An Exploration of Chaotic Behavior in Double Pendula

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Contents

1 Part 1: Introduction

1.1 Research Objective

The objective of this independent study encompasses the following:

- Design a computer model and simulation of the double pendulum system.
- Demonstrate the chaotic behavior exhibited a double pendulum using said simulation.
- Explore various methods of demonstrating characteristic chaotic behavior.

1.2 Review of Literature

Chaos theory

Chaos theory is quite frankly the branch of mathematics in which we try to understand the behaviors of chaotic systems. Oddly enough, the precise definition of a chaotic system, which possesses its characteristically chaotic behavior, has been the subject of considerable debate; the most widely accepted proposition defines them as dynamical systems that are highly sensitive to slight perturbations of initial conditions.

Double Pendulum

A pendulum is a weight hung from a fixed point. A double pendulum, on the other hand, is a dynamical system composed of two pendulums attached end to end. A simple pendulum exhibits relatively the same, predictable motion for different initial conditions, with the only real factor affecting its period being the length of the pendulum. But double pendula, despite simply being two conjoined simple pendula, have been said to exhibit chaotic behavior because any slight perturbation in initial conditions will be arbitrarily magnified over time and result in drastically different motions.

This characteristic of the double pendulum system means that it would be nearly impossible, practically speaking, to recreate the exact same situation twice. In other words, over time, two seemingly identical setups with the most minute difference in initial conditions will result in a magnified different end result. Hence, it would be concerningly frustrating to collect reproducible, meaningful data. Unless, of course, I use idealized computer models. Hence, double pendulum models and simulations were designed to study the chaotic behavior of double pendulums, which guarantees that we can predict the trajectory of double pendula.

2 Part 2: Designing the Simulation

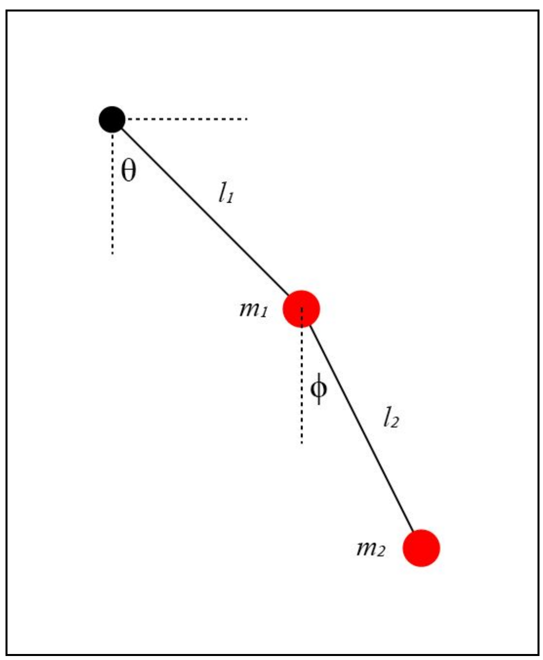
2.1 Assumptions

The assumptions when designing the simulation include the following:

- Massless strings
- No friction from air resistance or axles
- Point masses for the ends of the pendula

2.2 Developing the Mathematical Model

System Diagram



Wolfram Mathematica Code

Kinetic energy

```
ke = 1/2 m1 (L1 θ'[t])^2 + 1/2 m2 ((L2 (ϕ'[t]) + L1 θ'[t] Cos[ϕ[t] - θ[t]])^2 + (L1 θ'[t] Sin[ϕ[t] - θ[t]])^2);
```

Potential Energy

```
pe = -L1 m1 g Cos[θ[t]] - (L1 Cos[θ[t]] + L2 Cos[ϕ[t]]) m2 g;
```

Differential for θ

```
eq1 = D[D[ke - pe, θ'[t]], t] - D[ke - pe, θ[t]];
sol1 = First@Solve[eq1 == 0, θ''[t]];
```

Differential for ϕ

```
eq2 = D[D[ke - pe, ϕ'[t]], t] - D[ke - pe, ϕ[t]];
sol1 = First@Solve[eq2 == 0, ϕ''[t]];
```

