

THE BIRATIONAL GEOMETRY OF K-MODULI SPACES

JACOB KELLER

ABSTRACT. For C a smooth curve, the moduli space $U_C(2, \xi)$ of semistable vector bundles of rank two and determinant ξ is a Fano variety. We show that $U_C(2, \xi)$ is K-stable for a general curve $C \in \overline{\mathcal{M}}_g$. As a consequence, there are irreducible components of the moduli space of K-stable Fano varieties that are birational to $\overline{\mathcal{M}}_g$. In particular these components are of general type for $g \geq 22$.

CONTENTS

1. Introduction	1
2. Conformal Blocks and Degenerations of Moduli Spaces of Vector Bundles	3
3. Hecke Transformations	9
4. K-semistability of moduli spaces of vector bundles	20
5. The Luna Slice Theorem and GIT Stability	23
6. The Automorphism Group of the Toric Degenerations	25
7. Deformations of the Toric Varieties and Their Automorphisms	27
8. Proof of the Main Theorem	30
9. Open Questions	32
References	33

1. INTRODUCTION

The construction of moduli spaces parametrizing algebraic varieties is a central topic in algebraic geometry. One cannot expect to have reasonable moduli spaces for all varieties. Rather, attention should be restricted to varieties which are stable in some sense. Recently, the notion of K-stability has emerged as a powerful tool to construct moduli spaces of higher-dimensional algebraic varieties. All smooth canonically polarized varieties, and more generally, KSBA stable pairs are known to be K-stable, and there exist projective moduli spaces for these varieties ([29] and many others). On the other hand, not every smooth Fano variety is K-stable, and a lot of the research on K-stability centers on finding examples of K-stable Fano varieties.

Historically, it has been challenging to construct moduli spaces for Fano varieties since automorphism groups of Fano varieties are often positive-dimensional and non-reductive, in contrast to the general type case. However, recent advances in the theory of K-stability have allowed for the construction of projective moduli spaces that parameterize K-polystable Fano varieties ([35] finished the construction, but the proof is spread across many papers including but not limited to [49], [15], [2],