, and the non-linearity term $\Phi(y)$ are independent of μ, assumed to be *locally Lispchitz* and known as well. Without loss of generality, we set

$$C_o^\top := \begin{pmatrix} \mathrm{I}_m & 0_{m\times(n-m)} \end{pmatrix} \in \mathbb{R}^{m\times n}.$$

Given system (1) under the influence of the exosystem (2), the goal can be qualitatively expressed as finding a proper control input $u(t)$ to stabilize the closed-loop system and regulate the output $e(t)$ to zero.

To this end, we first split the dynamic matrix $A(\mu)$ into two parts as $A(\mu) = A_o + \tilde{A}(\mu)$, with

$$A_o := \begin{pmatrix} \mathbf{0} & \mathbf{I}_{n-1} \\ 0 & \mathbf{0} \end{pmatrix}$$

[1] In this note, we will drop the dependence of the variables on time t for the simplicity of notations.

being independent of μ. Obviously, the pair $\left(A_o, C_o^\top\right)$ is observable, which leads to the existence of vector $L_o \in \mathbb{R}^n$ such that $\bar{A} := A_o + L_o C_o^\top$ is Hurwitz. For future use, we take $A'(\mu) := A(\mu) + L_o C_o^\top = \bar{A} + \tilde{A}(\mu)$ and rewrite the dynamic of x as

$$\dot{x} = \bar{A}x + \tilde{A}(\mu)x + B(\mu)u + P(\mu)w + \bar{\Phi}(y), \tag{3}$$

where $\bar{\Phi}(y) := \Phi(y) - L_o y$ is also a known term.

Further, we postulate $B(\mu)$ admits the form of

$$B(\mu) := B_o\left(\mathbf{I}_m + \Delta(\mu)\right) \in \mathbb{R}^{n\times m}, \tag{4}$$

where $B_o \in \mathbb{R}^{n\times m}$ is a known control input matrix which is left-invertible and $\Delta(\mu)$ is a diagonal matrix given by

$$\Delta(\mu) = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m) \in \mathcal{R}^{m\times m} \tag{5}$$

Substituting the relation (4) into (3) yields

$$\begin{aligned} &\dot{x} = \bar{A}(\mu)x + \tilde{A}(\mu)x + B_o u + B_o\Delta(\mu)u + P(\mu)w + \bar{\Phi}(y), \\ &e = C_o^\top x - Qw. \end{aligned} \tag{6}$$

Now, we can list the assumptions made for the output regulation problem considered in this work:

Assumption II.1. There exists a μ-dependent matrix $\vartheta(\mu) \in \mathbb{R}^{m\times n}$ verifying

$$\tilde{A}(\mu) = B_o\vartheta(\mu) \tag{7}$$

for all $\mu \in \mathcal{P}$.

Assumption II.2. The minimal polynomial of matrix S that models the exosystem has one eigenvalue on the zero, and r pairs of simple eigenvalues on the imaginary axis, i.e., $0, \pm j2\pi f_i, i = 1, 2, \cdots, r$. Moreover, the frequency information f_i and its initial condition $w(0) \in \mathcal{W} \subset \mathbb{R}^{2r+1}$ is assumed to be known.

Remark II.1. Given Q and $w(0)$ known, Assumption II.2 implies that the reference signal $y_d = Qw$ to be tracked and the disturbance signal $d(t) := P(\mu)w$ to be rejected are both periodic. However, since $P(\mu)$ is completely unknown, this does not violate the general assumption that $d(t)$ is unmeasurable.

Assumption II.3. For any $\mu \in \mathcal{P}$, the triple $\left(A'(\mu), B_o, C_o^\top\right)$ is stabilizable and detectable, moreover, the family of *Francis equations*

$$\begin{aligned} \Pi(\mu)S &= A'(\mu)\Pi(\mu) + B_o\Psi(\mu) + P(\mu), \\ 0 &= C_o^\top\Pi(\mu) \end{aligned} \tag{8}$$

all have a μ-dependent solution pair $(\Pi(\mu), \Psi(\mu))$.

Assumption II.4. The sign of $\mathrm{I}_m + \Delta(\mu)$ is persistent and known for all $\mu \in \mathcal{P}$. In addition, the transfer function corresponding to the matrices pair $\left(A(\mu), B_o, C_o^\top\right)$ is strictly positive real (*SPR*).