Defining parameters for the simulation

```
pars = {L1 → 1, L2 → 1, m1 → 1, m2 → 1, g → 9.8};(*Parameters*)
ic = {θ[0] == 0, θ'[0] == 0, ϕ[0] == 101 π/90, ϕ'[0] == 0}; (*Initial Conditions*)
dur = 40(*Simulation Duration*)
```

Solving the differentials

```
sol = First@Solve[{eq1 == 0, eq2 == 0}, {θ''[t], ϕ''[t]}]; (*Decoupling the differentials above*)
numericalSolution = First@NDSolve[Flatten[{eq1 == 0, eq2 == 0, ic}] /. pars, {θ[t], ϕ[t]}, {t, 0, dur}];
(*Numerical solution for θ(t) and ϕ(t)*)

θn[t_] := Evaluate[θ[t] /. numericalSolution];
x1[t_] := Values[pars][[1]] * Sin[θn[t]]
y1[t_] := -Values[pars][[1]] * Cos[θn[t]]

ϕn[t_] := Evaluate[ϕ[t] /. numericalSolution];
x2[t_] := Values[pars][[1]] * Sin[θn[t]] + Values[pars][[2]] * Sin[ϕn[t]]
y2[t_] := -Values[pars][[1]] * Cos[θn[t]] - Values[pars][[2]] * Cos[ϕn[t]]
```

3 Part 3: Findings and Conclusions

3.1 Data Collection

Simulation Parameters

- L_1 and L_2 have length of 1 meter
- m_1 and m_2 have mass of 1 kilogram
- Initial velocities of m_1 and m_2 were 0 m/s

Data Collection Procedure

- Initial θ and ϕ were both varied by 2-degree intervals. $\theta : [0, \pi]$, $\phi : [0, 2\pi]$
- The angles θ and ϕ were plotted in relation to time over the course of 40 seconds.

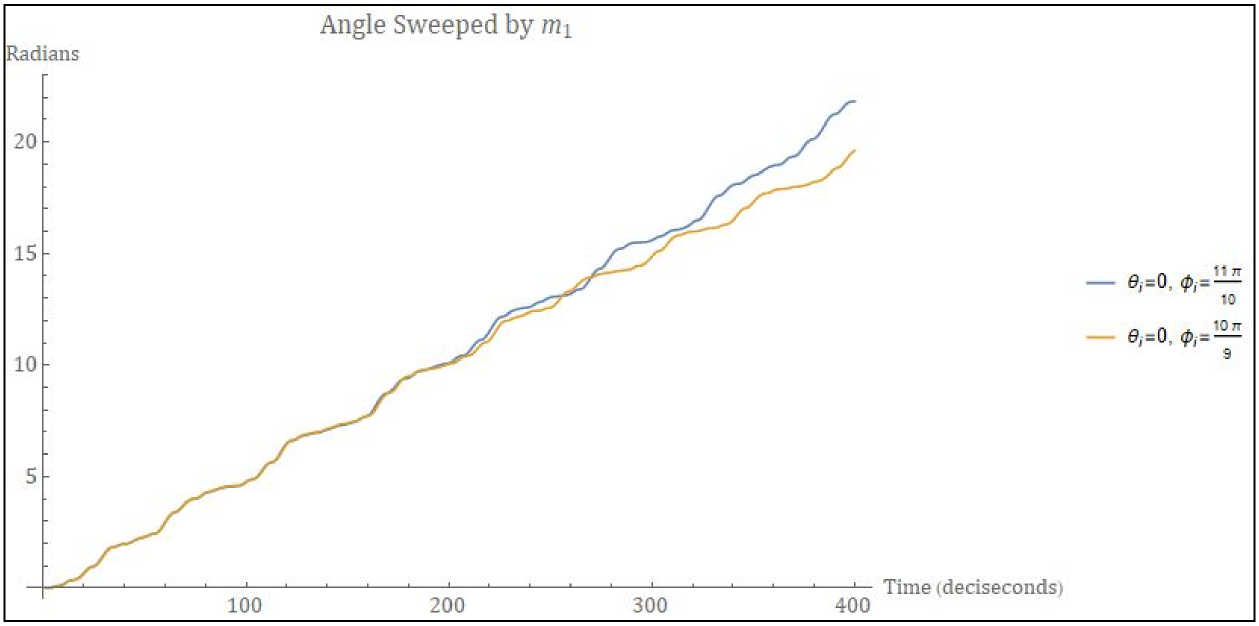
3.2 How can Chaotic Behavior be Demonstrated?

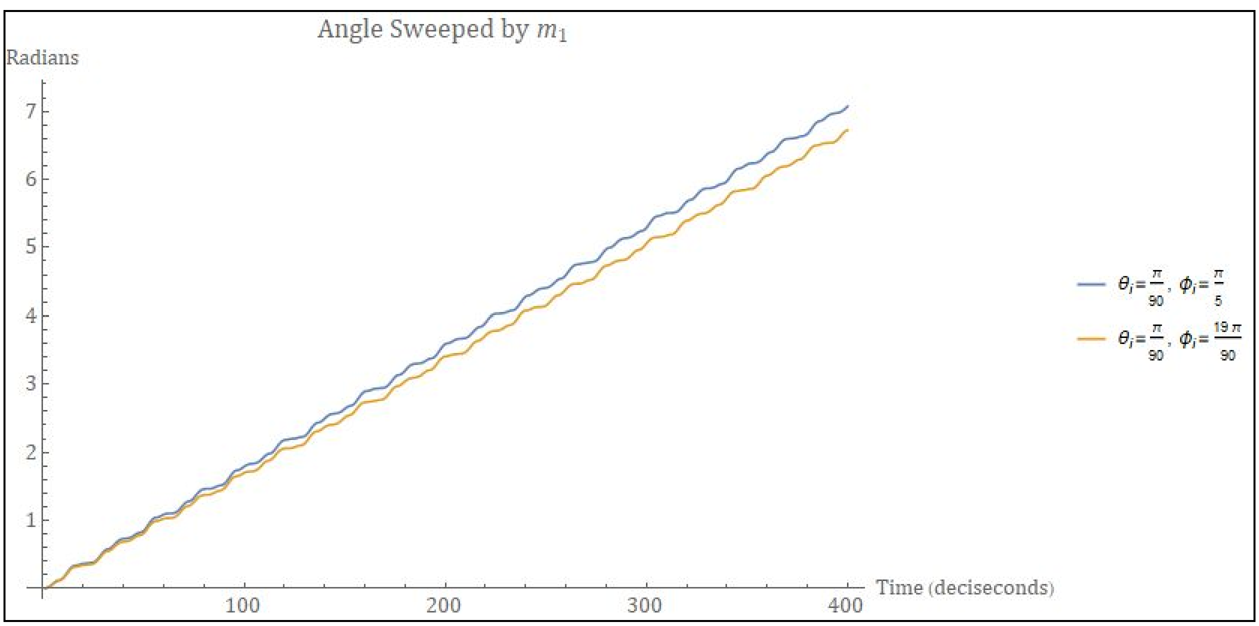
In the study of chaos theory, it's a well-accepted idea that chaotic behavior cannot be proven. Rather, chaotic behavior can be demonstrated. One method of doing so is known as the 'iterative process.' Iteration, as utilized in computer science, is the method of repeating a process using each previous result as the input for the subsequent iterations. So, the question remains: how can this iterative process be implemented to demonstrate the chaotic behavior of double pendula? Essentially, the simulation itself utilizes the iterative process, feeding in the positions, velocities, and accelerations of the masses in the antecedent time-frame to compute the movement and trajectories of the masses in the subsequent time-frames.