Remark II.2. Technically speaking, Assumption II.3 shall be made for the original matrices pair $\left(A(\mu), B(\mu), C_o^\top\right)$. Here, for simplicity of the forthcoming analysis, we directly make this assumption on the primary subsystem that will be obtained by the additive decomposition later.

Now, the problem addressed in this paper can be formally stated as follows:

Problem II.1. Suppose Assumptions II.1-II.4 hold for system (1) under the effect of a single sinusoidal disturbance generated by the exosystem (2), find a controller in the form of

$$\begin{aligned} \dot{x}_c &= f_c(x_c, e), \quad x_c(0) \in \mathcal{X}_c \subset R^{n_c} \\ u &= f_c(x_c, e) \end{aligned} \tag{9}$$

such that the trajectories of the closed-loop system originating from all initial condition $\big(w(0), x(0), x_c(0)\big) \in \mathcal{W} \times \mathcal{X} \times \mathcal{X}_c$ are bounded, and $\lim_{t\to\infty} e(t) = 0$. ◁

Instead of directly solving the issue via the celebrated internal model-based technique, here we first introduce an additive decomposition to system (6). Splitting the variables into two parts $x := x_p + x_s$, $u := u_p + u_s$, $e := e_p + e_s$ and $y := y_p + y_s$, where the dynamic of the 'primary' state x_p and 'secondary' state x_s are given by

$$\Sigma_p : \begin{cases} \dot{x}_p = \bar{A}x_p + \tilde{A}(\mu)x + B_o u_p + B_o \Delta(\mu)u + P(\mu)w, \\ y_p = C_o^\top x_p, \\ e_p = C_o^\top x_p, \quad x_p(0) = x_0 \end{cases} \tag{10}$$

and

$$\Sigma_s : \begin{cases} \dot{x}_s = A_o x_s + B_o u_s + \Phi(y), \\ y_s = C_o^\top x_s, \\ e_s = C_o^\top x_s - Qw, \quad x_s(0) = 0 \end{cases} \tag{11}$$

respectively. Note that, given the model of the 'secondary' system (11) is completely known with zero initial condition, the state x_s and auxiliary output y_s can be directly estimated by

$$\begin{cases} \dot{\hat{x}}_s = A_o \hat{x}_s + B_o u_s + \Phi(y), \\ \hat{y}_s = C_o^\top \hat{x}_s, \\ \hat{e}_s = C_o^\top \hat{x}_s - Qw, \quad \hat{x}_s(0) = 0 \end{cases} \tag{12}$$

It is trivial to see that $\hat{x}_s \equiv x_s$ and $\hat{y}_p = y - \hat{y}_s \equiv y_p$.

Remark II.3. Additive State Decomposition(ASD) was first introduced by [18]. Via ASD, we transform the control problem of the nonlinear uncertain system into a few simpler subproblems. In this way, the challenging tracking problem becomes more tractable, and by taking full advantage of the prior knowledge, we might be able to significantly improve the performance when the uncertainties and disturbances are relatively small.

For the decomposed systems, Problem II.1 can be recast as follows:

Problem II.2. Considering the 'primary' system Σ_p defined in (10), and the 'secondary' system Σ_s defined in (11), given Assumptions II.1-II.4 hold, the control objective is specified for each subsystem as follows:

1) Find a dynamic output feedback control law u_p such that the (10) is stable and $\lim_{t\to\infty} e_p(t) = 0$.
2) Develop a full state feedback control law u_s, such that the secondary subsystem (11) is stable and $\lim_{t\to\infty} e_s(t) = 0$.

Eventually, summing up u_p and u_s to obtain the control law u, we solve the robust output regulation problem for the original system (1) . ◁

III. Controller Design

In this section, we present the design of internal model-based regulators for subsystems (10) and (11) that solves Problem II.2 .

A. *Stabilization regulator design for the primary system* (10)

To this end, rewrite primary system (10) as

$$\begin{aligned} \dot{x}_p &= \bar{A}x_p + \tilde{A}(\mu)(x_p + x_s) + B_o u_p + B_o \Delta u + P(\mu)w \\ &= A'(\mu)x_p + \tilde{A}(\mu)x_s + B_o u_p + B_o \Delta u + P(\mu)w \\ &= A'(\mu)x_p + B_o(u_p + \Delta(\mu)u + \vartheta(\mu)x_s) + P(\mu)w \end{aligned} \tag{13}$$

where we have leveraged on the facts that $x = x_p + x_s$, $A'(\mu) = \bar{A} + \tilde{A}(\mu)$ and $\tilde{A}(\mu) = B_o \vartheta^\top(\mu)$. Define $\mathbf{H}^\top(\mu) := -\begin{pmatrix} \Delta(\mu) & \vartheta^\top(\mu) \end{pmatrix} \in \mathbb{R}^{m\times(n+m)}$ and $\bar{\xi} := \begin{pmatrix} u^\top & x_s^\top \end{pmatrix}^\top \in \mathbb{R}^{(n+m)\times 1}$, we have

$$\dot{x}_p = A'(\mu)x_p + B_o(u_p - \mathbf{H}^\top \bar{\xi}) + P(\mu)w \tag{14}$$

Naturally, proposing

$$u_p = \hat{\mathbf{H}}^\top \bar{\xi} + \nu_p, \tag{15}$$

where $\hat{\mathbf{H}}$ being the estimate of $\mathbf{H}$ to be determined later, it follows that

$$\begin{aligned} \dot{x}_p &= A'(\mu)x_p + B_o \nu_p + P(\mu)w + B_o \tilde{\mathbf{H}}^\top \bar{\xi} \\ y_p &= e_p = C_o^\top x_p \end{aligned} \tag{16}$$

with $\tilde{\mathbf{H}} := \hat{\mathbf{H}} - \mathbf{H}$ standing for the estimation error matrix. Now, in virtue of Assumptions II.3 and II.4 we can construct a robust regulator to achieve the output zeroing for the primary system Σ_p. The possibility of constructing a controller that solves above problem reposes on the preliminary result presented in [19].

Let $\phi(\lambda) = s_0\lambda + \cdots + s_{d-1}\lambda^{d-1} + \lambda^d$ be the minimal polynomial of the matrix S with $d := 2r+1$, and let $\Phi_p \in \mathbb{R}^{md\times md}$ be defined by

$$\Phi_p = \begin{pmatrix} \mathbf{0} & \mathbf{I}_m & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_m & \cdots & \mathbf{0} \\ . & . & . & \ddots & . \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_m \\ \mathbf{0} & -s_0\mathbf{I}_m & \cdots & \cdots & -s_{d-1}\mathbf{I}_m \end{pmatrix}$$

and set $G = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I}_m \end{pmatrix}^\top$. where $\mathbf{I}_m$ being an $m\times m$ identity matrix. Given G and Φ_p constructed in this way, the internal model unit is designed as follows

$$\dot{\eta} = \Phi_p \eta + G y_p, \tag{17}$$

Now, combined with the internal model unit (17), we consider a stabilizing controller fulfilling the following structure,

$$\begin{aligned} \dot{\eta} &= \Phi_p \eta + G y_p \\ \dot{\xi} &= A_{s\eta}\eta + A_s \xi + B_s y_p \\ \nu_p &= C_{c\eta}\eta + C_s \xi + D_s y_p \end{aligned} \tag{18}$$

The system defined features the following important lemma:

Lemma III.1. Suppose Assumptions II.2 and II.3 hold, there exist matrices $A_{s\eta}, A_s, C_{c\eta}, C_s, B_s$ and D_s such that the matrix

$$\mathbf{A}'_{cl}(\mu) := \begin{pmatrix} A'(\mu) + B_o D_s C_o^\top & B_o C_{c\eta} & B_o C_s \\ G C_o^\top & \Phi_p & \mathbf{0} \\ B_s C_o^\top & A_{s\eta} & A_s \end{pmatrix}$$

is Hurwitz for all $\mu \in \mathcal{P}$.