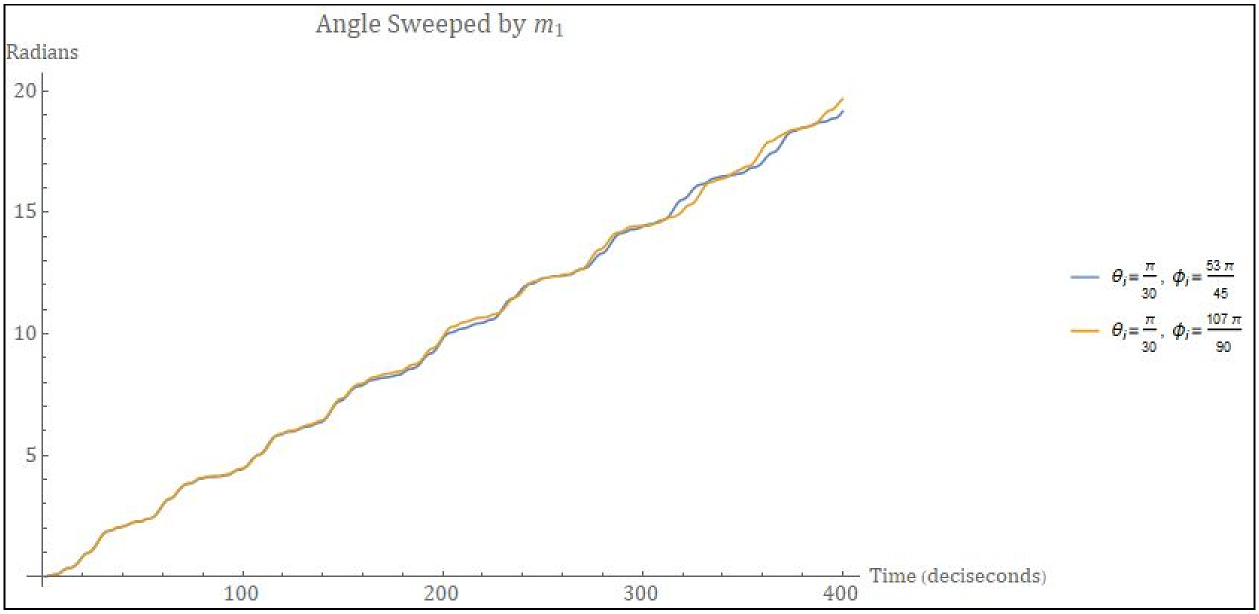
Using the data collected, I was able to find the cumulative sum of the angle swept by m_1 and plotted this cumulative sum for two different distinct but nearly identical initial conditions over the course of the simulation duration.