Proof. First, recall that the matrix S is neutrally stable and is not affected by parameter uncertainty under Assumption II.2. Then, according to [19], the augmented system of x and η is robustly stabilizable and detectable if and only if the uncertain matrices pair $\left(A'(\mu), B_o, C_o^\top\right)$ is robustly stabilizable by means of a $\mu-$independent controller in the form of (18) and the non-resonance condition

$$\operatorname{rank}\begin{pmatrix} A'(\mu) & B_o \\ C_o & 0 \end{pmatrix} = n + m, \qquad \forall \lambda \in \sigma(S) \tag{19}$$

holds for all $\mu \in \mathcal{P}$. Those two conditions are guaranteed by Assumption II.3. As a consequence, there exist matrices $A_{s\eta}, A_s, C_{c\eta}, C_s, B_s$ and D_s such that $\mathbf{A}'_{cl}(\mu)$ is Hurwitz for all $\mu \in \mathcal{P}$. □

Substituting the regulator (18) into the primary system (10), along with the control law (15), we have the closed-loop system as

$$\dot{\mathbf{x}} = \mathbf{A}'_{cl}(\mu)\mathbf{x} + \mathbf{B}'_{cl}\tilde{\mathbf{H}}^\top \bar{\xi} + \mathbf{P}(\mu) w \tag{20}$$

where we denote $\mathbf{x} := \begin{pmatrix} x_p^\top & \eta^\top & \xi^\top \end{pmatrix}^\top$, $\mathbf{P}(\mu) := \begin{pmatrix} P(\mu)^\top & \mathbf{0} & \mathbf{0} \end{pmatrix}^\top$, $\mathbf{B}'_{cl} := \begin{pmatrix} B_o^\top & \mathbf{0} & \mathbf{0} \end{pmatrix}^\top$. Now, for the primary system Σ_p, we have the following result.

Theorem III.1. Under Assumption II.1-II.4, the controller (15) composed of internal model-based regulator (18) and the compensation term $\hat{\mathbf{H}}^\top \bar{\xi}$ generated by the following adaptive law as

$$\dot{\hat{\mathbf{H}}} = -\Gamma\, \bar{\xi}\, y_p, \quad \hat{\mathbf{H}}(0) \in \mathbb{R}^{(n+m)\times m} \tag{21}$$

with Γ being a positive definite and symmetric matrix, is able to stabilize the primary plant (10), in the sense that the trajectories of the closed-loop system (20) are bounded and $\lim_{t\to\infty} e_p(t) = 0$.

Proof. The closed-loop system (20) possesses the complementary invariant subspace: a stable invariant subspace, and a center invariant subspace. The later, in particular, is the graph of a μ-dependent linear map

$$w \mapsto \begin{pmatrix} x_p \\ \eta \\ \xi \end{pmatrix} := \begin{pmatrix} \Pi_x(\mu) \\ \Pi_\eta(\mu) \\ \Pi_\xi(\mu) \end{pmatrix} w,$$

in which $\mathbf{\Pi}(\mu) := \left(\Pi_x^\top(\mu), \Pi_\eta^\top(\mu), \Pi_\xi^\top(\mu)\right)^\top$ is the unique solutions of a *Slyvester Equation* having the form of

$$\mathbf{\Pi}(\mu) S = \mathbf{A}'_{cl}(\mu)\mathbf{\Pi}(\mu) + \mathbf{P}(\mu) \tag{22}$$

To proceed, define

$$\mathbf{z} := \begin{pmatrix} x_p \\ \eta \\ \xi \end{pmatrix} - \begin{pmatrix} \Pi_x(\mu) \\ \Pi_\eta(\mu) \\ \Pi_\xi(\mu) \end{pmatrix} w. \tag{23}$$

This leads to

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{A}'_{cl}(\mu) z + \mathbf{B}'_{cl}\tilde{\mathbf{H}}^\top \bar{\xi}, \\ e_p &= y_p = \mathbf{C}'^\top_{cl}\mathbf{z}, \end{aligned} \tag{24}$$

with $C'^\top_{cl} = \begin{pmatrix} C_o^\top & \mathbf{0} & \mathbf{0} \end{pmatrix}$. Since the matrix $\mathbf{A}'_{cl}(\mu)$ is Hurwitz according to Lemma III.1, we consider the following Lyapunov candidate function

$$V_1(\mathbf{z}, \tilde{\mathbf{H}}) = \mathbf{z}^\top P_z \mathbf{z} + \operatorname{trace}(\tilde{\mathbf{H}}^\top \Gamma^{-1}\tilde{\mathbf{H}}), \tag{25}$$

in which, according to *SPR* condition in Assumption II.4, the matrix P_z is a positive definite solution of the following *Kalman-Yakubovich-Popov (KYP)* equation [20, Lemma 3.5.2]:

$$\begin{aligned} P_z \mathbf{A}'_{cl} + \mathbf{A}'^\top_{cl} P_z &= -Q_z \\ P_z \mathbf{B}'_{cl} &= \mathbf{C}'_{cl}, \end{aligned} \tag{26}$$

where Q_z is a positive definite and symmetric matrix.

Differentiating (25) gives

$$\dot{V}_1 = -\mathbf{z}^\top Q_z \mathbf{z} + 2\mathbf{z}^\top P_z \mathbf{B}'_{cl}\tilde{\mathbf{H}}^\top \bar{\xi} + 2\operatorname{trace}(\tilde{\mathbf{H}}^\top \Gamma^{-1}\dot{\tilde{\mathbf{H}}}),$$

Thanks to (26) and the adaptive law $\hat{\mathbf{H}}$ selected as (21), one obtains $\dot{V}_1 \le -\mathbf{z}^\top Q_z \mathbf{z}$. Thus, $\mathbf{z}, \dot{\mathbf{z}}, \tilde{\mathbf{H}} \in \mathcal{L}_\infty$, $\mathbf{z} \in \mathcal{L}_2$. Using the *Barbalat's lemma* , we have $\mathbf{z} \to 0$, as $t \to \infty$, which indicates $x_p \to \Pi_x w$, $\lim_{t\to\infty} y_p(t) = 0$, $\lim_{t\to\infty} e_p(t) = 0$. Thus ending the proof. □

B. Tracking controller design for secondary system (11)

For secondary system given in (11), the problem becomes even simpler as the full state x_s in (11) and full state w of the exosystem are both available, which is the so-called full information output regulation problem. In what follows, we propose an internal-model-based full information regulator to achieve trajectory tracking.

Let us construct the static regulator in the form of

$$u_s = K\hat{x}_s + L(w) \tag{27}$$

where $\hat{x}_s$ is estimate of x_s from (12), K is designed such that the matrix $A_o + B_o K$ is Hurwitz and L will be determined later. Again, we define $\pi_s(w)$ and resort to the nonlinear regulator equation:

$$\begin{aligned} \frac{\partial \pi_s(w)}{\partial w} S w &= f(\pi_s(w)) + B_o \psi_s(w) \\ 0 &= C_o{}^\top \pi_s(w) - Q w \end{aligned} \tag{28}$$

where we denote $f(\pi_s(w)) := A_o \pi_s(w) + \Phi(C_o^\top \pi_s(w))$, $\pi_s(w), \psi_s(w)$ is the solution pair of (28), which will be exemplified by in the following simulation.

Now, we are ready to present our second result:

Theorem III.2. For the secondary system (11), the regulator (27) based on the full-state observer (12), is able to stabilize the subsystem, and achieve $\lim_{t\to\infty} y_s = y_d$, if $L(w)$ is designed as

$$L(w) = \psi_s(w) - K\pi_s(w),$$

where $\pi_s(w), \psi_s(w)$ is the solution pair of (28).

Proof. Substituting (27) into the subsystem (11), the corresponding closed-loop system is

$$\begin{aligned} \dot{x}_s &= (A_o + B_o K)x_s + B_o \psi_s(w) - B_o K \pi_s(w) + \Phi(y), \\ e_s &= C_o^\top x_s - Q w, \end{aligned} \tag{29}$$

where we have utilized the relation $x_s \equiv \hat{x}_s$. Then, introducing the coordinate change $\tilde{x}_s := x_s - \pi_s(w)$, and bearing (29) and the choice of $L(w)$, observe that, in the new coordinates,

$$\dot{\tilde{x}}_s = (A_o + B_o K)\tilde{x}_s + \Phi(C_o^\top x) - \Phi(C_o^\top \pi_s(w)). \tag{30}$$

Define the candidate Lyapunov function $V_2 = \lambda_2 V_1 + \tilde{x}_s^\top P_s \tilde{x}_s$, where P_s is the positive definite and symmetric solution of the Lyapunov equation $(A_o + B_o K)^\top P_s + P_s(A_o + B_o K) = -\lambda_1 \mathbf{I}_n$ with λ_1 and λ_2 to be selected later.