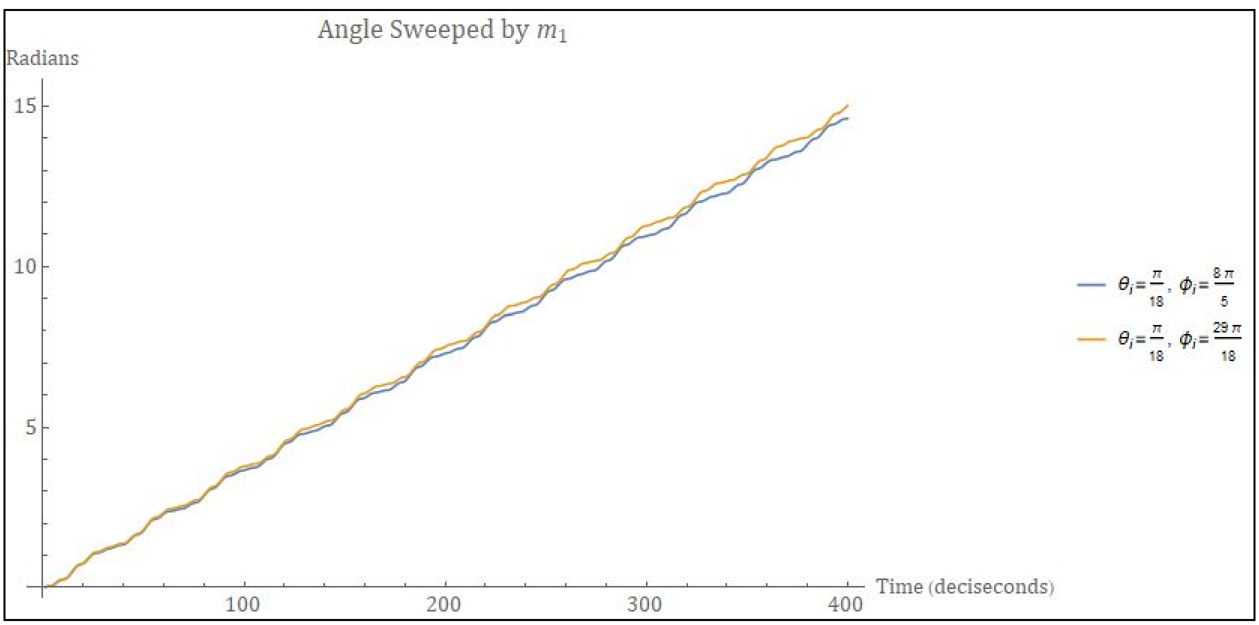
3.3 Results

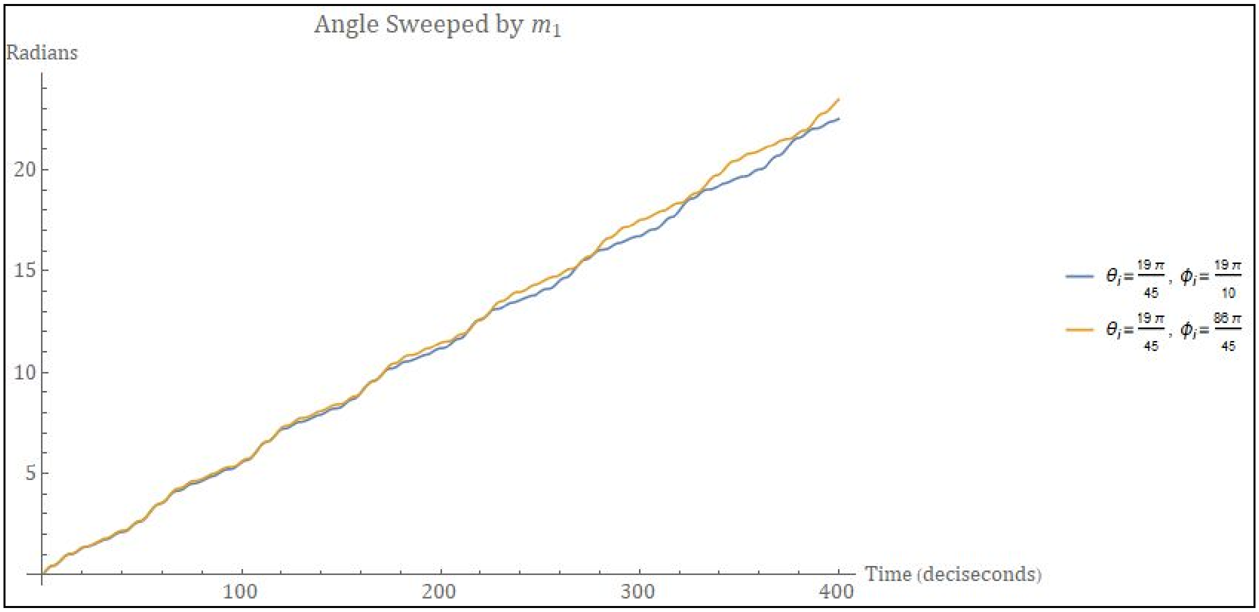
A total of 16,471 distinct initial conditions were tested, and several of the results are excerpted below.






Angle Sweeped by m_1
Radians
20
15
10
5
100
200
300
400
Time (deciseconds)
$\theta_i = \frac{\pi}{30}$, $\phi_i = \frac{53\pi}{45}$
$\theta_i = \frac{\pi}{30}$, $\phi_i = \frac{107\pi}{90}$


Angle Sweeped by m_1
Radians
15
10
5
100
200
300
400
Time (deciseconds)
$\theta_i = \frac{\pi}{18}$, $\phi_i = \frac{8\pi}{5}$
$\theta_i = \frac{\pi}{18}$, $\phi_i = \frac{29\pi}{18}$



3.4 Conclusion

Based on the multitudinous data that was collected and analyzed, chaotic behavior of double pendula systems are characteristic of their sensitivity to initial conditions. As seen in the previously depicted graphs representing the angle swept by m_1, despite having nearly identical conditions, as shown by very closely overlapping lines for earlier times, the two different initial conditions will sooner or later result in drastic differences later on. Note that only a small portion of the data collected was shown above for the sake of clarity and conservation of space. Nonetheless, observing each of the graphs gives us valuable insight into the behaviors of chaotic systems. The graphs and the fact that the curves diverge and split over time, demonstrate and solidifies the quintessential characteristic of chaotic behavior: sensitivity to initial conditions.