The following upper bound on the maximum solution P of the Lyapunov equation $X^\top P + P X = -Q$, for X Hurwitz and $Q \succ 0$ (see [21]): $\sigma_{\max}(P) \le \frac{1}{2}\sigma_{\max}(-Q X^{-1})$, similarily, one obtains the upper bound on the maximum solution P_s in question, i.e.,

$$\sigma_{\max}(P_s) \le \frac{\lambda_1}{2}\sigma_{\max}(-(A_o + B_o K)^{-1}).$$

Due to the fact that all the eigenvalues of matrix $-(A_o + B_o K)$ are positive, one derives

$$\sigma_{\max}(-(A_o + B_o K)) = \frac{1}{\underline{\sigma}} \tag{31}$$

where we denote $\underline{\sigma} := \sigma_{\min}(-(A_o + B_o K)) > 0$, then it holds

$$\sigma_{\max}(P_s) \leq \frac{\lambda_1}{2\underline{\sigma}} \tag{32}$$

which will be utilized later.

Thanks to the *locally lispchtiz* property of nonlinear term $\Phi(.)$, we have

$$\|\Phi(C_o^\top x) - \Phi(C_o^\top x_s)\| \leq k_1 \|C_o^\top x_p\|,$$
$$\|\Phi(C_o^\top x_s) - \Phi(C_o^\top \pi_s(w))\| \leq k_2 \|C_o^\top \tilde{x}_s\|,$$

with k_1, k_2 being positive constants, then the derivative of V_2 is

$$\begin{aligned}\dot{V}_2 &= \lambda_2 \dot{V}_1 - \lambda_1 \|\tilde{x}_s\|^2 \\ &\quad + 2\tilde{x}_s^\top P_s(\Phi(C_o^\top x) - \Phi(C_o^\top \pi_s(w)) \pm \Phi(C_o^\top x_s)) \\ &\leq -\lambda_2 \sigma_{\min}(Q_z)\|\mathbf{z}\|^2 - \lambda_1\|\tilde{x}_s\|^2 + k_3\|\tilde{x}_s\|^2 + k_4\|\tilde{x}_s\|\|x_p\| \\ &\leq -(\lambda_2\sigma_{\min}(Q_z) - \frac{k_4}{2})\|\mathbf{z}\|^2 - (\lambda_1 - k_3 - \frac{k_4}{2})\|\tilde{x}_s\|^2\end{aligned}$$

where we repose upon the fact that $\|x_p\| \leq \|\mathbf{z}\|$ and $k_3 := 2k_2\|P_s\|\|C_o^\top\|$, and $k_4 := 2k_1\|P_s\|\|C_o^\top\|$.

Now, if designing K such that the minimum eigenvalue of $-(A_o + B_o K)$ fulfills the inequality

$$\underline{\sigma} \geq \frac{2k_2 + k_1}{2}\|C_o^\top\|,$$

from (32), one derives

$$k_3 + \frac{k_4}{2} \leq (2k_2 + k_1)\|C_o^\top\|\frac{\lambda_1}{2\underline{\sigma}} \leq \lambda_1,$$

and chooses a sufficiently large λ_2 to satisfy

$$\lambda_2 > \frac{k_4}{2\ \sigma_{\min}(Q_z)}.$$

It can be easily deduce that $\lim_{t\to\infty} \tilde{x}_s = 0$. Then, $e_s(t) = C_o^\top \tilde{x}_s(t) + (C_o^\top \Pi_s - Q)w(t)$. Therefore, utilizing the second equation of (29), one derives $\lim_{t\to\infty} e_s = 0$. □

IV. Numerical Example

In this section, we will compare our proposed control method with two internal model-based methods in [7] and [12] to show the robustness with respect to plant parameter uncertainties.

Consider the nonlinear uncertain system in the form of

$$\begin{aligned}\dot{x}_1 &= 1.6\mu_1 x_1 + (1.6\mu_2 + 1)x_2 + 1.6(1+\mu_3)u + \frac{x_1^2}{1+x_1^2} \\ \dot{x}_2 &= \mu_1 x_1 + \mu_2 x_2 + (1+\mu_3)u + d\end{aligned} \tag{33}$$

with the measurable output $y = x_1$ to be regulated towards the tracking signal $y_d = 0.5$, and the disturbance d to be $d(t) = 2\sin(2t)$. The parameter vector $\mu = \begin{pmatrix}\mu_1 & \mu_2 & \mu_3\end{pmatrix}^\top$ is assumed to range over $\mu \in \{0 \leq \mu_1 \leq 5, 0 \leq \mu_2 \leq 2, -0.8 \leq \mu_3 \leq 1\}$.

The initial condition of the system ranges over a given compact set $\{x \in \mathbb{R}^2 : |x_i(0)| \leq 30, i = 1, 2\}$. In what follows, we will follow the procedures of designing the proposed controller under the framework of additive state decomposition.

The system is first decomposed into the primary system (10) and secondary system (11). For the primary system, we design the first regulator u_p in (15) by modeling

$$\begin{aligned}\dot{\eta} &= \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -4 & 0\end{pmatrix}\eta + \begin{pmatrix}0\\0\\1\end{pmatrix}y_p, \\ \dot{\xi} &= \begin{pmatrix}-1 & -1 & -1\end{pmatrix}\eta - 2\xi - 10y_p, \\ \nu_p &= \begin{pmatrix}-1 & -3 & -6\end{pmatrix}\eta - 10y_p,\end{aligned}$$

such that the eigenvalues of A'_{cl} in Lemma III.1 are placed at the left half plane and select the tunning parameter Γ in the adaptive law (21) as $\Gamma = \mathbf{I}_3$. As for the secondary system (11), we design the second regulator u_s by computing the solution pair of the nonlinear regulator equation as $\psi_s(w) = 0, \quad \pi_s(w) = \begin{pmatrix}w_1 & -\frac{w_1^2}{1+w_1^2}\end{pmatrix}^\top$, and select the control gain vector K such that the eigenvalues of $A_o - B_o K$ are assigned at $-1, -0.8$. Now, the controller u for system (33) is proposed as $u = u_p + u_s$.

The state x of the plant (33) is initialized as $x(0) = \begin{pmatrix}-10 & 10\end{pmatrix}^\top$ and the initial conditions of dynamics of the regulator η, ξ in (18), and adaptive law $\hat{H}$ in (21) are set to zero.

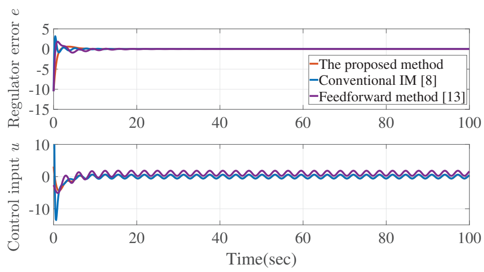

Fig. 1. Time history of input $u(t)$ and regulator error $e(t)$ when $\vartheta(\mu) = \begin{pmatrix}0 & 0\end{pmatrix}$ and $\Delta(\mu) = 0$.

To illustrate the robustness of our proposed method with respect to parameter uncertainties, we compare the results with the conventional internal model-based method in [7] and the recently developed robust internal model-based controller with the forwarding technique in [12]. The controller in [7] is designed in the form of

$$\begin{aligned}\dot{\xi} &= \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -4 & 0\end{pmatrix}\xi - 5\begin{pmatrix}3\\2\\1\end{pmatrix}e \\ u &= \begin{pmatrix}1 & 0 & 0\end{pmatrix}\xi - 5e\end{aligned}$$

with the initial condition of ξ set to be zero. Then, the controller in [12] is designed in the form of $u = \alpha(x) + \theta(x, \xi)$, in which the pre-stabilizer $\alpha(x)$ is chosen to be $\alpha(x) = -2x_1 + 0.2x_2$ and the forward stabilizer $\theta(x, \xi)$ is designed as $\theta(x, \xi) = -0.0435x_1 - 0.1222x_2 - 0.15\xi_1 + 0.093\xi_2 - 0.129\xi_3$. The internal model ξ has the form of

$$\dot{\xi} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 2\\ 0 & -2 & 0\end{pmatrix}\xi + \begin{pmatrix}1\\0\\1\end{pmatrix}e$$

with zero initial condition.

The simulation results are shown in Fig. 1-3. Fig. 1 presents the performance of the regulated error e and control input u under the absence of parameter uncertainty. It is worth noting that the significant difference in amplitude scales between the proposed method and the approaches presented in [8] and [13]. Figs. 2-3 demonstrate the robustness with parameter uncertainties by comparing three regulation performances for two different values of the unknown parameter set μ. As depicted in Figs. 2-3, the

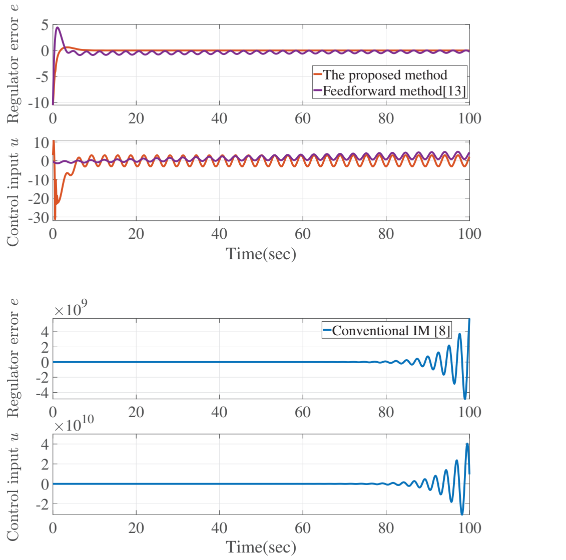

Fig. 2. Time history of input $u(t)$ and output $y(t)$ when $\vartheta(\mu) = \begin{pmatrix}0 & 0\end{pmatrix}$ and $\Delta(\mu) = -0.8$.

proposed controller u drives the output $y(t) \to 0.5$ as $t \to \infty$ in our method. However, Fig. 2 reveals that in the presence of large uncertainty in the input matrix, the conventional internal model methods exhibit a rapid divergence, undermining their effectiveness. Similarly, Fig. 3 highlights that when significant uncertainty is present in the system matrix as well, both the methods proposed in [7] and [12] experience immediate divergence. The comparison of three different regulation performances in Figs. 1-3 shows a noticeable improvement in the robustness achieved by the proposed method, with respect to large parameter uncertainties.

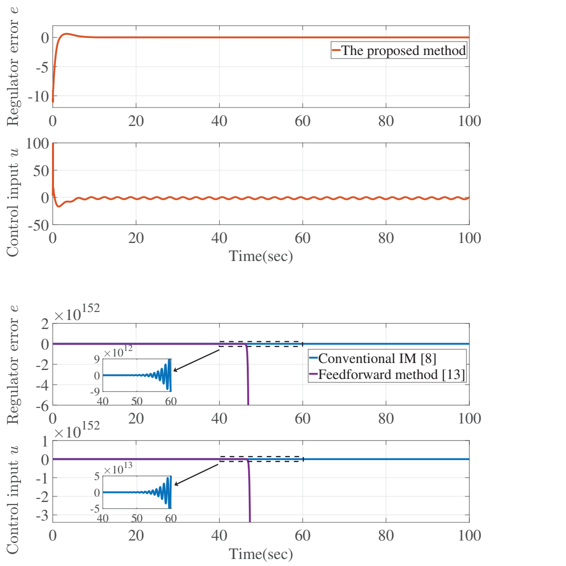

Fig. 3. Time history of input $u(t)$ and output $y(t)$ when $\vartheta(\mu) = \begin{pmatrix}5 & 2\end{pmatrix}$ and $\Delta(\mu) = -0.8$..

V. Conclusions

In this paper, we propose a novel internal model-based regulator under the framework of *additive state decomposition* for nonlinear MIMO systems. By decomposition, we design the internal model-based regulators separately for two subsystems, which simplifies the design and increases the flexibility of the designed controller. The extension to nonminimum phase system will be our main task, which is an intriguing challenge.

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